Lecture 10: Functions of A Random Variable

CPE251 Probability Methods In Engineering

Dr. Zaid Ahmad
Advisor IEEE Student Branch
COMSATS University Islamabad, Lahore Campus

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Function of a random variable

A function $Y = g(X)$ of a random variable X is another random variable.

The pmf $p_Y(y)$ or pdf $f_Y(y)$ can be derived from the pmf $p_X(x)$ or pdf $f_X(x)$ and $g(X)$.

$$p_Y(y) = \sum_{x|g(x)=y} p_X(x)$$

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Example

A parcel shipping company offers a charging plan: \$1.00 for the first pound, \$0.90 for the second pound, etc., down to \$0.60 for the fifth pound, with rounding up for a fraction of a pound. For all packages between 6 and 10 pounds, the shipper will charge \$5.00 per package. (It will not accept shipments over 10 pounds.) Find a function $Y = g(X)$ for the charge in cents for sending one package.

$$Y = g(X) = \begin{cases} 105X - 5X^2 & X = 1, 2, 3, 4, 5 \\ 500 & X = 6, 7, 8, 9, 10. \end{cases}$$

suppose all packages weigh 1, 2, 3, or 4 pounds with equal probability.

$$P_X(x) = \begin{cases} 1/4 & x = 1, 2, 3, 4, \\ 0 & \text{otherwise.} \end{cases}$$

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Expected value of a function of random variable

$$E(Y) = E\big(g(x)\big) = \sum_{x} g(X) p_X(x)$$

$$E(Y) = E\big(g(x)\big) = \int_{x=x_1}^{x_2} g(X) f_X(x) dx$$

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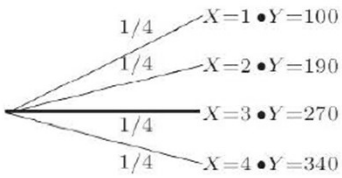
Example



$$P_Y(y) = \begin{cases} 1/4 & y = 100, 190, 270, 340, \\ 0 & \text{otherwise.} \end{cases}$$

The expected shipping bill is

$$\begin{aligned} \mathrm{E}[Y] &= \frac{1}{4}(100 + 190 + 270 + 340) \\ &= 225 \text{ cents.} \end{aligned}$$

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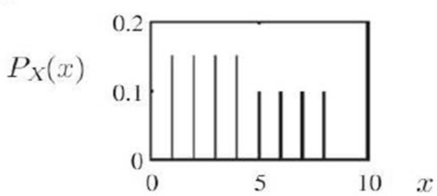
Example

Suppose the probability model for the weight in pounds X of a package in Example is



$$P_X(x) = \begin{cases} 0.15 & x = 1, 2, 3, 4, \\ 0.1 & x = 5, 6, 7, 8, \\ 0 & \text{otherwise.} \end{cases}$$

For the pricing plan given in Example 3.25, what is the PMF and expected value of Y, the cost of shipping a package?

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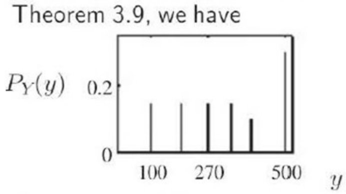
For this situation we need the more general view of the PMF of Y, given by Theorem 3.9. In particular, $y_6 = 500$, and we have to add the probabilities of the outcomes $X = 6$, $X = 7$, and $X = 8$ to find $P_Y(500)$. That is,

$$P_Y(500) = P_X(6) + P_X(7) + P_X(8) = 0.30. \qquad (3.57)$$

The steps in the procedure are illustrated in the diagram of Figure 3.1. Applying Theorem 3.9, we have



$$P_Y(y) = \begin{cases} 0.15 & y = 100, 190, 270, 340, \\ 0.10 & y = 400, \\ 0.30 & y = 500, \\ 0 & \text{otherwise.} \end{cases}$$

For this probability model, the expected cost of shipping a package is

$$E[Y] = 0.15(100 + 190 + 270 + 340) + 0.10(400) + 0.30(500) = 325 \text{ cents.}$$

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Expected values of a function of random variable

$$E(aX + b) = aE(X) + b$$

$$VAR(aX + b) = a^2 VAR(X)$$

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Derived Gaussian (Normal) Random Variable

If X is Gaussian random variable i.e. $X \sim \mathcal{N}(\mu, \sigma)$, then $Y = aX + b \sim \mathcal{N}(a\mu + b, a\sigma)$ i.e. Y is also Gaussian.

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References

1. Walpole, R.E., Myers, R.H., Myers, S.L. and Ye, K. (2007) *Probability & Statistics for Engineers & Scientists.* 9th Edition, Pearson Education, Inc.
2. Leon-Garcia, A. (2008). *Probability, Statistics, and Random Processes for Electrical Engineering*. 3rd Edition, Pearson/Prentice Hall.

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