# Hilton-Milnor Splitting in HoTT

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Title okay?

#### Abstract

We show in homotopy type theory that for pointed types X,Y we have a pointed equivalence:  $\Omega(X\vee Y)\simeq\Omega X\times\Omega Y\times\Omega\Sigma(\Omega X\wedge\Omega Y)$  known as the Hilton-Milnor splitting.

Abstract needs to be fleshed out

#### 1 Introduction

something about classical homotopy theory

## 2 Homotopy pullbacks and descent

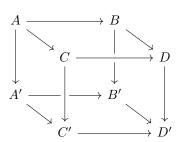
**Lemma 2.1** (Splitting lemma). Let  $f: X \to_* Y$  be a pointed map. Let  $g: \Omega Y \to_* \Omega X$  be a section of  $\Omega f$ . Then the composition of pointed maps

$$\Omega B \times \Omega \mathsf{hfiber}(f) \xrightarrow{g \times \mathsf{pr}_1}_* \Omega A \times \Omega A \xrightarrow{\mathsf{concat}}_* \Omega A$$

is an equivalence of pointed types.

Theorem 2.2 (Descent).

Statement and version for pointed types



Lemma 2.3 (Two-pullbacks lemma). Let

Needs proof and proper statement. Also a note about pointed version.

$$\begin{array}{cccc}
A & \longrightarrow & B & \longrightarrow & C \\
\downarrow & & \downarrow & & \downarrow \\
D & \longrightarrow & E & \longrightarrow & F
\end{array}$$

**Definition 2.4** (Wedge sum). Let X, Y be pointed types. Define the *wedge sum* (or simply *wedge*) of X and Y, denoted  $X \vee Y$  to be the following pushout:

$$\begin{array}{ccc}
1 & \longrightarrow X \\
\downarrow & & \downarrow \\
Y & \longrightarrow X \lor Y
\end{array}$$

**Definition 2.5** (Wedge inclusion). There is a map  $\omega: X \vee Y \to X \times Y$  called the wedge inclusion. Defined by wedge recursion:

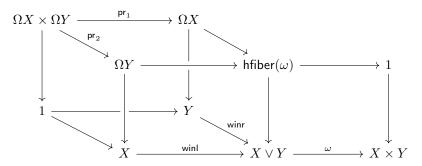
Wedge induction, recursion, "constructors"

- $winl(x) \mapsto (x,*)$
- $winr(y) \mapsto (*,y)$
- $\mathsf{wglue} \mapsto \mathsf{refl}_{(*,*)}$

Lemma 2.6 (Fiber of wedge inclusion). There is a pointed equivalence

$$\mathsf{hfiber}(\omega) \simeq \Omega X * \Omega Y$$

*Proof.* We have zero maps  $\Omega X \to Y$  and  $\Omega Y \to X$ . We have maps  $\Omega Y \to \mathsf{hfiber}(\omega)$  which takes  $p: \Omega Y$  and maps it to  $(*,\mathsf{pair}^=(\mathsf{refl},p))$  and likewise for  $\Omega X \to \mathsf{hfiber}(\omega)$ . Observe that the following diagram commutes:



Now the left and back faces of the cube are pullbacks. The bottom face is a pushout by definition. The front and right faces are pullbacks since the commutative square joined with the right square is a pullback by definition, therefore

by the pullback lemma they are pullbacks. Hence by descent the top square is a pushout. Observe that the pushout of the top square is the join  $\Omega X * \Omega Y$  hence  $\mathsf{hfiber}(\omega) \simeq \Omega X * \Omega Y$ .

**Lemma 2.7** (Join as a suspended smash). We have the following pointed equivalence for pointed types X and Y:

$$X * Y \simeq \Sigma(X \wedge Y)$$

**Remark 2.8.** Thus  $\mathsf{hfiber}(\omega) \simeq \Sigma(\Omega X \wedge \Omega Y)$ 

### 3 The Hilton-Milnor splitting

**Theorem 3.1** (Hilton-Milnor splitting). Let X,Y be pointed types. Then there is a pointed equivalence:

$$\Omega(X\vee Y)\simeq \Omega X\times \Omega Y\times \Omega \Sigma(\Omega X\wedge \Omega Y)$$