Alexander-Sadiku Fundamentals of Electric Circuits

Chapter 16

Applications of the Laplace Transform

Application of the Laplace Transform
Chapter 16

16.1 Circuit Element Models
16.2 Circuit Analysis
16.3 Transfer Functions
16.4 State Variables

16.1 Circuit Element Models (1)

Steps in Applying the Laplace Transform:

1. Transform the circuit from the time domain to the s-domain
2. Solve the circuit using nodal analysis, mesh analysis, source transformation, superposition, or any circuit analysis technique with which we are familiar
3. Take the inverse transform of the solution and thus obtain the solution in the time domain.

16.1 Circuit Element Models (2)

For an inductor,

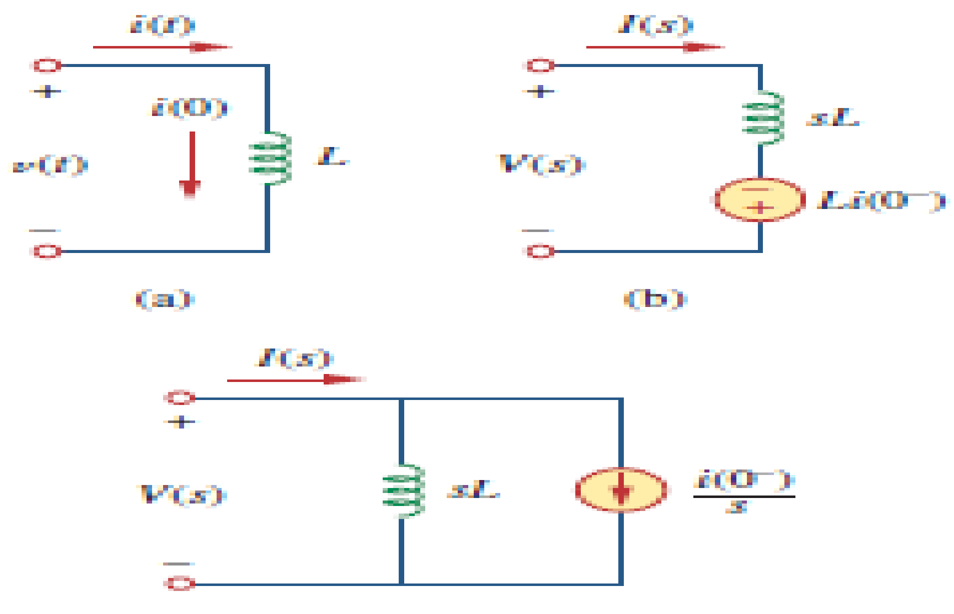
$$v(t) = L\frac{di(t)}{dt}$$

Taking the Laplace transform of both sides gives

$$V(s) = L[sI(s) - i(0^-)] = sLI(s) - Li(0^-)$$

or

$$\boxed{I(s) = \frac{1}{sL}V(s) + \frac{i(0^-)}{s}}$$



(a) (b) (c)

16.1 Circuit Element Models (3)

For a capacitor,

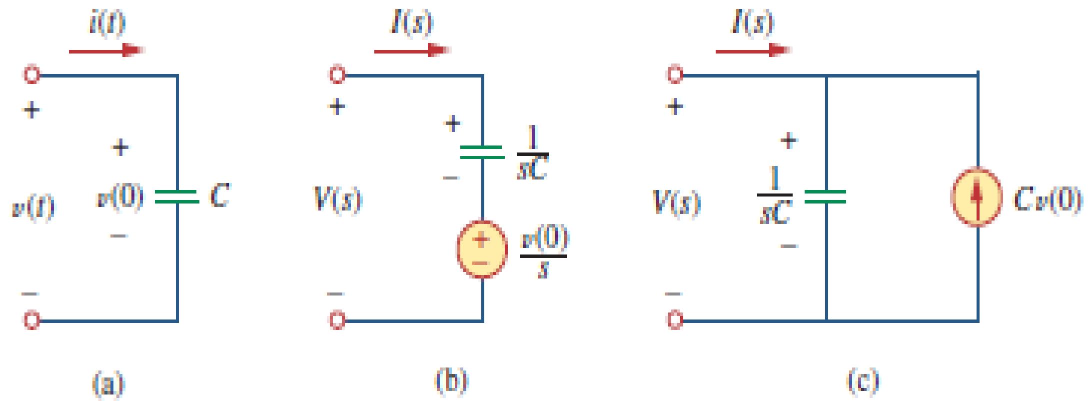
$$i(t) = C\frac{dv(t)}{dt}$$

which transforms into the s-domain as

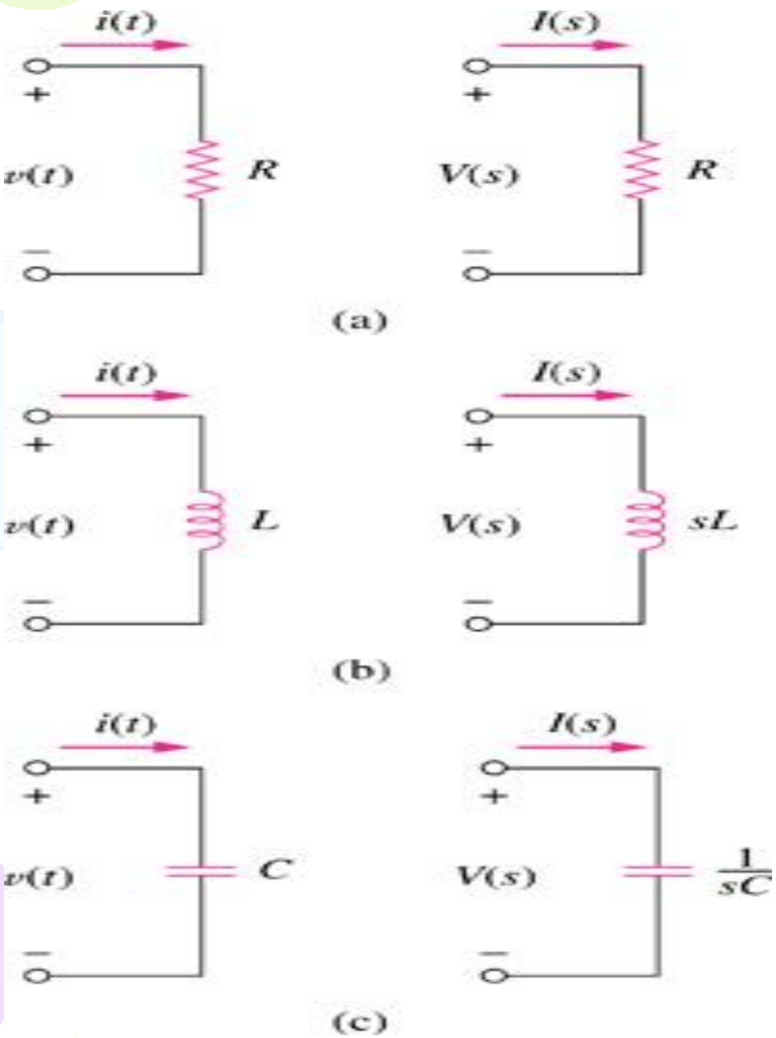
$$I(s) = C[sV(s) - v(0^-)] = sCV(s) - Cv(0^-)$$

or

$$\boxed{V(s) = \frac{1}{sC}I(s) + \frac{v(0^-)}{s}}$$



16.1 Circuit Element Models (4)



Time-domain and s-domain representations of passive elements under zero initial conditions.

Assume zero initial condition for the inductor and capacitor,

Resistor : V(s)=RI(s)

Inductor: V(s)=sLI(s)

Capacitor: V(s) = I(s)/sC

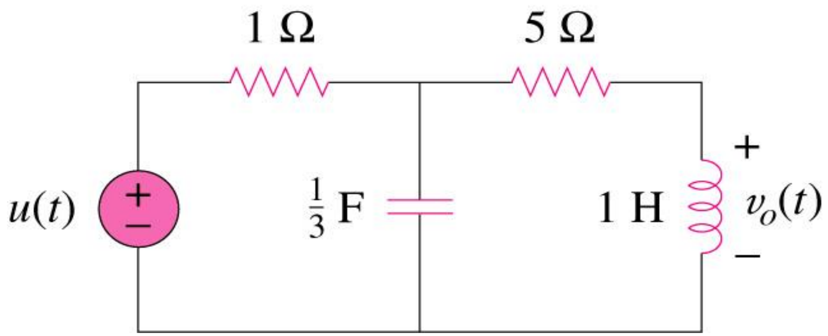
The impedance in the s-domain is defined as Z(s) = V(s)/I(s)

The admittance in the s-domain is defined as Y(s) = I(s)/V(s)

16.1 Circuit Element Models (5)

Example 1:

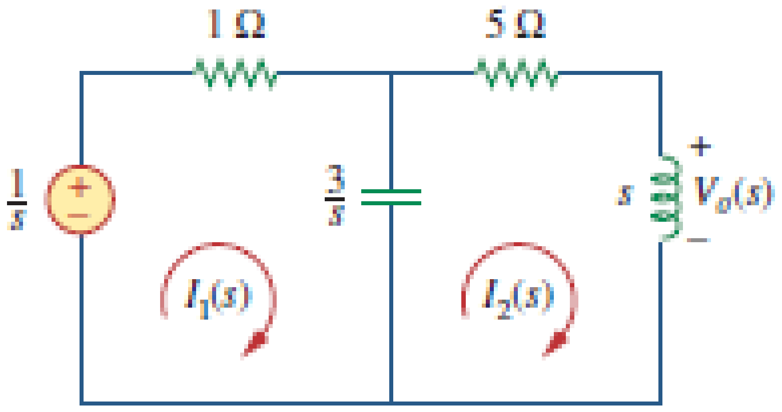
Find $v_0(t)$ in the circuit shown below, assuming zero initial conditions.



$$u(t) \Rightarrow \frac{1}{s}$$

$$1\,\mathrm{H} \Rightarrow sL = s$$

$$\frac{1}{3}\,\mathrm{F} \Rightarrow \frac{1}{sC} = \frac{3}{s}$$



16.1 Circuit Element Models (6)

Solution:

$$\frac{1}{s} = \left(1 + \frac{3}{s}\right)I_1 - \frac{3}{s}I_2 \qquad (16.1.1)$$

For mesh 2,

$$0 = -\frac{3}{s}I_1 + \left(s + 5 + \frac{3}{s}\right)I_2$$

or

$$I_1 = \frac{1}{3}(s^2 + 5s + 3)I_2 \qquad (16.1.2)$$

Substituting this into Eq. (16.1.1),

$$\frac{1}{s} = \left(1 + \frac{3}{s}\right)\frac{1}{3}(s^2 + 5s + 3)I_2 - \frac{3}{s}I_2$$

Multiplying through by $3s$ gives

$$3 = (s^3 + 8s^2 + 18s)I_2 \quad \Rightarrow \quad I_2 = \frac{3}{s^3 + 8s^2 + 18s}$$

$$V_o(s) = sI_2 = \frac{3}{s^2 + 8s + 18} = \frac{3}{\sqrt{2}}\frac{\sqrt{2}}{(s+4)^2 + (\sqrt{2})^2}$$

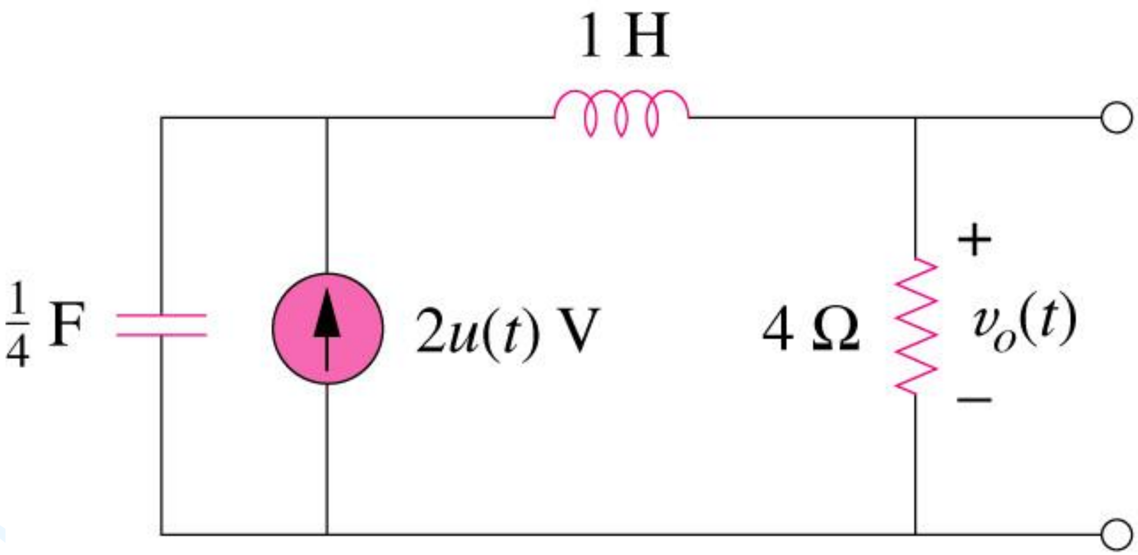
Taking the inverse transform yields

$$v_o(t) = \frac{3}{\sqrt{2}}e^{-4t}\sin\sqrt{2}t \text{ V}, \qquad t \geq 0$$

16.1 Circuit Element Models (7)

Example 2:

Determine $v_0(t)$ in the circuit shown below, assuming zero initial conditions.



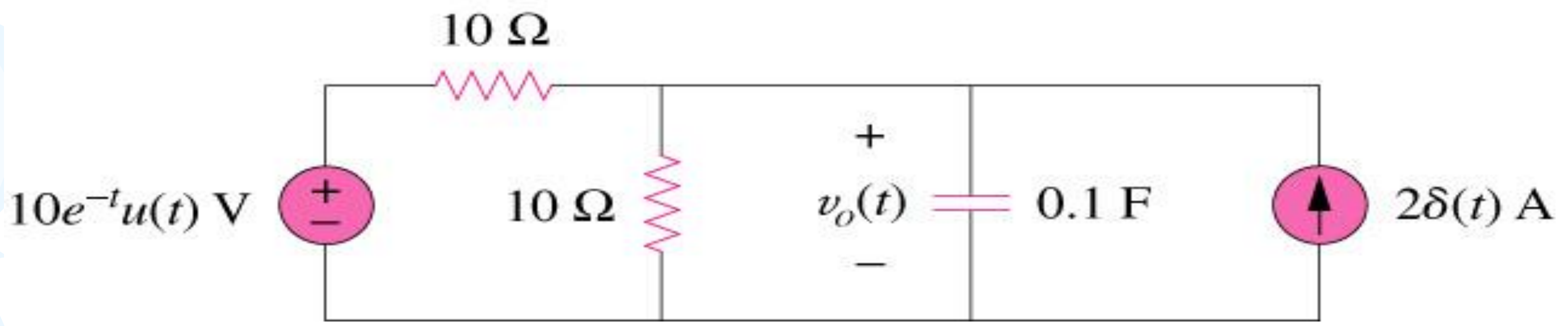
*Refer to in-class illustration, textbook

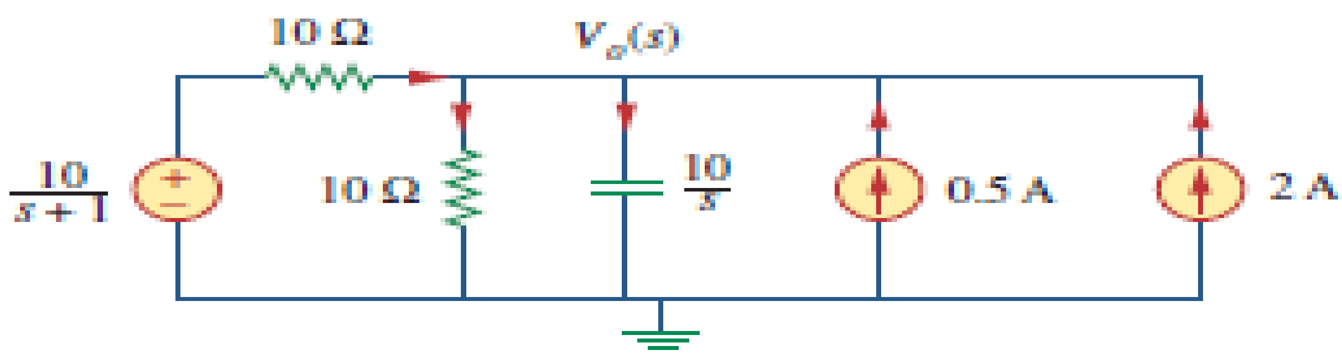
$\text{Ans}: \quad 8(1-e^{-2t}-2te^{-2t})u(t) \quad \text{V}$

16.1 Circuit Element Models (8)

Example 3:

Find $v_0(t)$ in the circuit shown below. Assume $v_0(0)=5V$.





$$\text{Ans}: \quad v_0(t) = (10e^{-t} + 15e^{-2t})u(t) \quad \text{V}$$

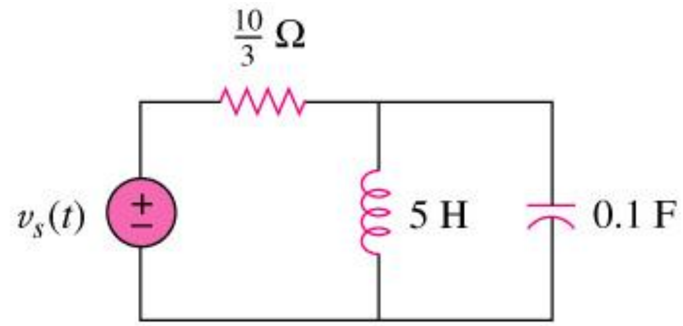
16.2 Circuit Analysis (1)

- Circuit analysis is relatively easy to do in the s-domain.
- By transforming a complicated set of mathematical relationships in the time domain into the s-domain where we convert operators (derivatives and integrals) into simple multipliers of s and 1/s.
- This allow us to use algebra to set up and solve the circuit equations.
- In this case, all the circuit theorems and relationships developed for dc circuits are perfectly valid in the s-domain.

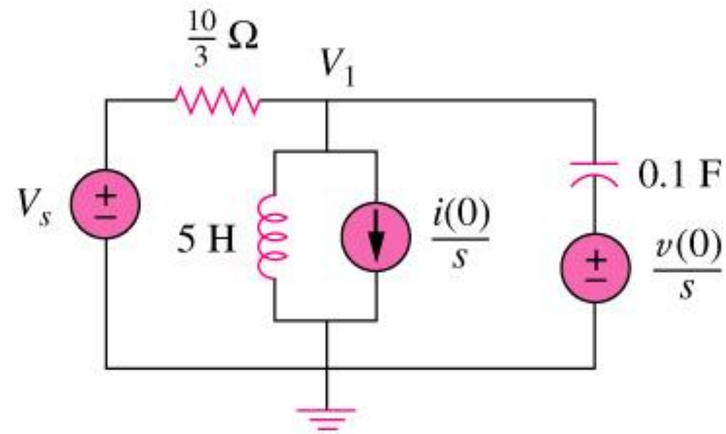
16.2 Circuit Analysis (2)

Example 5:

Consider the circuit below. Find the value of the voltage across the capacitor assuming that the value of $v_s(t)=10u(t)$ V and assume that at t=0, -1A flows through the inductor and +5 is across the capacitor.



(a)



(b)

16.2 Circuit Analysis (3)

Solution:

Transform the circuit from time-domain (a) into s-domain (b) using Laplace Transform. On rearranging the terms, we have

$$\boxed{V_1 = \frac{35}{s+1} - \frac{30}{s+2}}$$

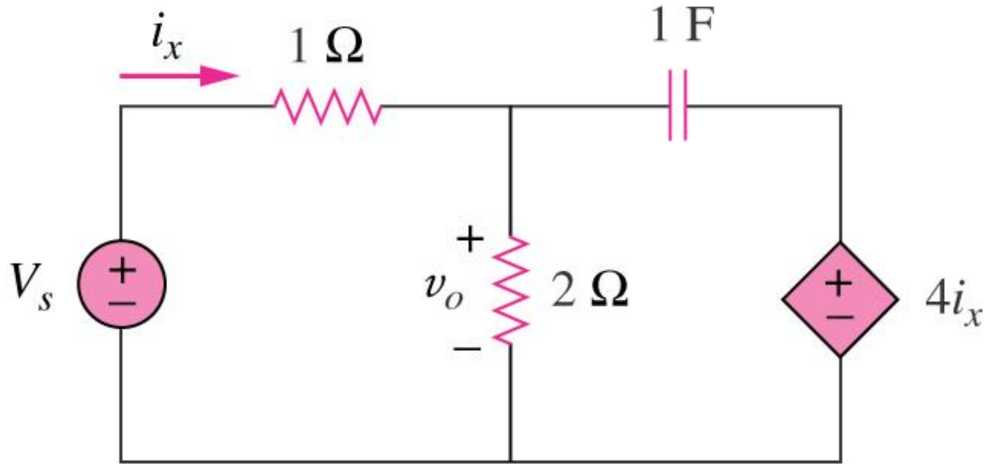
By taking the inverse transform, we get

$$\boxed{v_1(t) = (35e^{-t} - 30e^{-2t})u(t) \quad \text{V}}$$

16.2 Circuit Analysis (4)

Example 6:

The initial energy in the circuit below is zero at *t=0*. Assume that $v_s=5u(t)$ V. (a) Find V0(s) using the thevenin theorem. (b) Apply the initial- and final-value theorem to find $v_0(0)$ and $v_0(\infty)$. (c) Obtain $v_0(t)$.



Ans: (a) $V_0(s) = 4(s+0.25)/(s(s+0.3))$ (b) 4,3.333V, (c) (3.333+0.6667e-0.3t)u(t) V.

*Refer to in-class illustration, textbook

16.3 Transfer Functions (1)

- The transfer function H(s) is the ratio of the output response Y(s) to the input response X(s), assuming all the initial conditions are zero.

$$\boxed{H(s) = \frac{Y(s)}{X(s)}}$$

, **h(t) is the impulse response function**.

- Four types of gain:

1. H(s) = voltage gain = $V_0(s)/V_i(s)$
2. H(s) = Current gain = $I_0(s)/I_i(s)$
3. H(s) = Impedance = V(s)/I(s)
4. H(s) = Admittance = I(s)/V(s)

16.3 Transfer Function (2)

Example 7:

The output of a linear system is $y(t)=10e^{-t}\cos 4t$ when the input is $x(t)=e^{-t}u(t)$. Find the transfer function of the system and its impulse response.

Solution:

Transform y(t) and x(t) into s-domain and apply H(s)=Y(s)/X(s), we get

$$\boxed{H(s)=\frac{Y(s)}{X(s)}=\frac{10(s+1)^2}{(s+1)^2+16}=10-40\frac{4}{(s+1)^2+16}}$$

Apply inverse transform for H(s), we get

$$\boxed{h(t)=10\delta(t)-40e^{-t}\sin(4t)u(t)}$$

16.3 Transfer Function (3)

Example 8:

The transfer function of a linear system is

$$\boxed{H(s) = \frac{2s}{s+6}}$$

Find the output y(t) due to the input e-3tu(t) and its impulse response.

$$\boxed{\text{Ans}: \ -2e^{-3t} + 4e^{-6t}, \ \mathrm{t} \geq 0; \ 2\delta(\mathrm{t})\text{-}12\mathrm{e}^{\text{-}6\mathrm{t}}u(t)}$$

*Refer to in-class illustration, textbook

16.4 State Variables (1)

- it is a physical property that characterizes the state of a system, regardless of how the system got to that state.

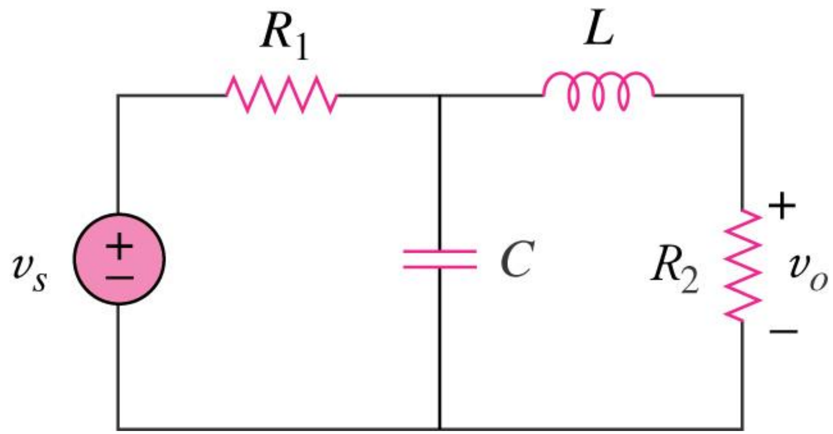
- Steps to apply the State Variable Method to Circuit Analysis

  1. Select the inductor current i and capacitor voltage v as the state variables, making sure they are consistent with the passive sign convention.
  2. Apply KCL and KVL to the circuit and obtain circuit variables in terms of state variables. This should lead to a set of first order differential equations necessary and sufficient to determine all the state variables.
  3. Obtain the output equations and put the final result in a state-space representation.

16.4 State Variables (2)

Example 9:

Obtain the state variable model for the circuit shown below. Let R_1=1Ω, R_2=2 Ω,C=0.5F and L=0.2H and obtain the transfer function.



Ans $\begin{bmatrix} \dot{v} \\ \dot{i} \end{bmatrix} = \begin{bmatrix} \frac{1}{R_1 C} & \frac{-1}{C} \\ \frac{1}{L} & \frac{-R_2}{L} \end{bmatrix} \begin{bmatrix} v \\ i \end{bmatrix} + \begin{bmatrix} \frac{1}{R_1 C} \\ 0 \end{bmatrix} v_s, \quad v_0 = \begin{bmatrix} 0 & R_2 \end{bmatrix} \begin{bmatrix} v \\ i \end{bmatrix} \quad ; \quad \mathrm{H}(s) = \frac{20}{s^2 + 12s + 30}$

*Refer to in-class illustration, textbook