1 Category theory

1.1 Introduction

We will introduce basic category theory. Good references are: [1, 2]

Definition 1.1.1. A **category** $\mathcal{C}$ consists of:

- A class $\mathrm{Ob}(\mathcal{C})$ (usually simply denoted $\mathcal{C}$ without ambiguity) of **objects**.
- For each object $A, B \in \mathcal{C}$ a set $\hom_{\mathcal{C}}(A, B)$ of **morphisms** called a **homset** (written hom when the context is clear). When writing $f \in \hom(A, B)$ we usually denote this $f : A \to B$.
- For each object $A \in \mathcal{C}$ a morphism $1_A : A \to A$ called the **identity**.
- For each object $A, B, C \in \mathcal{C}$, and for each $f : A \to B$ and $g : B \to C$ there is a function (written infix or sometimes simply omitted $(fg \equiv f \circ g)$

$$- \circ - : \hom(B, C) \times \hom(A, B) \to \hom(A, C)$$

  called **composition**.

Such that the following hold:

- (Identity) For each $A, B \in \mathcal{C}$ and $f : A \to B$ we have $f \circ 1_A = f$ and $1_B \circ f = f$.
- (Associativity) For all $A, B, C, D \in \mathcal{C}$ and $f : A \to B$, $g : B \to C$, $h : C \to D$. We have: $h \circ (g \circ f) = (h \circ g) \circ f$.

We now give some examples:

Example 1.1.2. The **category of sets** denoted **Set** is the category whose objects are small[1] sets and morphisms are functions between sets. Composition is given by composition of functions. This is a very important category in category theory for reasons we shall come across later.

Choosing the direction in which our arrows point was arbitrary, but it does also mean that if we had chosen the other way we would also get a category. So every category we make canonically comes with a friend.

Example 1.1.3. For any category $\mathcal{C}$, there is another category called the **opposite category** $\mathcal{C}^{\mathrm{op}}$ whose objects are the same as $\mathcal{C}$ however the homsets are defined as follows: $\hom_{\mathcal{C}^{\mathrm{op}}}(x, y) := \hom_{\mathcal{C}}(y, x)$. Composition is defined using the composition from the original category.

Definition 1.1.4. We call a category **small** if its class of objects is really a set.

Definition 1.1.5. Let $\mathcal{C}, \mathcal{D}$ be categories. A **functor** F from $\mathcal{C}$ to $\mathcal{D}$ (written $F : C \to D$) consists of:

- An object $F(A) \in \mathcal{D}$, for all $A \in \mathcal{C}$ (also denoted FA).
- For each $A, B \in \mathcal{C}$, a function $F_{A,B} : \hom_{\mathcal{C}}(A, B) \to \hom_{\mathcal{D}}(FA, FB)$ (also denoted F).

[1] due to Russellian paradoxes we must distinguish between "all sets" and "enough sets". See appendix for details.

- For each $A \in \mathcal{C}$, $F(1_A) = 1_{FA}$.
- For each $A, B, C \in \mathcal{C}$, $f : A \to B$, $g : B \to C$, we have

$$F(g \circ f) = F(g) \circ F(f)$$

Remark 1.1.6. Historically in category theory, one would define covariant, as defined above, and contravariant functors. Contravariant functors mean to swap the order of composition when the functor is applied. In modern category theory texts, this is completely dropped as a contravariant functor from $\mathcal{C}$ to $\mathcal{D}$ is simply a covaraint functor from $\mathcal{C}^{\text{op}}$ to $\mathcal{D}$. Henceforth, we shall not mention co(tra)variance of functors and refer to them simply a functors.

Remark 1.1.7. Given two functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ we can make a new functor by composition of functors. Where $G \circ F$ is defined as the functor which takes an object A of $\mathcal{C}$

Now that we have 'morphisms' between categories we can define another category:

Example 1.1.8. The category of small categories **Cat** has objects small categories and morphisms functors. Composition is given by composition of functors.

References

[1] Saunders Mac Lane. *Categories for the working mathematician.* Graduate texts in mathematics ; 5. Springer, New York, 2nd ed. edition, 1998.

[2] J.J. Rotman. *An Introduction to Homological Algebra.* Universitext. Springer New York, 2008.