Lecture 2: Concepts in Axiomatic Probability

CPE251 Probability Methods in Engineering

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Random Experiments

Random Experiment: A process that generates data with uncertainty. A random experiment is an experiment repeated under the same conditions.

Observation: Recording and information

Experiment	Observation
Launching a missile	Velocity at specified times
Transmitting data bits	Number of errors
Tossing a coin	Number of heads
Throwing a dice	An even number

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Random Experiments

Outcome or Sample Point: Any possible observation ζ that cannot be decomposed into other results – *finest grain*

Sample Space: Set S of *all* outcomes of a random experiment; or alternatively, we can say: $\zeta \in S$.

Discrete vs Continuous Sample Space: *Discrete* if *countable*, *Continuous* if *Uncountable*

Finite vs Infinite Sample Space: *Finite* if the number of outcomes is finite, *infinite* if the number of outcomes are not finite

$S_1 = \{HH, HT, TH, TT\}$ Discrete and Finite
$S_2 = \{1,2,3, \dots\}$ Discrete and Infinite
$S_3 = \{x | 0 \leq x \leq 1\}$ Continuous and Infinite

Visualizing the Sample Space

Tables

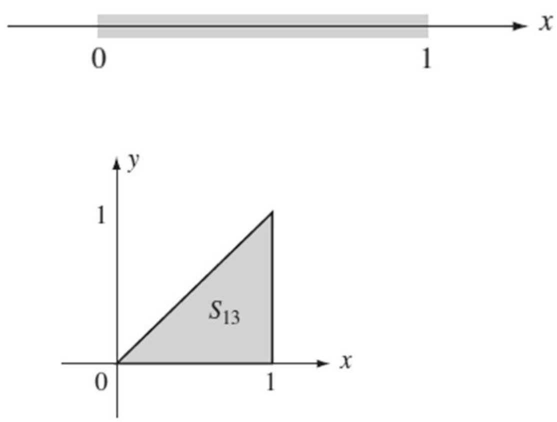
Tree Diagrams

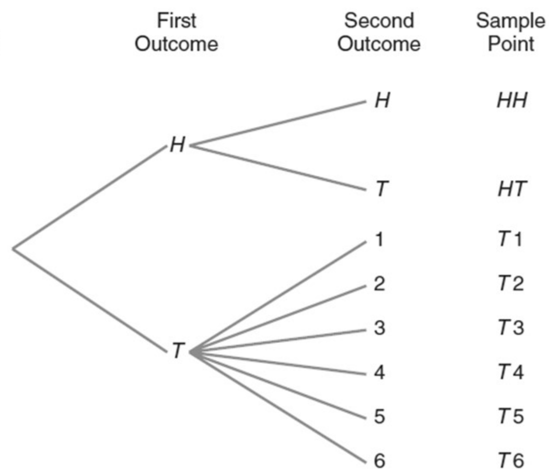
Intervals on real line

Regions of planes

Segments of three-dimensional spaces

Visualizing the Sample Space





Defining Probability

Statistical Regularity: Averages obtained in long sequences of repetitions (trials) of random experiments consistently yield approximately the same value

Prediction: Useful models help to predict future behaviour of systems

Relative Frequency: When a random experiment is repeated *n* times, and its outcome ζ is repeated k times, the relative frequency is defined by

$$f_{\zeta(n)} = \frac{k}{n}$$

Probability: If the statistical regularity exists, i.e., $n \to \infty$, $f_{\zeta(n)}$ varies less and less about a constant value, the *relative frequency* can *predict* the chance of future occurrent of ζ, and hence known as *probability* p_k of ζ.

$$\lim_{n\to\infty} f_{\zeta(n)} = p_k$$

Events

Event: a subset A of a sample space S, such that $\zeta \in A \subset S$

A *certain* event: S

A *null* or *impossible* event: $\varnothing$

Complementary Event: the subset A^c of all outcomes of S *not* in A.

Mutually Exclusive or Disjoint Events: Two events A and B are mutually exclusive, or disjoint, if $A \cap B = \varnothing$

Collectively Exhaustive: When $A_1 \cup A_2 \cup \cdots \cup A_n = S$, then $A_1, A_2, \ldots$ are known as collectively exhaustive.

Probability of an Event

The *probability* of an *event* A is the sum of the probabilities (weights) of all outcomes in it. It is denoted by $P[A]$, and satisfies the following axioms:

$P[A] \geq 0$

$P[S] = 1$

For a countable collection of mutually exclusive events A_k, $k = 1,2,3, \ldots$

$$P[\cup_{k=1}^{\infty} A_k] = \sum_{k=1}^{\infty} P[A_k]$$

$$\boxed{P[\varnothing] \leq P[A] \leq P[S]}$$

Probability of an event: Discrete vs. Continuous Sample Space

DISCRETE SAMPLE SPACE

For a discrete S with N *equally likely* outcomes, an event A occurring with n outcomes has probability $P[A]$ given by

$$P[A] = \frac{n}{N}$$

CONTINUOUS SAMPLE SPACE

The probability that the outcome falls in a subinterval $[a, b]$ of the continuous S is equal to the length of the subinterval

$$P[[a, b]] = b - a, \quad 0 \le a \le b \le 1$$

Examples

1. A coin is tossed twice. What is the probability that at least 1 head occurs?

2. A die is loaded in such a way that an even number is twice as likely to occur as an odd number.
   a) If E is the event that a number less than 4 occurs on a single toss of the die, find $P(E)$.
   b) Let A be the event that an even number turns up and let B be the event that a number divisible by 3 occurs. Find $P(A \cup B)$ and $P(A \cap B)$.

Additive Rules

For any two events A and B,

$$P[A \cup B] = P[A] + P[B] - P[A \cap B]$$

For any three events A, B and C

$$P[A \cup B \cup C] = P[A] + P[B] + P[C] - P[A \cap B] - P[B \cap C] - P[A \cap C] + P[A \cap B \cap C]$$

For a pair of complementary events A and A^c,

$$P\left[A^c\right] = 1 - P[A]$$

Examples

1. What is the probability of getting a total of 7 or 11 when a pair of fair dice is tossed?
2. If the probabilities are, respectively, 0.09, 0.15, 0.21, and 0.23 that a person purchasing a new automobile will choose the colour green, white, red, or blue, what is the probability that a given buyer will purchase a new automobile that comes in one of those colours?

Practice Exercises: Representing Sample Spaces and Events

1. Set of all possible outcomes if three items are selected at random from a manufacturing process. Each item is inspected and classified defective, *D*, or non-defective, *N*.
2. The set of all points (x, y) on the boundary or the interior of a circle of radius 2 with center at the origin.
3. The event *A* that the component fails before the end of the fifth year, given the sample space $S = \{t | t \geq 0\}$.
4. The set of integers between 1 and 50 divisible by 8.

Practice Exercises: Representing Sample Spaces and Events

5. An experiment consists of tossing a die and then flipping a coin once if the number on the die is even. If the number on the die is odd, the coin is flipped twice. Using the notation 4*H*, for example, to denote the outcome that the die comes up 4 and then the coin comes up heads, and 3*HT* to denote the outcome that the die comes up 3 followed by a head and then a tail on the coin, construct a tree diagram to show the 18 elements of the sample space *S*.

Practice Exercises

6. Indicate whether each of the following statement is true or false:
   (a) If $P[4] = 2P[A^c]$, then $P[A] = \frac{1}{2}$.
   (b) For all A and B , $P[AB] < P\,[A]P\,[B]$.
   (c) If $P[A] < P[B]$, then $P[AB] < P[B]$.
   (d) If $P[A \cap B] = P[A]$, then $P[A] > P[B]$.

References

1. Walpole, R.E., Myers, R.H., Myers, S. L. and Ye, K. (2007) *Probability & Statistics for Engineers & Scientists.* 9th Edition, Pearson Education, Inc.
2. Leon-Garcia, A. (2008). *Probability, Statistics, and Random Processes for Electrical Engineering*. 3rd Edition, Pearson/Prentice Hall.