Probability

TUE

C

Set :-

A set is a collection of things

e.g.

$A = \{$All the students registered in sect. A of PME$\}$

$B = \{$All the students in section A of PME with CGPA $> 3.0\}$

$C = \{$Amjad, Akram, Bill$\}$

$D = \{x^2 / x = 1, 2, 3, 4\}$.

OR

$D = \{1, 4, 9, 16\}$ ← Tabular method

$D = \{x^2 | x = 1, 2, 3 \ldots\}$ ← Set builder form or Property method.

Subset :-

$A \subseteq A$

If $A \subset B$ and $A \neq B$, then we say that "A is a proper subset of B".

$A \subset B$

If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$

Equal Set :-

$A = B$ if and only if $B \subseteq A$ and $A \subseteq B$.

Universal Set :-

All sets under instigation in any problem of set theory are assumed to be contained in some large fixed set called "universal set" or "universe of discourse".

denoted by :- "U or S".

e.g. if we are investigating the grades of current students of ECE in PME, then

S = {All students of ECE who are study PME}.

Set of Numbers :-

N = Natural numbers or positive Integers
{1, 2, 3, 4, ...}

Z = All integers, -ve or +ve and 0.
= {... -2, -1, 0, 1, 2, ...}

Q = Rational Numbers
= $\{p/q \mid p, q \in Z, q \neq 0\}$

I = Irrational Numbers
= $\{x \mid x \in R \wedge x \notin Q\}$

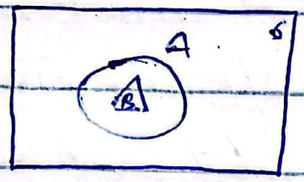
R = Real Number

R = Q ∪ I

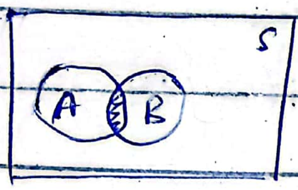
Note that:
$N \subseteq Z \subseteq Q \subseteq R$

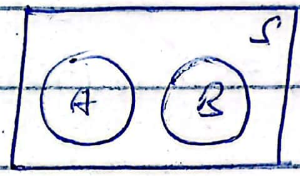
Venn Diagram:

$B \subseteq A$



$A \cap B$



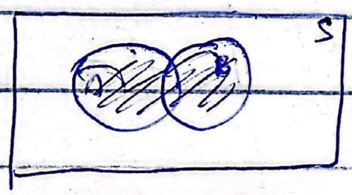


$A \cap B = \phi$ Disjoint set

Union :-

$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$.

$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$.

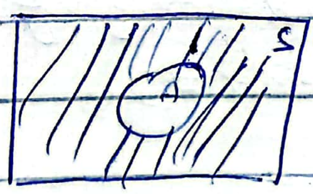


$A \cup B$

Complement :-

Absolute complement or simply complement of a set A is denoted by A^c or A' $\bar{A}$.

$A^c = \{x \mid x \in S, x \notin A\}$.



A^c

Note

$S^c = \phi$

$(A^c)^c = A$

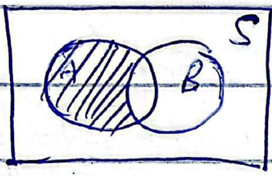
Set Difference:-

The difference b/w sets A and B is denoted by $A-B$, A/B or $A\sim B$.

$A-B$ contains all elements of A that are not elements of B.

$$A-B=\{x \mid x\in A,\ x\notin B\}.$$

$A-B$ may be considered as relative compliment of set B w.r.t set A.



Note that

$$\boxed{A-B=A\cap B^c} \qquad A^c=S-A$$

$$A-B\subseteq A$$

$$\bigcup_{i=1}^{n} A_i=A_1\cup A_2\cup A_3\cdots\cup A_n$$

$$\bigcap_{i=1}^{n} A_i=A_1\cap A_2\cap A_3\cdots\cap A_n$$

Mutually Exclusive Set:-

A collection of sets $A_1, A_2, \dots, A_n$ is mutually exclusive if and only if

$$A_i\cap A_j=\phi \quad \forall\ i\neq j$$

Collectively Exhaustive Sets:-

A collection of sets $A_1, A_2, \dots A_n$ is collectively exhaustive if and only if

$$A_1 \cup A_2 \cup \dots \cup A_n = S$$

Principle of duality:-

The dual E^* of E is the equation obtained by replacing each occurence of $\cup, \cap, S, \emptyset$ in E by $\cap, \cup, \emptyset, S$ respectively.

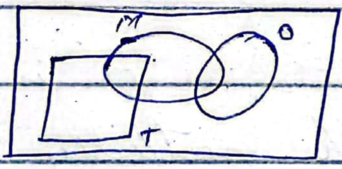
e.g. the dual of

$$(S \cap A) \cup (B \cap A) = A$$

$$(\emptyset \cup A) \cap (B \cup A) = A$$

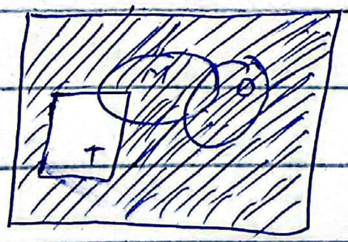
[Quiz 1.1 (Solve) (H.w)]

Quiz 1.1:-

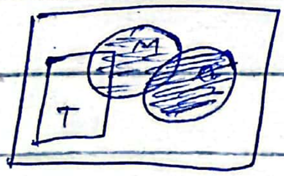


$S = \{T, R\}$

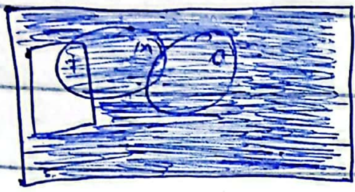
(i) R:-



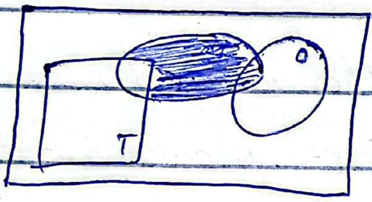
(ii) $M \cup O$



(4) $R \cup M$

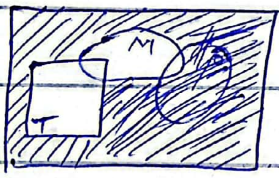


(5) $R \cap M$



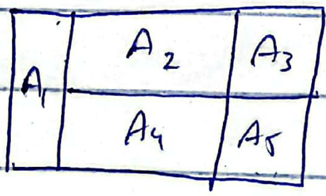
(6) $T^c - M$

$T^c - M = \{x \mid x \in T^c, x \notin M\}$.



Partition:-

A collection of sets $A_1, A_2, \ldots A_n$ is a Partition if it is both mutually exclusive and collectively exhausted



Experiment

Toss a coin - Procedure

Observe head or tail - observation

Model - Head or tail

(ii) Outcome

(iii) Sample Space

The Sample Space of an experiment is it mutually exclusive, collectively exhausted set of all possible outcomes.

$$S = \{0, 1, 2\}.$$

if sequence - $S = \{TT, TH, HT, HH\}$.

Event :-

An event is a set of outcomes of an experiment.

$S = \{1, 2, 3, 4, 5, 6\}$.

Each subset of a sample space is an event.

$E_1 = \{4, 5, 6\}$. $E_1 \subseteq S$

$E_2 = \{$The roll is even$\}$.

$E_2 = \{2, 4, 6\}$. $E_2 \subset S$

$E_3 = \{$The roll is the square of a integer$\}$.

$E_3 = \{1, 4\}$.

$S = \{x \mid x \geq 0\}$
$E = \{x \mid x > 5\}$.

Set Algebra	Probability
Universal Set	Sample space
Element	outcome
Subsets	Event.

Ex $S = \{rrr, rrg, rgr, rgg, grr, grg, ggr, ggg\}$.

if we let

$r = 1, g = 0$.

then

0	0	0
0	0	1
0	1	0
0	1	1
1	0	0
1	0	1
1	1	0
1	1	1

$S = \{ggg, ggr, grg, grr, rgg, rgr, rrg, rrr\}$.

$R_1 = \{rgg, rgr, rrg, rrr\}$.
$G_3 = \{ggg, grg, rgg, rrg\}$.

Quiz 1.2:-

$v = 1, \ d = 0$

$S = \{ddd, ddv, dvd, dvv, vdd, vdv, vvd, vvv\}$

(i) $A_1 = \{$first call is a voice call$\}$.
$A_1 = \{vdd, vdv, vvd, vvv\}$.

(ii) $B_1 = \{$first call is a data call$\}$.
$B_1 = \{ddd, ddv, dvd, dvv\}$

(iii) $A_2 = \{$Second call is a voice call$\}$.

$A_2 = \{dvd, dvv, vvd, vvv\}$.

(iv) $B_2 = \{$Second call is a data call$\}$.

$B_2 = \{ddd, ddv, vdd, vdv\}$.

(v) $A_3 = \{$all calls are the same$\}$.

$A_3 = \{ddd, vvv\}$.

(vi) $B_3 = \{$voice and data alternate$\}$.

$B_3 = \{dvd, vdv\}$.

(vii) $A_4 = \{$one or more voice calls$\}$.

$A_4 = \{ddv, dvd, dvv, vdd, vdv, vvd, vvv\}$.

(viii) $B_4 = \{$Two or more data calls$\}$.

$B_4 = \{ddd, ddv, dvd, vdd\}$.

(i) A_1 and B_1

Yes, it is mutually exclusive and collectively exhaus

	Mutually Exclusive	Collectively Exhaustive
A_1 and B_1	YES	YES
A_2 and B_2	YES	YES
A_3 and B_3	YES	NO
A_4 and B_4	NO	YES

Points:-

1. Outcome is also called sample point.
2. The event $\{a\}$ consisting of a single point $a \in S$ is called elementary event.
3. ϕ and S are both subsets of S.
   ϕ is sometimes called "null" or "Impossible" event.
   S is called "sure" or "certain" events.
   $$S = \{T, H\}.$$
4. Union of events is the event $(A \cup B)$ iff A occurs or B occurs or both occurs.
5. $(A \cap B)$ is the event that occurs iff A occurs and B occurs.
6. A^c is ~~also~~ the event that occurs iff A does not occurs.
7. Events A & B are called mutually exclusive if they are disjoint. i.e; $A \cap B = \phi$.
8. Three or more events are mutually exclusive if every two of them are mutually exclusive.

Probability Axioms

$P[A]$ $P(A)$

The probability of an event is the proportion of the time that event is observed in a larged number of runs of the experiment

- A probability measure $P[\cdot]$ is a type of function that maps events in the sample space to Real numbers.

Such that

① For any event A

$$P[A] \geq 0$$

② $P[S] = 1$: probability of the Sample Space is 1

e.g. $S = \{h, t\}$

$$P[h] + P[t] = P[S] = 1.$$

For any countable collection

③ A_1, A_2 -- of <u>mutually exclusive</u> events.

$$P[A_1 \cup A_2 \cup \cdots] = P[A_1] + P[A_2] + \cdots$$

$$P[A_1 \cup A_2] = P[A_1] + P[A_2].$$

$$A_1 \cap A_2 = \phi$$

$$P[A_1 \cap A_2] = P[A_1 A_2] \overset{(or)}{=} P[A_1, A_2]$$

Theorem :-

(1) If $A = A_1 \cup A_2 \cup \cdots \cup A_m$ and $A_i \cap A_j = \phi$ for $i \neq j$; then

$$P[A] = \sum_{i=1}^{m} P[A_i].$$

(2) The ~~probodi~~ probability of an event $B = \{s_1, s_2, \cdots s_m\}$ is the sum of the probabilities of the outcomes contained in the event

$$P[B] = \sum_{i=1}^{m} P[s_i]$$

For an experiment with sample space $S = \{s_1, s_2, \cdots s_n\}$, in which each outcome is equally likely

$$P[s_i] = 1/n \qquad 1 \leq i \leq n$$

$$P[s_1] = P[s_2] = \cdots = P[s_n].$$

$$P[s_1] + P[s_2] + \cdots P[s_n] = 1$$

$$x + x + x \cdots + x = 1$$

$$nx = 1$$

$$\boxed{x = 1/n}$$

Example :-

$S = \{1, 2, 3, 4, 5, 6\}$

$P[1] = P[2] = P[3] = P[4] = P[5] = P[6] = P[x]$

$P[1] + P[2] + \cdots + P[6] = 1$

$P[x] + P[x] + \cdots + P[x] = 1$

$6 \times P[x] = 1$

$\boxed{P[x] = 1/6}$

(a) Roll 4 or higher :-

$P[x \geq 4] = P[4 \leq x \leq 6] = P[4, 5, 6]$

$= P[4] + P[5] + P[6] = 1/6 + 1/6 + 1/6$

$= \frac{3}{6} = \boxed{1/2}$

(b) Roll an even number :-

$P[2, 4, 6] = P[2] + P[4] + P[6] = 1/6 + \frac{1}{6} + 1/6$

$= \boxed{1/2}$

(c) Roll the square of an integer :-

$P[1, 4] = P[1] + P[4] = \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$

Quiz #

$0 \leq T \leq 100$

$0 \sim 100 \approx s_0, s_1, \ldots, s_{100}$

$90 \sim 100 \rightarrow A$

$80 \sim 89 \rightarrow B$

$70 \sim 79 \rightarrow C$

$60 \sim 69 \rightarrow D$

$T < 60 \rightarrow F$

51 - 100 are equally likely

$T \leq 50$ never occurs

Find

① $P[\{s_{79}\}] = P[79]$

② $P[\{s_{100}\}] = P[100]$

③ $P[A]$ ④ $P[F]$ ⑤ $P[T \geq 80]$

⑥ $P[T < 90]$ ⑦ $P[C \text{ or better}]$

⑧ $P[\text{student Passes}]$

$T = \{51, 52, \ldots, 100\}$

$S = \{s_{51}, s_{52}, \ldots, s_{100}\}$.

$P[s_i] = \frac{1}{50} = 0.02$

① $P[\{s_{79}\}] = P[T = 79]$.

$= \frac{1}{50} = 0.02 = 2\%$ probability

(2) $P[\{O_{100}\}] = P[T=100]$

$= \frac{1}{50} = 0.02 = 2\%$

(3) $P[A] = P[90] + P[91] + \cdots P[100]$.

$= \frac{11}{50} = 11 \times 0.02 = 0.22 = 22\%$

(4) $P[F] = P[50 < T < 60]$.

$= P[51] + P[52] + \cdots P[59]$

$= \frac{9}{50} = 9 \times 0.02 = 18\%$

(5) $P[T \geq 80]$

$= P[80] + P[81] + \cdots P[100]$.

$= 21 \times 0.02 = 0.42 = 42\%$

(6) $P[T < 90]$

$= P[51] + \cdots P[89]$ $\boxed{\rightarrow (89-51)+1}$

$= 39 \times 0.02 = 0.78 = 78\%$

(7) $P[\text{C or better}]$

$P[T \geq 70]$

$P[70] + \cdots + P[100]$

$= 31 \times 0.02 = 0.62 = 62\%$

(8) $P[\text{student passes}]$.

$P[T \geq 60] = P[60] + \cdots P[100]$

$= 41 \times 0.02 = 0.82 = 82\%$

Rules:-

$P[\phi] = 0, \quad P[S] = 1$

$P[A] + P[A^c] = 1.$

$A \cup A^c = S$

$P[A \cup A^c] = P[S]$

$P[A] + P[A^c] = 1.$

$\boxed{P[A^c] = 1 - P[A]} \rightarrow$ Rule

Theorem / Addition Rule

$P[A \cup B] = P[A] + P[B] - P[A \cap B].$

$\boxed{P[A \cup B] = P[A] + P[B] - P[AB].}$

If $A \cap B$ is ϕ, then

$P[A \cap B] = P[A] + P[B] - P[\phi].$

$= P[A] + P[B].$

Theorem:-

If $A \subset B$, then

$P[A] \leq P[B].$

Ex# 1.14:-

l, b $\{v, d, f\}$.

$t > 3 \to l$

$t < 3 \to b$

$S = \{lv, bv, ld, bd, lf, bf\}$

	v	d	f	
l	0.3	0.12	0.15	0.57
B	0.2	0.08	0.15	0.43
	0.50	0.20	0.30	

Adding all results 1.

$S = \{lv, bv, ld, bd, lf, bf\}$

$S = \{0.3, 0.2, 0.12, 0.08, 0.15, 0.15\}$

$P[L] = P[LV] + P[LD] + P[LF]$

$= 0.3 + 0.12 + 0.15 = 57\%$

$P[B] = P[BV] + P[BD] + P[BF]$

$= 0.2 + 0.08 + 0.15$

$= 0.43 = 43\%$

$P[L] = 1 - P[B]$

$P[B] = 1 - P[L]$

$P[L] + P[B] = 1$

Quiz 1.4 :-

$P[L] = 0.6$

$P[VL] = 0.35$

$P[V] = 0.7$

(i) $P[DL]$ or $P[D \cap L]$ (ii) $P[D \cup L]$

(iii) $P[VB]$ (iv) $P[V \cup L]$ (v) $P[V \cup D]$

(vi) $P[LB]$

	V	D	
L	0.35	0.65	0.60
B	0.65	0.35	0.40
	0.70	0.30	

$P[L] + P[B] = 1$

$P[B] = 1 - P[L]$

$P[B] = 0.4$

$P[V] + P[D] = 1$

$0.70 + P[D] = 1$

$P[D] = 0.30$

	V	D	
L	0.35	0.25	0.60
B	0.35	0.05	0.40
	0.70	0.30	

$P[D \cap L] = P[D] + P[L] - P[DL]$.

$= 0.30 + 0.60 - 0.25$

$= 0.65$

$P[D \cup L] = \cancel{0.30}\ 0.35 + 0.25 + 0.05 = 0.65$

$P[VB] =$

Conditional Probability

$$P[A|B] = \frac{P[A \cap B]}{P[B]} = \frac{P[AB]}{P[B]}$$

$P[AB] = P[A|B] \times P[B]$

1 Month of 30 days

B Clouds 7-Days

A Rain 2 Days

$$P[B] = \frac{7}{30}$$

$$P[R] = \frac{2}{30} = 0.067$$

$A \cap B = 2$

$$P[A \cap B] = \frac{2}{30}$$

$$P[A|B] = \frac{P[AB]}{P[B]} = \frac{2/30}{7/30} = \frac{2}{7}$$

$P[A|B] = 0.285$

CamScanner

Theorem axioms

1. $P[A|B] \geq 0$
2. $P[B|B] = 1$
3. If $A = A_1 \cup A_2 \cup \cdots$
   with $A_i \cap A_j = \phi \; \forall, i \neq j$;
   then
   $$P[A|B] = P[A_1|B] + P[A_2|B] + \cdots$$

Ex# 1.15
1.16

Example #1.16:-

$B = \{rr, ra\}$
$A = \{rr, ar\}$.
$P[rr] = 0.01$, $P[ra] = 0.01$, $P[ar] = 0.01$
$P[aa] = 0.97$.

A = 2nd chip rej. $\quad A \cap B = [rr]$
B = 1st chip rej.

$P[A] = P[rr] + P[ar] = 0.01 + 0.01 = 0.02$
$P[B] = P[rr] + P[ra] = 0.01 + 0.01 = 0.02$

$$P[A|B] = \frac{P[A \cap B]}{P[B]} = \frac{P[rr]}{P[B]} = \frac{0.01}{0.02} = \frac{1}{2}$$

$= 0.5 = 50\%$

Law of Total Probability :-

For a partition $\{B_1, B_2, \cdots, B_m\}$ with $P[B_i] > 0 \ \forall i$,

$$P[A] = \sum_{i=1}^{m} P[A \mid B_i] \times P[B_i]$$

i.e; $P[A] = P[A \mid B_1] \times P[B_1] + P[A \mid B_2] \times P[B_2] + \cdots P[A \mid B_m] \times P[B_m]$.

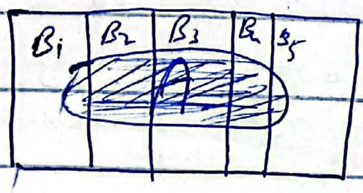
$$P[A] = P[A \cap B_1] + P[A \cap B_2] + \cdots P[A \cap B_n]$$
$$P[A] = P[A \mid B_1] \times P[B_1] + \cdots P[A \mid B_n] \times P[B_n].$$

Theorem :-

For any event A and partition $\{B_1, B_2, \cdots, B_m\}$

$$P[A] = \sum_{i=1}^{m} P[A \cap B_i].$$

$$P[A] = P[A \cap B_1] + P[A \cap B_2] + \cdots + P[A \cap B_m]$$



Law of Total Probability:-

For any partition $\{B_1, B_2, \ldots B_m\}$.
with $P[B_i] > 0 \;\forall\, i$

$$P[A] = \sum_{i=1}^{m} P[A|B_i] \times P[B_i]$$

$$P[A] = P[A|B_1] \times P[B_1] + P[A|B_2] \times P[B_2] + \cdots + P[A|B_m] \times P[B_m].$$

Example #1.19:-

resistors 1k

let,

A = {Resistor is within set of nominal value}.

= 950 - 1050

$P[A|B_1] = 0.8$

$P[A|B_2] = 0.9$

$P[A|B_3] = 0.6$

Total resistors = 3k + 4k + 3k = 10k

$$P[B_1] = \frac{3000}{10,000} = 0.3$$

$P[B_2] = 0.4$

$P[B_3] = 0.3$

$P[A] = P[A|B_1] \times P[B_1] + P[A|B_2] \times P[B_2] + P[A|B_3] \times P[B_3]$

$P[A] = 0.8 \times 0.3 + 0.9 \times 0.4 + 0.6 \times 0.3 = 0.78$

$P[A] = 0.78$

Baye's Theorem:-

$$P[A|B] = \frac{P[AB]}{P[B]}$$

$$P[B|A] = \frac{P[BA]}{P[A]} = \frac{P[A|B] \times P[B]}{P[A]}$$

$$P[B|A] = \frac{P[A|B] \times P[B]}{P[A]}$$

Ex#1.20:-

$P[B_3] = 0.3$

$P[A] = 0.78$

$P[A|B_3] = 0.6$

what is the ~~propabibility~~ probability that an ~~an~~ comps for B_3?

$P[B_3|A] = ?$

$$P[B_3|A] = \frac{P[A|B_3] P[B_3]}{P[A]}$$

$$= \frac{[0.6][0.3]}{0.78} = \boxed{0.23 = P[B_3|A]}$$

Quiz #1.5:-

(1) $P[N_V = 2]$ (2) $P[N_V \geq 1]$

(3) $P[\{vvd\} N_V = 2]$ (4) $P[\{ddv\} N_V \geq 2]$.

(5) $P[N_V = 2 | N_V \geq 1]$ (6) $P[N_V \geq 1 | N_V = 2]$

$S = \{vvv, vvd, vdv, vdd, dvv, dvd, ddv, ddd\}$

$P[vvv] = 0.2$

$P[ddd] = 0.2$

$P[vvd] = P[vdv] = P[vdd] = P[dvv] = P[dvd]$
$= P[ddv] = 0.1$

$N_V = \{0, 1, 2, 3\}$

(i) $P[N_V = 2]$

Events with exactly two voice calls:

$S_{V=2} = \{vvd, vdv, dvv\}$

$P[N_V = 2] = P[vvd] + P[vdv] + P[dvv] = 0.3$

(2) $P[N_V \geq 1]$

$P[N_V \geq 1] = P[vvv] + P[vvd] + P[vdv] + P[vdd]$
$+ P[dvv] + P[dvd] + P[ddv]$
$= 0.8$

(3) $P[\{vvd\} | N_v = 2]$

$$P[\{vvd\} | N_v = 2] = \frac{P[\{vvd\} \cap \{N_v = 2\}]}{P[N_v = 2]} = \frac{\cancel{(0.1)(0.3)}}{\cancel{0.3}}$$

$$P = \frac{[\{vvd\} \cap \{vvd, vdv, dvv\}]}{P[N_v = 2]} = \frac{P\{vvd\}}{P[N_v = 2]} = \frac{0.1}{0.3}$$

$$P[\{vvd\} | N_v = 2] = 0.33 = \frac{1}{3}$$

(4) $P[ddv | N_v = 2] = P[ddv] \cap [N_v = 2] = \frac{0}{P[N_v = 2]} = 0$

$P[\{ddv\} \cap [N_v = 2] = \{ddv\} \cap [vvd, vdv, dvv\}$.

(5) $P[N_v = 2 | N_v \geq 1] = \frac{P[N_v = 2]}{P[N_v \geq 1]} = \frac{0.3}{0.8} = \frac{3}{8}$

(6) $P(N_v \geq 1 \ \& \ N_v = 2] = \frac{P[N_v = 2]}{P\{N_v = 2]} = 1$

Independent Events:-

Events A and B are independent if and only if

$$P[A \cap B] = P[A] \, P[B].$$

If $P[B|A] = P[B]$

or $P[A|B] = P[A]$

Independence is a symmetric relationship

$$P[A \cap B] = P[A]\, P[B]$$

$$P[B|A] = P[B] \text{ and } P[A|B] = P[A].$$

(i) Independent and disjoint are NOT synonyms

(ii) Disjoint events are not Independent unless one of them has zero probability

Suppose A and B are disjoint and also independent at the same time. Then, as disjoint events;

$$A \cap B = \phi$$

$$P[A \cap B] = P[\phi] = 0 \quad (1)$$

As Independent events:

$$P[A \cap B] = P[A]\, P[B] \quad (2)$$

Comparing (1) and (2)

$$P[A]\, P[B] = 0$$

Therefore;

either $P[A] = 0$ or $P[B] = 0$.

$$P[A \mid B] = \frac{P[A \cap B]}{P[B]}$$

$$P[B \mid A] = \frac{P[A \mid B] \times P[B]}{P[A]}.$$

RANDOM VARIABLES:-

A random variable is a real number assigned to each outcome in a sample space of a random experiment.

- A discrete random variable is a random variable with a finite range and continous random variable is a variable with an interval of real numbers of ranges.

Example:-

Two balls 3 Red balls, 4 Black balls:-

Sample Space

= {RR, RB, BR, BB}.

Y # No. of Red balls:-

$\boxed{Y = 2, 1, 0}$

Example #

A stockroom clerks return, three steel mill employees, If Smith, Jones and Brown

S.S = {SJB, SBJ, BSJ, JBS, JSB, BJS}.

Y # correct matches

SJB → 3

SBJ → 1

$SJB \to 3$

$JBJ \to 1$

$BSJ \to 0$

$JBS \to 0$

$JSB \to \cancel{0}1$

$BJS \to \cancel{0}1$

Example

$$SS = \{\underset{2}{PP}, \underset{1}{PF}, \underset{1}{FP}, \underset{0}{FF}\}$$

Independent Events:

Independence of three events:-

Three events A_1, A_2 & A_3 are independent if and only if:

(a) They are Pairwise Independent

i.e; $P[A_1 \cap A_2] = P[A_1] \cdot P[A_2]$.

$P[A_1 \cap A_3] = P[A_1] \cdot P[A_3]$

$P[A_2 \cap A_3] = P[A_2] \cdot P[A_3]$

(b) $P[A_1 \cap A_2 \cap A_3] = P[A_1] \cdot P[A_2] \cdot P[A_3]$.

∴ Examples of Independent events (H.W)

Quiz #1.6:-

$P[v] = 0.8$

$P[d] = 0.2$

, $C_i = V$; $C_1 = v$

$C_2 = d$

$N_V = \{0, 1, 2)$

~~(a) $\{N_V = 2\}$ and $\{N_V \geq 1\}$:-~~

$S.S = \{vv, vd, dv, dd\}$

$P[vv] = P[v][v] = 0.8 \times 0.8 = 0.64.$

$P[vd] = 0.8 \times 0.2 = 0.16$

$P[dv] = 0.8 \times 0.2 = 0.16$

$P[dd] = 0.2 \times 0.2 = 0.04$

(a) $\{\underset{A}{N_V = 2}\}$ and $\{\underset{B}{N_V \geq 1}\}$:-

L.H.S. $P[A \cap B] = P[A] \times P[B]$,

$P[A \cap B] = P[vv] \cap P[vd, dv, vv]$

$= P[vv] = 0.64.$

R.H.S

$[N_V \geq 1] =$ ~~$P[A] \exp P[B]$~~

$[N_V \geq 1] = 0.64 + 0.16 + 0.16 = 0.96$

$\{N_V = 2\} \cap \{N_V \geq 1\} = \{vv\} = \{N_V = 2\} = 0.64$ (1)

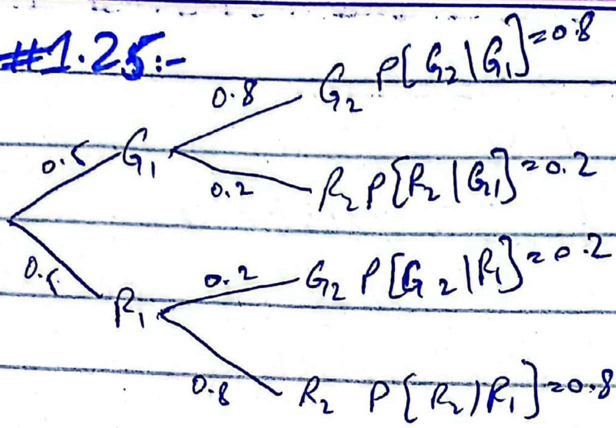
$P[N_V = 2] \times P[N_V \geq 1] = 0.64 \times 0.96 = 0.6144$ — (2)

From (1) and (2), we observe that

$P[N_V = 2 \cap N_V \geq 1] \neq P[N_V = 2] \times P[N_V \geq 1].$

Hence, the events are Not independent.

Example #1.25:-



$$P[A|B] = \frac{P[AB]}{P[B]}$$

$$P[G_1 G_2] = P[G_2|G_1] \times P[G_2]$$
$$= 0.8 \times 0.5 = 0.40$$
$$P[G_1 R_2] = P[R_2|G_1] \times P[G_1] = 0.2 \times 0.5 = 0.10$$
$$P[R_1 G_2] = P[G_2|R_1] \times P(R_1] = 0.2 \times 0.5 = 0.10$$
$$P[R_1 R_2] = P[R_2|R_1] \times P[R_1] = 0.8 \times 0.5 = 0.40$$

① $P[G_2] = P[G_1 G_2] + P[R_1 G_2]$
$$= 0.40 + 0.10 = 0.50$$

② $P[W] = P[G_1 R_2] + P[R_1 G_2] + P[R_1 R_2]$
$$= 0.10 + 0.10 + 0.40$$
$$\boxed{P[W] = 0.60}$$

③ $P[G_1|R_2] = ?$
$$P[G_1|R_2] = \frac{P[G_1 R_2]}{P[R_2]}$$
$$= \frac{0.10}{0.50} = \boxed{0.20}$$

$\because$ $P(R_2] = P[G_1 R_2] + P[R_1 R_2]$

$\therefore P[R_2] = ?$
$$P[G_2] + P[R_2] = 1$$
$$P[R_2] = 1 - P[G_2]$$
$$P[R_2] = 1 - 0.50$$
$$= 0.50$$

Function

Let A and B be two non empty sets. A function "f" is a relations from A to B such that f is a set of ordered pairs and "f" is a subset of $A \times B$.

$$f : A \to B$$

Let (x, y) be an element of f. Then x is element of a and y is element of b.

$$x \in A \text{ and } y \in B.$$

$$y = f(x)$$

y is called the value of f at x or image of x under f.

Domain:

The set of the first elements of the ordered pairs forming a function is called it's domain

Range:

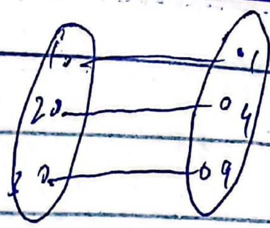
The set of the second elements of the ordered pairs forming a function is called its range.

Example:-

Let $A = \{1, 2, 3\}$ and $B = \{1, 4, 9\}$

$f = \{(1,1), (2,4), (3,9)\}$.

$f=\{(x,y) \mid y=x^2, x\in A\}$



$f(1)=1, \quad f(3)=9$

$f(2)=4$

domain $f = A$

range $f = B$

$A\times B=\{(1,1),(1,4),(1,9),(2,1),(2,4),(2,9),(3,1),(3,4),(3,9)\}.$

Example:-

$N=\{1,2,3,\ldots\}.$

$g=\{(x,y) \mid y=x^2, x\in N\}$

$g(1)=1$

$g(2)=4$

$g(4)=16$

$g(0)=$ undefind

$g(2.5)=$ undefind

~~Dom~~ Domain $g=N$

Range $g=\{1,4,9,16,25 \ldots\}.$

$N\times B=\{(1,1),(1,4)(1,9),\ldots(2,3),(2,4),\ldots\}.$

Example #

$h = \{(x,y) \mid y = x^2, x \in R\}$.

$h(0) = 0$

$h(1.5) = 2.25$

Domain = R

Range = $B(R)$

$R \times B = \{$

Random Variable:

A ~~r~~random variable X is a function that assigns a real number to each ~~number~~ outcome in the sample space of an experiment.

Example:

Flip a fair coin twice. Number of heads N is a random variable

$S = \{HH, HT, TH, TT\}$.

$N = \ \ 2 \ \ \ 1 \ \ \ 1 \ \ \ 0$

$N = \{0, 1, 2\}$.

Sample Space	No. of heads
H, H	2
H, T	1
T, H	1
T, T	0

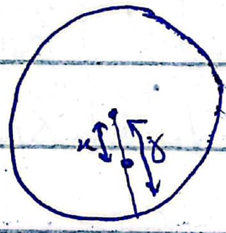
N	0	1	2
$P[N=n]$	$1/4$	$1/2$	$1/4$

$$P[N=n] = \begin{cases} 1/4, & n=0,2 \\ 1/2, & n=1 \\ 0 & \text{otherwise} \end{cases}$$

Probability Mass Function (PMF)

Continous Random Variable :-

$S_X = \{0, r\}$



PMF

Example #

X is the number of voice calls

Y is the number of data calls.

$R = XY$

$S = \{ddd, ddv, dvd, dvv, vdd, vdv, vvd, vvv\}$

outcomes	ddd	ddv	dvd	dvv	vdd	vdv	vvd	vvv
	$1/8$	$1/8$	$1/8$	$1/8$	$1/8$	$1/8$	$1/8$	$1/8$
X	0	1	1	2	1	2	2	3
Y	3	2	2	1	2	1	1	0
R	0	2	2	2	2	2	2	0

X	0	1	2	3
$P[X=x]$	$1/8$	$3/8$	$3/8$	$1/8$

Y	0	1	2	3
$P[Y=y]$	$1/8$	$3/8$	$3/8$	$1/8$

R	0	2
$P[R=r]$	$2/8$	$6/8$

$$P[X=x] = P_x(x)$$

$$P[X=0] = P_x(0) = 1/8$$

$$P_x(x) = \begin{cases} 1/8 & , x = 0, 3 \\ 3/8 & , x = 1, 2 \\ 0 & \text{otherwise} \end{cases}$$

$$P_Y(y) = \begin{cases} 1/8 & y = 0, 3 \\ 3/8 & y = 1, 2 \\ 0 & \text{otherwise} \end{cases}$$

$$P_R(r) = \begin{cases} 1/4 & r = 0 \\ 3/4 & r = 2 \\ 0 & \text{otherwise} \end{cases}$$

Example#

$S = \{\overset{1/4}{gg}, \overset{1/4}{gb}, \overset{1/4}{bg}, \overset{1/4}{bb}\}$

$S_x = \{0, 1, 2\}$

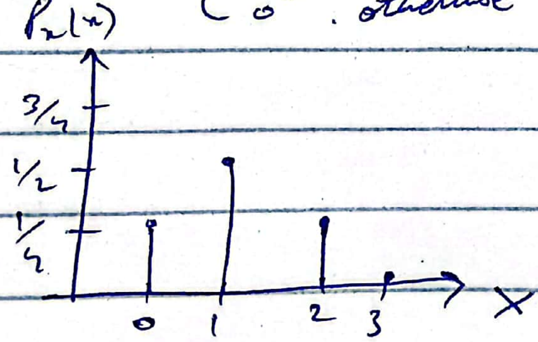
X	0	1	2
$P_x(x)$	$1/4$	$2/4$	$1/4$

$P[x=0] = P_x(0) = 1/4$

$P[X=1] = P_x(1) = 2/4 = 1/2$

$P[x=2] = P_x(2) = 1/4$

$$P_x(x) = \begin{cases} 1/4 & x = 0, 2 \\ 1/2 & x = 1 \\ 0 & \text{otherwise} \end{cases}$$



For a descrete random variable X with PMF $P_x(x)$ and range S_x

(a) For any X, $P_x(x) \geq 0$

(b) $\sum_{x \in S_x} P_x(x) = 1$

(c) For any event $B \subset S_x$ $P[B] = \sum_{x \in B} P_x(x)$

Bernoulli Distribution: (when we have only Two outcomes)

If X is a Bernoulli random variable it's PMF has the form,

$$P_x(x) = \begin{cases} 1-p & ; x=0 \\ p & ; x=1 \\ 0 & ; \text{otherwise} \end{cases}$$

Probability circuit is rejected $= p$

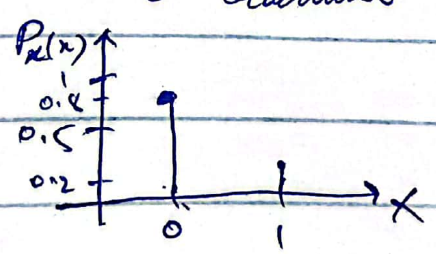
~~upto the first oc~~

X is a Bernoulli (p) random variable

$X \sim \text{Bernoulli}(p)$

If $P = 0.2$

$$P_x(x) = \begin{cases} 0.8 & x=0 \\ 0.2 & x=1 \\ 0 & \text{otherwise} \end{cases}$$



Geometric Distribution :-

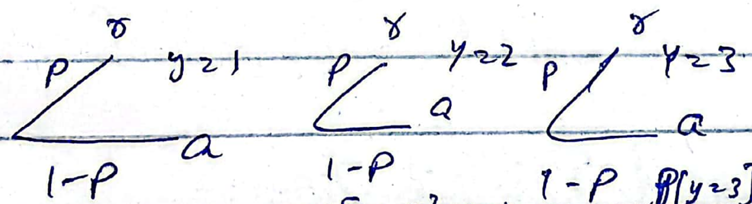
X is a geometric random variable if the PMF of X has the form;

$X \sim$ geometric

$$P_X(x) = \begin{cases} p(1-p)^{x-1}, & x = 1, 2, 3, \dots \\ 0 & \text{otherwise} \end{cases}$$

It is used to model the number of trials upto and including the first occurrence of an event.

$$\left.\begin{array}{ll} x=1 & p \\ x=2 & p(1-p) \\ x=3 & p(1-p)^2 \\ x=n & p(1-p)^{n-1} \end{array}\right\} \quad a, ar, ar^2, ar^{n-1}$$



$P[y=2] = p(1-p)$ $\quad P[y=3] = (1-p)^2 p$

$P[y=1] = p$

$$P_y(y) = \begin{cases} (0.2)(0.8)^{x-1} & x = 1, 2, 3 \dots \\ 0 & \text{otherwise} \end{cases}$$

Binomial Distribution:-

X is a binomial (n, p) random variable, if the PMF of x has the form;

$$P_X(x) = \binom{n}{x} p^x (1-p)^{n-x} \quad x = 0, 1, 2, \ldots n$$

$$X \sim \text{binomial}(n, p)$$

Binomial random variable is the sum of 'n' independent and identical distributed (iid) bernoulli random variable.

For $n=1$, $x=0$ $\quad P_x(0) = \binom{1}{0} p^0 (1-p)^1 = 1-p$

For $n=1$, $x=1$ $\quad P_x(1) = \binom{1}{1} p^1 (1-p)^0 = p$

Pascal Distribution:-

X is a pascal (k, p) random variable if the PMF of x has the form;

$$P_x(x) = \binom{x-1}{k-1} p^k (1-p)^{x-k}$$

K is an integer such that $K \geq 1$ and $x = K, K+1, \ldots$

when

$K=1$, then $n=1,2,3,\ldots$ and

then

Pascal distribution reduces to geometric distribution.

Pascal distribution is also called negative binomial distribution or Polya distribution.

Discrete Uniform Distribution:-

X is a discrete uniform Distribution (k, l) random variable if the PMF of X has the form,

$$P_x(x) = \begin{cases} \dfrac{1}{l-k+1}, & x = k, k+1, \ldots l \\ 0, & \text{otherwise} \end{cases}$$

K and l are integers such that $K < l$

probability of all outcomes is constant.

Example of Binomial Distribution:-

$p = 0.2$

$n = 10$, $x = k$

$$P_x(k) = \binom{10}{k} (0.2)^k (0.8)^{10-k}$$

Probability of k Rejects?

Pascal:-

$p = 0.2$

we see 4 defective items $k=4$, $x=l$

$$P_L(l) = \binom{l-1}{3} (0.2)^4 (0.8)$$

Poisson (α) Random Variable :-

X is a poisson random variable if PMF of X has the form

$$P_X(x) = \begin{cases} \dfrac{\alpha^x e^{-\alpha}}{x!} & x = 0, 1, 2, \dots \\ 0 & \text{otherwise} \end{cases}$$

where $\alpha > 0$

Poisson distribution expresses the probability of a given number of events occuring in a fixed interval of time or space if ~~there exists~~ these events occur with a known constant rate and independent of the time since last event.

Also called arrival of the event.

(a) Average rate, λ arrivals/sec.

(b) Time interval, T.

$$\boxed{\alpha = \lambda T}$$

Example

$\lambda = 2$ hits/sec

$T = 0.25$ secs

$\alpha = 2 \times 0.25$

$$\boxed{\alpha = 0.5 \text{ hits}}$$

(a) let,

no. of hits $= H = 0.50$ hits.

$$P_H(h) = \frac{\alpha^h e^{-\alpha}}{h!} = \frac{(0.5)^h e^{-0.50}}{h!}$$

$$P[H=0] = P_H(0) = \frac{(0.5)^0 e^{-0.25}}{0!} = e^{-0.50/2}$$

$$\boxed{P[H=0] = 0.607}$$

(b) Let

Number of hits = J

T = 1 secs

$\lambda = 2$ hits/sec

$\alpha = \lambda T = 2$ hits

$$P_J(j) = \frac{\alpha^j e^{-2}}{j!} = \frac{(2)^j e^{-2}}{j!} \; ; j = 0, 1, 2, \ldots$$

$$P[J \leq 2] = P[J=0] + P[J=1] + P[J=2]$$

$$= \frac{(2)^0 e^{-2}}{0!} + \frac{(2)^1 e^{-2}}{1!} + \frac{(2)^2 e^{-2}}{2!}$$

$$= 0.1353 + 2e^{-2} + 5e^{-2}$$

$$\boxed{P[J \leq 2] = 0.677}$$

Quiz #2.3:-

(i) $X \sim \text{geometric}(p)$

$$P_X(x) = \begin{cases} p(1-p)^{x-1} & , x=1,2,\dots \\ 0 & , \text{otherwise} \end{cases}$$

(ii) $x = 10$? , $p = 0.1$

$$P_X(x) = (0.1)(0.9)^{9} = 0.0387$$

Probability of $X \geq 10$

$$P[X \geq 10] = 1 - P[X < 10]$$

$$\because P[X < n] + P[n \leq X] = 1$$

$$P[x \geq 10] = (1-p)^{x-1} = (0.9)^{9}$$

$$P[x \geq 10] = \cancel{0.03874} = 0.3874$$

(iii) 100 bits, PMF of Y = ? no. of errors?

$n = 100$

Y = no. of errors

$$P_Y(y) = \binom{100}{y} p^{y} (1-p)^{100-y} \quad , y = 0,1,2,\dots$$

(iv) if $P = 0.01$, 100 bits, $Y = 2$ errors at the receiver

Probability that $Y \leq 2$:-

(a) $P = 0.01$ $n = 100$, $P[Y=2]$.

$$P[Y=2] = P_Y(2) = \binom{100}{2} (0.01)^2 (0.99)^{98}$$

$$\boxed{P_Y(y) = 0.1849}$$

(b) $P[P_Y \leq 2] = P[P_Y = 0] + P[P_Y = 1] + P[Y=2]$

$$P[Y \leq 2] = \binom{100}{0} (0.01)^0 (0.99)^{100} + \binom{100}{1} (0.01)^1 (0.99)^{99} + \binom{100}{2} (0.01)^2 (0.99)^{98}$$

$$\boxed{P[Y \leq 2] = 0.9206}$$

(v) Three errors, PMF of Z :-

$$P_N(n) = \binom{n-1}{k-1} p^k (1-p)^{n-k}, \quad n = k, k+1 \ldots$$

$$P_Z(z) = \binom{z-1}{2} p^3 (1-p)^{z-3}, \quad z = 3, 4, 5 \ldots$$

(vi) $P = 0.25$, $P_Z(12) = ?$

$$P_Z(12) = \binom{11}{2} (0.25)^3 (0.75)^9$$

$$\boxed{P_Z(12) = 0.0645}$$

Cumulative Distribution Function (CDF)

$$F_X(x) = P[X \le x]$$

If X is a discrete random variable

$$F_X(x) = P[X \le x] = \sum_{x_i \le x} P_X(x_i)$$

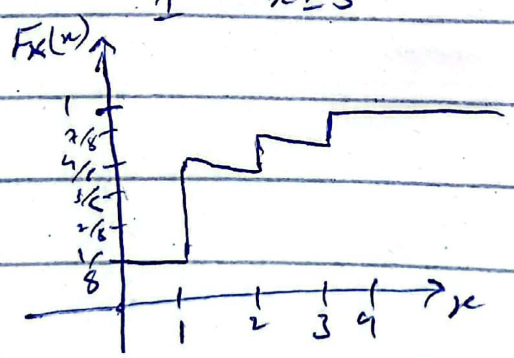
$$S_X = \{0, 1, 2, 3\}$$

X	0	1	2	3
$P_X(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$
$P[X \le x]$	$\frac{1}{8}$	$\frac{4}{8}$	$\frac{7}{8}$	$\frac{8}{8} = 1$

Last value in CDF must be 1.

$$P_X(x) = \begin{cases} 1/8, & x = 0, 3, \ldots \\ 3/8, & x = 1, 2 \\ 0 & \text{otherwise} \end{cases}$$

$$F_X(x) = \begin{cases} 0 & x < 0 \\ 1/8 & 0 \le x < 1 \\ 4/8 & 1 \le x < 2 \\ 7/8 & 2 \le x < 3 \\ 1 & x \ge 3 \end{cases}$$



From CDF to PMF:

$P[x=0] = P[x \leq 0] - P[x < 0] = 1/8 - 0 = 1/8$

$P[x=1] = P[x \leq 1] - P[x < 1] = 4/8 - 1/8 = 3/8$

$P[x=2] = P[x \leq 2] - P[x < 2] = \frac{7}{8} - \frac{4}{8} = \frac{3}{8}$

$P[x=3] = P[x \leq 3] - P[x < 3] = 1 - 7/8 = 1/8$

$$P_x(x) = \begin{cases} 1/8 & x = 0, 3 \\ 3/8 & x = 1, 2 \\ 0 & \text{otherwise} \end{cases}$$

observations:

(a) $\lim_{x \to -\infty} F_x(x) = 0$

$\lim_{x \to +\infty} F_x(x) = 1$

(b) For all $x' \geq x$, $F_X(x') \geq F_x(x)$

(c) For $x_i \in S_X$

$F_x(x_i) - F_X(x_i - \epsilon) = P_X(x_i)$

ϵ is a small number.

(d) $F_X(x) = F_X(x_i)$ $\forall$ x, $x_i \leq x < x_i + 1$

For all $b \geq a$

$F_x(b) - F_X(a) = P[a < x \leq b]$.

$P[1 < x \leq 2] = F_X(2) - F_X(1) = \frac{7}{8} - \frac{4}{8} = \frac{3}{8}$

Expected Value

$$E[X] = \mu_x$$

$$E[X] = \mu_x = \sum_{i \in S} x_i P_x(x_i)$$

$$P_x(x) = \begin{cases} 1/8 & x = 0, 3 \\ 3/8 & x = 1, 2 \\ 0 & \text{otherwise} \end{cases}$$

$$E[X] = 0\left(\tfrac{1}{8}\right) + 3 \times \tfrac{1}{8} + 1 \times \tfrac{3}{8} + 2 \times \tfrac{3}{8}$$

$$= 0 + \frac{3}{8} + \frac{3}{8} + \frac{6}{8} = \frac{12}{8} = \frac{3}{2} = 1.5$$

For Bernoulli random variable;

$$P_x(x) = \begin{cases} 1-p & ; x = 0 \\ p & , x = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$E[X] = \mu_x = (1-p) \times 0 + p \times 1$$

$$\boxed{E[X] = p}$$

Distribution	Expected Value	Variance
(i) Bernoulli	p	$p(1-p)$
(ii) Geometric	$1/p$	$(1-p)/p^2$
(iii) Poisson	α	α
(iv) Binomial	np	$np(1-p)$
(v) Pascal	k/p	$k(1-p)/p^2$
(vi) Discrete Uniform	$(k+l)/2$	$(l-k)(l-k+2)/12$

Variance σ^2

Standard deviation σ

Standard deviation $= \sqrt{\text{Variance}}$

$$Var[X] = \sigma_X^2 = E[(X-\mu_x)^2]$$

$$VAR[X] = \sum_{i \in S} (x_i - \mu_{x_i})^2 P_X(x_i)$$

Variance of r

$$P_X(x) = \begin{cases} \frac{1}{8} & x = 0, 3 \\ 3/8 & x = 1, 2 \\ 0 & \text{other} \end{cases}$$

$$VAR[X] = \left(0-\frac{3}{2}\right)^2 \times \frac{1}{8} + \left(3-\frac{3}{2}\right)^2 \times \frac{1}{8} + \left(1-\frac{3}{2}\right)^2 \times \frac{3}{8}$$

$$+ \left(2-\frac{3}{2}\right)^2 \times \left(\frac{3}{8}\right)$$

$$= \left(\frac{9}{4}\right)\left(\frac{1}{8}\right) + \cdot - - -$$

$$= \frac{3}{4} = 0.75$$

$$VAR[X] = \sigma_X^2 = \frac{3}{4} = 0.75$$

$$STD[X] = \sigma_X = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2} = 0.866$$

$$E[X+Y] = E[X] + E[Y]$$

$$\boxed{E[ax+b] = E[ax] + E[b] = aE[x] + b}$$

1

$$VAR[x] = E\left[(x-\mu_x)^2\right].$$

$$VAR[x] = E\left[x^2 + \mu_x^2 - 2\mu_x x\right]$$
$$= E[x^2] + E[\mu_x^2] - E[2\mu_x x].$$
$$= E[x^2] + \mu_x^2 - 2\mu_x E[x].$$

But

$$E[x] = \mu_x$$

$$VAR[x] = E[x^2] + \mu_x^2 - 2\mu_x \mu_x$$

$$= E[x^2] - \mu_x^2$$

$$\boxed{\begin{aligned} VAR[x] &= \sigma_x^2 = E[x^2] - \mu_x^2 \\ VAR[x] &= E[x^2] - (E[x])^2 \end{aligned}}$$

$$E[x^2] = \sum_{i \in S} x_i^2 \, P_x(x_i)$$

$$E[x^2] = (0)^2 \times \frac{1}{8} + 3^2 \times \frac{1}{8} + 1^2 \times \frac{3}{8} + 2^2 \times \frac{3}{8}$$

$$= 0 + \frac{9}{8} + \frac{3}{8} + \frac{12}{8}$$

$$\boxed{E[x^2] = 3}$$

$$VAR[x] = \sigma_x^2 = E[x^2] - (E[x])^2$$
$$= 3 - \left(\frac{3}{2}\right)^2 = 3 - 9/4 = 3/4$$

$VAR[x] = \frac{3}{4} = 0.75$

$STD\{x\} = \sigma_x = \frac{\sqrt{3}}{2} = 0.866$

Moments of a Random Variable :-

(a) The n^{th} moment of a Random Variable is

$$E\{x^n\} = \sum_{i \in S} x_i^n P_x(x_i)$$

(b) The n^{th} central moment of Random Variable X is

$$E[(x - \mu_x)^n] = \sum_{i \in S} (x_i - \mu_x)^n \times P_x(x_i)$$

$$VAR[x] = \sigma_x^2 = E[(x - \mu_x)^2]$$

Example # 2.30

$$P_V(v) = \begin{cases} 1/7 & , v = -3, -2, \dots 3 \\ 0 & \text{otherwise} \end{cases}$$

Let $Y = V^2/2$, Find $P_Y(y) = ?$

v	-3	-2	-1	0	1	2	3
$P_V(v)$	$1/7$	$1/7$	$1/7$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$
$y = \frac{v^2}{2}$	$\frac{9}{2}$	2	$\frac{1}{2}$	0	$\frac{1}{2}$	2	$9/2$
$P_Y(y)$							

$$P_Y(y) = \begin{cases} \frac{1}{7} & y = 0 \\ \frac{2}{7} & y = \frac{9}{2}, 2, 1/2 \\ 0 & \text{otherwise} \end{cases}$$

v	-3	-2	-1	0	1	2	3
$P_v(v)$	$1/7$	$1/14$	$1/14$	$\frac{1}{7}$	$1/7$	$1/7$	$2/7$
$Y = \frac{v^2}{2}$	$\frac{9}{2}$						

$$P_x(x) = \begin{cases} 1/8 & x = 0, 2 \\ 3/8 & x = 1, 3 \\ 0 & \text{otherwise} \end{cases}$$

let $Y = 2x + 3$

x	0	1	2	3
$P_x(x)$	$1/8$	$3/8$	$1/8$	$3/8$
$Y = 2x+3$	3	5	7	9

$$P_Y(y) = \begin{cases} 1/8 & y = 3, 7 \\ 3/8 & y = 5, 9 \\ 0 & \text{otherwise} \end{cases}$$

For Expected value

$$P_x(x) = \begin{cases} 1/8 & x = 0, 3 \\ 3/8 & x = 1, 2 \\ 0 & \text{otherwise} \end{cases}$$

$$E[x] = \mu_x = 3/8 = 0.75$$

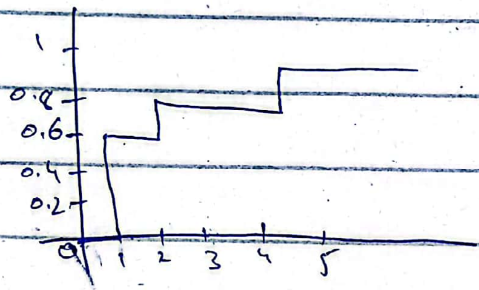
$$E[x - \mu_x] = \sum_{i \in S} (x_i - \mu_x)\, P_x(x_i)$$

$$= \left(0 - \frac{3}{2}\right)\left(\frac{1}{8}\right) + \left(3 - \frac{3}{2}\right)\frac{1}{8} + \left(1 - \frac{3}{2}\right)\left(\frac{3}{8}\right) + \left(2 - \frac{3}{2}\right)\left(\frac{3}{8}\right)$$

$$E[x-\mu_x] = \frac{1}{8}\left[\left(0-\frac{3}{2}\right)+\left(3-\frac{3}{2}\right)\right] + \frac{3}{8}\left[\left(1-\frac{3}{2}\right)+\left(2-\frac{3}{2}\right)\right]$$

$$= \frac{1}{8}\left[0-\frac{3}{2}+3-\frac{3}{2}\right] + \frac{3}{8}\left[-\frac{1}{2}+\frac{1}{2}\right] = 0.$$

Quiz # 2.4



(1) $P[Y<1]$ (2) $P[Y\le 1]$

(3) $P[Y>2]$ (4) $P[Y\ge 2]$

(5) $P[Y=1]$ (6) $P[Y=3]$

$$F_Y(y) = \begin{cases} 0 & y<1 \\ 0.6 & 1\le y<2 \\ 0.8 & 2\le y<4 \\ 1 & y\ge 4 \end{cases}$$

(1) $P[Y<1] = 0$

(2) $P[Y\le 1] = F_Y(1) = 0.6$

(3) $P[Y>2] = 1 - P[Y\le 2]$

$= 1 - F_Y(2) = 1 - 0.8 = 0.2$

(4) $P[Y\ge 2] = 1 - P[Y<2]$

$= 1 - 0.6 = 0.4$

(5) $P[Y=1] = P[Y\le 1] - P[Y<1] = 0.6 - 0 = 0.6$

(b) $P[Y=3] = P[Y\le 3] - P[Y<3]$
$= 0.8 - 0.8 = 0$

PMF of this CDF

The size of jump at single numbers like at 2 and 4 the jump is 0.2

$$P_X(x) = \begin{cases} 0.6 & y=1 \\ 0.2 & y=2,4 \\ 0 & \text{otherwise} \end{cases}$$

$g(y) = X = 2Y+3$

PMF of X

Y	1	2	4
$P_Y(y)$	0.6	0.2	0.2
X	5	7	11

$$P_X(X) = \begin{cases} 0.6 & x=5 \\ 0.2 & x=7,11 \\ 0 & \text{other..} \end{cases}$$

$E[X] = 5(0.6) + 7(0.2) + 11(0.2)$
$= 3 + 1.4 + 2.2$

$$\boxed{E[x] = 6.6}$$

~~the~~ we can also find Expected value of 'x' from PMF of y but the formula will be :-

$$E[x] = \sum_{i \in S} g(y)_S P_Y(y)_S$$

$$E[x] = [2(1)+3] \times 0.6 + [2(2)+3] \times 0.2] + [2(4)+3] \times 0.2$$

$$\boxed{E[x] = 6.6}$$

Ch#2
(Problems)

2.2.1, 2.2.2, 2.2.3, 2.2.4,
2.2.9, 2.3.1, 2.3.2, 2.3.5,
2.3.6, 2.3.7, 2.4.1, 2.4.2,
2.4.3, 2.4.8, 2.5.4, 2.5.5,
2.5.6, 2.5.7, 2.6.2, 2.6.3
2.7.4, 2.8.1, 2.8.2, 2.8.3
2.8.4

Probability of Interval

$$P[a < Y \leq b] = F_Y(b) - F_Y(a)$$

$$P[1 < Y \leq 2] = 0.2$$

from CDF $= F_Y(2) - F_Y(1)$

$= 0.8 - 0.6 = 0.2$

$P[1 < Y \leq 4] = 0.2 + 0.2 = 0.4$ (from PMF)

$= F_Y(4) - F_Y(1) = 1 - 0.6 = 0.4$ (from CDF)

If the Probability that tube ligth has a useful life of atleast 800 hours is 0.9 find the Probabilities that amoung 20 such ligth:

(a) Exactly 18 will have a useful life of atleast 800 hours.

(b) Atleast 15 will have a useful life of atleast 800 hours.

(c) Atleast 2 will not have a useful life of atleast 800 hours.

$P[TL] = 0.9$

Probability of

$p = 0.9$, $q = 1 - 0.9 = 0.1$

$\boxed{n = 20}$

$P[X = x] = p(x) = {}^{20}C_x (0.9)^x (0.1)^{20-x}$

$x = 1 \ldots 20$

$= P(X \leq 18) = 1 - P(X > 18)$

$= 1 - [P[X = 19] + P(X = 20)]$

$= 1 - [{}^{20}C_{19} (0.9)^{19} (0.1) + {}^{20}C_{20} (0.9)^{20}]$

$\boxed{M = 0.1544}$

Continous Random Variable

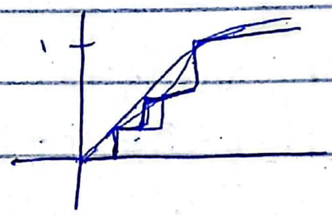
X

$S = \{1, 2, 3, 4, 5, 6\}$

Probability Density Function (PDF)

CDF $\quad$ PDF

$F_X(x) \quad f_X(x)$



$$\boxed{f_X(x) = \frac{d}{dx} F_X(x)}$$

Example :-

$S_V = \{v \mid -5 < v < 5\}$

$S_A = \{a \mid 0 \leq a \leq 2\pi\}$

$S_T = \{t \mid -5 \leq t < 5\}$

CDF

$F_X(x) = P[X \leq x]$.

Theorem :-

For any random variable X,

(a) $F_X(-\infty) = 0$

(b) $F_X(+\infty) = 1$

(c) $P[x_1 < x \leq x_2] = F_X(x_2) - F_X(x_1)$

Quiz #31 :-

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ y/4 & 0 \leq y \leq 4 \\ 1 & y > 4 \end{cases}$$

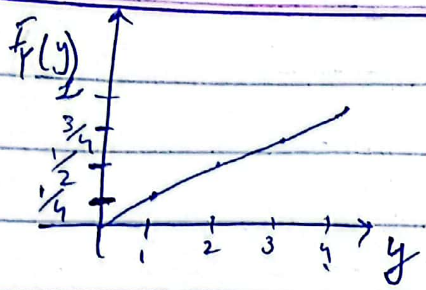
Sketch the CDF.

(a) $P[Y \leq -1]$

(b) $P[Y \leq 1]$

(c) $P[2 < Y \leq 3]$

(d) $P[Y > 1.5]$



y	$y/4$
0	0
1	$1/4$
2	$2/4 = 1/2$
3	$3/4$
4	1

(a) $P\{Y \le -1\} = F_Y(-1) = 0$

(b) $P[Y \le 1] = F_Y(1) = 1/4$

(c) $P[1 < Y \le 3] = F_Y(3) - F_Y(2) = 3/4 - 1/2 = 1/4$

(d) $P[Y \ge 1.5] = 1 - P[Y \le 1.5]$

$= 1 - F_Y(1.5)$ $\qquad \because F_Y(1.5) = \frac{1.5}{4}$

$= 1 - \frac{3}{8} = \boxed{\frac{5}{8}}$

$$P_X(x) = \begin{cases} 1/4 & x = 0, 1 \\ 1/8, & x = 2, 3, 4, 5 \\ 0, & \text{otherwise} \end{cases}$$

$P[1 < X \le 3] = P_X(2) + P_X(3) = \frac{1}{8} + \frac{1}{8} = \frac{2}{8} = 1/4$

$P[1 \le x \le 3] = P_X(1) + P_X(2) + P_X(3)$

$= \frac{1}{4} + \frac{1}{8} + \frac{1}{8} = \frac{2}{4} = \frac{1}{2}$

In terms of Continous Random Variable

$P\{X \le x\} = P[X < x]$

Probability Density Function (PDF):-

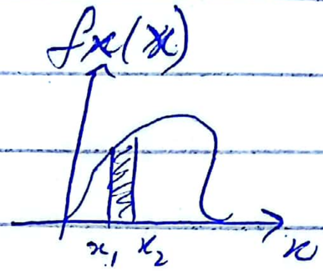
$$f_X(x) = \frac{d}{dx} F_X(x)$$

Theorem :-

(a) $f_X(x) \geq 0 \quad \forall x$

(b) $F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx$

(c) $\int_{-\infty}^{+\infty} f_X(x)\,dx = 1$

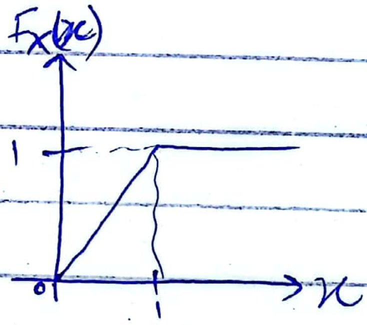


Theorem :-

$$P[x_1 < X < x_2] = F_X(x_2) - F_X(x_1)$$
$$= \int_{x_1}^{x_2} f_X(x)\,dx$$

$$P[X = a] = P[a < X \leq a] = \int_{a}^{a} f_X(x)\,dx = 0$$

Example #3.4 :-



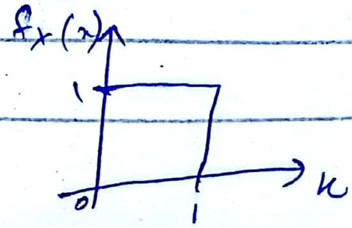
$$f_X(x) = \frac{d}{dx} F_X(0)$$

$$= \frac{d}{dx} x = 1$$

So;

$$f_X(x) = \begin{cases} 1, & 0 \leq x < 1 \\ 0, & \text{otherwise} \end{cases}$$

CDF :-



Using CDF:-

$$P[1/4 < X \le 3/4] = F_X\left(\frac{3}{4}\right) - F_X\left[\frac{1}{4}\right]$$
$$= \frac{3}{4} - \frac{1}{4} = \frac{2}{4} = \frac{1}{2}$$

Using PDF:-

$$P[1/4 < x \le 3/4] = \int_{1/4}^{3/4} (1)\, dx = x\Big|_{1/4}^{3/4} = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$$

(H.w)
Example 3.5

Quiz #3.2:-

$$f_X(x) = \begin{cases} cxe^{-x/2} & , x \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

(1) the constant c
(2) the CDF $F_X(x)$
(3) $P[0 \le x < 4]$
(4) $P[-2 \le x \le 2]$

L'Hopital Rule

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0}$$

or

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\pm\infty}{\pm\infty}$$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

$$\int_{-\infty}^{\infty} f_X(x)\, dx = 1$$

$$\int_{-\infty}^{0} f_X(x)\, dx + \int_{0}^{+\infty} f_X(x)\, dx = 1 \;\Big|\; \int_{0}^{\infty} f_X(x)\, dx = 1$$

$$\int_{0}^{\infty} \underbrace{cx}_{v}\, \underbrace{e^{-x/2}\, dx}_{dv} = 1$$

Integrating by parts;

$$\int_a^b v\,du = uv\Big|_a^b - \int_a^b v\,du$$

let

$$u = cx \rightarrow du = c\,dx$$

$$dv = e^{-x/2}dx, \quad v = \int e^{-x/2}dx$$
$$= -2e^{-x/2}$$

$$c\int_0^\infty x e^{-x/2}dx = (cx)(-2e^{-x/2})\Big|_0^\infty$$

$$0 - \int_0^\infty -2e^{-x/2}c\,dx = 1$$

$$+2c\int_0^\infty e^{-x/2}dx = 1$$

$$\therefore x e^{-x/2} = \frac{x}{e^{x/2}} = \frac{1}{2e^{x/2}}$$

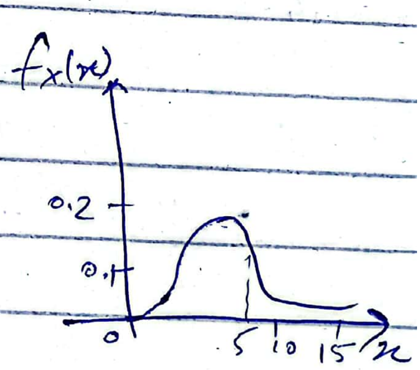
$$= (0-0) + 2C(-2)e^{-x/2}\Big|_0^\infty = 1$$

$$= -4C[0-1] = 1$$

$$\boxed{C = 1/4}$$

$$f_X(x) = \begin{cases} \frac{x}{4}e^{-x/2}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

x	$f_X(x)$
0	0
1	0.15
2	0.18
5	0.1026
10	0.016
20	2.26×10^{-4}



(2) CDF :-

$$F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx$$

$$F_X(x) = \int_0^x \underbrace{\frac{u}{4}}_{u} \underbrace{e^{-u/2}\,du}_{dv}$$

Using integration by parts

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

$$u = \frac{u}{4} \quad , \quad du = \frac{1}{4}\,dy$$

$$dv = e^{-y/2}\,dy \Rightarrow v = -2e^{-y/2}$$

$$\cancel{= \frac{u}{4}\left(-2e^{-y/2}\right)\Big|_0^x - \left(-2e^{-y/2}\right)\left(\tfrac{1}{4}\right)dy}$$

$$\cancel{= (0-0) + 2e^{-y/2}(2)\Big|_0^x}$$

$$\cancel{= 2e^{0}(2) - 2e^{-x/2}(2)}$$

$$\int_a u\,du = \frac{u}{4}\left(-2e^{-y/2}\right)\Big|_0^x - \int_0^x \left(-2e^{-y/2}\right)\left(\tfrac{1}{4}\right)du$$

$$= -\frac{u}{2}\,e^{-y/2}\Big|_0^x + \frac{1}{2}\left|-2e^{-y/2}\right|_0^x$$

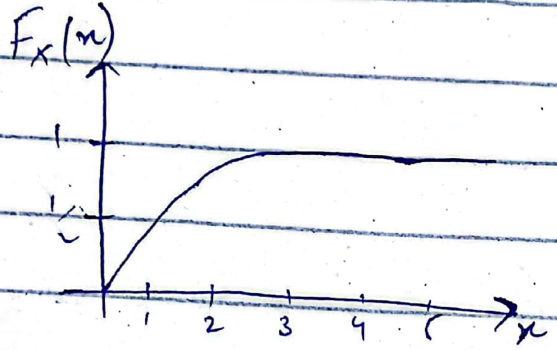
$$= -\frac{1}{2}\left[xe^{-x/2} - 0\right] - 1\left[e^{-x/2} - 1\right]$$

$$= -\frac{1}{2}xe^{-x/2} - e^{-x/2} + 1$$

$$= -\frac{x}{2}e^{-x/2} - e^{-x/2} + 1$$

$$= 1 - e^{-x/2}\left(1 + \frac{x}{2}\right)$$

$$F_X(x) = \begin{cases} 1 - e^{-x/2}\left(\frac{x}{2}+1\right), & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$



x	$F_X(x)$
0	
1	
2	
3	

(3) $P[0 \leq X \leq 4] = F_X(4) - F_X(0)$

$= 1 - e^{-2}(3) - [1 - e^{0}(1)]$

$= 1 - 3e^{-2}$

from PDF :-

$= \int_0^4 \frac{x}{4} e^{-x/2} dx$

(4) $P[-2 \leq x \leq 2] = F_X(2) - F_X(-2)$

$= 1 - e^{-1}(2) - 0$

$= \boxed{1 - 2e^{-1}}$

Expected Value :-

$$E[x] = \int_{-\infty}^{+\infty} x f_X(x) dx$$

$$E[g(x)] = \int_{-\infty}^{+\infty} g(x) f_X(x) dx = E[Y].$$

$$E[Y] = \int_{-\infty}^{+\infty} y f_Y(y) dy$$

$$E[X^2] = \int_{-\infty}^{+\infty} x^2 f_X(x)\,dx.$$

$$VAR[X] = \sigma_X^2 = E[(x-\mu_x)^2].$$

$$VAR[X] = E[X^2] - (E[X])^2.$$

$$\sigma_X^2 = \int_{-\infty}^{+\infty} (x-\mu_x)^2 f_X(x)\,dx$$

Theorem :-

(a) $E[X-\mu_x] = 0$

(b) $E[aX+b] = aE[X]+b$

(c) $VAR[X] = E[X^2] - \mu_x^2$

(d) $VAR[aX+b] = a^2 VAR[X]$

Quiz #3.3 :-

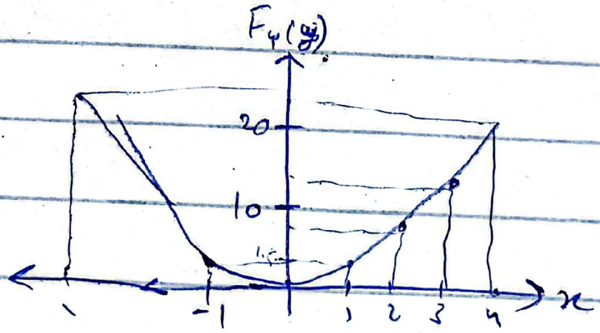
$$f_Y(y) = \begin{cases} \frac{3}{2}y^2 & -1 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Sketch PDF :-

(1) $E[Y]$ (2) $E[Y^2]$

(3) $VAR[Y]$ (4) STD :-

y	$f_Y(y) = \frac{3}{2}y^2$
0	0
1	$\frac{3}{2} = 1.5$
2	6
3	13.5
4	24



① $E[Y]$

$$E[Y] = \int_{-\infty}^{+\infty} y f_Y(y)\,dy.$$

$$= \int_{0}^{\infty} y\, f_Y(y)\,dy$$

$$= \int_{0}^{\infty} \frac{3}{2} y^2\,dy$$

$$= \frac{3}{2} \int_{0}^{y} y^2\,dy$$

$$= \frac{3}{2} \left. \frac{y^3}{3} \right|_{-\phi}^{y}$$

$$= \frac{3}{2}\left(\frac{-\phi}{3} + \frac{y^3}{3}\right)$$

$$= \frac{3}{2}\left(\frac{y^3}{3}\right) = \frac{y^3}{2} = 0$$

② $$E[Y^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx$$

$$= \int_{-1}^{1} \left(\frac{3}{2} y^2\right)^2 dx$$

$$= \frac{3}{2}\int_{1} y^4\,dx$$

$$= \frac{9}{4} \int \left. \frac{y^5}{5} \right|_{-1}^{1}$$

$$= \frac{3}{4}\left(\frac{-1}{5} - \frac{1}{5}\right) = \frac{3}{5}$$

$$VAR[Y] = E[Y^2] - (E[Y])^2$$

$$= \frac{3}{5} - 0 = \frac{3}{5}$$

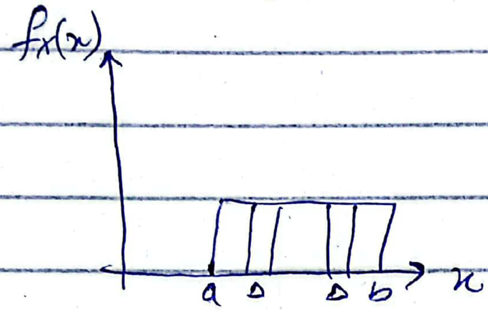
$$P[0 \le Y \le 1/2] = \int_{0}^{1/2} \frac{3}{2} y^2 = \frac{3}{2} \left. \frac{y^3}{3} \right|_{0}^{1/2}$$

$$P[0 \le Y \le 1] = \frac{2}{26}\int y^2\,dy = \frac{1}{2} = \frac{3}{2}\left(\frac{(1/2)^3}{3}\right) = \frac{3}{2}\left(\frac{1}{8}\right)$$

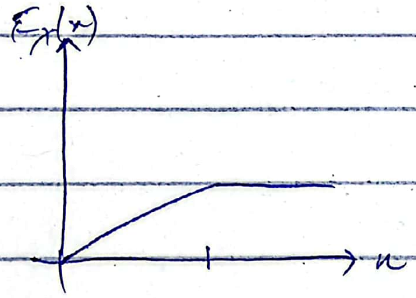
Continous Uniform Distribution:-

$X \sim \text{Uniform}(a, b)$

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & a \le x < b \\ 0, & \text{otherwise} \end{cases}$$



$$F_X(x) = \begin{cases} 0 & x \le a \\ \frac{x-a}{b-a}, & a < x \le b \\ 1, & x > b \end{cases}$$



$$E[X] = \frac{b+a}{2}$$

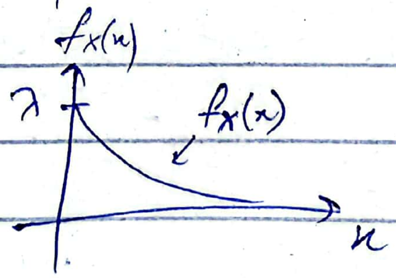
$$VAR[X] = \frac{(b-a)^2}{12}$$

Exponential Distribution:-

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \ge 0 \\ 0, & \text{otherwise} \end{cases}$$

$\lambda > 0$ is called rate parameters.

$$f_X(x) = \lambda e^{-\lambda x} u(x)$$



$$F_X(x) = \begin{cases} 1-e^{-\lambda x} & , x \geq 0 \\ 0 & , \text{otherwise} \end{cases}$$

$$E[X] = \frac{1}{\lambda}$$

It used to calculate how long will it take before the car breaks down?

$$VAR[X] = \frac{1}{\lambda^2}$$

n^{th} moment of X

$$E[X^n] = \frac{n!}{\lambda^n}$$

Example # 3.11:-

Uniform Distribution :-

$a = 0$

$b = 2\pi$

$$f_\theta(\theta) = \begin{cases} \frac{1}{2\pi} & , 0 \leq \theta \leq 2\pi \\ 0 & \text{otherwise} \end{cases}$$

CDF =:

$$F_\theta(\theta) = \begin{cases} 0 & , \theta \leq 0 \\ \frac{\theta - 0}{2\pi - 0} & , 0 < \theta < 2\pi \\ 1 & , \theta > 2\pi \end{cases}$$

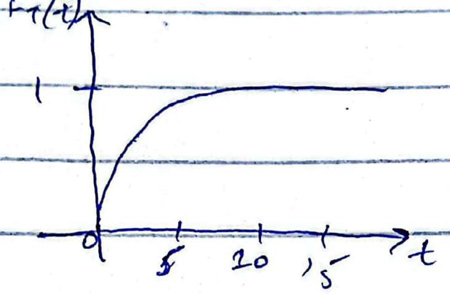
$$E[X] = \frac{b+a}{2} = \frac{2\pi + 0}{2} = \pi \text{ rad}$$

$$VAR[X] = \frac{(b-a)^2}{12} = \frac{(2\pi)^2}{12} = \frac{4\pi^2}{12} = \frac{\pi^2}{3} \text{ rad}^2$$

$$STD = \sqrt{\frac{\pi^2}{3}} = \frac{\pi}{\sqrt{3}} \text{ rad.}$$

Example #3.12:-

$$F_T = \begin{cases} 1-e^{-t/3} & , t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$



(a) PDF $f_T(t) = ?$

(b) $P[2 \leq t < 4] = ?$

$$f_T(t) = \frac{d}{dt}\left(1-e^{-t/3}\right)$$

$$= 0 - \left(\frac{1}{-3}\right) e^{-t/3} = \frac{1}{3} e^{-t/3}$$

$$f_T(t) = \begin{cases} \frac{1}{3} e^{-t/3} & , t \geq 0 \\ 0 & , \text{otherwise} \end{cases}$$

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & , x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\lambda = \frac{1}{3}$$

$$E[X] = \frac{1}{\lambda} = 3$$

(b) $P[2 \leq t < 4] = F_T(4) - F_T(2)$

$$= \left[1 - e^{-4/3}\right] - \left[1 - e^{-2/3}\right]$$

$$= \cancel{1} - e^{-4/3} - \cancel{1} + e^{-2/3}$$

$$P[2 \leq t < 4] = 0.2498$$

Example#3.13:-

$$E[X] = \frac{1}{\lambda} = 3$$

$$VAR[X] = \frac{1}{\lambda^2} = 9$$

$$STD = 3 = \sqrt{VAR}$$

$$E[T] = \int_{-\infty}^{+\infty} t f_T(t)\,dt$$

$$\sigma_T^2 = E[T^2] - (E[T])^2$$

$$P = [\mu - \sigma_T \leq T < \mu + \sigma_T]$$

$$= [3 - 3 \leq T < 3 + 3]$$

$$= [0 \leq T < 6]$$

∴ we can do it both ways

Erlang Distribution:-

The sum of a set of iid exponential random variable is an erlang random variable.

$$X = Y_1 + Y_2 + \cdots + Y_n$$

$$f_X(x) = \begin{cases} \dfrac{\lambda^n x^{n-1} e^{-\lambda x}}{(n-1)!}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$\lambda > 0$

$n \geq 1$ is an integer.

$$E[X] = \frac{n}{\lambda}$$

$$VAR[X] = \frac{n}{\lambda^2}$$

Quiz #3.4:-

$E[X] = 3$, $VAR[X] = 9$

(a) Exponential PDF = ?

$E[X] = \frac{1}{\lambda}$; $VAR[X] = \frac{1}{\lambda^2}$

$E[X] = \frac{1}{\lambda} = 3$, $VAR[X] = \frac{1}{\lambda^2} = 9$

$$\boxed{\lambda = 1/3}$$

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$f_X(x) = \begin{cases} \frac{1}{3} e^{-x/3}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

(2) Uniform PDF?

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & a \leq x \leq b \\ 0, & \text{otherwise} \end{cases}$$

$$E[X] = \frac{b+a}{2}$$

$$VAR[X] = \frac{(b-a)^2}{12}$$

Therefore;

$$\frac{b+a}{2} = 3$$

$$b+a = 6 \quad \text{(i)}$$

$$\frac{(b-a)^2}{12} = 9$$

$$(b-a)^2 = 108$$

$$b-a = \pm\sqrt{108} = \pm 10.39 \quad \text{(ii)}$$

when $b-a = +10.39$ then

$$b+a = 6$$

$$b-a = 10.39$$

$$\boxed{b = 8.195}$$

In this case;

$a = 6 - b = 6 - 8.195 = -2.195$

$\boxed{a = -2.195}$

Case - II :-

$b - a = -10.39$, then

$b + a = 6$

$b - a = -10.39$

$2b = -4.39$

$\boxed{b = -2.195}$

$a = 6 - b = 6 + 2.195$

$\boxed{a = 8.195}$

$\left.\begin{matrix} a = -2.195 \\ b = 8.195 \end{matrix}\right]$ $b > a$ ✓

$\left.\begin{matrix} a = 8.195 \\ b = -2.195 \end{matrix}\right]$ $b \not> a$

$$\frac{1}{b-a} = \frac{1}{8.195 + 2.195} = \frac{1}{10.39}$$

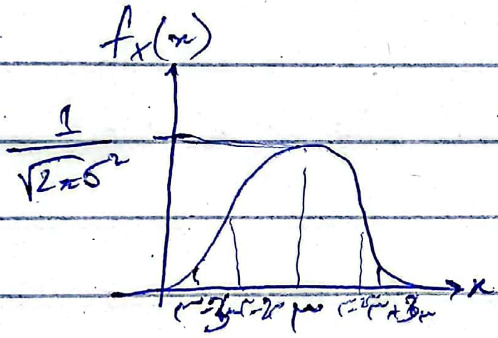
$$f_X(x) = \begin{cases} \frac{1}{10.39} & -2.195 \le x \le 8.195 \\ 0 & \text{otherwise} \end{cases}$$

Gaussian Distribution
Normal Distribution

$X \sim \text{Gaussian}(\mu, \sigma)$

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$



$E[X] = \mu \rightarrow$ Location Parameter

$VAR[X] = \sigma^2 \rightarrow$ Scale Parameter.

3 Sigma rule

68% of values are with in $\pm 1\sigma$ of the mean.

95% of Vales are with in $\pm 2\sigma$ of the mean.

99.7% of the values are with $\pm 3\sigma$ of the mean.

Standard Normal Distribution:-

The standard Normal random Variable Z is the Gaussian (0, 1) random variable.

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

$E[Z] = 0$

$VAR[Z] = 1$

$$\boxed{Z = \frac{x-\mu}{\sigma}}$$

$$\boxed{f_X(x) = \frac{1}{\sigma}\, \phi\left(\frac{x-\mu}{\sigma}\right)}$$

∴ Relation b/w Normal and Standard Normal Distribution.

$$\boxed{F_X(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)}$$

Important formula:- (Theorem)

The probability that x is in the interval (a, b) is ~~probability that~~

$$P[a < X \leq b] = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right) = F_X(b) - F_X(a)$$

Example #3.15:-

Gaussian (μ, σ)

$X \sim$ Gaussian (a, b)

$x = 46$ → Sample value

$Z = \frac{x-\mu}{\sigma} =$ for $\mu = 61$, $\sigma = 10$

$$z = \frac{46-61}{10} = -1.5$$

$$\boxed{z = -1.5}$$

Q-Function:

The Complement of the Standard Normal CDF is called the Q Function.

$$Q(z) = 1 - \Phi(z)$$

$$\Phi(z) = P[Z \leq z]$$

$$Q(z) = P[Z > z]$$

Theorem:-

$$\Phi(-z) = 1 - \Phi(z)$$

Example #3.16:-

$X \sim$ Guassian $(61, 10)$

$\mu = 61$ $\quad x = 46$

$\sigma = 10$ $\quad P[X \leq 46] = ?$

$$Z = \frac{x-\mu}{\sigma} = \frac{46-61}{10} = -1.5$$

$$P[X \leq 46] = F_X(46) = \Phi(-1.5) = 1 - \Phi(1.5)$$

Table# 3.1, 3.2

From table $\Phi(1.5) = 0.9332$

$$P[X \leq 46] = 1 - 0.9332 = 0.0668$$

$$P[X \leq 46] = 0.0668$$

$\mu = 61, \sigma = 10$

$P[51 < x \le 71] = ?$

$$Z_1 = \frac{x-\mu}{\sigma} = \frac{51-61}{10} = -1$$

$$Z_2 = \frac{71-61}{10} = 1$$

Therefore;

$$P[51 < x \le 71] = P[-1 < Z \le 1].$$

$= \Phi(1) - \Phi(-1)$

$= \Phi(1) - [1 - \Phi(1)].$

$= 2\Phi(1) = 2 \times 0.8413 - 1$

$= \boxed{0.6826}$

Example #3.18:-

$P[E] = Q\left(\sqrt{\frac{\gamma}{2}}\right)$

$\gamma = SNR = \frac{S}{N}$

min value of $\gamma = ?$, $Q(z) < 10^{-6}$

when $z \ge 4.76$, $\sqrt{\frac{\gamma}{2}} = 4.76$

$\cancel{\gamma^2 = 22.65}$ $\frac{\gamma}{2} = 22.6576$

$\gamma = 45.3152$

$P[E < 10^{-6}]$ when $\gamma \ge 45.3152$

Quiz #3.5:-

$\mu_x = 0 \quad , \sigma_x = 1$

① $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

Peak value of $f_X(x) = \frac{1}{\sqrt{2\pi}} = 0.3989$

$\pm 3\sigma = \pm 3$

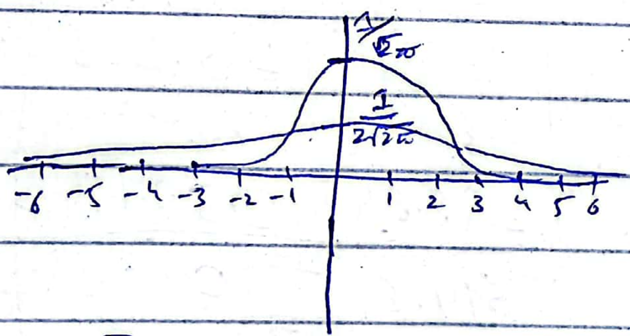
$y \sim$ Guassian $(0, 2)$

$\mu_y = 0, \quad \sigma_y = 2.$

$f_Y(y) = \frac{1}{2\sqrt{2\pi}} e^{-y^2/8}$

Peak value of $f_Y(y) = \frac{1}{2\sqrt{2\pi}} = 0.1995$

$\pm 3\sigma = \pm 6$



② $P[-1 < X \leq 1] = \phi(1) - \phi(-1)$

$= 2\phi(1) - 1 = 0.6826$

③ $P[-1 \leq Y \leq 1] = F_Y(1) - F_Y(-1) = \phi(\frac{1}{2}) - \phi(-\frac{1}{2})$

$z_1 = \frac{-1-0}{2} = -\frac{1}{2}, \quad z_2 = \frac{1-0}{2} = \frac{1}{2}$

$= \phi(\tfrac{1}{2}) - [1 - \phi(\tfrac{1}{2})]$ $Q(8) = 1 - \Phi(8)$

$= 2\phi(\tfrac{1}{2}) - 1 = 2 \times 0.6915 - 1$

$= \boxed{0.383}$

(4) $P[X > 3.5] = P[Z > 3.5] = Q(3.5)$

$Q(3.5) = 2.33 \times 10^{-4}$ (from table)

(5) $P[Y > 3.5] = Z = \frac{3.5 - 0}{2} = 1.75$

$P[Y > 3.5] = P[Z > 1.75] = Q(1.75)$

$= 1 - \phi(1.75) = 1 - 0.9599$

$= 0.0401$

X	$X - \bar{X}$	$(X - \bar{X})^2$	$Z = \frac{X - \bar{X}}{S}$	$(Z - \bar{Z})^2$
45	-15.8	249.64	-1.274	1.623
56	-4.8	23.04	-0.387	0.149
58	-2.8	7.84	-0.226	0.051
67	6.2	38.44	0.5	0.25
78	17.2	295.84	1.387	1.9237
$\Sigma = 304$		614.8	0	3.99
$\bar{X} = 60.8$			$\Sigma = 0$	$\Sigma 4$

$VAR = S^2 = \frac{\Sigma (X - \bar{X})^2}{n - 1}$

$= S^2 = \frac{614.8}{4} = 153.7$ $\boxed{S = 12.4}$

$F_X(x) = P[X \leq x]$

Joint Commulative Distribution Function:

$F_{X,Y}(x,y) = P[X \leq x, Y \leq y]$

Theorem:- Joint CDF

(a) $0 \leq F_{X,Y}(x,y) \leq 1$

(b) $F_X(x) = F_{X,Y}(x, \infty)$

(c) $F_Y(y) = F_{X,Y}(\infty, y)$

(d) $F_{X,Y}(-\infty, y) = F_{X,Y}(x, -\infty) = 0$

(e) If $x \leq x_1$ and $y \leq y_1$, then

$F_{X,Y}(x,y) \leq F_{X,Y}(x_1, y_1)$

(f) $F_{X,Y}(\infty, \infty) = 1$.

Join PMF

Joint Probability Mass Function

$P_{X,Y}(x,y) = P[X=x, Y=y]$.

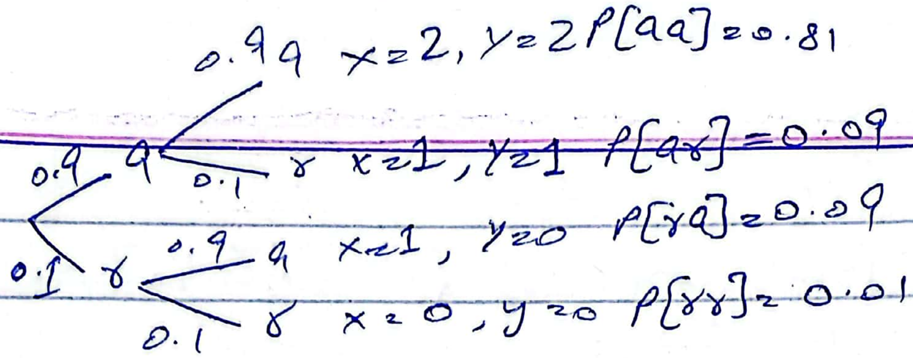
Example #4.1:-

$P[a] = 0.9$

$P[r] = 0.1$

No. of acceptable circuits $= X$

No. of ~~reject~~ successful tests before first reject $= Y$



$S=\{aa, ar, ra, rr\}$.

$$P_{X,Y}(x,y)=\begin{cases}0.81, & x=2, y=2\\ 0.09, & x=1, y=1\\ 0.09, & x=1, y=0\\ 0.01, & x=0, y=0\\ 0, & \text{otherwise}\end{cases}$$

$$\sum_{x\in S_x}\sum_{y\in S_y} P_{X,Y}(x,y)=1.$$

Marginal PMF:-

$P_{X,Y}(x,y)$	$y=0$	$y=1$	$y=2$	$P_X(x)$
$x=0$	0.01	0	0	0.01
$x=1$	0.09	0.09	0	0.18
$x=2$	0	0	0.81	0.81
	0.10	0.09	0.81	1

$P_Y(y)$

$$P_X(x)=\begin{cases}0.01 & x=0\\ 0.18 & x=1\\ 0.81 & x=2\\ 0 & \text{otherwise}\end{cases}$$

$$P_Y(y)=\begin{cases}0.10 & y=0\\ 0.09 & y=1\\ 0.81 & y=2\\ 0 & \text{otherwise}\end{cases}$$

Theorem :-

For discrete random variables X & Y and any set B in the x, y plane the Probability of the event $\{(x,y) \in B\}$ is

$$P[B] = P[(x,y) \in B] = \sum_{(x,y)\in B} P_{X,Y}(x,y)$$

$B : X = Y$

$$B = \{(0,0), (1,1), (2,2)\}.$$

$$P[B] = P_{X,Y}(0,0) = P_{X,Y}(1,1) + P_{X,Y}(2,2)$$

$$P[B] = 0.01 + 0.09 + 0.81 = 0.91$$

$$F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(u,v)\, dv\, du.$$

$$f_{X,Y}(x,y) = \frac{d^2 F_{X,Y}(x,y)}{dx\,dy}.$$

$$P[x_1 < X \leq x_2, y_1 < Y \leq y_2] = F_{X,Y}(x_2, y_2) - F_{X,Y}(x_2, y_1) - F_{X,Y}(x_1, y_2) + F_{X,Y}(x_1, y_1)$$

Theorem :-

A Joint PDF $f_{X,Y}(x,y)$ has the following Properties :-

(a) $f_{X,Y}(x,y) \geq 0 \quad \forall (x,y)$

(b) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = 1$

from PDF :-

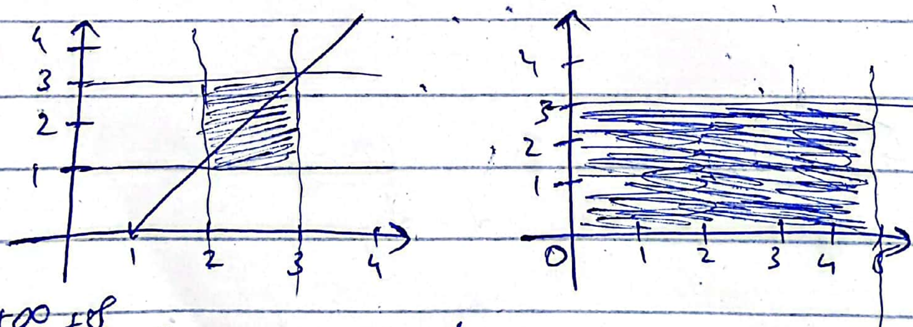
$$= \iint_{A} f_{X,Y}(x,y)\,dx\,dy = \int_{x_1}^{x_2}\int_{y_1}^{y_2} f_{X,Y}(x,y)\,dy\,dx$$

Ex 4.4

$$f_{X,Y}(x,y) = \begin{cases} c & ; \quad 0 \leq x \leq 5,\ 0 \leq y \leq 3 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

(a) Constant c

(b) $P[A] = P[2 \leq X < 3,\ 1 \leq Y < 3]$



$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dx\,dy = 1$$

$$\int_{0}^{3}\int_{0}^{5} c\,dx\,dy = 1$$

$$c \int_0^5 \int_0^3 dy\,dx = 1$$

$$c = \int_0^5 |y|_0^3 \, dx = 1$$

$$3c(5) = 1$$

$$15c = 1$$

$$\boxed{c = \frac{1}{15}}$$

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{15} & , 0 \le x \le 5,\ 0 \le y \le 3. \end{cases}$$

(5) $P[A] = P[2 \le X \le 3 \quad 1 \le Y \le 3]$.

$$= \frac{1}{15} \int_2^3 \int_1^3 dy\,dx$$

$$= \frac{1}{15} (3-1)(3-2)$$

$$= \frac{1}{15} (2)(1) = \frac{2}{15}$$

Example# 4.6:-

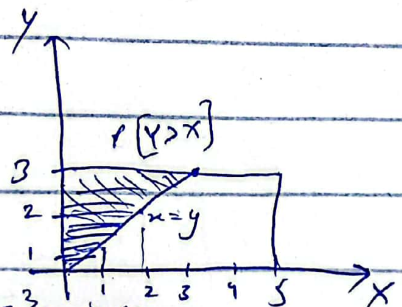
$$f_{X,Y}(x,y) = \begin{cases} 1/15 & 0 \le x \le 5,\ 0 \le y \le 3 \\ 0 & \text{otherwise.} \end{cases}$$

$P[A] = P[Y > X] = ?$

$$P[A] = \int_0^3 \left(\int_x^3 \frac{1}{15} \right) dy\,dx$$

$$= \frac{1}{15} \int_0^3 \int_0^y dy\,dx$$

$$= \frac{1}{15} \cdot 3(3)\,(y)|_0^3 = \frac{9}{30} = \frac{3}{10} \quad 0 < x < y.$$



$$\frac{2}{15}\int_0^3 |x|_0^y \, dy.$$

$$\frac{2}{15}\left.\frac{y^2}{2}\right]_0^3$$

$$= \frac{9}{30} = \frac{3}{10}$$

(or)

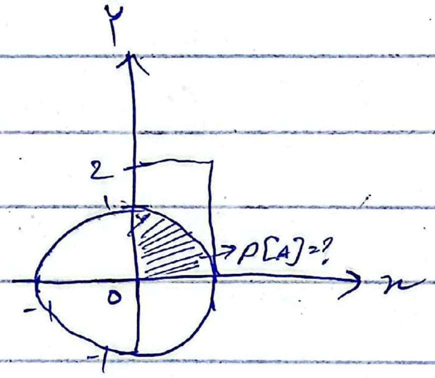
$$P[A] = \frac{1}{15} \int_0^3 \int_x^3 dy\,dx$$

Quiz #4.4:-

$$f_{X,Y}(x,y) = \begin{cases} cxy & 0 \le x \le 1,\ 0 \le y \le 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) Constant C ?

(b) $P[A] = P[X^2+Y^2 \le 1] = ?$



$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dx\,dy.$$

$$\int_0^2\int_0^1 cxy \, dx\,dy = 1$$

$$c\int_0^2 \left.\frac{x^2 y}{2}\right|_0^1 dy = 1$$

$$c\int_0^2 \frac{1}{2} y \, dy = 1$$

$$\boxed{c = 1}$$

$$\frac{c}{2} \left.\frac{y^2}{2}\right|_0^2 = 1 \Rightarrow \frac{c}{2}\,\frac{4}{2} = 1,\ \frac{c4}{4} = 1$$

$$f_{X,Y}(x,y) = \begin{cases} xy & 0\le x\le 1,\ 0\le y\le 2 \\ 0 & \text{otherwise} \end{cases}$$

(b)

$x^2+y^2=1$

$y^2=1-x^2$

$y=\pm\sqrt{1-x^2}$

$$P[A] = \int_0^1\int_0^{x} xy\,dy\,dx$$

$$P[A] = \int_0^1 \frac{y^2}{2}\Big|_0^{x^2} x\,dx$$

$$\cancel{= \frac{1}{2}\int_0^1 x^4\,dx = \frac{1}{2}\frac{x^5}{5}\Big|_0^1 = \frac{1}{2}\cdot\frac{1}{5} = \frac{1}{10}}$$

$$= \frac{1}{2}\int_0^1 x(1-x^2)\,dx$$

$$= \frac{1}{2}\int_0^1 (x-x^3)\,dx$$

$$= \frac{1}{2}\left[\frac{x^2}{2}-\frac{x^4}{4}\right]_0^1$$

$$= \frac{1}{2}\left[\frac{1}{2}-\frac{1}{4}\right] = \frac{1}{2}\left(\frac{1}{4}\right) = \frac{1}{8}$$

$$\boxed{P[A] = \frac{1}{8}}$$

Marginal PDF:-

$$\boxed{\begin{aligned} f_X(x) &= \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dy \\ f_Y(y) &= \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dx\,dy \end{aligned}}$$

$$f_X(x) = \frac{\partial}{\partial x} F_{X,Y}(x,\infty) = \frac{\partial}{\partial x} F_X(x)$$

$$f_Y(y) = \frac{\partial}{\partial y} F_{X,Y}(\infty,y) = \frac{\partial F_Y(y)}{\partial y}$$

Marginal P.D.F:

$f_X(x) = \int_0^2 xy\,dy$ → for x

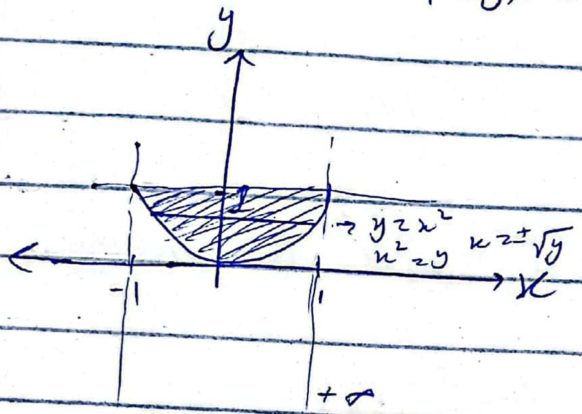
$f_Y(y) = \int_0^1 xy\,dx$ → for y.

Remaing Notes at the mid of the Register

Probability Notes (2)

$$f_{X,Y}(x,y) = \begin{cases} \frac{5}{4}y & -1 \le x \le 1,\ x^2 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$f_X(x) = ?$ $f_Y(y) = ?$



x	$y = x^2$
-1	1
$-1/2$	$1/4$
0	0
$1/2$	$1/4$
1	1

$$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\, dy$$

$$f_X(x) = \int_{x^2}^{1} \frac{5}{4} y\, dy$$

$$f_X(x) = \frac{5}{4} \frac{y^2}{2}\Big|_{x^2}^{1} = \frac{5}{4}\left(\frac{1}{2} - \frac{x^4}{2}\right)$$

$$= \frac{5}{8}\left[1 - x^4\right]$$

$$f_X(x) = \begin{cases} \frac{5}{8}(1-x^4) & -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx$$

$$= \int_{-\sqrt{y}}^{+\sqrt{y}} \frac{5}{4} y\, dx$$

$$= \frac{5}{4} y \left| x \right|_{-\sqrt{y}}^{+\sqrt{y}}$$

$$= \frac{5}{4}\left(\sqrt{y} \mp \sqrt{y}\right)$$

$$= \frac{5}{8} y \left[\frac{2}{8} y^{1/2}\right]$$

$$= \frac{5}{8} y^{3/2}$$

$$f_Y(y) = \begin{cases} \frac{5}{2} y^{3/2} & 0 \le y \le 1 \\ 0 , & \text{otherwise} \end{cases}$$

Co-Variance :-

$$CoV[X, Y] = \sigma_{XY} = E[(x-\mu_x)(y-\mu_y)]$$

continuous

$$CoV[X,Y] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (x-\mu_x)(y-\mu_y) f_{X,Y}(x,y)\,dx\,dy$$

discrete

$$CoV[X,Y] = \sum_x \sum_y (x-\mu_x)(y-\mu_y) P_{X,Y}(x,y)$$

Corelation :-

$$\gamma_{XY} = E[XY]$$

$$\gamma_{XY} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xy\, f_{X,Y}(x,y)\,dx\,dy$$

$$\gamma_{XY} = \sum\sum xy\; P_{X,Y}(x,y)$$

Theorem :-

★★ In exam

(a) $CoV[X,Y] = \gamma_{XY} - \mu_X \mu_Y$

$CoV[X,Y] = E[XY] - E[X]E[Y]$

$$E[X] = \int_{-\infty}^{+\infty} x\, f_X(x)\,dx$$

$$\boxed{E[X] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x\, f_{X,Y}(x,y)\,dx\,dy}$$

$$E[g(x,y)] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x,y)\, f_{X,Y}(x,y)\, dx\, dy$$

$$g(x,y) = x$$

$$E[g(x,y)] = E[X]$$

$$\boxed{E[Y] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\, f_{X,Y}(x,y)\, dx\, dy}$$

(b) If $X = Y$

$$CoV[X,Y] = CoV[X,X] = VAR[X] = \sigma_X^2$$

$$\boxed{CoV[X,Y] = E[X^2] - (E[X])^2}$$

Example # 4.12 :-

$P_{X,Y}(x,y)$	$y=0$	$y=1$	$y=2$	$P_X(x)$
$x=0$	0.01	0	0	0.01
$x=1$	0.09	0.09	0	0.18
$x=2$	0	0	0.81	0.81
	0.10	0.09	0.81	

$r_{x,y} = ?$ and $CoV[X,Y] = ?$

$$r_{xy} = E[XY] = \sum_{y \in S_Y}\sum_{x \in S_X} xy\, P_{X,Y}(x,y)$$

$$r_{xy} = (0)(0)(0.01) + (0)(1)(0) + (0)(2)(0)$$
$$(1)(0)(0.09) + (1)(1)(0.09) + (1)(2)(0) + (2)(0)(0)$$
$$+ (2)(1)(0) + (2)(2)(0.81)$$

$$\boxed{\gamma_{xy} = 3.33}$$

↳ corelation

$Cov[X, Y] = ?$

$Cov[X, Y] = \gamma_{xy} - \mu_x \mu_y$.

$E[X] = (0)(0.01) + (1)(0.18) + 2(0.81)$

$$\boxed{E[X] = 1.80}$$

$E[Y] = (0)(0.10) + (1)(0.09) + (2)(0.81)$

$E[Y] = 1.71$

$Cov[X, Y] = \sigma_{xy} = \gamma_{xy} - \mu_x \mu_y$

$= 3.33 - (1.80)(1.71)$

$$\boxed{Cov[X, Y] = 0.252}$$

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Orthogonal Random Variables :-

Radom Variables X and Y are orthogonal if $\gamma_{xy} = 0$

i.e;

$$E[XY] = 0$$

Uncorrelated Random Variables :-

RV'S X and Y are uncorrelated if

$$CoV[X,Y] = 0$$

$$\sigma_{xy} = E[XY] - E[X]E[Y]$$

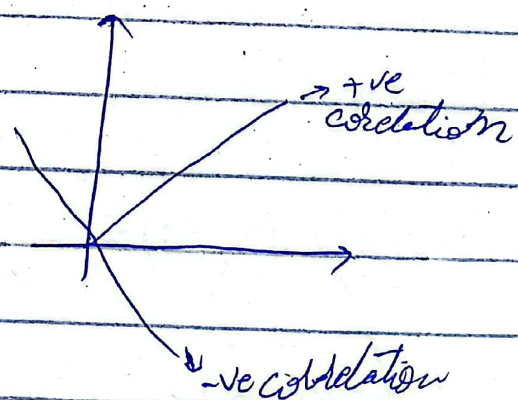
$$E[XY] = E[X]E[Y]$$

Correlation coefficient:

(video Lectures)

$$\rho_{XY} = \frac{CoV[X,Y]}{\sqrt{VAR[X]\,VAR[Y]}} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

$$-1 \leq \rho_{XY} \leq 1$$



Independent Random Variables

$$P[AB] = P[A] \times P[B]$$

RV'S X and Y are indpent if

$P_{X,Y}(x,y) = P_X(x)\,P_Y(y)$ (Discrete)

$f_{X,Y}(x,y) = f_X(x)\,f_Y(y)$ (Continous)

For independent RV's X and Y

$CoV[X,Y] = \sigma_{XY} = 0$.

1- $-1 \le \rho \le 1$
2- $\rho(X,Y) = \rho(Y,X)$
3- $\rho(X,X) = 1$
4- $\rho(X,-X) = -1$

Types of Transform :-

1- Characteristic Function
2- Probability Generating Function
3- Moment Generating Function (MGF)

MGF :-

$$\Phi_X(s) = E[e^{sX}]$$

(Continous) $$\Phi_X(s) = \int_{-\infty}^{+\infty} e^{sx} f_X(x)\,dx$$

(Discrete) $$\Phi_X(s) = \sum_{x_i \in S_x} e^{sx_i} P_X(x_i)$$

Theorem :-

A RV. X with MGF $\Phi_X(s)$ has n^{th} moment

$$E[X^n] = \frac{d^n}{ds^n}\Phi_X(s)\Big|_{s=0}$$

$$E[X^n] = \int_{-\infty}^{+\infty} x^n f_X(x)\, dx$$

$$E[X^n] = \sum_{x \in S_X} x_i^n P_X(x_i)$$

1- $\Phi_X(S)\big|_{S=0} = \Phi_X(0) = 1$

2- If X_1 and X_2 are two independent RV's with MGF's $\Phi_{X_1}(S)$ and $\Phi_{X_2}(S)$, then

$$\Phi_{X_1+X_2}(S) = \Phi_{X_1}(S)\, \Phi_{X_2}(S)$$

$$\boxed{Y = X_1 + X_2}$$

Example 1-

Derive the MGF for exponential distribution

Solution #

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\Phi_X(S) = E[e^{Sx}]$$

$$\Phi_X(S) = \int_{-\infty}^{+\infty} e^{Sx} f_X(x)\, dx$$

$$\Phi_X(s) = \int_0^\infty e^{sx} \left(\lambda e^{-\lambda x}\right) dx$$

$$= \lambda \int_0^\infty e^{-(\lambda - s)x} dx$$

$$\Phi_X(s) = \frac{-\lambda}{\lambda - s} \left. e^{(-\lambda - s)x} \right|_0^\infty$$

$$= \frac{-\lambda}{\lambda - s} [0 - 1]$$

$$\boxed{\Phi_X(s) = \frac{\lambda}{\lambda - s}}$$

$$E[X^n] = \frac{d^n}{ds^n} \Phi_X(s) \Big|_{s=0}$$

Therefore;

$$E[X] = \frac{d}{ds} \Phi_X(s) \Big|_{s=0} = \frac{d}{ds} \frac{\lambda}{\lambda - s} \Big|_{s=0}$$

$$= \frac{d}{ds} \left[\lambda (\lambda - s)^{-1}\right]$$

$$= -\lambda (\lambda - s)^{-2} (-1) \Big|_{s=0}$$

$$= \frac{\lambda}{(\lambda - s)^2} \Big|_{s=0} = \frac{1}{\lambda}$$

$$\boxed{E[X] = \frac{1}{\lambda}}$$

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$$\frac{d^2}{ds^2}\bar{\Phi}_x(s) = \frac{d}{ds}\left(\frac{\lambda}{(\lambda-s)^2}\right) = \frac{d}{ds}\lambda(\lambda-s)^{-2}$$

$$= -2\lambda(\lambda-s)^{-3}(-1) = \frac{2\lambda}{(\lambda-s)^3}$$

$$E[X^2] = \frac{d^2}{ds^2}\bar{\Phi}_x(s)\Big|_{s=0} = \frac{2\lambda}{\lambda^3} = \frac{2}{\lambda^2}$$

$$\boxed{E[X^2] = \frac{2}{\lambda^2}}$$

$$E[X^n] = \frac{d^n}{ds^n}\bar{\Phi}_x(s)\Big|_{s=0} = \frac{n!\lambda}{(\lambda-s)^{n+1}}\Big|_{s=0}$$

$$\boxed{E[X^n] = \frac{n!}{\lambda^n}}$$

J & K are independent RV's

$$P_J(j) = \begin{cases} 0.2 & j=1 \\ 0.6 & j=2 \\ 0.2 & j=3 \\ 0 & \text{otherwise} \end{cases} \qquad P_K(k) = \begin{cases} 0.5 & k=-1 \\ 0.5 & k=1 \\ 0 & \text{otherwise} \end{cases}$$

(a) MGF of $M = J+K$

(b) $E[M^3]$ (c) $P_M(m)$

Solv

$$\Phi_X(s) = E[e^{sX}]$$
$$= \sum_{x_i \in S_X} e^{s x_i} P_X(x_i)$$

$$\Phi_J(s) = \sum_{j \in S_j} e^{sj} P_J(j)$$
$$= (0.2)e^{s} + 0.6e^{2s} + 0.2e^{3s}$$

$$\Phi_K(s) = (0.5)e^{-s} + 0.5e^{s}$$

(a) $M = J + K$

$$\phi_M(s) = \Phi_J(s)\,\phi_K(s)$$

$$\phi_M(s) = \left(0.2e^{s} + 0.6e^{2s} + 0.2e^{3s}\right) \times \left(0.5e^{-s} + 0.5e^{s}\right)$$

$$\phi_M(s) = (0.1) + 0.3e^{s} + 0.1e^{2s} + 0.1e^{2s} + 0.3e^{3s} + 0.1e^{4s}$$

$$\phi_M(s) = 0.1 + 0.3e^{s} + 0.2e^{2s} + 0.3e^{3s} + 0.1e^{4s}$$

(b) $E[M^3] = ? = \frac{d^3}{ds^3} \Phi_M(s) \Big|_{s=0}$

$$\frac{d}{ds}\phi_M(s) = 0.3e^{s} + 0.4e^{2s} + 0.9e^{3s} + 0.4e^{4s}$$

$$\frac{d^2}{ds^2}\phi_M(s) = 0.3e^{s} + 0.8e^{2s} + 2.7e^{3s} + 1.6e^{4s}$$

$$\frac{d^3}{ds^3}\phi_M(s) = 0.3e^{s} + 1.6e^{2s} + 8.1e^{3s} + 6.4e^{4s}$$

$$E[X^3] = \frac{d^3}{ds^3}\phi_M(s)\Big|_{s=0} = 0.3 + 1.6 + 8.1 + 6.4$$

$$\boxed{E[X^3] = 16.4}$$

$$P_M(m) = \begin{cases} 0.1 & m=0 \\ 0.3 & m=1 \\ 0.2 & m=2 \\ 0.3 & m=3 \\ 0.1 & m=4 \\ 0 & \text{otherwise} \end{cases}$$

Theorem :- (YOU MUST KNOW ~~EXAM MEIN NHI ANA~~)

For a set of independent RV's $X_1, X_2, \ldots X_n$, MGF of

$$W = X_1 + X_2 + \cdots + X_n$$

is given by.

$$\phi_W(s) = \phi_{X_1}(s)\,\phi_{X_2}(s) \cdots \phi_{X_n}(s)$$

If RV are iid, then

$$\phi_M(s) = [\phi_X(s)]^n.$$

Central Limit Theorem (CLT):-

Independent : $X_1, X_2, \cdots, X_n$

$$X = X_1 + X_2 + \cdots + X_n = \sum_{i=1}^{n} X_i$$

$$E[X] = E[X_1] + E[X_2] + E[X_3] \cdots E[X_n]$$

$$\mu_X = \mu_1 + \mu_2 + \cdots \mu_n.$$

Variance of X :-

$$VAR[X] = \sigma_X^2 = \sigma_1^2 + \sigma_2^2 + \cdots \sigma_n^2$$

Central limit theorem states that

as 'n' increases, the distribution $F_X(x)$ of random variable X approaches a normal distribution or (Guassian).

$$F_X(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

$$f_X(x) = \frac{1}{\sigma_X\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2}$$

if

$$\mu_1 = \mu_2 = \mu_3 \cdots = \mu_n$$

$$\mu_2 = n\,\mu_{x2}$$

$$\boxed{\sigma^2 = n\sigma_x^2} \quad \boxed{\sigma = \sqrt{n}\,\sigma_x}$$

$$Z = \frac{x-\mu}{\sigma} = \sum_{i=1}^{n} \frac{X_i - n\mu_x}{\sqrt{n}\,\sigma_x}$$

Example#

$E[X_i] = p = 0.5$

$VAR[X_i] = p(1-p) = 0.25$

$n = 10^6 = 1000000$, $n = 10^6$

$W = X_1 + X_2 + \dots + X_n = \sum_{i=1}^{n} X_i$

$\mu_w = n\mu_x = 10^6 \times 0.5 = 5 \times 10^5$

$P_{X}(x) = \begin{cases} p, & x=1 \\ 1-p, & x=0 \\ 0, & \text{otherwise} \end{cases}$

$E[X] = p$

$E[X^2] = p$

$VAR[X] = E[X^2] - (E[X])^2$

$= p - p^2$

$= p(1-p)$

$\sigma_w^2 = VAR[W] = n\,VAR[X_i]$

$= 10^6 \times 0.25$

$\sigma_w^2 = 2.5 \times 10^5 = 250{,}000$

$\boxed{\sigma_w = 500}$

$P[W \geq 502000] = 1 - P[W < 502000]$

transform to W to Z:-

$= 1 - \Phi\left(\frac{w - \mu_w}{\sigma_w}\right)$

$= 1 - \Phi\left(\frac{502000 - 500000}{500}\right)$

$= 1 - \Phi(4) = Q(4)$

$Q(4) = 1 - \Phi(3)$

$= \boxed{P[W \geq 502000] = 3.17 \times 10^{-5}}$

$P[A] = P[499000 \le \omega \le 501000]$

Using same formula:

$$= F_\omega(501000) - F_\omega(499000)$$

$$= \Phi\left(\frac{501000 - 500000}{500}\right) - \Phi\left(\frac{499000 - 500000}{500}\right)$$

$$= \Phi(2) - \Phi(-2)$$

$$= \Phi(2) - [1 - \Phi(2)] = 2\Phi(2) - 1$$

$$= 2(0.97725) - 1$$

$$\boxed{P[A] = 0.9545}$$

(Ch # 07)

Sample Mean

RV $\rightarrow$ $M_n(X) = \frac{X_1 + X_2 + \cdots X_n}{n}$

Theorem:-

$$E[M_n(X)] = E[X]$$

$$VAR[M_n(X)] = \frac{VAR[X]}{n}$$

As n increases without bonds

$$\lim_{n \to \infty} VAR[M_n(x)] = \frac{VAR[X]}{n} = 0$$

Markov Inequality:-

For a RV X such that $P[X < 0] = 0$ and a constant $x > 0$, i.e; $f(x) = 0$ for $X < 0$,

$$P[X \geq K] = \frac{E[X]}{K} = \frac{\mu}{K}$$

Markov Inequality is valid only for non negative

Example:-

$E[X] = 5.5$

$P[X \geq 11] = \frac{5.5}{11} = \frac{1}{2}$

$$\boxed{P[X \geq 11] \leq 1/2}$$

Chebyshev Inequality:-

For an arbitrary random variable x and constant $k > 0$;

$$P[|X - \mu_x| \geq k] \leq \frac{VAR[X]}{k^2} = \frac{\sigma^2}{K^2} \quad \text{--- (1)}$$

let $k = n\sigma$ in (1)

$$P[|x - \mu_x| \geq n\sigma] \leq \frac{\sigma^2}{n^2\sigma^2} = \frac{1}{n^2}$$

$$P[|x - \mu_x| \geq n\sigma] \leq \frac{1}{n^2} \quad \text{--- (2)}$$

Multiply ① by (-1)

$$-P[|X-\mu_x| \geq k] > -\frac{\sigma^2}{k^2}$$

$$1-P[|X-\mu_x| \geq k] > 1-\frac{\sigma^2}{k^2}$$

However;

$$1-P[Y \geq y] = P[Y<y]$$

Therefore;

$$P[|X-\mu_x| < k] > 1-\frac{\sigma^2}{k^2} \quad -③$$

Similarly from ②

$$P[|X-\mu_x| < n\sigma] > 1-\frac{1}{n^2} \quad -④$$

$\pm(X-\mu_x) \geq k$

$(X-\mu_x) \geq k$ and $-x+\mu_x \geq k$

$x \geq \mu_x + k$ and $x \leq x-k$

Therefore;

$|X-\mu_x| \geq k \Rightarrow \mu_x+k \leq x \leq \mu_x-k$

$P[\bar{A}] = 1 - P[A]$

$= \dfrac{Var[A]}{nC^2}$

$= 0.0099$

$$P_d(D) = \begin{cases} 0.2 & D=1 \\ 0.8 & D=0 \\ 0 & \text{otherwise} \end{cases}$$

$E[X] = 0.2$ $\quad Var[X] = 0.2 - (0.2)^2$

$= 0.16$

$E[K_{100}] = 20$

$\sigma(K_{100}) = 0.16 \times 100 = \sqrt{16} = 4$

$$P_{M_m} = \begin{cases} 0.42 & m=-3 \\ 0.40 & m=-2 \\ & m=-1 \end{cases}$$

$$\int_a^b e^{sx} \frac{1}{b-a}$$

$$= \frac{e^{bs} - e^{as}}{s(b-a)}$$

$$P\left[|\hat{P}(A - P[A]| \geq 10^{-5}\right] \leq \frac{P[A][1-P[A]]}{nC^2}$$

$$= \frac{10^{-3}(1-10^{-3})}{n(10^{-5})^2} = \frac{9.99 \times 10^{-4}}{n10^{-10}}$$

$$= \frac{9.9 \times 10^{6}}{n}$$

$$[|\hat{P}_n(A) - P(A)| < 0.05] \geq 1 - \frac{Var[X_A]}{n\sigma^2}$$

$$\alpha = \frac{0.09}{90(0.05)^2}$$

$$0.1 = \frac{\sigma_x^2}{n(0.03)^2}$$

$$0.1 = \frac{(0.09)^2}{n(0.03)^2}$$

$$n = \frac{8.1 \times 10^3}{9 \times 10^4}$$

$$0.99 = 1 - \frac{(15)^2}{n\epsilon}$$