B

Date: 15-5-205 SN-standard Normal Day:

Def

3.11 SN Complementary CDF

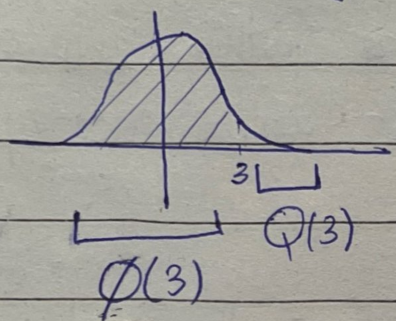
Q function

$$Q(z) = 1 - \Phi(z)$$

$$Q(\zeta) = P[Z > \zeta]$$

$$P[Z > \zeta] = 1 - P[Z \leq \zeta]$$

$$Q(\zeta) = 1 - \Phi(\zeta)$$



less than 3 is $\Phi(3)$

Ex 3.18 : → solve Q function Ex from last of the ch.

$$Q\left(\sqrt{\frac{\gamma}{2}}\right)$$

γ = SNR

$$Q(\zeta) < 10^{-6}$$

when $\zeta \geq 4.76$

$$\sqrt{\frac{\gamma}{2}} = 4.76$$

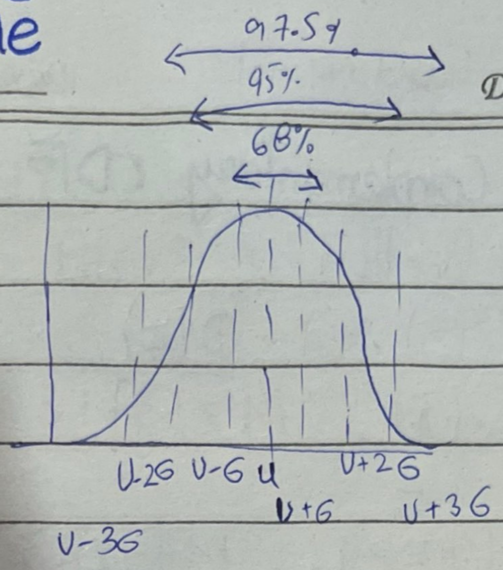
$$\frac{\gamma}{2} = 22.6576$$

$$\gamma = 45.3152$$

therefore $P[E] < 10^{-6}$, when $\gamma \geq 45.3152$

3 Sigma Rule

e:______ Day:______



68% value are within 1σ of mean
95% value are within 2σ of mean
97.5% values are within 3σ of mean

Quizz 3.5

$X \sim$ Gaussian$(0,1)$

$\mu = 0 \ , \ \sigma = 1$

$f_X(x) = \frac{1}{1\cdot\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-0}{1}\right)^2}$

$f_X(x) = \frac{1}{1\cdot\sqrt{2\pi}} e^{-x^2/2}$

peak value of $f_X(x) = \frac{1}{\sqrt{2\pi}} = 0.39 \approx 0.4$

$Y \sim$ Gaussian$(0,2)$

$\mu = 0, \ \sigma = 2$

$f_Y(y) = \frac{1}{2\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-0}{2}\right)^2}$

$= \frac{1}{2\sqrt{2\pi}} e^{-y^2/8}$

Date: ______ Day: ______

peak value of $f_Y(y) = \frac{1}{2\sqrt{2\pi}} = 0.1995 \approx 0.2$

Sketch

① $\mu = 0$

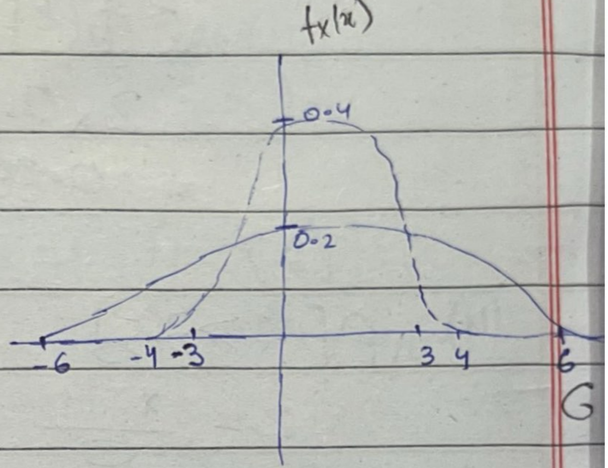
$f_X(x) = 0.4$ PV

$\sigma = 1 \times 3 = 3$

② $\mu = 0$

$f_Y(y) = 0.2$ PV

$\sigma = 2 \times 3 = 6$



(2)

$P[-1 < X \leq 1] = P[-1 < Z \leq 1]$

first find z_1 and z_2

$z_1 = \frac{x-\mu}{\sigma} = \frac{x-0}{1} = x = -1$

$z_2 = \frac{x-\mu}{\sigma} = \frac{x-0}{1} = x = 1$

$= \Phi(1) - \Phi(-1) = \Phi(1) - [1 - \Phi(1)]$

$= 2\Phi(1) - 1 = 2 \times 0.8413 - 1$

$= 0.6826$

(3) $P[-1 < Y \leq 1]$

$$z = \frac{x-\mu}{\sigma}$$

$$z_1 = \frac{-1-0}{2} = -\frac{1}{2}$$

$$z_2 = \frac{1-0}{2} = \frac{1}{2}$$

$$P\left[-\tfrac{1}{2} < Z \leq \tfrac{1}{2}\right] = \Phi\left(\tfrac{1}{2}\right) - \Phi\left(-\tfrac{1}{2}\right)$$

Day:

$$\Phi(1/2) - [1 - \Phi(1/2)]$$

$$= \Phi(1/2) - 1 + \Phi(1/2)$$

$$= 2\Phi(1/2) - 1$$

$$= 2 \times 0.6915 - 1$$

$$= 0.383$$

(4) $P[X > 3.5]$

$$z = \frac{x - \mu}{\sigma} = \frac{3.5 - 0}{1} = 3.5$$

$$P[z > 3.5] = Q(3.5) = 2.33 \times 10^{-4}$$

(5)

$P[Y > 3.5]$

$$z = \frac{x - \mu}{\sigma} = \frac{3.5 - 0}{2} = 1.75$$

$$P[Y > 3.5] = P[z > 1.75]$$

$$= Q(1.75)$$

$$= 1 - \Phi(1.75)$$

$$= 1 - 0.9599$$

$$= 0.0401$$

$F_X(x) = P[X \leq x]$

convert x to y CDF

Date: Day:

Example: 3.22

$Y = g(X)$

$$F_X(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$$

$Y = g(X) = 100X$

$F_Y(y) = P[Y \leq y] = [100X \leq y]$

$F_Y(y) = [X \leq y/100] = F_X(y/100)$

$$= \begin{cases} 0 & y/100 < 0 \\ y/100 & 0 \leq y/100 < 1, \\ 1 & y/100 \geq 1 \end{cases}$$

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ y/100 & 0 \leq y < 100 \\ 1 & y \geq 100 \end{cases}$$

$$f_Y(y) = \frac{d}{dy} F_Y(y)$$

$$f_Y(y) = \begin{cases} \frac{1}{100} & 0 < y \leq 100 \\ 0 & \text{otherwise} \end{cases}$$

$Y \sim \text{uniform}(\underset{a}{0}, \underset{b}{100})$

Day:

Theorem:

if $Y = aX$

where $a>0$, then Y

has the CDF

$$F_Y(y) = F_X(y/a)$$

$$f_Y(y) = \frac{1}{a} f_X(y/a)$$

Joint CDF of X, Y

Ex 3.32 3.33 3.34

$$F_{X,Y}(x,y) = P[X \le x,\ Y \le y]$$

$$F_X(x) = P[X \le x] = P[X \le x, Y < \infty]$$

$$= \lim_{y \to \infty} F_{X,Y}(x,y) = F_{X,Y}(x,y)$$

Theorem:

(a) $0 \le F_{X,Y}(x,y) \le 1$

(b) $F_X(x) = F_{X,Y}(x,\infty)$

(c) $F_Y(y) = F_{X,Y}(\infty, y)$

(d) $F_{X,Y}(-\infty, y) = F_{X,Y}(x, -\infty) = 0$

(e) If $x \le x_1$ and $y \le y_1$ then

$$F_{X,Y}(x,y) \le F_{X,Y}(x_1,y_1)$$

(f) $F_{X,Y}(\infty,\infty) = 1$

$$F_{X,Y}(\infty,\infty) = 1$$

$$F_{X,Y}(x,y) = F_{X,Y}(x,\infty)$$

Date: ________ Day: ________

Joint PMF

$P_{X,Y}(x,y) = P[X=x, Y=y]$

Ex 4.1

$p = 0.9$

Accept $= a$

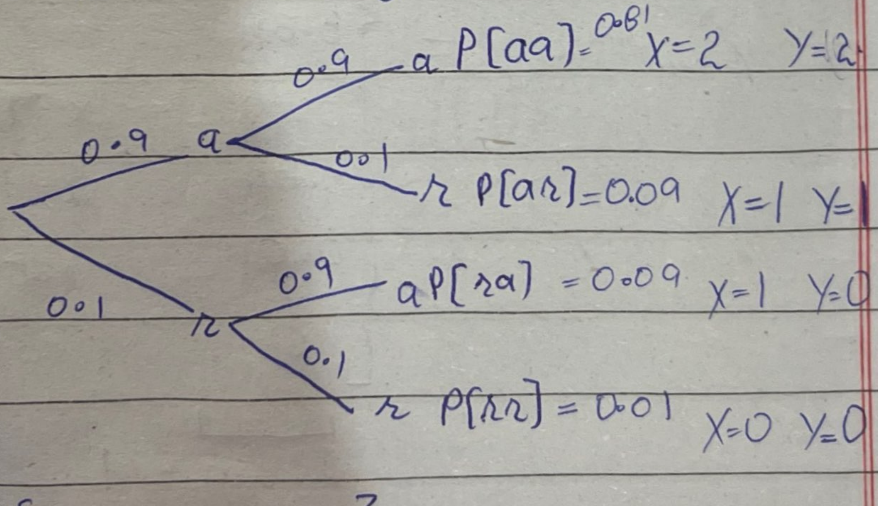
Reject $= r$

$P[a] = 0.9$

$P[r] = 0.1$

Number of acctaple circuit $= X$

Number of successful tests before first reject $= Y$

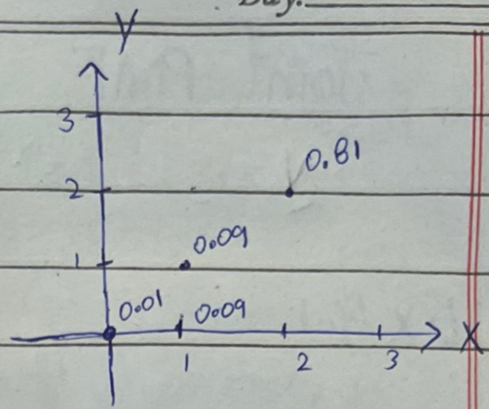


$S = \{aa, ar, ra, rr\}$

S	aa	ar	ra	rr
P(.)	0.81	0.09	0.09	0.01
X	2	1	1	0
Y	2	1	0	0

Day:

sketch



Table

	$y=0$	$y=1$	$y=2$	$P_X(x)$
$P_Y(y)$	0.10	0.09	0.81	
$X=0$	0.01	0	0	0.01
$X=1$	0.09	0.09	0	0.18
$X=2$	0	0	0.81	0.81

Joint PMF of RV

$$P_{X,Y}(x,y) = \begin{cases} 0.81 & x=2, y=2 \\ 0.09 & x=1, y=1 \\ 0.09 & x=1, y=0 \\ 0.01 & x=0, y=0 \\ 0 & \text{otherwise} \end{cases}$$

sum = 1

$$\sum_{x \in S_X} \sum_{y \in S_Y} P_{X,Y}(x,y) = 1$$

Date: Day:

$$P_X(x) = \begin{cases} 0.01 & x=0 \\ 0.18 & x=1 \\ 0.81 & x=2 \\ 0 & \text{otherwise} \end{cases}$$

$$P_Y(y) = \begin{cases} 0.10 & y=0 \\ 0.09 & y=1 \\ 0.81 & y=2 \\ 0 & \text{otherwise} \end{cases}$$

Ex 4.2 continuous Ex 4.1

$$P_{X,Y}(x,y) = \begin{cases} 0.81 & x=2, y=2 ✓ \\ 0.09 & x=1, y=1 ✓ \\ 0.09 & x=1, y=0 \times \\ 0.01 & x=0, y=0 ✓ \end{cases}$$

$$P[B] = P_{X,Y}(2,2) + P_{X,Y}(1,1) + P_{X,Y}(0,0)$$
$$= 0.81 + 0.09 + 0.01$$
$$= 0.91$$

Quizz 4.2:

$P_{Q,G}(q,g)$	$g=0$	$g=1$	$g=2$	$g=3$
$q=0$	0.06	0.18	0.24	0.12
$q=1$	0.04	0.12	0.16	0.08

Day:

(1) $P[Q=0]$

$$P[Q=0] = P_{Q,G}(0,0) + P_{Q,G}(0,1) + P_{Q}(0,2) + P_{Q,G}(0,3)$$

$$= 0.06 + 0.18 + 0.24 + 0.12$$

$$P[Q=0] = 0.6$$

(2) $P[Q=G]$

$$P[Q=G] = P_{Q,G}(0,0) + P_{Q,G}(1,1)$$

$$= 0.06 + 0.12$$

$$P[Q=G] = 0.18$$

(3) $P[G>1]$

$$P[G>1] = P_{G,Q}(2,0) + P_{G,Q}(2,1) + P_{G,Q}(3,0) + P_{G,Q}(3,1)$$

$$= 0.24 + 0.16 + 0.12 + 0.08$$

$$P[G>1] = 0.6$$

(4)

Quizz 4.3:

$P_{H,B}(h,b)$	$b=0$	$b=2$	$b=4$	$P_H(h)$
$h=-1$	0	0.4	0.2	0.6
$h=0$	0.1	0	0.1	0.2
$h=1$	0.1	0.1	0	0.2
$P_B(b)$	0.2	0.5	0.3	

$$P_H(h) = \begin{cases} 0.6 & h=-1 \\ 0.2 & h=0 \\ 0.2 & h=1 \\ 0 & \text{otherwise} \end{cases}$$

$$P_B(b) = \begin{cases} 0.2 & b=0 \\ 0.5 & b=2 \\ 0.3 & b=4 \\ 0 & \text{otherwise} \end{cases}$$

Day:

(1) $P[Q=0]$

$$P[Q=0] = P_{Q,G}(0,0) + P_{Q,G}(0,1) + P_{Q}(0,2) + P_{Q,G}(0,3)$$

$$= 0.06 + 0.18 + 0.24 + 0.12$$

$$P(Q=0) = 0.6$$

(2) $P[Q=G]$

$$P[Q=G] = P_{Q,G}(0,0) + P_{Q,G}(1,1)$$

$$= 0.06 + 0.12$$

$$P[Q=G] = 0.18$$

(3) $P[G>1]$

$$P[G>1] = P_{G,Q}(2,0) + P_{G,Q}(2,1) + P_{G,Q}(3,0) + P_{G,Q}(3,1)$$

$$= 0.24 + 0.16 + 0.12 + 0.08$$

$$P[G>1] = 0.6$$

(4)

Date: ______ Day: ______

Quizz 4.3:

$P_{H,B}(h,b)$	$b=0$	$b=2$	$b=4$	$P_H(h)$
$h=-1$	0	0.4	0.2	0.6
$h=0$	0.1	0	0.1	0.2
$h=1$	0.1	0.1	0	0.2
$P_B(b)$	0.2	0.5	0.3	

$$P_H(h) = \begin{cases} 0.6 & h=-1 \\ 0.2 & h=0 \\ 0.2 & h=1 \\ 0 & \text{otherwise} \end{cases}$$

$$P_B(b) = \begin{cases} 0.2 & b=0 \\ 0.5 & b=2 \\ 0.3 & b=4 \\ 0 & \text{otherwise} \end{cases}$$

Day:

Joint PDF

$$F_{X,Y}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{X,Y}(u,v)\,du\,dv$$

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x\,\partial y}$$

$P[x_1 < X \leq x_2, y_1 < Y \leq y_2] =$

$F_{X,Y}(x_2,y_2) - F_{X,Y}(x_2 y_1) -$
$F_{X,Y}(x_1,y_2) + F_{X,Y}(x_1,y_1)$

Theorem:

(a) $f_{X,Y}(x,y) \geq 0 \quad \forall x,y$

(b) $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \underbrace{f_{X,Y}(x,y)}_{\text{volumn}}\,dx\,dy = 1$

$$P[x_1 < X \leq x_2,\ y_1 < Y \leq y_2] = \int_{x_1}^{x_2}\int_{y_1}^{y_2} f_{X,Y}(x,y)\,dy\,dx$$

Day:

Joint PDF

$$F_{X,Y}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{X,Y}(u,v)\,du\,dv$$

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x\,\partial y}$$

$P[x_1 < X \leq x_2, y_1 < Y \leq y_2] =$

$$F_{X,Y}(x_2,y_2) - F_{X,Y}(x_2,y_1) - F_{X,Y}(x_1,y_2) + F_{X,Y}(x_1,y_1)$$

Theorem:

(a) $f_{X,Y}(x,y) \geq 0 \quad \forall x,y$

(b) $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \underbrace{f_{X,Y}(x,y)}_{\text{volumn}}\,dx\,dy = 1$

$$P[x_1 < X \leq x_2, y_1 < Y \leq y_2] = \int_{x_1}^{x_2}\int_{y_1}^{y_2} f_{X,Y}(x,y)\,dy\,dx$$

Date: ______ Day: ______

Ex 4.4

$$f_{X,Y}(x,y) = \begin{cases} c & 0 \le x \le 5, 0 \le y \le 3 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find $c = ?$

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dx\,dy = 1$$

$$\int_0^5\int_0^3 c\,dy\,dx = 1$$

$$c\int_0^5 |y|_0^3\,dx = 1$$

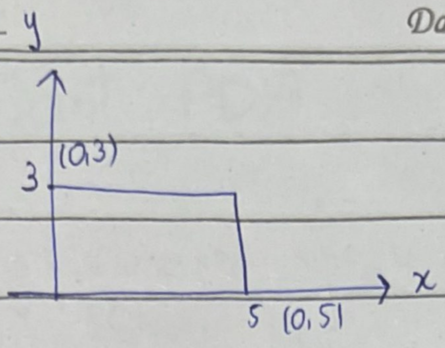
$$3c\int_0^5 dx = 1$$

$$15c = 1$$

$$\boxed{c = 1/15}$$

$$f_{X,Y}(x,y) = \begin{cases} 1/15 & 0 \le x < 5, 0 \le y \le 3 \\ 0 & \text{otherwise} \end{cases}$$

Day:



(b)

$$P[A] = \int_2^3 \int_1^3 \frac{1}{15}\, dy\, dx$$

$$= \frac{1}{15} \int_2^3 |y|_1^3 \, dx$$

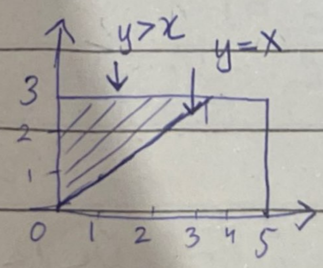
$$\frac{2}{15} \int_2^3 dx = \frac{2}{15} |x|_2^3$$

$$= \frac{2}{15}$$

Ex 4.6

$P[A] = P[Y > X]$

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{15} & 0 \le x \le 5,\ 0 \le y \le 3 \\ 0 & \text{otherwise} \end{cases}$$



Date: Day:

$$P[A] = \int_{y=0}^{3}\int_{x=0}^{y} \frac{1}{15}\, dx\, dy$$

$$= \frac{1}{15}\int_{0}^{3} y\, dy$$

$$= \frac{1}{15\times 2} \left| y^2 \right|_0^3$$

$$= \frac{1}{30}(9-0)$$

$$= \frac{9}{30} = \frac{3}{10}$$

Day:

Quizz 4.4

$$f_{X,Y}(x,y) = \begin{cases} cxy & 0 \le x \le 1,\ 0 \le y \le 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find c

$$\int_{y=0}^{2}\int_{x=0}^{1} cxy \, dx\, dy = 1$$

$$\times \; \cancel{c\left|\frac{x^2}{2}\right|_0^1 \int_0^2 y\, dy = 1}$$

$$c\int_0^2 \left|\frac{x^2}{2}\right|_0^1 y\, dy = 1$$

$$\frac{1}{2}c \int_0^2 y\, dy = 1$$

$$\frac{1}{2}c \left|\frac{y^2}{2}\right|_0^2 = 1$$

$$\frac{1}{2}c \left|\frac{4-0}{2}\right| = 1$$

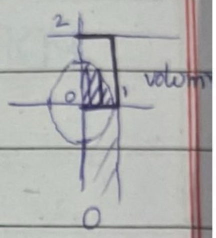
$$\frac{1}{2}c\,|2| = 1$$

$$\boxed{c=1}$$

Date: Day:

$$f_{X,Y}(x,y) = \begin{cases} xy & 0 \le x \le 1,\ 0 \le y \le 2 \\ 0 & \text{otherwise} \end{cases}$$

(b) $P[A] = P[X^2 + Y^2 \le 1]$ $X^2+Y^2=1$



$$P[A] = \int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} xy\, dy\, dx$$

$X^2+Y^2=1$

$Y^2 = 1-X^2$

$Y = \pm\sqrt{1-X^2}$

$Y = \sqrt{1-X^2}$

Marginal PDF

y ko elliminate kar raha hai • x rah jay ga

$$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\, dy$$

$$f_Y(y) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\, dx$$

$$f_X(x) = \frac{\partial F_{X,Y}(x,y)}{\partial x}$$

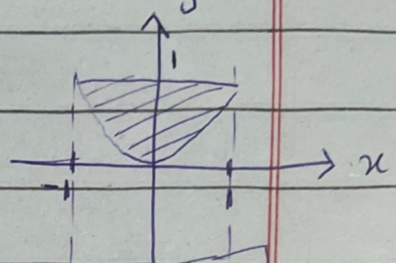
$$F_Y(y) = \frac{\partial F_{X,Y}(x,y)}{\partial y}$$

Day:

Ex 4.7

$$f_{X,Y}(x,y) = \begin{cases} 5y/4 & -1 \le x \le 1,\ x^2 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

PDF X



$$f_X(x) = \int_{y=x^2}^{1} \frac{5}{4} y \, dy$$

$$= \frac{5}{8} \left| y^2 \right|_{x^2}^{1}$$

$$= \frac{5}{8}(1 - x^4)$$

$y = x^2$
$x = \pm\sqrt{y}$
$x = +\sqrt{y}$
$x = -\sqrt{y}$

for $f_Y(y)$

Limit x ke hai

Therefore

$$f_X(x) = \begin{cases} \frac{5}{8}(1-x^4) & -1 \le x \le 1 \\ 0 & \text{otherwis} \end{cases}$$

PDF Y

$$f_Y(y) = \int_{-\sqrt{y}}^{\sqrt{y}} \frac{5}{4} y \, dx$$

$$= \frac{5}{4} y \left| x \right|_{-\sqrt{y}}^{\sqrt{y}} = \frac{5}{4} y (\sqrt{y} + \sqrt{y})$$

$$= \frac{5}{4} y \times 2\sqrt{y} = \frac{5}{2} y\sqrt{y}$$

Quizz 4.5

Date: Day:

$$\frac{5}{2} y^{3/2}$$

$$f_Y(y) = \begin{cases} \frac{5}{2} y^{3/2} & 0 \le y \le 1, \\ 0 & \text{otherwise} \end{cases}$$

Expected value:

continuous RV

$$W = g(x,y) = x$$

$$* \; E[X] = \int_{-\infty}^{+\infty} x f_X(x) \, dx$$

$$* \; E[X] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x f_{X,Y}(x,y) \, dx \, dy$$

$$* \; E[W] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x,y) f_{X,Y}(x,y) \, dx \, dy$$

discrete RV

$$* \; E[W] = \sum_{x \in S_X} \sum_{y \in S_Y} g(x,y) P_{X,Y}(x,y)$$

$$VAR[X] = E[X^2] - (E[X])^2$$

$$E[X^2] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x^2 f_{X,Y}(x,y) \, dx \, dy$$

Day:

Theorem:

$$E[g_1(X,Y) + g_2(X,Y) + \cdots + g_n(X,Y)] = E[g_1(X,Y)] + E[g_2(X,Y)] + \cdots + E[g_n(X,Y)]$$

Theorem

$$E[X+Y] = E[X] + E[Y]$$

Theorem

$$VAR[X+Y] = VAR[X] + VAR[Y] + 2E[(X-\mu_x)(Y-\mu_x)]$$

$$VAR[X+Y] = \sigma_x^2 + \sigma_y^2 + 2\sigma_{x,y}^2$$

Covariance:

about center

For two random variables X and Y the covariance is defined as

$$Cov[X,Y] = \sigma_{xy} = E[(X-\mu_x)(Y-\mu_y)]$$

CRV

$$cov[X,Y] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (x-\mu_x)(y-\mu_x) f_{XY}(x,y)\,dx\,dy$$

DRV

$$cov[X,Y] = \sum_{y \in S_y} \sum_{x \in S_x} (x-\mu_x)(y-\mu_y) f_{X,Y}(x,y)$$

Corelation:

- first moment about the region

corelation of RVs X and Y

$$r_{xy} = E[XY]$$

Date: Day:

Theorem:

(a) $\text{cov}[X,Y] = r_{xy} - \mu_x \mu_y$

$\text{cov}[X,Y] = E[XY] - E[X]E[Y]$

(b) if $X = Y$

$\text{cov}[X,Y] = E[X^2] - (E[X])^2$

$= VAR[X] = VAR[Y]$

Ex4112

$P_{X,Y}(x,y)$	$y=0$	$y=1$	$y=2$	$P_X(x)$
$x=0$	0.01	0	0	0.01
$x=1$	0.09	0.09	0	0.18
$x=2$	0	0	0.81	0.81
$P_Y(y)$	0.10	0.09	0.81	

(a) r_{xy}

$$r_{xy} = E[XY] = \sum_{x=0}^{2} \sum_{y=0}^{2} xy P_{XY}(x,y)$$

$$= (0)(0)(0.01) + (0)(1)(0) + (0)(2)(0)$$
$$(1)(0)(0.09) + (1)(1)(0.09) + (1)(2)(0)$$
$$(2)(0)(0) + (2)(1)(0) + (2)(2)(0.81)$$

$$r_{xy} = 0.09 + 3.24 = 3.33$$

Day:

(b) cov [X, Y]

$E[X] = (0)(0.01) + (1)(0.18) + (2)(0.81)$

$E[X] = 1.80$

$E[Y] = (0)(0.10) + (1)(0.09) + (2)(0.81)$

$E[Y] = 1.71$

$\text{cov}[X,Y] = \sigma_{XY} = \gamma_{XY} - \mu_X \mu_Y$

$= 3.33 - (1.80)(1.71)$

$= 0.252$

$\text{cov}[X,Y] = 0.252$

$E[X] = \sum_{y \in S_y} \sum_{x \in S_x} x\, f_{XY}(x,y)$

Date: Day:

Corelation

$$\gamma_{xy} = E[XY]$$

RV linear relationship

Covariance

$$\text{cov}[X.Y] = E[XY] - E[X]E[Y]$$
$$= \gamma_{xy} - \mu_x \mu_y$$

Orthogonal random Variables

$$\gamma_{xy} = 0$$
$$E[XY] = 0$$

Uncorrelated RV

$$\text{cov}[X,Y] = 0$$
$$\sigma_{XY} = E[XY] - E[X]E[Y] = 0$$
$$E[XY] = E[X]E[Y]$$

Correlation Coefficient

$$\rho_{XY} = \frac{\text{cov}[X,Y]}{\sqrt{VAR[X]VAR[Y]}} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

* $-1 \leq \rho_{XY} \leq +1$
* Dimension less, no unit
* X, Y ka darmiyan kitna relationship han. quantify karta hai.
* X increase, Y increase positive coralati

Day:

+ X increase Y decrease vice versa
negative corelation.

Quizz 4.7:

Discrete

$P_{LT}(l,t)$	$t=40s$	$t=60s$	$P_L(l)$
$l=1p$	0.15	0.1	0.25
$l=2p$	0.30	0.2	0.5
$l=3p$	0.15	0.1	0.25
$P_T(t)$	0.6	0.4	

$$P_L(l) = \begin{cases} 0.25 & l=1,3 \\ 0.5 & l=2 \\ 0 & \text{otherwise} \end{cases}$$

$$P_T(t) = \begin{cases} 0.6 & t=40 \\ 0.4 & t=60 \\ 0 & \text{otherwise} \end{cases}$$

(a)

1

$$E(L) = (1)(0.25)+(3)(0.25)+(0.5)(2)$$

$$\boxed{E(L) = 2p}$$

$$VAR(L) = E[L^2] - (E[L])^2$$

Date: Day:

$$E[L^2] = (1)^2(0.25) + (3)^2(0.25) + (2)^2(0.5)$$

$$E[L^2] = 4.5$$

$$VAR[L] = E[L^2] - (E[L])^2$$

$$= 4.5 - (2)^2$$

$$= 4.5 - 4$$

$$\boxed{VAR[L] = 0.5}$$

2

$$\boxed{E[T] = 48s}$$

$$VAR[T] = E[T^2] - (E[T])^2$$

$$= 2400 - (48)^2$$

$$\boxed{VAR[T] = 96}$$

3 $\gamma_{L,T} = E[LT]$

$$= \sum_{l=1}^{3} \sum_{t=40,60} l \times t \; P_{LT}(l,t)$$

$$= (1)(40)(0.15) + (1)(60)(0.1) + (2)(40)(0.30) + (2)(60)(0.2) + (3)(40)(0.15) + (3)(60)(0.1)$$

$$= 96$$

$$\boxed{\gamma_{L,T} = 96 = E[LT]}$$

Day:

(4)

$$\mathrm{cov}[L,T] = \sigma_{L,T} = E[LT] - E[L]E[T]$$
$$= 96 - 2\times 48 = 0$$
$$\mathrm{cov}[L,T] = 0$$

(5)

$$\rho_{L,T} = \frac{\mathrm{cov}[L,T]}{\sqrt{\mathrm{Var}[L]\,\mathrm{var}[T]}} = 0$$

continuous

(b)

$$f_{X,Y}(x,y) = \begin{cases} xy & 0 \le x \le 1,\ 0 \le y \le 2 \\ 0 & \text{otherwise} \end{cases}$$

(1)

$E[X]$, $VAR[X]$

Marginal PDF

$$f_X(x) = \int_{-\infty}^{+\infty} f_{XY}(x,y)\,dy$$

$$f_X(x) = \int_0^2 xy\,dy = \frac{1}{2}x\left|y^2\right|_0^2$$

$$f_X(x) = 2x$$

$$f_X(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Date: Day:

$$f_Y(y) = \int_{-\infty}^{+\infty} f_{XY}(x,y)\,dx$$

$$= \int_0^1 xy\,dx$$

$$= \frac{1}{2} y \left| x^2 \right|_0^1$$

$$= \frac{1}{2} y$$

$$f_Y(y) = \begin{cases} y/2 & 0 \le y \le 2 \\ 0 & \text{otherwise} \end{cases}$$

(1)

$$E[X] = \int_{-\infty}^{+\infty} x\, f_x(x)\,dx$$

$$= \int_0^1 x\,(2x)\,dx$$

$$= \frac{2}{3} \left| x^3 \right|_0^1 = \frac{2}{3}$$

$$E[X] = 2/3$$

2nd method:

$$E[X] = \int_{y=0}^{2} \int_{x=0}^{1} x(xy)\,dx\,dy$$

Day:

$$VAR[X] = E[X^2] - (E[X])^2$$

$$E[X^2] = \int_{x=0}^{1} x^2 \, 2x \, dx = \frac{1}{2}$$

$$VAR[X] = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{1}{18}$$

$$\boxed{VAR[X] = \frac{1}{18}}$$

2

$$E[Y] = \int_{-\infty}^{+\infty} y f_Y(y) \, dy$$

$$E[Y] = \int_0^2 y (y/2) \, dy$$

$$E[Y] = 4/3$$

$$VAR[Y] = E[Y^2] - (E[Y])^2$$

$$E[Y^2] = \int_{-\infty}^{+\infty} y^2 f_Y(y) \, dy$$

$$E[Y^2] = \int_0^2 y^2 (y/2) \, dy = 2$$

$$VAR[Y] = 2 - \left(4/3\right)^2 = 2/9$$

$$\boxed{VAR[Y] = 2/9}$$

Date: Day:

③

$\gamma_{xy} = E[X, Y]$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xy f_{xy}(x,y)\,dx\,dy$$

$$= \int_0^2\int_0^1 (xy)^2\,dx\,dy$$

$$= \int_0^2 \frac{y^2}{3}\left|x^3\right|_0^1 dy = \frac{1}{3}\int_0^2 y^2\,dy$$

$$= \frac{1}{9}\left|y^3\right|_0^2 = \frac{8}{9}$$

$$\boxed{\gamma_{xy} = E[XY] = \frac{8}{9}}$$

④ $\mathrm{Cov}[X,Y]$

$$\mathrm{COV}[X,Y] = \gamma_{xy} - \mu_x \mu_y$$

$$= \frac{8}{9} - \left(\frac{2}{3}\right)\left(\frac{4}{3}\right) = 0$$

$$\boxed{\mathrm{COV}[X,Y] = 0}$$

⑤ $$\rho_{X,Y} = \frac{\mathrm{COV}[X,Y]}{\sqrt{\mathrm{VAR}[X]\,\mathrm{VAR}(Y)}} = 0$$

Day:

Independent RV

$$P[AB] = P[A] \times P[B]$$

random variables X and Y are independent if and only if

$$P_{X,Y}(x,y) = P_X(x) \times P_Y(y) \quad \text{Discrete.}$$

$$f_{X,Y}(x,y) = f_X(x) f_Y(y) \quad \text{Continuous.}$$

Example:

continuous IRV

$$f_{X,Y}(x,y) = \begin{cases} 4xy & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Marginal PDF

$$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dy$$

$$= \int_0^1 4xy\,dy$$

$$= 4x \int_0^1 y\,dy = 4x \left| \frac{y^2}{2} \right|_0^1$$

$$\boxed{f_X(x) = 2x}$$

$$f_Y(y) = \int_{-\infty}^{+\infty} f_{XY}(x,y)\,dx$$

$$= \int_0^1 4xy\,dx$$

Date: Day:

$$= 4y \left| \frac{x^2}{2} \right|_0^1$$

$$\boxed{f_Y(y) = 2y}$$

$$f_X(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}, \quad f_Y(y) = \begin{cases} 2y & 0 \le y \\ 0 & \text{o} \end{cases}$$

check

$$f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$$

$$4xy = (2x)(2y)$$

$$4xy = 4xy$$

Therefore
X and Y are independent.

Properties of Independent RV

Theorems:

(a) $E[g(x)h(y)] = E[g(x)]E[h(y)]$

(b) $\operatorname{cov}[X,Y] = \sigma_{X,Y} = 0$

(c) $E[XY] = E[X]E[Y]$

(d) $VAR[X+Y] = VAR[X] + VAR[Y] + 2\operatorname{cov}(X$

Day:

As $\operatorname{cov}[X,Y] = 0$

$$VAR[X+Y] = VAR[X] + VAR[Y]$$

(e) $E[X \mid Y = y] = E[X]$

(f) $E[Y/X = x] = E[Y]$

while $\operatorname{cov}[X,Y]=0$ is a necessary property for independent, it is not coefficient

Discrete IRV

Ex:

$P_{XY}(x,y)$	$y=-1$	$y=0$	$y=1$	$P_X(x)$
$x=-1$	0	0.25	0	0.25
$x=1$	0.25	0.25	0.25	0.75
$P_Y(y)$	0.25	0.5	0.25	

(a) Are X and Y independent

(b) X and Y uncorrelated

(a)

$$P_{XY}(x,y) = P_X(x) \cdot P_Y(y)$$

$$P_X(1) = 0.75$$

$$P_Y(1) = 0.25$$

$$P_{X,Y}(1,1) = 0.25$$

$$P_X(1) \cdot P_Y(1) = 0.75 \times 0.25 = 0.18 \neq P_{X,Y}(1,1)$$

Date: Day:

hence

X and Y are not independent

(b) $Cov[X,Y] = E[XY] - E[X]E[Y]$

$$E[XY] = \sum_{x=-1,1} \sum_{y=-1}^{1} xy\, P_{XY}(x,y)$$

$E[XY] = 0$

this is orthogonal RV

$E[X] = 0.50$

$E[Y] = 0$

$Cov[X,Y] = 0 - (0.50)(0) = 0$

X and Y are uncorrealeted

Day:

Theorem:

$X_1, X_2, \ldots, X_n$

$W = X_1 + X_2 + \cdots + X_n$

$$E[W] = E[X_1] + E[X_2] + \cdots + E[X_n]$$

$$VAR[W] = \sum_{i=1}^{n} VAR[X_i] + 2\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} cov(X_i, X_j)$$

If $X_1, X_2, \ldots, X_n$ are Uncorrelated

$$VAR[W] = \sum_{i=1}^{n} VAR[X_i] = VAR[X_1] + VAR[X_2] + \cdots + VAR[X_n]$$

1- Characteristic function

✓ 2- Moment Generating function MGF

3- Probability Generating function

Date: Day:

MGF:

$$\Phi_X(s) = E[e^{sX}]$$

continuous random variable

$$\Phi_X(s) = \int_{-\infty}^{+\infty} e^{sx} f_X(x)\,dx$$

Discrete random variables

$$\Phi_X(s) = \sum_{x \in S_X} e^{s x_i} P_X(x)$$

Theorem:

$$E[X^n] = \frac{d^n}{ds^n} \Phi_X(s) \Big|_{s=0}$$

1- $\Phi_X(s) \Big|_{s=0} = \Phi_X(0) = 1$

2- $Y = X_1 + X_2$

$$\Phi_Y(s) = \Phi_{X_1}(s)\,\Phi_{X_2}(s)$$

Drive MGF of Exponential RV

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Day:

$$\Phi_X(s) = E[e^{sx}]$$

$$\Phi_X(s) = \int_{-\infty}^{+\infty} e^{sx} f_X(x)\,dx$$

$$= \int_0^{\infty} e^{sx} (\lambda e^{-\lambda x})\,dx$$

$$= \lambda \int_0^{\infty} e^{-(\lambda - s)x}\,dx$$

$$= \frac{-\lambda}{\lambda - s} \left| e^{-(\lambda - s)} \right|_0^{\infty}$$

$$= \frac{-\lambda}{\lambda - s} [e^{-\infty} - e^{0}]$$

$$= \frac{-\lambda}{\lambda - s} [0 - 1] = \frac{-\lambda}{\lambda - s} (-1)$$

$$\boxed{\Phi_X(s) = \frac{\lambda}{\lambda - s}}$$

$\therefore$ if $s = 0$ then $\Phi(s) = 1$

Date: Day:

n^{th} Moment

$$E[X^n] = \frac{d^n}{ds^n} \Phi_X(s) \Big|_{s=0}$$

$$\frac{d}{ds} \Phi_X(s) = \frac{d}{ds} \lambda(\lambda-s)^{-1}$$

$$= -\lambda(\lambda-s)^{-2}(-1)$$

$$\frac{d}{ds} \Phi_X(s) = \lambda(\lambda-s)^{-2}$$

$$\frac{d^2}{ds^2} \Phi_X(s) = -2\lambda(\lambda-s)^{-3}(-1)$$

$$\frac{d^2}{ds^2} \Phi_X(s) = \frac{2\lambda}{(\lambda-s)^3}$$

$$\boxed{\frac{d^n}{ds^n} \Phi_X(s) = \frac{n!\,\lambda}{(\lambda-s)^{n+1}}}$$

$$E[X] = \frac{\lambda}{(\lambda-s)^2}\Big|_{s=0} = \frac{\lambda}{\lambda^2} = \frac{1}{\lambda}$$

$$E[X^2] = \frac{2\lambda}{(\lambda-s)^3}\Big|_{s=0} = \frac{2\lambda}{\lambda^3} = \frac{2}{\lambda^2}$$

Day:

$$E[X^n] = \frac{n!\,\lambda}{(\lambda-s)^{n+1}}\Bigg|_{s=0}$$

$$\boxed{E[X^n] = \frac{n!}{\lambda^n}}$$

Theorem:-

For a set of independent random variables $X_1, X_2, \ldots X_n$, Let $W = X_1 + X_2 + \cdots + X_n$
These distribution may be different not identical.

O
ED
UD
BD

$$\Phi_W(s) = \Phi_{X_1}(s)\,\Phi_{X_2}(s) \cdots \Phi_{X_n}(s)$$

When $X_1, X_2, \ldots, X_n$ are (iid) identical

$$\Phi_{X_1}(s) = \Phi_{X_2}(s) = \Phi_{X_n}(s)$$

$$\Phi_W(s) = [\Phi_X(s)]^n$$

Date:

Day:

Example:

$$P_J(j) = \begin{cases} 0.2 & j=1 \\ 0.6 & j=2 \\ 0.2 & j=3 \\ 0 & \text{otherwise} \end{cases}$$

$$P_k(k) = \begin{cases} 0.5 & k=-1 \\ 0.5 & k=1 \\ 0 & \text{otherwise} \end{cases}$$

$\boxed{\text{Theorem 2}}$

(a) MGF of $M = J + K$

Discrete RV

$\therefore \quad \Phi_X(s) = \sum_{x_i \in S_X} e^{sx_i} P_X(x_i)$

Solve

$$\Phi_J(s) = e^{s}(0.2) + e^{2s}(0.6) + e^{3s}(0.2)$$

$$\Phi_k(s) = e^{-s}(0.5) + e^{s}(0.5)$$

$M = J + k$

$$\Phi_M(s) = \Phi_J(s) \cdot \Phi_k(s)$$

$$= \left((0.2)e^{s} + (0.6)e^{2s} + (0.2)e^{3s}\right) \times \left(e^{-s}(0.5) + e^{s}(0.\right.$$

Day:

$= 0.1 + 0.1e^{2s} + 0.3e^{s} + 0.3e^{3s} + 0.1e^{2s} + 0.1e^{4s}$

$= 0.1 + 0.3e^{s} + 0.2e^{2s} + 0.3e^{3s} + 0.1e^{4s}$

b) $E[M^3]$

$$\frac{d}{ds}\Phi_M(s) = 0.3e^{s} + 0.4e^{2s} + 0.9e^{3s} + 0.4e^{4s}$$

$$\frac{d^2}{d^2s}\Phi_M(s) = 0.3e^{s} + 0.8e^{2s} + 2.7e^{3s} + 1.6e^{4s}$$

$$\frac{d^3}{d^3s}\Phi_M(s) = 0.3e^{s} + 1.6e^{2s} + 8.1e^{3s} + 6.4e^{4s}$$

$$E[M^3] = \frac{d^3}{d^3s}\Phi_M(s)\Big|_{s=0}$$

$$= 0.3 + 1.6 + 8.1 + 6.4$$

$$\boxed{E[M^3] = 16.4}$$

(c) $P_M(m)$

$$\Phi_M(s) = 0.1 + 0.3e^{s} + 0.2e^{2s} + 0.3e^{3s} + 0.1e^{4s}$$

for

$m = 0$, 0.1
$m = 1$, 0.3
$m = 2$, 0.2
$m = 3$, 0.3
$m = 4$, 0.1

$$P_M(m) = \begin{cases} 0.1 & m = 0, 4 \\ 0.3 & m = 1, 3 \\ 0.2 & m = 2 \\ 0 & \text{otherwise} \end{cases}$$

$$\boxed{MGF \rightleftarrows PMF}$$

2nd Method for find Expected value

$$E[M^3] = (0)^3 \times (0.1) + (4)^3 (0.1) + (1)^3 (0.3)$$
$$(3)^3 (0.3) + (2)^3 (0.2)$$

$$E[M^3] = 16.4$$

Day:

Theorem:

Let $X_1, X_2, \ldots$ be a collection of iid random variables each with MGF $\Phi_X(s)$

Let N be a non-negative integer valued random variable that is independent of $X_1, X_2 \ldots$

that random sum

$$R = X_1 + X_2 + \cdots + X_n$$

has the MGF

$$\Phi_R(s) = \Phi_N(\ln \Phi_X(s))$$

$\boxed{\text{Table 6.1}}$ see

Central Limit Theorem:

Theorem

Given n independent random variables X_i

let

$$X = X_1 + X_2 + \cdots + X_n = \sum_{i=1}^{n} X_i$$

Date: Day:

X is a random variables with mean μ

$$\mu = \mu_1 + \mu_2 + \cdots + \mu_n$$

and variance σ^2

$$\sigma^2 = \sigma_1^2 + \sigma_2^2 + \cdots + \sigma_n^2$$

According to central limit theorem, as n increases, the distribution $F_X(x)$ of X approaches a Normal distributio with the same mean and variance

$$F_X(x) \simeq \Phi\left(\frac{x-\mu}{\sigma}\right)$$

if X_i are of continuous type

$$f_X(x) \approx \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$\boxed{\text{iid}}$

if X_i are iid

$$\mu = E[X] = n\mu_x$$

$$VAR[X] = \sigma^2 = n\sigma_x^2$$

$$\sigma = \sqrt{n}\,\sigma_x$$

Day:

$$z = \frac{X-\mu}{\sigma} = \frac{\sum_{i=1}^{n} X_i - n\mu_x}{\sqrt{n}\,\sigma_x}$$

Ex 6.14

Let,

X_i be one bit which is 0 or 1

$X_i \sim \text{Bernoulli}(0.5)$

$p = 0.5$

$$P_X(x) = \begin{cases} 0.5 & x=1 \\ 0.5 & x=0 \\ 0 & \text{otherwise} \end{cases}$$

$\mu_x = E[X_i] = p = 0.5$

$VAR[X_i] = P(1-P)$

$\sigma_x^2 = (0.5)(0.5)$

$= 0.25$

Number of bits $= n = 10^6$

$$W = X_1 + X_2 + \dots + X_n = \sum_{i=1}^{n=10^6} X_i$$

As X_i are $\boxed{\text{iid}}$, therefore

$\mu = E[W] = n\mu_x = 10^6 \times 0.5 = 5 \times 10^5$

$\mu = 500000$

$\sigma^2 = VAR[W] = n\,VAR[X_i]$

$= 10^6 \times 0.25 = 250000$

Standard deviation $= \sigma = \sqrt{250000} = 500$

Date: $P[W \geq 502000] = 1 - P[W \leq 502000]$
$= 1 - \Phi(4)$ Day:

$$P[W \geq 502000] = 1 - \Phi(4) = Q(4)$$

$$z = \frac{W - \mu}{\sigma} = \frac{502000 - 500000}{500}$$

$$z = 4$$

$$P[W \geq 502000] = Q(4) = 3.17 \times 10^{-5}$$

Ex.6.15

$$499000 \leq W \leq 501000$$

$P[a < x < b]$
$F(b) - F(a)$

$\Phi\left(\frac{x-\mu}{\sigma}\right)$

$$P(A) = P[499000 \leq W \leq 501000]$$
$$= F_W(501000) - F_W(499000)$$
$$= \Phi\left(\frac{501000 - 500000}{500}\right) - \Phi\left(\frac{499000 - 500000}{500}\right)$$
$$= \Phi(2) - \Phi(-2) = \Phi(2) - (1 - \Phi(2))$$

$$P(A) = 2\Phi(2) - 1 = 2 \times 0.97725 - 1$$
$$\boxed{P(A) = 0.9545}$$

CH:7 Sample Mean:

$$M_n(X) = \frac{X_1 + X_2 + \cdots + X_n}{n} = \frac{\text{Sample value}}{\text{Total value}}$$

Sample Mean

$M_n(X)$ is a random variable.

Day:

Theorem

$$E[M_n(X)] = E[X]$$

$$VAR[M_n(X)] = \frac{VAR[X]}{n}$$

Markov Inequality:

For a random variable X such that

PDF

$$P[X<0] = 0 \Rightarrow f_X(x) = 0 \text{ for } X<0$$

(for negative values proabiliity is zero.)

and a constant $k>0$,

$$P[X \geq k] \leq \frac{E[X]}{k} = \frac{\mu_x}{k}$$

Ex 7.1

$E[X] = 5.5$ feet

$k = 11$ feet

$$P[X \geq 11] \leq \frac{5.5}{11} = \frac{1}{2}$$

$$P[X \geq 11] \leq \frac{1}{2}$$

we don't know the proabiliity distribution

this loss bound

$1/2$ sy proabiliity zada nahi ho sakti.

Date: Day:

Theorem 73

upper bound

Chebyshev Inequality

For arbitrary random variable, X constant $k>0$,

(1) $P[|X-\mu_x| \geq k] \leq \frac{VAR[X]}{K^2} = \frac{\sigma^2}{k^2}$ — (1)

let $k = n\sigma$ in (1)

$$P[|X-\mu_x| \geq n\sigma] \leq \frac{\sigma^2}{n^2\sigma^2} = \frac{1}{n^2}$$

(2) $P[|X-\mu_x| \geq n\sigma] \leq \frac{1}{n^2}$ — (2)

Multiply (1) by -1

$$-P[|X-\mu_x| \geq k] > -\frac{\sigma^2}{k^2}$$

Add 1 to both side

$$1 - P[|X-\mu_x| \geq k] > 1 - \frac{\sigma^2}{k^2} \quad \text{— (3)}$$

However,

$$\boxed{1 - P[Y \geq y] = P[Y < y]}$$

eq (3) become

(3) $P[|X-\mu_x| < k] > 1 - \frac{\sigma^2}{k^2}$

In $k = n\sigma$

$$P[|X-\mu_x| < k] > 1 - \frac{\sigma^2}{k^2}$$

Day:

4 $P[|X-\mu_x| < n\sigma] > 1-\frac{1}{n^2}$

$|X-\mu_x| \geq k$

$\pm(X-\mu_x) \geq k$

$(X-\mu_x) \geq k$ and $-(X-\mu_x) \geq k$

$|X-\mu_x| \geq k$ implies $\mu_x+k \leq X \leq \mu_x-k$

$P[\mu_x+k < X < \mu_x-k] \leq \frac{\sigma^2}{k^2}$ — 1(B)

$P[\mu_x+n\sigma \leq X < \mu-n\sigma] \leq \frac{1}{n^2}$ — 2(B)

$P[\mu_x-k \leq X < \mu_x+k] > 1-\frac{\sigma^2}{k^2}$ — 3(B)

$P[\mu_x-n\sigma \leq X < \mu_x+n\sigma] > 1-\frac{1}{n^2}$ — 4(B)

Ex:

$E[X] = \mu_x = 5.5$ feet

$k=11 \;,\; \sigma = 1$

for using CI

$P[x \geq 11] = P[X-\mu_x \geq 11-\mu_x]$

$= P[|X-\mu_x| \geq 11-5.5]$

$= P[|X-\mu_x| \geq 5.5]$

$P[|X-\mu_x| \geq k] \leq \frac{\sigma^2}{k^2}$

$P[x \geq 11] = P[|X-\mu_x| \geq 5.5] \leq \frac{1}{(5.5)^2} = 0.033058$

Date: Day:

Comparing code Segment

Different inst. have Diff. CPI

IC

↓

Inst. Count

J I R

Seq 1	Seq 2
① $IC = 5$	$IC = 6$
② CPU CC 10	CPU CC 9

$P[X \geq 11] \leq 0.033058$

Markov's Inequality:

$$P[X \geq k] \leq \frac{E[X]}{k}$$

If $k < E[X]$ then

$$P[X \geq k] > 1$$

In such cases Markov inequality does not provide useful information.

CIE 7.44 7.45 7.46 Quizz 7.4

Quizz:

$$E[X] = R$$

$$k = \frac{3}{4} R$$

Therefore

$$P[X \geq \tfrac{3}{4}R] \leq \frac{R}{\frac{3}{4}R} = \frac{4}{3}$$

$$P[X \geq \tfrac{3}{4}R] \leq 4/3$$

But $P[\cdot] \leq 1$ always therefore

$$P[X \geq \tfrac{3}{4}R] \leq 1$$ (Maximum 1 ho sakti hai)

Proability

Indicator Random Variables

Let

Event A

Proability : $P[A]$

Indicator RV : X_A

$$P_{X_A}(X_A) = \begin{cases} 1 - P[A] & X_A = 0 \\ P[A] & X_A = 1 \\ 0 & \text{otherwise} \end{cases}$$

$X_1 + X_2 + X_3 + \cdots + X_{100}$

event of intrest is head

$$\hat{p}[A] = \frac{X_1 + X_2 + X_3 \cdots + X_{100}}{100} \nearrow \text{point estimate}$$

- Point estimate
- Interval estimate

Properties of Point Estimate:

1- Bias

2- Consistency

3- Accuracy

Bias:

Actual parameter r

Estimate Parameter $\hat{R}$

expected value $E[\hat{R}]$

$$\text{Bias} = E[\hat{R}] - r$$

In case, the estimate is unbiased

$$\text{Bias} = E[\hat{R}] - r = 0$$

$$\boxed{E[\hat{R}] = r}$$

Consistency:

The sequence of estimate $\hat{R}_1, \hat{R}_2, \ldots$ of parameter r is consistent if for any $\epsilon > 0$,

$$\lim_{n \to \infty} P[|\hat{R}_n - r| \geq \epsilon] = 0$$

$$|\hat{R}_n - r| \leq \epsilon$$

$$\hat{R}_n \simeq r$$

Mean Square Error

To measure the accuracy of estimate

The MSE of estimator $\hat{R}$ of r is

$$e = E[(\hat{R} - r)^2] \quad \text{—— (1)}$$

if $\hat{R}$ is unbiased estimator of r, then

$$E[\hat{R}] = r \quad \text{—— (2)}$$

$$e = E[(\hat{R} - E[\hat{R}])^2] \quad \text{—— (3)}$$

$E[(X-\mu_X)]$

MSR is the variance of estimated error

Root Mean Square root (RMSE)

if $\hat{R}$ is unbiased

$$RMSE = \sqrt{e} = \sqrt{E[(\hat{R} - E[\hat{R}])^2]}$$

Standard error

if biased

$$RMSE = \sqrt{e} = \sqrt{E[(\hat{R} - r)^2]}$$

Theorem:

The sample mean $M_n(X)$ is an unbiased estimate of $E[X]$

Theorem:

The sample mean estimator $M_n(X)$ has mean square error

$$e_n = E[(M_n(X) - E[X])^2]$$

$$= VAR[M_n(X)]$$

$$= \frac{VAR[X]}{n}$$

Given

Number of trials $= n$

Estimator $\hat{P}_n[A]$

$P[A]$

$\sqrt{e} < 0.1$

Solve:

$$\hat{P}_n[A] = M_n(X_A) = \frac{X_{A_1} + X_{A_2} + \cdots + X_{A_n}}{n}$$

MSE of $M_n(X_A)$ is given by

$$e_n = E\left[\left(M_n(X_A) - E[M_n(X_A)]\right)^2\right]$$

But

unbiased estimator

point estimate of expected value

$$E[M_n(X_A)] = E[X_A]$$

Therefore

$$e_n = E[(M_n(X_A) - E[X_A])^2]$$

$$= VAR[M_n(X_A)]$$

point Estimate of variance

$$= \frac{VAR[X_A]}{n}$$

$$P_{X_A}(X_A) = \begin{cases} 1-P(A) & x=0 \\ P(A) & x=1 \\ 0 & \text{otherwise} \end{cases}$$

$$E[X_A] = 0 \times (1-P(A)) + 1 \times P[A] = P[A]$$

$E[X_A^2] = 0 + (1)^2 \times P[A] = P[A]$

$VAR[X_A] = E[X_A^2] - (E[X_A])^2$

$= P[A] - (P[A])^2$

$VAR[X_A] = P[A](1 - P[A])$

$$e_n = \frac{VAR[X_A]}{n} = \frac{P[A](1-P[A])}{n}$$

$\sqrt{e_n} < 0.1$

$e_n < 0.01$

$$\frac{P[A](1-P[A])}{n} < 0.01$$

Let worst case:

$P[A] = 1 - P[A]$

$2P[A] = 1$

$P[A] = \frac{1}{2} = 0.5$

2nd method for worst case

let $P(A) = x$

$P[A](1-P[A]) = x(1-x)$

$f = x - x^2$

$\frac{df}{dx} = 1 - 2x = 0$

$1 - 2x = 0$

$x = \frac{1}{2} = 0.5 = P[A]$

$\frac{d^2f}{dx^2} = -2 < 0$ (check)

$$\frac{P[A](1-P[A])}{n} < 0.01$$

$$\frac{(0.5)(0.5)}{n} < 0.01$$

$$\frac{0.25}{n} < 0.01$$

$$n > \frac{0.25}{0.01} = 25$$

$$\boxed{n > 25}$$

Confidence Intervals:

(a) confidence interval

(b) confidence coefficient

$$P[|Y-\mu_Y| \geq c] \leq \frac{VAR[Y]}{c^2}$$

OR

$$P[|Y-\mu_Y| < c] \geq 1 - \frac{VAR[Y]}{c^2}$$

17-6-2025

Point Estimate of Expected value

$$E[M_n(x)] = E[X]$$

Point Estimate of variance.

$$VAR[M_n(x)] = \frac{VAR[X]}{n}$$

Theorem:

According to chebysher inequality

(a) $P[|Y-\mu_Y| \geq c] \leq \dfrac{VAR[Y]}{c^2}$

OR (b) $P[|Y-\mu_Y| < c] \geq 1-\dfrac{VAR[Y]}{c^2}$

Therefore for any constant $c>0$

(a) $P[|M_n(X)-\mu_x| \geq c] \leq \dfrac{VAR[M_n(X)]}{c^2}$

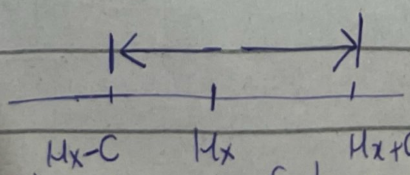
$$= \frac{VAR[X]}{nc^2} = \alpha$$

(b) $P[|M_n(X)-\mu_x| < c] \geq 1-\dfrac{VAR[M_n(X)]}{c^2}$

$$= 1-\frac{VAR[X]}{nc^2} = \underbrace{1-\alpha}_{\text{Condifen co-efficien}}$$

$\mu_x - C < M_n(x) < \mu_x + C$ (confidence I)

$|M_n(x)-\mu_x|$



$\mu_x + C - (\mu_x - C) = 2C$ confidence interval

$P[|M_n(X) - \mu_x| < C] \geq 1-\alpha$

states that the proability that the sample mean is in the confidence interval 2C is at least $1-\alpha$

Ex 7.6:

$P[A]$, $\hat{P}_n[A]$

Length of interval $= 2C = 0.02$

$C = 0.01$

$1-\alpha = 0.999$

According to chebyshev inequality

$$P[|\hat{P}_n(A) - P[A]| < C] \geq 1 - \frac{VAR[\hat{P}_n(A)]}{C^2} \quad (1)$$

However,

$\hat{P}(A) = M_n(X_A)$

Therefore, (point estimate of variance)

$$VAR[\hat{P}_n(A)] = VAR[M_n(X_A)] = \frac{VAR(X_A)}{n}$$

$X_A \sim$ Bernoulli;

but, $VAR[X_A] = P[A](1-P[A])$

$$VAR[\hat{P}_n(A)] = \frac{P[A](1-P[A])}{n}$$

Therefore eq ①

$$P[|P_n^{\wedge}(A) - P[A]| < c] \geq 1 - \frac{P[A](1-P[A])}{nc^2} = 1-\alpha$$

Hence

$$1 - \frac{P[A](1-P[A])}{nc^2} = 1-\alpha = 0.999 \quad — ②$$

Solve

$$P[A] = (1-P[A]) \Rightarrow P(A)+P[A] = 1$$
$$2P[A] = 1$$
$$P[A] = \frac{1}{2} = 0.5$$

Now

$$P[A]((1-P[A]) = 0.5(1-0.5) = 0.25 = \frac{1}{4}$$

put in eq ②

$$1 - \frac{1}{4nc^2} = 0.999$$

$$\frac{1}{4nc^2} = 1-0.999$$

$$\frac{1}{4n(0.01)^2} = 0.001$$

$$4n(0.01)^2 = 1000$$

$$n = \frac{1000}{4(0.01)^2} = 2.5 \times 10^6$$

Number of trials $n \geq 2.5 \times 10^6$

Ex 7.9

Gaussian RV

$$X_i = b + Z_i$$

$$2c = 0.2 \implies c = 0.1$$

$$1 - \alpha = 0.99$$

$$P[|M_n(X) - b| < 0.1] \geq 1 - \frac{VAR[M_n(X)]}{(0.1)^2}$$

$$= 1 - \frac{VAR[X_i]}{n(0.1)^2} = 1 - \alpha$$

As b and Z_i are independent

however $VAR[X_i] = VAR[b + Z_i]$

$$= VAR[b] + VAR[Z_i]$$ constant variance is zero

$$= 0 + VAR[Z_i]$$

$$VAR[X_i] = VAR[Z_i] = 1$$

Hence

$$P[M_n(X) - 0.1 < b < M_n(X) + 0.1] \geq 1 - \frac{1}{n(0.1)^2} = 0.99$$

$$1 - \frac{1}{n(0.1)^2} = 0.99$$

$$1 - \frac{100}{n} = 0.99$$

$$\frac{100}{n} = 1 - 0.99 = 0.01$$

$$n = 10000 \qquad n \geq 10000$$ No. of trials.

Theorem:

Let X be a Gaussian (μ, σ) random variable. A confidence interval estimate of μ of the form

$$|M_n(X) - \mu| < c$$

$$M_n(x) - c \leq \mu < M_n(x) + c$$

has confidence interval

$1 - \alpha$,

where

$$\frac{\alpha}{2} = Q\left(\frac{c\sqrt{n}}{\sigma}\right) = 1 - \Phi\left(\frac{c\sqrt{n}}{\sigma}\right)$$

Ex 7.10

$$2c = 0.2$$

$$c = 0.1$$

$$VAR[X_i] = VAR[Z_i] = 1$$

$$X_i = b + Z_i$$

we use this formula because this is iid otherwise formula is different

$$\leftarrow VAR[X_i] = VAR[b] + VAR[Z_i]$$

$$VAR[X_i] = VAR[Z_i]$$

$$\sigma = 1$$

As Z_i is a Gaussian

random variable

$$\frac{\alpha}{2} = Q\left(\frac{c\sqrt{n}}{\sigma}\right)$$

$$\frac{\alpha}{2} = Q\left(\frac{0.1\sqrt{n}}{1}\right)$$

$$\frac{\alpha}{2} = Q\frac{\sqrt{n}}{10} = 1 - \Phi\left(\frac{\sqrt{n}}{10}\right) \quad (1)$$

But

$$1 - \alpha = 0.99$$

$$\alpha = 0.01$$

$$\frac{\alpha}{2} = 0.05 \quad (2)$$

compare (1) and (2)

$$Q\left(\frac{\sqrt{n}}{10}\right) = 1 - \Phi\left(\frac{\sqrt{n}}{10}\right) = 0.005$$

$$\Phi\left(\frac{\sqrt{n}}{10}\right) = 0.995$$

from the table

$$\Phi(z) \approx 0.995 \text{ when}$$

$$z = 2.58$$

$$\frac{\sqrt{n}}{10} = 2.58$$

$$\sqrt{n} = 25.8$$

$$n = 666$$

$$n \geq 666$$

7.11

$$1-\alpha = 0.99$$

$$\alpha = 0.01$$

$$\sigma_Y^2 = 10$$

$$n = 100$$

$$M_{100}(Y) = 33.2$$

Now

$$M_{100}(Y) = 33.2$$

Now

$$P\left[|M_n(Y) - \mu_Y| < c\right] \geq 1-\alpha$$

$$P\left[M_n(Y) - c \leq \mu_Y < M_n(Y) + c\right] \geq 1-\alpha$$

where

$$\frac{\alpha}{2} = Q\left(\frac{c\sqrt{n}}{\sigma}\right) = 1 - \Phi\left(\frac{c\sqrt{n}}{\sigma}\right)$$

$$\boxed{\frac{\sqrt{100}}{\sqrt{10}} = \frac{10}{\sqrt{10}} = \sqrt{10}}$$

$$\frac{0.01}{2} = Q\left(\frac{c \times \sqrt{100}}{\sqrt{10}}\right) = 1 - \Phi\left(\frac{c \times \sqrt{100}}{\sqrt{10}}\right)$$

$$0.05 = 1 - \Phi(\sqrt{10}\,c)$$

$$\Phi(\sqrt{10}\,c) = 0.995$$

$\sqrt{10}\ c = 2.58$

$c = \frac{2.58}{\sqrt{10}}$

$c = 0.8159$

$M_{100}(Y) - c = 33.2 - 0.8159 = 32.3841$

$M_{100}(Y) + c = 33.2 + 0.8159 = 34.0149$

hence our interval estimate of the expected value is

$32.3841 \leq \mu \leq 34.0159$

with confidence 0.99

2nd way for explain:

there is 0.99 proability that μ lies within 33.2 ± 0.8159 between 32.3841 and 34.0159

Theorem:

If X has finite variance then the sample mean $M_n(X)$ is a sequence of consistent estimate.

$$\lim_{n\to\infty} VAR[M_n(X)] = \lim_{n\to\infty} \frac{VAR[X]}{n} = 0$$

$M_n(X) = M_1(X), M_2(X), \ldots$

Weak law of large Numbers

If random variable X has finite variance then for any constant non zero $c > 0$,

(a) $\lim\limits_{n \to \infty} P[|M_n(X) - \mu_x| \geq c] = 0$

(b) $\lim\limits_{n \to \infty} P[|M_n(X) - \mu_x| < c] = 1$

$$P[X \leq x] = P$$

$$P[X > x] = 1 - P[X \leq x]$$

$$= 1 - P$$

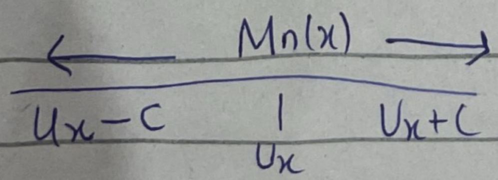
$$P[X > x] + P[X \leq x] = 1$$

$$|M_n(X) - \mu_x| < C$$

$$\mu_x - C < M_n(X) < \mu_x + C$$

$$\mu_x \pm C$$

$c \to$ koi be arbitatory constant



Strong law of Large Number:

if X has finite variance, then

$$P\left[\lim_{n\to\infty} M_n(x) = \mu_x\right] = 1$$ 8. Sure event

Quizz

$E[X] = R$

$k = \frac{4}{3}R$

$$P[X \geq k] \leq \frac{E[X]}{k}$$

$$= \frac{\cancel{R}}{4/3\cancel{R}} = \frac{3}{4}$$

$P[X \geq 144] \leq 0.75$