Introduction to dependent type theory

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1 Category theory

1.1 Introduction

1.2 Categories

We begin with the notion of a category. This can be thought of as a place mathematical objects live and interact with eachother.

Definition 1.2.1. A *category* $\mathcal{C}$ consists of:

- A set $\mathrm{Ob}(\mathcal{C})$, elements of which are called *objects* of $\mathcal{C}$.
- For each $X, Y \in \mathrm{Ob}(\mathcal{C})$, a set $\mathcal{C}(X, Y)$, called the *homset* from X to Y.
- For each $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$, a function $\circ_{X,Y,Z} : \mathcal{C}(Y, Z) \times \mathcal{C}(XY) \to \mathcal{C}(X, Z)$, called the *composite function*.
- For each $X \in \mathrm{Ob}(\mathcal{C})$, an element $\iota_X \in \mathcal{C}(X, X)$, called the identity map, sometimes written $\iota_X : 1 \to \mathcal{C}(X, X)$.

Such that the following axioms hold:

- *Associativity*: For every $f \in \mathcal{C}(X,Y)$, $g \in \mathcal{C}(Y,Z)$ and $h \in \mathcal{C}(Z,W)$, one has $(h \circ g) \circ f = h \circ (g \circ f)$.
- Identity: For every $f \in \mathcal{C}(X,Y)$, one has $f \circ \iota_X = f = \iota_Y \circ f$.

Remark 1.2.2. One typically writes composition as juxtaposition and omits the symbol $\circ$.

Example 1.2.3. The category of sets denoted **Set** is the category whose objects are small[1] sets and morphisms are functions between sets. Composition is given by composition of functions.

Choosing the direction in which our arrows point was rather arbitrary. This suggests that if we had chosen the other way we would have also gotten a category. So any category we can come up with has an associated dual.

Example 1.2.4. For any category $\mathcal{C}$, there is another category called the **opposite category** $\mathcal{C}^{\mathrm{op}}$ whose objects are the same as $\mathcal{C}$ however the homsets are defined as follows: $\mathcal{C}^{\mathrm{op}}(x,y) := \mathcal{C}(y,x)$. Composition is defined using the composition from the original category.

Remark 1.2.5. It is a simple exercise to check that for any category $\mathcal{C}$

[TODO: Note on commutative diagrams]

1.3 Functors

Definition 1.3.1. Given categories $\mathcal{C}$ and $\mathcal{D}$, a *functor* H from $\mathcal{C}$ to $\mathcal{D}$, written $H : \mathcal{C} \to \mathcal{D}$, consists of

- A function $\mathrm{Ob}(H) : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$, with notation typically abbreivated to H
- For each $X, Y \in \mathrm{Ob}(\mathcal{C})$ a function $H_{X,Y} : \mathcal{C}(X,Y) \to \mathcal{D}(HX, HY)$

Such that the following diagrams commute in the category **Set**:

- *H respects composition*:

$$\begin{tikzcd}
\mathcal{C}(Y,Z) \times \mathcal{C}(X,Y) \arrow[r, "H \times H"] \arrow[d, "\circ"'] & \mathcal{D}(HY,HZ) \times \mathcal{D}(HX,HY) \arrow[d, "\circ"] \\
\mathcal{C}(X,Z) \arrow[r, "H"] & \mathcal{D}(HX,HZ)
\end{tikzcd}$$

- *H respects units*:

$$\begin{tikzcd}
1 \arrow[r, "\iota_X"] \arrow[dr, "\iota_{HX}"'] & \mathcal{C}(X,X) \arrow[d, "H"] \\
& \mathcal{D}(HX,HX)
\end{tikzcd}$$

[1]due to Russellian paradoxes we must distinguish between "all sets" and "enough sets". See appendix for details.

1.4 Natural transformations

Definition 1.4.1. Given categories $\mathcal{C}$ and $\mathcal{D}$ and functors $H, K : \mathcal{C} \to \mathcal{D}$, a *natural transformation* $\alpha : H \to K$ consists of

For each $X \in \mathrm{Ob}(\mathcal{C})$, a map $\alpha_X : HX \to KX$.

Such that for each map $f : X \to Y$ in $\mathcal{C}$, the following diagram commutes:

$$\begin{tikzcd} HX \arrow[r, "\alpha_X"] \arrow[d, "Hf"'] & KX \arrow[d, "Kf"] \\ HY \arrow[r, "\alpha_Y"] & KY \end{tikzcd}$$

1.5 Having a left adjoint

This is arguably the most important definition we will study in category theory, we will see later on many concepts are special case of the following definition:

Definition 1.5.1. A functor $U : \mathcal{C} \to \mathcal{D}$ has a left adjoint if for all $X \in \mathcal{D}$, there exists an $FX \in C$ and $\eta_X : X \to UFX$ in $\mathcal{D}$ such that for all $A \in \mathcal{D}$ and forall $f : X \to UA$, exists a **unique** map $g : FX \to A$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, "f"'] & UFX \arrow[d, "Ug"] \\ & UA \end{tikzcd}$$

2 !!old stuff

We will introduce basic category theory. Good references are: [**?** 1, 3, 5]

Category theory will allow us to model the desired behaviour of dependent types.

2.1 Categories

Definition 2.1.1. A **category** $\mathcal{C}$ consists of:

- A class $\mathrm{Ob}(\mathcal{C})$ (usually simply denoted $\mathcal{C}$ without ambiguity) of **objects**.
- For each object $A, B \in \mathcal{C}$, a set $\mathcal{C}(A, B)$ of **morphisms** or **arrows** called a **homset**. When writing $f \in \mathcal{C}(A, B)$ we usually denote this $f : A \to B$.
- For each object $A \in \mathcal{C}$ a morphism $1_A : A \to A$ called the **identity**.
- For each object $A, B, C \in \mathcal{C}$, and for each $f : A \to B$ and $g : B \to C$ there is a function (written infix or sometimes simply omitted $(gf \equiv g \circ f)$

  $$- \circ - : \mathcal{C}(B, C) \times \mathcal{C}(A, B) \to \mathcal{C}(A, C)$$

  called **composition**.

Such that the following hold:

- (Identity) For each $A, B \in \mathcal{C}$ and $f : A \to B$ we have $f \circ 1_A = f$ and $1_B \circ f = f$.
- (Associativity) For all $A, B, C, D \in \mathcal{C}$ and $f : A \to B$, $g : B \to C$, $h : C \to D$. We have: $h \circ (g \circ f) = (h \circ g) \circ f$.

Remark 2.1.2. There are many similar and mostly equivalent definitions of category in mathematics. They mostly fall into two camps, differing by how they treat their collection of morphisms. The two sorts of definitions are equivalent in the usual foundations of mathematics but each has their own advantages. In books such as [4] a collection of morphisms is used. This approach lends itself more naturally to the notion of an *internal category* which will be an important concept later on. The other definition uses a family of collections of morphisms which lends itself to easily generalise to the notion of an *enriched category*, the definitive reference for which is [2].

The reason we care is that it cannot be swept under the rug so easily. This is because the issue of size is of fundamental importance in category theory. Depending on what definition we chose, it will effect how easily we can talk about it. For an introduction to category theory, these ideas would mostly confuse the reader, hence we will simply point to [6] for a survey on how size issues are treated in category theory.

We now give some examples:

Example 2.1.3. The **category of sets** denoted **Set** is the category whose objects are small[2] sets and morphisms are functions between sets. Composition is given by composition of functions. This is a very important category in category theory for reasons we shall come across later.

Choosing the direction in which our arrows point was arbitrary, but it does also mean that if we had chosen the other way we would also get a category. So every category we make canonically comes with a "friend".

Example 2.1.4. For any category $\mathcal{C}$, there is another category called the **opposite category** $\mathcal{C}^{\mathrm{op}}$ whose objects are the same as $\mathcal{C}$ however the homsets are defined as follows: $\mathcal{C}^{\mathrm{op}}(x, y) := \mathcal{C}(y, x)$. Composition is defined using the composition from the original category.

[NEEDS REWORDING AND REORGANISING] Size is a common issue in category theory with many similar ways of dealing with it. It can however cause much confusion and hoop-jumping to be correct. For our purposes we will safely ignore these issues. A formal treatment can be found in the appendix. [TODO: Add formal treatment of size].

Definition 2.1.5. We call a category **small** if its class of objects is really a set.

[2]due to Russellian paradoxes we must distinguish between "all sets" and "enough sets". See appendix for details.

2.2 Functors and natural transformations

Definition 2.2.1. Let $\mathcal{C}, \mathcal{D}$ be categories. A **functor** F from $\mathcal{C}$ to $\mathcal{D}$ (written $F : C \to D$) consists of:

- An object $F(A) \in \mathcal{D}$, for all $A \in \mathcal{C}$ (also denoted FA).
- For each $A, B \in \mathcal{C}$, a function $F_{A,B} : \mathcal{C}(A, B) \to \mathcal{D}(FA, FB)$ (also denoted F).
- For each $A \in \mathcal{C}$, $F(1_A) = 1_{FA}$.
- For each $A, B, C \in \mathcal{C}$, $f : A \to B$, $g : B \to C$, we have

$$F(g \circ f) = F(g) \circ F(f)$$

Remark 2.2.2. Historically in category theory, one would define covariant, as defined above, and contravariant functors, as a result this terminology has crept into uses of category in certain fields [REFERENCE pretty much any homological algebra book before 80s]. Contravariant functors mean to swap the order of composition when the functor is applied. In modern category theory texts, this is completely dropped as a contravariant functor from $\mathcal{C}$ to $\mathcal{D}$ is simply a covariant functor from $\mathcal{C}^{\text{op}}$ to $\mathcal{D}$. Henceforth, we shall not mention co(tra)variance of functors and refer to them simply a functors.

Remark 2.2.3. Given two functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ we can make a new functor $G \circ F$ called its **composite**, by first applying F then applying G on objects or morphisms. It is simple to check that this is indeed a functor.

2.2.1 Properties of functors

Here we list a few properties a given functor may possess. We mostly include them here for reference and it will not be so essential to understand what they mean.

[TODO: Do we even care about essentially surjective functors? Is it something category theory ought to study? Can't we get away with fully faithful upto equivalence of categories?]

Definition 2.2.4. A functor $F : \mathcal{C} \to \mathcal{D}$ is

- **full** if for each $x, y \in \mathcal{C}$, the map $\mathcal{C}(x, y) \to \mathcal{D}(Fx, Fy)$ is surjective;
- **faithful** if for each $x, y \in \mathcal{C}$. the map $\mathcal{C}(x, y) \to \mathcal{D}(Fx, Fy)$ is injective;
- **essentially surjective** if for each object $y \in \mathcal{D}$, there exists an object $x \in \mathcal{C}$ and an isomorphism $F(x) \cong y$ in $\mathcal{D}$. ("Surjective on objects upto isomorphism").

Remark 2.2.5. Being full or faithful are local conditions, this means that they don't hold globally. For example a functor that is full is not necesserily surjective on all morphisms and a faithful is not necesserily injective on all morphisms, this is because we know nothing of how the functor acts on objects.

Definition 2.2.6. A functor $F : \mathcal{C} \to \mathcal{D}$ is

- **fully faithful** if it is full and faithful.
- an **embedding** if it is faithful and injective on objects.
- a full embedding if it is full and an embedding.

The domain of a full embedding $F : \mathcal{C} \to \mathcal{D}$ defines a **full subcategory** $\mathcal{C}$ of $\mathcal{D}$.

Now that we have 'morphisms' between categories we can define another category:

Example 2.2.7. The category of small categories **Cat** has objects small categories and morphisms functors. Composition is given by composition of functors.

Definition 2.2.8 (Definition of natural transformation)**.**

Example 2.2.9. Given two categories $\mathcal{C}$ and $\mathcal{D}$ we can from a category $[\mathcal{C}, \mathcal{D}]$ called the functor category between $\mathcal{C}$ and $\mathcal{D}$. It's objects are functors $\mathcal{C} \to \mathcal{D}$ and morphisms are natural transformations between functors.

Special cases of this example include:

Example 2.2.10. A functor $\mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ is typically called a **presheaf** in geometry and logic. They live in the functor category $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ which we will call the **category of presheaves**. This is an interesting construction as it acts like the category $\mathcal{C}$ in some ways with some nice properties from **Set**.

[CHECK THIS] One of the first theorems that is proven in category theory is the **Yoneda lemma**. It says if an object acts like a certain object in every possible way, then it must be isomorphic to that object. Akin to how particles are discovered in particle accelerators by bombarding them with

Lemma 2.2.11. Let $\mathcal{C}$ be a category. There is an embedding $\mathbf{y} : \mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$. Where $\mathbf{y}(A) := \mathcal{C}(A, -)$, maps each object to its contravariant hom functor. Presheaves that arise this way are called **representable presheaves**.

Remark 2.2.12. We haven't actually proven that the "Yoneda embedding" is an embedding however this is a corollary of the Yoneda lemma which will prove now.

[PICTURES]

Theorem 2.2.13. Yoneda lemma Let $\mathcal{C}$ be a category. For all $X \in [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$, there is a natural isomorphism between the following functors:

$$[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}](\mathbf{y}(-), X) \cong X(-)$$

Proof. $\square$

Remark 2.2.14. The set of natural transformations between $\mathbf{y}(A)$ and a presheaf X is bijective to the sections of X at A.

References

[1] Michael Barr and Charles Wells. *Category Theory for Computing Science*. Prentice-Hall, Inc., Upper Saddle River, NJ, USA, 1990.

[2] M. Kelly. *Basic Concepts of Enriched Category Theory*. Lecture note series / London mathematical society. Cambridge University Press, 1982.

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[5] J.J. Rotman. *An Introduction to Homological Algebra*. Universitext. Springer New York, 2008.

[6] Michael A. Shulman. Set theory for category theory. *arXiv e-prints*, page arXiv:0810.1279, October 2008.