Lecture 6: Cumulative Distribution Function and Its Properties

CPE251 Probability Methods in Engineering

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Cumulative Distribution Function of Discrete Random Variable

The cumulative distribution function $F_X(x)$ of a random variable X is given by

$$F_X(x) = \sum_{t \leq x} p_X(t)$$

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Example

The probability distribution of X, the number of imperfections per 10 meters of a synthetic fabric in continuous rolls of uniform width, is given by

x	0	1	2	3	4
$f(x)$	0.41	0.37	0.16	0.05	0.01

Construct the cumulative distribution function of X.

Cumulative Distribution Function of Continuous Random Variable

The cumulative distribution function $F_X(x)$ of a continuous random variable X is defined by

$$F_X(x) = P[X \le x] = \int_{-\infty}^{x} f(t)dt\,, \quad -\infty \le x \le \infty$$

Example

The Department of Energy (DOE) puts projects out on bid and generally estimates what a reasonable bid should be. Call the estimate b. The DOE has determined that the density function of the winning (low) bid is

$$f(y) = \begin{cases} \frac{5}{8b}, & \frac{2}{5}b \le y \le 2b, \\ 0, & \text{elsewhere.} \end{cases}$$

Find $F(y)$ and use it to determine the probability that the winning bid is less than the DOE's preliminary estimate b.

For $2b/5 \le y \le 2b$,

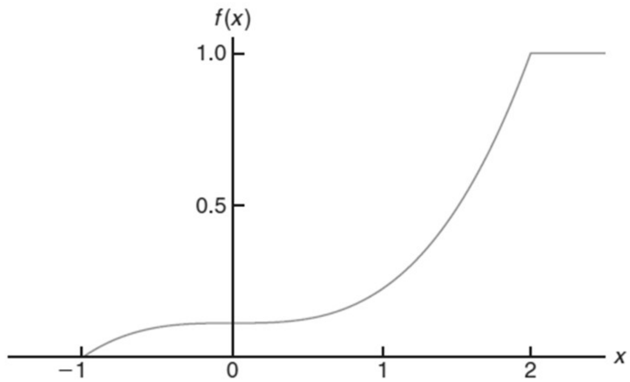
$$F(y) = \int_{2b/5}^{y} \frac{5}{8b} dy = \left. \frac{5t}{8b} \right|_{2b/5}^{y} = \frac{5y}{8b} - \frac{1}{4}.$$

Thus,

$$F(y) = \begin{cases} 0, & y < \frac{2}{5}b, \\ \frac{5y}{8b} - \frac{1}{4}, & \frac{2}{5}b \le y < 2b, \\ 1, & y \ge 2b. \end{cases}$$

To determine the probability that the winning bid is less than the preliminary bid estimate b, we have

$$P(Y \le b) = F(b) = \frac{5}{8} - \frac{1}{4} = \frac{3}{8}.$$



Representing cdf

Algebraic: conditions for range below lower bound and the range above upper bound

Graphical: using a 2D plot

EXERCISE 1

On a laboratory assignment, if the equipment is working, the density function of the observed outcome, X, is

$$f(x) = \begin{cases} 2(1-x), & 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Calculate $P(X \leq 1/3)$.
(b) What is the probability that X will exceed 0.5?
(c) Given that $X \geq 0.5$, what is the probability that X will be less than 0.75?

EXERCISE 2

Suppose a certain type of small data processing firm is so specialized that some have difficulty making a profit in their first year of operation. The probability density function that characterizes the proportion Y that make a profit is given by

$$f(y) = \begin{cases} ky^4(1-y)^3, & 0 \leq y \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) What is the value of k that renders the above a valid density function?
(b) Find the probability that at most 50% of the firms make a profit in the first year.
(c) Find the probability that at least 80% of the firms make a profit in the first year.

Properties of CDF

For any random variable X,

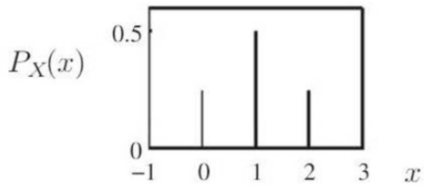
(a) $F_X(-\infty) = 0$ *(b)* $F_X(\infty) = 1$

(c) $P[x_1 < X \leq x_2] = F_X(x_2) - F_X(x_1)$

For all $x' \geq x$, $F_X(x') \geq F_X(x)$.

For a discrete random variable X, there is a jump (discontinuity) at each value of $x_i \in S_X$. The height of the jump at x_i is $P_X(x_i)$.

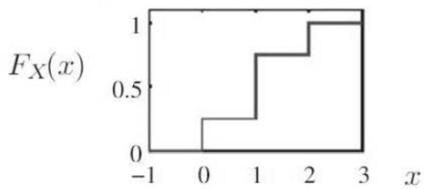
Example



$$P_X(x) = \begin{cases} 1/4 & x = 0, \\ 1/2 & x = 1, \\ 1/4 & x = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Find and sketch the CDF of random variable X.

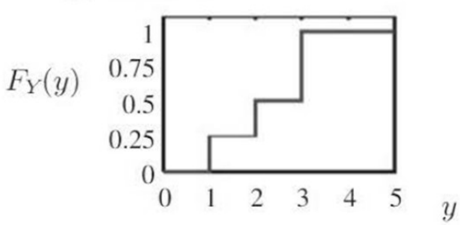
Referring to the PMF $P_X(x)$, we derive the CDF of random variable X:



$$F_X(x) = \mathrm{P}[X \le x] = \begin{cases} 0 & x < 0, \\ 1/4 & 0 \le x < 1, \\ 3/4 & 1 \le x < 2 \\ 1 & x \ge 2. \end{cases}$$

Exercise

Discrete random variable Y has the CDF $F_Y(y)$ as shown:



Use the CDF to find the following:

(a) $\mathrm{P}[Y < 1]$ and $\mathrm{P}[Y \le 1]$ (b) $\mathrm{P}[Y > 2]$ and $\mathrm{P}[Y \ge 2]$

(c) $\mathrm{P}[Y = 3]$ and $\mathrm{P}[Y > 3]$ (d) $P_Y(y)$

Exercise

The cumulative distribution function of the random variable Y is

$$F_Y(y) = \begin{cases} 0 & y < 0, \\ y/4 & 0 \leq y \leq 4, \\ 1 & y > 4. \end{cases}$$

Sketch the CDF of Y and calculate the following probabilities:

(a) $P[Y \leq -1]$

(b) $P[Y \leq 1]$

(c) $P[2 < Y \leq 3]$

(d) $P[Y > 1.5]$

References

1. Yates, Goodman, *Probability and Stochastic Processes: A Friendly Introduction for Electrical and Computer Engineers*, 3rd edition