Lec # 3

FOURIER SERIES REPRESENTATION OF A PERIODIC SIGNAL :-

→ $$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega t}$$ Synthesis Eq.

→ $$\left.\begin{aligned} a_k &= \frac{1}{T}\int_T x(t)\, e^{-jk\omega t}\, dt \\ a_0 &= \frac{1}{T}\int_T x(t)\, dt \end{aligned}\right\}$$ Analysis Equation

Exp :-

$$x(t) = \underbrace{1}_{\substack{k=0 \\ 0^{th}\text{ harmonic}}} + \underbrace{2\cos\omega_0 t}_{\substack{k=1 \\ 1^{st}\text{ harmonic}}} + \underbrace{\cos(2\omega_0 t + \pi/4)}_{\substack{k=2 \\ 2^{nd}\text{ harmonic}}}$$

$$x(t) = 1 + \cancel{2}\left(\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{\cancel{2}}\right) + \left(\frac{e^{j(2\omega_0 t + \pi/4)} + e^{-j(2\omega_0 t + \pi/4)}}{2}\right)$$

$$x(t) = 1 + e^{j\omega_0 t} + e^{-j\omega_0 t} + \frac{1}{2} e^{j(2\omega_0 t)} \cdot e^{j\pi/4}$$

$$+ \frac{1}{2} e^{-j(2\omega_0 t)} \cdot e^{-j\pi/4}$$

$$= \underbrace{1}_{k=0} + \underbrace{e^{j\omega_0 t}}_{k=1} + \underbrace{e^{-j\omega_0 t}}_{k=-1} + \underbrace{\left(\frac{1}{2} e^{j\pi/4}\right) e^{j2\omega_0 t}}_{k=2} + \underbrace{\left(\frac{1}{2} e^{-j\pi/4}\right) e^{-j2\omega_0 t}}_{k=-2}$$

$\therefore$

$$a_0 = 1$$
$$a_1 = 1$$
$$a_{-1} = 1$$
$$a_2 = \frac{1}{2} e^{j\pi/4}$$
$$a_{-2} = \frac{1}{2} e^{-j\pi/4}$$

Magnitude & Phase Spectrum :-

Spectrum of FS are always in the form of line.

that's why it is called "Line Spectrum"

Magnitude Spectrum :-

$$|a_k| = \sqrt{Re^2 + Img^2}$$

Phase Spectrum :-

$$\angle a_k = \tan^{-1}\left(\frac{Img}{Re}\right)$$

Magnitude Spectrum :-

$a_0 = 1$ $\quad |a_0| = \sqrt{1^2+0^2} = \sqrt{1^2} = 1$

$a_1 = 1$ $\quad |a_1| = \sqrt{1^2+0^2} = 1$

$a_{-1} = 1$ $\quad |a_{-1}| = \sqrt{1^2+0^2} = 1$

$a_2 = \frac{1}{2} e^{j\pi/4}$

$a_{-2} = \frac{1}{2} e^{-j\pi/4}$

$|a_2| = ???$

$$a_2 = \frac{1}{2} e^{j\pi/4}$$

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$e^{j\pi/4} = \underbrace{\cos\pi/4}_{\frac{1}{\sqrt{2}}} + j\underbrace{\sin\pi/4}_{\frac{1}{\sqrt{2}}}$$

$$a_2 = \frac{1}{2}\left[\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right]$$

$$= \underbrace{\frac{1}{2\sqrt{2}}}_{Re} + j\underbrace{\frac{1}{2\sqrt{2}}}_{Img}$$

$$|a_2| = \sqrt{\left(\frac{1}{2\sqrt{2}}\right)^2 + \left(\frac{1}{2\sqrt{2}}\right)^2}$$

$$|a_2| = \frac{1}{2}$$

Similarly

$$a_{-2} = \frac{1}{2} e^{-j\pi/4}$$

$$e^{-j\pi/4} = \underbrace{\cos\pi/4}_{\frac{1}{\sqrt{2}}} - j\frac{1}{\sqrt{2}}\sin\pi/4$$

$$= \frac{1}{2}\left(\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right)$$

$$|a_{-2}| = \sqrt{\left(\frac{1}{2\sqrt{2}}\right)^2 + \left(\frac{1}{2\sqrt{2}}\right)^2} = 1/2.$$

$\therefore$

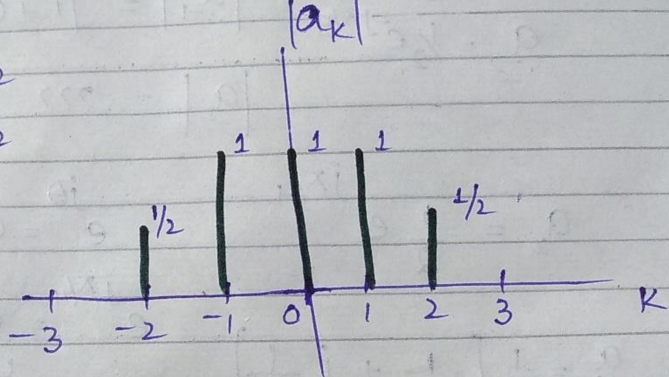
$|a_0|$ 1

$|a_1|$ 1

$|a_{-1}| = 1$

$|a_2| = 1/2$

$|a_{-2}| = 1/2$



Phase Spectrum :-

$a_0 = 1$ $\qquad \angle a_0 = \tan^{-1}\left(\frac{0}{1}\right) = 0$

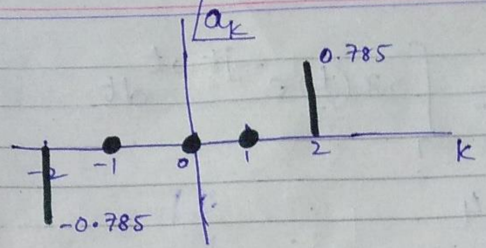
$a_1 = 1$ $\qquad \angle a_1 = 0$

$a_{-1} = 1$ $\qquad \angle a_{-1} = 0$

$a_2 = 1/2\, e^{j\pi/4} \longrightarrow \frac{1}{2\sqrt{2}} + j\frac{1}{2\sqrt{2}}$

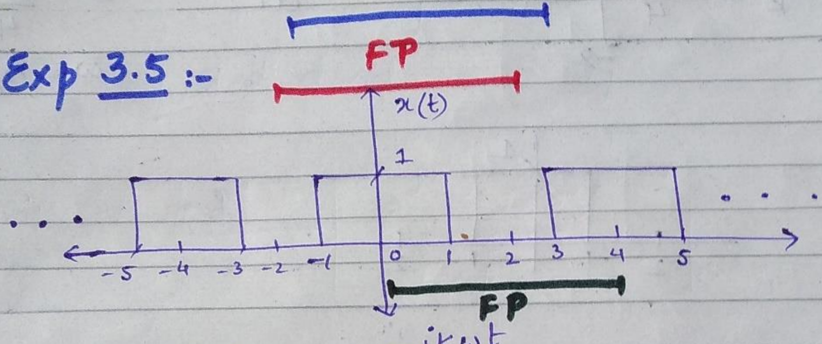
$$\angle a_2 = \tan^{-1}\left(\frac{1/2\sqrt{2}}{1/2\sqrt{2}}\right) = 0.785 \text{ rad}.$$

$$a_{-2} = 1/2\, e^{-j\pi/4} = \tan^{-1}\left(\frac{-1/2\sqrt{2}}{1/2\sqrt{2}}\right) = -0.785 \text{ rad}$$



FS of Graphical Signals :-

Exp 3.5 :-



$$a_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega t}\, dt$$

Fundamental periods ranges :-

$-2 \leftrightarrow 2$ * Select this one will bring simplicity in Calculations

$-1 \leftrightarrow 3$

$0 \leftrightarrow 4$

$-3 \leftrightarrow 1$

$2-(-2) = 2+2 = 4$

$3-(-1) = 3+1 = 4$

$4-0 = 4$

$1-(-3) = 4$

$\therefore$ Fundmental period $= 4 = T$.

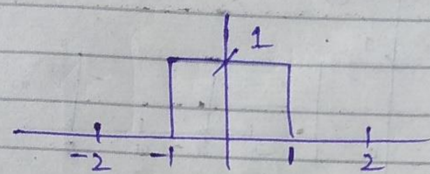
$$a_k = \frac{1}{4}\int_T x(t)\, e^{-jk\omega t}\, dt$$

$T = 4$

$T = 4$.
$\omega = 2\pi f$.
$\omega = \frac{2\pi}{T}$
$\omega = \frac{2\pi}{4}$
$\omega = \pi/2$

Take FP range from -2 to 2.

$$a_k = \frac{1}{4}\int_{-2}^{2} x(t)\, e^{-jk\omega t}\, dt$$



$\therefore$

$$a_k = \frac{1}{4}\int_{-1}^{1} (1)\, e^{-jk\omega t}\, dt$$

$$= \frac{1}{4}\left.\frac{e^{-jk\omega t}}{-jk\omega}\right|_{-1}^{1}$$

$$= \cancel{\tfrac{1}{4}}\ \frac{1}{T(-jk\omega)}\left[e^{-jk\omega(1)} - e^{-jk\omega(-1)}\right]$$

$$= \frac{1}{-jk\omega T}\left(e^{-jk\omega} - e^{+jk\omega}\right)$$

$$= \frac{1}{-jk\omega T}\left(-e^{jk\omega} + e^{-jk\omega}\right).$$

Take minus common.

$$a_k = \frac{(-1)}{-jk\omega T}\left[e^{jk\omega} - e^{-jk\omega}\right] \quad \text{— (1)}$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

Multiply and divide with 2j.

$$a_k = \frac{2j}{jk\omega T}\left[\frac{e^{jk\omega} - e^{-jk\omega}}{2j}\right]$$

$$a_k = \frac{2}{k\omega T}\sin(k\omega)$$

Now put values

$T = 4$

$\omega = \pi/2$

$$= \frac{2 \cdot \sin(k\pi/2)}{k(\pi/2)(4)}$$

$$\boxed{a_k = \frac{[illegible]\,\sin(\frac{k\pi}{2})}{k\pi}}$$

8

$$a_k = \frac{\text{[redacted]} \sin(\frac{k\pi}{2})}{k\pi}$$

Now to draw spectrums, values of k can be put

$$k = \cancel{0}\ \pm 1, \pm 2, \pm 3, \ldots$$

$$\therefore \quad a_0 = \frac{1}{T}\int_{-1}^{1} x(t)\, dt$$

$$a_0 = \frac{1}{4}\int_{-1}^{1} (1)\, dt$$

$$= \frac{1}{4}\left(1-(-1)\right)$$

$$= \frac{1}{4}(2) = \frac{1}{2}$$

$$\therefore \quad a_0 = \frac{1}{2}$$

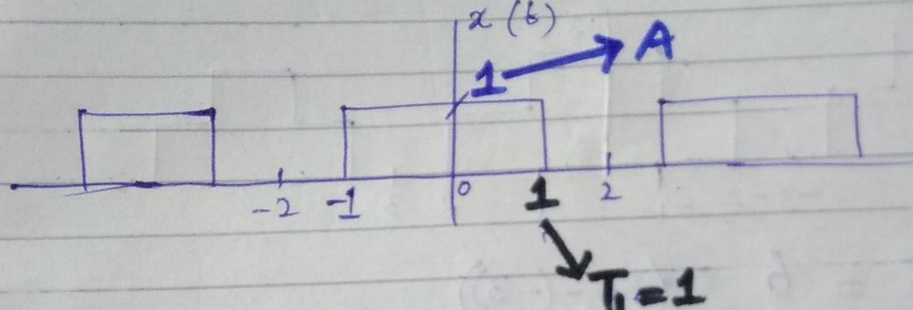
$$a_k = \frac{\text{[redacted]} \sin(\frac{k\pi}{2})}{k\pi} \qquad K = \pm 1, \pm 2, \ldots$$

$$a_1 = a_{-1} = \frac{1}{\pi}$$

$$a_2 = a_{-2} = 0$$

$$a_3 = a_{-3} = -\frac{1}{3\pi}$$

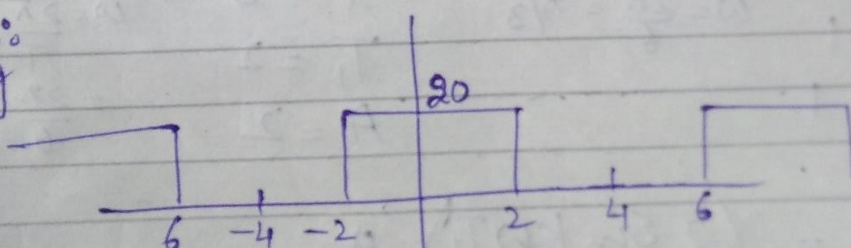
Important Result :-



FP $\Rightarrow$ $T=4$

$$a_k = \left(\frac{2\sin k\omega T_1}{k\omega T}\right) A \longrightarrow$$ for any rect signal.

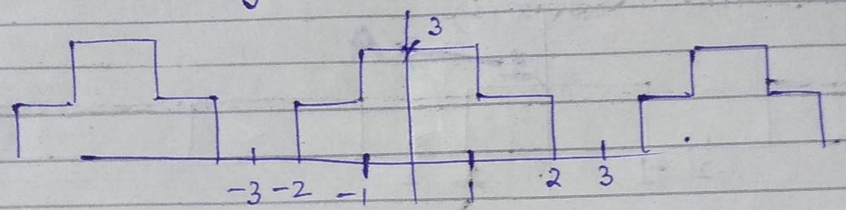
e.g.



$T_1 = 2$, $A = 20$, $T = 8$

$$a_k = \left(\frac{2\sin\omega(2)}{k\omega(8)}\right)20$$

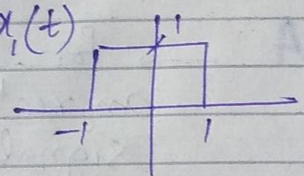
Similarly,

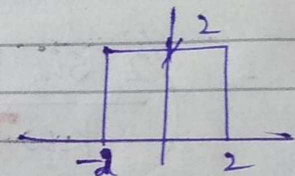


$T = 6 \quad (3-(-3))$.

It can be broken into two rect signals.

$x_1(t)$





$x_2(t)$

$T = 6$
$T_1 = 1$
$A = 1$
$\omega = \frac{2\pi}{6} = \pi/3$

$T = 6$
$T_1 = 2$
$A = 2$
$\omega = \frac{2\pi}{T} = \frac{2\pi}{6} = \pi/3$

$$\therefore \quad a_k = \left(\frac{2 \sin k\omega T_1}{k\omega T}\right) A$$

$x_1(t)$

$$a_k = \frac{2 \sin k(\pi/3)(1)}{k(\pi/3)(6)}$$

$$x_2(t) = \frac{2 \sin k(\pi/3)\, 2}{k(\pi/3)(6)}$$