Alexander-Sadiku Fundamentals of Electric Circuits

Chapter 15

Introduction to the Laplace Transform

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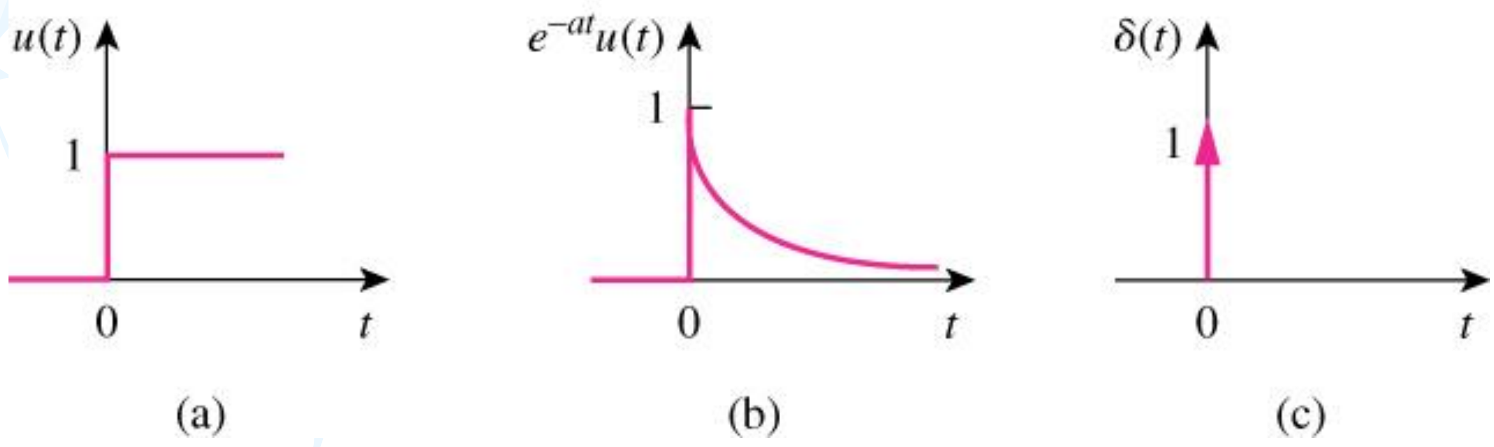
Chapter 15

15.1 Definition of Laplace Transform (2)

Example 1

Determine the Laplace transform of each of the following functions shown below:



15.1 Definition of Laplace Transform (3)

Solution:

a) The Laplace Transform of unit step, *u(t)* is given by

$$\boxed{L[u(t)] = F(s) = \int_0^{\infty} 1e^{-st}\,dt = \frac{1}{s}}$$

15.1 Definition of Laplace Transform (4)

Solution:

b) The Laplace Transform of exponential function, $e^{-\alpha t}u(t), \alpha > 0$ is given by

$$\boxed{L\left[u(t)\right] = F(s) = \int_0^{\infty} e^{\alpha t} e^{-st}\,dt = \frac{1}{s+\alpha}}$$

15.1 Definition of Laplace Transform (5)

Solution:

c) The Laplace Transform of impulse function, $\delta(t)$ is given by

$$\boxed{L\left[u(t)\right] = F(s) = \int_0^{\infty} \delta(t) e^{-st} dt = 1}$$

Laplace Transformation Table

S.no	$f(t)$	$\mathcal{L}\{f(t)\}$	S.no	$f(t)$	$\mathcal{L}\{f(t)\}$
1	1	$\frac{1}{s}$	11	$e^{at}\sinh bt$	$\frac{b}{(s-a)^2-b^2}$
2	e^{at}	$\frac{1}{s-a}$	12	$e^{at}\cosh bt$	$\frac{s-a}{(s-a)^2-b^2}$
3	t^n	$\frac{n!}{s^{n+1}}$	13	$t\cos at$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
4	$\sin at$	$\frac{a}{s^2+a^2}$	14	$t\sin at$	$\frac{2as}{(s^2+a^2)^2}$
5	$\cos at$	$\frac{s}{s^2+a^2}$	15	$f'(t)$	$sF(s)-f(0)$
6	$\sinh at$	$\frac{a}{s^2-a^2}$	16	$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
7	$\cosh at$	$\frac{s}{s^2-a^2}$	17	$\int_0^t f(u)du$	$\frac{1}{s}F(s)$
8	$e^{at}t^n$	$\frac{n!}{(s-a)^{n+1}}$	18	$t^n f(t)$ Where $n = 1,2,3,..$	$(-1)^n\frac{d^n}{ds^n}\{F(s)\}$
9	$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$	19	$\frac{1}{t}\{f(t)\}$	$\int_s^\infty F(s)ds$
10	$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$	20	$e^{at}f(t)$	$F(s-a)$

15.2 Properties of Laplace Transform (1)

Linearity:

If $F_1(s)$ and $F_2(s)$ are, respectively, the Laplace Transforms of $f_1(t)$ and $f_2(t)$

$$\boxed{L\left[a_1 f_1(t) + a_2 f_2(t)\right] = a_1 F_1(s) + a_2 F_2(s)}$$

Example:

$$\boxed{L\left[\cos(\omega t)u(t)\right] = L\left[\frac{1}{2}\left(e^{j\omega t} + e^{-j\omega t}\right)u(t)\right] = \frac{s}{s^2+\omega^2}}$$

15.2 Properties of Laplace Transform (2)

Scaling:

If *F (s)* is the Laplace Transforms of *f (t)*, then

$$\boxed{L[f(at)] = \frac{1}{a} F(\frac{s}{a})}$$

Example:

$$\boxed{L[\sin(2\omega t)u(t)] = \frac{2\omega}{s^2 + 4\omega^2}}$$

15.2 Properties of Laplace Transform (3)

Time Shift:

If *F (s)* is the Laplace Transforms of *f (t)*, then

$$L\left[f(t-a)u(t-a)\right]=e^{-as}F(s)$$

Example:

$$L\left[\cos(\omega(t-a))u(t-a)\right]=e^{-as}\frac{s}{s^2+\omega^2}$$

15.2 Properties of Laplace Transform (4)

Frequency Shift:

If $F(s)$ is the Laplace Transforms of $f(t)$, then

$$\boxed{L\left[e^{-at} f(t) u(t)\right] = F(s+a)}$$

Example:

$$\boxed{L\left[e^{-at} \cos(\omega t) u(t)\right] = \frac{s+a}{(s+a)^2 + \omega^2}}$$

15.2 Properties of Laplace Transform (5)

Time Differentiation:

If $F(s)$ is the Laplace Transforms of $f(t)$, then the Laplace Transform of its derivative is

$$\boxed{L\left[\frac{df}{dt}u(t)\right] = sF(s) - f(0^-)}$$

Example:

$$\boxed{L[\sin(\omega t)u(t)] = \frac{\omega}{s^2 + \omega^2}}$$

15.2 Properties of Laplace Transform (6)

Time Integration:

If $F(s)$ is the Laplace Transforms of $f(t)$, then the Laplace Transform of its integral is

$$\boxed{L\left[\int_0^t f(t)dt\right] = \frac{1}{s}F(s)}$$

Example:

$$\boxed{L\left[t^n\right] = \frac{n!}{s^{n+1}}}$$

15.2 Properties of Laplace Transform (7)

Frequency Differentiation:

If *F(s)* is the Laplace Transforms of *f (t)*, then the derivative with respect to s, is

$$\boxed{L\left[tf(t)\right] = -\frac{dF(s)}{ds}}$$

Example:

$$\boxed{L\left[te^{-at}u(t)\right] = \frac{1}{(s+a)^2}}$$

15.2 Properties of Laplace Transform (8)

Initial and Final Values:

The initial-value and final-value properties allow us to find the initial value *f(0)* and *f(∞)* of *f(t)* directly from its Laplace transform *F(s)*.

$$\boxed{f(0) = \lim_{s\to\infty} sF(s)}$$

Initial-value theorem

$$\boxed{f(\infty) = \lim_{s\to 0} sF(s)}$$

Final-value theorem

15.3 The Inverse Laplace Transform (1)

Suppose F(s) has the general form of

$$\boxed{F(s)=\frac{N(s)\text{......numerator polynomial}}{D(s)\text{...denominator polynomial}}}$$

The finding the inverse Laplace transform of *F(s)* involves two steps:

1. Decompose F(s) into simple terms using partial fraction expansion.
2. Find the inverse of each term by matching entries in Laplace Transform Table.

15.3 The Inverse Laplace Transform (2)

Example 2

Find the inverse Laplace transform of

$$F(s) = \frac{3}{s} - \frac{5}{s+1} + \frac{6}{s^2+4}$$

Solution:

$$\boxed{\begin{aligned} f(t) &= L^{-1}\left(\frac{3}{s}\right) - L^{-1}\left(\frac{5}{s+1}\right) + L^{-1}\left(\frac{6}{s^2+4}\right) \\ &= (3 - 5e^{-t} + 3\sin(2t)u(t), \quad \mathrm{t} \geq 0 \end{aligned}}$$

15.4 The Convolution Integral (1)

- It is defined as $y(t)=\int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)d\lambda$ or $y(t)=x(t)*h(t)$

- Given two functions, $f_1(t)$ and $f_2(t)$ with Laplace Transforms $F_1(s)$ and $F_2(s)$, respectively

$$\boxed{F_1(s)F_2(s)=L\left[f_1(t)*f_2(t)\right]}$$

- Example: $y(t)=4e^{-t}$ and $h(t)=5e^{-2t}$

$$\boxed{h(t)*x(t)=L^{-1}\left[H(s)X(s)\right]=L^{-1}\left[\left(\frac{5}{s+2}\right)\left(\frac{4}{s+1}\right)\right]=20(e^{-t}-e^{-2t}),\quad \mathrm{t}\geq 0}$$

15.5 Application to Integro-differential Equation (1)

- The Laplace transform is useful in solving linear integro-differential equations.
- Each term in the integro-differential equation is transformed into s-domain.
- Initial conditions are automatically taken into account.
- The resulting algebraic equation in the s-domain can then be solved easily.
- The solution is then converted back to time domain.

15.5 Application to Integro-differential Equation (2)

Example 3:

Use the Laplace transform to solve the differential equation

$$\frac{d^2v(t)}{dt^2}+6\frac{dv(t)}{dt}+8v(t)=2u(t)$$

Given: $v(0) = 1$; $v'(0) = -2$

15.5 Application to Integro-differential Equation (3)

Solution:

Taking the Laplace transform of each term in the given differential equation and obtain

$$\left[s^2V(s)-sv(0)-v'(0)\right]+6\left[sV(s)-v(0)\right]+8V(s)=\frac{2}{s}$$

Substituting $v(0)=1; v'(0)=-2$, we have

$$(s^2+6s+8)V(s)=s+4+\frac{2}{s}=\frac{s^2+4s+2}{s}\Rightarrow V(s)=\frac{\frac{1}{4}}{s}+\frac{\frac{1}{2}}{s+2}+\frac{\frac{1}{4}}{s+4}$$

By the inverse Laplace Transform,

$$v(t)=\frac{1}{4}(1+2e^{-2t}+e^{-4t})u(t)$$