# Lecture 4: Conditional Probability

CPE251 Probability Methods in Engineering

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## Conditional Probability

The probability of an event $B$ occurring when an event $A$ has occurred, stated as probability of B given A has occurred.

$$P[B|A] = \frac{P[A \cap B]}{P[A]}, \qquad P[A] \neq 0$$

$$P[A|B] = \frac{P[A \cap B]}{P[B]}, \qquad P[B] \neq 0$$

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# Example

You receive a laptop gift from a company which is either black ($B$) or white ($W$) in color. The laptops come with three sizes of RAM installed: small ($S$), medium ($M$), and large ($L$). If you get a laptop chosen randomly,

A. Write down the sample space of your observations assuming equally likely outcomes;

Compute:

B. $P\,[B]$

C. $P[S|B]$

D. $P[L \cup W]$

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# Check your solution

A. $S = \{BS, BM, BL, WS, WM, WL\}$

B. $P[B] = \frac{3}{6} = \frac{1}{2}$
$P[S \cap B] = \frac{1}{6}$

C. $P[L \cup W] = [S|B] = \frac{P[S \cap B]}{P[B]} = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}$

D. $P[L] + P[W] - P[L \cap W] = \frac{2}{6} + \frac{3}{6} - \frac{1}{6} = \frac{2}{3}$

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# Multiplicative or Product Rule

$$P[A \cap B] = P\,[A|B]\,P[B]$$

$$P[B \cap A] = P\,[B|A]\,P[A]$$

$$P\,[A|B]\,P[B] = P\,[B|A]\,P[A]$$

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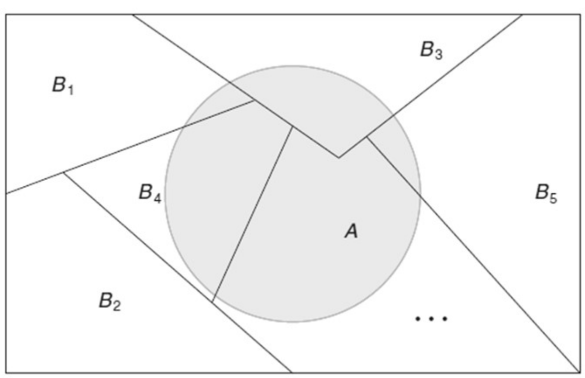
# Partitioning and Total Probability

Breakdown of sample space into mutually exclusive collectively exhaustive events

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# Partitioning and Total Probability

For *n* partitions $B_k$ of a sample space $S$,

$$P[A] = \sum_{k=1}^{n} P[A \cap B_k] = \sum_{k=1}^{n} P\,[A|B_k]\,P\,[B_k]\,, \qquad P\,[B_i] \neq 0\ \forall i \in [1, k]$$

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# Bayes' Rule

$$P\,[B_j|A] = \frac{P[A \cap B_j]}{P[A]} = \frac{P\,[A|B_j]\,P\,[B_j]}{\sum_{k=1}^{n} P\,[A|B_k]\,P\,[B_k]}, \qquad \text{j} = 1,2,3, \dots, k$$

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# Independence of Events

Two events $A$ and $B$ are independent iff

$P[B|A] = P[B]$ or $P[A|B] = P[A]$ or $P[A \cap B] = P[A]P[B]$

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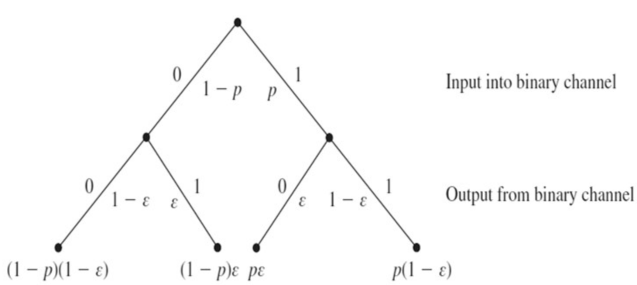
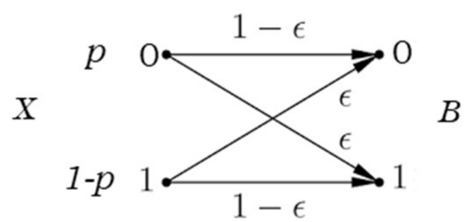
# Binary Communication System

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# Example

In the binary communication system in Example 2.26, find which input is more probable given that the receiver has output a 1. Assume that, a priori, the input is equally likely to be 0 or 1.

Solution:

Let $A_k$ be the event that the input was $k, k = 0, 1$, then $A_0$ and $A_1$ are a partition of the sample space of input-output pairs. Let $B_1$ be the event "receiver output was a 1." The probability of $B_1$ is

$$P[B_1] = P[B_1|A_0]P[A_0] + P[B_1|A_1]P[A_1]$$
$$= \varepsilon\left(\frac{1}{2}\right) + (1 - \varepsilon)\left(\frac{1}{2}\right) = \frac{1}{2}.$$

Applying Bayes' rule, we obtain the a posteriori probabilities

$$P[A_0|B_1] = \frac{P[B_1|A_0]P[A_0]}{P[B_1]} = \frac{\varepsilon/2}{1/2} = \varepsilon$$

$$P[A_1|B_1] = \frac{P[B_1|A_1]P[A_1]}{P[B_1]} = \frac{(1 - \varepsilon)/2}{1/2} = (1 - \varepsilon).$$

Thus, if $\varepsilon$ is less than 1/2, then input 1 is more likely than input 0 when a 1 is observed at the output of the channel.

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# Example

In a certain assembly plant, three machines, $B_1$, $B_2$, and $B_3$, make 30%, 45%, and 25%, respectively, of the products. It is known from past experience that 2%, 3%, and 2% of the products made by each machine, respectively, are defective. Now, suppose that a finished product is randomly selected. What is the probability that it is defective?

Solution:

Consider the following events:

$A$: the product is defective,

$B_1$: the product is made by machine $B_1$,

$B_2$: the product is made by machine $B_2$,

$B_3$: the product is made by machine $B_3$.

Applying the rule of elimination, we can write

$$P(A) = P(B_1)P(A|B_1) + P(B_2)P(A|B_2) + P(B_3)P(A|B_3).$$

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# Example

With reference to Example 2.41, if a product was chosen randomly and found to be defective, what is the probability that it was made by machine $B_3$?

Solution:

Using Bayes' rule to write

$$P(B_3|A) = \frac{P(B_3)P(A|B_3)}{P(B_1)P(A|B_1) + P(B_2)P(A|B_2) + P(B_3)P(A|B_3)},$$

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# Example

A manufacturing process produces a mix of "good" memory chips and "bad" memory chips. The lifetime of good chips follows the exponential law introduced in Example 2.13, with a rate of failure $\alpha$. The lifetime of bad chips also follows the exponential law, but the rate of failure is $1000\alpha$. Suppose that the fraction of good chips is $1 - p$ and of bad chips, $p$. Find the probability that a randomly selected chip is still functioning after $t$ seconds.

Let $C$ be the event "chip still functioning after $t$ seconds," and let $G$ be the event "chip is good," and $B$ the event "chip is bad." By the theorem on total probability we have

$$\begin{aligned} P[C] &= P[C|G]P[G] + P[C|B]P[B] \\ &= P[C|G](1 - p) + P[C|B]p \\ &= (1 - p)e^{-\alpha t} + pe^{-1000\alpha t}, \end{aligned}$$

where we used the fact that $P[C|G] = e^{-\alpha t}$ and $P[C|B] = e^{-1000\alpha t}$.

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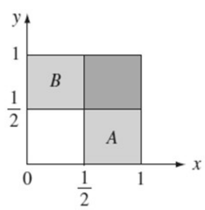
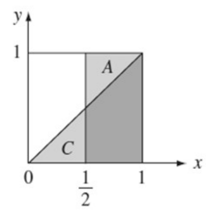
# Example

Two numbers $x$ and $y$ are selected at random between zero and one. Let the events $A$, $B$, and $C$ be defined as follows:

$$A = \{x > 0.5\}, \qquad B = \{y > 0.5\}, \qquad \text{and } C = \{x > y\}.$$

Are the events $A$ and $B$ independent? Are $A$ and $C$ independent?

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# Example

Figure 2.13 shows the regions of the unit square that correspond to the above events. Using Eq. (2.32a), we have

$$P[A|B] = \frac{P[A \cap B]}{P[B]} = \frac{1/4}{1/2} = \frac{1}{2} = P[A],$$

so events $A$ and $B$ are independent. Again we have that the "proportion" of outcomes in $S$ leading to $A$ is equal to the "proportion" in $B$ that lead to $A$.

Using Eq. (2.32b), we have

$$P[A|C] = \frac{P[A \cap C]}{P[C]} = \frac{3/8}{1/2} = \frac{3}{4} \neq \frac{1}{2} = P[A],$$

so events $A$ and $C$ are not independent. Indeed from Fig. 2.13(b) we can see that knowledge of the fact that $x$ is greater than $y$ increases the probability that $x$ is greater than 0.5.

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# References


1. Walpole, R.E., Myers, R.H., Myers, S.L. and Ye, K. (2007) *Probability & Statistics for Engineers & Scientists.* 9$^{th}$ Edition, Pearson Education, Inc.
2. Leon-Garcia, A. (2008). *Probability, Statistics, and Random Processes for Electrical Engineering*. 3rd Edition, Pearson/Prentice Hall.