

Probability

TUE

Set :-

A set is a collection of things

e.g.

$A = \{ \text{All the students registered in sec. A of PME} \}$

$B = \{ \text{All the students in section A of PME with CGPA} > 3.0 \}$

$C = \{ \text{Ajay, Abram, Rish} \}$

$D = \{ n^2 | n = 1, 2, 3, 4 \}$

OR

$D = \{ 1, 4, 9, 16 \} \leftarrow \text{Tabular method}$

$D = \{ n^2 | n = 1, 2, 3, \dots \} \leftarrow \text{Set builder form or Property method.}$

Subset :-

$$A \subseteq A$$

If $A \subset B$ and $A \neq B$, then we say that "A is a proper subset of B".

$$A \subset B$$

If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$

Equal Set :-

$A = B$ if and only if $B \subseteq A$ and $A \subseteq B$.

Universal Set :-

All sets under investigation in any problem of set theory are assumed to be

Contained in some large ~~set~~ fixed set called "universal Set" or "universe of discourse".

denoted by:- U or S .

e.g. if we are investigating the ~~the~~ grades of ~~current~~ students of ECI in PME, then

$S = \{ \text{All students of ECI who are study PME} \}$.

Set of Numbers :-

N = Natural numbers or positive Integers
 $\{ 1, 2, 3, 4, \dots \}$

Z = All integers, -ve or +ve and 0.
 $= \{ \dots -2, -1, 0, 1, 2, \dots \}$

Q = Rational Numbers
 $= \{ p/q \mid p, q \in Z, q \neq 0 \}$

I = Irrational Numbers
 $= \{ x \mid x \in R \wedge x \notin Q \}$

R = Real Numbers

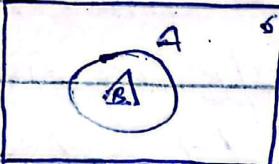
$R = Q \cup I$

Note that:

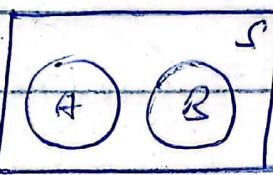
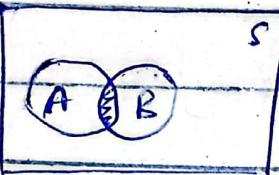
$$N \subseteq Z \subseteq Q \subseteq R$$

Venn Diagram:

$B \cap A$



$A \cap B$



$$A \cap B = \emptyset \text{ Disjoint set}$$

Union :-

$$A \cup B = \{x | x \in A \text{ or } x \in B\}.$$

$$A \cap B = \{x | x \in A \text{ and } x \in B\}.$$

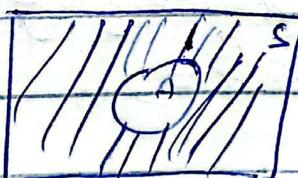


$A \cup B$

Complement :-

Absolute complement or simply complement of a set A is denoted by A^c or A' .

$$A^c = \{x | x \in S, x \notin A\}.$$



A^c

Note

$$S^c = \emptyset$$

$$(A^c)^c = A$$

Set Difference:-

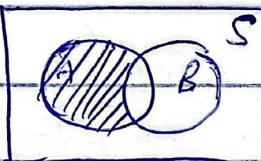
The difference b/w sets A and B is

denoted by $A - B$, A/B or $A \setminus B$.

$A - B$ contains all elements of A that are not elements of B.

$$A - B = \{x | x \in A, x \notin B\}.$$

$A - B$ may be considered as relative compliment of set B w.r.t set A.



Note that

$$\boxed{A - B = A \cap B^c} \quad | \quad A^c = S - A$$
$$A - B \subseteq A$$

$$\bigcup_{i=1}^n A_i = A_1 \cup A_2 \cup A_3 \dots \cup A_n$$

$$\bigcap_{i=1}^n A_i = A_1 \cap A_2 \cap A_3 \dots \cap A_n$$

Mutually Exclusive Sets:-

A collection of sets A_1, A_2, \dots, A_n is mutually exclusive if and only if

$$A_i \cap A_j = \emptyset \quad i \neq j$$

Collectively Exhaustive Sets:-

A collection of sets A_1, A_2, \dots, A_n is collectively exhaustive if and only if

$$A_1 \cup A_2 \cup \dots \cup A_n = S$$

Principle of duality:-

The dual E^* of E is the equation obtained by replacing each occurrence of \cup, \cap, S, \emptyset in E by \cap, \cup, \emptyset, S respectively.

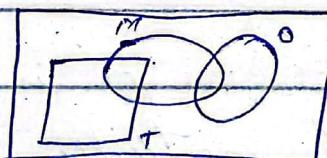
e.g. The dual of

$$(S \cap A) \cup (B \cap A) = A$$

$$(\emptyset \cup A) \cap (B \cup A) = A$$

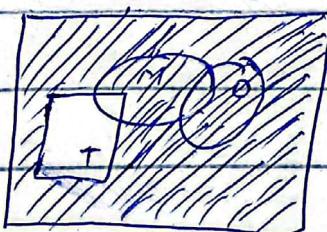
[Quiz 1.1 (H.W)
Quiz 1.1 (Solve)]

Quiz 1.1:-

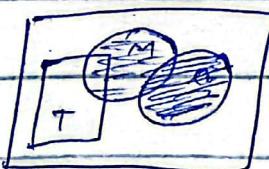


(i) $R =$

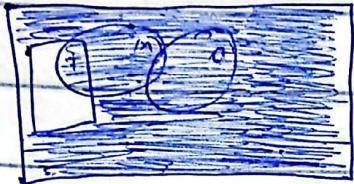
$$S \setminus \{T, R\}$$



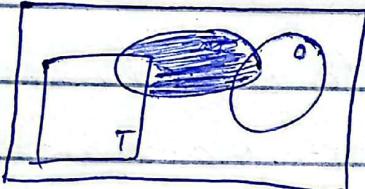
(ii) $M \cup O$



(4) RUM

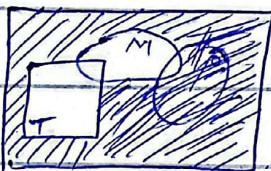


(5) RM



(6) $T^c - M$

$$T^c - M = \{x | x \in T^c, x \notin M\}.$$



Partition:-

A collection of sets A_1, A_2, \dots, A_n is a

Partition if it is both mutually exclusive
and collectively exhausted

A_1	A_2	A_3
	A_4	A_5

Experiment

Toss a coin - Procedure

Observe head or tail - observation

Model - Head or tail

(i) Outcome

(ii) Sample Space

The Sample Space of an experiment is a mutually exclusive, collectively exhausted set of all possible outcomes.

$$S = \{0, 1, 2\}.$$

if sequence = $S = \{TT, TH, HT, HH\}$.

Event :-

An event is a set of outcomes of an experiment.

$$S = \{1, 2, 3, 4, 5, 6\}.$$

Each subset of a sample space is an event.

$$E_1 = \{4, 5, 6\}. E_1 \subseteq S$$

$E_2 = \{\text{The roll is even}\}$.

$$E_2 = \{2, 4, 6\}. E_2 \subseteq S$$

$E_3 = \{\text{The roll is the square of a integer}\}$.

$$E_3 = \{1, 4, 9\}.$$

$$S_E = \{x | x \geq 0\}$$

$$E = \{x | x \geq 5\}.$$

Set Algebra	Probability
Universal Set	Sample Space
Element	outcome
Subsets	Event.

$$S = S = \{xxx, xxy, xyx, yxx, yyx, yxy, yyx, yyy\}.$$

if we let

$$x=1, y=0.$$

then

$$\begin{array}{r} 000 \\ 001 \\ 010 \\ 011 \\ 100 \\ 101 \\ 110 \\ 111 \end{array}$$

$$SS = \{ 000, 001, 010, 011, 100, 101, 110, 111 \}.$$

$$R_1 = \{yyy, yxy, xyx, xyy\}.$$

$$G_3 = \{000, 001, 010, 011\}.$$

Quiz 1-2:-

$$V=1, d=0$$

$$S = \{ddd, ddv, dv\bar{d}, \bar{d}\bar{v}v, vdd, \bar{v}dv, vvv\}$$

$$(i) A_1 = \{\text{first call is a voice call}\}.$$

$$A_1 = \{vdd, vd\bar{v}, v\bar{v}d, vvv\}.$$

$$(ii) B_1 = \{\text{first call is a data call}\}.$$

$$B_1 = \{dd\bar{d}, \bar{d}d\bar{v}, dv\bar{d}, dv\bar{v}\}$$

(iii) $A_2 = \{\text{Second call is a voice call}\}$.

$A_2 = \{dVd, dVv, vvd, vvv\}$.

(iv) $B_2 = \{\text{Second call is a data call}\}$.

$B_2 = \{ddd, ddv, vdd, vdV\}$.

(v) $A_3 = \{\text{all calls are voice}\}$.

$A_3 = \{ddd, vvv\}$.

(vi) $B_3 = \{\text{voice and data alternate}\}$.

$B_3 = \{dVd, vdV\}$.

(vii) $A_4 = \{\text{one or more voice calls}\}$.

$A_4 = \{dvv, dvv, dvv, vdd, vdV, vvd, vvv\}$.

(viii) $B_4 = \{\text{Two or more data calls}\}$.

$B_4 = \{ddd, ddv, vdd, vdV\}$.

i) A_1 and B_1

Yes, it is mutually exclusive and collectively exhaustive.

	Mutually Exclusive	Collectively Exhaustive
A_1 and B_1	YES	YES
A_1 and B_2	YES	YES
A_2 and B_2	YES	No
A_1 and B_4	No	YES

Points:-

- ① Outcome is also called Sample point.
- ② The event $\{a\}$ consisting of a single point $a \in S$ is called elementary event.
- ③ \emptyset and S are both subsets of S .

\emptyset is sometimes called "null" or "impossible" event.

S is called "sure" or "certain" events.

$$S = \{T, H\}$$

- ④ Union of events is the event $(A \cup B)$ iff A occurs or B occurs or both occurs.
- ⑤ $(A \cap B)$ is the event that occurs iff A occurs and B occurs.
- ⑥ A^c is also the event that occurs iff A does not occur.
- ⑦ Events A & B are called mutually exclusive if they are disjoint. i.e; $A \cap B = \emptyset$.
- ⑧ Three or more events are mutually exclusive if every two of them are mutually exclusive.

Probability Axioms

$$P[A] \quad P(A)$$

The probability of an event is the proportion of the time that event is observed in a large number of runs of the experiment.

- A probability measure $P[\cdot]$ is a type of function that maps events in the sample space to Real numbers.

Such that

- ① For any event A

$$P[A] \geq 0$$

- ② $P[S] = 1 \rightarrow$ probability of the Sample Space is 1
e.g. if $S = \{h, t\}$

$$P[h] + P[t] = P[S] = 1.$$

For any countable collection

- ③ A_1, A_2, \dots of mutually exclusive events.

$$P[A_1 \cup A_2 \cup \dots] = P[A_1] + P[A_2] + \dots$$

$$P[A_1 \cup A_2] = P[A_1] + P[A_2]$$

$$A_1 \cap A_2 = \emptyset$$

$$P[A_1 \cap A_2] = P[A_1, A_2] \stackrel{\text{(or)}}{=} P[A_1, A_2]$$

Theorem :-

- ① If $A = A_1 \cup A_2 \cup \dots \cup A_m$ and $A_i \cap A_j = \emptyset$ for $i \neq j$; then

$$P[A] = \sum_{i=1}^m P[A_i].$$

- ② The probabilitiy of an event

$B = \{s_1, s_2, \dots, s_m\}$ is the sum of the probabilities of the outcomes contained in the event

$$P[B] = \sum_{i=1}^m P[s_i]$$

For an experiment with sample space
 $S = \{s_1, s_2, \dots, s_n\}$, in which each outcome is equally likely

$$P[s_i] = \frac{1}{n} \quad 1 \leq i \leq n$$

$$P[s_1] = P[s_2] = \dots = P[s_n].$$

$$P[s_1] + P[s_2] + \dots + P[s_n] = 1$$
$$x + x + x + \dots + x = 1$$

$$nx = 1$$

$$\boxed{x = \frac{1}{n}}$$

Example :-

$$S = \{1, 2, 3, 4, 5, 6\}$$

$$P[1] = P[2] = P[3] = P[4] = P[5] = P[6] = P[x]$$

$$P[1] + P[2] + \dots + P[6] = 1$$

$$P[x] + P[x] + \dots + P[x] = 1$$

$$6 \times P[x] = 1$$

$$\boxed{P[x] = \frac{1}{6}}$$

(a) Roll 4 or higher -

$$P[x \geq 4] = P[4 \leq x \leq 6] = P[4, 5, 6]$$

$$= P[4] + P[5] + P[6] = \frac{1}{6} + \frac{1}{6} + \frac{1}{6}$$

$$= \frac{3}{6} = \boxed{\frac{1}{2}}$$

(b) Roll an even number -

$$P[2, 4, 6] = P[2] + P[4] + P[6] = \frac{1}{6} + \frac{1}{6} + \frac{1}{6}$$

$$= \boxed{\frac{1}{2}}$$

(c) Roll the square of an integer -

$$P[1, 4] = P[1] + P[4] = \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$$

Quiz

$$0 \leq T \leq 100$$

$$0 \sim 100 \approx S_0, S_1, \dots, S_{100}$$

$$80 \sim 100 \rightarrow A$$

$$80 \sim 89 \rightarrow B$$

$$70 \sim 79 \rightarrow C$$

$$60 \sim 69 \rightarrow D$$

$$T < 60 \rightarrow F$$

51-100 are equally likely
 $T \leq 50$ never occurs

Find

$$\textcircled{1} \quad P[\{D_{79}\}] = P[79]$$

$$\textcircled{2} \quad P[\{D_{100}\}] = P[100]$$

$$\textcircled{3} \quad P[A] \quad \textcircled{4} \quad P[F] \quad \textcircled{5} \quad P[T \geq 80]$$

$$\textcircled{6} \quad P[T < 90] \quad \textcircled{7} \quad P[C \text{ or better}]$$

$$\textcircled{8} \quad P[\text{student Passes}]$$

$$T = \{S_1, S_2, \dots, S_{100}\}$$

$$S = \{P_{S_1}, P_{S_2}, \dots, P_{S_{100}}\}$$

$$P[P_i] = \frac{1}{50} = 0.02$$

$$\textcircled{1} \quad P[\{D_{79}\}] = P[T=79].$$

$$= \frac{1}{50} = 0.02 = 2\% \text{ probability}$$

$$\textcircled{2} P[\{Q_{100}\}] = P[T=100] \\ = \frac{1}{50} = 0.02 = 2\%$$

$$\textcircled{3} P[A] = P[Q_0] + P[Q_1] + \dots + P[Q_{100}] \\ = \frac{11}{50} = 11 \times 0.02 = 0.22 = 22\%$$

$$\textcircled{4} P[F] = P[50 < T < 60] \\ = P[S_1] + P[S_2] + \dots + P[S_9] \\ = \frac{9}{50} = 9 \times 0.02 = 18\%$$

$$\textcircled{5} P[T \geq 80] \\ = P[80] + P[81] + \dots + P[100] \\ = 21 \times 0.02 = 0.42 = 42\%$$

$$\textcircled{6} P[T < Q_0] \\ = P[S_1] + \dots + P[S_9] \quad \boxed{\rightarrow (89 - 51) + 1} \\ = 39 \times 0.02 = 0.78 = 78\%$$

$$\textcircled{7} P[\text{C or better}] \\ P[T \geq 70] \\ P[70] + \dots + P[100] \\ = 31 \times 0.02 = 0.62 = 62\%$$

$$\textcircled{8} P[\text{Student passes}] \\ P[T \geq 60] = P[60] + \dots + P[100] \\ = 41 \times 0.02 = 0.82 = 82\%$$

Rules :-

$$P[\emptyset] = 0, \quad P[S] = 1$$

$$P[A] + P[A^c] = 1.$$

$$A \cup A^c = S$$

$$P[A \cup A^c] = P[S]$$

$$P[A] + P[A^c] = 1.$$

$$\boxed{P[A^c] = 1 - P[A]} \rightarrow \text{rule}$$

Theorem / Addition Rule

$$P[A \cup B] = P[A] + P[B] - P[A \cap B].$$

$$\boxed{P[A \cup B] = P[A] + P[B] - P[AB].}$$

If $A \cap B$ is \emptyset , then

$$P[A \cap B] = P[A] + P[B] - P[\emptyset].$$
$$= P[A] + P[B].$$

Theorem :-

If $A \subset B$, then
 $P[A] \leq P[B]$.

Ex# 7.14 :-

L, b

$t > 3 \rightarrow L$

$t < 3 \rightarrow b$

$\{V, d, f\}$.

$$S = \{LV, BV, LD, b, d, LF, BF\}$$

	V	d	f	
L	0.3	0.12	0.15	0.57
B	0.2	0.08	0.15	0.43
	0.50	0.20	0.30	

Add all results 1.

$$S = \{LV, BV, LD, b, d, LF, BF\}$$

$$S = \{0.3, 0.2, 0.12, 0.08, 0.15, 0.15\}$$

$\sum S =$

$$\begin{aligned} P[L] &= P[LV] + P[LD] + P[LF] \\ &= 0.3 + 0.12 + 0.15 = 57\% \end{aligned}$$

$$\begin{aligned} P[B] &= P[BV] + P[BD] + P[BF] \\ &= 0.2 + 0.08 + 0.15 \\ &= 0.43 = 43\% \end{aligned}$$

$$P[L] = 1 - P[B]$$

$$P[B] = 1 - P[L]$$

$$P[L] + P[B] = 1$$

Quiz 1.4 :-

$$P[L] = 0.6$$

$$P[VL] = 0.35$$

$$P[V] = 0.7$$

$$(i) P[DL] \text{ or } P[D \cap L] \quad (ii) P[D \cup L]$$

$$(iii) P[VB] \quad (iv) P[V \cup L] \quad (v) P[V \cap D]$$

$$(vi) P[LB]$$

	V	D	
L	0.35	0.25	0.60
B	0.68	0.35	0.40
	0.70	0.30	

$$P[L] + P[B] = 1$$

$$P[B] = 1 - P[L]$$

$$P[B] = 0.4$$

$$P[V] + P[D] = 1$$

$$0.70 + P[D] = 1$$

$$P[D] = 0.30$$

	V	D	
L	0.35	0.25	0.60
B	0.35	0.15	0.40
	0.70	0.30	

$$P[D \cap L] = P[D] + P[L] - P[DL].$$

$$= 0.30 + 0.60 - 0.25$$

$$= 0.65$$

$$P[D \cup L] = 0.30 + 0.35 + 0.25 + 0.05 = 0.65$$

$$P[VB] =$$

Conditional Probability

$$P[A|B] = \frac{P[A \cap B]}{P[B]} = \frac{P[AB]}{P[B]}$$

$$P[AB] = P[A|B] \times P[B]$$

1 Month of 30 days

B Clouds 7 Days

A Rain 2 Days

$$P[B] = \frac{7}{30}$$

$$P[R] = \frac{2}{30} = 0.067$$

$$A \cap B = 2$$

$$P[A \cap B] = \frac{2}{30}$$

$$P[A|B] = \frac{P[AB]}{P[B]} = \frac{\frac{2}{30}}{\frac{7}{30}} = \frac{2}{7}$$

$$P[A|B] = 0.285$$

Theorem axioms

$$\textcircled{1} \quad P[A|B] \geq 0$$

$$\textcircled{2} \quad P[B|B] = 1$$

\textcircled{3} If $A = A_1 \cup A_2 \cup \dots$

with $A_i \cap A_j = \emptyset \forall i \neq j$:

then

$$P[A|B] = P[A_1|B] + P[A_2|B] + \dots$$

Ex # 1.15

1.16

Example #1.16:-

$$B = \{\text{xx}, \text{xa}\}$$

$$A = \{\text{xx}, \text{ax}\}$$

$$P[\text{xx}] = 0.01, P[\text{xa}] = 0.01, P[\text{ax}] = 0.01$$

$$P[\text{aa}] = 0.97.$$

$$A = 2^{\text{nd}} \text{ chip } \text{xx}. \quad A \cap B = [\text{xx}]$$

$$B = 1^{\text{st}} \text{ chip } \text{xx}.$$

$$P[A] = P[\text{xx}] + P[\text{ax}] = 0.01 + 0.01 = 0.02$$

$$P[B] = P[\text{xx}] + P[\text{xa}] = 0.01 + 0.01 = 0.02$$

$$P[A|B] = \frac{P[A \cap B]}{P[B]} = \frac{P[\text{xx}]}{P[B]} = \frac{0.01}{0.02} = \frac{1}{2}$$

$$= 0.5 = 50\%$$

Law of Total Probability :-

For a Partition $\{B_1, B_2, \dots, B_m\}$ with $P[B_i] > 0 \forall i$,

$$P[A] = \sum_{i=1}^m P[A|B_i] \times P[B_i]$$

i.e.; $P[A] = P[A|B_1] \times P[B_1] + P[A|B_2] \times P[B_2] + \dots + P[A|B_m] \times P[B_m]$.

$$P[A] = P[A \cap B_1] + P[A \cap B_2] + \dots + P[A \cap B_m]$$

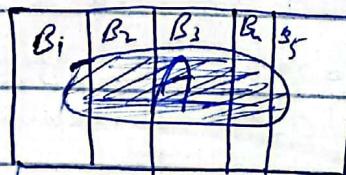
$$P[\bar{A}] = P[A|B_1] \times P[B_1] + \dots + P[A|B_m] \times P[B_m].$$

Theorem :-

For any event A and partition $\{B_1, B_2, \dots, B_m\}$

$$P[A] = \sum_{i=1}^m P[A \cap B_i].$$

$$P[A] = P[A \cap B_1] + P[A \cap B_2] + \dots + P[A \cap B_m].$$



Law of Total Probability:-

For any partition $\{B_1, B_2, \dots, B_m\}$ with $P[B_i] > 0 \forall i$

$$P[A] = \sum_{i=1}^m P[A|B_i] \times P[B_i]$$

$$\begin{aligned} P[A] &= P[A|B_1] \times P[B_1] + P[A|B_2] \times P[B_2] + \dots \\ &\quad + P[A|B_m] \times P[B_m]. \end{aligned}$$

Example #1.19:-

Resistors 1K

Let,

$A = \{\text{Resistor is within set of nominal value}\}$.

$$= 950 - 1050$$

$$P[A|B_1] = 0.8$$

$$P[A|B_2] = 0.9$$

$$P[A|B_3] = 0.6$$

$$\text{Total resistors} = 3k + 4k + 3k = 10k$$

$$P[B_1] = \frac{3000}{10000} = 0.3$$

$$P[B_2] = 0.4$$

$$P[B_3] = 0.3$$

$$P[A] = P[A|B_1] \times P[B_1] + P[A|B_2] \times P[B_2] + P[A|B_3] \times P[B_3]$$

$$P[A] = 0.8 \times 0.3 + 0.9 \times 0.4 + 0.6 \times 0.3 = 0.78$$

$$P[A] = 0.78$$

Baye's Theorem :-

$$P[A|B] = \frac{P[AB]}{P[B]}$$

$$P[B|A] = \frac{P[BA]}{P[B]} = \frac{P[A|B] \times P[B]}{P[A]}$$

$$P[B|A] = \frac{P[A|B] \times P[B]}{P[A]}$$

Ex #1.20 :-

$$P[B_3] = 0.3$$

$$P[A] = 0.78$$

$$P[A|B_3] = 0.6$$

What is the probability that an error occurs for B_3 ?

$$P[B_3|A] = ?$$

$$P[B_3|A] = \frac{P[A|B_3] \times P[B_3]}{P[A]}$$

$$= \frac{0.6 \times 0.3}{0.78} = \boxed{0.23 = P[B_3|A]}$$

Quiz #1.5 :-

$$(1) P[N_V = 2]$$

$$(2) P[N_V \geq 1]$$

$$(3) P[(vvd) N_V = 2]$$

$$(4) P[(ddv) N_V \geq 2].$$

$$(5) P[N_V = 2 | N_V \geq 1] \quad (6) P[N_V \geq 1 | N_V \geq 2]$$

$S = \{vvv, vvd, vdv, vdd, dvv, dvd, ddv, ddd\}$

$$P[vv] = 0.2$$

$$P[dd] = 0.2$$

$$P[vvd] = P[vdv] = P[vdd] = P[dvv] = P[dvd] = P[ddv] = 0.1$$

$$N_V = \{0, 1, 2, 3\}$$

$$(i) P[N_V = 2]$$

Events with exactly two voice calls:

$$S_{V=2} = \{vvd, vdv, dvv\}$$

$$P[N_V = 2] = P[vvd] + P[vdv] + P[dvv] = 0.3$$

$$(2) P[N_V \geq 1]$$

$$P[N_V \geq 1] = P[vv] + P[vvd] + P[vdv] + P[vdd] + P[dvv] + P[dvd] + P[ddv]$$

$$= 0.8$$

$$③ P[(vvd) | N_V = 2]$$

$$P[(vvd) | N_V = 2] = \frac{P[(vvd) \cap \{N_V = 2\}]}{P\{N_V = 2\}} = \frac{(0.1)(0.3)}{0.3}$$

$$\frac{P_2[(vvd) \cap \{vvd, vdv, dd\}]}{P\{N_V = 2\}} \geq P[vvd] = \frac{0.1}{0.3}$$

$$P[(vvd) | N_V = 2] = 0.33 = \frac{1}{3}$$

$$④ P[ddv | N_V = 2] = P[ddv] \cap \{N_V = 2\} = \frac{0}{0.3} = 0$$

$$P[(ddv) \cap \{N_V = 2\}] = (ddv) \cap \{vvd, vdv, ddv\}.$$

$$⑤ P[N_V = 2 | N_V \geq 1] = \frac{P[N_V = 2]}{P\{N_V \geq 1\}} = \frac{0.3}{0.8} = \frac{3}{8}$$

$$⑥ P(N_V \geq 1 \neq N_V = 2) = \frac{P\{N_V \geq 2\}}{P\{N_V \geq 2\}} = 1$$

Independent Events:-

Events A and B are independent if and only if

$$P[A \cap B] = P[A] P[B].$$

$$\text{If } P[B|A] = P[B]$$

$$\text{or } P[A|B] = P[A]$$

Independence is a symmetric relationship

$$P[A \cap B] = P[A] P[B]$$

$$P[B|A] = P[B] \text{ and } P[A|B] = P[A].$$

(i) Independent and disjoint are NOT synonymous

(ii) Disjoint events are not independent unless one of them has zero probability

Suppose A and B are disjoint and also independent at the same time. Then, as disjoint events:

$$A \cap B = \emptyset$$

$$P[A \cap B] = P[\emptyset] = 0 \quad \textcircled{1}$$

As independent events

$$P[A \cap B] = P[A] P[B] \quad \textcircled{2}$$

Comparing \textcircled{1} and \textcircled{2}

$$P[A] P[B] = 0$$

Therefore;

$$\text{either } P[A] = 0 \text{ or } P[B] = 0.$$

$$P[A|B] = \frac{P[A \cap B]}{P[B]}$$

$$P[B|A] = \frac{P[A|B] \times P(B)}{P(A)}$$

RANDOM VARIABLES:-

A random variable is a real number assigned to each outcome in a sample space of a random experiment.

- A discrete random variable is a random variable with a finite range and continuous random variable is a variable with an interval of real numbers of ranges.

Example :-

Two balls 3 red balls, 4 black balls -

Sample Space.

$$= \{RR, RB, BR, BB\}.$$

$Y \#$ No. of Red balls.

$$\boxed{Y = 2, 1, 0}$$

Example #

A stockroom clerks return three steel mill employees, If Smith, Jones and Brown

$$S.S = \{ SJR, SBJ, RSJ, JRS, JSR, RJS \}.$$

$Y \#$ correct matches

$$SJ\bar{B} \rightarrow 3$$

$$S\bar{B}J \rightarrow 1$$

$SJB \rightarrow 3$
 $SJSB \rightarrow 1$
 $B SJ \rightarrow 0$

$SBS \rightarrow 0$
 $SJSB \rightarrow 1$
 $BJS \rightarrow 0$

Example

$$SS = \{ \underset{2}{P}P, \underset{1}{P}F \underset{1}{F}P, \underset{0}{F}F \}$$

Independent Events :-

Independence of three events :-

Three events A_1, A_2 & A_3 are independent if and only if:

(a) They are pairwise independent

$$\text{i.e. } P[A_1 \cap A_2] = P[A_1] \cdot P[A_2]$$

$$P[A_1 \cap A_3] = P[A_1] \cdot P[A_3]$$

$$P[A_2 \cap A_3] = P[A_2] \cdot P[A_3]$$

(b) $P[A_1 \cap A_2 \cap A_3] = P[A_1] \cdot P[A_2] \cdot P[A_3]$

Explain of Independent events (H.W)

Quiz #1.6 :-

$$P[V] = 0.8$$

$$P[d] = 0.2$$

$$C_i = V ; C_1 = V$$

$$C_2 = d$$

$$N_V = \{0, 1, 2\}$$

(a) $\{N_V = 2\}$ and $\{N_V \geq 1\}$

$$S.S = \{vv, vd, dv, dd\}$$

$$P[vv] = P[v]P[v] = 0.8 \times 0.8 = 0.64.$$

$$P[vd] = 0.8 \times 0.2 = 0.16$$

$$P[dv] = 0.8 \times 0.2 = 0.16$$

$$P[dd] = 0.2 \times 0.2 = 0.04$$

(a) $\{N_V = 2\}$ and $\{N_V \geq 1\}$,

A.S. $P[A \cap B] = P[A] \times P[B]$,

$$P[A \cap B] = P[vv] \cap P[vd, dv, vv]$$

$$P_{A.S.} = P[vv] = 0.64.$$

$$\overline{\{N_V \geq 1\}} = P[A] \times P[B]$$

$$\overline{\{N_V \geq 1\}} = 0.64 \times 0.16 + 0.16 = 0.96$$

$$\{N_V = 2\} \cap \{N_V \geq 1\} = \{vv\} = \{N_V = 2\} = 0.64 \quad (1)$$

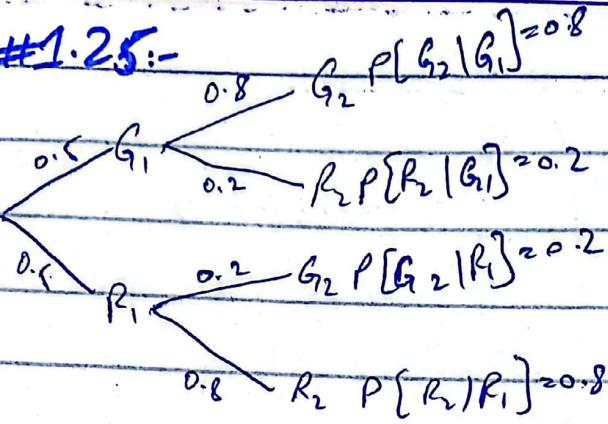
$$P[N_V = 2] \times P[N_V \geq 1] = 0.64 \times 0.96 = 0.6144 \quad (2)$$

From (1) and (2), we observe that

$$P[N_V = 2 \cap N_V \geq 1] \neq P[N_V = 2] \times P[N_V \geq 1].$$

Hence, the events are Not independent.

Example #1.25:-



$$P[A|B] = \frac{P[AB]}{P[B]}$$

$$\begin{aligned} P[G_1, G_2] &= P[G_2|G_1] \times P[G_1] \\ &= 0.8 \times 0.5 = 0.40 \end{aligned}$$

$$P[G_1, R_2] = P[R_2|G_1] \times P[G_1] = 0.2 \times 0.5 = 0.10$$

$$P[R_1, G_2] = P[G_2|R_1] \times P[R_1] = 0.2 \times 0.5 = 0.10$$

$$P[R_1, R_2] = P[R_2|R_1] \times P[R_1] = 0.8 \times 0.5 = 0.40$$

$$\begin{aligned} \textcircled{1} \quad P[G_2] &= P[G_1, G_2] + P[R_1, G_2] \\ &= 0.40 + 0.10 = 0.50 \end{aligned}$$

$$\begin{aligned} \textcircled{2} \quad P[\omega] &= P[G_1, R_2] + P[R_1, G_2] + P[R_1, R_2] \\ &= 0.10 + 0.10 + 0.40 \end{aligned}$$

$$\boxed{P[\omega] = 0.60}$$

$$\textcircled{3} \quad P[G_1|R_2] = ?$$

$$P[G_1|R_2] = \frac{P[G_1, R_2]}{P[R_2]} = \frac{0.10}{0.50} = \boxed{0.20}$$

$$\therefore P[R_2] = P[G_1, R_2] + P[R_1, R_2]$$

$$\therefore P[R_2] = ?$$
$$P[G_1] + P[R_1] = 1$$

$$P[R_2] = 1 - P[R_1]$$

$$P[R_2] = 1 - 0.50$$
$$= 0.50$$

Function

let A and B be two non empty sets.
A function "f" is a relation from A to B
such that f is a set of ordered pairs and
'f' is a subset of $A \times B$.

$$f: A \rightarrow B$$

let (n, y) be an element of f. Then
 n is element of A and y is element of B .
 $n \in A$ and $y \in B$.

$$y = f(n)$$

y is called the value of f at n or image
of n under f .

Domain

The set of the first elements
of the ordered pairs forming a function
is called its domain.

Range

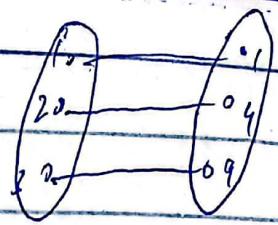
The set of the second elements of
the ordered pairs forming a function is called
its range.

Example :-

Let $A = \{1, 2, 3\}$ and $B = \{1, 4, 9\}$

$$f = \{(1, 1), (2, 4), (3, 9)\}$$

$$f = \{(x, y) | y = x^2, x \in A\}$$



$$\begin{aligned} f(1) &= 1, & f(3) &= 9 \\ f(2) &= 4 \end{aligned}$$

Domain $f = A$

Range $f = B$

$$A \times B = \{(1, 1), (1, 4), (1, 9), (2, 1), (2, 4), (2, 9), (3, 1), (3, 4), (3, 9)\}.$$

Example :-

$$N = \{1, 2, 3, \dots\}.$$

$$g = \{(x, y) | y = x^2, x \in N\}$$

$$g(1) = 1$$

$$g(2) = 4$$

$$g(4) = 16$$

$$g(0) = \text{undefined}$$

$$g(2.5) = \text{undefined}$$

Domain $g = N$

$$\text{Range } g = \{1, 4, 9, 16, 25, \dots\}.$$

$$N \times B = \{(1, 1), (1, 4), (1, 9), \dots, (2, 1), (2, 4), \dots\}.$$

Example

$$h = \{(x, y) \mid y = x^2, x \in \mathbb{R}\}.$$

$$h(0) = 0$$

$$h(1.5) = 2.25$$

$$\text{Domain} = \mathbb{R}$$

$$\text{Range} = \mathbb{B}(\mathbb{R})$$

$$\mathbb{R} \times \mathbb{B} = \{$$

Random Variable:

A random variable X is a function that assigns a real number to each ~~number~~ outcome in the sample space of an experiment.

Example:

Flip a fair coin twice. Number of heads N is a random variable

$$\Omega = \{\text{HH}, \text{HT}, \text{TH}, \text{TT}\}.$$

$$N = \begin{matrix} 2 & 1 & 1 & 0 \end{matrix}$$

$$N = \{0, 1, 2\}.$$

Sample Space	No. of heads
H, H	2
H, T	1
T, H	1
T, T	0

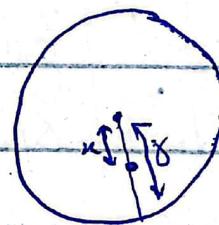
N	0	1	2
$P(N_{2n})$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

$P(N_{2n}) = \begin{cases} \frac{1}{4}, & n=0, 2 \\ \frac{1}{2}, & n=1 \\ 0, & \text{otherwise} \end{cases}$

Probability Mass Function
(PMF)

Continuous Random Variable :-

$$S_x = \{0, \infty\}$$



Example #

 X is the number of voice calls Y is the number of data calls.

$$R = X + Y$$

$$S = \{ddd, ddv, dvd, dvv, vdd, vdv, vvd, vvv\}$$

outcomes	ddd	ddv	dvd	dvv	vdd	vdv	vvd	vvv
X	$\frac{1}{8}$							
Y	0	1	1	2	1	2	1	0
R	0	2	2	2	2	2	2	0

X	0	1	2	3
$P[X=x]$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Y	0	1	2	3
$P[Y=y]$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

R	0	2
$P[R=r]$	$\frac{1}{8}$	$\frac{6}{8}$

$$P[X=x] = P_x(x)$$

$$P[X=0] = P_x(0) = \frac{1}{8}$$

$$P_x(x) = \begin{cases} \frac{1}{8}, & x = 0, 3 \\ \frac{3}{8}, & x = 1, 2 \\ 0 & \text{otherwise} \end{cases}$$

$$P_Y(y) = \begin{cases} \frac{1}{4} & y = 0, 3 \\ \frac{3}{4} & y = 1, 2 \\ 0 & \text{otherwise} \end{cases}$$

$$P_R(\gamma) = \begin{cases} \frac{2}{3} & \gamma = 0 \\ \frac{1}{3} & \gamma = 2 \\ 0 & \text{otherwise} \end{cases}$$

Example #

$$S_2 = \left\{ \overset{\gamma_1}{gg}, \overset{\gamma_1}{gb}, \overset{\gamma_2}{bg}, \overset{\gamma_2}{bb} \right\}$$

$$S_{\gamma_n} = \{0, 1, 2\}$$

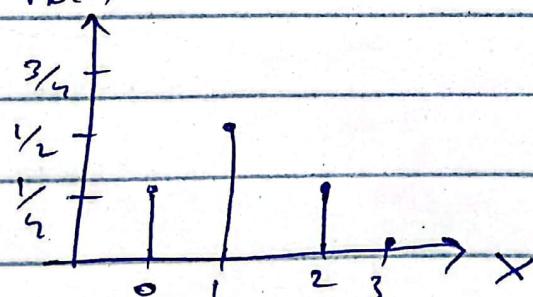
	0	1	2
$P_n(x)$	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{4}$

$$P[x=0] = P_n(0) = \frac{1}{4}$$

$$P[x=1] = P_n(1) = \frac{2}{4} = \frac{1}{2}$$

$$P[x=2] = P_n(2) = \frac{1}{4}$$

$$P_n(x) = \begin{cases} \frac{1}{4} & x=0, 2 \\ \frac{1}{2} & x=1 \\ 0 & \text{otherwise} \end{cases}$$



For a discrete random variable X with PMF $P_n(x)$ and range S_x

(a) For any x , $P_n(x) \geq 0$

$$(b) \sum_{x \in S_x} P_x(x) = 1$$

(c) For any event $B \subset S_x$ $P[B] = \sum_{x \in B} P_n(x)$

Bernoulli Distribution: (when we have only two outcomes)

If X is a Bernoulli random variable it's PMF has the form,

$$P_n(x) = \begin{cases} 1-p, & x=0 \\ p, & x=1 \\ 0, & \text{otherwise} \end{cases}$$

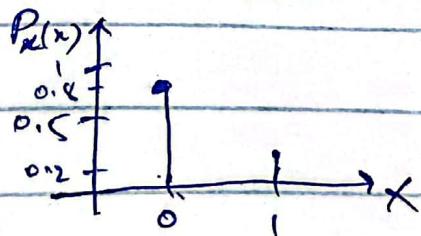
Probability circuit is rejected = p
upto the first oc.

X is a Bernoulli (p) random variable

$X \sim \text{Bernoulli}(p)$

If $P = 0.2$

$$P_x(x) = \begin{cases} 0.8, & x=0 \\ 0.2, & x=1 \\ 0, & \text{otherwise} \end{cases}$$



Geometric Distribution :-

X is a geometric random variable if the PMF of X has the form;

$$P_n(x) = \begin{cases} p(1-p)^{n-1}, & n=1, 2, 3, \dots \\ 0 & \text{otherwise} \end{cases} \quad x \sim \text{geometric}$$

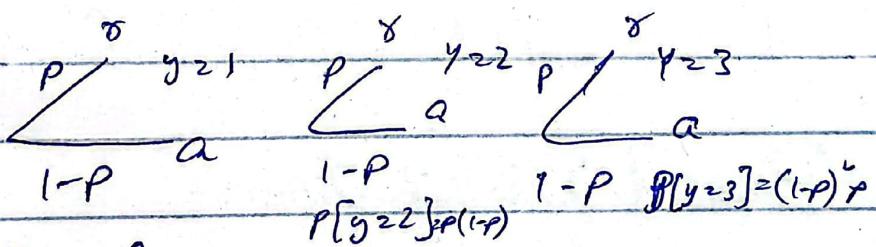
It is used to model the number of trials upto and including the first occurrence of an event.

$$n=1 \quad p$$

$$n=2 \quad p(1-p) \quad \left. \begin{array}{c} \\ p(1-p) \end{array} \right\} a, a\delta, a\delta^2, a\delta^{n-1}$$

$$n=3 \quad p(1-p)^2 \quad \left. \begin{array}{c} \\ p(1-p)^2 \end{array} \right\}$$

$$n=n \quad p(1-p)^{n-1}$$



$$p[y_21] = p$$

$$p_y(y) = \begin{cases} (0.2)(0.8)^{n-1}, & n=1, 2, 3, \dots \\ 0 & \text{otherwise} \end{cases}$$

Binomial Distribution:-

X is a binomial (n, p) random variable, if the PMF of X has the form;

$$P_X(x) = \binom{n}{x} p^x (1-p)^{n-x} \quad x=0, 1, 2, \dots, n$$

$$X \sim \text{binomial}(n, p)$$

Binomial random variable is the sum of ' n ' independent and identical distributed (iid) Bernoulli random variable.

$$\text{For } n=1, x=0 \quad P_X(0) = \binom{1}{0} p^0 (1-p)^1 = 1-p$$

$$\text{For } n=1, x=1 \quad P_X(1) = \binom{1}{1} p^1 (1-p)^0 = p$$

Pascal Distribution:-

X is a pascal (k, p) random variable if the PMF of X has the form;

$$P_X(n) = \binom{n-1}{k-1} p^k (1-p)^{n-k}$$

k is an integer such that $k \geq 1$ and $n = k, k+1, \dots$

when

$k=1$, then $n=1, 2, 3, \dots$ and
then

Pascal distribution reduces to geometric distribution.

Pascal distribution is also called negative binomial distribution or Polya distribution.

Discrete Uniform Distribution :-

X is a discrete uniform Distribution (k, l)
Random variable if the PMF of X has the form,

$$P_n(x) = \begin{cases} \frac{1}{l-k+1}, & x=k, k+1, \dots, l \\ 0, & \text{otherwise} \end{cases}$$

k and l are integers such that $k < l$

probability of all outcomes is constant.

Example of Binomial Distribution :-

$$p=0.2$$

$$n=10, n=k$$

$$P_n(k) = \binom{10}{k} (0.2)^k (0.8)^{10-k}$$

Probability of Re Rejects?

$$p=0.2$$

Pascal

we see 4 defective items $k=4, n=10$

$$P_n(l) = \binom{l-1}{3} (0.2)^4 (0.8)^6$$

Poisson (α) Random Variable:-

X is a poisson random variable if $\Pr\{T\}$ of X has the form

$$\Pr(x) = \begin{cases} \frac{\alpha^n e^{-\alpha}}{n!} & n = 0, 1, 2, \dots \\ 0 & \text{otherwise} \end{cases}$$

where $\alpha > 0$

Poisson distribution expresses the probability of a given number of events occurring in a fixed interval of time or space if ~~these~~ events occur with a known constant rate and independently of the time since last event.

Also called arrival of the event.

- (a) Average rate, λ arrivals/sec.
- (b) Time interval, T .

$$\boxed{\alpha = \lambda T}$$

Example

$$\lambda = 2 \text{ hits/sec}$$

$$T = 0.25 \text{ secs}$$

$$\alpha = 2 \times 0.25$$

$$\boxed{\alpha = 0.5 \text{ hits}}$$

(a) Let,

$$\text{no. of hits} = n = 0.50 \text{ hits.}$$

$$P_H(4) = \frac{\alpha^4 e^{-\lambda}}{4!} = \frac{(0.5)^4}{4!} e^{-0.5}$$

$$P[H=0] = P_H(0) = \frac{(0.5)^0}{0!} e^{-0.5} = e^{-0.5}$$

$$\boxed{P[H=0] = 0.677}$$

(b) Let

Number of hits = J

$$T = 1 \text{ secs}$$

$$\lambda = 2 \text{ hits/sec}$$

$$\alpha = 2 T = 2 \text{ hits}$$

$$P_J(j) = \frac{\alpha^j e^{-2}}{j!} = \frac{(2)^j e^{-2}}{j!} ; j=0, 1, 2, \dots$$

$$P[J \leq 2] = P[J=0] + P[J=1] + P[J=2]$$

$$= \frac{(2)^0 e^{-2}}{0!} + \frac{(2)^1 e^{-2}}{1!} + \frac{(2)^2 e^{-2}}{2!}$$

$$= 0.1353 + 2e^{-2} + 5e^{-2}$$

$$\boxed{P[J \leq 2] = 0.677}$$

Quiz #2-3:-

(i) $X \sim \text{geometric}(p)$

$$P_n(x) = \begin{cases} p(1-p)^{x-1}, & n=1, 2, \dots \\ 0, & \text{otherwise} \end{cases}$$

(ii) $n=20$, $p=0.1$

$$P_n(n) = (0.1)(0.9)^9 = 0.0387$$

Probability of $X \geq 10$

$$P[X \geq 10] = 1 - P[X < 10]$$

$$\therefore P[X < n] + P[n \leq X] = 1$$

$$P[n \geq 10] = (1-p)^{n-1} = (0.9)^9$$

$$P[n \geq 10] = 0.3874 = 0.3874$$

(iii) 100 bits, PMF of Y ? no. of errors?

$$n=100$$

Y = no. of errors

$$P_Y(y) = \binom{100}{y} p^y (1-p)^{100-y}, y=0, 1, 2, \dots$$

(iv) if $P = 0.01$, 100 bits, $\gamma = 2$ errors at the receiver

Probability that $Y \leq 2$:-

$$(a) P = 0.01 \quad n = 100, P[Y=2]$$

$$P[Y=2] = P_Y(2)^2 \binom{100}{2} (0.01)^2 (0.99)^{98}$$

$$[P_Y(2) = 0.1849]$$

$$(b) P[P_f \leq 2] = P[P_f = 0] + P[P_f = 1] + P[P_f = 2]$$

$$P[P_f \leq 2] = \binom{100}{0} (0.01)^0 (0.99)^{100} + \binom{100}{1} (0.01)^1 (0.99)^{99} \\ + \binom{100}{2} (0.01)^2 (0.99)^{98}$$

$$[P[P_f \leq 2] = 0.9206]$$

(v) Three errors, PMF of Z :-

$$P_n(n) = \binom{n-1}{k-1} P^k (1-P)^{n-k}, \quad n=2k, 2k+1, \dots$$

$$P_3(3) = \binom{8-1}{2} P^3 (1-P)^{8-3}, \quad 3=3, 4, 5, \dots$$

(vi) $P = 0.25$, $P_2(12) = ?$

$$P_2(12) = \binom{11}{2} (0.25)^3 (0.75)^9$$

$$P_2(12) = 0.0645$$

Cumulative Distribution Function (CDF)

$$F_X(x) = P[X \leq x]$$

If X is a discrete random variable

$$F_X(n) = P[X \leq n] = \sum_{x_i \leq n} P_x(x_i)$$

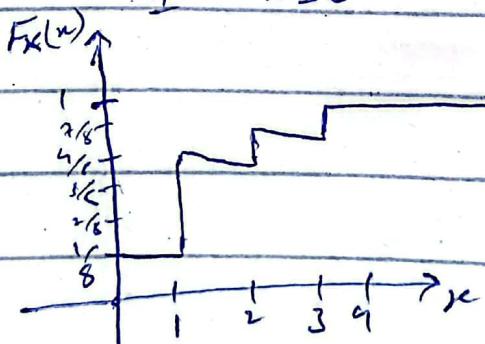
$$S_X = \{0, 1, 2, 3\}$$

x	0	1	2	3
$P_x(n)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{2}{8}$
$P[X \leq n]$	$\frac{1}{8}$	$\frac{4}{8}$	$\frac{7}{8}$	$\frac{8}{8} = 1$

Last value in CDF must be 1.

$$P_X(n) = \begin{cases} \frac{1}{8}, & n = 0, 3, - \\ \frac{3}{8}, & n = 1, 2 \\ 0 & \text{otherwise} \end{cases}$$

$$F_X(n) = \begin{cases} 0 & n < 0 \\ \frac{1}{8} & 0 \leq n < 1 \\ \frac{4}{8} & 1 \leq n < 2 \\ \frac{7}{8} & 2 \leq n < 3 \\ 1 & n \geq 3 \end{cases}$$



From CDF to PDF

$$P[x=0] = P[x \leq 0] - P[x < 0] = \frac{1}{8} - 0 = \frac{1}{8}$$

$$P[x=1] = P[x \leq 1] - P[x < 1] = \frac{4}{8} - \frac{2}{8} = \frac{3}{8}$$

$$P[x=2] = P[x \leq 2] - P[x < 2] = \frac{7}{8} - \frac{4}{8} = \frac{3}{8}$$

$$P[x=3] = P[x \leq 3] - P[x < 3] = 1 - \frac{7}{8} = \frac{1}{8}$$

$$P_n(x) = \begin{cases} \frac{1}{8} & x = 0, 1, 2 \\ \frac{3}{8} & x = 1, 2 \\ 0 & \text{otherwise} \end{cases}$$

Observations:

(a) $\lim_{n \rightarrow \infty} F_n(x) = 0$

$$\lim_{n \rightarrow +\infty} F_n(x) = 1$$

(b) For all $x' \geq n$, $F_X(x') \geq F_n(x)$

(c) For $x_i \in S_x$

$$F_n(x_i) = F_X(x_i - \epsilon) = P_X(x_i)$$

ϵ is a small number.

(d) $F_X(x) = F_X(x_i) \quad \forall x, x_i \leq x \leq x_i + 1$

For all $b \geq a$

$$F_X(b) - F_X(a) = P[a \leq x \leq b].$$

$$P[1 \leq x \leq 2] = F_X(2) - F_X(1) = \frac{7}{8} - \frac{4}{8} = \frac{3}{8}$$

Expected Value

$$E[X] = \mu_x$$

$$E[X] = \mu_x = \sum_{i \in S} x_i p_n(x_i)$$

$$p_n(x) = \begin{cases} \frac{1}{8} & x=0, 3 \\ \frac{3}{8} & x=1, 2 \\ 0 & \text{otherwise} \end{cases}$$

$$E[X] = 0 \left(\frac{1}{8}\right) + 3 \times \frac{1}{8} + 1 \times \frac{3}{8} + 2 \times \frac{3}{8}$$

$$= 0 + \frac{3}{8} + \frac{3}{8} + \frac{6}{8} = \frac{12}{8} = \frac{3}{2} \approx 1.5$$

For Bernoulli random variable:

$$p_n(x) = \begin{cases} 1-p & ; x=0 \\ p & ; x=1 \\ 0 & \text{otherwise} \end{cases}$$

$$E[X] = \mu_x = (1-p)x_0 + p \times 1$$

$$\boxed{E[X] = p}$$

Distribution	Expected Value	Variance
i) Bernoulli	p	$p(1-p)$
ii) Geometric	$\frac{1}{p}$	$(1-p)/p^2$
iii) Poisson	λ	λ
iv) Binomial	np	$np(1-p)$
v) Pascal	$\frac{k}{p}$	$k(1-p)p^k$
vi) Discrete Uniform	$(k+l)/2$	$\frac{(l-k)(l+k+2)}{12}$

Variance σ^2

Standard deviation σ

Standard deviation = $\sqrt{\text{Variance}}$

$$\text{Var}[X] = \sigma^2 = E[(x - \mu_x)^2]$$

$$\text{VAR}[x] = \sum_{i \in S} (x_i - \mu_{x_i})^2 P_X(x_i)$$

Variance of x

$$P_X(n) = \begin{cases} \frac{1}{8} & n=0, 3 \\ \frac{3}{8} & n=1, 2 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} \text{VAR}[x] &= \left(0 - \frac{3}{2}\right)^2 \times \frac{1}{8} + \left(3 - \frac{3}{2}\right)^2 \times \frac{1}{8} + \left(1 - \frac{3}{2}\right)^2 \times \frac{3}{8} \\ &\quad + \left(2 - \frac{3}{2}\right)^2 \times \left(\frac{3}{8}\right) \end{aligned}$$

$$= \left(\frac{9}{4}\right)\left(\frac{1}{8}\right) + \dots$$

$$= \frac{9}{32} = 0.28125$$

$$\text{VAR}[x] = \sigma^2 = \frac{3}{4} = 0.75$$

$$\text{STD}[x] = \sigma_x = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2} = 0.866$$

$$E[x+y] = E[x] + E[y]$$

$$[E[ax+b] = E[ax] + E[b] = aE[x] + b]$$

(1)

$$VAR[x] = E[(x - \mu_x)^2].$$

$$\begin{aligned} VAR[x] &= E[x^2 + \mu_x^2 - 2\mu_x x] \\ &= E[x^2] + E[\mu_x^2] - E[2\mu_x x]. \\ &= E[x^2] + \mu_x^2 - 2\mu_x E[x]. \end{aligned}$$

But

$$\begin{aligned} E[x] &= \mu_x \\ VAR[x] &= E[x^2] + \mu_x^2 - 2\mu_x \mu_x \\ &= E[x^2] - \mu_x^2 \end{aligned}$$

$$VAR[x] = \sigma_x^2 = E[x^2] - \mu_x^2$$

$$VAR[x] = E[x^2] - (E[x])^2$$

$$E[x^2] = \sum_{i=1}^n x_i^2 P(x_i)$$

$$\begin{aligned} E[x^2] &= (0)^2 \times \frac{1}{8} + 3^2 \times \frac{1}{8} + 1^2 \times \frac{3}{8} + 2^2 \times \frac{3}{8} \\ &= 0 + \frac{9}{8} + \frac{3}{8} + \frac{12}{8} \end{aligned}$$

$$\boxed{E[x^2] = 3}$$

$$\begin{aligned} VAR[x] &= \sigma_x^2 = E[x^2] - (E[x])^2 \\ &= 3 - \left(\frac{3}{2}\right)^2 = 3 - \frac{9}{4} = \frac{3}{4} \end{aligned}$$

$$VAR[x] = \frac{3}{4} = 0.75$$

$$STD[x] = \sqrt{\frac{3}{4}} = 0.866$$

Moments of a Random Variable :-

(a) The n^{th} moment of a Random Variable is

$$E[x^n] = \sum_{i \in S} x_i^n P_n(x_i)$$

(b) The n^{th} central moment of Random Variable X is

$$E[(x - \mu_n)^n] = \sum_{i \in S} (x_i - \mu_n)^n \times P_n(x_i)$$

$$VAR[X] = \sigma_x^2 = E[(x - \mu_n)^2]$$

Example # 2.30

$$P_v(v) = \begin{cases} \frac{1}{7}, & v = -3, -2, -1, 0, 1, 2, 3 \\ 0 & \text{otherwise} \end{cases}$$

Let $y = \frac{v^2}{2}$, Find $P_Y(y) = ?$

v	-3	-2	-1	0	1	2	3
$P_v(v)$	$\frac{1}{7}$						
$y = \frac{v^2}{2}$	$\frac{9}{2}$	2	$\frac{1}{2}$	0	$\frac{1}{2}$	2	$\frac{9}{2}$
$P_Y(y)$							

$P_Y(y) = \begin{cases} \frac{1}{7} & y = 0 \\ \frac{2}{7} & y = \frac{1}{2}, 2 \\ 0 & \text{otherwise} \end{cases}$

v	-3	-2	-1	0	1	2	3
$P_{X(V)}$	$\frac{1}{7}$	$\frac{1}{14}$	$\frac{1}{14}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{2}{7}$
$y = 2x + 3$	$\frac{3}{2}$						

$$P_n(x) = \begin{cases} \frac{1}{8} & n = 0, 2 \\ \frac{3}{8} & n = 1, 3 \\ 0 & \text{otherwise} \end{cases}$$

let $y = 2x + 3$

n	0	1	2	3
$P_n(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	$\frac{3}{8}$
$y = 2x + 3$	3	5	7	9

$$P_y(y) = \begin{cases} \frac{1}{8} & y = 3, 7 \\ \frac{3}{8} & y = 5, 9 \\ 0 & \text{otherwise} \end{cases}$$

For Expected value

$$P_n(n) = \begin{cases} \frac{1}{8} & n = 0, 3 \\ \frac{3}{8} & n = 1, 2 \\ 0 & \text{otherwise} \end{cases}$$

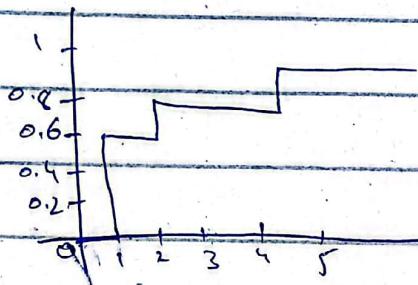
$$E[n] = \mu_n = \frac{3}{8} = 0.75$$

$$E[x - \mu_n] = \sum_{i \in S} (n_i - \mu_n) P_n(n_i)$$

$$= \left(0 - \frac{3}{2}\right) \left(\frac{1}{8}\right) + \left(3 - \frac{3}{2}\right) \frac{1}{8} + \left(1 - \frac{3}{2}\right) \left(\frac{3}{8}\right) + \left(2 - \frac{3}{2}\right) \left(\frac{3}{8}\right)$$

$$\begin{aligned}
 E[X - \mu_x] &= \frac{1}{8} \left[\left(0 - \frac{3}{2}\right) + \left(3 - \frac{3}{2}\right) \right] + \frac{3}{8} \left[\left(1 - \frac{3}{2}\right) + \left(2 - \frac{3}{2}\right) \right] \\
 &= \frac{1}{8} \left[0 - \frac{3}{2} + 3 - \frac{3}{2} \right] + \frac{3}{8} \left[1 - \frac{3}{2} + 2 - \frac{3}{2} \right] = 0.
 \end{aligned}$$

Quiz # 2.4



(1) $P[Y < 1]$ (2) $P[Y \leq 1]$

(3) $P[Y > 2]$

(5) $P[Y = 1]$

(4) $P[Y \geq 2]$

(6) $P[Y = 3]$

$$F_Y(y) = \begin{cases} 0 & y < 1 \\ 0.6 & 1 \leq y < 2 \\ 0.8 & 2 \leq y < 3 \\ 1 & y \geq 3 \end{cases}$$

(1) $P[Y < 1] = 0$

(2) $P[Y \leq 1] = F_Y(1) = 0.6$

(3) $P[Y > 2] = 1 - P[Y \leq 2]$

$$= 1 - P_Y(2) = 1 - 0.8 = 0.2$$

(4) $P[Y \geq 2] = 1 - P[Y < 2]$

$$= 1 - 0.6 = 0.4$$

(5) $P[Y = 1] = P[Y \leq 1] - P[Y < 1] = 0.6 - 0 = 0.6$

$$\textcircled{6} \quad P[Y=3] = P[Y \leq 3] - P[Y < 3]$$

$$= 0.8 - 0.8 = 0$$

PMF of H in CDF

The size of jump at sign
numbers like at 2 and 4
the jump is 0.2

$$P_H(y) = \begin{cases} 0.6 & y=1 \\ 0.2 & y=2, 4 \\ 0 & \text{otherwise} \end{cases}$$

$$g(y) = 2y + 3$$

PMF of X

X	1	2	4
P _y (y)	0.6	0.2	0.2
X	5	7	11

$$P_X(x) = \begin{cases} 0.6 & x=5 \\ 0.2 & x=7, 11 \\ 0 & \text{other.} \end{cases}$$

$$E[X] = 5(0.6) + 7(0.2) + 11(0.2)$$

$$= 3 + 1.4 + 2.2$$

$$\boxed{E[X] = 6.6}$$

we can also find Expected value from PMF but the formula

will be :-

$$E[X] = \sum_{y \in S} g(y) P_Y(y)$$

$$E[X] = [2(1) + 3] \times 0.6 + [2(2) + 3] \times 0.2 + [2(4) + 3] \times 0.2$$

$$\boxed{E[X] = 6.6}$$

Ch #2
(Problems)

- 2.2.1, 2.2.2, 2.2.3, 2.2.4,
2.2.9, 2.3.1, 2.3.2, 2.3.5,
2.3.6, 2.3.7, 2.4.1, 2.4.2,
2.4.3, 2.4.8, 2.5.4, 2.5.5,
2.5.6, 2.5.7, 2.6.2, 2.6.3
2.7.5, 2.8.1, 2.8.2, 2.8.3
2.8.4

Probability of Interval

$$P[a < Y \leq b] = F_Y(b) - F_Y(a)$$

$$P[1 < Y \leq 2] = 0.2$$

from CDF = $F_Y(2) - F_Y(1)$

$$= 0.8 - 0.6 = 0.2$$

$$P[1 < Y \leq 4] = 0.2 + 0.2 = 0.4 \text{ (from prob)}$$

$$= F_Y(4) - F_Y(1) = 1 - 0.6 = 0.4 \text{ (from CDF)}$$

If the Probability that tube length has a useful life of atleast 800 hours is 0.9 find the Probabilities that among 20 such length

- (a) Exactly 18 will have a useful life of atleast 800 hours.
- (b) Atleast 15 will have a useful life of atleast 800 hours.
- (c) Atleast 2 will not have a useful life of atleast 800 hours.

$$P[TL] = 0.9$$

Probability of

$$p = 0.9, q = 1 - 0.9 = 0.1$$

$$\boxed{n = 20}$$

$$P[X = k] = P(n) = {}^{20}C_k (0.9)^k (0.1)^{20-k}$$

$$n = 1 \dots 20$$

$$= P(X \leq 18) = 1 - P(X > 18)$$

$$= 1 - [P(X = 19) + P(X = 20)]$$

$$= 1 - \left[{}^{20}C_{19} (0.9)^9 (0.1)^1 + {}^{20}C_{20} (0.9)^{20} \right]$$

$$\boxed{M = 0.1514}$$

Continuous Random Variable

X

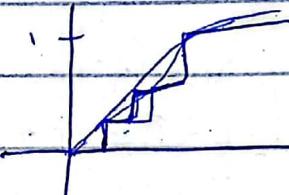
$$S = \{1, 2, 3, 4, 5, 6\}$$

Probability Density Function (PDF)

CDF PDF

$$F_X(x) \quad f_X(x)$$

$$\boxed{f_X(x) = \frac{d}{dx} F_X(x)}$$



Example :-

$$S_V = \{v \mid -5 \leq v \leq 5\} \quad | \quad \text{CDF: } F_X(x) = P[X \leq x]$$

$$S_A = \{a \mid 0 \leq a \leq 2\pi\}$$

$$S_T = \{t \mid -5 \leq t \leq 5\}$$

Theorem :-

For any random variable X,

$$(a) F_X(-\infty) = 0$$

$$(b) F_X(+\infty) = 1$$

$$(c) P[x_1 < x \leq x_2] = F_X(x_2) - F_X(x_1)$$

Quiz #3.1 :-

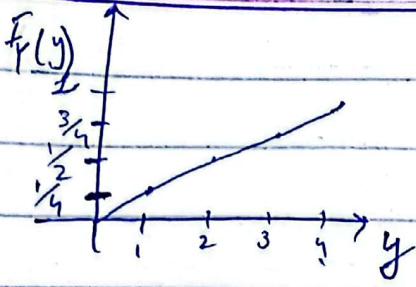
$$f_Y(y) = \begin{cases} 0 & y < 0 \\ \frac{y}{4} & 0 \leq y \leq 4 \\ 1 & y > 4 \end{cases} \quad \text{Sketch the CDF!}$$

$$(a) P[X \leq -1]$$

$$(b) P[Y \leq 1]$$

$$(c) P[2 < Y \leq 3]$$

$$(d) P[Y > 1.5]$$



y	$F_Y(y)$
0	0
1	$\frac{1}{4}$
2	$\frac{1}{2} = \frac{2}{4}$
3	$\frac{3}{4}$
4	1

$$(a) P[Y \leq -1] = F_Y(-1) = 0$$

$$(b) P[Y \leq 1] = F_Y(1) = \frac{1}{4}$$

$$(c) P[1 < Y \leq 3] = F_Y(3) - F_Y(2) = \frac{3}{4} - \frac{1}{2} = \frac{1}{4}$$

$$(d) P[Y \geq 1.5] = 1 - P[Y \leq 1.5]$$

$$= 1 - F_Y(1.5) \quad \therefore F_Y(1.5) = \frac{1.5}{4}$$

$$= 1 - \frac{3}{8} = \boxed{\frac{5}{8}}$$

$$P_X(x) = \begin{cases} \frac{1}{4}, & x = 1 \\ \frac{1}{8}, & x = 2, 3, 4 \\ 0, & \text{otherwise} \end{cases}$$

$$P[1 \leq X \leq 3] = P_X(2) + P_X(3) = \frac{1}{8} + \frac{1}{8} + \frac{2}{8} = \frac{1}{2}$$

$$P[1 \leq X \leq 3] = P_X(1) + P_X(2) + P_X(3)$$

$$= \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8} = \frac{1}{2}$$

In terms of Continuous Random Variable

$$P[X \leq n] = P[X < n]$$

Probability Density Function (PDF):-

$$f_X(x) = \frac{d}{dx} F_X(x)$$

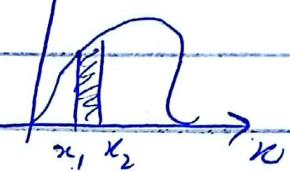
Theorem :-

(a) $f_x(x) \geq 0 \quad \forall x$

(b) $F_x(x) = \int_{-\infty}^x f_x(u) du$

(c) $\int_{-\infty}^{\infty} f_x(u) du = 1$

$$f_x(x)$$

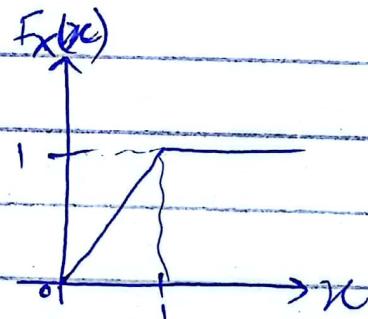


Theorem :-

$$\begin{aligned} P[x_1 < X < x_2] &= F_x(x_2) - F_x(x_1) \\ &= \int_{x_1}^{x_2} f_x(u) du \end{aligned}$$

$$P[X=a] = P[a < X \leq a] = \int_a^a f_x(u) du = 0$$

Example #3.4:-



$$f_x(x) = \frac{d}{dx} F_x(x)$$

$$= \frac{d}{dx} x = 1$$

for:

$$f_x(x) = \begin{cases} 1, & 0 \leq x < 1 \\ 0, & \text{otherwise} \end{cases}$$

CDF :-

$$F_x(x)$$



Using CDF

$$P\left[\frac{X}{4} < x \leq \frac{3}{4}\right] = F_x\left(\frac{3}{4}\right) - F_x\left(\frac{1}{4}\right)$$
$$= \frac{3}{4} - \frac{1}{4} = \frac{2}{4} = \frac{1}{2}$$

Using PDF

$$P\left[\frac{X}{4} < x \leq \frac{3}{4}\right] = \int_{\frac{1}{4}}^{\frac{3}{4}} (1) dx = x \Big|_{\frac{1}{4}}^{\frac{3}{4}} = \frac{3}{4} - \frac{1}{4} = \frac{2}{4} = \frac{1}{2}$$

(H.W)
Example #3.5

Quiz #3.2 :-

$$f_x(n) = \begin{cases} cn e^{-\frac{n^2}{2}}, & n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

- (1) the constant C
- (2) the CDF $F_x(n)$
- (3) $P[0 \leq n \leq 4]$
- (4) $P[-2 \leq n \leq 2]$

L'Hopital Rule

$$\lim_{n \rightarrow a} \frac{f(n)}{g(n)} = \frac{0}{0}$$

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\pm \infty}{\pm \infty}$$

$$\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = \lim_{n \rightarrow \infty} \frac{f'(n)}{g'(n)}$$

$$\int_{-\infty}^{\infty} f_x(n) dn = 1$$

$$\int_{-\infty}^{\infty} f_x(n) dn + \int_0^{\infty} f_x(n) dn = 1 \quad \left[\int_{-\infty}^{\infty} f_x(n) dn = 1 \right]$$

$$\int_0^{\infty} cn e^{-\frac{n^2}{2}} dn = 1$$

Integrating by parts;

$$\int_a^b uv du = uv \Big|_a^b - \int_a^b v du$$

let

$$u = cx \Rightarrow du = cdx$$

$$dv = e^{-\frac{x}{2}} dx, v = \int e^{-\frac{x}{2}} dx = -2e^{-\frac{x}{2}}$$

$$c \int_0^\infty n e^{-\frac{x}{2}} dn = (cn) (-2e^{-\frac{x}{2}}) \Big|_0^\infty$$

$$= - \int_0^\infty 2e^{-\frac{x}{2}} c dn = 1$$

$$+ 2c \int_0^\infty e^{-\frac{x}{2}} dn = 1$$

$$= ne^{-\frac{x}{2}} \frac{x}{e^{\frac{x}{2}}} = \frac{1}{2e^{\frac{x}{2}}}$$

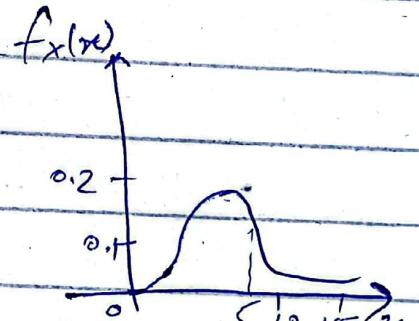
$$= (0 - 0) + 2c(-2)e^{-\frac{x}{2}} \Big|_0^\infty = 1$$

$$= 4c[0 - 1] = 1$$

$$\boxed{c = \frac{1}{4}}$$

$$f_x(x) = \begin{cases} \frac{1}{4} e^{-\frac{x}{2}}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

x	$f_x(x)$
0	0
1	0.15
2	0.18
3	0.1026
10	0.016
20	2.26×10^{-5}



② CDF is

$$F_x(u) = \int_{-\infty}^u f_x(x) dx$$

$$F_x(u) = \int_0^u \frac{u}{4} e^{-u/2} du$$

Using integration by parts

$$\int u dv = uv \Big|_a^b - \int v du$$

$$u = \frac{u}{4}, \quad du = \frac{1}{4} dy$$

$$dv = e^{-y/2} dy \Rightarrow v = -2e^{-y/2}$$

$$= \frac{u}{4} (-2e^{-y/2}) \Big|_0^n - (-2e^{-y/2}) \left(\frac{u}{4}\right) dy$$

$$= (0 - 0) + 2e^{-y/2} \Big|_0^n$$

$$= 2e^{-0} (2) + -2e^{-n/2} (2)$$

$$\Leftrightarrow$$

$$\int_u^n du = \frac{u}{4} (-2e^{-y/2}) \Big|_0^n - \int_u^n (-2e^{-y/2}) \left(\frac{u}{4}\right) du$$

$$= \frac{u}{2} e^{-y/2} \Big|_0^n + \frac{1}{2} \int_u^n -2e^{-y/2} dy$$

$$= -\frac{1}{2} [u e^{-y/2} - 0] \Big|_0^n - \frac{1}{2} [-2e^{-y/2}] \Big|_0^n$$

$$= -\frac{1}{2} x e^{-y/2} \Big|_0^n - \frac{1}{2} e^{-y/2} \Big|_0^n + 1$$

$$= -\frac{x}{2} e^{-n/2} - \frac{1}{2} e^{-n/2} + 1$$

$$= 1 - e^{-n/2} \left(1 + \frac{n}{2}\right)$$

$$F_X(x) = \begin{cases} 1 - e^{-\frac{x}{2}} (x+1), & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$F_X(x)$$

1

1/2

1/4

1/8

1/16

1/32

1/64

1/128

1/256

1/512

1/1024

1/2048

1/4096

1/8192

1/16384

1/32768

1/65536

1/131072

1/262144

1/524288

1/1048576

1/2097152

1/4194304

1/8388608

1/16777216

1/33554432

1/67108864

1/134217728

1/268435456

1/536870912

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$$E[x^2] = \int_{-\infty}^{+\infty} x^2 f_x(x) dx.$$

$$VAR[x] = \sigma_x^2 = E[(x - \mu_x)^2].$$

$$VAR[x] = E[x^2] - (E[x])^2.$$

$$\sigma_x^2 = \int_{-\infty}^{+\infty} (x - \mu_x)^2 f_x(x) dx$$

Theorem :-

$$(a) E[x - \mu_x] = 0$$

$$(b) E[ax + b] = aE[x] + b$$

$$(c) VAR[x] = E[x^2] - \mu_x^2$$

$$(d) VAR[ax + b] = a^2 VAR[x]$$

Quiz #3.3 :-

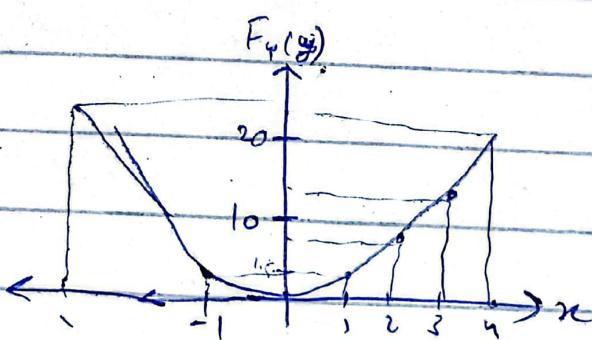
$$f_y(y) = \begin{cases} \frac{3}{2}y^2 & -1 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Sketch PDF :-

$$(1) E[y] \quad (2) E[y^2]$$

$$(3) VAR[y] \quad (4) STD :-$$

y	$f_y(y) = \frac{3}{2}y^2$
0	0
1	$\frac{3}{2} = 1.5$
2	6
3	13.5
4	24



$$\textcircled{1} E[Y]$$

$$E[Y] = \int_{-\infty}^{+\infty} y f_Y(y) dy.$$

$$\text{VAR}[Y] = E[Y^2] - (E[Y])^2$$
$$= \frac{3}{5} - 0 = \frac{3}{5}$$

$$\begin{aligned} &= \int_0^{\infty} y f_Y(y) dy \\ &= \int_0^{\infty} \frac{3}{2} y^2 dy \\ &= \frac{3}{2} \int_0^{\infty} y^2 dy \end{aligned}$$

$$= \frac{3}{2} \int_0^{\infty} y^3 dy$$

$$= \frac{3}{2} \left(\frac{y^4}{4} + \frac{y^3}{3} \right)$$

$$= \frac{3}{2} \left(\frac{y^3}{3} \right) = \frac{y^3}{2} = 0$$

$$\textcircled{2} E[Y^2] = \int_{-\infty}^{\infty} y^2 f_Y(y) dy$$

$$= \int_{-1}^1 (2y)^2 dy$$

$$= \frac{9}{2} \int_{-1}^1 y^4 dy$$

$$= \frac{9}{4} \int_{-1}^1 y^5 dy$$

$$= \frac{9}{4} \left(\frac{-1}{5} - \frac{1}{5} \right) = \frac{3}{5}$$

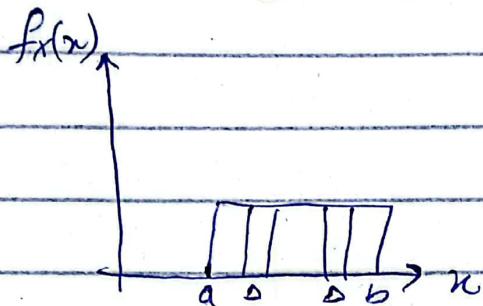
$$P[0 \leq Y \leq \frac{1}{2}] = \int_0^{\frac{1}{2}} \frac{3}{2} y^2 dy = \frac{3}{2} \frac{y^3}{3} \Big|_0^{\frac{1}{2}}$$

$$P[0 \leq Y \leq 1] = \int_0^1 5^2 dy = \frac{1}{2} = \frac{3}{2} \left(\frac{1}{2} \right)^2 = \frac{3}{2} \left(\frac{1}{3} \right)^2 = \frac{3}{2} \left(\frac{1}{8} \right)$$

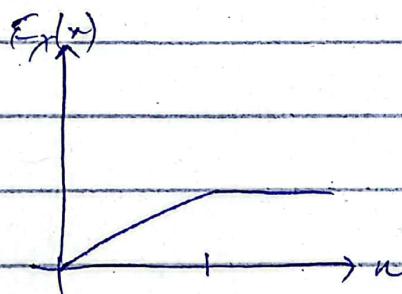
Continuous Uniform Distribution:-

$X \sim \text{Uniform}(a, b)$

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & a \leq x \leq b \\ 0, & \text{otherwise} \end{cases}$$



$$F_X(x) = \begin{cases} 0, & x < a \\ \frac{x-a}{b-a}, & a \leq x \leq b \\ 1, & x > b \end{cases}$$



$$E[X] = \frac{b+a}{2}$$

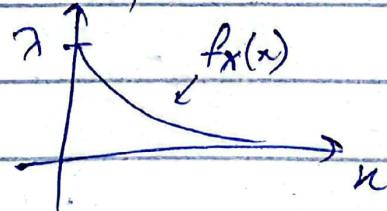
$$\text{VAR}[X] = \frac{(b-a)^2}{12}$$

Exponential Distribution:-

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$\lambda > 0$ is called rate parameter.

$$f_X(x) = \lambda e^{-\lambda x} u(x)$$



$$F_x(n) = \begin{cases} 1 - e^{-\lambda n}, & n \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$E[x] = \frac{1}{\lambda}$$

$$\text{VAR}[x] = \frac{1}{\lambda^2}$$

It used to calculate how long will it take before the car breaks down?

n^{th} moment of X

$$E[x^n] = \frac{n!}{\lambda^n}$$

Example # 3.11:-

Uniform Distribution :-

$$a = 0$$

$$b = 2\pi$$

$$f_\theta(\phi) = \begin{cases} \frac{1}{2\pi}, & 0 \leq \phi \leq 2\pi \\ 0, & \text{otherwise} \end{cases}$$

CDF :-

$$F_\theta(\phi) = \begin{cases} 0, & \phi \leq 0 \\ \frac{\phi - 0}{2\pi - 0}, & 0 < \phi < 2\pi \\ 1, & \phi \geq 2\pi \end{cases}$$

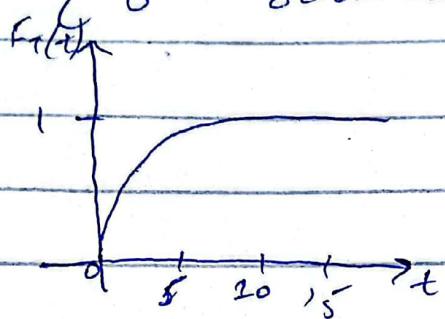
$$E[x] = \frac{b+a}{2} = \frac{2\pi+0}{2} = \pi \text{ rad}$$

$$\text{VAR}[x] = (b-a)^2 = \frac{(2\pi)^2}{12} = \frac{4\pi^2}{12} = \frac{1}{3}\pi^2 = \frac{\pi^2}{3} \text{ rad}^2$$

$$\text{STD} = \sqrt{\frac{\pi^2}{3}} = \frac{\pi}{\sqrt{3}} \text{ rad.}$$

Example #3.12 -

$$F_T = \begin{cases} 1 - e^{-t^{1/3}}, & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$



(a) PDF $f_T(t) = ?$

(b) $P[2 \leq t < 4] = ?$

$$f_T(t) = \frac{d}{dt} (1 - e^{-t^{1/3}})$$

$$= 0 - \left(-\frac{1}{3}\right) e^{-t^{1/3}} = \frac{1}{3} e^{-t^{1/3}}$$

$$f_T(t) = \begin{cases} \frac{1}{3} e^{-t^{1/3}}, & t \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\lambda = \frac{1}{3}$$

$$E[X] = \frac{1}{\lambda} = 3$$

$$\begin{aligned} (b) P[2 \leq t < 4] &= F_T(4) - F_T(2) \\ &= [1 - e^{-4^{1/3}}] - [1 - e^{-2^{1/3}}] \\ &= 1 - e^{-4^{1/3}} - 1 + e^{-2^{1/3}} \end{aligned}$$

$$P[2 \leq t < 4] = 0.2498$$

Example #3.13:-

$$E[X] = \frac{1}{2} = 3$$

$$VARE[X] = \frac{1}{2^2} = 9$$

$$STD = 3 = \sqrt{VAR}$$

$$E[T] = \int_{-\infty}^{+\infty} t f_T(t) dt$$

$$\sigma_T^2 = E[T^2] - (E[T])^2$$

$$\begin{aligned} P &= \{ \mu - \sigma_T \leq T \leq \mu + \sigma_T \} \\ &= \{ 3 - 3 \leq T \leq 3 + 3 \} \\ &= \{ 0 \leq T \leq 6 \} \end{aligned}$$

∴ we can do it both ways

Erlang Distribution:-

The sum of a set of iid exponential random variable is an Erlang random variable.

$$X = Y_1 + Y_2 + \dots + Y_n$$

$$f_X(x) = \begin{cases} \frac{\lambda^n x^{n-1} e^{-\lambda x}}{(n-1)!}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\lambda > 0$$

$n \geq 1$ is an integer.

$$E[X] = \frac{n}{\lambda}$$

$$E[\text{VAR}[X]] = \frac{n}{\lambda^2}$$

Quiz #3.45

$$E[X] = 3, \text{ VAR}[X] = 9$$

(a) Exponential PDF?

$$E[X] = \frac{1}{\lambda} = 3 ; \text{ VAR}[X] = \frac{1}{\lambda^2}$$

$$E[X] = \frac{1}{\lambda} = 3, \text{ VAR}[X] = \frac{1}{\lambda^2} = 9$$

$$\boxed{\lambda = \frac{1}{3}}$$

$$f_X(x) = \begin{cases} x e^{-x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$f_X(x) = \begin{cases} \frac{1}{3} e^{-\frac{x}{3}}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

② Uniform PDF?

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & a \leq x \leq b \\ 0, & \text{otherwise} \end{cases}$$

$$E[X] = \frac{b+a}{2}$$

$$\text{VAR}[X] = \frac{(b-a)^2}{12}$$

Therefore:

$$\frac{b+a}{2} = 3$$

$$b+a = 6 \quad \text{---(i)}$$

$$\frac{(b-a)^2}{12} = 9$$

$$(b-a)^2 = 108$$

$$b-a = \pm \sqrt{108} = \pm 10.39 \quad \text{---(ii)}$$

when $b-a = +10.39$ then

$$\begin{aligned} b+a &= 6 \\ b-a &= 10.39 \end{aligned} \quad \boxed{b = 8.195}$$

In this case;

$$a = b - b = 6 - 8 \cdot 195 = -2 \cdot 195$$

$$\boxed{a = -2 \cdot 195}$$

Case - II :-

$$b - a = -10.39, \text{ then}$$

$$b + a = 6$$

$$b - a = -10.39$$

$$\underline{2b = -4.39}$$

$$\boxed{b = -2.195}$$

$$a = 6 - b = 6 + 2.195$$

$$\boxed{a = 8.195}$$

$$a = -2.195 \quad b > a \checkmark$$

$$b = 8.195$$

$$a = 8.195 \quad b \neq a$$

$$b = -2.195$$

$$\frac{1}{b-a} = \frac{1}{8.195 + 2.195} = \frac{1}{10.39}$$

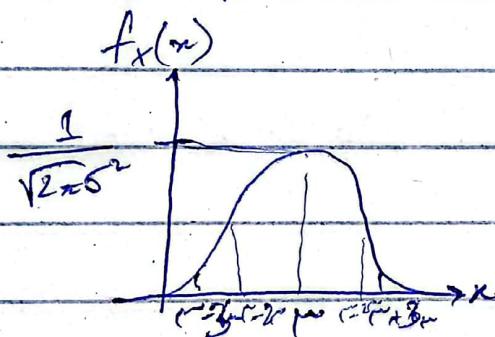
$$f_X(n) \approx \begin{cases} \frac{1}{10.39} = 2.195 \text{ in } [8.195 \\ 0 \quad \text{otherwise} \end{cases}$$

Gaussian Distribution Normal Distribution

$X \sim \text{Gaussian}(\mu, \sigma)$

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$



$E[X] = \mu \rightarrow$ location Parameter

$\text{VAR}[X] = \sigma^2 \rightarrow$ Scale Parameter

3 Sigma rule

68% of values are within $\mu \pm 1\sigma$ of the mean.

95% of values are within $\mu \pm 2\sigma$ of the mean.

99.7% of the values are within $\pm 3\sigma$ of the mean.

Standard Normal Distribution:-

The standard normal random variable Z is the Gaussian $(0, 1)$ random variable.

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

$$E[Z] = 0$$

$$\text{VAR}[Z] = 1$$

$$Z = \frac{x - \mu}{\sigma}$$

$$f_x(x) = \frac{1}{\sigma} \phi\left(\frac{x - \mu}{\sigma}\right)$$

Relation b/w Normal and Standard Normal Distribution.

$$F_x(u) = \Phi\left(\frac{u - \mu}{\sigma}\right)$$

Important formula:- (Theorem)

The probability that x is in the interval (a, b) is probability that

$$P[a < x \leq b] = \Phi\left(\frac{b - \mu}{\sigma}\right) - \Phi\left(\frac{a - \mu}{\sigma}\right) = F_x(b) - F_x(a)$$

Example #3.15:-

Gaussian (μ, σ)

$x \sim \text{Gaussian } (\mu, \sigma)$

$x = 46$: Sample value

$$Z = \frac{x - \mu}{\sigma} \quad \text{for } \mu = 61, \sigma = 10$$

$$Z = \frac{46 - 61}{10} = -1.5$$

$$\boxed{Z = -1.5}$$

Φ -Function:

The Complement of the Standard Normal CDF is called the Φ Function.

$$\Phi(z) = 1 - \bar{\Phi}(z)$$

$$\bar{\Phi}(z) = P[z \leq z]$$

$$\Phi(z) = P[z > z]$$

Theorem:-

$$\bar{\Phi}(-z) = 1 - \bar{\Phi}(z)$$

Example #376:-

$$X \sim \text{Gaussian}(61, 10)$$

$$\mu = 61 \quad \sigma = \sqrt{10}$$

$$\sigma = 10 \quad P[X \leq 46] = ?$$

$$Z = \frac{x - \mu}{\sigma} = \frac{46 - 61}{10} = -1.5$$

$$P[X \leq 46] = \bar{\Phi}(46) = \bar{\Phi}(-1.5) = 1 - \bar{\Phi}(1.5)$$

Table #3.1, 3.2

From table $\bar{\Phi}(1.5) = 0.9332$

$$P[X \leq 46] = 1 - 0.9332 = 0.0668$$

$$P[X \leq 46] = 0.0668$$

$$\mu = 61, \sigma = 10$$

$$P[51 < X \leq 71] = ?$$

$$Z_1 = \frac{x - \mu}{\sigma} = \frac{71 - 61}{10} = 1$$

$$Z_2 = \frac{51 - 61}{10} = -1$$

Therefore;

$$P[51 < X \leq 71] = P[-1 < Z \leq 1].$$

$$= \Phi(1) - \Phi(-1)$$

$$= \Phi(1) - [1 - \Phi(1)]$$

$$\Rightarrow 2\Phi(1) = 2 \times 0.8413 - 1$$

$$= 0.6826$$

Example #3.18:-

$$P[E] = Q(\frac{\gamma}{\sqrt{2}})$$

$$\gamma = SNR = \frac{S}{N}$$

min value of γ such that $Q(z) \leq 10^{-6}$

$$\text{when } z \geq 4.76, \frac{\gamma}{\sqrt{2}} \geq 4.76$$

$$\frac{\gamma^2 - 22.65}{2} \geq 22.6576$$

$$\gamma^2 \geq 45.3152$$

$$P[E \leq 10^{-6}] \text{ when } \gamma \geq 45.3152$$

Quiz #3.5 :-

$$\mu_x = 0, \sigma_x = 1$$

$$① f_x(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}$$

$$f_x(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

Peak value of $f_x(x) = \frac{1}{\sqrt{2\pi}} \approx 0.3989$

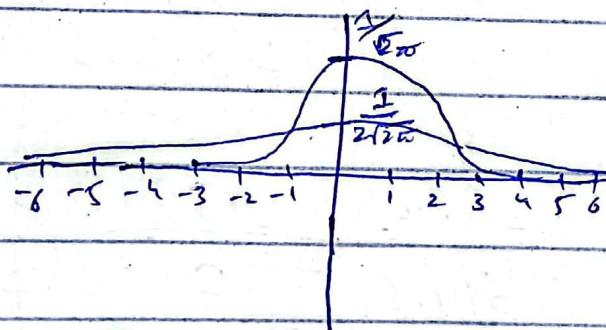
$$\pm 3\sigma = \pm 3$$

$y \sim \text{Gaussian}(0, 2)$

$$\mu_y = 0, \sigma_y = \sqrt{2}$$

$$f_y(y) = \frac{1}{2\sqrt{2\pi}} e^{-\frac{y^2}{8}}$$

Peak value of $f_y(y) = \frac{1}{2\sqrt{2\pi}} \approx 0.1925$



$$② P[-1 \leq X \leq 1] = \Phi(1) - \Phi(-1) \\ \approx 2\Phi(1) - 1 = 0.6826$$

$$③ P[-1 \leq Y \leq 1] = F_Y(1) - F_Y(-1) = \Phi(\frac{1}{\sqrt{2}}) - \Phi(-\frac{1}{\sqrt{2}}) \\ Z_1 = \frac{-1 - 0}{\sqrt{2}} = -\frac{1}{\sqrt{2}}, Z_2 = \frac{1 - 0}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$= \phi(\frac{z}{\sqrt{2}}) [1 - \phi(\frac{z}{\sqrt{2}})]$$

$$\Phi(z)^2 [1 - \Phi(z)]$$

$$= 2\phi(\frac{1}{\sqrt{2}}) - 1 = 2 \times 0.6915 - 1 \\ = 0.383$$

(4) $P[X > 3.5] = P[Z > 3.5] = \Phi(3.5)$
 $\Phi(3.5) = 2.33 \times 10^{-4}$ (from table)

(5) $P[Y > 3.5] = P[Z > \frac{3.5 - 0}{2}] = 1.75$

$$P[Y > 3.5] = P[Z > 1.75] = \Phi(1.75) \\ = 1 - \Phi(1.75) = 1 - 0.9599 \\ = 0.0401$$

X	$X - \bar{X}$	$(X - \bar{X})^2$	$Z = \frac{X - \bar{X}}{\sigma}$	$(Z - \bar{Z})^2$
45	-15.8	249.64	-1.274	1.623
56	-4.8	23.04	-0.387	0.199
58	-2.8	7.84	-0.226	0.051
67	6.2	38.44	0.5	0.25
78	17.2	298.84	1.387	1.9237
$\Sigma X = 304$		$\Sigma (X - \bar{X})^2 = 614.8$	$\Sigma Z = 0$	$\Sigma (Z - \bar{Z})^2 = 3.99$
$\bar{X} = 260.8$			$\bar{Z} = 0$	$\bar{Z} = 0$

$$V A R = s^2 = \frac{\sum (X - \bar{X})^2}{n-1}$$

$$s^2 = \frac{614.8}{4} = 153.7 \quad | \quad [s = 12.4]$$

$$F_x(x) = P[X \leq x]$$

Joint Cumulative Distribution Function:

$$F_{x,y}(x,y) = P[X \leq x, Y \leq y]$$

Theorem:-

Joint CDF

(a) $0 \leq F_{x,y}(x,y) \leq 1$

(b) $F_x(x) = F_{x,y}(x,\infty)$

(c) $F_y(y) = F_{x,y}(\infty, y)$

(d) $F_{x,y}(\infty, y) = F_{x,y}(x, -\infty) = 0$

(e) If $x_1 \leq x_2$ and $y_1 \leq y_2$, then

$$F_{x,y}(x_1, y_1) \leq F_{x,y}(x_2, y_2)$$

(f) $F_{x,y}(\infty, \infty) = 1$.

Joint PMF

Joint Probability Mass Function

$$P_{x,y}(x,y) = P[X=x, Y=y]$$

Example #4.1:-

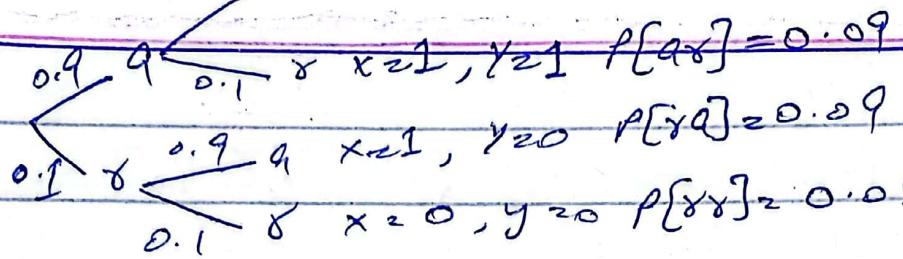
$$P[A] = 0.9$$

$$P[Y] = 0.1$$

No. of acceptable circuits $\approx X$

No. of ~~at least~~ successful tests before first reject $\approx Y$

$$0.9 \quad x=2, y=2 \quad P[aa] = 0.81$$



$$S = \{aa, ax, ya, xy\}$$

$$P_{x,y}(x,y) = \begin{cases} 0.81, & x=2, y=2 \\ 0.09, & x=2, y=1 \\ 0.09, & x=1, y=2 \\ 0.01, & x=0, y=2 \\ 0, & \text{otherwise} \end{cases}$$

$$\sum_{x \in S_n} \sum_{y \in S_y} P_{x,y}(x,y) = 1.$$

Marginal PMF:-

$P_{x,y}(x)$	$y=0$	$y=1$	$y=2$	$P_x(x)$
$x=0$	0.01	0	0	0.01
$x=1$	0.09	0.09	0	0.18
$x=2$	0	0	0.81	0.81
	0.10	0.09	0.81	1

$$P_x(y)$$

$$P_x(x) = \begin{cases} 0.01 & x=0 \\ 0.18 & x=1 \\ 0.81 & x=2 \\ 0 & \text{otherwise} \end{cases}$$

$$P_y(y) = \begin{cases} 0.10 & y=0 \\ 0.09 & y=1 \\ 0.81 & y=2 \\ 0 & \text{otherwise} \end{cases}$$

Theorem :-

For discrete random variables X & Y
and any set B in the x, y plane, the
Probability of the event $\{(x, y) \in B\}$ is

$$P[B] = P[(x, y) \in B] = \sum_{(x, y) \in B} P_{X,Y}(x, y)$$

$$B : X = Y$$

$$B = \{(0, 0), (1, 1), (2, 2)\}.$$

$$P[B] = P_{X,Y}(0, 0) = P_{X,Y}(1, 1) + P_{X,Y}(2, 2)$$

$$P[B] = 0.01 + 0.09 + 0.81 = 0.91$$

$$E_{X,Y}(m, y) = \int_{-\infty}^{\infty} \int_{-\infty}^y f_{X,Y}(u, v) dv du.$$

$$E_{X,Y}(x, y) = \frac{d^2 F_{X,Y}(x, y)}{dx dy}.$$

$$P[x_1 < x \leq x_2, y_1 < y \leq y_2] = F_{X,Y}(x_2, y_2) - \\ F_{X,Y}(x_1, y_1) = F_{X,Y}(x_1, y_2) + F_{X,Y}(x_2, y_1)$$

Theorem :-

A joint PDF $f_{x,y}(x,y)$ has the following properties :-

$$(a) f_{x,y}(x,y) \geq 0 \quad \forall (x,y)$$

$$(b) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{x,y}(x,y) dx dy = 1$$

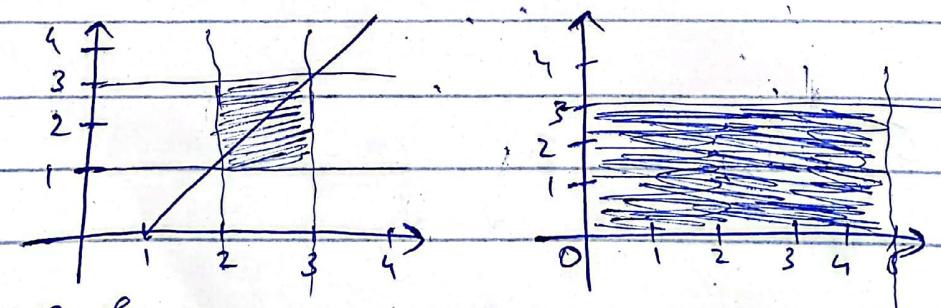
from PDF :-

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{x,y}(x,y) dx dy = \int_{x_1}^{x_2} \int_{y_1}^{y_2} f_{x,y}(x,y) dy dx$$

Ex. ^{Ans.} $f_{x,y}(x,y) = \begin{cases} c, & 0 \leq x \leq 5, 0 \leq y \leq 3 \\ 0, & \text{otherwise} \end{cases}$

(a) Constant c

$$(b) P[A] = P[2 \leq x \leq 3, 1 \leq y \leq 3]$$



$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_{x,y}(x,y) dx dy = 1$$

$$\int_0^5 \int_0^3 c dy dx = 1$$

$$c \int_0^5 \int_0^y dy dx = 1$$

$$c = \int_0^5 (y)_0^3 dy = 1$$

$$3c(5) = 1.$$

$$15c = 1$$

$$\boxed{c = \frac{1}{15}}$$

$$f_{x,y}(x,y) = \begin{cases} \frac{1}{15} & 0 \leq x \leq 5, 0 \leq y \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

$$(5) P[A] = P[2 \leq x \leq 3, 1 \leq y \leq 2].$$

$$= \frac{1}{15} \int_2^3 \int_1^2 dy dx$$

$$= \frac{1}{15} (3-1)(3-2)$$

$$= \frac{1}{15} (2)(1) = \frac{2}{15}$$

Example 4.6 :-

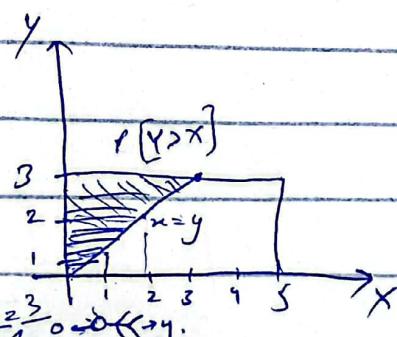
$$f_{x,y}(x,y) = \begin{cases} \frac{1}{15} & 0 \leq x \leq 5, 0 \leq y \leq 3 \\ 0 & \text{otherwise.} \end{cases}$$

$$P[A] = P[Y > X] = ?$$

$$P[A] = \int_0^3 \left(\int_x^3 \frac{1}{15} dy \right) dx$$

$$= \frac{1}{15} \int_0^3 \int_x^3 dy dx$$

$$= \frac{1}{15} \cdot 3 \cdot \int_0^3 (y)_0^3 dx = \frac{9}{10}$$



$$\int_0^2 \int_0^y dy$$

$$\int_0^2 \frac{y^2}{2} dy$$

$$= \frac{9}{30} = \frac{3}{10}$$

(or)

$$P[A] = \frac{1}{15} \cdot \int_0^3 \int_0^y dy dx$$

Quiz #4.4 :-

$$f_{x,y}(x,y) = \begin{cases} Cxy & 0 \leq x \leq 1, 0 \leq y \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) Constant C?

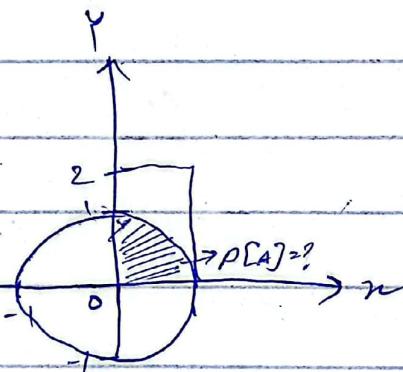
$$(b) P[A] = P[X^2 + Y^2 \leq 1] = ?$$

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_{x,y}(x,y) dx dy$$

$$\int_0^2 \int_0^y Cxy dx dy = 1$$

$$C \int_0^2 \frac{x^2 y}{2} \Big|_0^1 dy = 1$$

$$C \cdot \int_0^2 \frac{1}{2} y dy = 1$$



$$\boxed{C=1}$$

$$\frac{C}{2} \int_0^2 \frac{y^2}{2} dy = 1 \Rightarrow \frac{C}{2} \cdot \frac{4}{2} = 1, \frac{CY_2}{4}$$

$$f_{X,Y}(x,y) = \begin{cases} ny & 0 \leq x \leq 1, 0 \leq y \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

(b)

$$x^2 + y^2 \leq 1$$

$$y^2 \leq 1 - x^2$$

$$y \leq \sqrt{1 - x^2}$$

$$P[A] = \int_0^1 \int_0^{\sqrt{1-x^2}} ny dy dx$$

$$P[A] = \int_0^1 \int_0^{\sqrt{1-x^2}} x dx$$

~~$$= \int_0^1 \int_0^{\sqrt{1-x^2}} n(1-x^2) dx dy$$~~

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} (n - n x^2) dx dy$$

$$= \frac{1}{2} \left[\frac{n}{2} x^2 - \frac{n}{3} x^3 \right]_0^1$$

$$= \frac{1}{2} \left[\frac{1}{2} - \frac{1}{3} \right] = \frac{1}{2} \left(\frac{1}{6} \right) = \frac{1}{12}$$

$$\boxed{P[A] = \frac{1}{12}}$$

Marginal PDF:-

$$f_x(u) = \int_{-\infty}^{+\infty} f_{x,y}(u, y) dy$$
$$f_y(y) = \int_{-\infty}^{+\infty} f_{x,y}(u, y) du dy$$

$$f_x(u) = \frac{\partial}{\partial u} F_{x,y}(u, \infty) = \frac{\partial}{\partial u} F_x(u)$$

$$f_y(y) = \frac{\partial}{\partial y} F_{x,y}(\infty, y) = \frac{\partial}{\partial y} F_y(y)$$

Marginal PDF :-

$$f_x(u) = \int_1^u n y dy \rightarrow \text{for } x$$

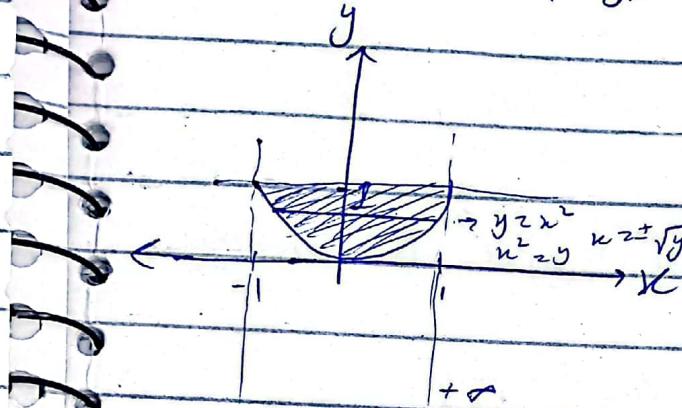
$$f_y(y) = \int_0^n u y du \rightarrow \text{for } y.$$

Remaining Notes
at the mid of the
Register

Probability Notes (2)

$$f_{X,Y}(x,y) = \begin{cases} \frac{5}{4}y & -1 \leq x \leq 1, x^2 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_X(x) = ? \quad f_Y(y) = ?$$



x	y=x^2
-1	1
0	0
1	1

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) dy.$$

$$f_X(x) = \int_{x^2}^1 \frac{5}{4}y dy.$$

$$\begin{aligned} f_X(x) &= \frac{5}{4} \int_{x^2}^1 y^2 dy = \frac{5}{4} \left(\frac{1}{2} - \frac{x^4}{2} \right) \\ &= \frac{5}{8} [1 - x^4] \end{aligned}$$

$$f_X(x) = \begin{cases} \frac{5}{8}(1-x^4) & -1 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) dx$$

$$= \int_{-\sqrt{y}}^{\sqrt{y}} \frac{5}{4}y dx$$

$$= \frac{5}{4}y \left[x \right]_{-\sqrt{y}}^{\sqrt{y}}$$

$$\begin{aligned} &= \frac{5}{4}y (\sqrt{y} + \sqrt{y}) \\ &= \frac{5}{2}y^{3/2} \\ &= \frac{5}{8}y^4 \end{aligned}$$

$$f_Y(y) = \begin{cases} \frac{1}{2} y^2 & 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Covariance :-

$$\text{Cov}[X, Y] = \sigma_{XY} = E[(x - \mu_x)(y - \mu_y)]$$

Continuous

$$\text{Cov}[X, Y] = \iint_{-\infty}^{+\infty} (x - \mu_x)(y - \mu_y) f_{X,Y}(x, y) dx dy$$

discrete

$$\text{Cov}[X, Y] = \sum_n \sum_m (x - \mu_x)(y - \mu_y) P_{X,Y}(x, y)$$

Calculation :-

$$\gamma_{XY} = E[XY]$$

$$\gamma_{XY} = \iint_{-\infty}^{+\infty} xy f_{X,Y}(x, y) dx dy$$

$$\gamma_{XY} = \sum \sum xy P_{X,Y}(x, y)$$

Theorem :-

$$(a) \text{Cov}[X, Y] = \gamma_{XY} - \mu_x \mu_y \quad \text{Subtract}$$

$$\Leftrightarrow \text{Cov}[X, Y] = E[XY] - E[X]E[Y]$$

$$E[X] = \int_{-\infty}^{+\infty} x f_X(x) dx$$

$$E[X] = \iint_{-\infty}^{+\infty} x f_{X,Y}(x, y) dx dy$$

$$E[g(x, y)] = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} g(x, y) f_{x,y}(x, y) dx dy$$

$$g(x, y) = x$$

$$E[g(x, y)] = E[x]$$

$$E[Y] = \int_{-\infty}^{+\infty} y f_{x,y}(x, y) dx dy$$

(b) If $X = Y$

$$\text{Cov}[X, Y] = \text{Cov}[X, X] = \text{Var}[X] = \sigma_x^2$$

$$\text{Cov}[X, Y] = E[X^2] - (E[X])^2$$

Example # 4.12 :-

$P_{x,y}(x,y)$	$y=0$	$y=1$	$y=2$	$P_x(x)$
$x=0$	0.01	0	0	0.01
$x=1$	0.09	0.09	0	0.18
$x=2$	0	0	0.81	0.81
	0.10	0.09	0.81	

$$\delta_{xy} = ? \text{ and } \text{Cov}[X, Y] = ?$$

$$\delta_{xy} = E[XY] = \sum_{y \in S_Y} \sum_{x \in S_X} xy P_{x,y}(x, y)$$

$$\delta_{xy} = (0)(0)(0.01) + (0)(1)(0) + (0)(2)(0)$$

$$(1)(0)(0.09) + (1)(1)(0.09) + (1)(2)(0) + (2)(0)(0)$$

$$+ (2)(1)(0) + (2)(2)(0.81)$$

$$\boxed{\sigma_{xy} = 3.33}$$

→ correlation

$$\text{Cov}[X, Y] \text{?}$$

$$\text{Cov}[X, Y] = \sigma_{xy} - \mu_x \mu_y$$

$$E[X] = (0)(0.01) + (1)(0.18) + (2)(0.81)$$

$$\boxed{E[X] = 1.80}$$

$$E[Y] = (0)(0.10) + (1)(0.09) + (2)(0.81)$$

$$\boxed{E[Y] = 1.71}$$

$$\begin{aligned}\text{Cov}[X, Y] &= \sigma_{xy} = \sigma_{xy} - \mu_x \mu_y \\ &= 3.33 - (1.80)(1.71)\end{aligned}$$

$$\boxed{\text{Cov}[X, Y] = 0.252}$$

Orthogonal Random Variables :-

Random Variables X and Y are orthogonal if $\text{Cov}[X, Y] = 0$

i.e;

$$E[XY] = 0$$

Uncorrelated Random Variables :-

RV's X and Y are uncorrelated

if

$$\text{Cov}[X, Y] = 0$$

$$\text{Cov}[X, Y] = E[XY] - E[X]E[Y]$$

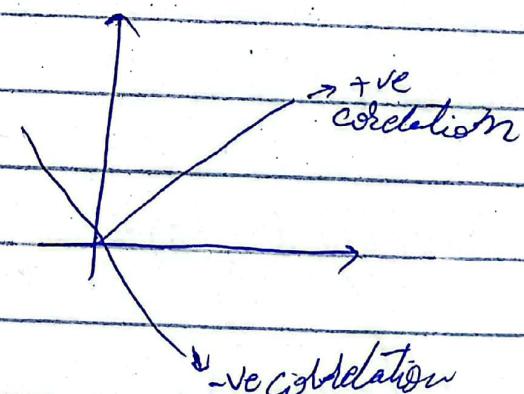
$$E[XY] = E[X]E[Y]$$

Correlation coefficient:

(video lectures)

$$\rho_{XY} = \frac{\text{Cov}[X, Y]}{\sqrt{\text{Var}[X]\text{Var}[Y]}} = \frac{\text{Cov}[X, Y]}{\sigma_X \sigma_Y}$$

$$-1 \leq \rho_{XY} \leq 1$$



Independent Random Variables

$$P[AB] = P[A] \times P[B]$$

RV's X and Y are independent if

be

$$P_{X,Y}(x,y) = P_x(x) P_y(y) \quad (\text{Discrete})$$

$$f_{X,Y}(x,y) = f_x(x) f_y(y) \quad (\text{Continuous})$$

For independent RV's X and Y

$$\text{Cov}[X, Y] = \sigma_{XY} = 0$$

- 1 - $-1 \leq P \leq 1$
- 2 - $P(X, Y) = P(XY)$
- 3 - $P(X|X) = 1$
- 4 - $P(X, -X) = 0$

Types of Transform :-

- 1 - Characteristic Function
- 2 - Probability Generating Function
- 3 - Moment Generating Function (MGF)

MGF :-

$$\Phi_x(s) = E[e^{sx}]$$

$$(\text{Continuous}) \Phi_x(s) = \int_{-\infty}^{+\infty} e^{sx} f_x(n) dx$$

$$(\text{Discrete}) \Phi_x(s) = \sum_{n \in \mathbb{N}} e^{sx_i} P_x(n_i)$$

Theorem :-

A RV X with MGF $\Phi_x(s)$ has

n^{th} moment

$$E[X^n] = \frac{d^n}{ds^n} \Phi_x(s) \Big|_{s=0}$$

$$E[X^n] = \int_{-\infty}^{+\infty} x^n f_x(x) dx$$

$$E[X^n] = \sum_{x \in S_X} x^n P_x(x)$$

$$1 - \Phi_X(s)|_{s=0} = \Phi_X(0) = 1$$

2 = If X_1 and X_2 are two independent RV's with MGF $\Phi_{X_1}(s)$ and $\Phi_{X_2}(s)$, then

$$\Phi_{X_1+X_2}(s) = \Phi_{X_1}(s) \Phi_{X_2}(s)$$

$$Y = X_1 + X_2$$

Example :-

Derive the MGF for exponential distribution

Solution :-

$$f_X(n) = \begin{cases} \lambda e^{-\lambda x} & n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\Phi_X(s) = E[e^{sx}]$$

$$\Phi_X(s) = \int_{-\infty}^{+\infty} e^{sx} f_X(x) dx$$

$$\mathbb{E}_x(s)^2 \int_0^\infty e^{sx} (\lambda e^{-\lambda x}) dx$$

$$= \lambda \int_0^\infty e^{-(\lambda-s)x} dx$$

$$\mathbb{E}_x(s)^2 = \frac{-\lambda}{\lambda-s} \left| e^{(\lambda-s)x} \right|_{0}^{\infty}$$

$$= \frac{-\lambda}{\lambda-s} [0-1]$$

$$\boxed{\mathbb{E}_x(s)^2 = \frac{\lambda}{\lambda-s}}$$

$$E[X^n] = \frac{d^n}{ds^n} \mathbb{E}_x(s) \Big|_{s=0}$$

Therefore;

$$E[X] = \frac{d}{ds} \mathbb{E}_x(s) \Big|_{s=0} = \frac{d}{ds} \frac{\lambda}{\lambda-s} \Big|_{s=0}$$

$$= \frac{d}{ds} [\lambda(\lambda-s)^{-1}]$$

$$= -\lambda(\lambda-s)^{-2}(1) \Big|_{s=0}$$

$$= \frac{\lambda}{(\lambda-s)^2} \Big|_{s=0} = \frac{1}{\lambda}$$

$$\boxed{E[X] = \frac{1}{\lambda}}$$

$$\frac{d^2}{ds^2} \bar{\mathbb{E}}_x(s)^2 \frac{d}{ds} \left(\frac{A}{(A-s)^2} \right) = \frac{d}{ds} A(A-s)^{-2}$$

$$= -2A(A-s)^{-3}(-1) = \frac{2A}{(A-s)^3}$$

$$E[x]^2 = \frac{d^2}{ds^2} \bar{\mathbb{E}}_x(s) \Big|_{s=0} = \frac{2A}{A^3} = \frac{2}{A^2}$$

$$\boxed{E[x]^2 = \frac{2}{A^2}}$$

$$E[x^n] = \frac{d^n}{ds^n} \bar{\mathbb{E}}_x(s) \Big|_{s=0} = \frac{n!A}{(A-s)^{n+1}} \Big|_{s=0}$$

$$\boxed{E[x^n] = \frac{n!}{A^n}}$$

J & K are independent RV's

$$P_J(j) = \begin{cases} 0.2 & j=1 \\ 0.6 & j=2 \\ 0.2 & j=3 \\ 0 & \text{otherwise} \end{cases}$$

$$P_K(k) = \begin{cases} 0.5 & k=-1 \\ 0.5 & k=2 \\ 0 & \text{otherwise} \end{cases}$$

(a) MGF of $M = J + K$

(b) $E[M^3]$ (c) $P_M(m)$

Solv

$$\mathbb{E}_x(s) = E[e^{sx}] \\ = \sum_{x_i \in s_x} e^{sx_i} P_x(x_i)$$

$$\mathbb{E}_y(s) = \sum_{j \in s_y} e^{sj} P_j(j)$$

$$= (0.4)e^s + 0.6e^{2s} + 0.2e^{3s}$$

$$\mathbb{E}_K(s) = (0.5)e^s + 0.5e^{2s}$$

$$(a) M = J + K$$

$$\mathbb{E}_M(s) = \mathbb{E}_J(s) \mathbb{E}_K(s)$$

$$\mathbb{E}_M(s) = (0.2e^s + 0.6e^{2s} + 0.2e^{3s}) \\ \times (0.5e^s + 0.5e^{2s})$$

$$\mathbb{E}_M(s) = (0.1) + 0.3e^s + 0.1e^{2s} + 0.1e^{3s} + 0.3e^{4s}$$

$$\mathbb{E}_M(s) = 0.1 + 0.3e^s + 0.2e^{2s} + 0.3e^{3s} + 0.1e^{4s}$$

$$(b) E[M^3] = ? = \frac{d^3}{ds^3} \mathbb{E}_M(s) |_{s=0}$$

$$\frac{d}{ds} \mathbb{E}_M(s) = 0.3e^s + 0.4e^{2s} + 0.9e^{3s} + 0.4e^{4s}$$

$$\frac{d^2}{ds^2} \mathbb{E}_M(s) = 0.3e^s + 0.8e^{2s} + 2.7e^{3s} + 1.6e^{4s}$$

$$\frac{d^3}{ds^3} \phi_{X_1}(s) = 0.3e^s + 1.6e^{2s} + 8.1e^{3s} + 6.4e^{4s}$$

$$E[X^3] = \frac{d^3}{ds^3} \phi_{X_1}(s) \Big|_{s=0} = 0.3 + 1.6 + 8.1 + 6.4$$

$$[E[X^3] = 16.4]$$

$$P_M(m) = \begin{cases} 0.1 & m=0 \\ 0.3 & m=1 \\ 0.2 & m=2 \\ 0.3 & m=3 \\ 0.1 & m=4 \\ 0 & \text{otherwise} \end{cases}$$

Theorem :- (YOU MUST KNOW EXAMINENTHANA)

For a set of independent RV's

$$X_1, X_2, \dots, X_n, MGF of$$

$$W = X_1 + X_2 + \dots + X_n$$

is given by.

$$\phi_W(s) = \phi_{X_1}(s) \phi_{X_2}(s) \dots \phi_{X_n}(s)$$

If RV are iid, then

$$\phi_W(s) = [\phi_X(s)]^n$$

Central Limit Theorem (CLT) :-

Independent : x_1, x_2, \dots, x_n

$$X = x_1 + x_2 + \dots + x_n = \sum_{i=1}^n x_i$$

$$E[X] = E[x_1] + E[x_2] + E[x_3] + \dots + E[x_n]$$

$$\mu_X = \mu_1 + \mu_2 + \dots + \mu_n$$

Variance of X :-

$$\text{VAR}[X] = \sigma_X^2 = \sigma_1^2 + \sigma_2^2 + \dots + \sigma_n^2$$

Central limit theorem states
that

as 'n' increases, the distribution
 $F_X(n)$ of random variable X approaches
a normal distribution or (Gaussian).

$$F_X(n) = \Phi\left(\frac{n - \mu}{\sigma}\right)$$

$$f_X(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2} \left(\frac{x-\mu}{\sigma}\right)^2}$$

if

$$\mu_1 = \mu_2 = \mu_3 = \dots = \mu_n$$

$$\sigma_x^2 = \frac{1}{n} \sum_{i=1}^n \sigma_i^2$$

$$\sigma = \sqrt{n} \sigma_x$$

$$Z = \frac{\bar{X} - \mu}{\sigma} = \frac{\sum_{i=1}^n \bar{X}_i - n\mu}{\sqrt{n} \sigma_x}$$

Example

$$E[X] = p = 0.5$$

$$\text{VAR}[X] = p(1-p) = 0.25$$

$$n = 10^6 = 1,000,000$$

$$W = X_1 + X_2 + \dots + X_n = \sum_{i=1}^n X_i$$

$$E[W] = n \cdot p = 10^6 \cdot 0.5 = 5 \times 10^5$$

$$P(X=x) = \begin{cases} p, & x=1 \\ 1-p, & x=0 \\ 0, & \text{otherwise} \end{cases}$$

$$E[X] = p$$

$$E[X^2] = p$$

$$\text{VAR}[X] = E[X^2] - (E[X])^2$$

$$= p - p^2 \\ = p(1-p)$$

$$\sigma_w^2 = \text{VAR}[W] = n \text{VAR}[X_i]$$

$$= 10^6 \cdot 0.25$$

$$\sigma_w^2 = 2.5 \times 10^5 = 250,000$$

$$\boxed{\sigma_w = 500}$$

$$P[W \geq 502,000] = 1 - P[W \leq 502,000]$$

transform to $w \rightarrow z$

$$= 1 - \Phi\left(\frac{w - \mu_w}{\sigma_w}\right)$$

$$= 1 - \Phi\left(\frac{502,000 - 500,000}{500}\right)$$

$$= 1 - \Phi(4) \approx Q(4)$$

$$Q(4) \approx 1 - \Phi(3)$$

$$= \boxed{P[W \geq 502,000], 3.17 \times 10^{-8}}$$

$$P[A] = P[499000 \leq \omega \leq 501000]$$

Using same formula:

$$= F_w(501000) - F_w(499000)$$

$$= \bar{\Phi}\left(\frac{501000 - 500000}{500}\right) - \bar{\Phi}\left(\frac{499000 - 500000}{500}\right)$$

$$= \bar{\Phi}(2) - \bar{\Phi}(-2)$$

$$= \bar{\Phi}(2) - [1 - \bar{\Phi}(2)] = 2\bar{\Phi}(2) - 1$$

$$= 2(0.97725) - 1$$

$$\boxed{P[A] = 0.9545}$$

(Ch # 07)

RV \rightarrow Sample Mean

$$M_n(x) = \frac{x_1 + x_2 + \dots + x_n}{n}$$

Theorems:-

$$E[M_n(x)] = E[x]$$

$$\text{VAR}[M_n(x)] = \frac{\text{VAR}[x]}{n}$$

As n increases without bound

$$\lim_{n \rightarrow \infty} \text{VAR}[M_n(x)] = \frac{\text{VAR}[x]}{n} = 0$$

Markov Inequality:-

For a RV x such that $P[x < 0] = 0$ and a constant $c > 0$, i.e; $f(x) = 0$ for $x < 0$,

$$P[X \geq K] \leq \frac{E[X]}{K} = \frac{\mu}{K}$$

Markov Inequality is valid only for non negative

Example :-

$$E[X] = 5.5$$

$$P[X \geq 11] = \frac{5.5 - 1}{11} = \frac{1}{2}$$

$$\boxed{P[X \geq 11] \leq \frac{1}{2}}$$

Chebyshev Inequality:-

For an arbitrary random variable X

and constant $k > 0$;

$$P[|X - \mu| \geq k] \leq \frac{\text{VAR}[X]}{k^2} = \frac{\sigma^2}{k^2} \quad (1)$$

let $R = n\sigma$ in (1)

$$P[|X - \mu_n| \geq n\sigma] \leq \frac{\sigma^2}{n^2\sigma^2} = \frac{1}{n^2}$$

$$P[|X - \mu_n| \geq n\sigma] \leq \frac{1}{n^2} \quad (2)$$

Multiply ① by -1

$$-P[|X - \mu_x| \geq k] > -\frac{\sigma^2}{k^2}$$

$$1 - P[|X - \mu_x| \geq k] > 1 - \frac{\sigma^2}{k^2}$$

However;

$$1 - P[Y \geq y] = P[Y < y]$$

Therefore;

$$P[|X - \mu_x| < k] > 1 - \frac{\sigma^2}{k^2} \quad \text{--- (3)}$$

Similarly from ②

$$P[|X - \mu_x| < h] > 1 - \frac{1}{n^2} \quad \text{--- (4)}$$

$$\pm (X - \mu_x) \geq k$$

$$(X - \mu_x) \geq k \quad \text{and} \quad -X + \mu_x \geq k$$

$$X \geq \mu_x + k \quad \text{and} \quad X \leq \mu_x - k$$

Therefore;

$$|X - \mu_x| \geq k \Rightarrow \mu_x + k \leq X \leq \mu_x - k$$

$$P(A) = 1 - P(\bar{A})$$

$$\frac{\text{Var}[A]}{nC^2}$$

$$0.0099$$

$$P(D) \geq \begin{cases} 0.2 & D=1 \\ 0.8 & D=0 \\ 0 & \text{otherwise} \end{cases}$$

$$E[X] = 0.2 \quad \text{Var}[X] = 0.2 - (0.2)^2 \\ \approx 0.16$$

$$E[k_{100}] = 20$$

$$\sigma(k_{100}) \approx 0.16 \times 100 = \sqrt{16} = 4$$

$$P_{Mm} = \begin{cases} 0.42 & m=3 \\ 0.40 & m=2 \\ 0.18 & m=1 \end{cases}$$

$$\int_a^b \frac{1}{e^{-\frac{x^2}{2}}}$$

$$= \frac{e^{-\frac{b^2}{2}} - e^{-\frac{a^2}{2}}}{2(b-a)}$$

$$P[(P(A) - P(\bar{A})) \geq 10^{-5}] \leq \frac{P(\bar{A})(1 - P(\bar{A}))}{nC^2}$$

$$\geq \frac{10^{-3}(1 - 10^{-3})}{n(10^{-5})^2} = \frac{9.99 \times 10^{-6}}{n10^{-10}} \\ \approx \frac{9.99 \times 10^{-6}}{n}$$

$$[P_n(A) - P(A) < 0.05] \geq 1 - \frac{\text{Var}[X_A]}{n\sigma^2}$$

$$\alpha = \frac{0.09}{9(0.03)^2}$$

$$0.1 = \frac{\sigma_x^2}{n(0.03)^2}$$

$$0.1 = \frac{(0.09)^2}{n(0.03)^2}$$

$$n = \frac{8.1 \times 10^3}{9 \times 10^4}$$

$$0.99 = 1 - (15)$$

n &