1 Theories and models

[NOTE: This is a rough outline of what the document ought to look like, not even worthy of being a draft]
[TODO: Find references for these]

Definition 1.0.1. A theory asserts data and axioms. A model is a particular example of a theory.

For example a model of "the theory of groups" in the category of sets is simply a group. A model of "the theory of groups" in the category of topological spaces is a topological group. A model of "the theory of groups" in a the category of manifolds is a Lie group.

Categorical semantics is a general procedure to go from "a theory" to the notion of an internal object in some category.

The internal objects of itnerest is a model of the theory in a cateagory.

Then anything we prove formally about the theory is true for all models of the theory in any category.

For each kind of "type theory" there is a corresponding kind of "structured category" in which we consider models.

- Lawvere theories $\leftrightarrow$ Category with finite products
- Simply typed lambda calculus $\leftrightarrow$ Cartesian closed category
- Dependent type theory $\leftrightarrow$ Locally CC category

A doctrine specifies: - A collection of type constructors - A categorical structure realizing these constructors as operations.

Once we fix a doctrine $\mathbb{D}$, then a $\mathbb{D}$-theory specifies "generaing" or "axiomatic" types and terms. A $\mathbb{D}$-category is one pocessing the specified structure. A model of a $\mathcal{D}$-theory T in a $\mathcal{D}$-category C realizes the types and terms in T as objects and morphisms of C.

A finite-product theory is a type theory with unit and cartesian product as te only type constructors. Plus any number of axioms.

Example:

The theory of magmas has one axiomatic type M, and axiomatic terms $\vdash e : M$ and $x : M, y : M \vdash xy : M$. For monoids and groups we will need equality axioms.

Let T be a finite-product theory, C a category with finite products

A mdoel of T in C assigns:

1. To each type A in T, an object $[\![A]\!]$ in C
2. To each judgement derivable in T:

$$x_1 : A_1, \ldots, x_n : A_n \vdash b : B$$

   A morphism in C

$$[\![A_1]\!] \times \cdots \times [\![A_n]\!] \xrightarrow{[\![b]\!]} [\![B]\!]$$

3. Such taht $[\![A \times B]\!] = [\![A]\!] \times [\![B]\!]$

References