The Inverse Z-Transform

$$x[n] = \frac{1}{2\pi j} \int_{|z|=r} X(z)\, z^{n-1}\, dz$$

Example 10.9

Given that $X(z) = \dfrac{3 - \frac{5}{6} z^{-1}}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{3}z^{-1}\right)}$, $|z| > 1/3$

Remember

before partial fraction, function $X(z)$ should be in proper fraction. In this example, it is in proper form.

$$\frac{3 - \frac{5}{6} z^{-1}}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{3}z^{-1}\right)} = \frac{A}{1-\frac{1}{4}z^{-1}} + \frac{B}{1-\frac{1}{3}z^{-1}}$$

$$= \frac{1}{1-\frac{1}{4}z^{-1}} + \frac{2}{1-\frac{1}{3}z^{-1}} \qquad \begin{pmatrix} A = 1 \\ B = 2 \end{pmatrix} \text{ by partial fraction}$$

Since $X(z)$ with ROC $|z| > 1/3$ & the fact $\overset{P_2}{1/3} > \overset{P_1}{1/4}$, so both are RSS. Therefore

$$\frac{1}{1-\frac{1}{4}z^{-1}} \xleftrightarrow{z^{-1}} \left(\tfrac{1}{4}\right)^n u[n] \quad \text{with ROC: } |z| > 1/4$$

$$\frac{1}{1-\frac{1}{3}z^{-1}} \xleftrightarrow{z^{-1}} \left(\tfrac{1}{3}\right)^n u[n] \quad \text{with ROC: } |z| > 1/3$$

so $\boxed{x[n] = \left(\tfrac{1}{4}\right)^n u[n] + 2\left(\tfrac{1}{3}\right)^n u[n]}$ → Unique solution for given ROC $|z| > 1/3$

Example 10.10

$$X(z) = \frac{3-\frac{5}{6}z^{-1}}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{3}z^{-1}\right)}, \frac{1}{4}<|z|<\frac{1}{3}$$

$\frac{1}{4}<|z|$ ← RSS

$|z|<\frac{1}{3}$ ← LSS

$$X(z) = \frac{1}{1-\frac{1}{4}z^{-1}} + \frac{2}{1-\frac{1}{3}z^{-1}}$$

$$\frac{1}{1-\frac{1}{4}z^{-1}}, |z|>\frac{1}{4} \underset{RSS}{\overset{z^{-1}}{\longleftrightarrow}} \left(\frac{1}{4}\right)^n u[n]$$

$$\frac{1}{1-\frac{1}{3}z^{-1}}, |z|<\frac{1}{3} \underset{LSS}{\overset{z^{-1}}{\longleftrightarrow}} -\left(\frac{1}{3}\right)^n u[-n-1]$$

Overall $\Rightarrow$ $\boxed{x[n] = \left(\frac{1}{4}\right)^n u[n] - 2\left(\frac{1}{3}\right)^n u[-n-1]}$

Example 10.11

$$X(z) = \frac{3-\frac{5}{6}z^{-1}}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{3}z^{-1}\right)}, |z|<\frac{1}{4}$$

$P_1 = \frac{1}{4}$

$P_2 = \frac{1}{3}$

since $\frac{1}{4}<\frac{1}{3}$

so both LSS

$$X(z) = \frac{1}{1-\frac{1}{4}z^{-1}} + \frac{2}{1-\frac{1}{3}z^{-1}}$$

$$\frac{1}{1-\frac{1}{4}z^{-1}} \underset{LSS}{\overset{z^{-1}}{\longleftrightarrow}} -\left(\frac{1}{4}\right)^n u[-n-1]$$

$$\frac{1}{1-\frac{1}{3}z^{-1}} \underset{LSS}{\overset{z^{-1}}{\longleftrightarrow}} -\left(\frac{1}{3}\right)^n u[-n-1]$$

Overall $\Rightarrow$ $\boxed{x[n] = -\left(\frac{1}{4}\right)^n u[-n-1] - \left(\frac{1}{3}\right)^n u[-n-1]}$

NOTE: If ROC is not provided, then there can be more than 1 solutions. In that case, you need to find all possibe $x[n]$ for given $X(z)$.

PROPERTIES OF Z-TRANSFORM

1 Linearity

if $x_1[n] \xleftrightarrow{z} X_1(z)$, ROC = R_1

& $x_2[n] \xleftrightarrow{z} X_2(z)$, ROC = R_2

then

$$a\, x_1[n] + b\, x_2[n] \xleftrightarrow{z} a_1 X_1(z) + b X_2(z) \;, \quad ROC = R_1 \cap R_2$$

2 Time Shifting

$$x[n] \xleftrightarrow{z} X(z) \;, \quad ROC = R$$

$$x[n-n_0] \xleftrightarrow{z} z^{-n_0} X(z) \;, \quad ROC = R$$

other than pole-zero cancellation possibility

3 Scaling in z-domain

$$x[n] \xleftrightarrow{z} X(z) \;, \quad ROC = R$$

$$z_0^n\, x[n] \xleftrightarrow{z} X\left(\frac{z}{z_0}\right) \;, \quad ROC = |z_0| R$$

for $r_0 < 1$, ROC compress

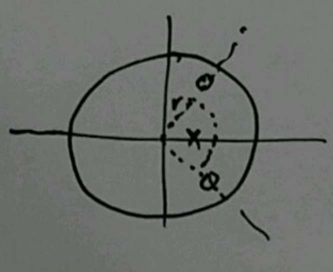
for $r_0 > 1$, ROC expands

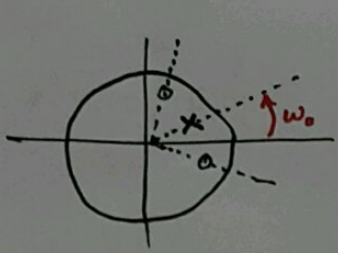
if $z_0^n = r_0^n e^{j\omega_0 n}$

& $r_0^* = 1 \Rightarrow z_0^n = e^{j\omega_0 n}$, so

$$e^{j\omega_0 n} . \, x[n] \longleftrightarrow X\left(e^{-j\omega_0 n} z\right) \;, \quad ROC = R$$

Shift of pole location due to ω_0



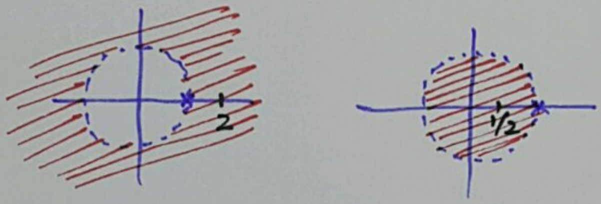


4 Time Reversal

$$x[n] \xleftrightarrow{z} X(z), \quad ROC = R \rightarrow \text{(If RSS)}$$

$$x[-n] \xleftrightarrow{z} X\left(\frac{1}{z}\right), \quad ROC = \frac{1}{R} \quad \text{Inverted ROC}$$

$\rightarrow$ LSS



$X(z)$ $\quad$ $X\left(\frac{1}{z}\right)$

5 The Convolution Property

$$x_1[n] \xleftrightarrow{z} X_1(z), \quad ROC = R_1$$

$$x_2[n] \xleftrightarrow{z} X_2(z), \quad ROC = R_2$$

then

$$x_1[n] * x_2[n] \xleftrightarrow{z} X_1(z)\,X_2(z), \quad ROC = R_1 \cap R_2$$

6 Differentiation in z-Domain

$$x[n] \xleftrightarrow{z} X(z), \quad ROC = R$$

$$n\,x[n] \xleftrightarrow{z} -z\frac{d}{dz}X(z), \quad ROC = R$$

Example 10.18

$x[n] = ?$ for $\quad X(z) = \dfrac{az^{-1}}{(1-az^{-1})^2}, \quad |z| > |a|$

Since $\quad a^n u[n] = \dfrac{1}{1-az^{-1}}, \quad |z| > |a|$

$$n a^n u[n] = -z\frac{d}{dz}\left(\frac{1}{1-az^{-1}}\right) = -\cancel{z}\,\frac{-a/z^{\cancel{2}}}{(1-az^{-1})^2} = +\frac{az^{-1}}{(1-az^{-1})^2}$$

so

$$n a^n u[n] \xleftrightarrow{z} \frac{az^{-1}}{(1-az^{-1})^2}, \quad |z| > |a|$$