SC42030 Control for high resolution imaging
Homework Assignment 1 : Controller design

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1 Evaluating H_2-norm of a system

The objective of this section is to compute the H_2-norm of the expression outlined in Equation 1.

$$x(k+1) = Ax(k) + Bw(k)$$
$$y(k) = Cx(k) \tag{1}$$

Where, A = state matrix, B = input matrix, w(k) = input, C = output matrix.
x is the state and y is the output.

If G(s) is the transfer function for the causal DT LTI expression presented in Equation 1 and G(k) its impulse response then H_2 norm of G is a measure of the energy of impulse response.

State response of an LTI system at time instant k to an initial state x(0) is given by the expression presented in Equation 2[3][Eq.9.3, Pg-294].

$$x(k) = A^k x(0) + \sum_{i=0}^{k-1} A^{k-i-1} Bw(i) \tag{2}$$

Using the expression in Equation 2 the impulse response of the LTI system, is presented in Equation 3.

$$y(k) = CA^k x(0) + \sum_{i=0}^{k-1} CA^{k-i-1} Bw(i)$$

For impulse response Only i=0 contributes to the result.
The zero initial state impulse response of a DT LTI is: (3)

$$y(k) = CA^{k-1} Bw(k)$$

Impulse response of the system is thus represented by:

$$G(k-1) = CA^{k-1} B$$

H_2 norm of the transfer function G is the L_2 norm of its impulse response, this is elaborated with the aid of Equation 4.

$$
\begin{aligned}
||G||_2 &= \sqrt{\sum_{k=0}^{\infty} tr(G(k)G(k)^T)} \\
&= \sqrt{\sum_{k=0}^{\infty} tr(CA^kBB^T(A^k)^TC^T)} \\
\text{Let, P be the controllability Gramian} &= \sum_{k=0}^{\infty} A^kBB^T(A^k)^T \\
\text{Thus, } ||G||_2 &= \sqrt{\sum_{k=0}^{\infty} tr(CPC^T)}
\end{aligned}
\tag{4}
$$

The solution to controllability Gramian P can be found using the solution to Lyapunov Equation 5.

$$
\begin{aligned}
&AP + PA^T + BB^T = 0 \\
&\text{or via} \\
&APA^T - P + BB^T = 0
\end{aligned}
\tag{5}
$$

Once P is obtained solution to H_2 norm follows from the final expression in Equation 4.

2 Effect of Strehl ratio on H_2-norm

When aiming at reducing the aberrations in a closed loop control strategy, if the variance of the phase error, σ_t^2, over the aperture is smaller than 2 radians then the Strehl ratio may be reduced by the Marechal approximation as shown in Equation 6[1][Eq-2.46, pg-35].

$$S \approx e^{-\sigma_t^2} \tag{6}$$

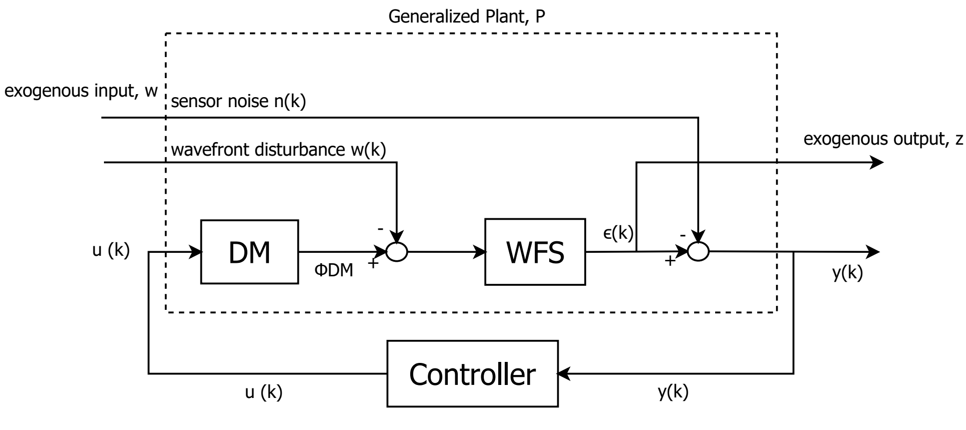

Figure 1: Generalised plant of the system under observation.

A generalised plant of the system is outlined with the aid of Figure 1. The objective is to demonstrate that maximising the Strehl ratio in Equation 6 is equivalent to minimising the H_2 norm of the transfer function from wavefront disturbance input w(k) to phase error ϵ (k). u(k) is the controller actions, which is input to the Deformable Mirror/actuator.

From Equation 6 one can observe that minimising the variance of phase error σ_t^2, leads to maximising the Strehl ratio. The task is to prove that minimising the H_2 norm of the transfer function between z and w(k) is equivalent to minimizing the variance of phase error σ_t^2 i.e. maximising the Strehl ratio, S.

For a generalised plant the transfer function N connecting the exogenous output z to exogenous input w is expressed as z = Nw. Where, z = $\in (k)$ and w = $w(k)$ in the above scenario. N is given by the linear fractional transformation (LFT) of plant P with controller K as the parameter. i.e. N $\triangleq F_l(P, K)$[2][Eq.3.102 pg.103].

The standard H_2 optimal control problem is to find a stabilizing controller K which mimizes the expression in Equation 7.

$$||F(s)||_2 = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} tr[F(j\omega)F(j\omega)^H]d\omega}; \; F \triangleq F_l(P, K) \tag{7}$$

If the general exogenous input w is white noise of unit intensity i.e. $E\{w(k)w(k)^T\} = I\delta(t-\tau)$ then the expected power of the phase error exogenous signal z = $\in (k)$ can be written as shown in Equation 8.

$$\begin{aligned}
&E\left\{\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} z(t)^T z(t)dt\right\} \\
&tr(E\ z(t)z(t)^H) = \frac{1}{2\pi}\int_{-\infty}^{\infty} tr[F(j\omega)F(j\omega)^H]d\omega \\
&\text{By Parseval's Theorem} \\
&||F||_2^2 = ||F_l(P, K)||_2^2
\end{aligned} \tag{8}$$

From the above expression it can be concluded that minimizing the H_2 norm, leads to minimizing the output power of the generalised system due to a unit intensity white noise input. This refers to minimizing the variance of z = $\in (k)$ and maximising the Strehl ratio[2][pg.371-372].

3 Estimating Controller matrix C and matrix Q

The objective of this exercise is to find proper relation between the controller matrix C $\in R^{ns\times na}$ and the matrix Q defined in the signal model in Equation 9.

$$E(k) = (I - QW)\phi(k) + Q\cdot n(k) \tag{9}$$

Using the model expressed in Figure 2 as guide, the expression for the signal model is outlined in Equation 10.

$$\begin{aligned}
E(k) &= \phi(k) - DCn(k) - DCWE(k) \\
E(k) &= (I + DCW)^{-1}(\phi(k) - DCn(k))
\end{aligned} \tag{10}$$

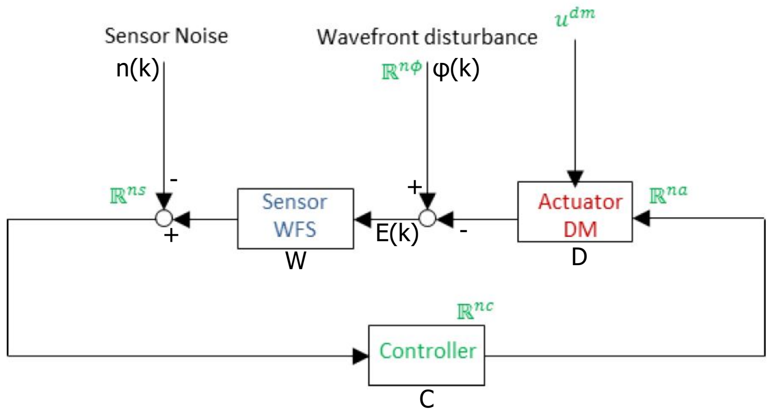

Figure 2: Overview of the system under observation, with designated signal nomenclature.

Comparing the expressions in Equations 9 and 10, the equality expressed in Equation 11 can be deduced.

$$\begin{aligned} Q &= -(I + DCW)^{-1}DC \\ (I - QW) &= (I + DCW)^{-1} \end{aligned} \tag{11}$$

The first expression in Equation 11 can be simplified to obtain Equation 12.

$$\begin{aligned} (I + DCW)Q &= -DC \\ (I + DCW)QW &= -DCW \\ QW &= -DCW(I + QW) \\ QW(I + QW)^{-1} &= -DCW \\ DCW &= -QW(I + QW)^{-1} \end{aligned} \tag{12}$$

The simplified expression in Equation 12 can be used obtain the expression for C. This is presented in Equation 13. Note that "matrix$^+$" indicates the pseudo inverse.

$$C = -D^+QW(I + QW)^{-1}W^+ \tag{13}$$

Now that the expression for C is derived, the next step involves deriving the expression for matrix Q, such that the cost function outlined in Equation 14 is minimized.

$$\begin{aligned} &\text{min} E\left[E(k)E(k)^T\right] = E\Big[[(I - QW)\phi(k) + Qn(k)] \cdot [(I - QW)\phi(k) + Qn(k)]^T\Big] \\ &E\left[(I - QW)\phi\phi^T(I - QW)^T + (I - QW)\phi(k)n(k)^TQ^T + Qn(k)\phi(k)^T(I - QW)^T + Qnn^TQ^T\right] \end{aligned} \tag{14}$$

Since, ϕ and n are uncorrelated, this means E[ϕn]=0. Let, E[$\phi\phi^T$] = P and E[nn^T]=L. The minimization expression is further simplified as outlined in Equation 15.

$E\left[E(k)E(k)^T\right] = (I - QW)P(I - QW)^T + QLQ^T$
Using the expression outlined in Equation 4.22 in [3][pg.114]
$E[...] = \begin{bmatrix} I & -Q \end{bmatrix} \cdot \begin{bmatrix} P & PW^T \\ WP & WPW^T + L \end{bmatrix} \cdot \begin{bmatrix} I \\ -Q^T \end{bmatrix}$
With the aid of Lemma 2.3 [3][pg.19], the central matrix can be expanded and E(...) be expressed as:
$\begin{bmatrix} P & PW^T \\ WP & WPW^T + L \end{bmatrix} = \begin{bmatrix} I & PW^T(WPW^T + L)^{-1} \\ 0 & I \end{bmatrix} \cdot \begin{bmatrix} P - PW^T(WPW^T + L)^{-1}WP & 0 \\ 0 & WPW^T + L \end{bmatrix} \cdots$
$\cdots \cdot \begin{bmatrix} I & 0 \\ (WPW^T + L)^{-1}WP & I \end{bmatrix}$
With the aid of completion of squares, E(...) is minimized when :
$$Q = PW^T(WPW^T + L)^{-1} \tag{15}$$

References

[1] MICHEL VERHAEGEN, PAULO POZZI, O. S. G. V. D. W. *Control for High Resolution Imaging*. Lecture notes for the course SC42030 TU Delft, 2017.

[2] SKOGESTAD, S., AND POSTLETHWAITE, I. *MULTIVARIABLE FEEDBACK CONTROL Analysis and design 2nd Edition*. JOHN WILEY SONS, 2001.

[3] VERHAEGEN, M., AND VERDULT, V. *Filtering and Subspace Identification : A least squares approach.* Cambridge University Press, 2007.