Hilton-Milnor Splitting in HoTT

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December 26, 2019

Title okay?

Abstract

We show in homotopy type theory that for pointed types X, Y we have a pointed equivalence: $\Omega(X \vee Y) \simeq \Omega X \times \Omega Y \times \Omega\Sigma(\Omega X \wedge \Omega Y)$ known as the Hilton-Milnor splitting.

Abstract needs to be fleshed out

1 Introduction

[1]

something about classical homotopy theory

2 Homotopy pullbacks and descent

Lemma 2.1 (Splitting lemma)**.** Let $f : X \to_* Y$ be a pointed map. Let $g : \Omega Y \to_* \Omega X$ be a section of Ωf. Then the composition of pointed maps

$$\Omega B \times \Omega\mathsf{hfiber}(f) \xrightarrow{g \times \mathsf{pr}_1}_* \Omega A \times \Omega A \xrightarrow{\mathsf{concat}}_* \Omega A$$

is an equivalence of pointed types.

Theorem 2.2 (Descent)**.**

Statement and version for pointed types

$$\begin{tikzcd}
A \arrow[rr] \arrow[dr] \arrow[dd] & & B \arrow[dr] \arrow[dd] & \\
& C \arrow[rr] \arrow[dd] & & D \arrow[dd] \\
A' \arrow[rr] \arrow[dr] & & B' \arrow[dr] & \\
& C' \arrow[rr] & & D'
\end{tikzcd}$$

Lemma 2.3 (Two-pullbacks lemma)**.** Let

Needs proof and proper statement. Also a note about pointed version.

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[d] \\ D \arrow[r] & E \arrow[r] & F \end{tikzcd}$$

Definition 2.4 (Wedge sum)**.** Let X, Y be pointed types. Define the *wedge sum* (or simply *wedge*) of X and Y, denoted $X \vee Y$ to be the following pushout:

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d] \\ Y \arrow[r] & X \vee Y \end{tikzcd}$$

Wedge induction, recursion, ”constructors”

Definition 2.5 (Wedge inclusion)**.** There is a map $\omega : X \vee Y \to X \times Y$ called the wedge inclusion. Defined by wedge recursion:

- $\mathsf{winl}(x) \mapsto (x, *)$
- $\mathsf{winr}(y) \mapsto (*, y)$
- $\mathsf{wglue} \mapsto \mathsf{refl}_{(*,*)}$

Lemma 2.6 (Fiber of wedge inclusion)**.** There is a pointed equivalence

$$\mathsf{hfiber}(\omega) \simeq \Omega X * \Omega Y$$

Proof. We have zero maps $\Omega X \to Y$ and $\Omega Y \to X$. We have maps $\Omega Y \to \mathsf{hfiber}(\omega)$ which takes $p : \Omega Y$ and maps it to $(*, \mathsf{pair}^{=}(\mathsf{refl}, p))$ and likewise for $\Omega X \to \mathsf{hfiber}(\omega)$. Observe that the following diagram commutes:

$$\begin{tikzcd} \Omega X \times \Omega Y \arrow[rr, "\mathsf{pr}_1"] \arrow[dr, "\mathsf{pr}_2"] \arrow[dd] & & \Omega X \arrow[dr] \arrow[dd] & & \\ & \Omega Y \arrow[rr] \arrow[dd] & & \mathsf{hfiber}(\omega) \arrow[r] \arrow[dd] & 1 \arrow[dd] \\ 1 \arrow[rr] \arrow[dr] & & Y \arrow[dr, "\mathsf{winr}"] & & \\ & X \arrow[rr, "\mathsf{winl}"] & & X \vee Y \arrow[r, "\omega"] & X \times Y \end{tikzcd}$$

Now the left and back faces of the cube are pullbacks. The bottom face is a pushout by definition. The front and right faces are pullbacks since the commutative square joined with the right square is a pullback by definition, therefore by the pullback lemma they are pullbacks. Hence by descent the top square is a pushout. Observe that the pushout of the top square is the join $\Omega X * \Omega Y$ hence $\mathsf{hfiber}(\omega) \simeq \Omega X * \Omega Y$. $\square$

Lemma 2.7 (Join as a suspended smash)**.** We have the following pointed equivalence for pointed types X and Y:

$$X * Y \simeq \Sigma(X \wedge Y)$$

Remark 2.8. Thus $\mathsf{hfiber}(\omega) \simeq \Sigma(\Omega X \wedge \Omega Y)$

3 The Hilton-Milnor splitting

Theorem 3.1 (Hilton-Milnor splitting)**.** Let X, Y be pointed types. Then there is a pointed equivalence:

$$\Omega(X \vee Y) \simeq \Omega X \times \Omega Y \times \Omega\Sigma(\Omega X \wedge \Omega Y)$$

References

[1] The Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics*. Institute for Advanced Study: `https://homotopytypetheory.org/book`, 2013.