Lecture 9: Families of Continuous Random Variable

CPE251 Probability Methods In Engineering

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1

Families of Continuous Random Variable

Continuous Uniform Random Variable

Exponential Random Variable

Gaussian (Normal) Random Variable

2

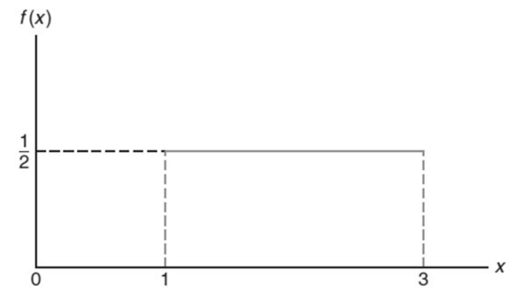
Continuous Uniform Random Variable

The pdf of continuous uniform random variable X on the interval $[a, b]$ is

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & a \le x \le b \\ 0 & \text{elsewhere} \end{cases}$$

$E(X) = \frac{a+b}{2}$

$VAR(X) = \frac{(b-a)^2}{12}$



3

Exponential Random Variable

The continuous random variable X is exponential with parameter λ if its pdf is given by

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0, \lambda > 0 \\ 0 & \text{otherwise} \end{cases}$$

Where λ is the arrival rate.

$$E(X) = \frac{1}{\lambda}$$

$$VAR(X) = \frac{1}{\lambda^2}$$

4

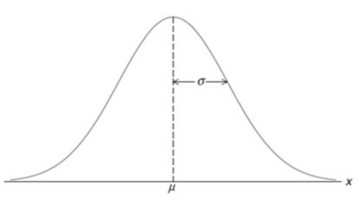
Gaussian (Normal) Random Variable

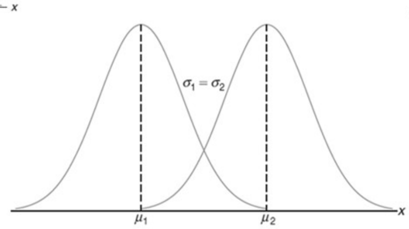
The pdf of the normal random variable is X with mean μ and variance σ^2, is

$$\mathcal{N}(\mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} \quad -\infty < x < \infty$$

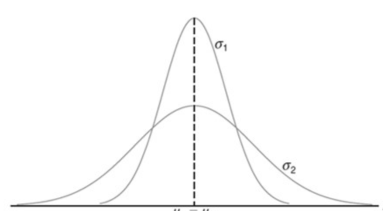
5

Gaussian (Normal) Random Variable





Normal curves with $\mu_1 < \mu_2$ and $\sigma_1 = \sigma_2$

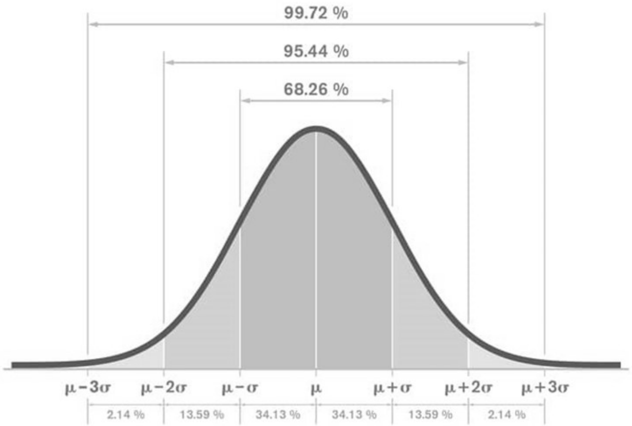


Normal curves with $\mu_1 = \mu_2$ and $\sigma_1 < \sigma_2$

6

Gaussian (Normal) Random Variable

Symmetry of Gaussian Random Variable



7

Gaussian (Normal) cdf

$$F_X(x) = \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt$$

where $t = \frac{u-\mu}{\sigma}$. This is a closed form expression and evaluated empirically using the following function:

$$F_X(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

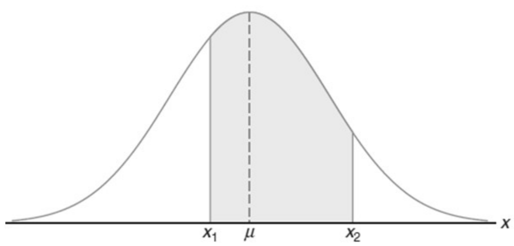
When $\mu = 0$ and $\sigma = 1$, this reduces to $\Phi(x)$

8

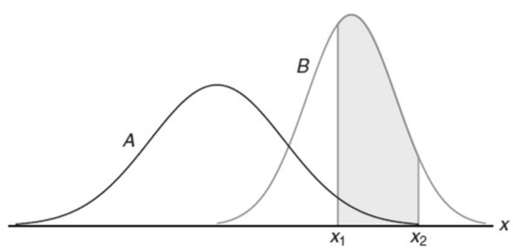
Intervals of Gaussian Random Variable

Probability of an interval of Gaussian random variable is computed using the area under the normal curve:

$$P[x_1 < x < x_2] = \frac{1}{\sqrt{2\pi}\sigma}\int_{x_1}^{x_2} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}\,dx = \Phi\left(\frac{x_2-\mu}{\sigma}\right) - \Phi\left(\frac{x_1-\mu}{\sigma}\right)$$



$P(x_1 < X < x_2) =$ area of the shaded region



$P(x_1 < X < x_2)$ for different normal curves

9

Standard Normal Random Variable

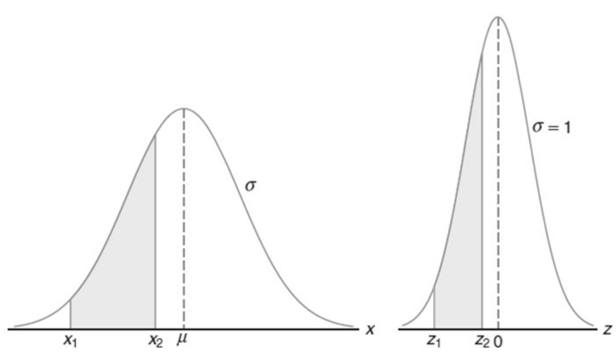
The area under the curve for all values of μ and σ cannot be evaluated.

The normal random variable with zero mean ($\mu = 0$) and unit variance ($\sigma^2 = 1$):

$$Z = \frac{x-\mu}{\sigma}$$

and

$$F_Z(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{z^2}{2}}\,dz = \Phi(z)$$



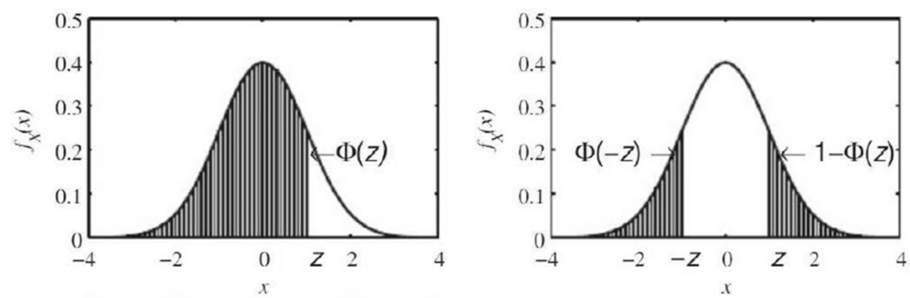
The original and transformed normal distributions

10

Standard Normal Random Variable

Symmetry of Standard Gaussian Random Variable

$$\Phi(-z) = 1 - \Phi(z)$$



11

Complimentary Standard Gaussian cdf

In electrical engineering, Q-function is used as a matter of convention, which is defined by:

$$Q(z) = 1 - \Phi(z) = \frac{1}{\sqrt{2\pi}} \int_z^{\infty} e^{-\frac{z^2}{2}} dz$$

Q-function is also known as the tail of the standard normal distribution.

$$Q(-z) = 1 - Q(z) = \Phi(z)$$

Gaussian cdf and complimentary cdf

12

Intervals of Standard Normal Random Variable

$$P[z_1 < z < z_2] = \frac{1}{\sqrt{2\pi}\sigma}\int_{z_1}^{z_2} e^{-\frac{z^2}{2}} dz = \Phi(z_2) - \Phi(z_1)$$

$$P[z_1 < z < z_2] = Q(z_2) - Q(z_1)$$

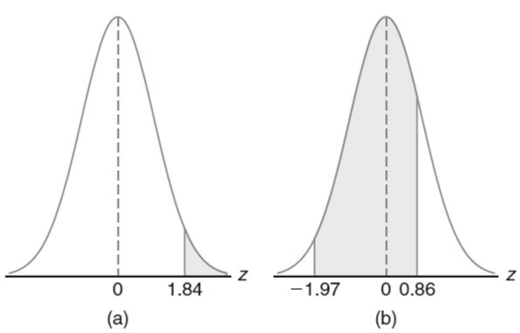
13

Example

Given a standard normal distribution, find the area under the curve that lies

(a) to the right of $z = 1.84$ and

(b) between $z = -1.97$ and $z = 0.86$.



14

Example

X is the Gaussian $(0, 1)$ random variable and Y is the Gaussian $(0, 2)$ random variable. Sketch the PDFs $f_X(x)$ and $f_Y(y)$ on the same axes and find:

(a) $P[-1 < X \leq 1]$,

(b) $P[-1 < Y \leq 1]$,

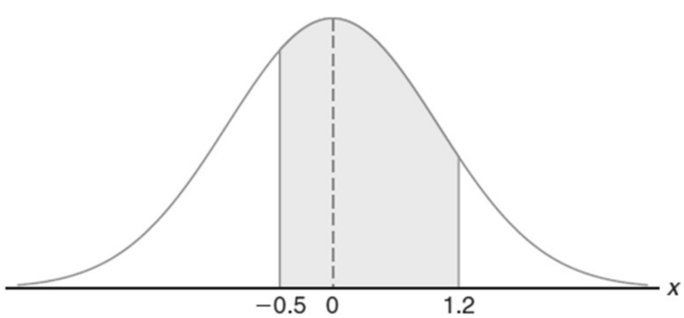
(c) $P[X > 3.5]$,

(d) $P[Y > 3.5]$.

15

Example

Given a random variable X having a normal distribution with $\mu = 50$ and $\sigma = 10$, find the probability that X assumes a value between 45 and 62.



16

Example

In an optical fiber transmission system, the probability of a bit error is $Q(\sqrt{\gamma/2})$, where γ is the signal-to-noise ratio. What is the minimum value of γ that produces a bit error rate not exceeding 10^{-6}?

17

Example

A communication system accepts a positive voltage V as input and outputs a voltage $Y = \alpha V + N$, where $\alpha = 10^{-2}$ and N is a Gaussian random variable with parameters $m = 0$ and $\sigma = 2$. Find the value of V that gives $P[Y < 0] = 10^{-6}$.

The probability $P[Y < 0]$ is written in terms of N as follows:

$$P[Y < 0] = P[\alpha V + N < 0]$$

$$= P[N < -\alpha V] = \Phi\left(\frac{-\alpha V}{\sigma}\right) = Q\left(\frac{\alpha V}{\sigma}\right) = 10^{-6}.$$

From Table 4.3 we see that the argument of the Q-function should be $\alpha V/\sigma = 4.753$. Thus $V = (4.753)\sigma/\alpha = 950.6$.

18

Example

A certain type of storage battery lasts, on average, 3.0 years with a standard deviation of 0.5 year. Assuming that battery life is normally distributed, find the probability that a given battery will last less than 2.3 years.

19

References

1. Walpole, R.E., Myers, R.H., Myers, S.L. and Ye, K. (2007) *Probability & Statistics for Engineers & Scientists.* 9th Edition, Pearson Education, Inc.
2. Leon-Garcia, A. (2008). *Probability, Statistics, and Random Processes for Electrical Engineering*. 3rd Edition, Pearson/Prentice Hall.

20