Lecture 12: Covariance, Correlation, Orthogonality and Independence

CPE251 Probability Methods in Engineering

Dr. Zaid Ahmad, SMIEEE
Advisor IEEE CUI Lahore
COMSATS University Islamabad, Lahore Campus

1

Covariance

Discrete Case	Continuous Case
$\sigma_{XY} = COV[X,Y]$ $= E[(X-\mu_X)(Y-\mu_Y)]$ $= \sum_x \sum_y (x-\mu_X)(Y-\mu_Y)p_{X,Y}(x,y)$	$\sigma_{XY} = COV[X,Y]$ $= E[(X-\mu_X)(Y-\mu_Y)]$ $= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x-\mu_X)(Y-\mu_Y)f_{X,Y}(x,y)\,dxdy$

2

Covariance

Positive Covariance: large X → large Y or small X → small Y

Negative Covariance: large X → small Y or small X → large Y

3

Correlation

*The **correlation** of X and Y is* $r_{X,Y} = \mathrm{E}[XY]$

Theorems

For any two random variables X and Y, $\mathrm{E}[X+Y] = \mathrm{E}[X] + \mathrm{E}[Y]$.

$\mathrm{Cov}[X,Y] = r_{X,Y} - \mu_X \mu_Y$.

$\mathrm{Var}[X+Y] = \mathrm{Var}[X] + \mathrm{Var}[Y] + 2\,\mathrm{Cov}[X,Y]$.

If $X = Y$, $\mathrm{Cov}[X,Y] = \mathrm{Var}[X] = \mathrm{Var}[Y]$ *and* $r_{X,Y} = \mathrm{E}[X^2] = \mathrm{E}[Y^2]$.

4

Example

Find $r_{X,Y}$ and $\text{Cov}[X,Y]$ when the probability model for X and Y is given by the following matrix.

$P_{X,Y}(x,y)$	$y=0$	$y=1$	$y=2$	$P_X(x)$
$x=0$	0.01	0	0	0.01
$x=1$	0.09	0.09	0	0.18
$x=2$	0	0	0.81	0.81
$P_Y(y)$	0.10	0.09	0.81	

5

Orthogonal vs Uncorrelated Random Variables

Orthogonal Random Variables

*Random variables X and Y are **orthogonal** if $r_{X,Y}=0$.*

Uncorrelated Random Variables

*Random variables X and Y are **uncorrelated** if $\text{Cov}[X,Y]=0$.*

6

Correlation Coefficient

The ***correlation coefficient*** *of two random variables X and Y is*

$$\rho_{X,Y} = \frac{\mathrm{Cov}[X,Y]}{\sqrt{\mathrm{Var}[X]\,\mathrm{Var}[Y]}} = \frac{\mathrm{Cov}[X,Y]}{\sigma_X \sigma_Y}.$$

$$-1 \le \rho_{X,Y} \le 1$$

If X and Y are random variables such that $Y = aX + b$,

$$\rho_{X,Y} = \begin{cases} -1 & a < 0, \\ 0 & a = 0, \\ 1 & a > 0. \end{cases}$$

7

Correlation Coefficient

When $\rho_{X,Y} > 0$, we say that X and Y are *positively correlated*, and when $\rho_{X,Y} < 0$ we say X and Y are *negatively correlated*. If $|\rho_{X,Y}|$ is close to 1, say $|\rho_{X,Y}| \ge 0.9$, then X and Y are *highly correlated*. Note that high correlation can be positive or negative.

8

Independent Random Variable

*Random variables X and Y are **independent** if and only if*

$$\textit{Discrete:} \quad P_{X,Y}(x,y) = P_X(x)\,P_Y(y)\,;$$

$$\textit{Continuous:} \quad f_{X,Y}(x,y) = f_X(x)\,f_Y(y)\,.$$

9

Properties of Independent Random Variables

For independent random variables X and Y,

$\mathrm{E}[g(X)h(Y)] = \mathrm{E}[g(X)]\,\mathrm{E}[h(Y)]$,

$r_{X,Y} = \mathrm{E}[XY] = \mathrm{E}[X]\,\mathrm{E}[Y]$,

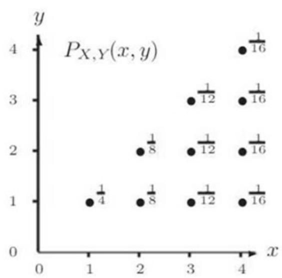
$\mathrm{Cov}[X,Y] = \rho_{X,Y} = 0$,

$\mathrm{Var}[X+Y] = \mathrm{Var}[X] + \mathrm{Var}[Y]$,

10

Example

The random variables X and Y have joint PMF



Find

(a) The expected values $\mathrm{E}[X]$ and $\mathrm{E}[Y]$,

(b) The variances $\mathrm{Var}[X]$ and $\mathrm{Var}[Y]$,

(c) The correlation, $r_{X,Y} = \mathrm{E}[XY]$,

(d) The covariance, $\mathrm{Cov}[X,Y]$,

(e) The correlation coefficient, $\rho_{X,Y}$.

11

References

1. Walpole, R.E., Myers, R.H., Myers, S.L. and Ye, K. (2007) *Probability & Statistics for Engineers & Scientists.* 9th Edition, Pearson Education, Inc.
2. Leon-Garcia, A. (2008). *Probability, Statistics, and Random Processes for Electrical Engineering*. 3rd Edition, Pearson/Prentice Hall.

12