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An internal Lorentz symmetry induces the background Lorentz symmetry in the dissipative dynamics

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Abstract We show that a dissipative field theory with background Lorentz symmetry underlies the field theory with global $U(1) \times SO(1,1)$ symmetry constructed on a hyperbolic ring; the theory represents a dissipative model for a bipartite system compound of Klein-Gordon fields with different masses; the infrared limit corresponds to the usual dissipative field theory with a constant dissipative parameter, and with broken background Lorentz symmetry; in the ultraviolet limit the fields behave as free fields with unobservable dissipative effects. In this hyperbolic ring-based formulation, the observables correspond to Hermitian quantities, encoding two real quantities, which are appropriate for describing bipartite system; thus, the Lagrangian is constructed as a Hermitian $U(1)\times SO(1,1)$ invariant quantity, and the two real potentials are identified with the subsystem-plus-reservoir system. The potentials can be identified with elliptic and hyperbolic paraboloids by adjusting a real parameter that is interpolating between pure $U(1)$ and pure $SO(1,1)$ symmetries. At the end we address the problem of constructing a propagator on the hyperbolic ring.

1 Introduction

The current interest in the study of dissipative field theories covers from the high energy physics to condensed matter physics. In the former the motivations come from the experimental results of the Relativistic Heavy Collider, and of the Large Hadron Collider; in particular such results suggest that the dissipative effects determine dramatically the dynamics of the quark-gluon plasma, specifically the transport properties, the thermalization and even the dynamics of the system at early times in the evolution. In this context several approaches have been developed, for example in [1] a non-equilibrium master equation is derived from the Lindblad operators, leading to a stochastic nonlinear Schrodinger equation. Additionally finite temperature holographic techniques have been developed; in [2] the transport coefficients for such a plasma are estimated by invoking the AdS/CFT correspondence, and in [3], by using a BTZ black hole as holographic model, the Brownian dynamics of a heavy quark was studied by considering the Langevin equation.

In the AdS/CFT correspondence, the dissipative versions of conformal field theories (and of other dissipative systems associated with open-systems approaches) are constructed by analogy with the classical and quantum descriptions of the damped harmonic oscillator (see [4] for a revision), which lead immediately to three problems in the formulation, namely, the Lorentz covariant breaking, loss of unitarity, and the fact that it is not defined globally in time. This situation generates difficulties in the holographic realizations of a system compound of two interacting conformal field theories; in fact it is not clear whether such a realization is even though possible ([5, 6]).

With these antecedents we have developed recently in [7] a hyperbolic ring-based formulation for classical and quantum field theories in order to circumvent those difficulties coming from the standard formulation; in that approach the loss of unitary and the lack of a global definition in time were circumvented, and in this letter we return to that hyperbolic formulation with the purpose of addressing the restoration of the Lorentz symmetry in the dissipative dynamics of field theories.

It is undeniable the influential role of the harmonic oscillators physics in field theory, whose damped version is described basically by the equation $\ddot{q}+2\alpha\dot{q}+\omega^2 q = 0$, where ω is a constant frequency, and α a constant friction coefficient; by analogy the corresponding equation of motion for a massive scalar field ϕ will read

$$(\partial_t^2 - \nabla^2)\phi + \underbrace{\alpha\partial_t\phi} + m^2\phi = 0; \quad \omega^2 = k^2 - \alpha^2 + m^2; \tag{1}$$

where the dispersion relation for a damped plane wave, $\phi = e^{-\alpha t}e^{i(wt+kx)}$, has been displayed. The presence of the *velocity term* (with underbrace) leads directly to a breaking of the Lorentz covariance under boost transformations; thus, dissipation determines a natural preferred frame in the standard scheme. Although the Lagrangian formulation for the dissipative field theory is not a direct

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issue, by considering the doubling formalism in which the (sub-)system of interest (ϕ) will have a mirror image (ψ) that evolves with the time direction reversed, one can construct the following (real) Lagrangian involving both fields, and with explicit Lorentz symmetry breaking,

$$L = \int d^n x \left[\partial_\mu \phi \partial^\mu \psi + \frac{\alpha}{2}(\phi \partial_t \psi - \psi \partial_t \phi) - m^2 \phi \psi \right]; \tag{2}$$

the variations with respect to the field ϕ leads to the equation of motion (1), and the variations with respect to the field ψ leads to the equation (1) with the time reversed $t \to -t$, for the field ψ. At this point is timely to put in perspective our strategy for overcoming such a symmetry breaking; namely, the real mass and the solution will be extended on a hyperbolic complex plane, leading, without a velocity term, to a Lorentz invariant description for the dissipation. In fact, the linear operator $(\partial_t^2 - \nabla^2)$ will turn out to be a natural invariant under hyperbolic rotations.

Symmetries in the so called hyperbolic Hilbert space have been studied; specifically the hermiticity of the Poincaré mass operator has allowed to show that the corresponding sixteen dimensional structure encodes the space-time Lorentz symmetry, and an internal symmetry that is equivalent to the original form of the Pati-Salam model; this model unifies the fundamental hadrons and leptons into an irreducible representation [8, 9]. In these approaches the hermiticity is inspired from the fundamental principle of quantum mechanics, since the observables are represented by Hermitian operators. In the present approach the hermiticity will play also a crucial role, since the subset of Hermitian objects within the hyperbolic ring will allow to construct the Lagrangian, the Hamiltonian, Hermitian masses, etc, for describing bipartite systems, which are susceptible to undergo dissipation; thus, the approach at hand will be appropriate to describe open systems.

Hyperbolic ring-based formulations have been considered in supersymmetric scenarios; for example solutions with instantons and 3-branes coupled to scalar fields have been constructed in [10]; similarly instantons and seven-brane solutions of type IIB supergravity have been constructed in [11], specifically by using a hyperbolic complex plane with a complex unit with the property $j^2 = 1$. In the approach at hand we use basically the same hyperbolic complex structure with the idea of restoring the background Lorentz symmetry in the dissipative field theory.

In the next section we describe the basic elements of the hyperbolic ring that will be useful in the developments; in particular the subset of the so called Hermitian numbers are described in details, since they will play a crucial role. With the Hermitian objects at hand, in Sect. 3 we construct a Hermitian functional with $U(1) \times SO(1,1)$ global symmetry as an Lagrangian describing a bipartite system undergoing dissipation; it is the presence of the Lorentzian group $SO(1,1)$ that induces the background Lorentz symmetry in the description of the dissipative dynamics. In the same section we derive the equations of motion, whose solutions are described in terms of plane waves with damping effects by using hyperbolic complex exponentials. In Sects. 4 and 5 the dissipation is described from different Lorentz frames; the corresponding IR limits are compared with the usual scheme. In Sect. 6 the quadratic Hermitian potential is described in terms of real dynamical variables; by choosing a real parameter that encodes the extension of the compact symmetry $U(1)$ to the group $U(1) \times SO(1,1)$ on the hyperbolic ring, the real potentials that describe the subsystem and the environment can be bounded from below, or unbounded from both, above and below. In the conclusions we give the elements for addressing the issue of constructing the propagator on a hyperbolic ring.

2 Hermitian quantities on the hyperbolic ring: hermitian masses, and hermitian potentials

We outline briefly the basic aspects of the hypercomplex ring-based formalism that will be useful in the present treatment; for more mathematical details see [12]. A hypercomplex number has four components, $z = x + iy + jv + iju$, with $x, y, v, u \in R$, and with two complex units, namely the usual one with the properties $i^2 = -1$, and $\bar{i} = -i$, and the hyperbolic one with the properties $j^2 = 1$, and $\bar{j} = -j$; thus the complex conjugate reads $\bar{z} = x - iy - jv + iju$, and the norm $z\bar{z} = x^2 + y^2 - v^2 - u^2 + 2ij(xu - yv)$, is a $U(1) \times SO(1,1)$-invariant, i.e., an invariant quantity under the full rotation $z \to e^{i\theta} e^{j\chi} z$, where $e^{i\theta}$ is a $U(1)$-rotation, and $e^{j\chi}$ is a $SO(1,1)$-rotation; the full phase $e^{i\theta} e^{J\chi}$ is in itself a hypercomplex number. Note that the invariant norm is not in general a real quantity, rather it is an extension of the form $X + ijY$, with $X, Y, \in R$. These numbers are termed as *Hermitians* [12], and will play an important role in the scheme at hand for describing the dynamics of bipartite systems under dissipative interaction.

In particular, the mass of the bipartite system can be described, at Lagrangian level, by a Hermitian mass of the form $m_R^2 \pm ijm_H^2$, where m_R^2 is associated with the subsystem of interest, and m_H^2 is associated with the environment; in the scheme at hand the case with $+m_H^2$ can be identified with an absortive environment (and a nonabsorbent subsystem), and the case $-m_H^2$ leads to an interchange of roles of subsystem/reservoir; hence these masses will undergo dissipation as a dynamical effect; the sign of $+m_R^2$ is fixed in order to obtain the correct free fields limit. These expressions for a Hermitian mass are related by an inversion operation in the ring; since $(m_R^2 + ijm_H^2)(m_R^2 - ijm_H^2) = m_R^4 + m_H^4$, then we have $(m_R^2 + ijm_H^2)^{-1} = (m_R^2 - ijm_H^2)/M^4$, where $M^2 \equiv \sqrt{m_R^4 + m_H^4}$, can be interpreted as the effective full mass of the complete system; these expressions will appear in the developments below. Furthermore, the potential part of the Lagrangian is described also by an Hermitian expression of the form $V_R + ijV_H$; the real components (V_R, V_H) will correspond geometrically to elliptic and hyperbolic paraboloids for two scalar fields under dissipative interaction. Other physical quantities will have the form of Hermitian expressions that will allow to describe the full dynamics of the subsystem of interest, and of the environment. Other formal aspects of the formalism will be explained accordingly they appear in the developments.

Furthermore, the exponential function on z can be decomposed into the usual complex exponential, and a new pure hyperbolic exponential, $e^z = e^{x+iy+jv+iju} = e^{x+iy} \cdot e^{jv+iju}$, where the new part reads

$$e^{jv+iju} = e^{jv} \cdot e^{iju} = (J^+e^v + J^-e^{-v}) \cdot (\cos u + ij \sin u); \quad J^+ = \frac{1}{2}(1+j), \quad J^- = \frac{1}{2}(1-j); \tag{3}$$

hence, the hyperbolic part contains, as the usual exponential, a bounded part, and an unbounded part; the description for the bounded exponential e^{iju} is obtained through a series expansion; additionally in this expression the pair (J^+, J^-) corresponds to the idempotent basis with the annihilation property $J^+ \cdot J^- = 0$ (see [7] for more details). Furthermore, with the additional properties $J^+j = J^+$, and $J^-j = -J^-$, the hypercomplex exponential can be decomposed at the end as a combination of conventional complex exponentials, namely

$$e^z = J^+e^{i(y+u)+x+v} + J^-e^{i(y-u)+x-v}, \tag{4}$$

In the approach at hand, the dissipative dynamics will be described in terms of hypercomplex exponentials with oscillatory modes with damping effects, that will maintain intact the background Lorentz symmetry. Such exponentials will belong to the kernel for the D'alembertian operator, which includes a Hermitian mass described above, namely $(\partial^2 + m^2)$, where $\partial^2 = \partial_t^2 - \nabla^2$. Additionally we realize that ∂^2 can be rewritten as a real operator invariant under hyperbolic rotations, $\partial^2 = (\partial_t + j\nabla)(\partial_t - j\nabla)$; note that this invariance property can not to be established through a complexification with the standard complex unit i (neither with the hybrid complex unit ij).

3 A hermitian lagrangian as a dissipative system

From the general hypercomplex number z with four real components, we can obtain a number with two degrees of freedom with the identification $x = \gamma u$, and $y = \gamma v$, with $\gamma \in R$ (for more details see [12]), that we now call Ω,

$$\Omega = (\gamma + ij)u + (i\gamma + j)v, \quad \overline{\Omega} = (\gamma + ij)u - (i\gamma + j)v; \tag{5}$$

$$\Omega\overline{\Omega} = (\gamma^2 - 1)(u^2 + v^2) + 2ij\gamma(u^2 - v^2); \tag{6}$$

the norm (6) is still invariant under global $U(1) \times SO(1,1)$ rotations $\Omega \to e^{i\theta}e^{j\chi}\Omega$; this rotation has a nontrivial effect on the real components (u, v); in fact, the identication through the pararmeter γ described above, is the only one that allows us to retain the full symmetry with two degrees of freedom. The real parameter γ allows us to extend the usual $U(1)$-symmetry on the hyperbolic ring in order to incorporate the non-compact $SO(1,1)$-symmetry; in particular the pure $U(1)$-symmetry is realized with $\gamma = 0$, and the pure $SO(1,1)$-symmetry with $\gamma^2 = 1$. Note that in expression (6) the roles of the fields (u, v) are not interchangeable, rather we have the discrete symmetry $u \leftrightarrow v$ plus $\gamma \to -\gamma$.

In this scheme, we have with two complex units, the conventional one that allows to describe two charged fields $(\Omega, \overline{\Omega})$ with $U(1)$-charges, and the hyperbolic complex unit j that provides an enveloping algebra for describing the dynamics of the subsystem (Ω), and the environment $(\overline{\Omega})$.

We consider now the following Lagrangian constructed with the norm (6),

$$L(\Omega, \overline{\Omega}; \gamma) = \partial\Omega \cdot \partial\overline{\Omega} - m^2\Omega\overline{\Omega}, \tag{7}$$

and thus corresponds to a $U(1) \times SO(1,1)$ invariant; in general the mass corresponds to a Hermitian quantity, $m^2 = m_R^2 + ijm_H^2$, and hence we have two different masses (m_R^2, m_H^2) for two degrees of freedom (u, v).

With the incorporation of a quartic "$\lambda\phi^4$" term, the hypercomplex-based formulation for the spontaneous symmetry breaking mechanism, and the formation of topological defects were studied [13]; in this work we consider only the quadratic terms in the fields through a mass term with the purpose of revealing a model for the dissipative dynamics that preserves explicitly the Lorentz symmetry, as opposed to the usual formulation [14].

The equation of motion derived from the Lagrangian (7) reads,

$$(\partial^2 + m^2)\Omega = 0; \tag{8}$$

and its complex conjugate for the variable $\overline{\Omega}$; since the mass term is a Hermitian quantity, the operator $\partial^2 + m^2$ is Hermitian with $\partial^2 = \partial_t^2 - \nabla^2$. We show now that the dissipative dynamics between the fields $(\Omega, \overline{\Omega})$, with Lorentz symmetry can be described by hypercomplex fields belonging to the kernel of this operator.

We start from the standard phase $e^{i(wt - \vec{k}\cdot\vec{x})}$, where the pair $(w, \vec{k})$ are real quantities, and we construct a hypercomplex phase by promoting the real pair to Hermitian quantities, in similarity to the mass; hence the ansatz reads

$$w \to w_R + ijw_H, \quad \vec{k} \to \vec{k}_1 + ij\vec{k}_2, \quad \Omega = e^{i[(w_R+ijw_H)t - (\vec{k}_1+ij\vec{k}_2)\cdot\vec{x}]}; \tag{9}$$

where the pairs $(w_R, \vec{k}_1)$, and $(w_H, \vec{k}_2)$, are real quantities; as we shall see, the first pair represent as usual the oscillatory behavior for the fields, and the second pair will represent as we shall see the energy-momentum dissipation for the fields; the substitution of this expression into equations (8) leads to the following dispersion relations,

$$w_R^2 - \vec{k}_1 \cdot \vec{k}_1 - \left(w_H^2 - \vec{k}_2 \cdot \vec{k}_2\right) = m_R^2, \quad 2\left(w_R w_H - \vec{k}_1 \cdot \vec{k}_2\right) = m_H^2; \tag{10}$$

note that these expressions are Lorentz invariants constructed from the space-times vectors $(w_R, \vec{k}_1)$, and $(w_H, \vec{k}_2)$, namely, the first expression corresponds to the difference of the norms of the vectors, and the second one corresponds to the inner product of those vectors. Furthermore, if the extension in expressions (9) is made through the usual extensions iw_H, and ik_2, then the pair $(w_H, \vec{k}_2)$ will represent also damping effects, however, the corresponding dispersion relations imply that $m_H = 0$, restricting the possibility of describing two fields with different masses. Furthermore, if such an extension is made with the pure hyperbolic unit j, then there no exist damping effects, and the pair (w_H, k_2) will represent, such as the pair $(w_R, \vec{k}_1)$, a redundant oscillatory behavior for the fields. Thus the promotion of real objects to Hermitian ones has played again the crucial role.

With the two solutions $(\Omega, \overline{\Omega})$ we can construct a combination of the form $a\Omega_1 + \overline{b} \cdot \overline{\Omega}$, where a and b are arbitrary coefficients,

$$\Omega = \left[J^+ a \cdot e_1^{i(w_R t - \vec{k}\cdot\vec{x})} + J^- \overline{b} \cdot e_1^{-i(w_R t - \vec{k}\cdot\vec{x})}\right] e^{-w_H t + \vec{k}_2\cdot\vec{x}} + \left[J^- a \cdot e^{i(w_R t - \vec{k}_1\cdot\vec{x})} + J^+ \overline{b} \cdot e^{-i(w_R t - \vec{x}_1\cdot\vec{x})}\right] e^{w_H t - \vec{k}_2\cdot\vec{x}}; \tag{11}$$

where expressions (3), and (4) have been used; this general solution corresponds then to a linear combination of plane waves with decaying/growing factors, namely if $e^{-w_H t}$ corresponds to a decaying factor in time (for $w_R > 0$, and for $t > 0$), for the subsystem of interest, then $e^{w_H t}$ will correspond to a growing factor for the reservoir; we obtain similar conclusions for decaying/growing in space with the factors $e^{\pm\vec{k}_2\cdot\vec{x}}$; space decay appears for example in the propagation of electromagnetic waves in a conductor leading to the skin effect. Thus space decay may be a possible effect for scalar fields representing physical systems co-existing in the same geometrical space. On the other side the fields can to lie in different regions, for example two asymptotically AdS regions such that the fields lie on the respective asymptotic boundaries, and thus for these scenarios one must to choice $\vec{k}_2 = 0$ [6], since the spatial decay can not be present.

Furthermore, from expression (11) we see that the subsystem-frame can be defined with the constraints on the coefficients, $J^- a = 0$, and $J^+ \overline{b} = 0$, in such a way that the field Ω describes from the subsystem perspective only decaying (growing) fields through the global factor $e^{-w_H t + \vec{k}_2\cdot\vec{x}}$; similarly, the reservoir-frame can be defined by the constraints $J^+ a = 0$, and $J^- \overline{b} = 0$, and from the reservoir perspective we have only growing (decaying) fields. It is important to comment that an arbitrary hypercomplex number such as the coefficient a can be decomposed in the form $a = J^+ a + J^- a$, we mean in terms of orthogonal components, since $J^+ J^- = 0$, and similarly for the coefficient $\overline{b}$; thus the choice of the basis for the subsystem and for the reservoir in terms of the components for the coefficients $(a, \overline{b})$ turn out to be *orthogonal* in the hypercomplex plane.

4 A Lorentz frame with time decay/growing

If the subsystem and the reservoir are in different space locations, then one must choice $\vec{k}_2 = 0$, as commented above; on the other hand if the subsystem/reservoir coexist in the same region of space, then there exists a Lorentz transformation such that $(w_H, \vec{k}_2 = 0)$; we mean there exists a frame from which the dissipative dynamics is described only by time decay/growing effects. In this frame the solutions for Eq. (10) read

$$w_R = \pm \frac{1}{\sqrt{2}} \sqrt{m_R^2 + k_1^2 + \sqrt{(m_R^2 + k_1^2)^2 + m_H^4}}; \tag{12}$$

$$w_H = w_R \frac{\sqrt{(m_R^2 + k_1^2)^2 + m_H^4} - (m_R^2 + k_1^2)}{m_H^2}; \tag{13}$$

the first expression corresponds to a deformation of the usual dispersion relation due to dissipative effects; note that there no exist IR/UV cutoffs in this expression. Moreover, the variable w_H as dissipative parameter is depending on the momentum and the masses of the fields, and it has substituted to the constant dissipative parameter α from the standard scheme (see Eq. 1); note that once we have fixed the sign of w_R, the sign of m_H^2 determines the sign of w_H, and thus the direction of the energy-momentum dissipation in the bipartite system.

At the limit $m_H \to 0$, the above expressions reduce to,

$$\lim_{m_H \to 0} w_R = \pm\sqrt{m_R^2 + k_1^2}, \quad \lim_{m_H \to 0} w_H = 0, \tag{14}$$

hence, the second equation implies that there is not dissipation, and consistently the first expression represents the usual relativistic relation for a free field. Thus, in this frame, the mass m_H is the responsable of the dissipative effects; note that the role of the mass m_R is different, since at the limit of a vavishing mass there exist dissipative effects, $\lim_{m_R \to 0} w_H \neq 0$.

In the standard scheme one has a single dispersion relation of the form $w^2 = m^2 + k^2 - \alpha^2$ (see Eq. 1), as commented in the introduction; thus, in order to make a comparison we describe the IR limit of expressions (12), and (13) up to a quadratic dependence in the momentum,

$$2w_R^2 = \underbrace{m_R^2 + M^2} + \frac{m_R^2 + M^2}{M^2}k_1^2 + \cdots = \overbrace{m_R^2 + M^2 + \frac{m_R^2}{M^2}k_1^2} + k_1^2 + \cdots; \tag{15}$$

$$2w_H^2 = \underbrace{-m_R^2 + M^2} + \frac{m_R^2 - M^2}{M^2}k_1^2 + \cdots; \tag{16}$$

where $M^2 \equiv \sqrt{m_R^4 + m_H^4}$; in the first expression we have an effective mass, with underbrace, as an effect of the dissipation; additionally in this expression the coefficient for the momentum is deformed by the presence of the full mass M^2 of the system; in the second equality, the same expression (15) can be re-interpreted in terms of an effective mass modified by the momentum (with overbrace), showing in a manifest form the transference mass-momentum as an effect of the dissipation. In expression (16), the variable w_H has an IR cutoff defined by the full mass of the system, in order to take real values, $k_1^2 \leq M^2$; when this inequality is saturated, then $w_H \to 0$.

One may to enforce, from the above IR expressions, the usual scheme with a constant dissipative parameter by considering the zero-order term for w_H in expression (16) (with underbrace), and the quadratic expression in the momentum in Eq. (15); in this approximation the dissipation constant is depending on the masses, and the expression for w_R is a deformed version of the usual expression $w^2 = m^2 + k^2 - \alpha^2$, namely, $2w_R^2 = 2m_R^2 + \frac{m_R^2+M^2}{M^2}k_1^2 + (-m_R^2 + M^2)$.

Furthermore, the UV limits for expressions (12), and (13), are given by,

$$w_R = k_1 + \frac{m_R^2}{2k_1} + \frac{m_H^4 - m_R^4}{8k_1^3}\cdots, \qquad w_H = \frac{m_H^2}{2k_1} - \frac{m_R^2 m_H^2}{4k_1^3} + \cdots \tag{17}$$

we mean, the fields will behave as (free) massless fields with $w_R \approx k_1$, with an unobservable dissipative effect, $w_H \approx 0$.

5 A general Lorentz frame

From a general Lorentz frame the dissipative dynamics is viewed with both time and space decays, and described in terms of a general dissipative four-vector $(w_H, \vec{k}_2)$; the solutions for the equations (10) read

$$w_R = \pm\frac{1}{\sqrt{2}}\sqrt{m_R^2 + k_1^2 - k_2^2 + \sqrt{(m_R^2 + k_1^2 - k_2^2)^2 + (m_H^2 + 2\vec{k}_1 \cdot \vec{k}_2)^2}}; \tag{18}$$

$$w_H = w_R\frac{\sqrt{(m_R^2 + k_1^2 - k_2^2)^2 + (m_H^2 + 2\vec{k}_1 \cdot \vec{k}_2)^2} - (m_R^2 + k_1^2 - k_2^2)}{m_H^2 + 2\vec{k}_1 \cdot \vec{k}_2}; \tag{19}$$

where $k_1^2 \equiv \vec{k}_1 \cdot \vec{k}_1$, and $k_2^2 \equiv \vec{k}_2 \cdot \vec{k}_2$; note that these frequencies are real for all values of $(\vec{k}_1, \vec{k}_2)$, and under these conditions there will not exist UV/IR cutoff's. Furthermore, although $m_H \to 0$, a damping effect is observed provided that $\vec{k}_1 \cdot \vec{k}_2 \neq 0$, i.e., while the dissipation vector $\vec{k}_2$ has a nontrivial projection along the field momentum. Similarly in the frames in which the constraint $m_H^2 + 2\vec{k}_1 \cdot \vec{k}_2 = 0$ is satisfied, then the dissipation in time is unobservable since $w_H \to 0$.

One can obtain an IR limit as a quadratic approximation in the momenta, which can be compared with the conventional description of dissipative field theory outlined in the introduction;

$$2w_R^2 = M^2 + m_R^2 + 2\frac{m_H^2}{M^2}\vec{k}_1 \cdot \vec{k}_2 + \left(\frac{m_R^2}{M^2} + 1\right)\left(k_1^2 - k_2^2\right) + \cdots; \tag{20}$$

$$2w_H^2 = M^2 - m_R^2 + 2\frac{m_H^2}{M^2}\vec{k}_1 \cdot \vec{k}_2 + \left(\frac{m_R^2}{M^2} - 1\right)\left(k_1^2 - k_2^2\right) + \cdots; \tag{21}$$

where we recall that $M^2 = \sqrt{m_R^4 + m_H^4}$; note that in this approximation $w_H \to 0$ as $m_H \to 0$, irrespective of the value of $\vec{k}_1 \cdot \vec{k}_2$; note that the role of the mass m_R is different, since at the limit $m_R \to 0$, one has damping effect, $w_H \neq 0$. These expressions at the IR regime correspond to the generalization of Eqs. (15) and (16), and can be interpreted along the same lines in terms of shifted masses and momenta.

5.1 The massless dissipative case

The massless scenario is well defined within the scheme at hand; from expressions (19), we have that

$$\lim_{m\to 0} w_R = \pm\frac{1}{\sqrt{2}}\sqrt{k_1^2 - k_2^2 + \sqrt{\left(k_1^2 - k_2^2\right)^2 + \left(2\vec{k}_1\cdot\vec{k}_2\right)^2}};$$
$$\lim_{m\to 0} w_H = w_R\frac{\sqrt{(k_1^2 - k_2^2)^2 + (2\vec{k}_1\cdot\vec{k}_2)^2} - (k_1^2 - k_2^2)}{2\vec{k}_1\cdot\vec{k}_2}; \tag{22}$$

where $m \to 0$ stands for $m_R \to 0$, and $m_H \to 0$. As expected, for this case we have that $w_H \to 0$ in the frames where $\vec{k}_1\cdot\vec{k}_2 \to 0$; additionally in this limit one has that $w_R^2 = k_1^2 - k_2^2$, which shows the purely momentum dissipation.

6 The quadratic hybrid potential

Since the previous section has been developed in terms of the dynamical variables $(\Omega, \overline{\Omega})$, the parameter γ appearing in expressions (5), and (6) does not play any role; in this section we shall describe the quadratic potential $-m^2\Omega\overline{\Omega}$ in terms of the real variables (u, v), and that parameter will play a crucial role. Considering that the mass and the modulus (6) are Hermitian quantities, the potential is a Hermitian quantity of the form $V_R + ijV_H$, where

$$V_R = 2\left(\frac{\gamma^2-1}{2}m_R^2 + \gamma m_H^2\right)v^2 + 2\left(\frac{\gamma^2-1}{2}m_R^2 - \gamma m_H^2\right)u^2, \tag{23}$$

$$V_H = 2\left(\frac{\gamma^2-1}{2}m_H^2 - \gamma m_R^2\right)v^2 + 2\left(\frac{\gamma^2-1}{2}m_H^2 + \gamma m_R^2\right)u^2; \tag{24}$$

the potential V_R is transformed into the potential V_H through the discrete transformations $m_R \leftrightarrow m_H$, and $\gamma \to -\gamma$ (for more details see [12]). In the hypercomplex description at hand, the bipartite system constituted of the subsystem of interest and of the environment, is described by the hybrid potential $V_R + ijV_H$; the potential V_R describes the subsystem, and the potential V_H describes the environment. Thus, from the perspective of the subsystem the fields (v, u) will have effective masses, since they correspond to smeared out expressions of the original masses (m_R, m_H); the parameter γ, that extends the symmetry $U(1)$ on the ring in order to incorporate the additional symmetry $SO(1, 1)$, is the responsible for these effective expressions. A similar interpretation can be made for the masses from the perspective of the reservoir; note that the fields not necessarily will have the same effective masses from the different perspectives.

Now we are looking for configurations for the potential V_R that is bounded from below, and for the potential V_H such a configuration may be bounded from below, or unbounded from both, below and above, since the environment may to absorb and to transfer energy-momentum without limits.

The zero-energy point for both V_R, and V_H, corresponds to the point $(v = 0, u = 0)$, the unique critical point for the potentials; the Hessian determinants at this point read,

$$D_R = 16m_R^4\left[\left(\frac{\gamma^2-1}{2}\right)^2 - m^2\gamma^2\right], \qquad V_R = 0; \tag{25}$$

$$D_H = 16m_R^4\left[\left(\frac{\gamma^2-1}{2}\right)^2 m^2 - \gamma^2\right], \qquad V_H = 0; \tag{26}$$

where $m^2 \equiv (\frac{m_H^2}{m_R^2})^2$.

6.1 the case $m^2 > 1$

While $m^2 \neq 1$, there exists an unbalance between the masses of the subsystem and the environment, and it manifests by the presence of two curves for the determinants as functions on γ; we consider first the case when $m_H > m_R$.

The roots for D_R, and D_H are given in terms of the mass ratio m^2, and are illustrated in the above figure;

$$D_R: \quad r_4 = \sqrt{m^2+1} + m, \quad r_1 = -r_4; \quad r_3 = \sqrt{m^2+1} - m, \quad r_2 = -r_3; \tag{27}$$

$$D_H: \quad h_4 = \frac{\sqrt{m^2+1}+1}{m}, \quad h_1 = -h_4; \quad h_3 = \frac{\sqrt{m^2+1}-1}{m}, \quad h_2 = -h_3; \tag{28}$$

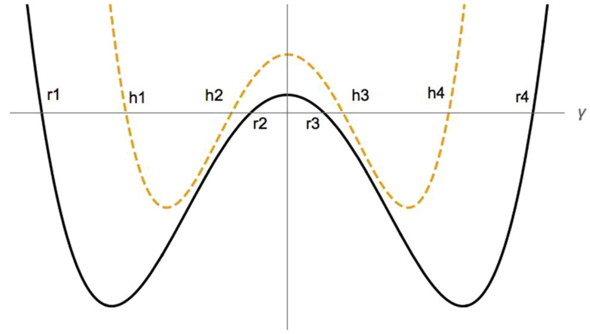

Fig. 1 The profiles of the determinants (25) and (26) as functions on γ, for $m^2 > 1$; the continuous curve represents D_R, and the dashed curve stands for D_H. For $m^2 > 1$ the continuous curve lies below the dashed curve, and for $m^2 < 1$ such a relative position is interchanged (see Fig. 2). For $m^2 = 1$, it means for $m_R = m_H$, the curves coincide to each other, and the full system is described by one curve

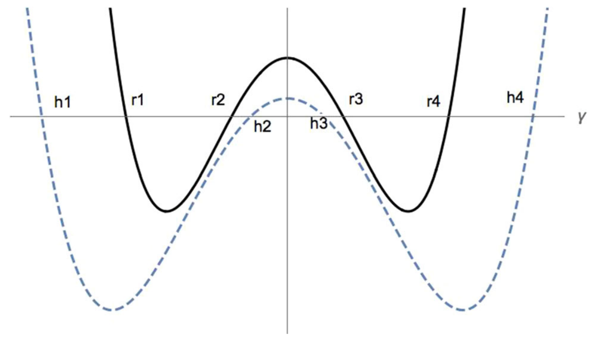

Fig. 2 The continuous curve (D_R) lies now above the dashed curve (D_H); V_R corresponds to an elliptic paraboloid, and V_H to a hyperbolic paraboloid. While $m^2 \neq 1$, there exists an unbalance between the masses of the subsystem and the environment, and such a difference manifests by the presence of two curves for the determinants. Thus, for $m^2 = 1$, the curves coincide to each other, and the full system is described by one curve

note that the roots (27) are transformed into the roots (28) through the duality transformation $m \to \frac{1}{m}$, which implies the interchange of the masses $m_R \leftrightarrow m_H$. Since D_R and D_H correspond basically to the product of the effective masses for the fields (u, v) in expressions (23), and (24), the vanishing of the determinants will have the following interpretation; if the effective mass for the field v in Eq. (23) vanishes, then the parameter γ will reduce to the roots r_1 and r_3; on the contrary, if the effective mass for the field u in Eq. (23) vanishes, then γ will reduce to the roots r_4 and r_2; similarly for the case D_H.

Within the ranges $(-\infty, r_1)$, and $(r_4, +\infty)$, D_R and D_H (and the second derivatives of the potentials respect to the field variables (u, v)), are positive, and then the zero-energy point corresponds to a (global) minimum for the potentials, which are represented then by elliptic paraboloids bounded from below.

Furthermore, although in the range (r_2, r_3), D_R, and D_H, are positive, the second derivatives of the potentials are negative, and then the zero-energy point corresponds to a (global) maximum, and the potentials corresponds also to elliptic paraboloids, but upside down, and thus bounded from above, and unbounded from below. Additionally within the ranges (r_1, r_2), and (r_3, r_4), D_R is negative, then the potential V_R is unbounded from below and from above, and the zero-energy point is a saddle point; if the parameter γ is chosen within the sub-interval (h_1, h_2), then V_H will have the same profile, and both potentials correspond to hyperbolic paraboloids; hence, the subsystem of interest and the bath can absorb and transfer energy-momentum without limits. Finally, if some of the effective masses vanishes, we mean γ reduces to some of the roots in expressions (27) and (28), then geometrically the paraboloids degenerate to parabolae.

6.2 The case $m^2 < 1$

As opposed to the previous case, in this case we have that $h_4 > r_4$, and $r_3 > h_3$; this fact will allow describe a scenario that can not be realized in the previous case. Within the intervals (h_1, r_1) and (r_4, h_4) the zero-energy point corresponds to a (global) minimum for V_R, which is represented again by an elliptic paraboloid, and the subsystem of interest is bounded from below. The new aspect in this case is that in such intervals D_H is negative, and the zero energy point corresponds to a saddle point for the potential V_H, which is represented by a hyperbolic paraboloid; thus, the bath can to absorb and transfer energy-momentum without limits.

6.3 The spontaneous symmetry breaking values for γ

It is evident that the curves in Figs. 1 and 2, represent spontaneous symmetry breaking scenarios in the γ-parameter space; thus, it is natural to ask for some physical meaning for the values of γ at the two mimima, which read,

$$D_R: \quad \gamma_R = \pm\sqrt{2m^2+1}; \qquad D_H: \quad \gamma_H = \pm\frac{\sqrt{m^2+2}}{m}; \tag{29}$$

note that these values are related by the duality transformation $m \to \frac{1}{m}$.

In Fig. 1, the values γ_R and γ_H are out of the ranges of interest $(-\infty, r_1)$, and $(r_4, +\infty)$, in which the subsystem is bounded from below. In contrast, in Fig. 2 the two values γ_H lie in the intervals of interest (h_1, r_1), and (r_4, h_4), but with an additional restriction on m^2, namely, $m^2 < \frac{\sqrt{5}-1}{2}$; this value that approximates to 0.618, corresponds to the intersection of the curves $r_4(m)$, and $\gamma_H(m)$, as functions of m; thus, within this sub-interval of the original restriction $m^2 < 1$, the SSB value $\gamma_H(m)$ can be selected. As an specific example, if we choice $m^2 = 1/2$, then one can choice for γ the value given by Eq. (29), $\gamma_H = \pm\sqrt{5}$. Hence, by choosing a global minimum value for the parameter γ, one can to ensure that the potential associated with the subsystem will have a global minimum, and for the environment a potential with a saddle point as vacuum.

7 Conclusions

We have restored the background Lorentz symmetry in the dissipative dynamics through the incorporation of an internal Lorentz symmetry to the usual compact symmetry considered in the standard description of scalar field theories; this is achieved by extending the usual complex plane with extra complex units. Although the generalization to local gauge symmetries of this hypercomplex formulation was constructed in [12], the underlying dissipative dynamics, and the possibility of describing open systems, were not identified; in particular the results obtained in that reference can be interpreted appropriately considering dissipative effects.

In the standard $U(1)$ (free) field theory there are two types of particles, namely, the particle and its anti-particle; they have opposite charges, but equal masses; the extension $U(1) \to U(1) \times SO(1,1)$ constructed here can be considered as the theory that contains also two types of particles, with opposite charges (see [7]), and with different masses. This feature is appropriate for describing open system under dissipative interaction, since the subsystem-reservoir is not compound necessarily from particle-antiparticle pairs.

Furthermore, the propagator for the standard $U(1)$ complex field reads $\frac{1}{k^2+m^2-i\epsilon}$, where $i\epsilon$ is inserted for defining a contour integration; similarly for the case at hand the formal inversion of the differential operator (8) defined on the hyperbolic ring will have exactly the same form, but with $k^2 = (w_R + ijw_H)^2 - (\vec{k}_1 + ij\vec{k}_2)^2$, and $m^2 = m_R^2 + ijm_H^2$. For this case ϵ can be generalized to a Hermitian quantity, $\epsilon = \epsilon_1 + ij\epsilon_2$, and thus $i\epsilon = i\epsilon_1 - j\epsilon_2$, which lies out of the Hermitian plane, on which k^2 is defined; furthermore, the possible poles lie on such a Hermitian plane, and the pair (ϵ_1, ϵ_2) will allow to define formally a contour integration in the hypercomplex space. However there exists a fundamental problem with the inversion of a differential operator on the hyperbolic ring, since the inversion is ill defined; furthermore, a Cauchy theorem on the hyperbolic scheme is not available, and it is not clear how to define a contour integration; more developments are clearly needed, and the construction of a propagator on the hyperbolic ring will be considered.

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