EXERCISES

(SEQUENCES AND SERIES)

Finding Terms of a Sequence

Each of Exercises 1–6 gives a formula for the nth term a_n of a sequence $\{a_n\}$. Find the values of a_1, a_2, a_3, and a_4.

1. $a_n = \dfrac{1-n}{n^2}$

2. $a_n = \dfrac{1}{n!}$

3. $a_n = \dfrac{(-1)^{n+1}}{2n-1}$

4. $a_n = 2 + (-1)^n$

5. $a_n = \dfrac{2^n}{2^{n+1}}$

6. $a_n = \dfrac{2^n - 1}{2^n}$

Solution:

Each of Exercises 7–12 gives the first term or two of a sequence along with a recursion formula for the remaining terms. Write out the first ten terms of the sequence.

7. $a_1 = 1, \quad a_{n+1} = a_n + (1/2^n)$

Solution:

7. $a_1 = 1$, $a_2 = 1 + \frac{1}{2} = \frac{3}{2}$, $a_3 = \frac{3}{2} + \frac{1}{2^2} = \frac{7}{4}$, $a_4 = \frac{7}{4} + \frac{1}{2^3} = \frac{15}{8}$, $a_5 = \frac{15}{8} + \frac{1}{2^4} = \frac{31}{16}$, $a_6 = \frac{63}{32}$, $a_7 = \frac{127}{64}$, $a_8 = \frac{255}{128}$, $a_9 = \frac{511}{256}$, $a_{10} = \frac{1023}{512}$

11. $a_1 = a_2 = 1, \quad a_{n+2} = a_{n+1} + a_n$

Solution: (Fibonachi Sequence)

11. $a_1 = 1$, $a_2 = 1$, $a_3 = 1 + 1 = 2$, $a_4 = 2 + 1 = 3$, $a_5 = 3 + 2 = 5$, $a_6 = 8$, $a_7 = 13$, $a_8 = 21$, $a_9 = 34$, $a_{10} = 55$

Finding a Sequence's Formula

In Exercises 13–22, find a formula for the nth term of the sequence.

16. The sequence $1, -\frac{1}{4}, \frac{1}{9}, -\frac{1}{16}, \frac{1}{25}, \ldots$

Reciprocals of squares of the positive integers, with alternating signs

Solution:

$$16.\quad a_n = \frac{(-1)^{n+1}}{n^2}, \; n = 1, 2, \ldots$$

19. The sequence $1, 5, 9, 13, 17, \ldots$

Every other odd positive integer

Solution:

$$19.\quad a_n = 4n - 3, \; n = 1, 2, \ldots$$

Finding Limits

Which of the sequences $\{a_n\}$ in Exercises 23–84 converge, and which diverge? Find the limit of each convergent sequence.

23. $a_n = 2 + (0.1)^n$

Solution:

23. $\lim_{n \to \infty} 2 + (0.1)^n = 2 \Rightarrow$ converges

26. $a_n = \dfrac{2n + 1}{1 - 3\sqrt{n}}$

Solution:

26. $\lim_{n \to \infty} \frac{2n+1}{1-3\sqrt{n}} = \lim_{n \to \infty} \dfrac{2\sqrt{n} + \left(\frac{1}{\sqrt{n}}\right)}{\left(\frac{1}{\sqrt{n}} - 3\right)} = -\infty \Rightarrow$ diverges

42. $a_n = \dfrac{\sin^2 n}{2^n}$

Solution:

42. $\lim_{n \to \infty} \frac{\sin^2 n}{2^n} = 0$ because $0 \le \frac{\sin^2 n}{2^n} \le \frac{1}{2^n}$

$\Rightarrow$ converges by the Sandwich Theorem for sequences

45. $a_n = \dfrac{\ln(n+1)}{\sqrt{n}}$

Solution:

$$45.\ \lim_{n \to \infty} \frac{\ln(n+1)}{\sqrt{n}} = \lim_{n \to \infty} \frac{\left(\frac{1}{n+1}\right)}{\left(\frac{1}{2\sqrt{n}}\right)}$$

$$= \lim_{n \to \infty} \frac{2\sqrt{n}}{n+1} = \lim_{n \to \infty} \frac{\left(\frac{2}{\sqrt{n}}\right)}{1+\left(\frac{1}{n}\right)} = 0 \ \Rightarrow \text{ converges}$$

49. $a_n = \left(1 + \dfrac{7}{n}\right)^n$

Solution:

$$49.\ \lim_{n \to \infty} \left(1 + \tfrac{7}{n}\right)^n = e^7 \ \Rightarrow \text{ converges}$$

58. $a_n = \sqrt[n]{3^{2n+1}}$

Solution:

$$58.\ \lim_{n \to \infty} \sqrt[n]{3^{2n+1}} = \lim_{n \to \infty} 3^{2+(1/n)}$$

$$= \lim_{n \to \infty} 3^2 \cdot 3^{1/n} = 9 \cdot 1 = 9 \ \Rightarrow \text{ converges}$$

65. $a_n = \left(\dfrac{3n+1}{3n-1}\right)^n$

Solution:

65. $\lim\limits_{n \to \infty} \left(\frac{3n+1}{3n-1}\right)^n = \lim\limits_{n \to \infty} \exp\left(n \ln\left(\frac{3n+1}{3n-1}\right)\right)$

$= \lim\limits_{n \to \infty} \exp\left(\dfrac{\ln(3n+1) - \ln(3n-1)}{\frac{1}{n}}\right)$

$= \lim\limits_{n \to \infty} \exp\left(\dfrac{\frac{3}{3n+1} - \frac{3}{3n-1}}{\left(-\frac{1}{n^2}\right)}\right) = \lim\limits_{n \to \infty} \exp\left(\dfrac{6n^2}{(3n+1)(3n-1)}\right)$

$= \exp\left(\frac{6}{9}\right) = e^{2/3} \Rightarrow$ converges

In Exercises 121–124, experiment with a calculator to find a value of N that will make the inequality hold for all $n > N$. Assuming that the inequality is the one from the formal definition of the limit of a sequence, what sequence is being considered in each case and what is its limit?

121. $|\sqrt[n]{0.5} - 1| < 10^{-3}$ **122.** $|\sqrt[n]{n} - 1| < 10^{-3}$

Solution:

$$121.\quad \left|\sqrt[n]{0.5} - 1\right| < 10^{-3} \Rightarrow -\frac{1}{1000} < \left(\frac{1}{2}\right)^{1/n} - 1 < \frac{1}{1000}$$

$$\Rightarrow \left(\frac{999}{1000}\right)^n < \frac{1}{2} < \left(\frac{1001}{1000}\right)^n \Rightarrow n > \frac{\ln\left(\frac{1}{2}\right)}{\ln\left(\frac{999}{1000}\right)} \Rightarrow n > 692.8$$

$$\Rightarrow N = 692;\ a_n = \left(\frac{1}{2}\right)^{1/n} \text{ and } \lim_{n\to\infty} a_n = 1$$

Solution:

$$122.\quad \left|\sqrt[n]{n} - 1\right| < 10^{-3} \Rightarrow -\frac{1}{1000} < n^{1/n} - 1 < \frac{1}{1000}$$

$$\Rightarrow \left(\frac{999}{1000}\right)^n < n < \left(\frac{1001}{1000}\right)^n \Rightarrow n > 9123 \Rightarrow N = 9123;$$

$$a_n = \sqrt[n]{n} = n^{1/n} \text{ and } \lim_{n\to\infty} a_n = 1$$

Finding nth Partial Sums

In Exercises 1–6, find a formula for the nth partial sum of each series and use it to find the series' sum if the series converges.

1. $2 + \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \cdots + \frac{2}{3^{n-1}} + \cdots$

2. $\frac{9}{100} + \frac{9}{100^2} + \frac{9}{100^3} + \cdots + \frac{9}{100^n} + \cdots$

3. $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots + (-1)^{n-1}\frac{1}{2^{n-1}} + \cdots$

4. $1 - 2 + 4 - 8 + \cdots + (-1)^{n-1}\, 2^{n-1} + \cdots$

Solution:

$$1.\quad s_n = \frac{a\,(1 - r^n)}{(1 - r)} = \frac{2\left(1 - \left(\frac{1}{3}\right)^n\right)}{1 - \left(\frac{1}{3}\right)} \Rightarrow \lim_{n \to \infty} s_n = \frac{2}{1 - \left(\frac{1}{3}\right)} = 3$$

5. $\frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \cdots + \frac{1}{(n+1)(n+2)} + \cdots$

6. $\frac{5}{1 \cdot 2} + \frac{5}{2 \cdot 3} + \frac{5}{3 \cdot 4} + \cdots + \frac{5}{n(n+1)} + \cdots$

Solution:

$$5.\quad \frac{1}{(n+1)(n+2)} = \frac{1}{n+1} - \frac{1}{n+2}$$

$$\Rightarrow s_n = \left(\tfrac{1}{2} - \tfrac{1}{3}\right) + \left(\tfrac{1}{3} - \tfrac{1}{4}\right) + \ldots + \left(\tfrac{1}{n+1} - \tfrac{1}{n+2}\right) = \tfrac{1}{2} - \tfrac{1}{n+2}$$

$$\Rightarrow \lim_{n \to \infty} s_n = \tfrac{1}{2}$$

Series with Geometric Terms

In Exercises 7–14, write out the first few terms of each series to show how the series starts. Then find the sum of the series.

7. $\sum_{n=0}^{\infty} \frac{(-1)^n}{4^n}$ **8.** $\sum_{n=2}^{\infty} \frac{1}{4^n}$

Solution:

7. $1 - \frac{1}{4} + \frac{1}{16} - \frac{1}{64} + \ldots$, the sum of this geometric series is $\frac{1}{1-\left(-\frac{1}{4}\right)} = \frac{1}{1+\left(\frac{1}{4}\right)} = \frac{4}{5}$

8. $\frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \ldots$, the sum of this geometric series is $\frac{\left(\frac{1}{16}\right)}{1-\left(\frac{1}{4}\right)} = \frac{1}{12}$