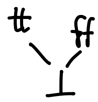
Definicija Domena $(D,\le)$ je delna urejenost $\le$ na množici D, za katero velja:

- obstaja najmanjši element $\bot_D$, ki mu pravimo dno, $\forall x\in D.\ \bot \le x$
- obstaja supremum vseh števnih naraščajočih verig $x_0\le x_1\le x_2\le \dots$ $\quad \bigvee_i x_i$

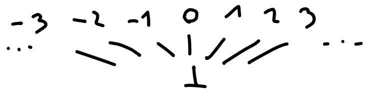
$(x_i)_{i\in\mathbb{N}}\in D,\ \forall i\in\mathbb{N}.\ x_i\le x_{i+1} \Rightarrow \bigvee_i x_i\in D$

Primer

- $([0,1],\le)$ ✓
- $((0,1],\le)$ ✗ nima dna
- $([0,1),\le)$ ✗ $\bigvee_i (1-\frac{1}{i}) \notin D$
- $([0,1)\cup\{2\},\le)$ ✓
- $\mathbb{B}_\bot$



- $\mathbb{Z}_\bot$



- $A_\bot = (A+\{\bot\},\le)$ $\quad x\le y \iff x=\iota_2(\bot) \lor x=y$

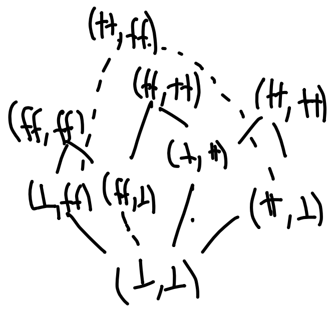
dvig A

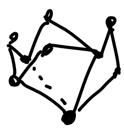
Definicija Za domeni $(D_1,\le_1)$ in $(D_2,\le_2)$ definiramo njun produkt kot $(D_1\times D_2,\le)$, kjer je

$$(x_1,x_2)\le(y_1,y_2) \iff x_1\le_1 y_1 \land x_2\le_2 y_2$$

Doma: to je res domena.

Primer $\mathbb{B}_\bot\times\mathbb{B}_\bot$





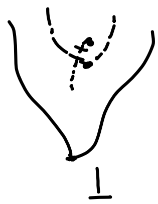
Definicija Vzemimo domeni $(D_1, \le_1)$ in $(D_2, \le_2)$. Preslikava $f: D_1 \to D_2$ je zvezna, če:

- je monotona $\quad x \le_1 x' \Rightarrow f(x) \le_2 f(x')$
- ohranja supremume verig

$$f\Big(\bigvee_i x_i\Big) = \bigvee_i f(x_i)$$

obstaja, ker je D_1 domena

obstaja, ker je D_2 domena in f monotona



Trditev Vzemimo domene $(D_1, \le_1)$, $(D_2, \le_2)$ in $(D_3, \le_3)$ ter zvezni preslikavi $f: D_1 \to D_2$ in $g: D_2 \to D_3$. Tedaj je $g \circ f: D_1 \to D_3$ zvezna.

Dokaz
- monotonost
$$x_1 \le_1 x_1' \Rightarrow f(x_1) \le_2 f(x_1') \Rightarrow g(f(x_1)) \le_3 g(f(x_1'))$$
- ohranjanje supremumov
$$(g \circ f)\Big(\bigvee_i x_i\Big) = g\Big(f\Big(\bigvee_i x_i\Big)\Big) = g\Big(\bigvee_i f(x_i)\Big)$$
$$= \bigvee_i \big(g(f(x_i))\big) = \bigvee_i (g \circ f)(x_i)$$

Izrek (Tarski) Naj bo $(D, \le)$ domena in $f: D \to D$ zvezna preslikava. Tedaj ima f najmanjšo fiksno točko.

Dokaz Definirajmo $x_0 = \bot$, $x_{i+1} = f(x_i)$.
Ker je $x_0 = \bot \le f(\bot) = x_1$ je $x_1 = f(x_0) \le f(x_1) = x_2$ ...
Zato dobimo verigo $x_0 \le x_1 \le x_2 \le \cdots$
Definirajmo $x = \bigvee_i x_i$. Tedaj velja
- $f(x) = f\big(\bigvee_i x_i\big) = \bigvee_i f(x_i) = \bigvee_i x_{i+1} = x$
- Naj velja $y = f(y)$. Tedaj velja $x_0 = \bot \le y$. Po indukciji velja tudi $x_{i+1} = f(x_i) \le f(y) = y$. Torej je $x_i \le y$ za vse i, zato je tudi $x = \bigvee_i x_i \le y$.

Definicija Vzemimo domeni $(D_1, \le_1)$ in $(D_2, \le_2)$ ter definirajmo domeno zveznih funkcij $[D_1 \to D_2]$ kot

$$\big(\{f: D_1 \to D_2 \mid f \text{ je zvezna}\}, \le\big), \text{ kjer je}$$

$$f \le g \iff \forall x \in D_1.\ f(x) \le_2 g(x).$$

Pokažimo, da res dobimo domeno.

$$\bot_{[D_1 \to D_2]} = x \mapsto \bot_{D_2} \quad ✓$$

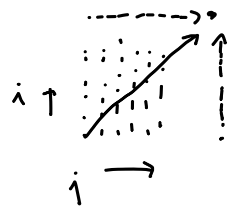
je zvezna ✓

Vzemimo verigo $f_1 \le f_2 \le \dots$ in pokažimo, da je $\bigvee_i f_i = f$, kjer je $f(x) := \bigvee_i f_i(x)$. Najprej pokažimo, da je f zvezna.

$$f\left(\bigvee_j x_j\right) = \bigvee_i f_i\left(\bigvee_j x_j\right) = \bigvee_i \bigvee_j f_i(x_j)$$

$$\overset{\checkmark}{=}$$

$$\bigvee_j f(x_j) = \bigvee_j \bigvee_i f_i(x_j) \qquad // \text{D.N.}$$



Pokažimo, da je f res supremum. Vzemimo $g \ge f_i$ in pokažimo, da je $g(x) \ge f(x)$ za vse $x \in D_1$.

$$f(x) = \bigvee_i f_i(x) \le \bigvee_i g(x) = g(x) \checkmark$$

Lema Naj bo $x_{i,j} : \mathbb{N}^2 \to D$, da je $x_{i,j} \le x_{i',j'}$ za vse $i \le i'$, $j \le j'$. Tedaj je $\bigvee_i \bigvee_j x_{ij} = \bigvee_k x_{kk}$.

Dokaz $(\le)$ $x_{ij} \le x_{\max(i,j),\max(i,j)} \le \bigvee_k x_{kk}$

$$\bigvee_j x_{ij} \le \bigvee_k x_{kk}$$

$$\bigvee_i \bigvee_j x_{ij} \le \bigvee_k x_{kk}$$

$(\ge)$ $x_{kk} \le \bigvee_j x_{kj} \le \bigvee_i \bigvee_j x_{ij}$

Trditev Preslikava $ev : [D_1 \to D_2] \times D_1 \to D_2$ je zvezna.

$$ev : (f, x) \longmapsto f(x)$$

Dokaz

- monotonost $\checkmark$
- $ev\left(\bigvee_i (f_i, x_i)\right)$

$$= ev\left(\bigvee_i f_i, \bigvee_j x_j\right)$$

$$= \left(\bigvee_i f_i\right)\left(\bigvee_j x_j\right)$$

$$= \bigvee_i \left(f_i\left(\bigvee_j x_j\right)\right)$$

$$= \bigvee_i \bigvee_j f_i x_j$$

$$= \bigvee_k f_k x_k$$

$$= \bigvee_k ev(f_k, x_k) \checkmark$$

Trditev Naj bodo D, D_1, D_2 domene. Tedaj je $f : D \to D_1 \times D_2$ zvezna natanko tedaj, kadar sta zvezni $\pi_1 \circ f : D \to D_1$ ter $\pi_2 \circ f : D \to D_2$.

Trditev Naj bodo D, D_1, D_2 domene. Tedaj je $f : D_1 \times D_2 \to D$ zvezna natanko tedaj, kadar sta zvezne

$$f_y^1 : D_1 \to D \qquad f_y^1(x) = f(x,y) \quad \text{za vse } y \in D_2$$

$$f_x^2 : D_2 \to D \qquad f_x^2(y) = f(x,y) \quad \text{za vse } x \in D_1$$

Vsak tip A bomo interpretirali z domeno $[\![A]\!]$

$$[\![\text{int}]\!] = \mathbb{Z}_\perp \qquad [\![\text{bool}]\!] = \mathbb{B}_\perp \qquad [\![A \to B]\!] = [[\![A]\!] \to [\![B]\!]]$$

Kontekst $\Gamma = x_1 : A_1, \dots, x_n : A_n$ bomo interpretirali s produktom $[\![A_1]\!] \times \dots \times [\![A_n]\!]$

Izraz $\Gamma \vdash M : A$ bomo interpretirali z zvezno preslikavo

$$[\![\Gamma \vdash M : A]\!] : [\![\Gamma]\!] \to [\![A]\!]$$

- $\underbrace{[\![x_1 : A_1, \dots, x_n : A_n \vdash x_i : A_i]\!]}_{f}(a_1, \dots, a_n) = a_i$

$(i \neq j) \quad f(x_1, \dots, \bigvee_j x_j, \dots, x_n) = x_i = \bigvee_j f(x_1, \dots, x_n)$

$(i = j) \quad f(x_1, \dots, \bigvee_i x_i, \dots, x_n) = \bigvee_i x_i = \bigvee_i f(x_1, \dots, x_n)$

- $[\![\Gamma \vdash M N : A]\!](\eta) = ([\![M]\!](\eta))([\![N]\!](\eta))$

$$= ev([\![M]\!](\eta), [\![N]\!](\eta))$$

- $[\![\Gamma \vdash \lambda x. M : A \to B]\!](\eta) = \underbrace{a \mapsto [\![\Gamma \vdash M : B]\!](\eta, a)}_{\text{zv. funkcij, ker } [\![M]\!] \text{ ohranja sup. v vsch komp.}}$

- $[\![\lambda x. M]\!](\bigvee_i \eta_i)(a) = [\![M]\!](\bigvee_i \eta_i, a)$

$$= \bigvee_i [\![M]\!](\eta_i, a)$$

$$= \bigvee_i ([\![\lambda x. M]\!](\eta_i)(a))$$

$$= (\bigvee_i [\![\lambda x. M]\!](\eta_i))(a)$$

torej je $[\![\lambda x. M]\!](\bigvee_i \eta_i) = \bigvee_i [\![\lambda x. M]\!](\eta_i)$ po ekstenzionalnosti.

- Ostale doma / na vajah / na izpitu.

<u>Trditev</u> Preslikava $\text{fix} : [D \to D] \to D$, ki vsaki zvezni preslikavi priredi njeno najmanjšo fiksno točko, je zvezna.

$\perp \leq f(\perp) \leq f(f(\perp)) \leq \dots$

$\text{fix}\, f = \bigvee_j f^j(\perp)$

<u>Dokaz</u>
- monotona ✓
- $\text{fix}(\bigvee_i f_i) = \bigvee_j (\bigvee_i f_i)^j(\perp)$

$$\stackrel{?}{=} \bigvee_j \bigvee_i f_i^j(\perp)$$

$$= \bigvee_i \bigvee_j f_i^j(\perp)$$

$$= \bigvee_i \text{fix}\, f_i$$

$$\left(\bigvee_i f_i\right)^0 (x) = x = \bigvee_i f_i^0 (x)$$

$$\begin{aligned}
\left(\bigvee_i f_i\right)^{n+1} (x) &= \left(\bigvee_i f_i\right) \left(\bigvee_j f_j\right)^n (x) \\
&= \bigvee_i f_i \left(\bigvee_j f_j^n (x)\right) \\
&= \bigvee_i \bigvee_j f_i \left(f_j^n (x)\right) \\
&= \bigvee_k f_k^{n+1} (x)
\end{aligned}$$

$$[[\![A]\!] \to [\![B]\!]] \to [[\![A]\!] \to [\![B]\!]]$$

$$[\![\Gamma \vdash \mathrm{rec}\, f\, x . M : A \to B]\!](\eta) = a \longmapsto \mathrm{fix}\left(f \mapsto [\![M]\!](\eta, f, a)\right)(a)$$