Lecture 19

Source Driven RC Circuits
(Step Response of an RC Circuit)

When the dc source of an *RC* circuit is suddenly applied, the voltage or current source can be modeled as a step function, and the response is known as a *step response*.

> The **step response** of a circuit is its behavior when the excitation is the step function, which may be a voltage or a current source.

The step response is the response of the circuit due to a sudden application of a dc voltage or current source.

Consider the RC circuit in Fig. 7.40(a) which can be replaced by the circuit in Fig. 7.40(b), where V_s is a constant dc voltage source. Again, we select the capacitor voltage as the circuit response to be determined. We assume an initial voltage V_0 on the capacitor, although this is not necessary for the step response. Since the voltage of a capacitor cannot change instantaneously,

$$v(0^-) = v(0^+) = V_0 \tag{7.40}$$

where $v(0^-)$ is the voltage across the capacitor just before switching and $v(0^+)$ is its voltage immediately after switching. Applying KCL, we have

$$C\frac{dv}{dt} + \frac{v - V_s u(t)}{R} = 0$$

or

$$\frac{dv}{dt} + \frac{v}{RC} = \frac{V_s}{RC}u(t) \tag{7.41}$$

where v is the voltage across the capacitor. For $t > 0$, Eq. (7.41) becomes

$$\frac{dv}{dt} + \frac{v}{RC} = \frac{V_s}{RC} \tag{7.42}$$

Rearranging terms gives

$$\frac{dv}{dt} = -\frac{v - V_s}{RC}$$

or

$$\frac{dv}{v - V_s} = -\frac{dt}{RC} \tag{7.43}$$

Integrating both sides and introducing the initial conditions,

$$\ln(v - V_s)\Big|_{V_0}^{v(t)} = -\frac{t}{RC}\Big|_0^t$$

$$\ln(v(t) - V_s) - \ln(V_0 - V_s) = -\frac{t}{RC} + 0$$

or

$$\ln\frac{v - V_s}{V_0 - V_s} = -\frac{t}{RC} \tag{7.44}$$

Taking the exponential of both sides

$$\frac{v - V_s}{V_0 - V_s} = e^{-t/\tau}, \qquad \tau = RC$$

$$v - V_s = (V_0 - V_s)e^{-t/\tau}$$

or

$$v(t) = V_s + (V_0 - V_s)e^{-t/\tau}, \qquad t > 0 \tag{7.45}$$

Thus

Thus,

$$\boxed{v(t) = \begin{cases} V_0, & t < 0 \\ V_s + (V_0 - V_s)e^{-t/\tau}, & t > 0 \end{cases}} \tag{7.46}$$

Classically there are two ways of decomposing this into two components. The first is to break it into a "natural response and a forced response" and the second is to break it into a "transient response and a steady-state response." Starting with the natural response and forced response, we write the total or complete response as

$$\boxed{\text{Complete response} = \underset{\text{stored energy}}{\text{natural response}} + \underset{\text{independent source}}{\text{forced response}}}$$

or

$$v = v_n + v_f \tag{7.50}$$

where

$$v_n = V_o e^{-t/\tau}$$

and

$$v_f = V_s(1 - e^{-t/\tau})$$

We are familiar with the natural response v_n of the circuit, as discussed in Section 7.2. v_f is known as the *forced* response because it is produced by the circuit when an external "force" (a voltage source in this case) is applied. It represents what the circuit is forced to do by the input excitation. The natural response eventually dies out along with the transient component of the forced response, leaving only the steady-state component of the forced response.

Another way of looking at the complete response is to break into two components—one temporary and the other permanent, i.e.,

$$\boxed{\text{Complete response} = \underbrace{\text{transient response}}_{\text{temporary part}} + \underbrace{\text{steady-state response}}_{\text{permanent part}}}$$

The *transient response* v_t is temporary; it is the portion of the complete response that decays to zero as time approaches infinity. Thus,

> The **transient response** is the circuit's temporary response that will die out with time.

The *steady-state response* v_{ss} is the portion of the complete response that remains after the transient reponse has died out. Thus,

> The **steady-state response** is the behavior of the circuit a long time after an external excitation is applied.

The first decomposition of the complete response is in terms of the source of the responses, while the second decomposition is in terms of the permanency of the responses. Under certain conditions, the natural response and transient response are the same. The same can be said about the forced response and steady-state response.

Whichever way we look at it, the complete response in Eq. (7.45) may be written as

$$\boxed{v(t) = v(\infty) + [v(0) - v(\infty)]e^{-t/\tau}} \tag{7.53}$$

where $v(0)$ is the initial voltage at $t = 0^+$ and $v(\infty)$ is the final or steady-state value. Thus, to find the step response of an RC circuit requires three things:

1. The initial capacitor voltage $v(0)$.
2. The final capacitor voltage $v(\infty)$.
3. The time constant τ.

We obtain item 1 from the given circuit for $t < 0$ and items 2 and 3 from the circuit for $t > 0$. Once these items are determined, we obtain

Example 7.10

The switch in Fig. 7.43 has been in position A for a long time. At $t = 0$, the switch moves to B. Determine $v(t)$ for $t > 0$ and calculate its value at $t = 1$ s and 4 s.

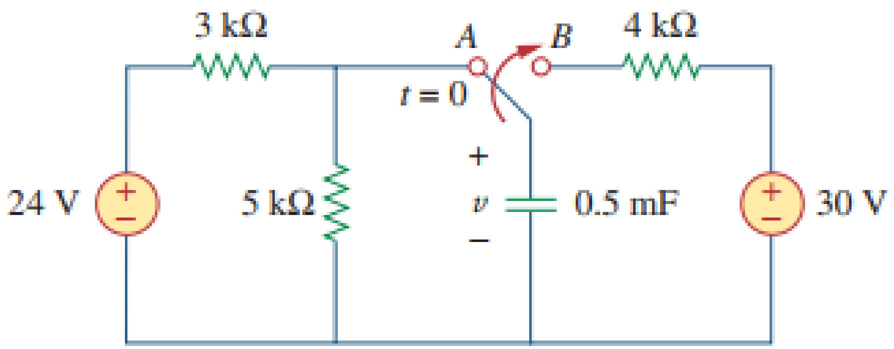


Figure 7.43
For Example 7.10.

Solution:

For $t < 0$, the switch is at position A. The capacitor acts like an open circuit to dc, but v is the same as the voltage across the 5-kΩ resistor. Hence, the voltage across the capacitor just before $t = 0$ is obtained by voltage division as

$$v(0^-) = \frac{5}{5 + 3}(24) = 15 \text{ V}$$

Using the fact that the capacitor voltage cannot change instantaneously,

$$v(0) = v(0^-) = v(0^+) = 15 \text{ V}$$

For $t > 0$, the switch is in position B. The Thevenin resistance connected to the capacitor is $R_{\text{Th}} = 4\ \text{k}\Omega$, and the time constant is

$$\tau = R_{\text{Th}}C = 4 \times 10^3 \times 0.5 \times 10^{-3} = 2\ \text{s}$$

Since the capacitor acts like an open circuit to dc at steady state, $v(\infty) = 30$ V. Thus,

$$\begin{aligned} v(t) &= v(\infty) + [v(0) - v(\infty)]e^{-t/\tau} \\ &= 30 + (15 - 30)e^{-t/2} = (30 - 15e^{-0.5t})\ \text{V} \end{aligned}$$

At $t = 1$,

$$v(1) = 30 - 15e^{-0.5} = 20.9\ \text{V}$$

At $t = 4$,

$$v(4) = 30 - 15e^{-2} = 27.97\ \text{V}$$

Practice Problem 7.10

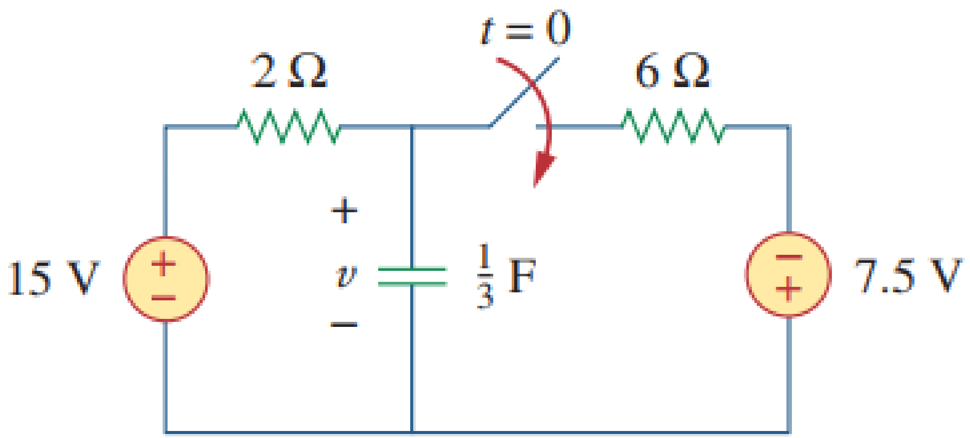


Figure 7.44
For Practice Prob. 7.10.

Find $v(t)$ for $t > 0$ in the circuit of Fig. 7.44. Assume the switch has been open for a long time and is closed at $t = 0$. Calculate $v(t)$ at $t = 0.5$.

Answer: $(9.375 + 5.625e^{-2t})$ V for all $t > 0$, 7.63 V.

Example 7.11

In Fig. 7.45, the switch has been closed for a long time and is opened at $t = 0$. Find i and v for all time.

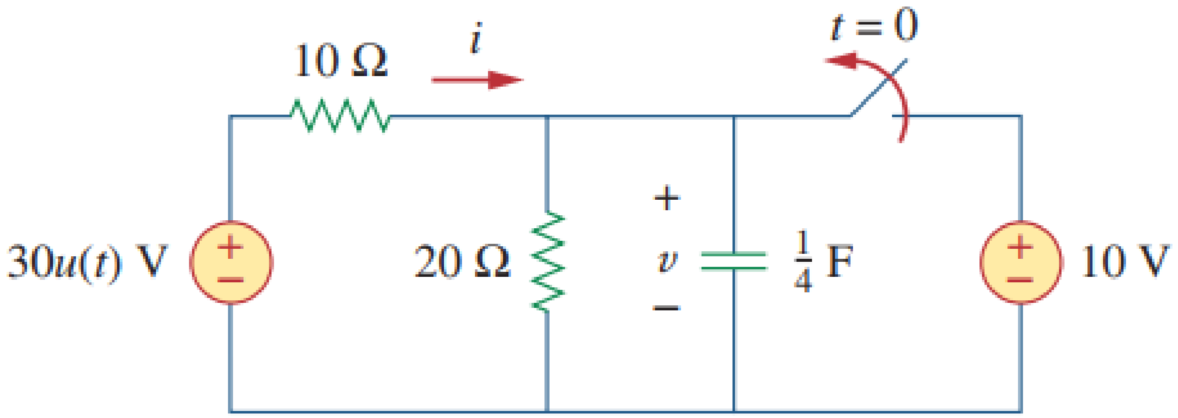
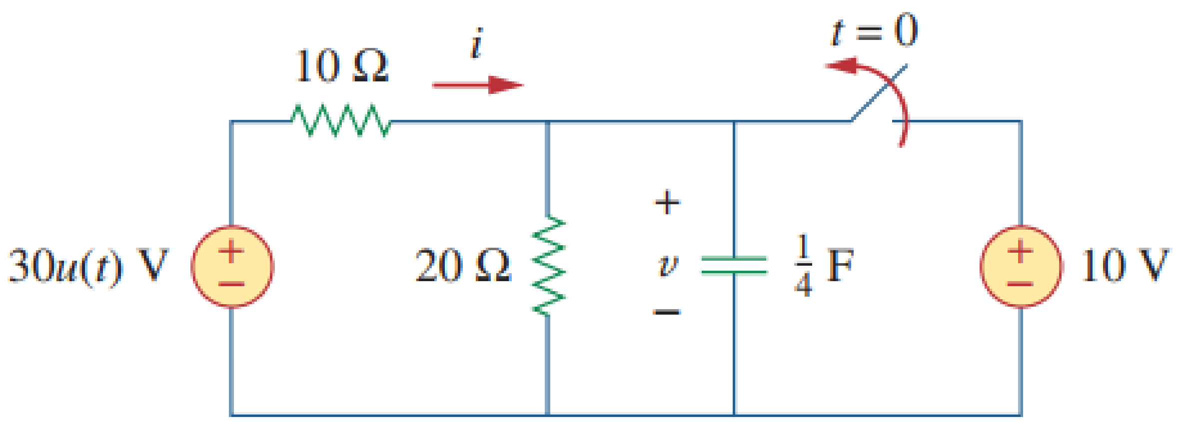
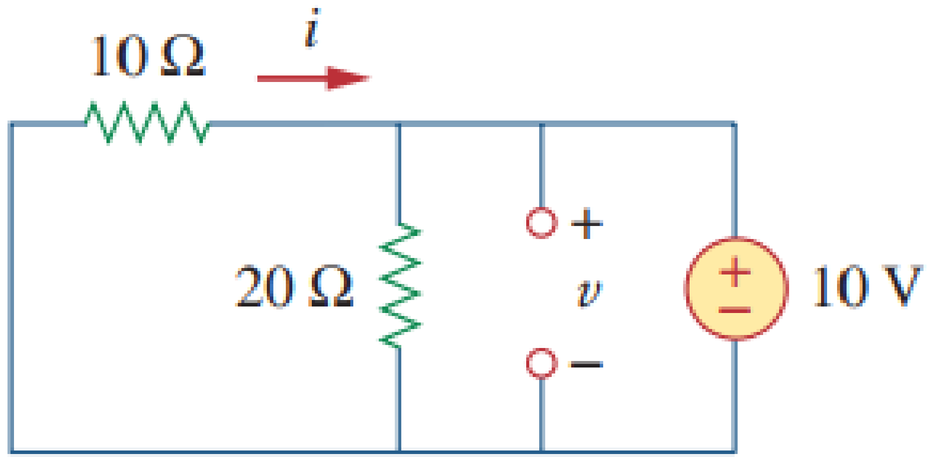


Figure 7.45
For Example 7.11.





(a)

By definition of the unit step function,

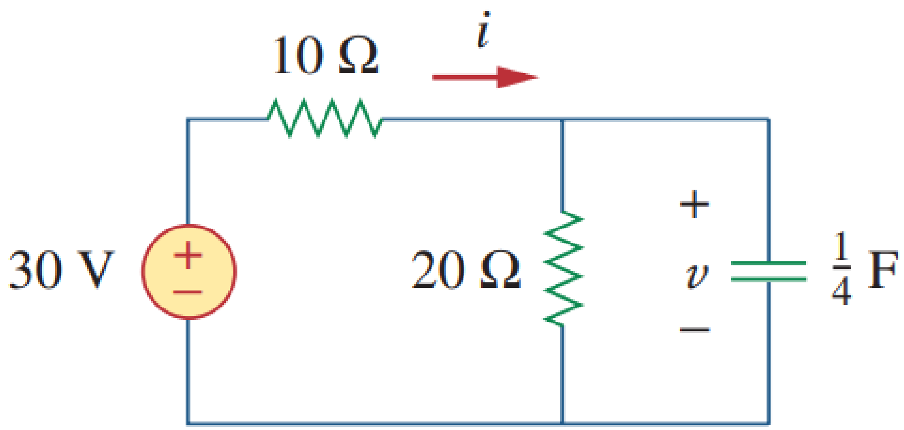
$$30u(t) = \begin{cases} 0, & t < 0 \\ 30, & t > 0 \end{cases}$$

For $t < 0$, the switch is closed and $30u(t) = 0$, so that the $30u(t)$ voltage source is replaced by a short circuit and should be regarded as contributing nothing to v. Since the switch has been closed for a long time, the capacitor voltage has reached steady state and the capacitor acts like an open circuit. Hence, the circuit becomes that shown in Fig. 7.46(a) for $t < 0$. From this circuit we obtain

$$v = 10 \text{ V}, \qquad i = -\frac{v}{10} = -1 \text{ A}$$

Since the capacitor voltage cannot change instantaneously,

$$v(0) = v(0^-) = 10 \text{ V}$$



For $t > 0$, the switch is opened and the 10-V voltage source is disconnected from the circuit. The $30u(t)$ voltage source is now operative, so the circuit becomes that shown in Fig. 7.46(b). After a long time, the circuit reaches steady state and the capacitor acts like an open circuit again. We obtain $v(\infty)$ by using voltage division, writing

$$v(\infty) = \frac{20}{20 + 10}(30) = 20 \text{ V}$$

The Thevenin resistance at the capacitor terminals is

$$R_{\text{Th}} = 10 \parallel 20 = \frac{10 \times 20}{30} = \frac{20}{3}\,\Omega$$

and the time constant is

$$\tau = R_{\text{Th}}C = \frac{20}{3} \cdot \frac{1}{4} = \frac{5}{3}\text{ s}$$

Thus,

$$\begin{aligned} v(t) &= v(\infty) + [v(0) - v(\infty)]e^{-t/\tau} \\ &= 20 + (10 - 20)e^{-(3/5)t} = (20 - 10e^{-0.6t}) \text{ V} \end{aligned}$$

To obtain i, we notice from Fig. 7.46(b) that i is the sum of the currents through the 20-Ω resistor and the capacitor; that is,

$$\begin{aligned} i &= \frac{v}{20} + C\frac{dv}{dt} \\ &= 1 - 0.5e^{-0.6t} + 0.25(-0.6)(-10)e^{-0.6t} = (1 + e^{-0.6t}) \text{ A} \end{aligned}$$

Notice from Fig. 7.46(b) that $v + 10i = 30$ is satisfied, as expected. Hence,

$$v = \begin{cases} 10 \text{ V}, & t < 0 \\ (20 - 10e^{-0.6t}) \text{ V}, & t \geq 0 \end{cases}$$

$$i = \begin{cases} -1 \text{ A}, & t < 0 \\ (1 + e^{-0.6t}) \text{ A}, & t > 0 \end{cases}$$

Practice Problem 7.11

The switch in Fig. 7.47 is closed at $t = 0$. Find $i(t)$ and $v(t)$ for all time. Note that $u(-t) = 1$ for $t < 0$ and 0 for $t > 0$. Also, $u(-t) = 1 - u(t)$.

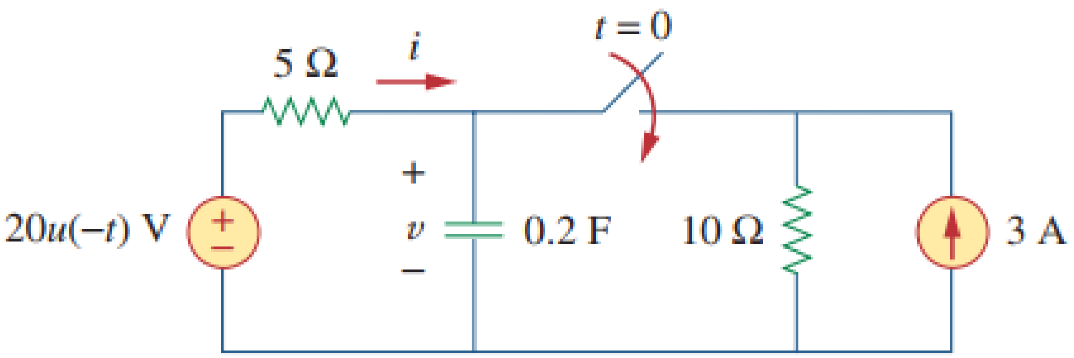
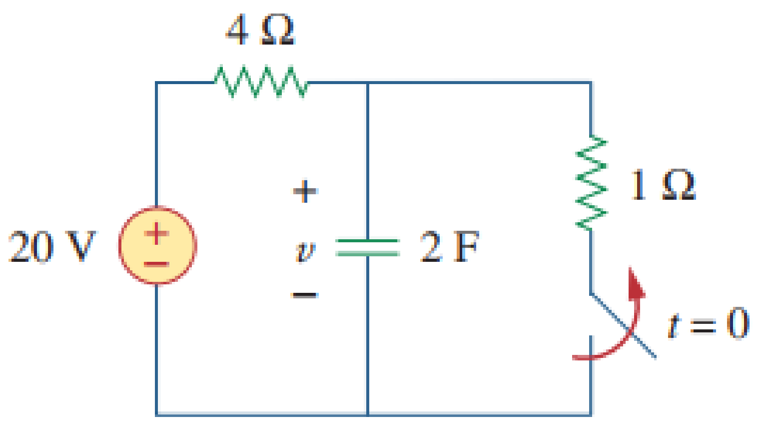


Figure 7.47
For Practice Prob. 7.11.

7.39 Calculate the capacitor voltage for $t < 0$ and $t > 0$ for each of the circuits in Fig. 7.106.



(a)

(a) Before t = 0,

$$v(t) = \frac{1}{4+1}(20) = \underline{\mathbf{4\ V}}$$

After t = 0,

$$v(t) = v(\infty) + \left[v(0) - v(\infty)\right] e^{-t/\tau}$$

$$\tau = RC = (4)(2) = 8, \quad v(0) = 4, \quad v(\infty) = 20$$

$$v(t) = 20 + (4 - 20)\,e^{-t/8}$$

$$v(t) = \underline{20 - 16e^{-t/8}\ \text{V}}$$

Thanks