Alexander-Sadiku
Fundamentals of Electric Circuits

Chapter 14
Frequency Response

Frequency Response
Chapter 14

14.1 Introduction

14.2 Transfer Function

14.3 Series Resonance

14.4 Parallel Resonance

14.5 Passive Filters

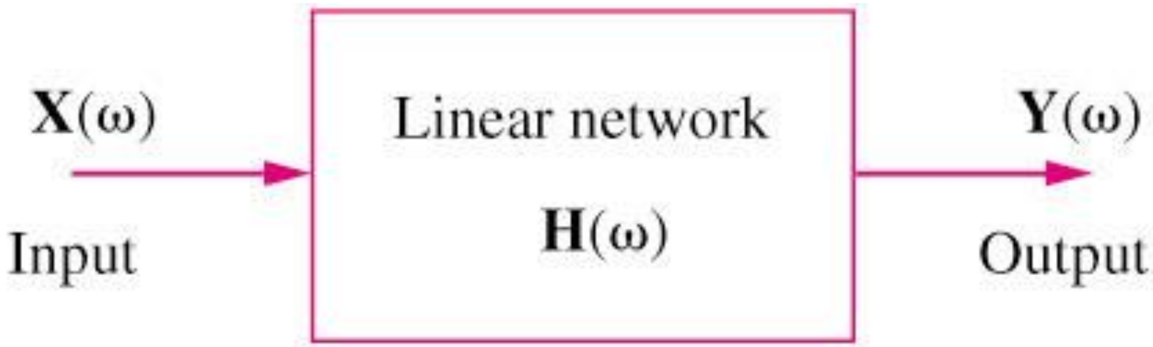
14.1 Introduction (1)

What is FrequencyResponse of a Circuit?

It is the variation in a circuit's behavior with change in signal frequency and may also be considered as the variation of the gain and phase with frequency.

14.2 Transfer Function (1)

- The transfer function H(ω) of a circuit is the **frequency-dependent ratio** of a phasor output Y(ω) (an element voltage or current) to a phasor input X(ω) (source voltage or current).



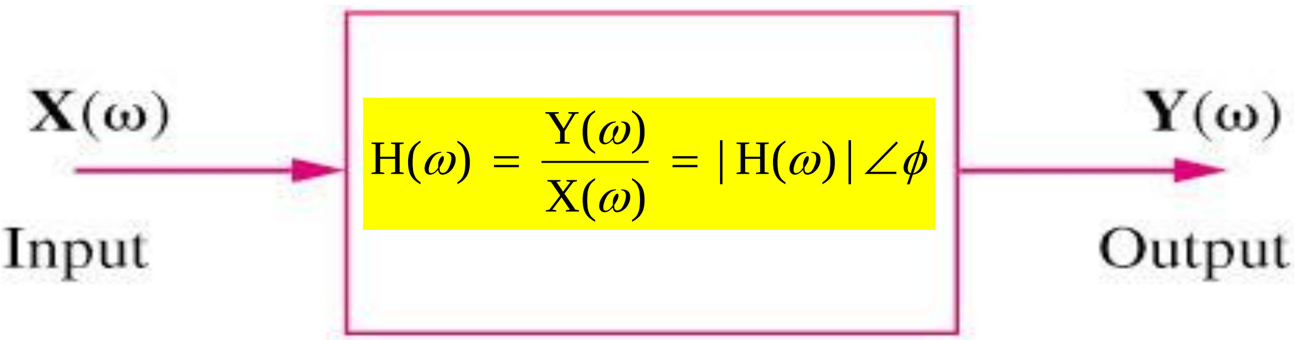
$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = |H(\omega)| \angle \phi$$

14.2 Transfer Function (2)

- Four possible transfer functions:

$$H(\omega) = \text{Voltage gain} = \frac{V_o(\omega)}{V_i(\omega)}$$

$$H(\omega) = \text{Transfer Impedance} = \frac{V_o(\omega)}{I_i(\omega)}$$



$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = |H(\omega)|\angle\phi$$

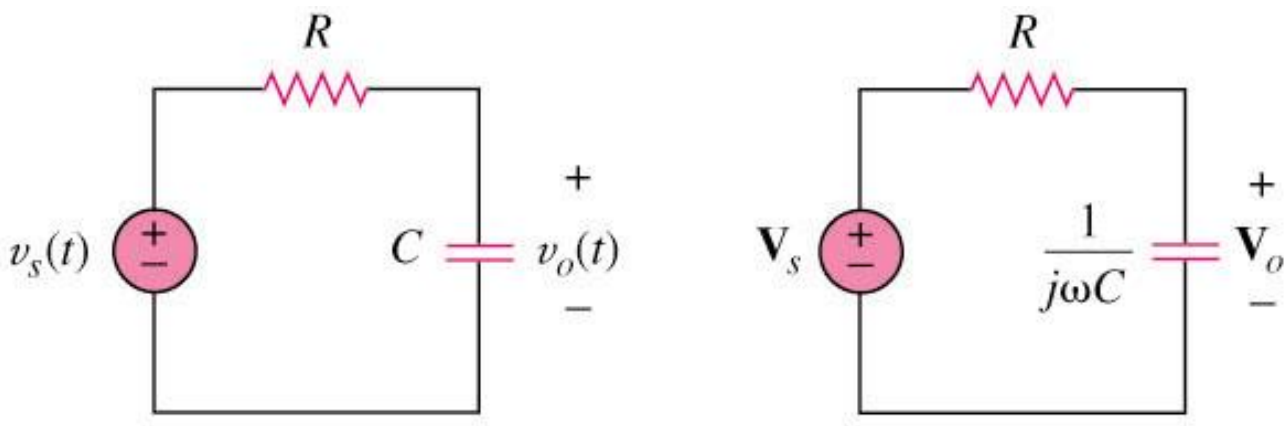
$$H(\omega) = \text{Current gain} = \frac{I_o(\omega)}{I_i(\omega)}$$

$$H(\omega) = \text{Transfer Admittance} = \frac{I_o(\omega)}{V_i(\omega)}$$

14.2 Transfer Function (3)

Example 1

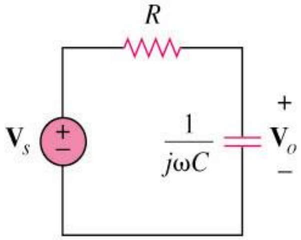
For the RC circuit shown below, obtain the transfer function Vo/Vs and its frequency response.
Let $v_s = V_m \cos\omega t$.



(a) (b)

14.2 Transfer Function (4)

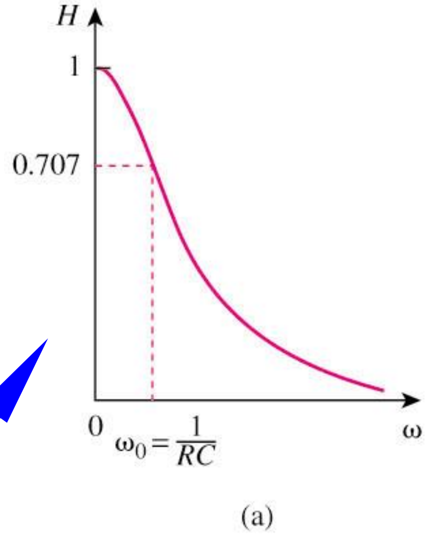
Solution:



The transfer function is

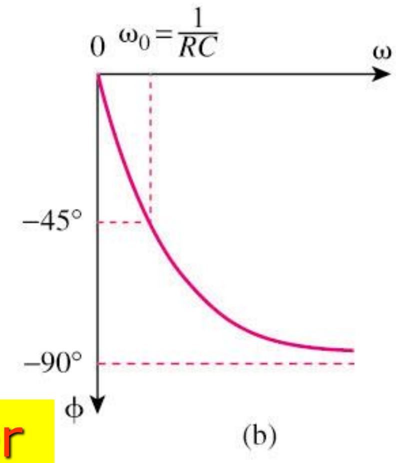
$$H(\omega) = \frac{V_o}{V_s} = \frac{\frac{1}{j\omega C}}{R + 1/j\omega C} = \frac{1}{1 + j\omega RC}$$

The magnitude is $|H(\omega)| = \frac{1}{\sqrt{1+(\omega/\omega_o)^2}}$



The phase is $\phi = -\tan^{-1}\frac{\omega}{\omega_o}$

$$\omega_o = 1/RC$$

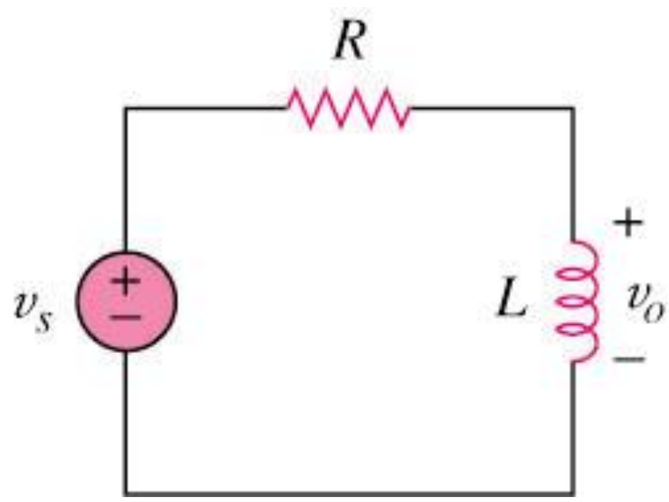


Low Pass Filter

14.2 Transfer Function (5)

Example 2

Obtain the transfer function Vo/Vs of the RL circuit shown below, assuming $v_s = V_m \cos\omega t$. Sketch its frequency response.

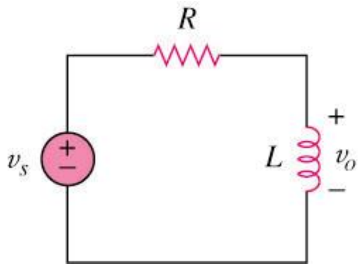


14.2 Transfer Function (6)

Solution:

The transfer function is

$$H(\omega) = \frac{V_o}{V_s} = \frac{j\omega L}{R + j\omega L} = \frac{1}{1 + \dfrac{R}{j\omega L}}$$



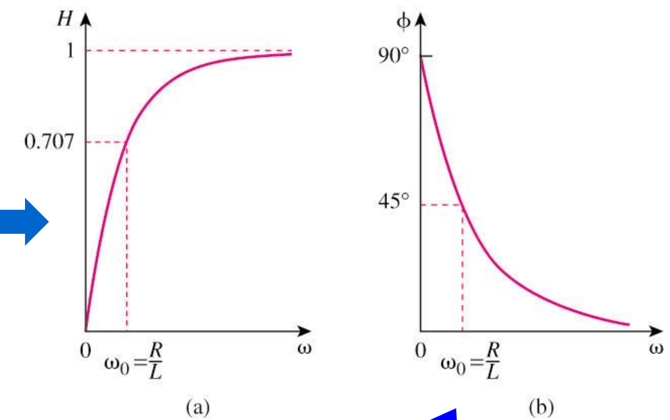
The magnitude is

$$H(\omega) = \frac{1}{\sqrt{1 + (\dfrac{\omega_o}{\omega})^2}}$$

The phase is $\phi = \angle 90° - \tan^{-1}\dfrac{\omega}{\omega_o}$

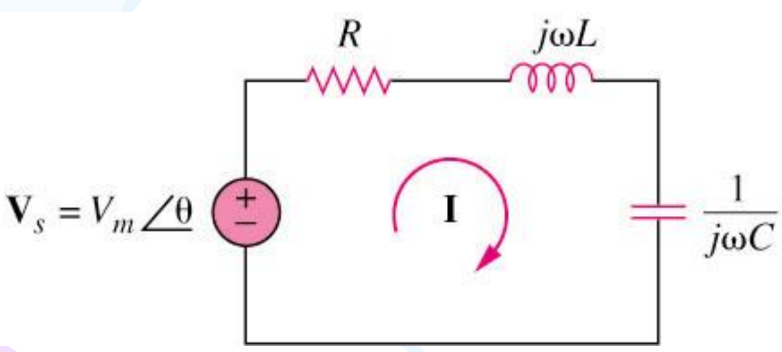
$$\omega_o = R/L$$

High Pass Filter



14.3 Series Resonance (1)

Resonance is a condition in an RLC circuit in which the capacitive and inductive reactance are equal in magnitude, thereby resulting in purely resistive impedance.



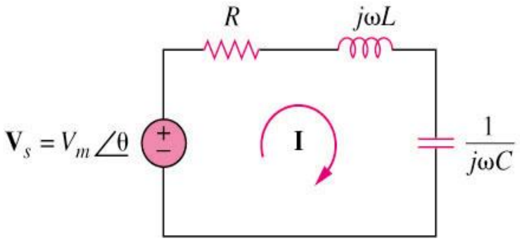
$$Z = R + j(\omega L - \frac{1}{\omega C})$$

Resonance frequency:

$$\omega_o = \frac{1}{\sqrt{LC}} \text{ rad/s} \qquad or$$

$$f_o = \frac{1}{2\pi\sqrt{LC}} \text{ Hz}$$

14.3 Series Resonance (2)



$$Z = R + j(\omega L - \frac{1}{\omega C})$$

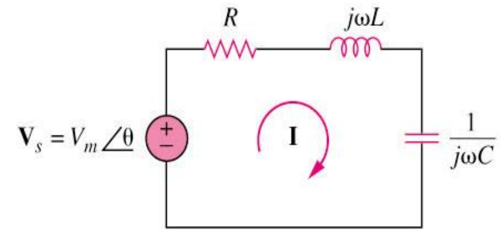
The features of series resonance:

The impedance is purely resistive, Z = R;

- The supply voltage Vs and the current I are in phase, so cos θ = 1;
- The magnitude of the transfer function H(ω) = Z(ω) is minimum;

14.3 Series Resonance (3)

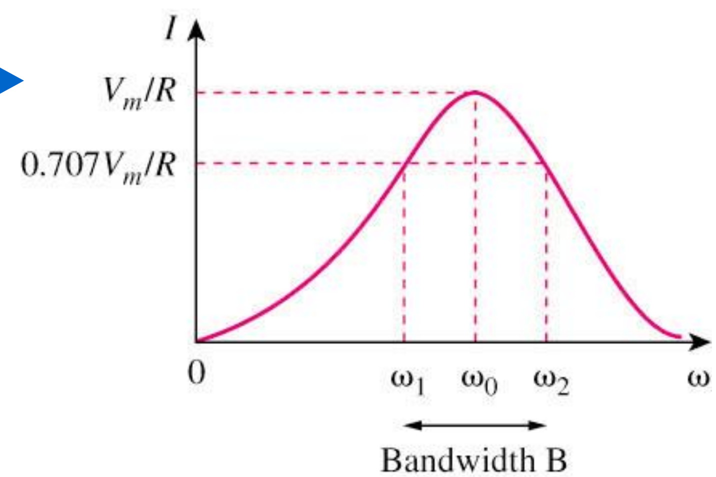
Bandwidth B



$$Z = R + j(\omega L - \frac{1}{\omega C})$$

The frequency response of the resonance circuit current is

$$I = |I| = \frac{V_m}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}}$$



The average power absorbed by the RLC circuit is

$$P(\omega) = \frac{1}{2} I^2 R$$

The highest power dissipated occurs at resonance:

$$P(\omega_o) = \frac{1}{2}\frac{V_m^2}{R}$$

14 3 Series Resonance (4)

Half-power frequencies ω_1 and ω_2 are frequencies at which the dissipated power is half the maximum value:

$$P(\omega_1) = P(\omega_2) = \frac{1}{2}\frac{(V_m/\sqrt{2})^2}{R} = \frac{V_m^2}{4R}$$

The half-power frequencies can be obtained by setting Z equal to $\sqrt{2}$ R.

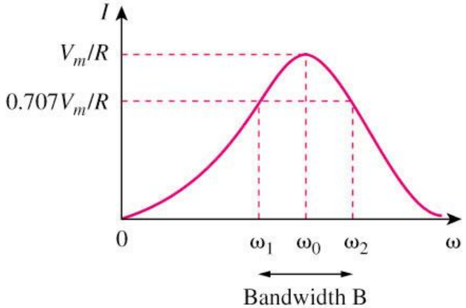
$$\omega_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

$$\omega_2 = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

$$\omega_o = \sqrt{\omega_1 \omega_2}$$

Bandwidth B

$$B = \omega_2 - \omega_1$$



14.3 Series Resonance (5)

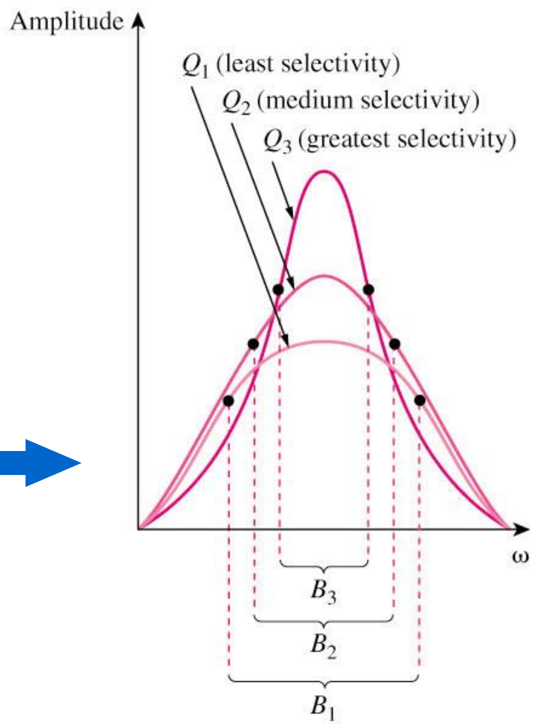
Quality factor,

$$Q = \frac{\text{Peak energy stored in the circuit}}{\text{Energy dissipated by the circuit in one period at resonance}} = \frac{\omega_o L}{R} = \frac{1}{\omega_o CR}$$

The relationship between the B, Q and ω_o:

$$B = \frac{R}{L} = \frac{\omega_o}{Q} = \omega_o^2 CR$$

- The quality factor is the ratio of its resonant frequency to its bandwidth.
- If the bandwidth is narrow, the quality factor of the resonant circuit must be high.
- If the band of frequencies is wide, the quality factor must be low.



14.3 Series Resonance (6)

Example 3

A series-connected circuit has $R = 4\ \Omega$ and $L = 25$ mH.

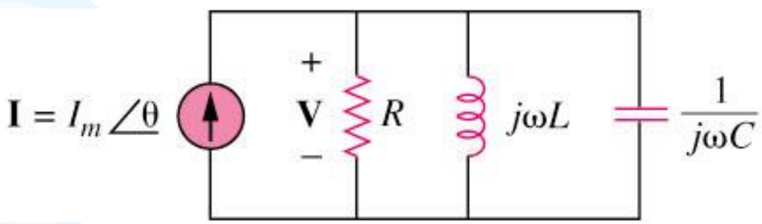
a. Calculate the value of C that will produce a quality factor of 50.

b. Find ω_1 and ω_2, and B.

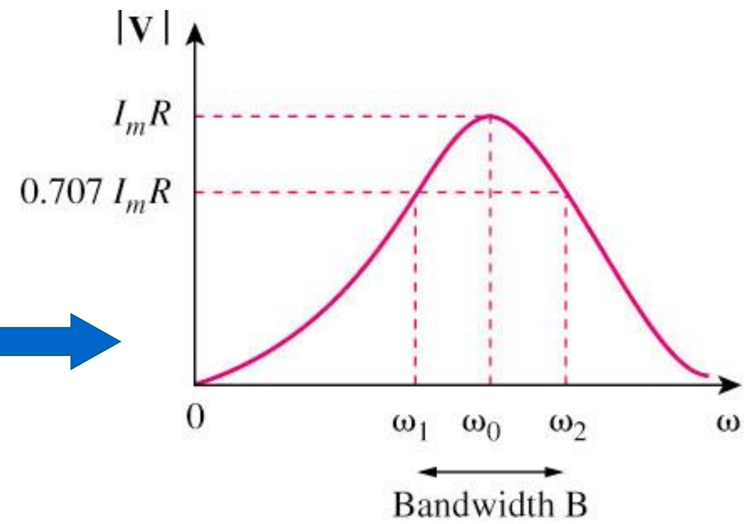
c. Determine the average power dissipated at $\omega = \omega_o, \omega_1, \omega_2$. Take $V_m = 100$V.

14.4 Parallel Resonance (1)

It occurs when imaginary part of Y is zero



$$Y = \frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right)$$



Resonance frequency:

$$\omega_o = \frac{1}{\sqrt{LC}} \text{ rad/s} \quad \text{or} \quad f_o = \frac{1}{2\pi\sqrt{LC}}\, Hz$$

14.4 Parallel Resonance (2)

$$\omega_1 = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}}$$

$$\omega_2 = \frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}}$$

$$B = \omega_2 - \omega_1 = \frac{1}{RC}$$

$$Q = \frac{\omega_0}{B} = \omega_0 RC = \frac{R}{\omega_0 L}$$

14.4 Parallel Resonance (2)

Summary of series and parallel resonance circuits:

characteristic	*Series circuit*	*Parallel circuit*
ω_o	$\frac{1}{\sqrt{LC}}$	$\frac{1}{\sqrt{LC}}$
Q	$\frac{\omega_o L}{R}$ or $\frac{1}{\omega_o RC}$	$\frac{R}{\omega_o L}$ or $\omega_o RC$
B	$\frac{\omega_o}{Q}$	$\frac{\omega_o}{Q}$
ω_1, ω_2	$\omega_o\sqrt{1+(\frac{1}{2Q})^2} \pm \frac{\omega_o}{2Q}$	$\omega_o\sqrt{1+(\frac{1}{2Q})^2} \pm \frac{\omega_o}{2Q}$
$Q \geq 10, \omega_1, \omega_2$	$\omega_o \pm \frac{B}{2}$	$\omega_o \pm \frac{B}{2}$

Example 14.8

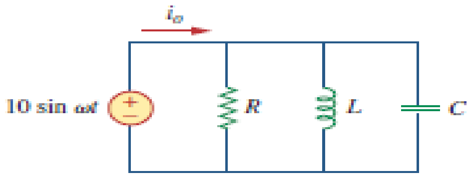


Figure 14.27
For Example 14.8.

In the parallel RLC circuit of Fig. 14.27, let $R = 8$ kΩ, $L = 0.2$ mH, and $C = 8\ \mu$F. (a) Calculate ω_0, Q, and B. (b) Find ω_1 and ω_2. (c) Determine the power dissipated at ω_0, ω_1, and ω_2.

Solution:

(a)

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.2 \times 10^{-3} \times 8 \times 10^{-6}}} = \frac{10^5}{4} = 25 \text{ krad/s}$$

$$Q = \frac{R}{\omega_0 L} = \frac{8 \times 10^3}{25 \times 10^3 \times 0.2 \times 10^{-3}} = 1{,}600$$

$$B = \frac{\omega_0}{Q} = 15.625 \text{ rad/s}$$

(b) Due to the high value of Q, we can regard this as a high-Q circuit, Hence,

$$\omega_1 = \omega_0 - \frac{B}{2} = 25{,}000 - 7.812 = 24{,}992 \text{ rad/s}$$

$$\omega_2 = \omega_0 + \frac{B}{2} = 25{,}000 + 7.812 = 25{,}008 \text{ rad/s}$$

(c) At $\omega = \omega_0$, $\mathbf{Y} = 1/R$ or $\mathbf{Z} = R = 8$ kΩ. Then

$$\mathbf{I}_o = \frac{\mathbf{V}}{\mathbf{Z}} = \frac{10\angle{-90^\circ}}{8{,}000} = 1.25\angle{-90^\circ} \text{ mA}$$

Since the entire current flows through R at resonance, the average power dissipated at $\omega = \omega_0$ is

$$P = \frac{1}{2}|\mathbf{I}_o|^2 R = \frac{1}{2}(1.25 \times 10^{-3})^2(8 \times 10^3) = 6.25 \text{ mW}$$

or

$$P = \frac{V_m^2}{2R} = \frac{100}{2 \times 8 \times 10^3} = 6.25 \text{ mW}$$

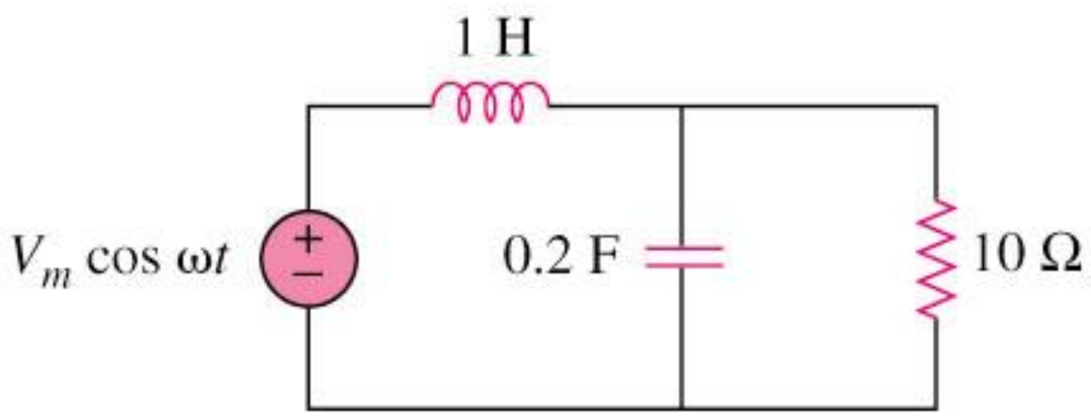
At $\omega = \omega_1, \omega_2$,

$$P = \frac{V_m^2}{4R} = 3.125 \text{ mW}$$

14.4 Parallel Resonance (3)

Example 4

Calculate the resonant frequency of the circuit in the figure shown below.



Answer: $\omega = \frac{\sqrt{19}}{2} = 2.179\,\text{rad/s}$

14.5 Passive Filters (1)

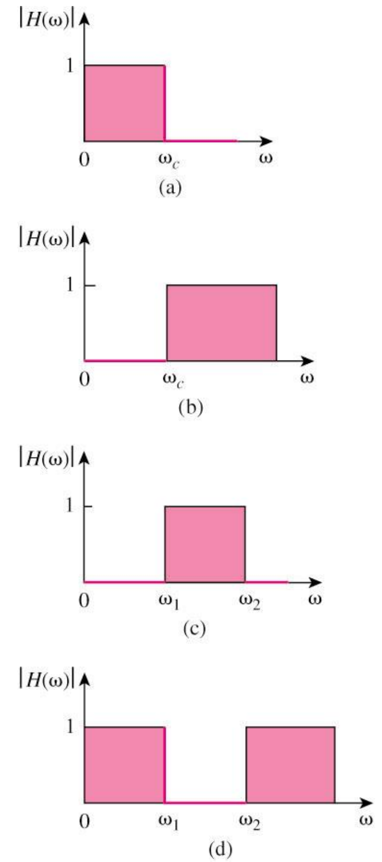
- A filter is a circuit that is designed to pass signals with desired frequencies and reject or attenuate others.
- Passive filter consists of only passive element R, L and C.
- There are four types of filters.

Low Pass

High Pass

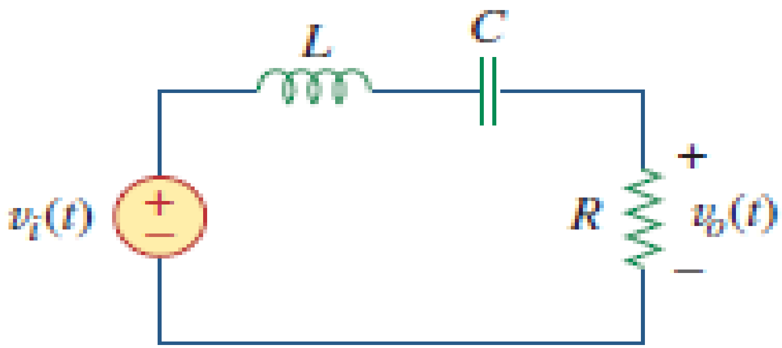
Band Pass

Band Stop

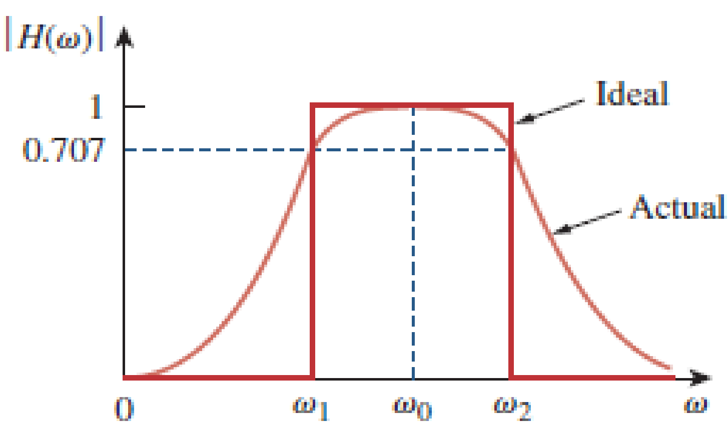


Bandpass and Bandstop Filters

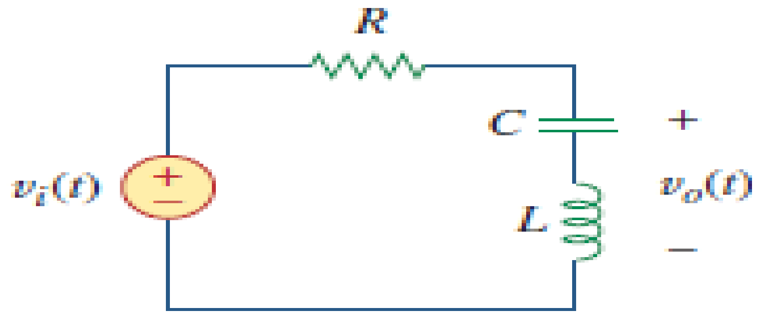
A **bandpass filter** is designed to pass all frequencies within a band of frequencies, $\omega_1 < \omega < \omega_2$.



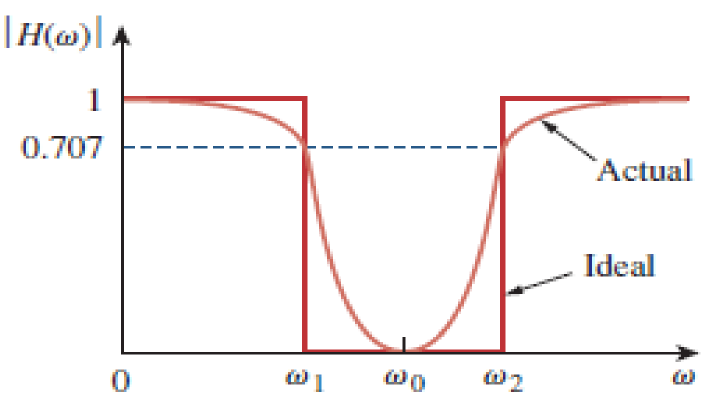
$$H(\omega) = \frac{V_o}{V_i} = \frac{R}{R + j(\omega L - 1/\omega C)}$$



A **bandstop filter** is designed to stop or eliminate all frequencies within a band of frequencies, $\omega_1 < \omega < \omega_2$.



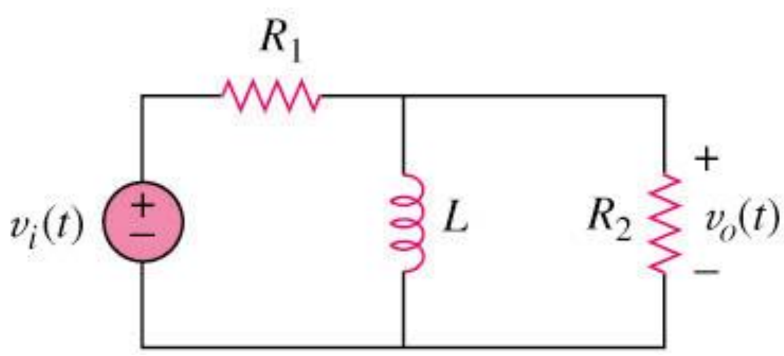
$$H(\omega) = \frac{V_o}{V_i} = \frac{j(\omega L - 1/\omega C)}{R + j(\omega L - 1/\omega C)}$$



14.5 Passive Filters (2)

Example 5

For the circuit in the figure below, obtain the transfer function Vo(ω)/Vi(ω). Identify the type of filter the circuit represents and determine the corner frequency. Take R1=100Ω =R2 and L =2mH.



Answer: $\omega = 25\,\text{krad/s}$

Bode Plots (1)

- Bode plots are in two parts: amplitude and phase
- The amplitude is plotted in dB (logarithmic)
- The frequency scale is also logarithmic
- Approximations to Bode plots can be drawn
- Bode plots are generated by setting $s = j\omega$ and finding magnitude and phase
- It's easy to generate approximations to a Bode plot for simple (1^{st} and 2^{nd} order) blocks

Bode Plots (2)

- Make sure the transfer function is in the form:

$$G(s) = K\frac{s^m(1+z_1 s)\left(1+\frac{2\zeta_{z2}}{\omega_{nz2}}s+\frac{1}{\omega_{nz2}^2}s^2\right)\ldots}{s^n(1+p_1 s)\left(1+\frac{2\zeta_{p2}}{\omega_{np2}}s+\frac{1}{\omega_{np2}^2}s^2\right)\ldots}$$

- This form ensures that each individual system block has a unity gain
- The 'gain constant' *K* will contribute a simple offset to the gain curve

Bode Plots (3)

- Differentiators and Integrators
  - Differentiators: $G(s) = s$

    Amplitude: $20\log_{10}|G(j\omega)| = 20\log_{10}\omega$

    Phase: $\angle G(j\omega) = \angle j = 90^\circ$

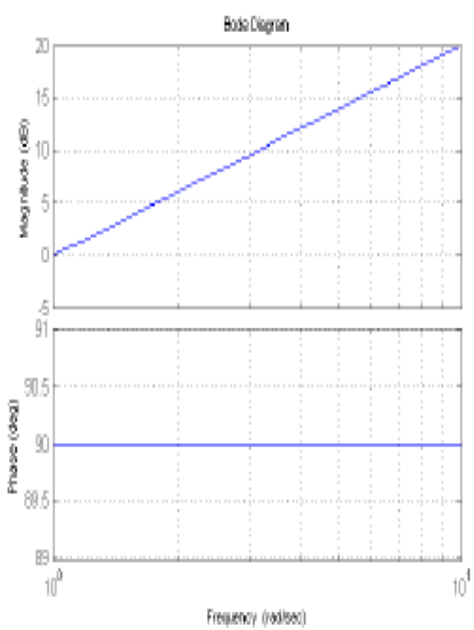
  - Integrators: $G(s) = \dfrac{1}{s}$

    Amplitude: $20\log_{10}\left|\dfrac{1}{j\omega}\right| = -20\log_{10}\omega$

    Phase: $\angle\dfrac{1}{j\omega} = \angle(-j) = -90^\circ$

Bode Plots (4)

First Order Approximations

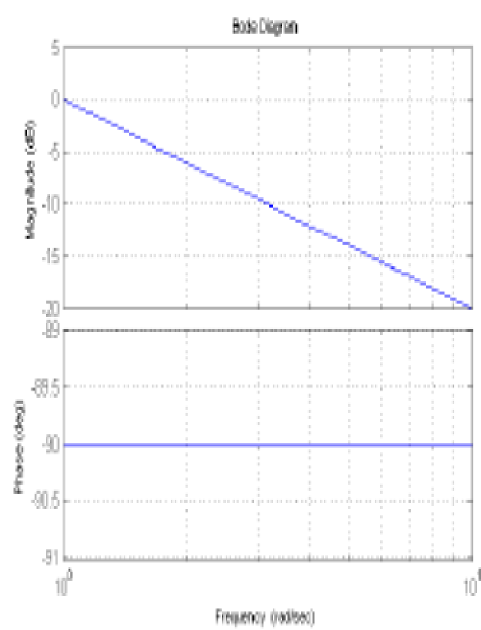


Differentiator:

20dB per decade upward slope in magnitude

90 degrees constant phase shift

First Order Approximations



Integrator:

20dB per decade downward slope in magnitude

-90 degrees constant phase shift

Bode Plots (5)

First Order Approximations

- First order poles and zeros
  - Zeros: $G(s) = 1 + Ts$

    Amplitude: $20\log_{10}\left|1 + j\omega T\right| = 20\log_{10}\sqrt{1+\omega^2 T^2}$

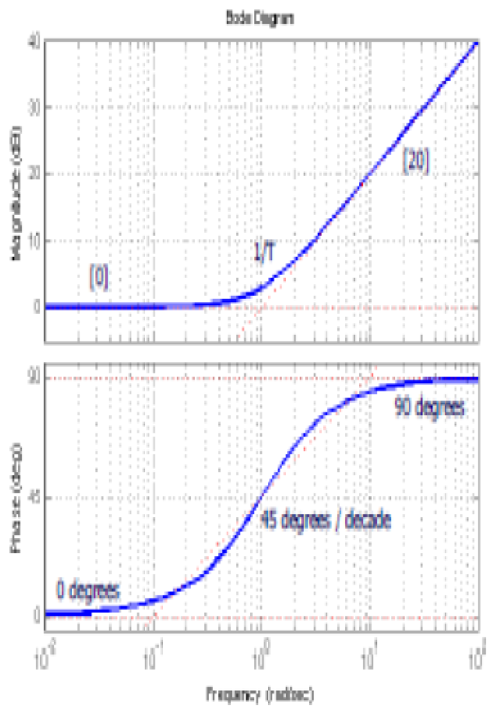
    Phase: $\angle(1 + j\omega T) = \tan^{-1}(\omega T)$
  - Poles: $G(s) = \dfrac{1}{1+Ts}$

    Amplitude: $20\log_{10}\left|\dfrac{1}{1+j\omega T}\right| = -20\log_{10}\sqrt{1+\omega^2 T^2}$

    Phase: $\angle\left(\dfrac{1}{1+j\omega T}\right) = -\tan^{-1}\omega T$

Bode Plots (6)

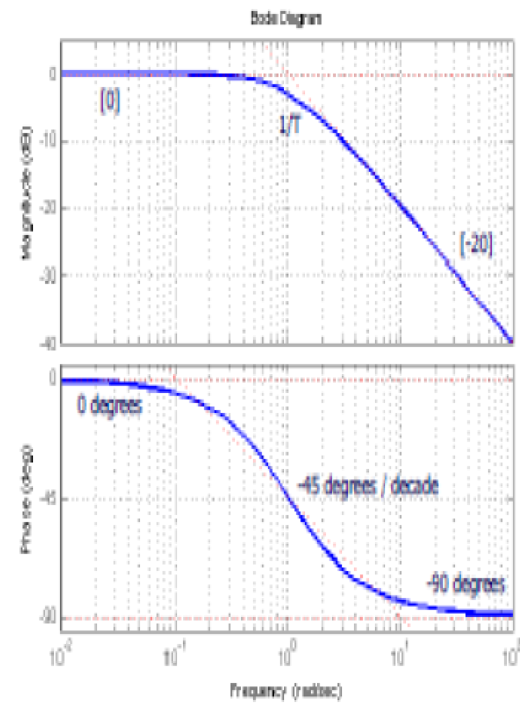
First Order Approximations



Zero:

20dB per decade upward slope in magnitude and 90 degrees phase shift after breakpoint at 1/T

First Order Approximations



Pole:

20dB per decade downward slope in magnitude and -90 degrees phase shift after breakpoint at 1/T

Bode Plots (7)

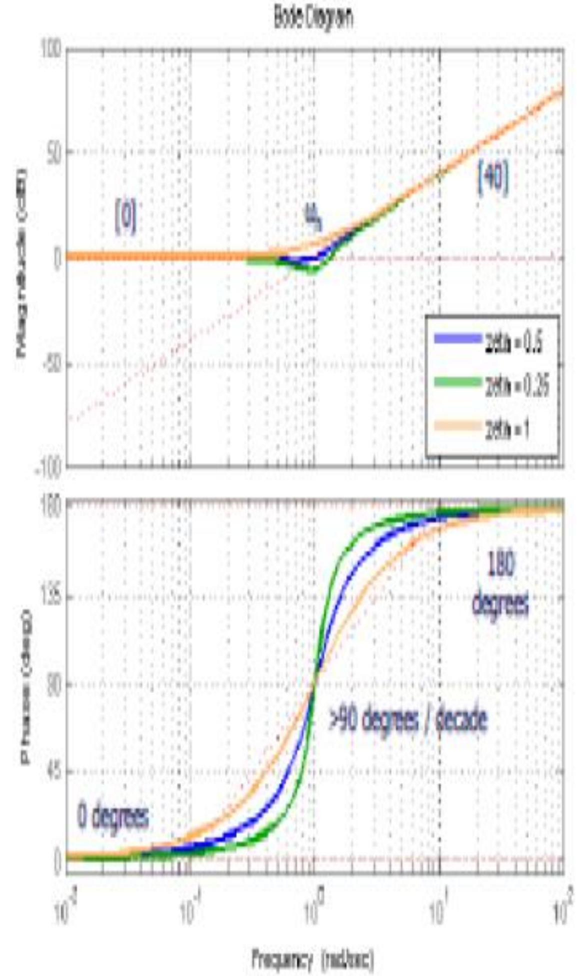
- Second order pairs of zeros:

$$G(s) = 1 + \frac{2\zeta}{\omega_n}s + \frac{1}{\omega_n^2}s^2$$

Amplitude: $$20\log_{10}|G(j\omega)| = 20\log_{10}\left|1 + j\frac{2\zeta}{\omega_n}\omega - \frac{1}{\omega_n^2}\omega^2\right|$$

$$= 20\log_{10}\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2 + \left(\frac{2\zeta\omega}{\omega_n}\right)^2}$$

Phase: $$\angle(G(j\omega)) = \tan^{-1}\left(\frac{\frac{2\zeta\omega}{\omega_n}}{1-\frac{\omega^2}{\omega_n^2}}\right)$$



Zeros:

40dB per decade upward slope in magnitude and 180 degrees phase shift after breakpoint at ω_n

Bode Plots (8)

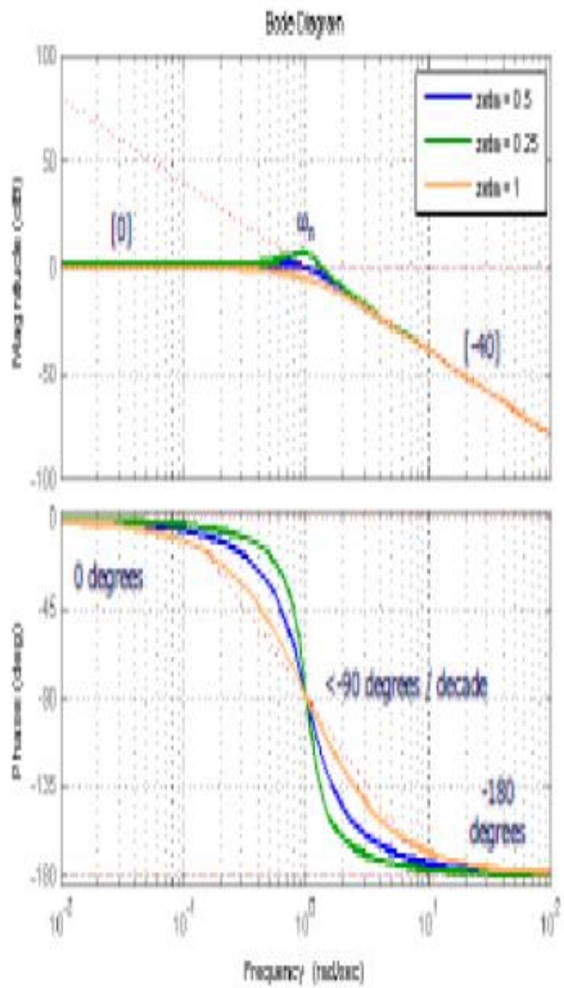
- Second order pairs of poles:

$$G(s) = \frac{1}{1 + \frac{2\zeta}{\omega_n}s + \frac{1}{\omega_n^2}s^2}$$

Amplitude: $$20\log_{10}|G(j\omega)| = -20\log_{10}\left|1 + j\frac{2\zeta}{\omega_n}\omega - \frac{1}{\omega_n^2}\omega^2\right|$$

$$= -20\log_{10}\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(\frac{2\zeta\omega}{\omega_n}\right)^2}$$

Phase: $$\angle(G(j\omega)) = -\tan^{-1}\left(\frac{\frac{2\zeta\omega}{\omega_n}}{1 - \frac{\omega^2}{\omega_n^2}}\right)$$



Poles:

40dB per decade downward slope in magnitude and -180 degrees phase shift after breakpoint at ω_n