Hilton-Milnor Splitting in HoTT

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Title okay?

Abstract

We show in homotopy type theory that for pointed types X,Y we have a pointed equivalence: $\Omega(X\vee Y)\simeq \Omega X\times \Omega Y\times \Omega \Sigma(\Omega X\wedge \Omega Y)$ known as the Hilton-Milnor splitting.

Abstract needs to be fleshed out

1 Introduction

[1]

something about classical homotopy theory

2 Homotopy pullbacks and descent

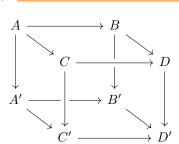
Lemma 2.1 (Splitting lemma). Let $f: X \to_* Y$ be a pointed map. Let $g: \Omega Y \to_* \Omega X$ be a section of Ωf . Then the composition of pointed maps

$$\Omega B \times \Omega \mathsf{hfiber}(f) \xrightarrow{g \times \mathsf{pr}_1}_* \Omega A \times \Omega A \xrightarrow{\mathsf{concat}}_* \Omega A$$

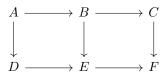
is an equivalence of pointed types.

Theorem 2.2 (Descent).

Statement and version for pointed types



Lemma 2.3 (Two-pullbacks lemma). Let _____



Needs proof and proper statement. Also a note about pointed version.

Definition 2.4 (Wedge sum). Let X, Y be pointed types. Define the wedge sum (or simply wedge) of X and Y, denoted $X \vee Y$ to be the following pushout:

$$\begin{array}{ccc}
1 & \longrightarrow & X \\
\downarrow & & \downarrow \\
Y & \longrightarrow & X \lor Y
\end{array}$$

Definition 2.5 (Wedge inclusion). There is a map $\omega: X \vee Y \to X \times Y$ called the wedge inclusion. Defined by wedge recursion:

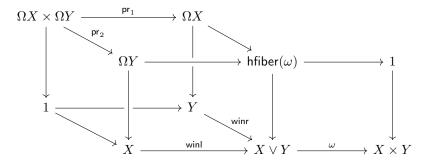
Wedge induction, recursion, "constructors"

- $winl(x) \mapsto (x,*)$
- $winr(y) \mapsto (*, y)$
- wglue $\mapsto \text{refl}_{(*,*)}$

Lemma 2.6 (Fiber of wedge inclusion). There is a pointed equivalence

$$\mathsf{hfiber}(\omega) \simeq \Omega X * \Omega Y$$

Proof. We have zero maps $\Omega X \to Y$ and $\Omega Y \to X$. We have maps $\Omega Y \to \mathsf{hfiber}(\omega)$ which takes $p: \Omega Y$ and maps it to $(*,\mathsf{pair}^=(\mathsf{refl},p))$ and likewise for $\Omega X \to \mathsf{hfiber}(\omega)$. Observe that the following diagram commutes:



Now the left and back faces of the cube are pullbacks. The bottom face is a pushout by definition. The front and right faces are pullbacks since the commutative square joined with the right square is a pullback by definition, therefore by the pullback lemma they are pullbacks. Hence by descent the top square is a pushout. Observe that the pushout of the top square is the join $\Omega X * \Omega Y$ hence $\mathsf{hfiber}(\omega) \simeq \Omega X * \Omega Y$.

Lemma 2.7 (Join as a suspended smash). We have the following pointed equivalence for pointed types X and Y:

$$X * Y \simeq \Sigma(X \wedge Y)$$

Remark 2.8. Thus $\mathsf{hfiber}(\omega) \simeq \Sigma(\Omega X \wedge \Omega Y)$

3 The Hilton-Milnor splitting

Theorem 3.1 (Hilton-Milnor splitting). Let X, Y be pointed types. Then there is a pointed equivalence:

$$\Omega(X \vee Y) \simeq \Omega X \times \Omega Y \times \Omega \Sigma(\Omega X \wedge \Omega Y)$$

References

[1] The Univalent Foundations Program. Homotopy Type Theory: Univalent Foundations of Mathematics. Institute for Advanced Study: https://homotopytypetheory.org/book, 2013.