Lecture 7

Nodal Analysis

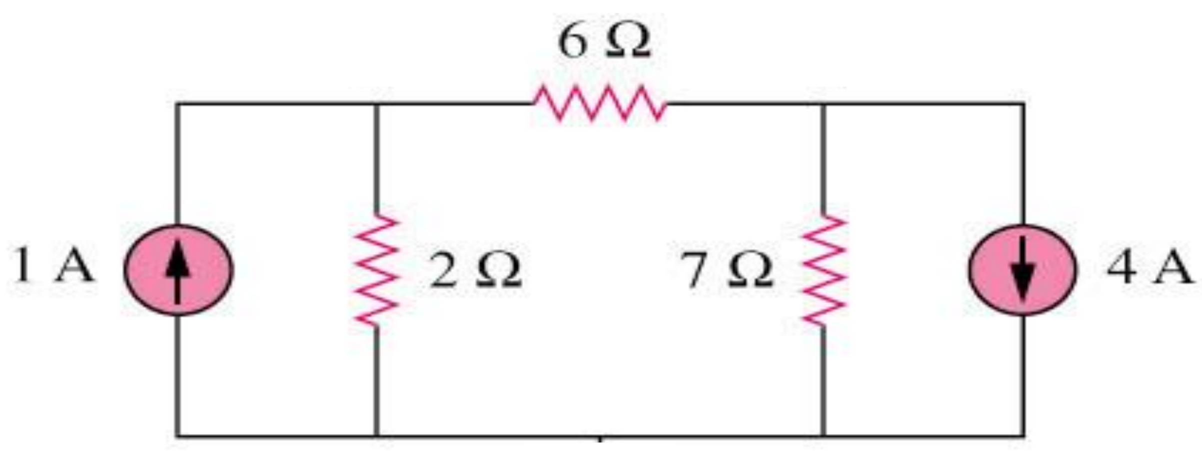
Nodal Analysis with Current Sources

Objectives of Lecture

- Provide step-by-step instructions for nodal analysis, which is a method to calculate node voltages and currents that flow through components in a circuit.

Motivation

- If you are given the following circuit, how can we determine (1) the voltage across each resistor, (2) current through each resistor. (3) power generated by each current source, etc.



What are the things which we need to know in order to determine the answers?

- Things we need to know in solving any resistive circuit with current and voltage sources only:
- Kirchhoff's Current Laws (KCL)
- Kirchhoff's Voltage Laws (KVL)
- Ohm's Law

How should we apply these laws to determine the answers?

Nodal Analysis

- **It provides a general procedure for analyzing circuits using <u>node voltages</u> as the circuit variables.**

  - First result from nodal analysis is the determination of node voltages (voltage at nodes referenced to ground).
    - These voltages are not equal to the voltage dropped across the resistors.
  - Second result is the calculation of the currents

Current flows from a higher potential to a lower potential in a resistor.

We can express this principle as

$$\boxed{i = \frac{v_{\text{higher}} - v_{\text{lower}}}{R}}$$

Steps in Nodal Analysis

- Steps to determine the node voltages:

1. Select a node as the reference node.
2. Assign voltages v1,v2,…,vn-1 to the remaining n-1 nodes. The voltages are referenced with respect to the reference node.
3. Apply KCL to each of the n-1 non-reference nodes. Use Ohm's law to express the branch currents in terms of node voltages.
4. Solve the resulting simultaneous equations to obtain the unknown node voltages.

Nodal Analysis

- Technique to find currents at a node using Ohm's Law and the potential differences betweens nodes.
  - First result from nodal analysis is the determination of node voltages (voltage at nodes referenced to ground).
    - These voltages are not equal to the voltage dropped across the resistors.
  - Second result is the calculation of the currents

How can we write resistor currents in terms of node voltages using ohm's law

Current flows from a higher potential to a lower potential in a resistor.

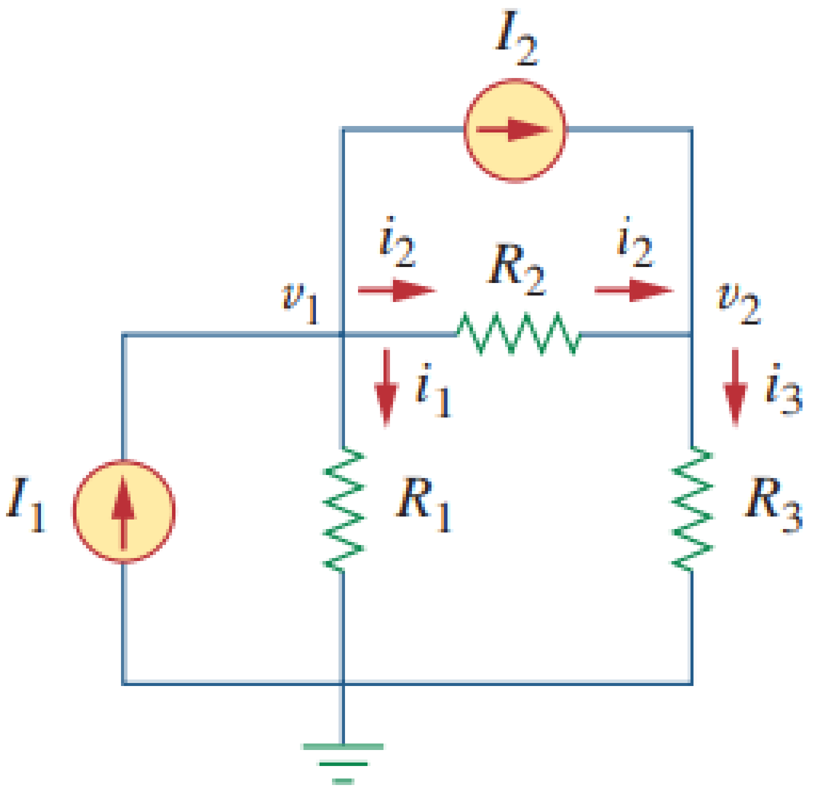
We can express this principle as

$$\boxed{i = \frac{v_{\text{higher}} - v_{\text{lower}}}{R}}$$

$$i_1 = \frac{v_1 - 0}{R_1}$$

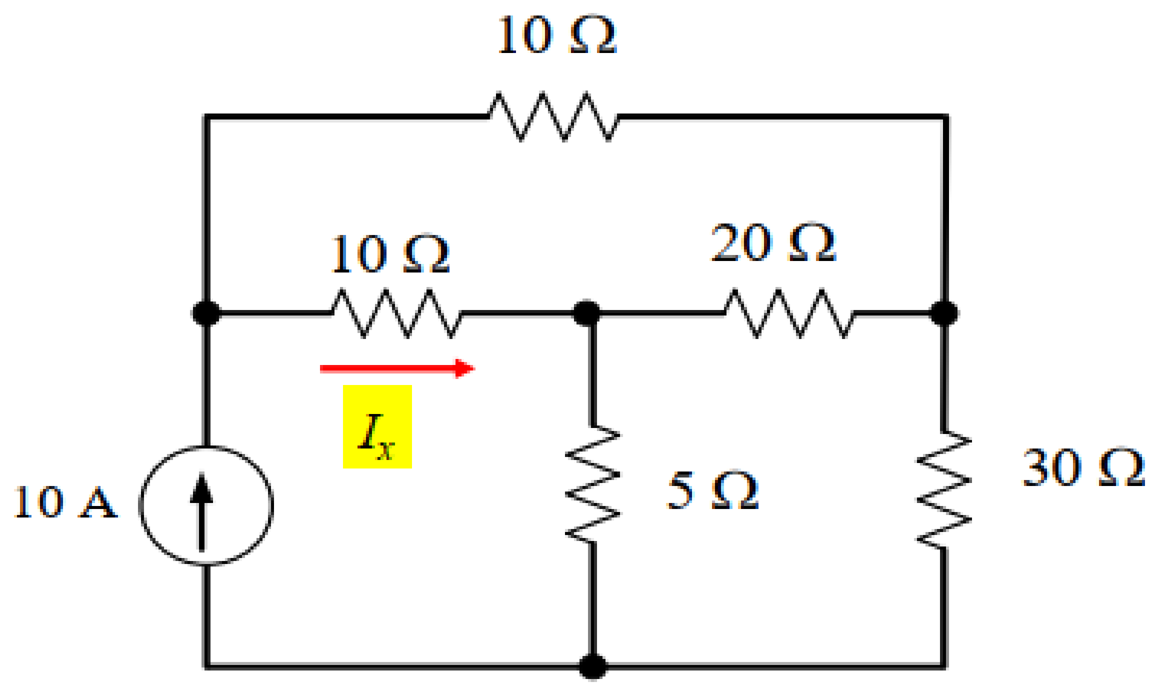
$$i_2 = \frac{v_1 - v_2}{R_2}$$

$$i_3 = \frac{v_2 - 0}{R_3}$$



Example 1

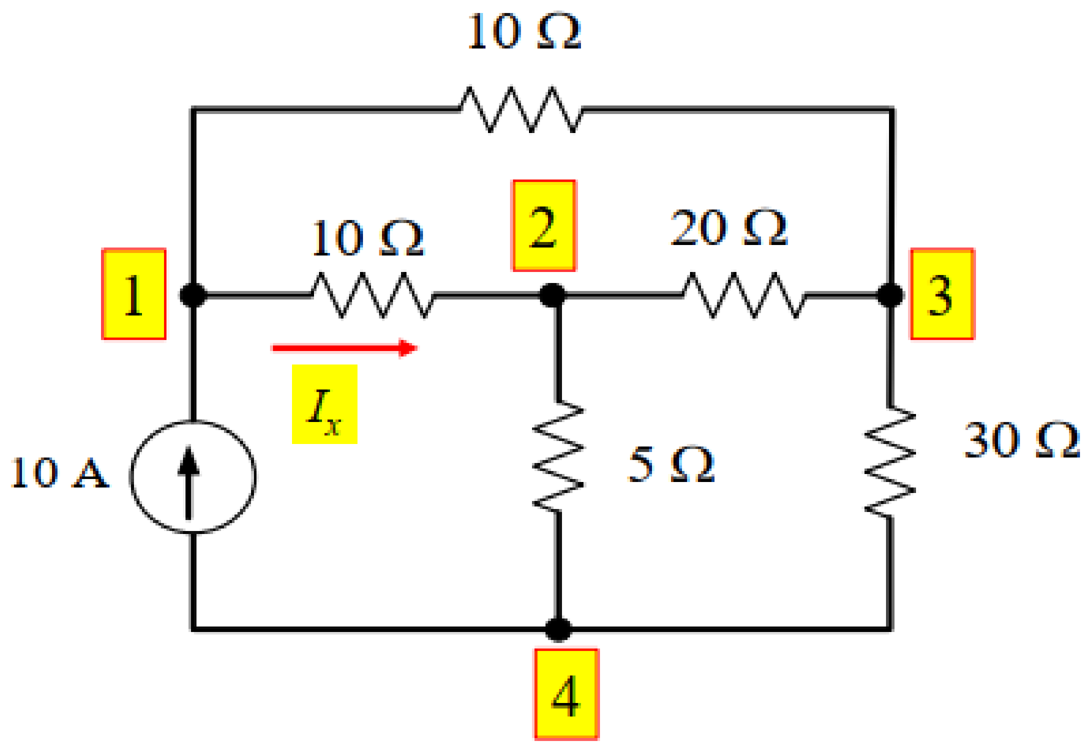
Suppose we have the following circuit



Suppose we want to find the current I_x

Step1: Identify nodes

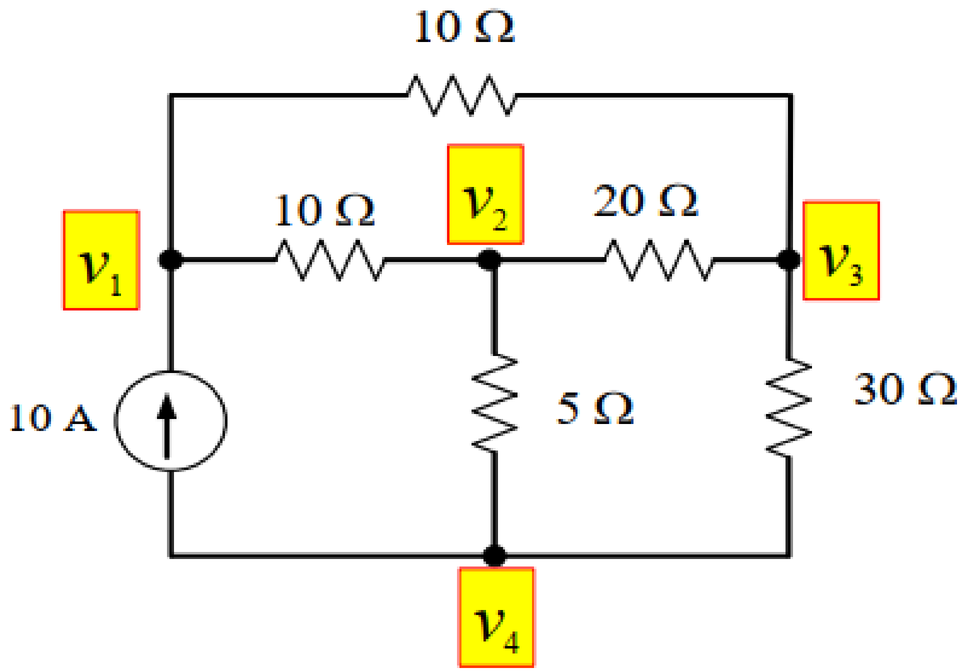
We can use a technique called the **nodal voltage method** as follows:



1- Identify **essential nodes** (connect 3 elements and above) ➡ 4 nodes

Step 2: Identify reference node

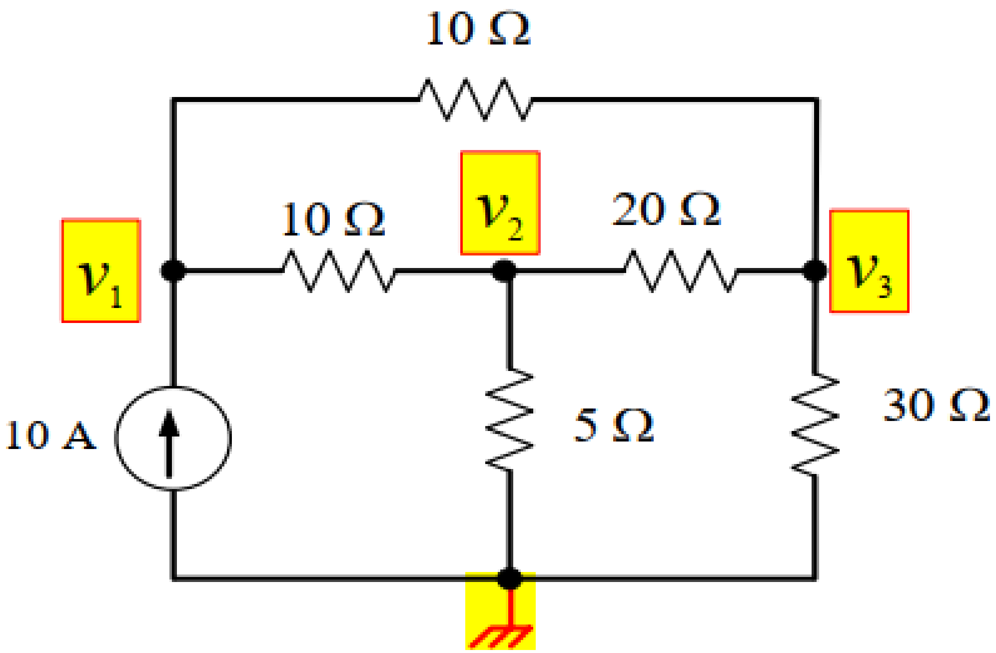
- Identify reference node and set its voltage to zero



Now if we set one of the nodes as a reference say to zero (ground) that will not change the voltage difference

We can select any of the nodes to be reference , example node 4 $v_4 = 0$

Selecting $v_4 = 0$, reduce the number of unknown voltage to three , v_1, v_2, v_3

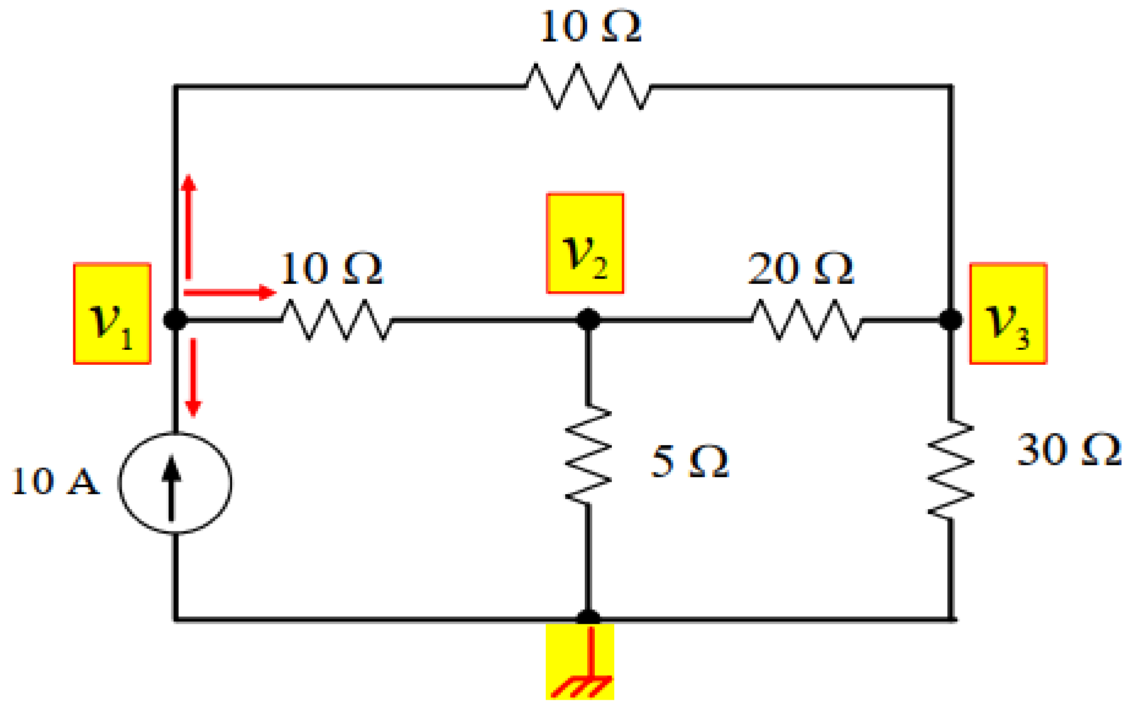


Now our objective is to find the nodes voltages v_1, v_2, v_3

We need 3 linear equations that relate v_1, v_2, v_3

We will apply KCL on each node and write the currents in terms of v_1, v_2, v_3

Step 3: KCL on node 1

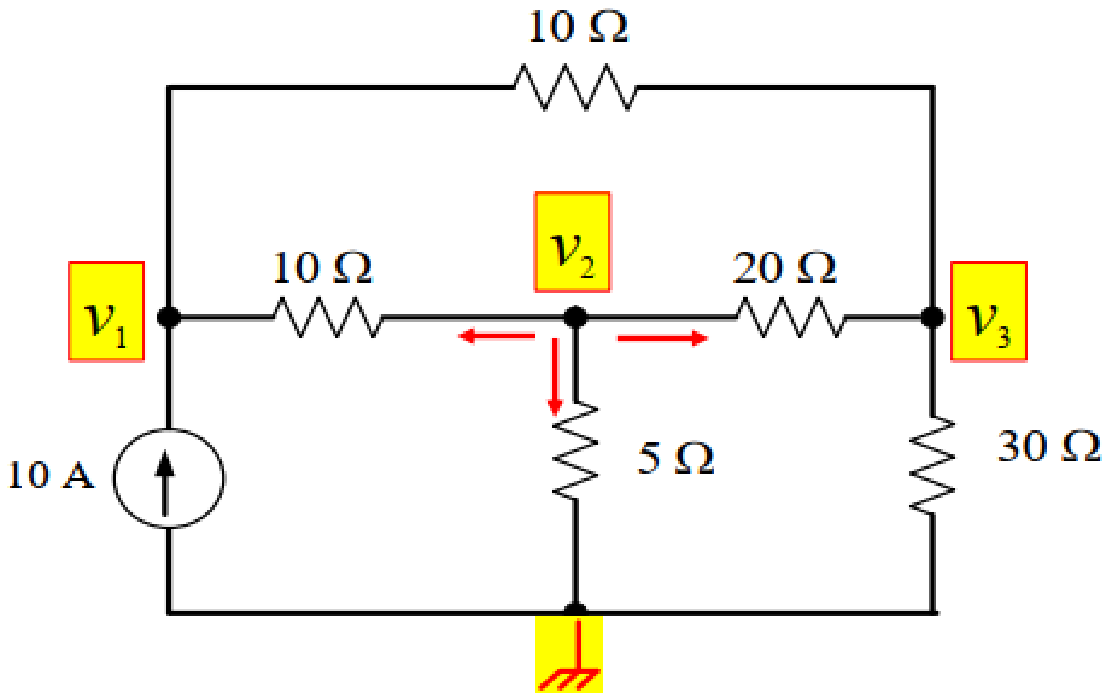


KCL on each node assuming currents leaving the node

KCL at node 1 $\dfrac{v_1 - v_2}{10} + \dfrac{v_1 - v_3}{10} + (-10) = 0$

Simplify $2v_1 - v_2 - v_3 = 100$ ———— (1)

KCL on node 2

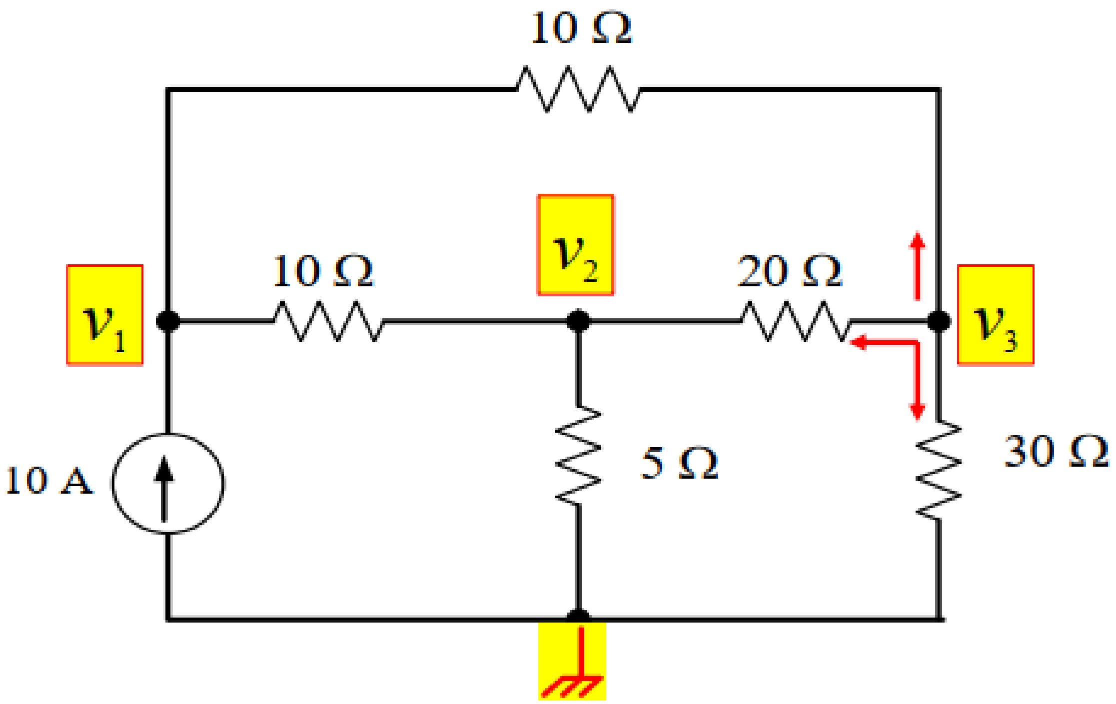


$$2v_1 - v_2 - v_3 = 100 \quad \text{—— (1)}$$

KCL at node 2 $\dfrac{v_2 - v_1}{10} + \dfrac{v_2 - v_3}{20} + \dfrac{v_2 - 0}{5} = 0$

Simplify $-2v_1 + 7v_2 - v_3 = 0$ —— (2)

KCL on node 3

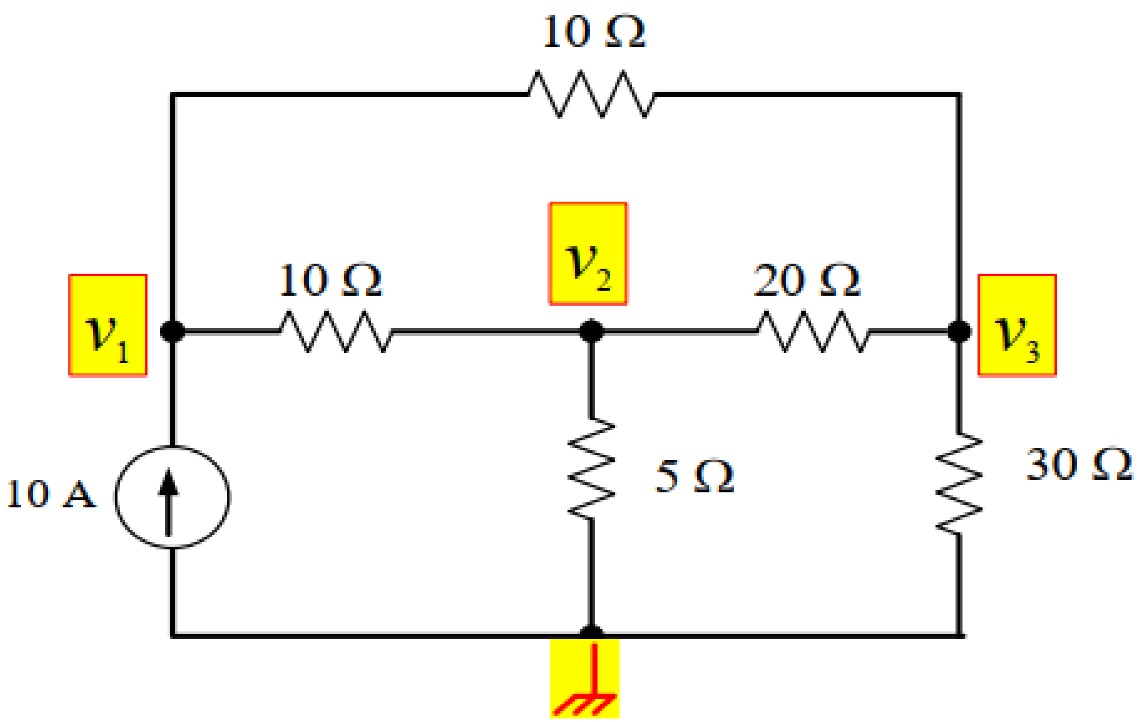


$$2v_1 - v_2 - v_3 = 100 \quad \text{———} \quad (1)$$

$$-2v_1 + 7v_2 - v_3 = 0 \quad \text{———} \quad (2)$$

KCL at node 3 $\quad \dfrac{v_3 - v_2}{20} + \dfrac{v_3 - v_1}{10} + \dfrac{v_3 - 0}{30} = 0$

Simplify $\quad -6v_1 - 3v_2 + 11v_3 = 0 \quad \text{———} \quad (3)$



$$2v_1 - v_2 - v_3 = 100 \quad \text{—— (1)}$$

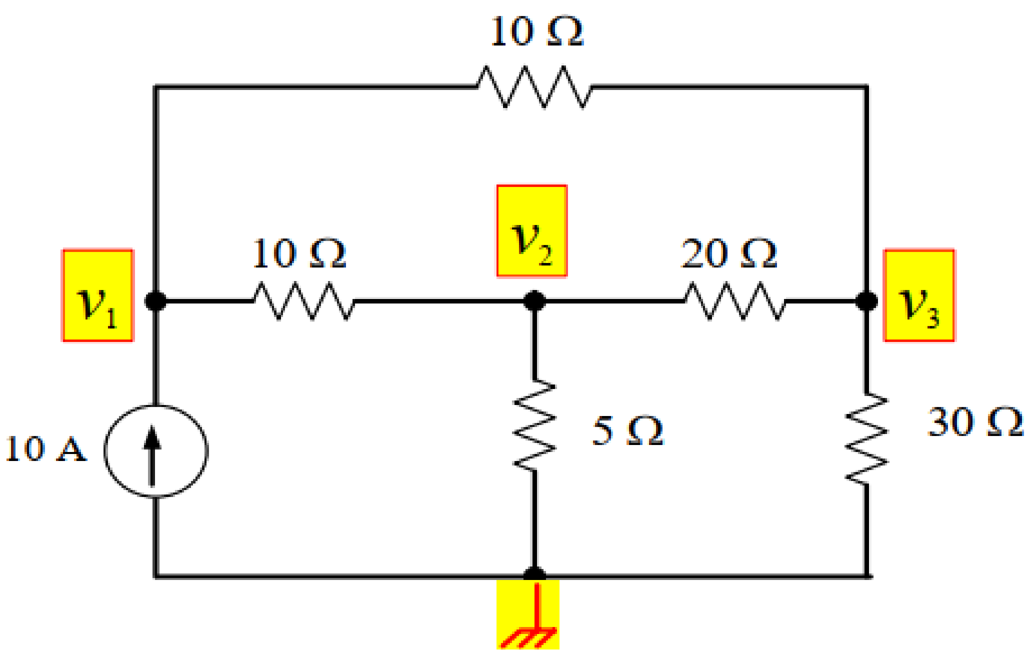
$$-2v_1 + 7v_2 - v_3 = 0 \quad \text{—— (2)}$$

$$-6v_1 - 3v_2 + 11v_3 = 0 \quad \text{—— (3)}$$

In Matrix form

$$\begin{bmatrix} 2 & -1 & -1 \\ -2 & 7 & -1 \\ -6 & -3 & 11 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \\ 0 \end{bmatrix}$$

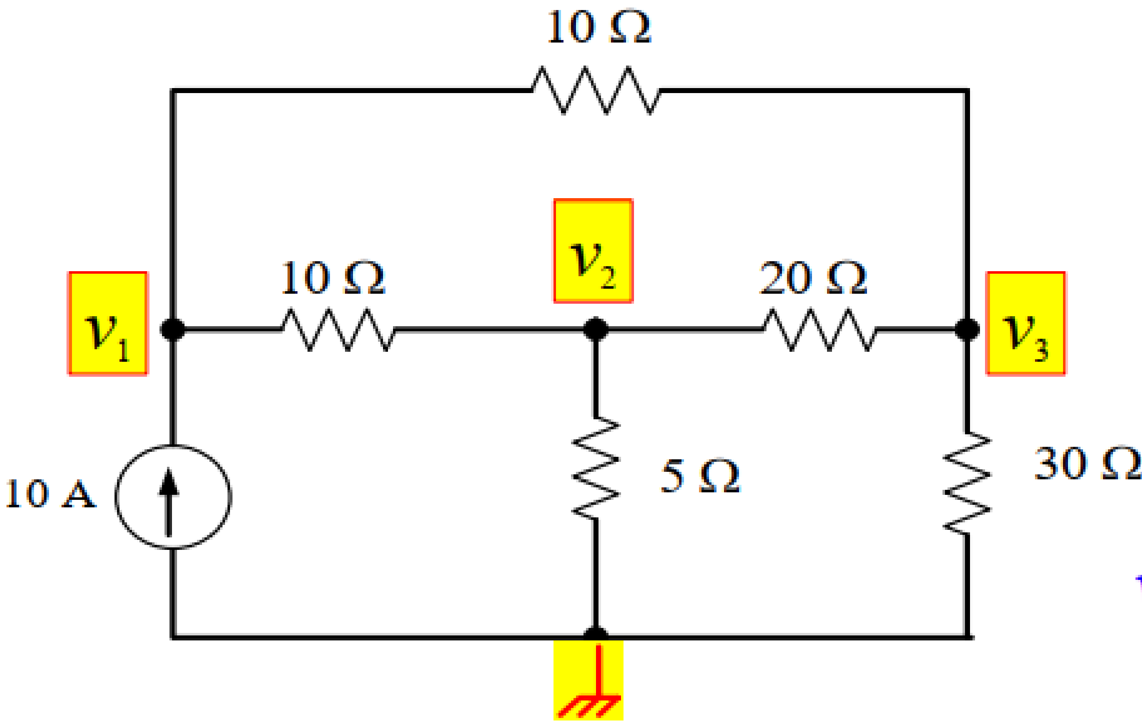
Solving for nodes voltages v_1, v_2, v_3



$$\begin{bmatrix} 2 & -1 & -1 \\ -2 & 7 & -1 \\ -6 & -3 & 11 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \\ 0 \end{bmatrix}$$

$$v_1 = \frac{\begin{vmatrix} 100 & -1 & -1 \\ 0 & 7 & -1 \\ 0 & -3 & 11 \end{vmatrix}}{\begin{vmatrix} 2 & -1 & -1 \\ -2 & 7 & -1 \\ -6 & -3 & 11 \end{vmatrix}} = \frac{7400}{72}$$

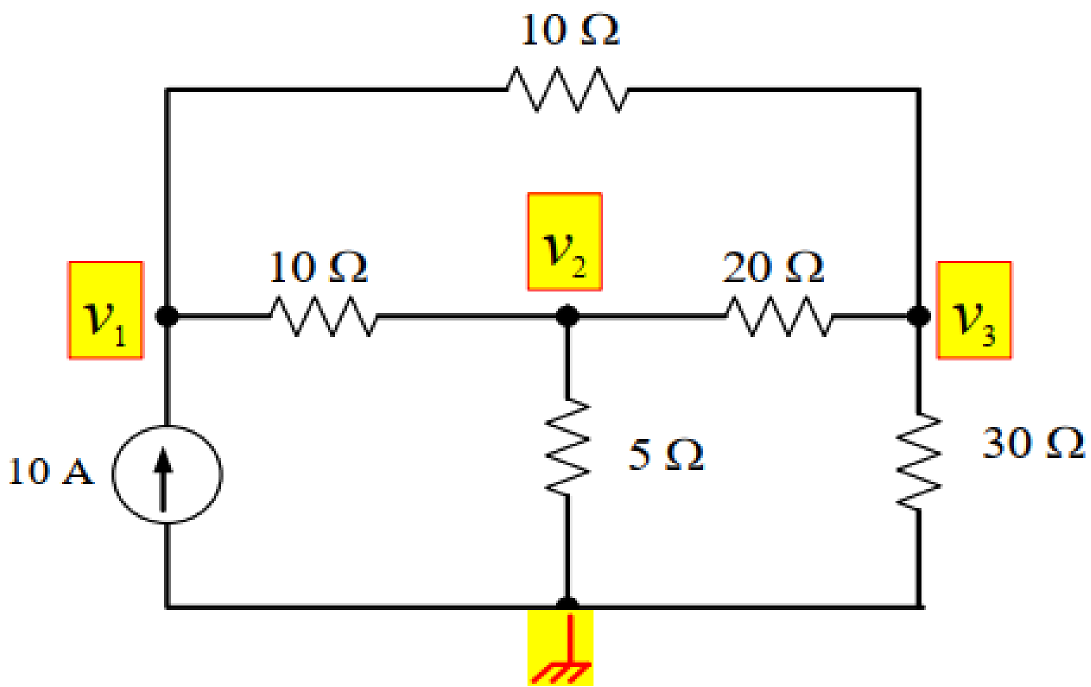
$$v_1 = 102.78 \text{ V}$$



$$\begin{bmatrix} 2 & -1 & -1 \\ -2 & 7 & -1 \\ -6 & -3 & 11 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \\ 0 \end{bmatrix}$$

$$v_2 = \frac{\begin{vmatrix} 2 & 100 & -1 \\ -2 & 0 & -1 \\ -6 & 0 & 11 \end{vmatrix}}{\begin{vmatrix} 2 & -1 & -1 \\ -2 & 7 & -1 \\ -6 & -3 & 11 \end{vmatrix}} = \frac{2800}{72}$$

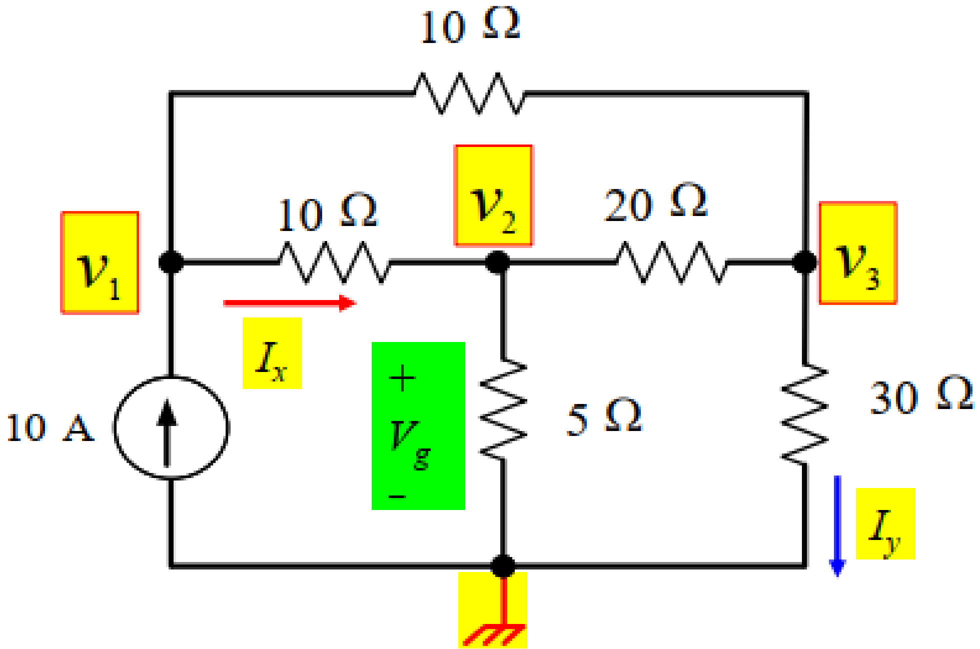
$$v_2 = 38.88 \text{ V}$$



$$\begin{bmatrix} 2 & -1 & -1 \\ -2 & 7 & -1 \\ -6 & -3 & 11 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \\ 0 \end{bmatrix}$$

$$v_3 = \frac{\begin{vmatrix} 2 & -1 & 100 \\ -2 & 7 & 0 \\ -6 & -3 & 0 \end{vmatrix}}{\begin{vmatrix} 2 & -1 & -1 \\ -2 & 7 & -1 \\ -6 & -3 & 11 \end{vmatrix}} = \frac{4800}{72}$$

$$\boxed{v_3 = 66.67 \text{ V}}$$



$v_1 = 102.78$ V

$v_2 = 38.88$ V

$v_3 = 66.67$ V

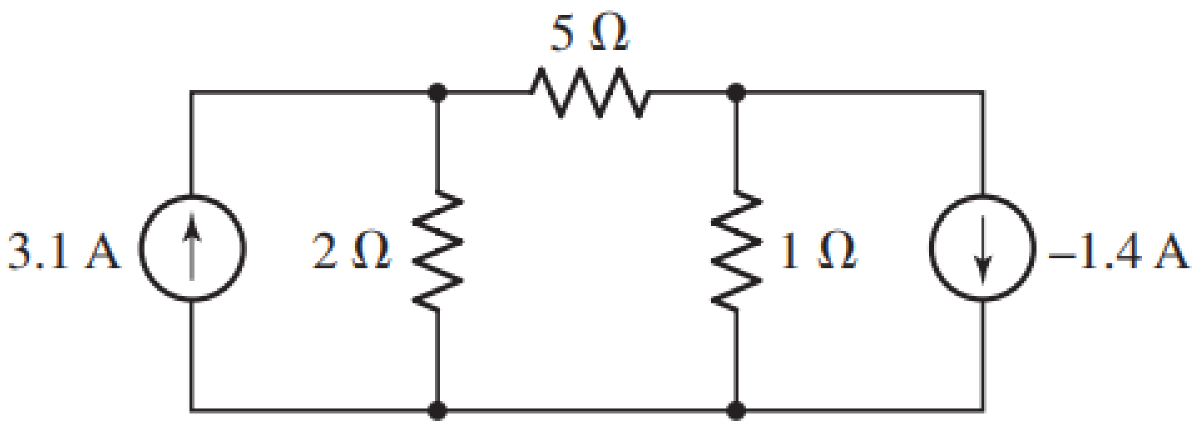
$$I_x = \frac{v_1 - v_2}{10} = \frac{102.78 - 38.88}{10} = 6.39 \text{ A}$$

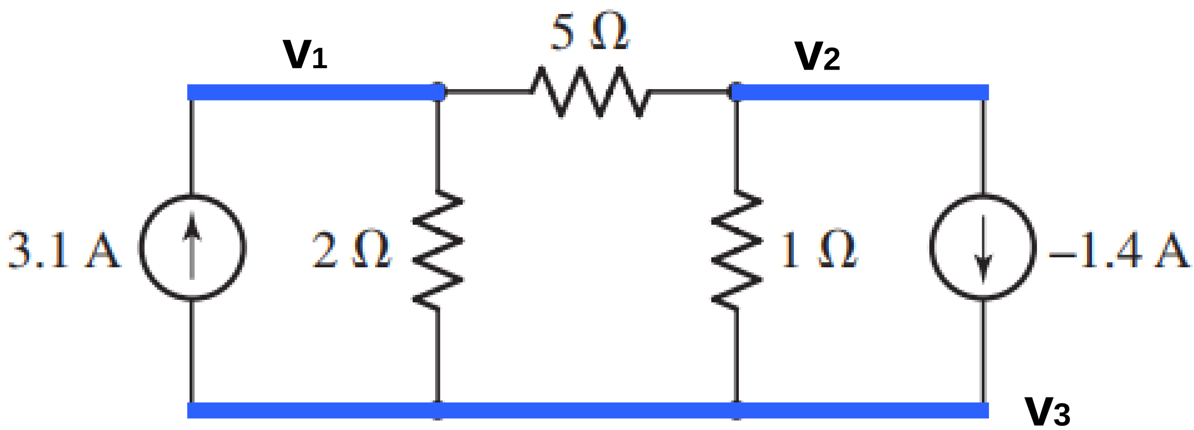
$$I_y = \frac{v_3 - 0}{30} = \frac{66.67 - 0}{30} = 2.22 \text{ A}$$

$$V_g = v_2 - 0 = 38.88 - 0 = 38.88 \text{ V}$$

Example 2

- Determine node voltages





$$\frac{v_1}{2} + \frac{v_1 - v_2}{5} = 3.1 \tag{1}$$

or

$$0.7v_1 - 0.2v_2 = 3.1 \tag{2}$$

At node 2 we obtain

$$\frac{v_2}{1} + \frac{v_2 - v_1}{5} = -(-1.4) \tag{3}$$

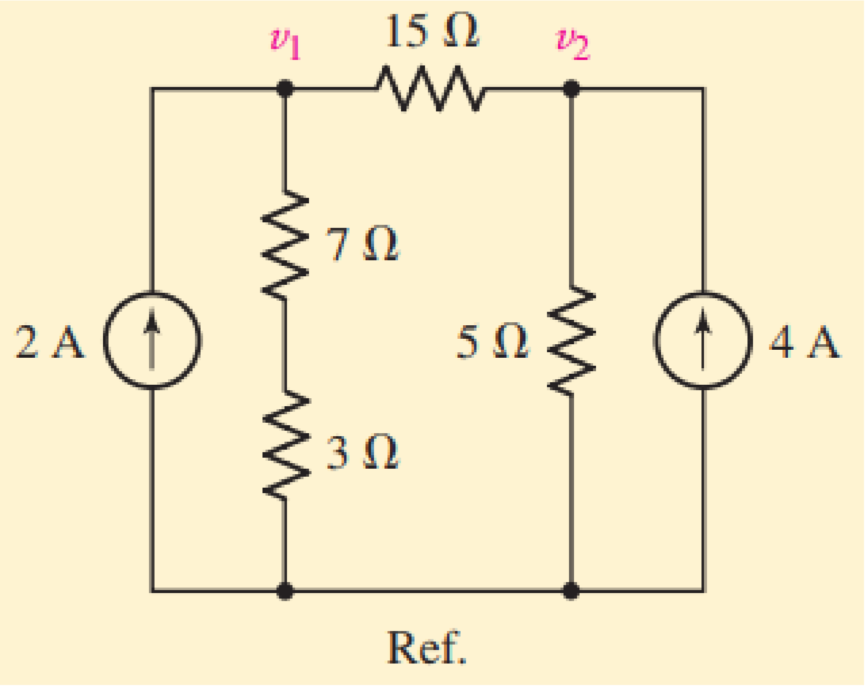
or

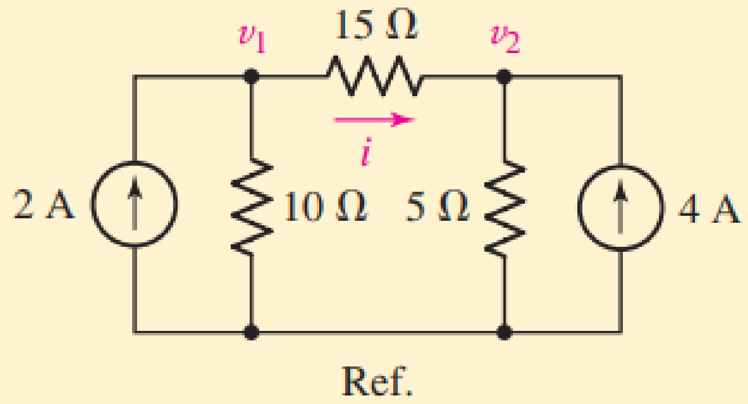
$$-0.2v_1 + 1.2v_2 = 1.4 \tag{4}$$

Equations [2] and [4] are the desired two equations in two unknowns, and they may be solved easily. The results are $v_1 = 5$ V and $v_2 = 2$ V.

Example 3

- Determine current flowing from left to right in 15Ω resister





Writing an appropriate KCL equation for node 1,

$$2 = \frac{v_1}{10} + \frac{v_1 - v_2}{15}$$

and for node 2,

$$4 = \frac{v_2}{5} + \frac{v_2 - v_1}{15}$$

Rearranging, we obtain

$$5v_1 - 2v_2 = 60$$

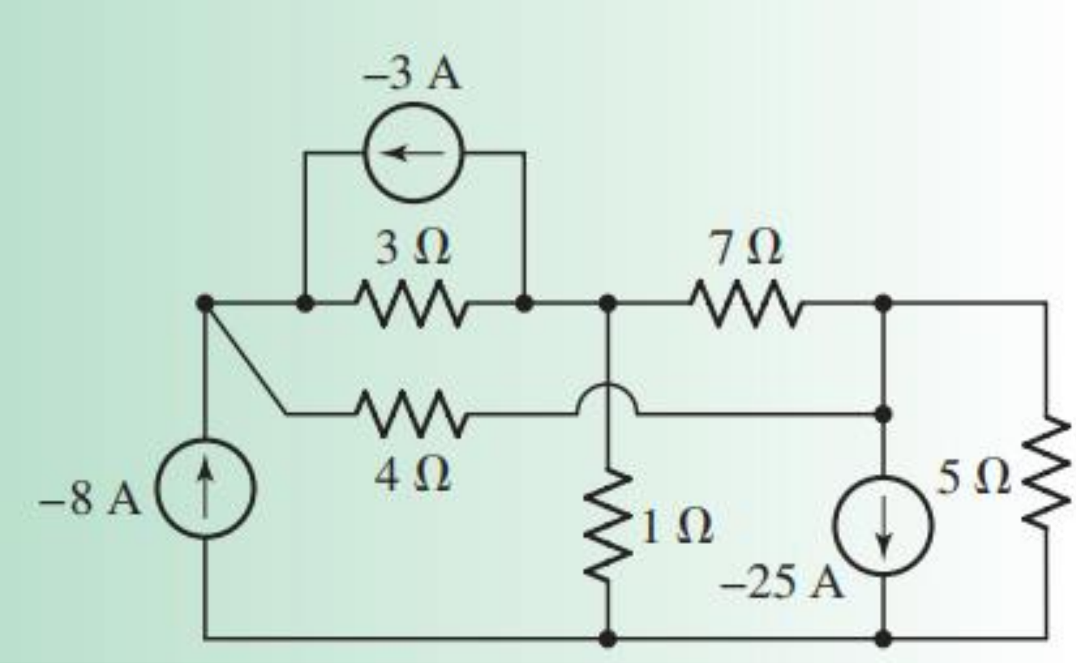
and

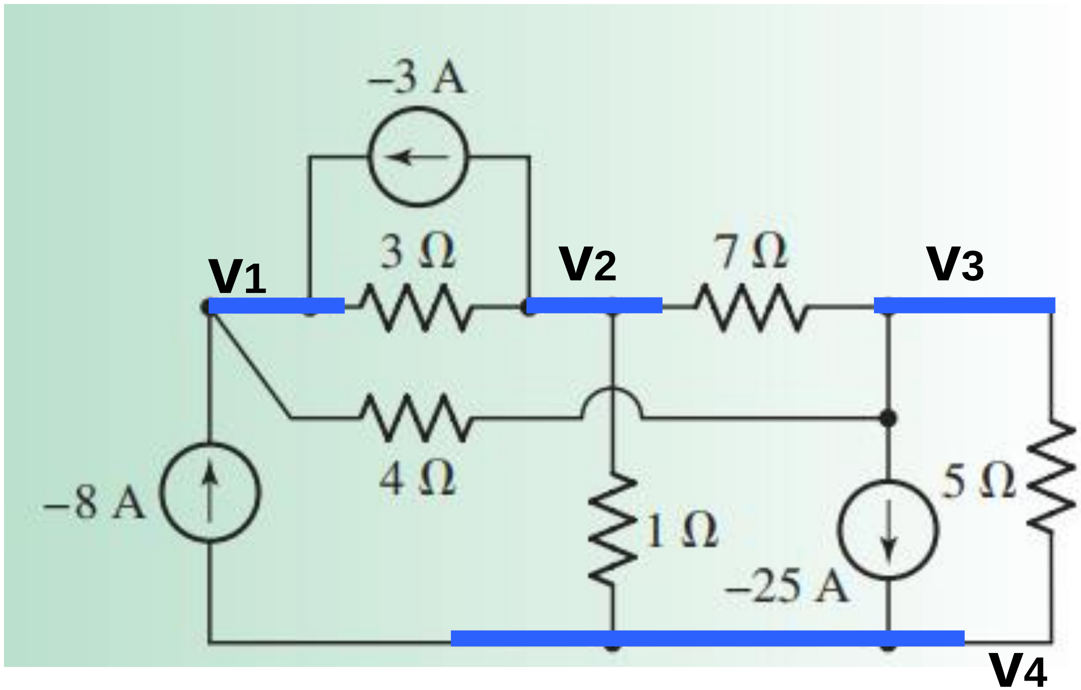
$$-v_1 + 4v_2 = 60$$

Solving, we find that $v_1 = 20$ V and $v_2 = 20$ V so that $v_1 - v_2 = 0$. In other words, ***zero current*** is flowing through the 15 Ω resistor

Example 4

- Determine nodal voltages for the circuit as referenced to the bottom node.





We begin by writing a KCL equation for node 1:

$$-8 - 3 = \frac{v_1 - v_2}{3} + \frac{v_1 - v_3}{4}$$

or

$$0.5833v_1 - 0.3333v_2 - 0.25v_3 = -11$$

At node 2:

$$-(-3) = \frac{v_2 - v_1}{3} + \frac{v_2}{1} + \frac{v_2 - v_3}{7}$$

or

$$-0.3333v_1 + 1.4762v_2 - 0.1429v_3 = 3$$

And, at node 3:

$$-(-25) = \frac{v_3}{5} + \frac{v_3 - v_2}{7} + \frac{v_3 - v_1}{4}$$

or, more simply,

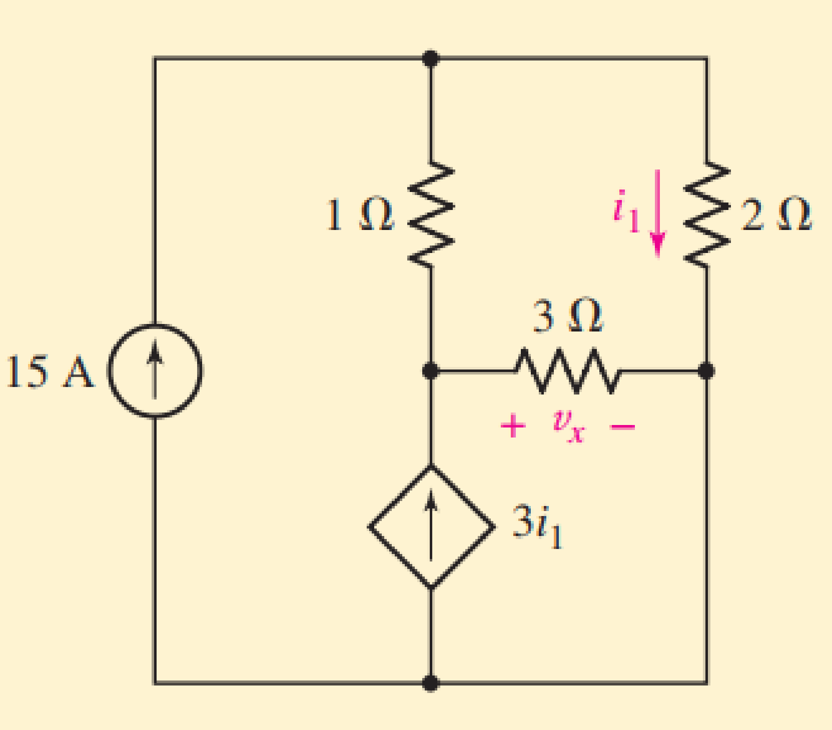
$$-0.25v_1 - 0.1429v_2 + 0.5929v_3 = 25$$

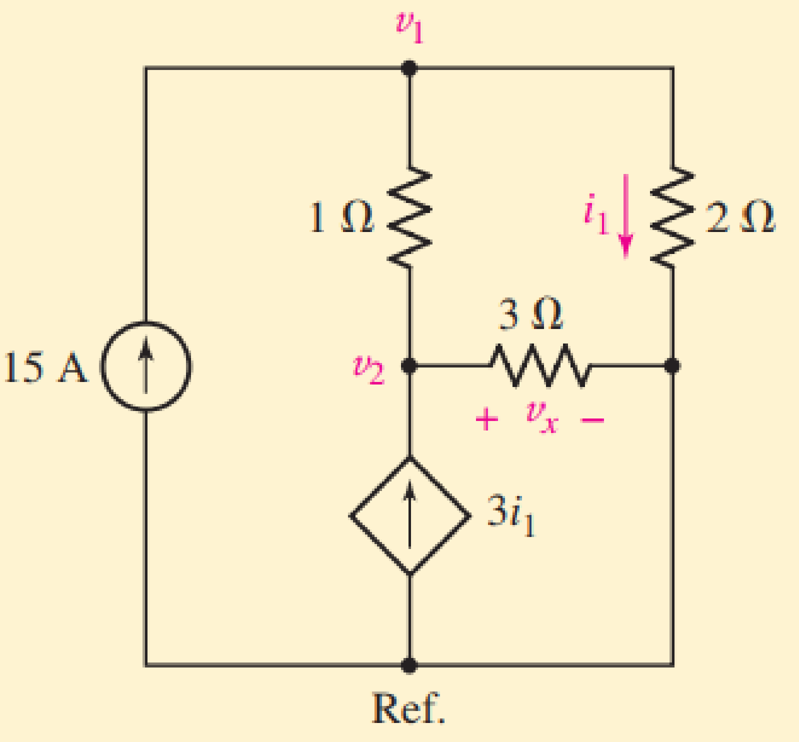
$$\begin{bmatrix} 0.5833 & -0.3333 & -0.25 \\ -0.3333 & 1.4762 & -0.1429 \\ -0.25 & -0.1429 & 0.5929 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -11 \\ 3 \\ 25 \end{bmatrix}$$

- V1 = 5.412 V, V2= 7.736 V, V3= 46.32 V

Example 5

- Determine the power supplied by the dependent source





At node 1, we write

$$15 = \frac{v_1 - v_2}{1} + \frac{v_1}{2}$$

and at node 2

$$3i_1 = \frac{v_2 - v_1}{1} + \frac{v_2}{3}$$

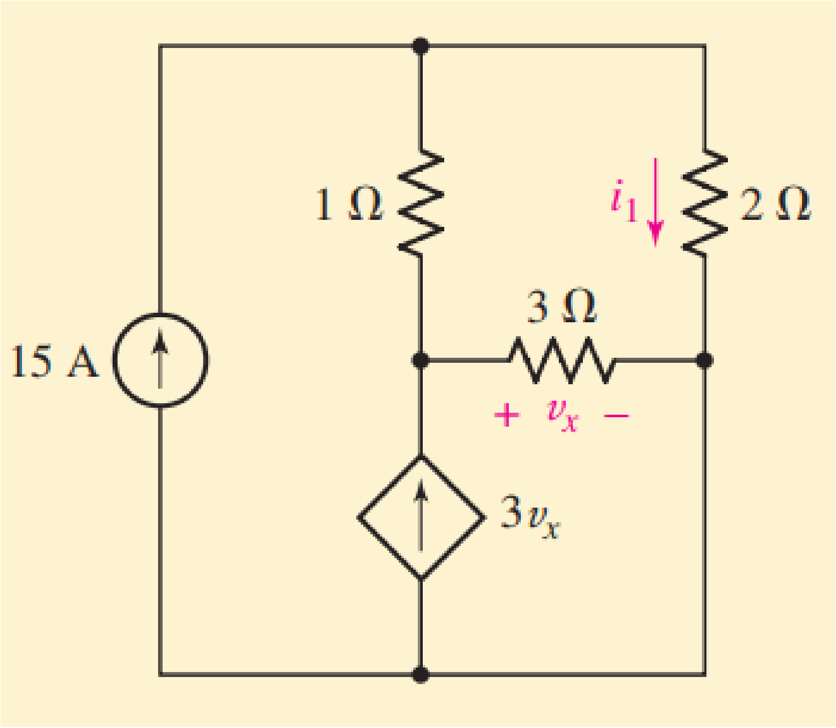
Unfortunately, we have only two equations but three unknowns; *this is a direct result of the presence of the dependent current source,*

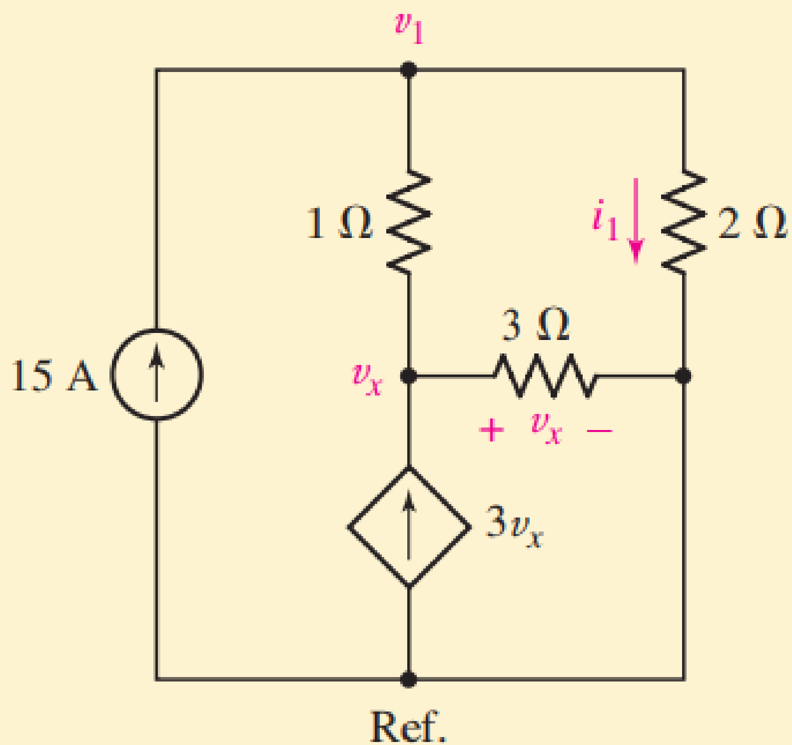
$$i_1 = \frac{v_1}{2}$$

Solving, we find that $v_1 = -40$ V, $v_2 = -75$ V, and $i_1 = 0.5v_1 = -20$ A. Thus, the power supplied by the dependent source is equal to $(3i_1)(v_2) = (-60)(-75) = 4.5$ kW.

Example 6

- Determine the power supplied by the dependent source





Our KCL equation for node 1 is

$$15 = \frac{v_1 - v_x}{1} + \frac{v_1}{2}$$

and for node x is

$$3v_x = \frac{v_x - v_1}{1} + \frac{v_x}{3}$$

Grouping terms and solving, we find that $v_1 = \frac{50}{7}$ V and $v_x = -\frac{30}{7}$ V. Thus, the dependent source in this circuit generates $(3v_x)(v_x) = 55.1$ W.

Summary

- Steps in Nodal Analysis
  1. Pick one node as a reference node
  2. Label the voltage at the other nodes
  3. Label the currents flowing through each of the components in the circuit
  4. Use Kirchoff's Current Law
  5. Use Ohm's Law to relate the voltages at each node to the currents flowing in and out of them.
  6. Solve for the node voltage
  7. Once the node voltages are known, calculate the currents.

Practice Problems

- PP 3.2

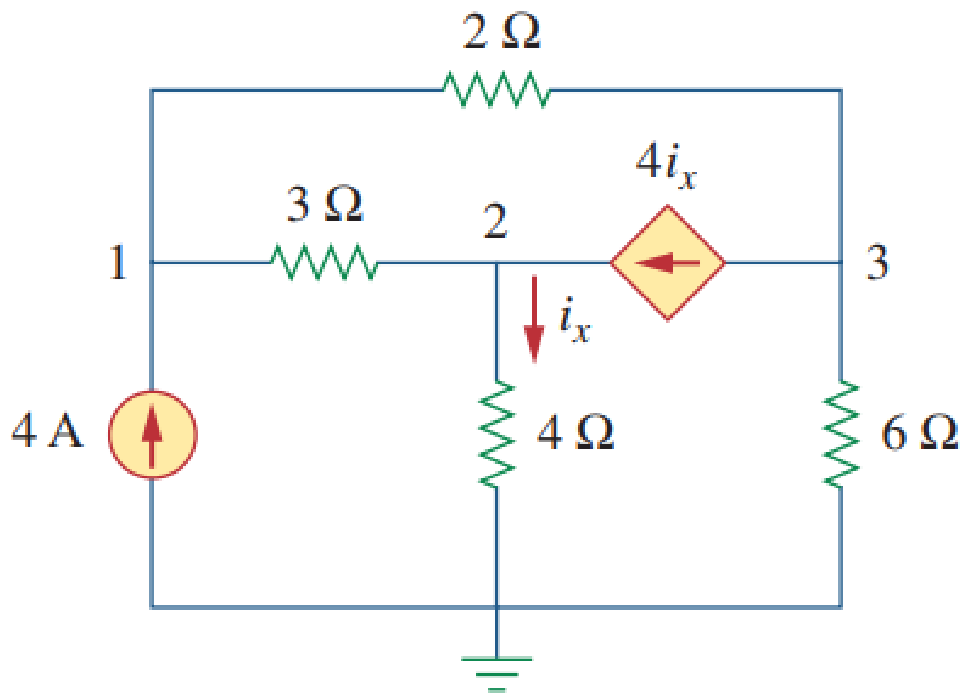


Figure 3.6
For Practice Prob. 3.2.

Find the voltages at the three nonreference nodes in the circuit of Fig. 3.6.

Answer: $v_1 = 32$ V, $v_2 = -25.6$ V, $v_3 = 62.4$ V.

Thank You