Lecture 20

1. Source Free Series RLC Circuit

Source Free Series RLC

Consider the series *RLC* circuit shown in Fig. 8.8. The circuit is being excited by the energy initially stored in the capacitor and inductor. The energy is represented by the initial capacitor voltage V_0 and initial inductor current I_0. Thus, at $t = 0$,

$$v(0) = \frac{1}{C}\int_{-\infty}^{0} i\,dt = V_0 \tag{8.2a}$$

$$i(0) = I_0 \tag{8.2b}$$

Applying KVL around the loop in Fig. 8.8,

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau = 0 \tag{8.3}$$

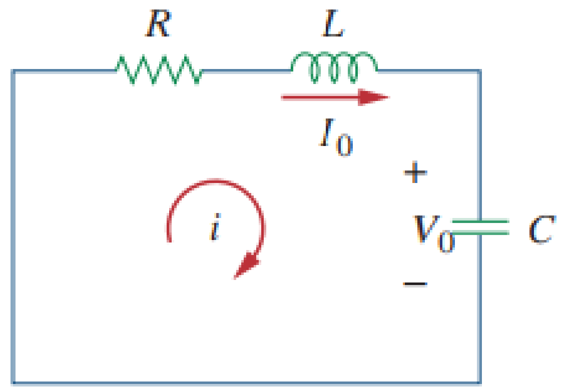


Figure 8.8
A source-free series *RLC* circuit.

To eliminate the integral, we differentiate with respect to t and rearrange terms. We get

$$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0 \tag{8.4}$$

This is a *second-order differential equation* and is the reason for calling the *RLC* circuits in this chapter second-order circuits. Our goal is to solve Eq. (8.4). To solve such a second-order differential equation requires that we have two initial conditions, such as the initial value of i and its first derivative or initial values of some i and v. The initial value of i is given in Eq. (8.2b). We get the initial value of the derivative of i from Eqs. (8.2a) and (8.3); that is,

$$Ri(0) + L\frac{di(0)}{dt} + V_0 = 0$$

or

$$\frac{di(0)}{dt} = -\frac{1}{L}(RI_0 + V_0) \tag{8.5}$$

With the two initial conditions in Eqs. (8.2b) and (8.5), we can now solve Eq. (8.4). Our experience in the preceding chapter on first-order circuits suggests that the solution is of exponential form. So we let

$$i = Ae^{st} \tag{8.6}$$

where A and s are constants to be determined. Substituting Eq. (8.6) into Eq. (8.4) and carrying out the necessary differentiations, we obtain

$$As^2e^{st} + \frac{AR}{L}se^{st} + \frac{A}{LC}e^{st} = 0$$

or

$$Ae^{st}\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right) = 0 \tag{8.7}$$

Since $i = Ae^{st}$ is the assumed solution we are trying to find, only the expression in parentheses can be zero:

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

This quadratic equation is known as the *characteristic equation* of the differential Eq. (8.4), since the roots of the equation dictate the character of i. The two roots of Eq. (8.8) are

$$s_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \tag{8.9a}$$

$$s_2 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \tag{8.9b}$$

A more compact way of expressing the roots is

$$\boxed{s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}, \qquad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}} \tag{8.10}$$

where

$$\boxed{\alpha = \frac{R}{2L}, \qquad \omega_0 = \frac{1}{\sqrt{LC}}} \tag{8.11}$$

The roots s_1 and s_2 are called *natural frequencies*, measured in nepers per second (Np/s), because they are associated with the natural response of the circuit; ω_0 is known as the *resonant frequency* or strictly as the *undamped natural frequency*, expressed in radians per second (rad/s); and α is the *neper frequency* or the *damping factor*, expressed in nepers per second. In terms of α and ω_0, Eq. (8.8) can be written as

$$s^2 + 2\alpha s + \omega_0^2 = 0 \tag{8.8a}$$

The variables s and ω_0 are important quantities we will be discussing throughout the rest of the text.

From Eq. (8.10), we can infer that there are three types of solutions:

1. If $\alpha > \omega_0$, we have the *overdamped* case.
2. If $\alpha = \omega_0$, we have the *critically damped* case.
3. If $\alpha < \omega_0$, we have the *underdamped* case.

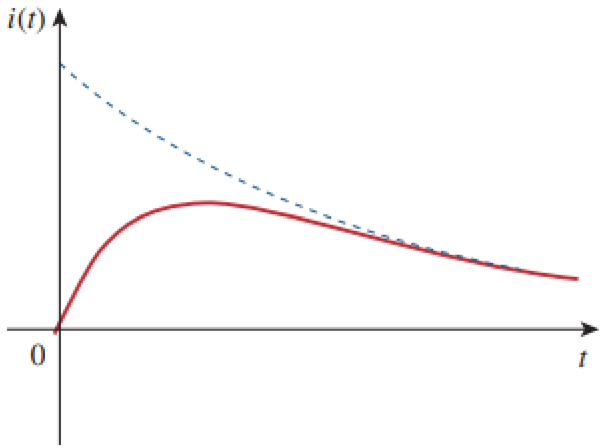
The response is *overdamped* when the roots of the circuit's characteristic equation are unequal and real, *critically damped* when the roots are equal and real, and *underdamped* when the roots are complex.

Overdamped Case ($\alpha > \omega_0$)

From Eqs. (8.9) and (8.10), $\alpha > \omega_0$ implies $C > 4L/R^2$. When this happens, both roots s_1 and s_2 are negative and real. The response is

$$\boxed{i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}} \tag{8.14}$$

which decays and approaches zero as t increases. Figure 8.9(a) illustrates a typical overdamped response.

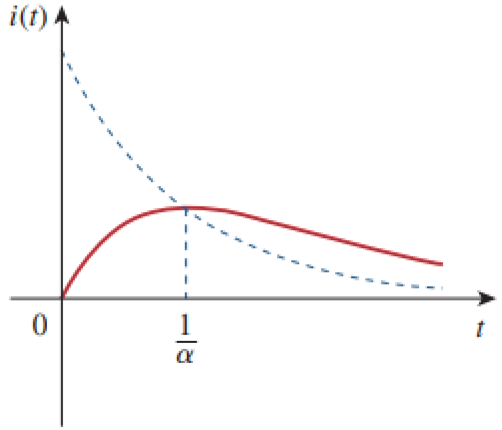


where the constants A_1 and A_2 are determined from the initial values $i(0)$ and $di(0)/dt$ in Eqs. (8.2b) and (8.5).

Critically Damped Case ($\alpha = \omega_0$)

When $\alpha = \omega_0$, $C = 4L/R^2$ and

$$s_1 = s_2 = -\alpha = -\frac{R}{2L} \tag{8.15}$$



Hence, the natural response of the critically damped circuit is a sum of two terms: a negative exponential and a negative exponential multiplied by a linear term, or

$$\boxed{i(t) = (A_2 + A_1 t)e^{-\alpha t}} \tag{8.21}$$

A typical critically damped response is shown in Fig. 8.9(b). In fact, Fig. 8.9(b) is a sketch of $i(t) = te^{-\alpha t}$, which reaches a maximum value of e^{-1}/α at $t = 1/\alpha$, one time constant, and then decays all the way to zero.

Underdamped Case ($\alpha < \omega_0$)

For $\alpha < \omega_0$, $C < 4L/R^2$. The roots may be written as

$$s_1 = -\alpha + \sqrt{-(\omega_0^2 - \alpha^2)} = -\alpha + j\omega_d \tag{8.22a}$$

$$s_2 = -\alpha - \sqrt{-(\omega_0^2 - \alpha^2)} = -\alpha - j\omega_d \tag{8.22b}$$

where $j = \sqrt{-1}$ and $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$, which is called the *damping frequency*. Both ω_0 and ω_d are natural frequencies because they help determine the natural response; while ω_0 is often called the *undamped natural frequency*, ω_d is called the *damped natural frequency*. The natural response is

$$\begin{aligned} i(t) &= A_1 e^{-(\alpha - j\omega_d)t} + A_2 e^{-(\alpha + j\omega_d)t} \\ &= e^{-\alpha t}(A_1 e^{j\omega_d t} + A_2 e^{-j\omega_d t}) \end{aligned} \tag{8.23}$$

Using Euler's identities,

$$e^{j\theta} = \cos\theta + j\sin\theta, \qquad e^{-j\theta} = \cos\theta - j\sin\theta \tag{8.24}$$

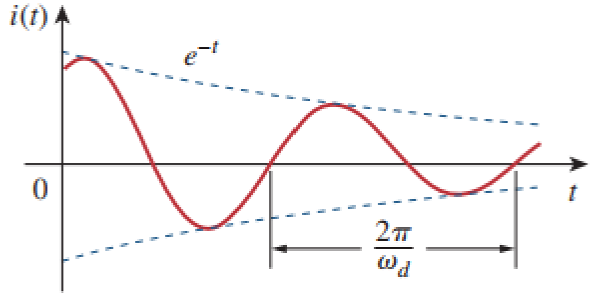
we get

$$\begin{aligned} i(t) &= e^{-\alpha t}[A_1(\cos\omega_d t + j\sin\omega_d t) + A_2(\cos\omega_d t - j\sin\omega_d t)] \\ &= e^{-\alpha t}[(A_1 + A_2)\cos\omega_d t + j(A_1 - A_2)\sin\omega_d t] \end{aligned} \tag{8.25}$$

Replacing constants $(A_1 + A_2)$ and $j(A_1 - A_2)$ with constants B_1 and B_2, we write

$$\boxed{i(t) = e^{-\alpha t}(B_1\cos\omega_d t + B_2\sin\omega_d t)} \tag{8.26}$$

With the presence of sine and cosine functions, it is clear that the natural response for this case is exponentially damped and oscillatory in nature.



Example 8.3

In Fig. 8.8, $R = 40\ \Omega$, $L = 4$ H, and $C = 1/4$ F. Calculate the characteristic roots of the circuit. Is the natural response overdamped, underdamped, or critically damped?

Solution:

We first calculate

$$\alpha = \frac{R}{2L} = \frac{40}{2(4)} = 5, \qquad \omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{4 \times \frac{1}{4}}} = 1$$

The roots are

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -5 \pm \sqrt{25 - 1}$$

or

$$s_1 = -0.101, \qquad s_2 = -9.899$$

Since $\alpha > \omega_0$, we conclude that the response is overdamped. This is also evident from the fact that the roots are real and negative.

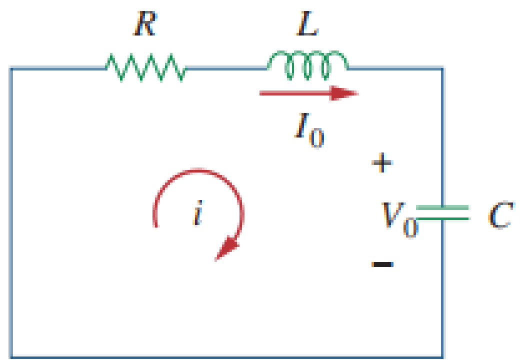


Figure 8.8
A source-free series *RLC* circuit.

Example 8.4

Find $i(t)$ in the circuit of Fig. 8.10. Assume that the circuit has reached steady state at $t = 0^-$.

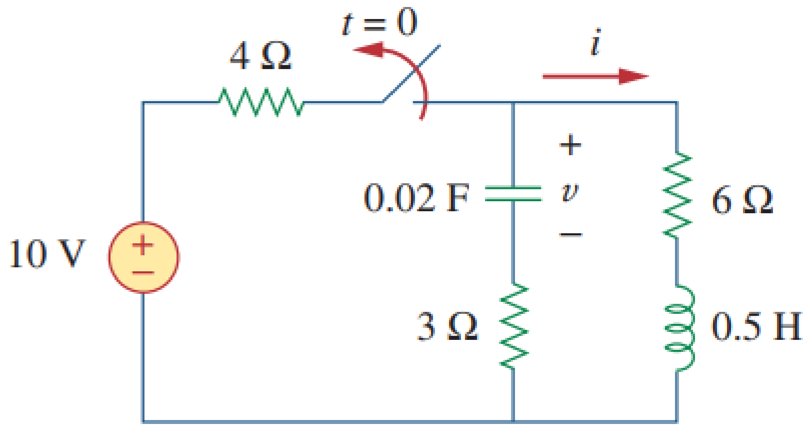


Figure 8.10
For Example 8.4.

Things to Remember !!!

A more compact way of expressing the roots is

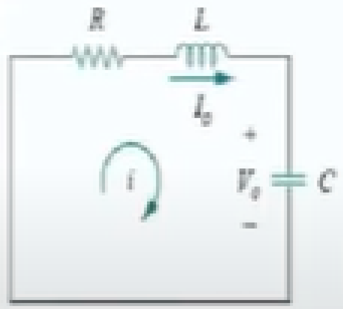
$$\boxed{s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}, \qquad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}} \tag{10}$$

where

$$\boxed{\alpha = \frac{R}{2L}, \qquad \omega_0 = \frac{1}{\sqrt{LC}}} \tag{11}$$

Overdamped Case ($\alpha > \omega_0$)

$$\boxed{i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}}$$



Critically Damped Case ($\alpha = \omega_0$)

$$\boxed{i(t) = (A_2 + A_1 t)e^{-\alpha t}}$$

Underdamped Case ($\alpha < \omega_0$)

$$\boxed{i(t) = e^{-\alpha t}(A_1 \cos \omega_d t + A_2 \sin \omega_d t)}$$

Problem solving strategy

1. For $t < 0$, **we have to** find the initial conditions $i(0)$ and $v(0)$

2. at t > 0 We write the KVL equation, and then putting t=0, we get

$$Ri(0) + L\frac{di(0)}{dt} + v(0) = 0$$ which is used to find $\left.\frac{di}{dt}\right|_{t=0}$

3. For $t > 0$, calculate α, ω_0.

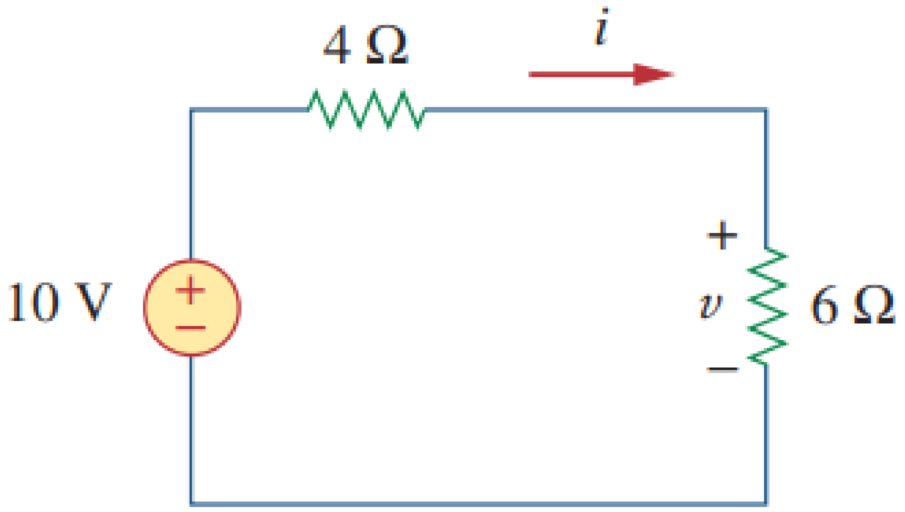
4. Based on the value of α and ω_0, choose the equation for current $i(t)$.

   (Calculate S1,S2, or Wd, if required bt the equation)

   and plug in values in it.

5. Calculate the value of A1 and A2, and plug it in to get the final answer.

- 1) For t<0, we have to find initial conditions $i(0)$ and $v(0)$



For $t < 0$, the switch is closed. The capacitor acts like an open circuit while the inductor acts like a shunted circuit. The equivalent circuit is shown in Fig. 8.11(a). Thus, at $t = 0$,

$$i(0) = \frac{10}{4 + 6} = 1 \text{ A}, \qquad v(0) = 6i(0) = 6 \text{ V}$$

- 2) For t>0, we have to find $\frac{di}{dt}$ at $t = 0$

Applying KVL to the circuit

$$Ri(t) + L\frac{di(t)}{dt} - V_C(t) = 0$$

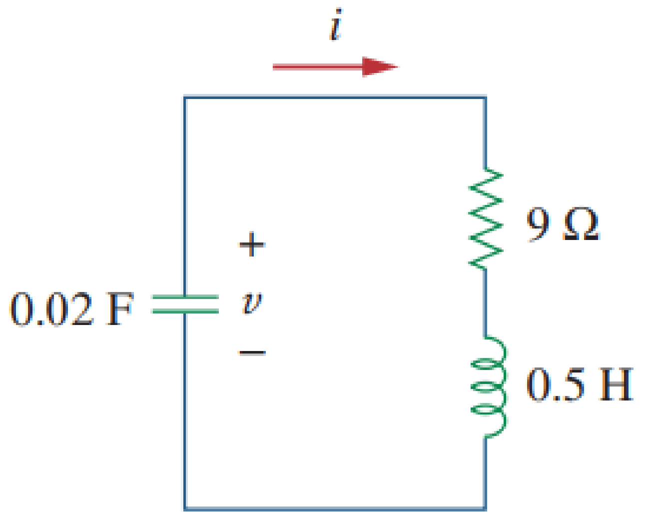
at t =0

$$Ri(0) + L\frac{di(0)}{dt} - V_0 = 0$$

$$9 \times 1 + 0.5\frac{di(0)}{dt} - 6 = 0$$

$$0.5\frac{di(0)}{dt} = -3$$

$$\frac{di(0)}{dt} = -6 \text{ A/s}$$



- **3) For t>0, we have to find α and ω_o**

$$\alpha = \frac{R}{2L} = \frac{9}{2(\frac{1}{2})} = 9, \qquad \omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\frac{1}{2} \times \frac{1}{50}}} = 10$$

4. Based on the value of α and ω_0, choose the equation for current $i(t)$.
(Calculate S1,S2, or Wd, if required bt the equation)
and plug in values in it.

Since $\alpha < \omega_o$ the case is underdamped. We will use

$$\boxed{i(t) = e^{-\alpha t}(B_1 \cos \omega_d t + B_2 \sin \omega_d t)}$$

Where

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2}, \quad = \sqrt{100 - 81} \quad = \sqrt{19} \quad = 4.359$$

Hence, the response is underdamped ($\alpha < \omega$); that is,

$$i(t) = e^{-9t}(A_1 \cos 4.359t + A_2 \sin 4.359t)$$

5. Calculate the value of A1 and A2, and plug it in to get the final answer.

$$i(t) = e^{-9t}(A_1 \cos 4.359t + A_2 \sin 4.359t) \quad (4.1)$$

Putting t= 0

$$i(t) = e^{-9t}(A_1 \cos 4.359t + A_2 \sin 4.359t)$$

$$i(0) = e^{0}(A_1 \cos 0 \quad + A_2 \sin 0 \quad)$$

or $i(0) = A_1$

or $i(0) = A_1$

But we found out earlier that i(0) = 1

Hence, $A_1 = 1$

We need another equation to find A2 , which we get by taking the derivative of $i(t)$

$$\frac{di}{dt} = -9e^{-9t}(A_1 \cos 4.359t + A_2 \sin 4.359t)$$
$$+ e^{-9t}(4.359)(-A_1 \sin 4.359t + A_2 \cos 4.359t)$$

Imposing the condition in Eq. (8.4.3) at $t = 0$ gives

$$-6 = -9(A_1 + 0) + 4.359(-0 + A_2)$$

But $A_1 = 1$ from Eq. (8.4.2). Then

$$-6 = -9 + 4.359A_2 \quad \Rightarrow \quad A_2 = 0.6882$$

Substituting the values of A_1 and A_2 in Eq. (8.4.1) yields the complete solution as

$$i(t) = e^{-9t}(\cos 4.359t + 0.6882 \sin 4.359t) \text{ A}$$

Practice Problem 8.4

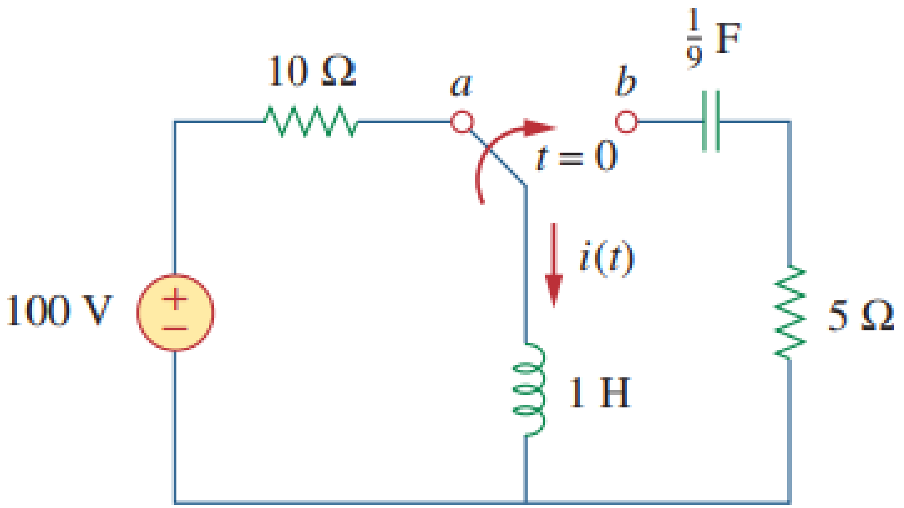


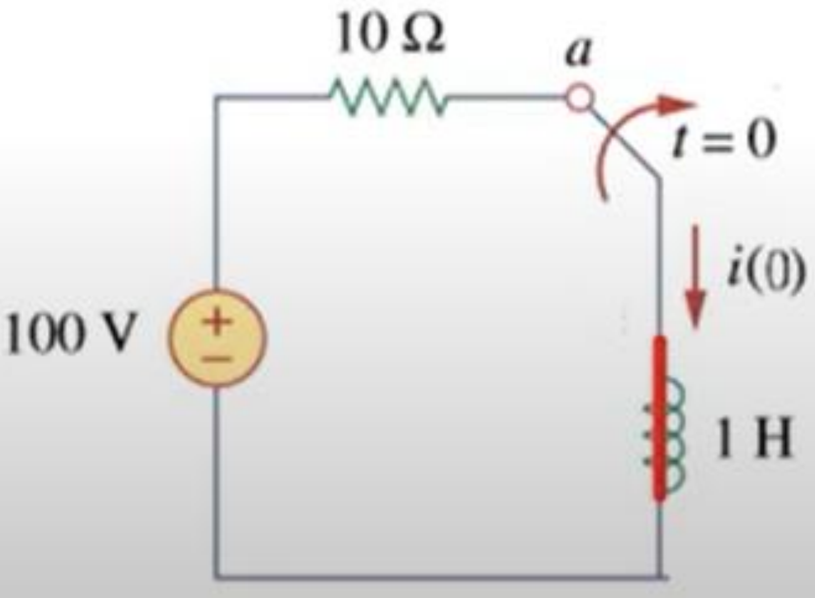
Figure 8.12
For Practice Prob. 8.4.

The circuit in Fig. 8.12 has reached steady state at $t = 0^-$. If the make-before-break switch moves to position b at $t = 0$, calculate $i(t)$ for $t > 0$.

Solution:

1. For $t < 0$, **we have to** find the initial conditions $i(0)$ and $v(0)$

at $t = 0^-$

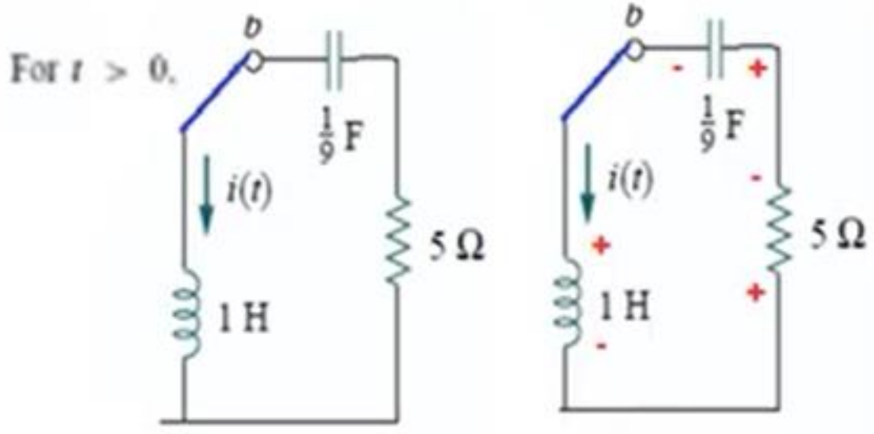


$$i(0) = \frac{100}{10} = 10A$$

$$v(0) = 0\ V$$

2. at t > 0 We write the KVL equation, and then putting t=0, we get

$$Ri(0) + L\frac{di(0)}{dt} + v(0) = 0$$ which is used to find $\left.\frac{di}{dt}\right|_{t=0}$



Applying KVL to the circuit

$$-L\frac{di(t)}{dt} - V_C(t) - Ri(t) = 0$$

$$Ri(t) + L\frac{di(t)}{dt} + V_C(t) = 0$$

Putting t = 0

$$Ri(0) + L\frac{di(0)}{dt} + V_0 = 0$$

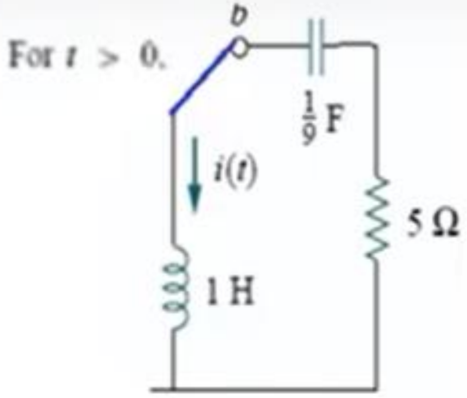
$$5 \times 10 + 1 \times \frac{di(0)}{dt} + 0 = 0$$

$$\frac{di(0)}{dt} = -50 \text{ A/s}$$

i(0) = 10A

v(0) = 0 V

3. For $t > 0$, calculate α , ω_0



$$\alpha = \frac{R}{2L} = \frac{5}{2 \times 1} = 2.5$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{1 \times \frac{1}{9}}} = 3$$

4. Based on the value of α and ω_0, choose the equation for current $i(t)$.

(Calculate S1,S2, or Wd, if required bt the equation)

and plug in values in it.

Since $\alpha < \omega_0$, we use

$$\boxed{i(t) = e^{-\alpha t}(A_1 \cos \omega_d t + A_2 \sin \omega_d t)}$$

Where $\omega_d = \sqrt{\omega_0^2 - \alpha^2} = \sqrt{9 - 6.25} = 1.6583$

$$i(t) = e^{-2.5t}(A_1 \cos 1.6583t + A_2 \sin 1.6583t)$$

We now obtain A_1 and A_2 using the initial conditions. At $t = 0$,

$$i(t) = e^{-2.5t}(A_1 \cos 1.6583t + A_2 \sin 1.6583t)$$

Putting t= 0

$$i(0) = e^{0}(A_1 \cos 0 \; + A_2 \sin 0 \;)$$

or $i(0) = A_1$

But we found out earlier that i(0) = **10A**

Hence, $A_1 = 10$

We need another equation to find A2 ,
which we get by taking the derivative of $i(t)$

Product Rule

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

$$\frac{di}{dt} = -2.5e^{-2.5t}(A_1 \cos 1.6583t + A_2 \sin 1.6583t) + e^{-2.5t}(1.6583)(-A_1 \sin 1.6583t + A_2 \cos 1.6583t)$$

$$\frac{di}{dt} = -2.5\, e^{-2.5t}\,(A_1 \cos 1.6583t + A_2 \sin 1.6583t)$$
$$+\, e^{-2.5t}\,(1.6583)\,(-A_1 \sin 1.6583\, t + A_2 \cos 1.6583t)$$

at $t = 0$

$$\left.\frac{di}{dt}\right|_{t=0} = -2.5\ (A_1 + 0) + (1.6583)\,(-0 + A_2)$$

But $\left.\frac{di}{dt}\right|_{t=0} = -50$ A/s (calculated earlier- step2..,)

Hence, $-50 = -2.5\,(A_1) + 1.6583(A_2)$

But $A_1 = 10$

Then $-50 = -2.5 \times 10 + 1.6583(A_2)$

$$\Rightarrow \quad A_2 = -15.07$$

Substituting the values of A_1 and A_2

$$i(t) = e^{-2.5t}(\ 10 \cos 1.6583t\ -15.07\ \sin 1.6583t)$$

Answer: $e^{-2.5t}(10 \cos 1.6583t - 15.076 \sin 1.6583t)$ A.

Thanks