Tipi

λ-račun s preprostimi tipi

(simply-typed λ-calculus / STLC)

izrazi $M, N ::= x \mid \lambda x. M \mid M\,N$
$\mid \underline{n} \mid M + N \mid M * N \mid -M$
$\mid M = N \mid M < N \mid M > N$
$\mid \mathsf{true} \mid \mathsf{false} \mid \mathsf{if}\ M\ \mathsf{then}\ N_1\ \mathsf{else}\ N_2$
$\mid \mathsf{rec}\ f\ x. M$

$(2+1) + \lambda x.\ \mathsf{if}\ \lambda y. y\ \mathsf{then}\ \mathsf{true}\ \mathsf{else}\ 42$

! sintaktično veljaven izraz !

$\mathsf{rec}\ f\ n.\ \mathsf{if}\ n = \underline{0}\ \mathsf{then}\ \underline{1}\ \mathsf{else}\ n * f(n-1)$

tipi $A, B ::= \mathsf{int} \mid \mathsf{bool} \mid A \to B$

$\boxed{M \rightsquigarrow M'}$

$V ::= \lambda x. M \mid \underline{n} \mid \mathsf{true} \mid \mathsf{false} \mid \mathsf{rec}\ f\ x. M$

$$\frac{M \rightsquigarrow M'}{M\,N \rightsquigarrow M'\,N} \qquad \frac{N \rightsquigarrow N'}{V\,N \rightsquigarrow V\,N'} \qquad \frac{}{(\lambda x. M)\,V \rightsquigarrow M[V/x]}$$

$$\frac{}{(\mathsf{rec}\ f\ x. M)\,V \rightsquigarrow M[V/x, (\mathsf{rec}\ f\ x. M)/f]}$$

$$\frac{M \rightsquigarrow M'}{M + N \rightsquigarrow M' + N} \qquad \frac{N \rightsquigarrow N'}{V + N \rightsquigarrow V + N'} \qquad \frac{}{\underline{m} + \underline{n} \rightsquigarrow \underline{m + n}}$$

podobno za $*, =, <, >$

$$\frac{M \rightsquigarrow M'}{-M \rightsquigarrow -M'} \qquad \frac{}{-\underline{n} \rightsquigarrow \underline{-n}}$$

$$\frac{M \rightsquigarrow M'}{\mathsf{if}\ M\ \mathsf{then}\ N_1\ \mathsf{else}\ N_2 \rightsquigarrow \mathsf{if}\ M'\ \mathsf{then}\ N_1\ \mathsf{else}\ N_2}$$

$$\frac{}{\mathsf{if}\ \mathsf{true}\ \mathsf{then}\ N_1\ \mathsf{else}\ N_2 \rightsquigarrow N_1} \qquad \frac{}{\mathsf{if}\ \mathsf{false}\ \mathsf{then}\ N_1\ \mathsf{else}\ N_2 \rightsquigarrow N_2}$$

$\boxed{\Gamma \vdash M : A}$

↑ kontekst $\quad x_1 : A_1, x_2 : A_2, \ldots, x_n : A_n$

izrazi $M, N ::= x \mid \lambda x. M \mid M\, N$
$\mid \underline{n} \mid M + N \mid M * N \mid -M$
$\mid M = N \mid M < N \mid M > N$
$\mid \mathrm{true} \mid \mathrm{false} \mid \mathrm{if}\ M\ \mathrm{then}\ N_1\ \mathrm{else}\ N_2$
$\mid \mathrm{rec}\ f x. M$

$$\frac{x : A \in \Gamma}{\Gamma \vdash x : A} \qquad \frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x. M : A \to B} \qquad \frac{\Gamma \vdash M : A \to B \qquad \Gamma \vdash N : A}{\Gamma \vdash M\, N : B}$$

$$\frac{}{\Gamma \vdash \underline{n} : \mathrm{int}} \qquad \frac{\Gamma \vdash M : \mathrm{int} \qquad \Gamma \vdash N : \mathrm{int}}{\Gamma \vdash M + N : \mathrm{int}}$$

podobno za $*, -, =, <, >$

$$\frac{}{\Gamma \vdash \mathrm{true} : \mathrm{bool}} \qquad \frac{}{\Gamma \vdash \mathrm{false} : \mathrm{bool}} \qquad \frac{\Gamma \vdash M : \mathrm{bool} \quad \Gamma \vdash N_1 : A \quad \Gamma \vdash N_2 : A}{\Gamma \vdash \mathrm{if}\ M\ \mathrm{then}\ N_1\ \mathrm{else}\ N_2 : A}$$

$$\frac{\Gamma, f : A \to B, x : A \vdash M : B}{\Gamma \vdash \mathrm{rec}\ f x. M : A \to B}$$

$$\frac{\dfrac{\dfrac{}{x : \mathrm{int} \vdash 2 : \mathrm{int}} \qquad \dfrac{}{x : \mathrm{int} \vdash x : \mathrm{int}}}{x : \mathrm{int} \vdash 2 * x : \mathrm{int}}}{\emptyset \vdash \lambda x. 2 * x : \mathrm{int} \to \mathrm{int}}$$

Izrek o varnosti

Napredek:

Če velja $\emptyset \vdash M : A$ (zaprt izraz), tedaj bodisi
- je M vrednost
- obstaja M', da velja $M \rightsquigarrow M'$

Ohranitev:

Če velja $\Gamma \vdash M : A$ in $M \rightsquigarrow M'$, tedaj velja tudi $\Gamma \vdash M' : A$

$$\frac{x:A\in\Gamma}{\Gamma\vdash x:A} \qquad \frac{\Gamma, x:A\vdash M:B}{\Gamma\vdash \lambda x.M : A\to B} \qquad \frac{\Gamma\vdash M:A\to B \quad \Gamma\vdash N:A}{\Gamma\vdash M\,N:B}$$

$$\frac{}{\Gamma\vdash \underline{n} : \mathrm{int}} \qquad \frac{\Gamma\vdash M:\mathrm{int} \quad \Gamma\vdash N:\mathrm{int}}{\Gamma\vdash M+N:\mathrm{int}}$$ podobno za $*, -, =, <, >$

$$\frac{}{\Gamma\vdash \mathrm{true}:\mathrm{bool}} \qquad \frac{}{\Gamma\vdash \mathrm{false}:\mathrm{bool}} \qquad \frac{\Gamma\vdash M:\mathrm{bool} \quad \Gamma\vdash N_1:A \quad \Gamma\vdash N_2:A}{\Gamma\vdash \mathrm{if}\ M\ \mathrm{then}\ N_1\ \mathrm{else}\ N_2 : A}$$

Dokaz (napredek)

Z indukcijo na $\vdash M:A$.

Če je bilo zadnje uporabljeno pravilo:

- $\dfrac{x:A\in\emptyset}{\emptyset\vdash x:A}$ // se ne more zgoditi, ker zahtevamo prazen kontekst.

- $\dfrac{\dots}{\emptyset\vdash \lambda x.M' : A'\to B}$ – $M=\lambda x.M'$ je vrednost ✓

- $\dfrac{\emptyset\vdash M_1 : A'\to A \quad \emptyset\vdash M_2 : A'}{\emptyset\vdash M_1 M_2 : A}$

  To ni vrednost, pokažimo, da $M_1 M_2$ lahko vedno naredi korak.

  Po indukcijski predpostavki za $\emptyset\vdash M_1 : A'\to A$ dobimo dva primera:

  1) M_1 je vrednost in ker ima tip $A'\to A$, velja $M_1=\lambda x.M_1'$ (ali $\mathrm{rec}\ f\,x.M_1'$)

     Po indukcijski predpostavki za $\emptyset\vdash M_2 : A'$ spet dobimo dva primera:

     1.1) M_2 je vrednost, recimo V_2

     Tedaj $M_1 M_2 = (\lambda x.M_1')V_2 \rightsquigarrow M_1'[V_2/x]$ ✓

     1.2) $M_2 \rightsquigarrow M_2'$

     Tedaj $M_1 M_2 = (\lambda x.M_1')M_2 \rightsquigarrow (\lambda x.M_1')M_2'$ ✓

  2) $M_1 \rightsquigarrow M_1'$

     Tedaj $M_1 M_2 \rightsquigarrow M_1' M_2$ ✓

Ostalo doma

Lema (o substituciji)

Če velja $\Gamma, x:A\vdash M:B$ in $\Gamma\vdash N:A$, tedaj $\Gamma\vdash M[N/x]:B$

Dokaz

Z indukcijo na $\Gamma, x:A\vdash M:B$.

Primer

$$\frac{\frac{}{\Gamma'\vdash x:\mathrm{bool}} \quad \frac{}{\Gamma'\vdash y:\mathrm{int}} \quad \frac{\frac{}{\Gamma'\vdash y:\mathrm{int}}}{\Gamma'\vdash -y:\mathrm{int}}}{\underbrace{y:\mathrm{int}, x:\mathrm{bool}}_{\Gamma'}\vdash \mathrm{if}\ x\ \mathrm{then}\ y\ \mathrm{else}\ -y : \mathrm{int}}$$

$$\frac{\vdots}{y:\mathrm{int}\vdash y<0:\mathrm{bool}}$$

$$\dfrac{\begin{array}{c}\vdots\\ \hline y:int \vdash y<0 : bool\end{array} \qquad \dfrac{}{\Gamma \vdash y : int} \qquad \dfrac{\dfrac{}{\Gamma \vdash y : int}}{\Gamma \vdash -y : int}}{y : int \vdash \text{if } (y<0) \text{ then } y \text{ else } -y : int}$$

$$\dfrac{M \rightsquigarrow M'}{M N \rightsquigarrow M' N} \qquad \dfrac{N \rightsquigarrow N'}{V N \rightsquigarrow V N'} \qquad \dfrac{}{(\lambda x. M)\, V \rightsquigarrow M[V/x]}$$

$$\dfrac{M \rightsquigarrow M'}{M + N \rightsquigarrow M' + N} \qquad \dfrac{N \rightsquigarrow N'}{V + N \rightsquigarrow V + N'} \qquad \dfrac{}{\underline{m} + \underline{n} \rightsquigarrow \underline{m+n}}$$

podobno za $*, =, <, >$

$$\dfrac{M \rightsquigarrow M'}{-M \rightsquigarrow -M'} \qquad \dfrac{}{-\underline{n} \rightsquigarrow \underline{-n}}$$

$$\dfrac{M \rightsquigarrow M'}{\text{if } M \text{ then } N_1 \text{ else } N_2 \rightsquigarrow \text{if } M' \text{ then } N_1 \text{ else } N_2}$$

$$\dfrac{}{\text{if true then } N_1 \text{ else } N_2 \rightsquigarrow N_1} \qquad \dfrac{}{\text{if false then } N_1 \text{ else } N_2 \rightsquigarrow N_2}$$

Dokaz (ohranitev)

Z indukcijo na $M \rightsquigarrow M'$.

Pogledamo primere.

- $$\dfrac{M_1 \rightsquigarrow M_1'}{M_1 M_2 \rightsquigarrow M_1' M_2}$$

  Če velja $\Gamma \vdash M : A$, tedaj velja tudi $\Gamma \vdash M_1 : A' \to A$ in $\Gamma \vdash M_2 : A'$.
  Po indukcijski predpostavki velja $\Gamma \vdash M_1' : A' \to A$.
  Torej je tudi $\Gamma \vdash M_1' M_2 : A$. ✓

- $$\dfrac{M_2 \rightsquigarrow M_2'}{V M_2 \rightsquigarrow V M_2'}$$ podobno kot prej. ✓

- $$\dfrac{}{(\lambda x. M_1)\, V \rightsquigarrow M_1[V/x]}$$

  Ko velja $\Gamma \vdash (\lambda x. M_1) V : A$, velja tudi $\Gamma \vdash (\lambda x. M_1) : A' \to A$ in $\Gamma \vdash V : A'$. Zato velja tudi $\Gamma, x : A' \vdash M_1 : A$ in po lemi o substituciji velja $\Gamma \vdash M_1[V/x] : A$. ✓

- ostalo doma.