Lecture 5: Introduction to Random Variables

CPE251 Probability Methods in Engineering

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Random Variable

A random variable is a function that assigns a real number $X(\zeta)$ to *each* outcome ζ of the *sample space* of a *random experiment*.

For example, when a coin is tossed thrice, two random variables can be associated with this experiment: $X =$ number of heads and $Y =$ number of tails.

The capital letters X and Y are the *labels* of the random variables, while small letters x and y denote the *values* of the random variables

$S = \{TTT, TTH, THT, THH, HTT, HTH, HHT, HHH\}$

$S_X = \{0,1,2,3\}$

$S_Y = \{0,1,2,3\}$

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Examples

Suppose a sampling plan involves sampling items from a process until a defective is observed. The evaluation of the process will depend on how many consecutive items are observed. In that regard, let X be a random variable defined by the number of items observed before a defective is found. With N a nondefective and D a defective, sample spaces are $S = \{D\}$ given $X = 1$, $S = \{ND\}$ given $X = 2$, $S = \{NND\}$ given $X = 3$, and so on. ▟

Let X be the random variable defined by the waiting time, in hours, between successive speeders spotted by a radar unit. The random variable X takes on all values x for which $x \geq 0$. ▟

Sample Problem

A fair coin is flipped three times and the number of heads X is observed. If the probability of occurring a head is p. Compute $P[X = 0]$ and $P[X = 2]$.

We are to find the probability of no head and exactly two heads

$P[X = 0] = P[\{TTT\}] = (1 - p)^3$

$P[X = 2]$

$= P[\{THH, HTH, HHT\}] = p^2(1 - p) + p^2(1 - p) + p^2(1 - p)$

$= 3p^2(1 - p)$

Discrete Random Variable

A random variable X is called a discrete random variable if S_X is countable.

Example: Coin tossed thrice, $X =$ number of heads, $p = \frac{1}{2}$

x	0	1	2	3
$P[X = x]$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Probability Mass Function

A function $p_X(x)$ is known as probability function, or probability mass function or probability distribution if

1. $p_X(x) \geq 0$
2. $\sum p_X(x) = 1$
3. $P[X = x] = p_X(x)$

Example

A shipment of 20 similar laptops to a retail outlet contains 3 that are defective. If a school makes a random purchase of 2 of these laptops, find the probability distribution for the number of defectives.

$D =$ defective, $N =$ not defective, $X =$ number of defectives picked by the school

The sample space for this case is $S = \{NN, ND, DN, DD\}$

For every outcome, there is a probability of number of defective and not defective laptops.

Let n_D is the total number of defective laptops selected and n_N is the total number of not defective laptops.

Then, probability that x number of defective laptops are picked by the school is:

$$p_X(x) = P[X = x] = \frac{\binom{n_D}{x}\binom{n_N}{2-x}}{\binom{20}{2}}$$

Cumulative Distribution Function (Discrete Random Variable)

The cumulative distribution function $F_X(x)$ of a random variable X is given by

$$F_X(x) = \sum_{t \le x} p_X(t)$$

Continuous Random Variable

When a random variable X can take on values on a continuous scale, it is called continuous random variable.

They represent *measured* data unlike *count* data (discrete random variable)

The probability of *exactly one value* of a discrete random variable is 0. Therefore, it cannot be tabulated like discrete random variable.

The probability of an interval of a continuous random variable is non-zero.

Probability Density Function or Density Function

The probability of a continuous random variable can be stated as a function $f(x)$.

$f(x)$ may or may not be continuous for all values. However, frequently used $f(x)$ are continuous.

Probability Density Function or Density Function

The function $f_X(x)$ is a probability density function (pdf) for the continuous random variable X, defined over the set of real numbers, if

1. $f_X(x) \geq 0, \forall x \in R$
2. $\int_{-\infty}^{\infty} f_X(x)dx = 1$
3. $P[a < X < b] = \int_a^b f_X(x)\, dx$

Cumulative Distribution Function (Continuous Random Variable)

The cumulative distribution function $F_X(x)$ of a continuous random variable X is

$$F_X(x) = P[X \leq x] = \int_{-\infty}^{x} f(t)dt\,, \quad -\infty \leq x \leq \infty$$

Example

The Department of Energy (DOE) puts projects out on bid and generally estimates what a reasonable bid should be. Call the estimate b. The DOE has determined that the density function of the winning (low) bid is

$$f(y) = \begin{cases} \frac{5}{8b}, & \frac{2}{5}b \leq y \leq 2b, \\ 0, & \text{elsewhere.} \end{cases}$$

Find $F(y)$ and use it to determine the probability that the winning bid is less than the DOE's preliminary estimate b.

For $2b/5 \le y \le 2b$,

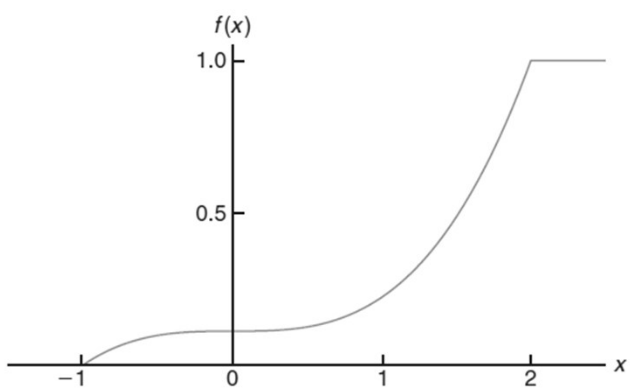
$$F(y) = \int_{2b/5}^{y} \frac{5}{8b} dy = \left.\frac{5t}{8b}\right|_{2b/5}^{y} = \frac{5y}{8b} - \frac{1}{4}.$$

Thus,

$$F(y) = \begin{cases} 0, & y < \frac{2}{5}b, \\ \frac{5y}{8b} - \frac{1}{4}, & \frac{2}{5}b \le y < 2b, \\ 1, & y \ge 2b. \end{cases}$$

To determine the probability that the winning bid is less than the preliminary bid estimate b, we have

$$P(Y \le b) = F(b) = \frac{5}{8} - \frac{1}{4} = \frac{3}{8}.$$



EXERCISE 1

On a laboratory assignment, if the equipment is working, the density function of the observed outcome, X, is

$$f(x) = \begin{cases} 2(1-x), & 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Calculate $P(X \leq 1/3)$.

(b) What is the probability that X will exceed 0.5?

(c) Given that $X \geq 0.5$, what is the probability that X will be less than 0.75?

EXERCISE 2

Suppose a certain type of small data processing firm is so specialized that some have difficulty making a profit in their first year of operation. The probability density function that characterizes the proportion Y that make a profit is given by

$$f(y) = \begin{cases} ky^4(1-y)^3, & 0 \leq y \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) What is the value of k that renders the above a valid density function?

(b) Find the probability that at most 50% of the firms make a profit in the first year.

(c) Find the probability that at least 80% of the firms make a profit in the first year.

References

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2. Leon-Garcia, A. (2008). *Probability, Statistics, and Random Processes for Electrical Engineering*. 3rd Edition, Pearson/Prentice Hall.