Chap # 9 -: Laplace Transform :-

→ Laplace transform is used for continuous time signals.

→ LT is the <u>general</u> form of Fourier Transform.

$$x(t) \overset{FT}{\rightleftharpoons} X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

→ $\mathcal{L}T$ is defined as;

$$f(t) \rightleftharpoons \mathcal{L}\{f(t)\} = F(s)$$

$$F(s) = \int_{-\infty}^{\infty} f(t)\, e^{-st}\, dt$$

complex variable

$$s = \sigma + j\omega$$

$$Re\{s\} = \sigma$$

$$Img\{s\} = j\omega$$

-: Classification of $\mathcal{L}T$:-

Bilateral $\mathcal{L}T$ (Two sided LT) $\int_{-\infty}^{\infty}$

Unilateral LT (Single sided $\mathcal{L}T$) $\int_{0}^{\infty}$

Important :-

$\mathcal{L}T$ is defined with region of Convergence (ROC). $F(s)$

* All values of $s = \sigma + j\omega$ for which $\mathcal{L}T$ Converges is known as ROC.

(study it later in lecture)

-: Relationship b/w FT & $\mathcal{L}T$:-

$$\mathcal{L}T\Big|_{s=j\omega} = FT$$

$$s = \sigma + j\omega$$
$$\sigma = 0$$

or $\sigma = 0$

$$\boxed{\text{If } \sigma = 0 \text{ then } LT = FT}$$

Condition for Existence of LT :-

LT of $f(t)$ = FT of $f(t)e^{-\sigma t}$

$$e^{-st}$$
$$e^{-(\sigma+j\omega)t}$$
$$e^{-\sigma t} \cdot e^{-j\omega t}$$

$$F(s) = \int_{-\infty}^{\infty} \underbrace{f(t)e^{-\sigma t}}_{f_1(t)} \cdot e^{-j\omega t} \, dt$$

$$F(s) = \int_{-\infty}^{\infty} f_1(t) \, e^{-j\omega t} \, dt$$

$F(s)$ will exists only if $f_1(t)$ is absolutely integrable.

$$\int_{-\infty}^{\infty} |f_1(t)| \, dt < \infty$$

$$\boxed{\int_{-\infty}^{\infty} \left| f(t) e^{-\sigma t} \right| dt < \infty}$$

range of σ is defining abs. integrability

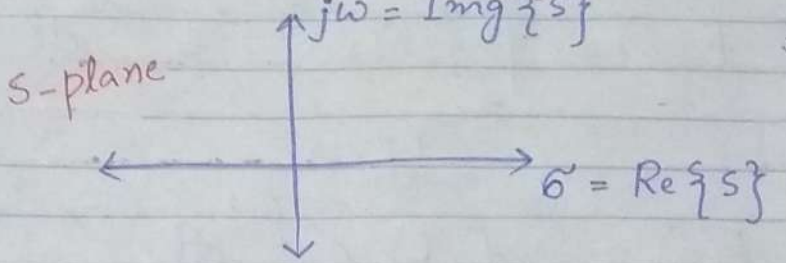
↓

ROC

-: ROC and its properties :- (4)

Range of 's' in s-plane for which LT is finite or Convergent.

$s = \sigma + j\omega$



Why ROC is important to mention alongwith the algebraic expression???

$\boxed{\text{Exp } 9.1}$

$$x(t) = e^{-at}u(t)$$

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt$$

$$= \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st}\,dt$$

$$= \int_{-\infty}^{\infty} u(t)e^{-(s+a)t}\,dt$$

$$= \int_{0}^{\infty} e^{-(s+a)t}\,dt$$

$$= \left.\frac{e^{-(s+a)t}}{-(s+a)}\right|_{0}^{\infty}$$

$$= \frac{1}{(s+a)} \longrightarrow \text{Same LT}$$

$\boxed{\text{Exp } 9.2}$

$$x(t) = -e^{-at}u(-t)$$

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt$$

$$= \int_{-\infty}^{\infty} -e^{-at}u(-t)e^{-st}\,dt$$

$$= -\int_{-\infty}^{\infty} u(-t)e^{-(s+a)t}\,dt$$

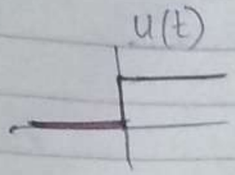
$$= -\int_{-\infty}^{0} e^{-(s+a)t}\,dt$$

$$= -\left.\frac{e^{-(s+a)t}}{-(s+a)}\right|_{-\infty}^{0}$$

$$= \frac{1}{s+a}$$

→ ROC is associated with range of σ (5)

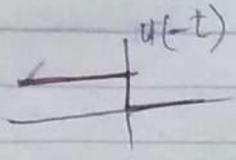
$Re\{s\} = \sigma$



$$\underbrace{e^{-at}\,u(t)}_{RSS} \longrightarrow \frac{1}{s+a} \quad \begin{matrix}\to \text{zero '0'} \\ \to \text{poles 'x'}\end{matrix}$$

$Re\{s\} + a > 0$ (1)

ROC : $Re\{s\} > -a$

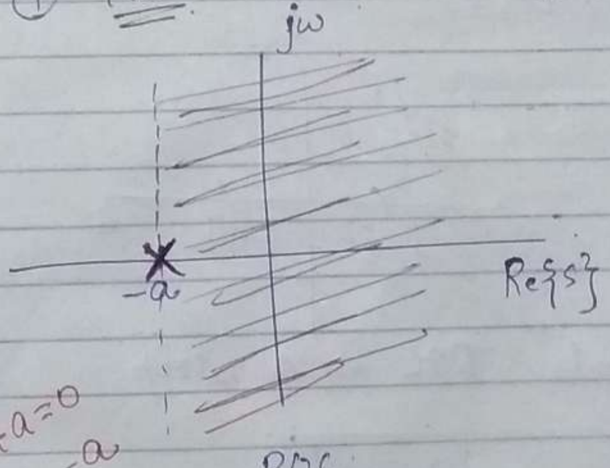


$$-\underbrace{e^{-at}\,u(-t)}_{LSS} \longrightarrow \frac{1}{s+a} \quad (2)$$

$Re\{s\} + a < 0$

ROC : $Re\{s\} < -a$

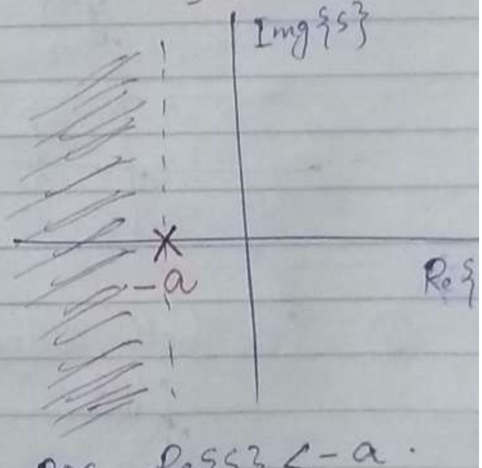
(1) Plot



$s+a=0$
$s=-a$

ROC
$Re\{s\} > -a$

$s+a=0$
$s=-a$



ROC: $Re\{s\} < -a$

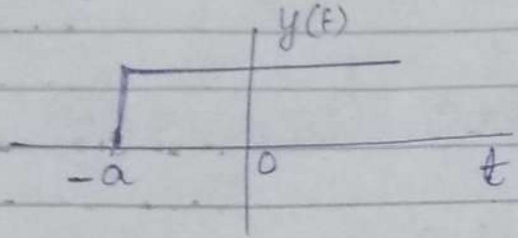
-: Properties :-

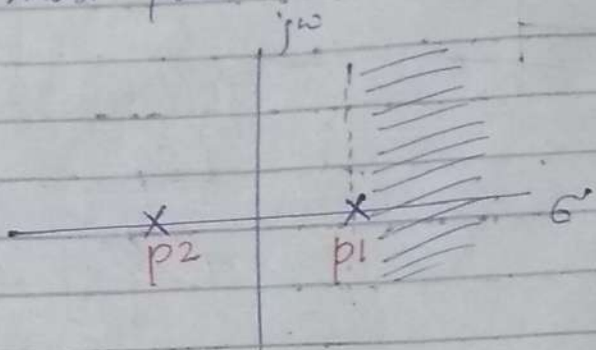
1. ROC does not include any pole.

$$F(s) = \frac{1 \rightarrow \text{zero}}{s+2 \rightarrow \text{pole}} \qquad s+2=0 \Rightarrow \quad s=-2$$

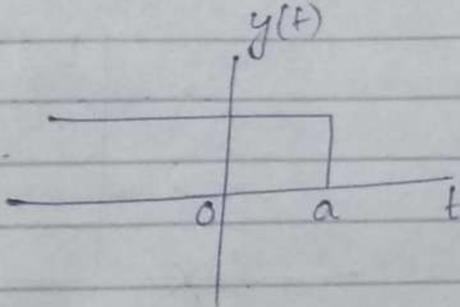
$$F(-2) = \frac{1}{-2+2} = \frac{1}{0} = \infty \text{ (Diverges)}.$$

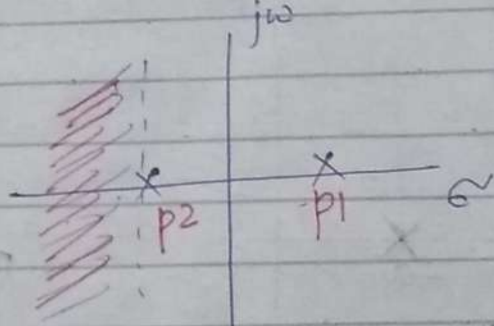
2. For right sided signals (RSS), ROC is right side to the right most pole.



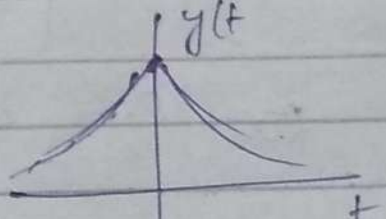


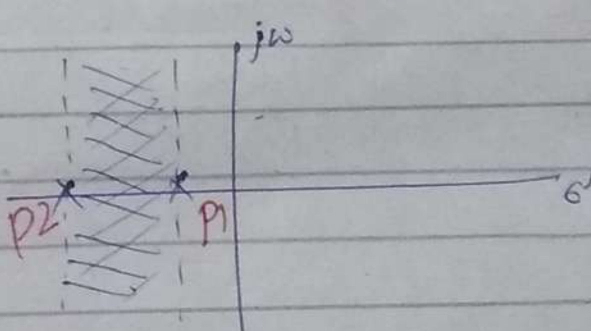
3. For left sided signals (LSS), ROC is left side to the left most pole.



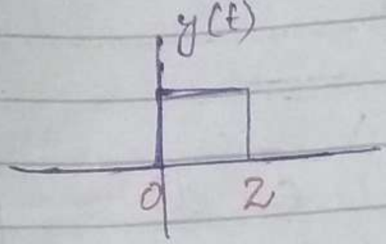


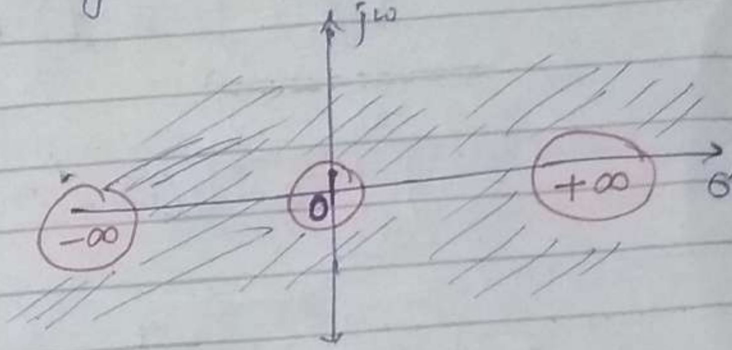
4. For both sided signals, ROC is a strip.





For finite duration signals, ROC is the entire plane. excluding $s = 0$ or/& $\pm\infty$





$$\left.\begin{array}{l} s = 0 \\ s = -\infty \\ s = +\infty \end{array}\right\}$$

Exp 9.3 :- $x(t) = 3e^{-2t}u(t) - 2e^{-t}u(t)$

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

$$= \int_{-\infty}^{\infty} \left(3e^{-2t}u(t) - 2e^{-t}u(t)\right) e^{-st}\, dt$$

$$= 3\int_{-\infty}^{\infty} e^{-2t}u(t)\, e^{-st}\, dt - 2\int_{-\infty}^{\infty} e^{-t}u(t)\, e^{-st}\, dt$$

$$= 3\int_{0}^{\infty} e^{-(s+2)t}\, dt - 2\int_{0}^{\infty} e^{-(s+1)t}\, dt$$

$$= 3 \left.\frac{e^{-(s+2)}}{-(s+2)}\right|_{0}^{\infty} - 2 \left.\frac{e^{-(s+1)}}{-(s+1)}\right|_{0}^{\infty}$$

$$X(s) = \frac{3}{s+2} - \frac{2}{s+1}$$

ROC: $Re\{s\} > -1$

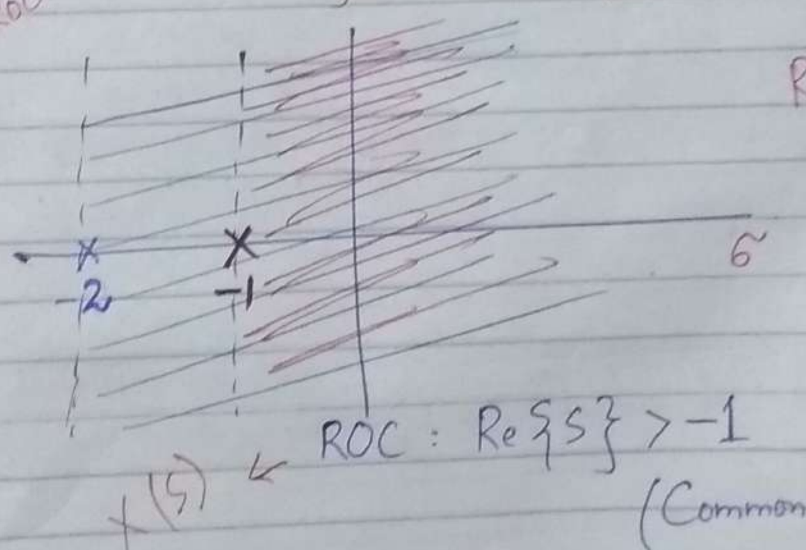
$s+2=0$
$s=-2$

$\frac{3}{s+2} \downarrow$
$u(t) \rightarrow RSS$
$Re\{s\}+2>0$
$Re\{s\}>-2$
ROC1

$\frac{2}{s+1} \downarrow$
$u(t) \rightarrow RSS$
$Re\{s\}+1>0$
$Re\{s\}>-1$
ROC2

$s+1=0$
$s=-1$

$ROC_1 \cap ROC_2$

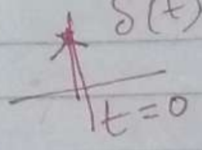


$X(s) \swarrow$ ROC : $Re\{s\} > -1$
(Common area)

Exp 9.5 :-

$$x(t) = \delta(t) - 4/3\, e^{-t} u(t) + \frac{1}{3} e^{2t} u(t)$$

$$\mathcal{L}\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t) e^{-st} dt$$



$$= e^{-st}\Big|_{t=0} = 1$$

$$\mathcal{L}\left\{\frac{4}{3} e^{-t} u(t)\right\} = \frac{4}{3}\int_{-\infty}^{\infty} e^{-t} u(t) e^{-st} dt$$

$$= 4/3 \int_{0}^{\infty} e^{-(s+1)t} dt$$

$$= 4/3 \left(\frac{1}{s+1}\right).$$

$$\mathcal{L}\left\{\frac{1}{3} e^{2t} u(t)\right\} = \frac{1}{3}\int_{-\infty}^{\infty} e^{2t} u(t) e^{-st} dt$$

$$= \frac{1}{3}\int_{0}^{\infty} e^{-(s-2)t} dt$$

$$= \frac{1}{3}\left(\frac{1}{s-2}\right).$$

$$\therefore\ X(s) = 1 + \frac{4}{3}\left(\frac{1}{s+1}\right) - \frac{1}{3}\left(\frac{1}{s-2}\right)$$

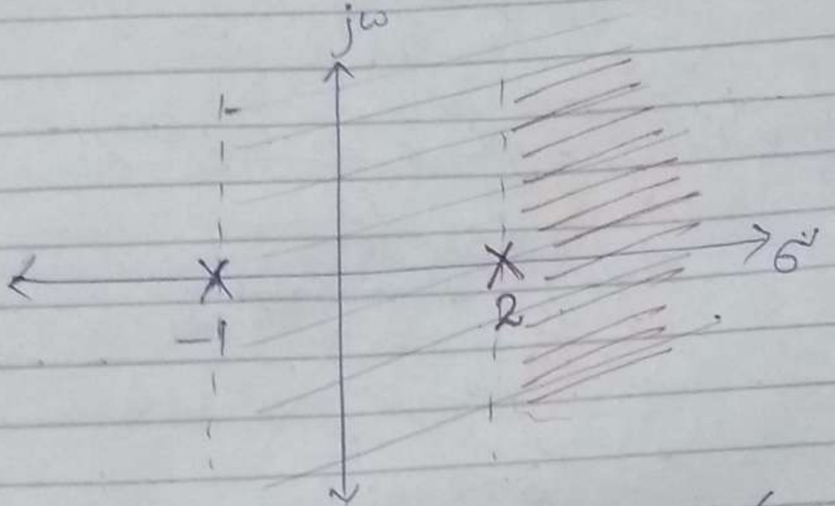
~~ROC Entire s-plane~~

(10)

$s+1=0 \quad s=-1 \qquad s-2=0 \quad s=2$

$$X(s) = 1 + \frac{4}{3}\left(\frac{1}{s+1}\right) + \frac{1}{3}\left(\frac{1}{s-2}\right)$$

ROC: Entire s-plane $\qquad Re\{s\}+1>0 \quad Re\{s\}>-1 \qquad Re\{s\}>2$



ROC: $Re\{s\} > 2$ (Common area)

$$\begin{cases} e^{-at}u(t) = \frac{1}{s+a} & Re\{s\} > -a \\ -e^{-at}u(-t) = \frac{1}{s+a} & Re\{s\} < -a \end{cases}$$

H.W.

$$\begin{cases} e^{at}u(t) = \frac{1}{s-a} & Re\{s\} > a \\ -e^{at}u(-t) = \frac{1}{s-a} & Re\{s\} < a \end{cases}$$