AE4878 - Mission Geometry and Orbit Design
Part 7 - Spherical geometry

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1 Why basic spherical geometry?

When it comes to designing the geometry of a satellite mission different co-ordinate systems are used. At first, the thought of not using a universal co-ordinate system for all mathematical problems might seem quite inconvenient. However, the aim of using different co-ordinate systems is to avoid complexities where-ever necessary; to provide insights (both mathematical and geometrical) that otherwise would not be observable. For example a 2D Cartesian co-ordinate is quite simple and provides an reliable overview into the equation of motions of a simple trolley problem. But, the same co-ordinate system cannot be used to provide proper insights into the equations of motions of a re-entry shuttle.

What are some advantages of using spherical geometry? It provides a relatively intuitive insight into the geometrical aspect of any satellite mission e.g. coverage data, eclipse situation etc. A quick overview of the spherical geometry is outlined with the aid of Figure1[2][Fig.6-10,pg.297].

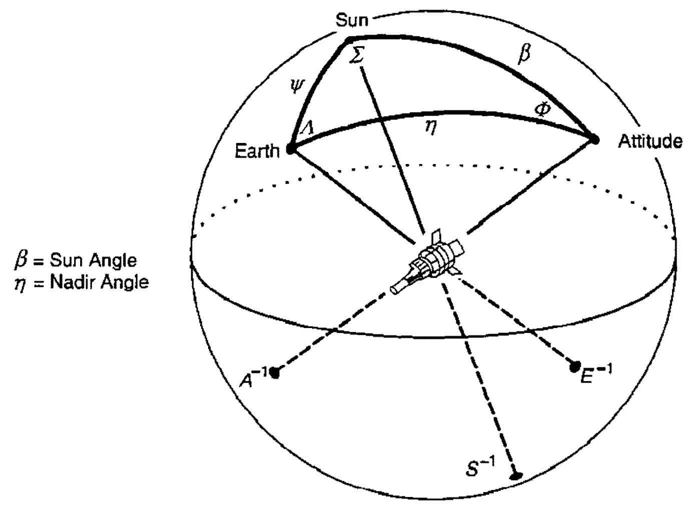

Fig. 6-10. The Spacecraft-Centered Celestial Sphere Specifies Direction in Space. The sides of the triangle are arc lengths. The angles of the triangle are rotation angles.

Figure 1: Overview of basic spherical geometry with unit vectors.

Three points indicate Sun, Earth and Attitude direction. Ψ, β and η indicate the arc lengths, which is the angular separation. And finally the rotation angles Σ, Λ and Φ indicate the angle between the planes[1][Slide21/40]. Azimuth and Elevation angles are indicated with the aid of Figure 2[2][Fig.6-14,pg.302]

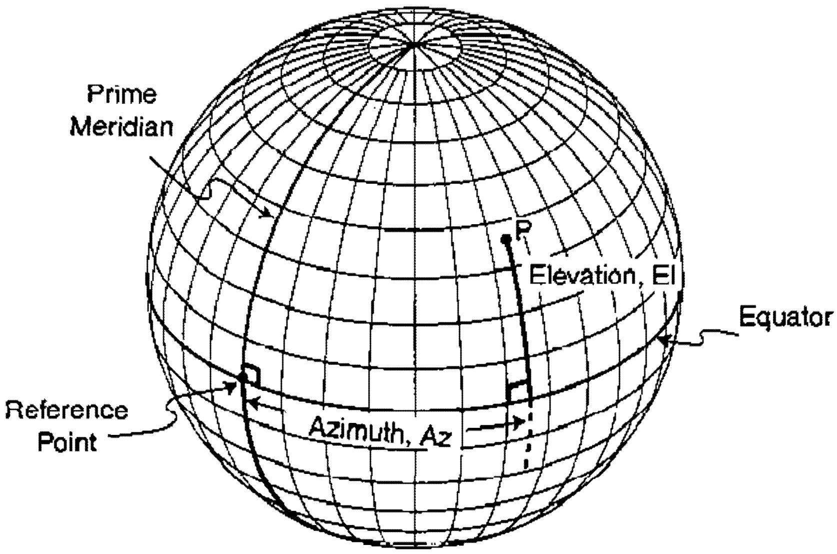

Figure 2: Overview of azimuth and elevation.

2 Problem Definition

The objective of this assignment is to compute angular distances, rotation angles (see Figure 1) and total area for spherical triangles given the azimuth and elevation angles of three points on the unit sphere (see Figure 2).

For given points P_i, i = 1, 2, 3 on a unit sphere with co-ordinates (α_i, δ_i) where α denotes the azimuth and δ denotes the elevation, the arc-length distance, θ_{12} between P_1 and P_2 is shown in Equation 1[2][A-1a,pg.770]. Similarly the angle between points P_1 and P_3, along with the angle between points P_2 and P_3 is also outlined in Equation 1.

$$\begin{aligned}
&cos(\theta_{12}) = cos(\theta_{21}) = sin(\delta_1) \cdot sin(\delta_2) + cos(\delta_1) \cdot cos(\delta_2) \cdot cos(\alpha_1 - \alpha_2) \\
&cos(\theta_{13}) = cos(\theta_{31}) = sin(\delta_1) \cdot sin(\delta_3) + cos(\delta_1) \cdot cos(\delta_3) \cdot cos(\alpha_1 - \alpha_3) \\
&cos(\theta_{23}) = cos(\theta_{32}) = sin(\delta_2) \cdot sin(\delta_3) + cos(\delta_2) \cdot cos(\delta_3) \cdot cos(\alpha_2 - \alpha_3) \\
&0 \leq \theta \leq 180^\circ
\end{aligned} \tag{1}$$

The rotation angle $\phi_1(P_1, P_2, P_3)$ from P_2 to P_3 around P_1, along with rotation angles ϕ_2, ϕ_3 is outlined with the aid of Equation 2[A-2a,pg.770].

$$\begin{aligned}
&\cos(\phi_1) = \frac{cos(\theta_{23}) - \cos(\theta_{21})\cos(\theta_{31})}{\sin(\theta_{21})\sin(\theta_{31})} \\
&\cos(\phi_2) = \frac{cos(\theta_{13}) - \cos(\theta_{12})\cos(\theta_{32})}{\sin(\theta_{12})\sin(\theta_{32})} \\
&\cos(\phi_3) = \frac{cos(\theta_{12}) - \cos(\theta_{13})\cos(\theta_{23})}{\sin(\theta_{13})\sin(\theta_{23})} \\
&0 \leq \Phi \leq 360^\circ
\end{aligned} \tag{2}$$

The inner and outer area covered by the spherical triangle(given inner rotation angles) can be estimated with the aid of Equation 3[2][Equation 6-3, pg.300]. "n" indicates the number of sides of any spherical polygon and $\sum$ is the sum of the rotation angles expressed in radians. $4\pi r^2$ is the area of a sphere. For a sphere of unit radius r = 1 and for a spherical triangle n=3.

Inner area of a spherical triangle is given by:

$$A = \sum -(n-2)\pi \quad [unit^2] \tag{3}$$

While the outer area of the spherical triangle is given by:

Outer Area : $4\pi r^2 - A \quad [unit^2]$

3 Results

Expressions outlined with the aid of Equations 1, 2 and 3 is scripted as shown in Section 4. The outcome of this is verified first. Verification results along with the solution to given three spherical points is tabulated with the aid of Table 1. The first shaded row, indicated the angular distances, rotation angles and area information of the spherical triangle. This can be verified against the benchmark data provided in Table 2[1][Slide 31/40]. The results are identical. However, to verify the inner and outer area one must refer to the second shaded row, which indicates an equilateral right triangle. For a unit sphere, the surface area is given by 4π[unit^2]. Area projected by an equilateral triangle is $\frac{1}{8}$th of the surface area i.e. $\frac{4\pi}{8} = 1.5708$ $[unit^2]$[1][Slide 26/40]. While the outer area is given by $\frac{7\cdot 4\pi}{8} = 4\pi\text{-}\frac{4\pi}{8} = 10.9956$ $[unit^2]$. Now that the results of angular distances, rotation angles and area are verified, the script can be used to obtain similar information for given data points. the results are tabulated in Table 1. For the last shaded row, outer area is asked for which is provided. The outer rotation angles can be obtain by subtracting the given angles from 360, i.e. [360-36.3693, 360-75.9897, 360-68.2365] = [323.6307°, 284.0103°, 291.7635°]. Since the upper limit for rotation angles is 360°.

Table 1: Estimated arc length, rotation angles along with inner and outer unit area.

	Angular Distances [deg]			**Rotation Angles [deg]**			**Area [$unit^2$]**	
Data [Azimuth, Elevation] [deg]	P_{12}	P_{13}	P_{23}	**Around** P_1	**Around** P_2	**Around** P_3	**Inner Area**	**Outer Area**
$P_1 = (10,15), P_2 = (45,70), P_3 = (110,32)$	59.0807	90.2917	51.6443	43.4153	118.7855	48.7544	0.5403	12.0261
$P_1 = (0,0), P_2 = (90,0), P_3 = (0,90)$	90	90	90	90	90	90	1.5708	10.9956
$P_1 = (0,0), P_2 = (30,0), P_3 = (0,90)$	30.0000	90	90	90	90	30.0000	0.5236	12.0428
$P_1 = (0,20), P_2 = (10,25), P_3 = (5,30)$	10.4996	10.9748	6.6816	36.3693	75.9897	68.2365	0.01039	12.5560
$P_1 = (5,0), P_2 = (30,30), P_3 = (5,90)$	38.2899	90	60.0000	36.2040	136.9969	25.0000	0.3177	12.2487
$P_1 = (0,-20), P_2 = (45,-20), P_3 = (0,90)$	42.1519	110	110	98.06340	98.06340	45.0000	1.06686	11.4995
$P_1 = (0,20), P_2 = (10,25), P_3 = (5,30)$	10.4996	10.9748	6.6816	36.3693	75.9897	68.2365	0.01039	12.5560

Table 2: Benchmark results from literature for angular distance and rotation angle algorithm verification.

	Angular Distances [deg]			**Rotation Angles [deg]**		
Data [Azimuth, Elevation] [deg]	P_{12}	P_{13}	P_{23}	**Around** P_1	**Around** P_2	**Around** P_3
$P_1 = (10,15), P_2 = (45,70), P_3 = (110,32)$	59.0807	90.2917	51.6443	43.4153	118.7855	48.7544

What can we conclude? Spherical geometry provides an intuitive overview in design of a satellite mission geometry. Information of three points on a spherical triangle (with azimuth,elevation information) can be used to generate rotation angles and arc lengths, which provide insight into area covered by the spherical triangle. The results can be generated with the aid of the script presented in Section 4.

4 Matlab Scipt

Following script is used to generate the above presented results.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%% AE4878 Mission Geometry and Orbit Design %%%%%%%%%%%
%%%% Assignment 7 - Week 4.3 - GEOMETRY 2 - Version 1 %%%%
%%%%% Author Info: Ali Nawaz; Student ID - 4276477 %%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% Data(Azimuth, Elevation) in radians
P1 = deg2rad([10 15; 0 0; 0 0; 0 20; 5 0; 0 -20; 0 20]); % [rad]
P2 = deg2rad([45 70; 90 0; 30 0; 10 25; 30 30; 45 -20; 10 25]); % [rad]
P3 = deg2rad([110 32; 0 90; 0 90; 5 30; 5 90; 0 90; 5 30]); % [rad]
```

```
% Arc length
theta12 = acos( sin(P1(:,2)).*sin(P2(:,2)) + cos(P1(:,2)).*cos(P2(:,2)).*cos(P1(:,1)-P2(:,1)) ...
    ); % [rad]
theta13 = acos( sin(P1(:,2)).*sin(P3(:,2)) + cos(P1(:,2)).*cos(P3(:,2)).*cos(P1(:,1)-P3(:,1)) ...
    ); % [rad]
theta23 = acos( sin(P2(:,2)).*sin(P3(:,2)) + cos(P2(:,2)).*cos(P3(:,2)).*cos(P2(:,1)-P3(:,1)) ...
    ); % [rad]
theta21 = theta12; % [rad]
theta31 = theta13; % [rad]
theta32 = theta23; % [rad]
% Accumulated arc lengths in [deg]
theta = rad2deg([theta12, theta13, theta23]);

% Rotation angle
%(P2,P3,P1) Around point 1 [rad]
phi1 = acos( ( cos(theta23) - cos(theta21).*cos(theta31) )./( sin(theta21).*sin(theta31)) );
%(P1,P3,P2) Around point 2 [rad]
phi2 = acos( ( cos(theta13) - cos(theta12).*cos(theta32) )./( sin(theta12).*sin(theta32)) );
% (P1,P2,P3) Around point 3 [rad]
phi3 = acos( ( cos(theta12) - cos(theta13).*cos(theta23) )./( sin(theta13).*sin(theta23)) );

phi = rad2deg([phi1,phi2,phi3]); % [deg]

n = 3; % number of sides of the spherical triangle
r = 1; % radius of sphere
InnerArea1 = [phi1+phi2+phi3] - (n-2)*pi; % [unit^2]
InnerArea2 = ([phi1+phi2+phi3] - pi)*r^2; % [unit^2]
OuterArea = 4*pi*(r^2) - InnerArea1;
```

References

[1] R. Noomen. *AE4878 Mission Geometry and Orbit Design, Mission Geometry v4-13*. TU Delft, 2017.

[2] James R. Wertz. *Orbit Constellation Design Management*. Springer, 2009.