Lecture 11: Random Variables

CPE251 Probability Methods in Engineering

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Multiple Random Variables

Outcomes of several random variables.

Examples:

2D sample space: Pressure and volume of a gas → outcome (p, v),

2D sample space: Hardness and tensile strength of copper wire → outcome (h, t)

3D sample space: HAT score, HSSC score, SSC score → outcome (t, h, s)

nD sample space: n samples of an audio signal → outcome $(s_1, s_2, \ldots, s_n)$

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Pair of Random Variables – Joint pmf

The function $f(x, y)$ is a **joint probability distribution** or **probability mass function** of the discrete random variables X and Y if

1. $f(x, y) \geq 0$ for all (x, y),
2. $\sum_x \sum_y f(x, y) = 1$,
3. $P(X = x, Y = y) = f(x, y)$.

For any region A in the xy plane, $P[(X, Y) \in A] = \sum \sum_A f(x, y)$.

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Example

Two ballpoint pens are selected at random from a box that contains 3 blue pens, 2 red pens, and 3 green pens. If X is the number of blue pens selected and Y is the number of red pens selected, find

(a) the joint probability function $f(x, y)$,

(b) $P[(X, Y) \in A]$, where A is the region $\{(x, y) | x + y \leq 1\}$.

Joint Probability Distribution for Example

	$f(x,y)$	x = 0	1	2	Row Totals
y	0	$\frac{3}{28}$	$\frac{9}{28}$	$\frac{3}{28}$	$\frac{15}{28}$
	1	$\frac{3}{14}$	$\frac{3}{14}$	0	$\frac{3}{7}$
	2	$\frac{1}{28}$	0	0	$\frac{1}{28}$
Column Totals		$\frac{5}{14}$	$\frac{15}{28}$	$\frac{3}{28}$	1

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Pair of Random Variables – Joint pdf

The function $f(x,y)$ is a **joint density function** of the continuous random variables X and Y if

1. $f(x,y) \geq 0$, for all (x,y),
2. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\ dx\ dy = 1$,
3. $P[(X,Y) \in A] = \int\int_A f(x,y)\ dx\ dy$, for any region A in the xy plane.

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Example

A privately owned business operates both a drive-in facility and a walk-in facility. On a randomly selected day, let X and Y, respectively, be the proportions of the time that the drive-in and the walk-in facilities are in use, and suppose that the joint density function of these random variables is

$$f(x,y) = \begin{cases} \frac{2}{5}(2x+3y), & 0 \leq x \leq 1, 0 \leq y \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\ dx\ dy = 1$

(b) Find $P[(X,Y) \in A]$, where $A = \{(x,y) \mid 0 < x < \frac{1}{2}, \frac{1}{4} < y < \frac{1}{2}\}$.

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Expected Values using Joint Distributions

Let X and Y be random variables with joint probability distribution $f(x, y)$. The mean, or expected value, of the random variable $g(X, Y)$ is

$$\mu_{g(X,Y)} = E[g(X,Y)] = \sum_x \sum_y g(x,y)f(x,y)$$

if X and Y are discrete, and

$$\mu_{g(X,Y)} = E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)f(x,y)\ dx\ dy$$

if X and Y are continuous.

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Marginal Distributions of a Joint Distribution

Discrete Case	Continuous Case
$p_X(x) = \sum_y p_{X,Y}(x,y)$	$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy$
$p_Y(y) = \sum_x p_{X,Y}(x,y)$	$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx$

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Expected Values using Marginal Distributions

Expected values of X of the marginal distributions in continuous and discrete case:

$$\mu_X = E(X) = \sum_x \sum_y x p_{X,Y}(x,y) = \sum_x x p_X(x)$$

$$\mu_X = E(X) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x f_{X,Y}(x,y) dx dy = \int_{-\infty}^{\infty} x f_X(x) dx$$

Same follows for Y.

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Examples

Suppose that X and Y have the following joint probability function:

$f(x,y)$		$x = 2$	$x = 4$
y	1	0.10	0.15
	3	0.20	0.30
	5	0.10	0.15

(a) Find the expected value of $g(X,Y) = XY^2$.

(b) Find μ_X and μ_Y.

Assume that two random variables (X,Y) are uniformly distributed on a circle with radius a. Then the joint probability density function is

$$f(x,y) = \begin{cases} \frac{1}{\pi a^2}, & x^2 + y^2 \le a^2, \\ 0, & \text{otherwise.} \end{cases}$$

Find μ_X, the expected value of X.

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References

1. Walpole, R.E., Myers, R.H., Myers, S.L. and Ye, K. (2007) *Probability & Statistics for Engineers & Scientists.* 9th Edition, Pearson Education, Inc.
2. Leon-Garcia, A. (2008). *Probability, Statistics, and Random Processes for Electrical Engineering*. 3rd Edition, Pearson/Prentice Hall.