Lecture 21

2. Source Free parallel RLC Circuit

Source Free parallel RLC circuit

Consider the parallel RLC circuit shown in Fig. 8.13. Assume initial inductor current I_0 and initial capacitor voltage V_0,

$$i(0) = I_0 = \frac{1}{L}\int_{\infty}^{0} v(t)\,dt \tag{8.27a}$$

$$v(0) = V_0 \tag{8.27b}$$

Since the three elements are in parallel, they have the same voltage v across them. According to passive sign convention, the current is entering each element; that is, the current through each element is leaving the top node. Thus, applying KCL at the top node gives

$$\frac{v}{R} + \frac{1}{L}\int_{-\infty}^{t} v(\tau)d\tau + C\frac{dv}{dt} = 0 \tag{8.28}$$

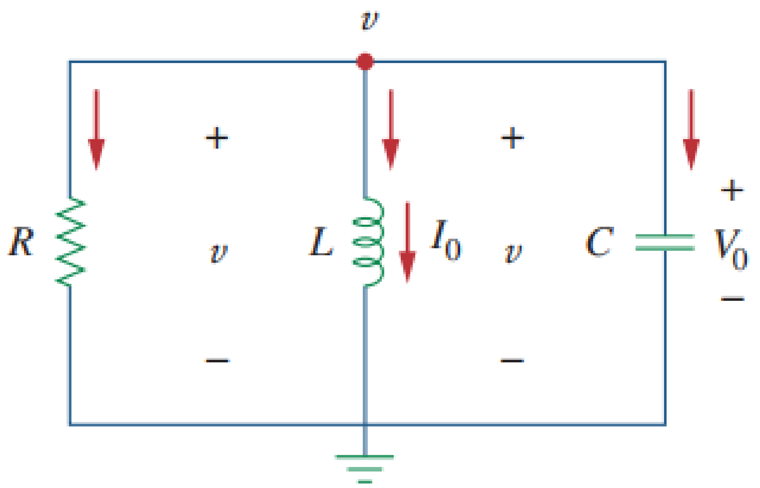

Figure 8.13
A source-free parallel RLC circuit.

Taking the derivative with respect to t and dividing by C results in

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC}v = 0 \tag{8.29}$$

We obtain the characteristic equation by replacing the first derivative by s and the second derivative by s^2. By following the same reasoning used in establishing Eqs. (8.4) through (8.8), the characteristic equation is obtained as

$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0 \tag{8.30}$$

The roots of the characteristic equation are

$$s_{1,2} = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

or

$$\boxed{s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}} \tag{8.31}$$

where

$$\boxed{\alpha = \frac{1}{2RC}, \qquad \omega_0 = \frac{1}{\sqrt{LC}}} \tag{8.32}$$

The names of these terms remain the same as in the preceding section, as they play the same role in the solution. Again, there are three possible solutions, depending on whether $\alpha > \omega_0$, $\alpha = \omega_0$, or $\alpha < \omega_0$. Let us consider these cases separately.

Overdamped Case ($\alpha > \omega_0$)

From Eq. (8.32), $\alpha > \omega_0$ when $L > 4R^2C$. The roots of the characteristic equation are real and negative. The response is

$$\boxed{v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}} \tag{8.33}$$

Critically Damped Case ($\alpha = \omega_0$)

For $\alpha = \omega_0$, $L = 4R^2C$. The roots are real and equal so that the response is

$$\boxed{v(t) = (A_1 + A_2 t)e^{-\alpha t}} \tag{8.34}$$

Underdamped Case ($\alpha < \omega_0$)

When $\alpha < \omega_0$, $L < 4R^2C$. In this case the roots are complex and may be expressed as

$$s_{1,2} = -\alpha \pm j\omega_d \tag{8.35}$$

where

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} \tag{8.36}$$

The response is

$$\boxed{v(t) = e^{-\alpha t}(A_1 \cos \omega_d t + A_2 \sin \omega_d t)} \tag{8.37}$$

The constants A_1 and A_2 in each case can be determined from the initial conditions. We need $v(0)$ and $dv(0)/dt$. The first term is known from Eq. (8.27b). We find the second term by combining Eqs. (8.27) and (8.28), as

The constants A_1 and A_2 in each case can be determined from the initial conditions. We need $v(0)$ and $dv(0)/dt$. The first term is known from Eq. (8.27b). We find the second term by combining Eqs. (8.27) and (8.28), as

$$\frac{V_0}{R} + I_0 + C\frac{dv(0)}{dt} = 0$$

or

$$\frac{dv(0)}{dt} = -\frac{(V_0 + RI_0)}{RC} \tag{8.38}$$

The voltage waveforms are similar to those shown in Fig. 8.9 and will depend on whether the circuit is overdamped, underdamped, or critically damped.

Problem solving Strategy:

1. t<0, $v(0)$ and $i(0)$
2. $\frac{dv(0)}{dt}$ (through KCL)
3. α , ω_0
4. $v(t)$ equation & $S_{1,2}$ or W_d etc
5. A1 and A2

Example 8.5

In the parallel circuit of Fig. 8.13, find $v(t)$ for $t > 0$, assuming $v(0) = 5$ V, $i(0) = 0$, $L = 1$ H, and $C = 10$ mF. Consider these cases: $R = 1.923\ \Omega$, $R = 5\ \Omega$, and $R = 6.25\ \Omega$.

Solution:

CASE 1 If $R = 1.923\ \Omega$,

$$\alpha = \frac{1}{2RC} = \frac{1}{2 \times 1.923 \times 10 \times 10^{-3}} = 26$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{1 \times 10 \times 10^{-3}}} = 10$$

Since $\alpha > \omega_0$ in this case, the response is overdamped. The roots of the characteristic equation are

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -2, -50$$

and the corresponding response is

$$v(t) = A_1 e^{-2t} + A_2 e^{-50t} \tag{8.5.1}$$

We now apply the initial conditions to get A_1 and A_2.

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$$v(0) = 5 = A_1 + A_2 \tag{8.5.2}$$

$$\frac{dv(0)}{dt} = -\frac{v(0) + Ri(0)}{RC} = -\frac{5 + 0}{1.923 \times 10 \times 10^{-3}} = -260$$

But differentiating Eq. (8.5.1),

$$\frac{dv}{dt} = -2A_1e^{-2t} - 50A_2e^{-50t}$$

At $t = 0$,

$$-260 = -2A_1 - 50A_2 \tag{8.5.3}$$

From Eqs. (8.5.2) and (8.5.3), we obtain $A_1 = -0.2083$ and $A_2 = 5.208$. Substituting A_1 and A_2 in Eq. (8.5.1) yields

$$v(t) = -0.2083e^{-2t} + 5.208e^{-50t} \tag{8.5.4}$$

CASE 2 When $R = 5\ \Omega$,

$$\alpha = \frac{1}{2RC} = \frac{1}{2 \times 5 \times 10 \times 10^{-3}} = 10$$

while $\omega_0 = 10$ remains the same. Since $\alpha = \omega_0 = 10$, the response is critically damped. Hence, $s_1 = s_2 = -10$, and

$$v(t) = (A_1 + A_2 t)e^{-10t} \qquad \textbf{(8.5.5)}$$

To get A_1 and A_2, we apply the initial conditions

$$v(0) = 5 = A_1 \qquad \textbf{(8.5.6)}$$

$$\frac{dv(0)}{dt} = -\frac{v(0) + Ri(0)}{RC} = -\frac{5 + 0}{5 \times 10 \times 10^{-3}} = -100$$

But differentiating Eq. (8.5.5),

$$\frac{dv}{dt} = (-10A_1 - 10A_2 t + A_2)e^{-10t}$$

At $t = 0$,

$$-100 = -10A_1 + A_2 \qquad \textbf{(8.5.7)}$$

From Eqs. (8.5.6) and (8.5.7), $A_1 = 5$ and $A_2 = -50$. Thus,

$$v(t) = (5 - 50t)e^{-10t}\ \text{V} \qquad \textbf{(8.5.8)}$$

CASE 3 When $R = 6.25\ \Omega$,

$$\alpha = \frac{1}{2RC} = \frac{1}{2 \times 6.25 \times 10 \times 10^{-3}} = 8$$

while $\omega_0 = 10$ remains the same. As $\alpha < \omega_0$ in this case, the response is underdamped. The roots of the characteristic equation are

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -8 \pm j6$$

Hence,

$$v(t) = (A_1 \cos 6t + A_2 \sin 6t)e^{-8t} \tag{8.5.9}$$

We now obtain A_1 and A_2, as

$$v(0) = 5 = A_1 \tag{8.5.10}$$

$$\frac{dv(0)}{dt} = -\frac{v(0) + Ri(0)}{RC} = -\frac{5 + 0}{6.25 \times 10 \times 10^{-3}} = -80$$

But differentiating Eq. (8.5.9),

$$\frac{dv}{dt} = (-8A_1 \cos 6t - 8A_2 \sin 6t - 6A_1 \sin 6t + 6A_2 \cos 6t)e^{-8t}$$

At $t = 0$,

$$-80 = -8A_1 + 6A_2 \tag{8.5.11}$$

From Eqs. (8.5.10) and (8.5.11), $A_1 = 5$ and $A_2 = -6.667$. Thus,

$$v(t) = (5 \cos 6t - 6.667 \sin 6t)e^{-8t} \tag{8.5.12}$$

Example 8.6

Find $v(t)$ for $t > 0$ in the *RLC* circuit of Fig. 8.15.

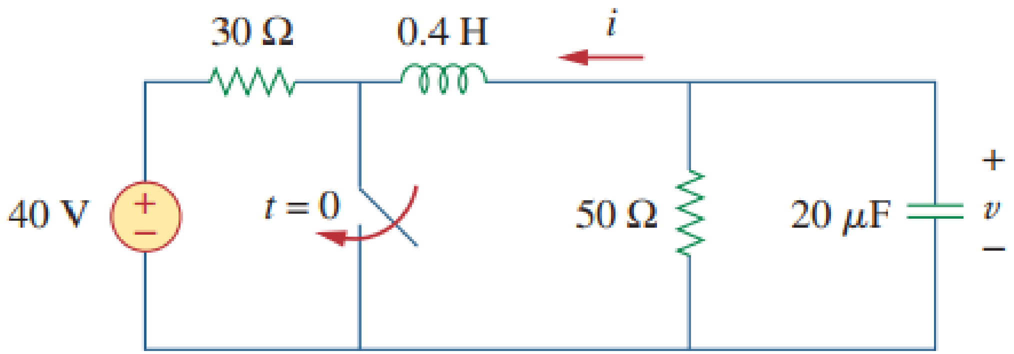
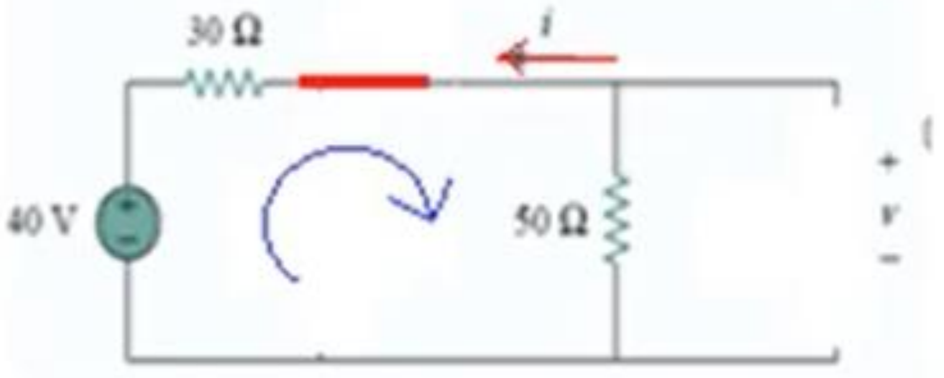


Figure 8.15
For Example 8.6.

Step1:

$t < 0$



The initial voltage across the capacitor is

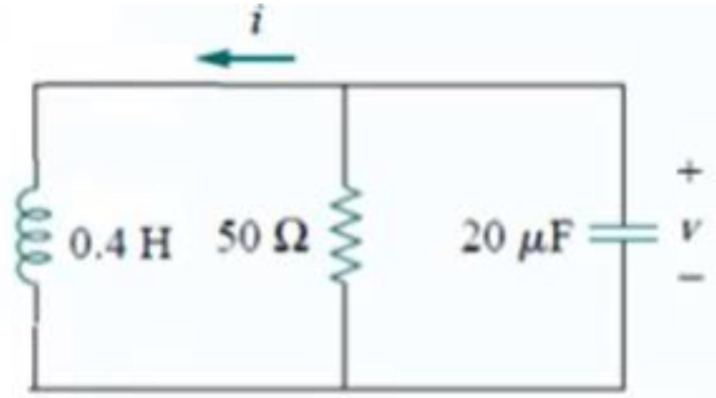
$$v(0) = \frac{50}{30+50}(40) = \frac{5}{8} \times 40 = 25\ \text{V} \qquad (1)$$

The initial current through the inductor is

$$i(0) = -\frac{40}{30+50} = -0.5\ \text{A}$$

Step 2:

$$\boxed{\textbf{2.}\ \frac{dv(0)}{dt} \quad \text{(through KCL)}}$$



$$\boxed{\begin{aligned} v(0) &= 25\text{ V} \\ i(0) &= -0.5\text{ A} \end{aligned}}$$

$$i_R(\text{t}) + i(\text{t}) + i_C(\text{t}) = 0$$

$$i_R(0) + i(0) + i_C(0) = 0$$

$$\frac{V_0}{R} + I_0 + C\frac{dv(0)}{dt} = 0$$

$$\frac{25}{50} - 0.5 + 20\,\mu\text{F} \times \frac{dv(0)}{dt} = 0$$

$$0.5 - 0.5 + 20\,\mu\text{F} \times \frac{dv(0)}{dt} = 0$$

$$\frac{dv(0)}{dt} = 0$$

Step 3:

$$\alpha = \frac{1}{2RC} = \frac{1}{2 \times 50 \times 20 \times 10^{-6}} = 500$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.4 \times 20 \times 10^{-6}}} = 354$$

$\alpha > \omega_0$ **Overdamped Case**

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

$$= -500 \pm \sqrt{250{,}000 - 124{,}997.6} \quad = -500 \pm 354$$

or $s_1 = -854, \qquad s_2 = -146$

$$v(t) = A_1 e^{-854t} + A_2 e^{-146t}$$

Step 4:

$$v(t) = A_1 e^{-854t} + A_2 e^{-146t}$$

At $t = 0$, the above eq becomes

$$v(0) = A_1 + A_2$$

or $$25 = A_1 + A_2$$

$$A_2 = 25 - A_1 \qquad (4)$$

$$\boxed{\begin{aligned} v(0) &= 25\text{ V} \\ i(0) &= -0.5\text{ A} \end{aligned}}$$

Problem solving **Strategy**:

1. t<0, $v(0)$ and $i(0)$
2. $\frac{dv(0)}{dt}$ (through KCL)
3. α , ω_0
4. $v(t)$ equation & $s_{1,2}$ or W_d etc
5. A1 and A2

Taking the derivative of $v(t)$

$$\frac{dv}{dt} = -854A_1 e^{-854t} - 146A_2 e^{-146t}$$

At $t = 0$,

$$\frac{dv(0)}{dt} = -854A_1 - 146A_2$$

$$\boxed{\frac{dv(0)}{dt} = 0}$$

$$0 = 854A_1 + 146A_2 \qquad (5)$$

Solving Eqs. (4) and (5) gives

$A_1 = -5.16, \qquad A_2 = 30.16$

$$v(t) = -5.156e^{-854t} + 30.16e^{-146t}\text{ V}$$

Practice Problem 8.6

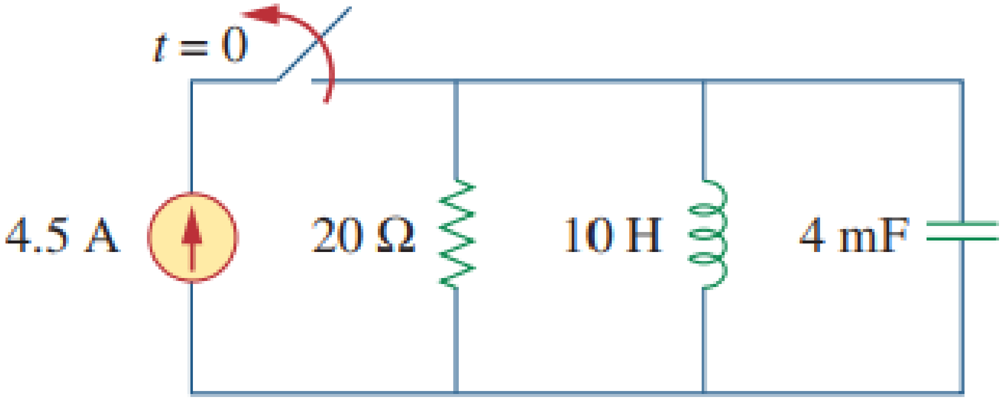
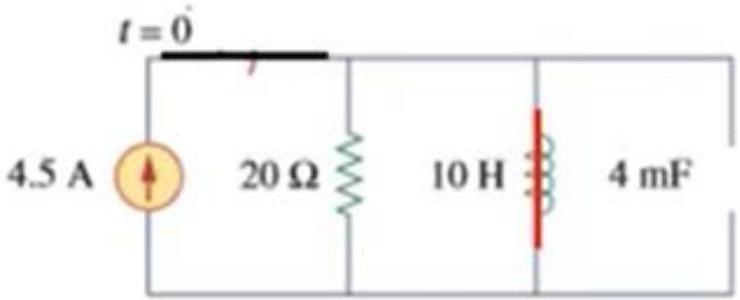


Figure 8.17
For Practice Prob. 8.6.

Refer to the circuit in Fig. 8.17. Find $v(t)$ for $t > 0$.

Solution:

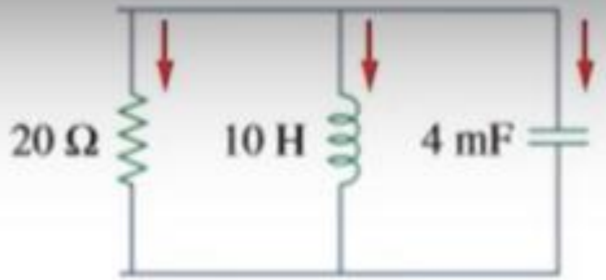
$t < 0$



$i(0) = 4.5$ A

$v(0) =$ **0** V

$\boxed{\textbf{2.}\ \frac{dv(0)}{dt}\ \text{(through KCL)}}$



$$i_R(\mathrm{t}) + i(\mathrm{t}) + i_C(\mathrm{t}) = 0$$

$$i_R(0) + i(0) + i_C(0) = 0$$

$$\frac{V_0}{R} + I_0 + C\frac{dv(0)}{dt} = 0$$

$$0 + 4.5 + 4\ \mathrm{mF} \times \frac{dv(0)}{dt} = 0$$

$$4\ \mathrm{mF} \times \frac{dv(0)}{dt} = -\ 4.5$$

$$\frac{dv(0)}{dt} = -\ 1125$$

$$\boxed{\begin{aligned} i(0) &= 4.5\ \mathrm{A} \\ v(0) &= \mathbf{0}\ \mathrm{V} \end{aligned}}$$

$$\alpha = \frac{1}{2RC} = \frac{1}{2 \times 20 \times 4 \times 10^{-3}} = 6.25$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{10 \times 4 \times 10^{-3}}} = 5$$

$\alpha > \omega_0$ **Overdamped Case**

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

$$= -6.25 \pm \sqrt{39.06 - 25} \qquad = -6.25 \pm 3.75$$

$$\text{or} \quad s_1 = -2.5\,, \qquad s_2 = -10$$

$$v(t) = A_1 e^{-2.5t} + A_2 e^{-10t}$$

$$v(t) = A_1 e^{-2.5t} + A_2 e^{-10t}$$

At $t = 0$, the above eq becomes

$$\boxed{\begin{array}{l} i(0) = 4.5 \text{ A} \\ v(0) = 0 \text{ V} \end{array}}$$

$$v(0) = A_1 + A_2$$

or $$0 = A_1 + A_2 \quad (4)$$

Taking the derivative of $v(t)$

$$\frac{dv}{dt} = -2.5 A_1 e^{-2.5t} - 10 A_2 e^{-10t}$$

At $t = 0$,

$$\frac{dv(0)}{dt} = -2.5 A_1 - 10 A_2$$

$$\boxed{\frac{dv(0)}{dt} = -1125}$$

$$-1125 = -2.5 A_1 - 10 A_2 \quad (5)$$

Solving Eqs. (4) and (5) gives

$$A_1 = -150, \qquad A_2 = 150$$

Thanks