EXERCISES 11.3

Determining Convergence or Divergence

Which of the series in Exercises 1–30 converge, and which diverge? Give reasons for your answers. (When you check an answer, remember that there may be more than one way to determine the series' convergence or divergence.)

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{10^n}$

2. $\displaystyle\sum_{n=1}^{\infty} e^{-n}$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+1}$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{5}{n+1}$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{3}{\sqrt{n}}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{-2}{n\sqrt{n}}$

7. $\displaystyle\sum_{n=1}^{\infty} -\frac{1}{8^n}$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{-8}{n}$

9. $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n}$

10. $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{\sqrt{n}}$

11. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{3^n}$

12. $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{4^n+3}$

13. $\displaystyle\sum_{n=0}^{\infty} \frac{-2}{n+1}$

14. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n-1}$

15. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n+1}$

16. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\left(\sqrt{n}+1\right)}$

17. $\displaystyle\sum_{n=2}^{\infty} \frac{\sqrt{n}}{\ln n}$

18. $\displaystyle\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n$

19. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(\ln 2)^n}$

20. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(\ln 3)^n}$

21. $\displaystyle\sum_{n=3}^{\infty} \frac{(1/n)}{(\ln n)\sqrt{\ln^2 n - 1}}$

22. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(1+\ln^2 n)}$

Exercises

23. $\sum_{n=1}^{\infty} n \sin \frac{1}{n}$

24. $\sum_{n=1}^{\infty} n \tan \frac{1}{n}$

25. $\sum_{n=1}^{\infty} \frac{e^n}{1 + e^{2n}}$

26. $\sum_{n=1}^{\infty} \frac{2}{1 + e^n}$

27. $\sum_{n=1}^{\infty} \frac{8 \tan^{-1} n}{1 + n^2}$

28. $\sum_{n=1}^{\infty} \frac{n}{n^2 + 1}$

29. $\sum_{n=1}^{\infty} \operatorname{sech} n$

30. $\sum_{n=1}^{\infty} \operatorname{sech}^2 n$

Theory and Examples

For what values of a, if any, do the series in Exercises 31 and 32 converge?

31. $\sum_{n=1}^{\infty} \left(\frac{a}{n + 2} - \frac{1}{n + 4} \right)$

32. $\sum_{n=3}^{\infty} \left(\frac{1}{n - 1} - \frac{2a}{n + 1} \right)$

33. a. Draw illustrations like those in Figures 11.7 and 11.8 to show that the partial sums of the harmonic series satisfy the inequalities

$$\ln (n + 1) = \int_1^{n+1} \frac{1}{x}\, dx \leq 1 + \frac{1}{2} + \cdots + \frac{1}{n}$$
$$\leq 1 + \int_1^{n} \frac{1}{x}\, dx = 1 + \ln n.$$

T **b.** There is absolutely no empirical evidence for the divergence of the harmonic series even though we know it diverges. The partial sums just grow too slowly. To see what we mean, suppose you had started with $s_1 = 1$ the day the universe was formed, 13 billion years ago, and added a new term every *second*. About how large would the partial sum s_n be today, assuming a 365-day year?

34. Are there any values of x for which $\sum_{n=1}^{\infty}(1/(nx))$ converges? Give reasons for your answer.

35. Is it true that if $\sum_{n=1}^{\infty} a_n$ is a divergent series of positive numbers then there is also a divergent series $\sum_{n=1}^{\infty} b_n$ of positive numbers with $b_n < a_n$ for every n? Is there a "smallest" divergent series of positive numbers? Give reasons for your answers.

36. (*Continuation of Exercise 35.*) Is there a "largest" convergent series of positive numbers? Explain.

37. The Cauchy condensation test The Cauchy condensation test says: Let $\{a_n\}$ be a nonincreasing sequence ($a_n \geq a_{n+1}$ for all n) of positive terms that converges to 0. Then $\sum a_n$ converges if and only if $\sum 2^n a_{2^n}$ converges. For example, $\sum(1/n)$ diverges because $\sum 2^n \cdot (1/2^n) = \sum 1$ diverges. Show why the test works.

38. Use the Cauchy condensation test from Exercise 37 to show that

a. $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges;

b. $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

39. Logarithmic *p*-series

a. Show that

$$\int_2^{\infty} \frac{dx}{x(\ln x)^p} \quad (p \text{ a positive constant})$$

converges if and only if $p > 1$.

b. What implications does the fact in part (a) have for the convergence of the series

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}\, ?$$

Give reasons for your answer.

40. (*Continuation of Exercise 39.*) Use the result in Exercise 39 to determine which of the following series converge and which diverge. Support your answer in each case.

a. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)}$

b. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{1.01}}$

c. $\sum_{n=2}^{\infty} \frac{1}{n \ln (n^3)}$

d. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$

41. Euler's constant Graphs like those in Figure 11.8 suggest that as n increases there is little change in the difference between the sum

$$1 + \frac{1}{2} + \cdots + \frac{1}{n}$$

and the integral

$$\ln n = \int_1^n \frac{1}{x}\, dx.$$

To explore this idea, carry out the following steps.

a. By taking $f(x) = 1/x$ in the proof of Theorem 9, show that

$$\ln (n + 1) \leq 1 + \frac{1}{2} + \cdots + \frac{1}{n} \leq 1 + \ln n$$

or

$$0 < \ln (n + 1) - \ln n \leq 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n \leq 1.$$

Thus, the sequence

$$a_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n$$

is bounded from below and from above.

b. Show that

$$\frac{1}{n + 1} < \int_n^{n+1} \frac{1}{x}\, dx = \ln (n + 1) - \ln n,$$

and use this result to show that the sequence $\{a_n\}$ in part (a) is decreasing.

Since a decreasing sequence that is bounded from below converges (Exercise 107 in Section 11.1), the numbers a_n defined in part (a) converge:

$$1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n \rightarrow \gamma.$$

The number γ, whose value is $0.5772\ldots$, is called *Euler's constant*. In contrast to other special numbers like π and e, no other expression with a simple law of formulation has ever been found for γ.

42. Use the integral test to show that

$$\sum_{n=0}^{\infty} e^{-n^2}$$

converges.