Domene

Def Domena (ω-CPO) je delno urejena množica $(D, \leq_D)$, da:

- obstaja najmanjši element $\bot_D \in D$, da velja $\forall x \in D.\ \bot \leq x$
- za vsako števno verigo $x_0 \leq x_1 \leq x_2 \leq \cdots$, t.j. zaporedje $\{x_i \in D\}_{i \in \mathbb{N}}$, da velja $\forall i.\ x_i \leq x_{i+1}$, obstaja supremum (najmanjša zgornja meja) $\bigvee_{i \in \mathbb{N}} x_i$.

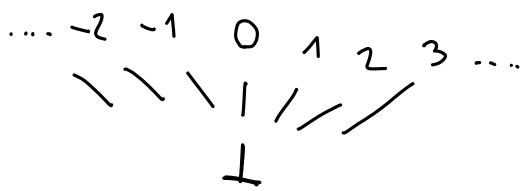
Primeri

- $(\mathcal{P}A, \subseteq)$ za poljubno množico A

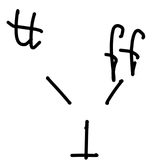
  $\bot = \emptyset$

  $\bigvee_i x_i = \bigcup_i x_i$

- $\mathbb{Z}_\bot = (\mathbb{Z} \cup \{\bot\}, \leq)$



- $\mathbb{B}_\bot$



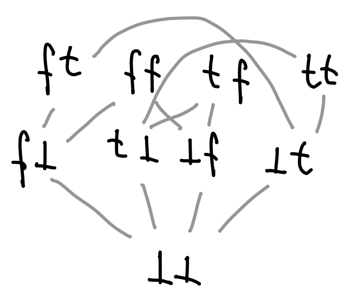
- $\mathbb{S}$

- $(D_1 \times D_2, \leq)$ za poljubni domeni $(D_1, \leq_1)$, $(D_2, \leq_2)$

$$(x_1, x_2) \leq (y_1, y_2) \iff x_1 \leq_1 y_1 \wedge x_2 \leq_2 y_2$$

$\mathbb{B}_\bot \times \mathbb{B}_\bot$

Na krajše pišimo npr. $f\bot = (ff, \bot)$



Def Umejino domeni $(D_1, \leq_1)$ in $(D_2, \leq_2)$.

Preslikava $f : D_1 \to D_2$ je Scottovo zvezna, če:

- je monotona $x \leq_1 y \Rightarrow fx \leq_2 fy$
- ohranja supremume verig

$$f\left(\bigvee_i x_i\right) = \bigvee_i f x_i$$

ta supremum obstaja, ker je f monotona in je $fx_0 \leq fx_1 \leq fx_2 \leq \cdots$ veriga

Trditev

Naj bodo $(D_1, \leq_1)$, $(D_2, \leq_2)$, $(D_3, \leq_3)$ domene in $f : D_1 \to D_2$, $g : D_2 \to D_3$ zvezni preslikavi.

Tedaj je $g \circ f : D_1 \to D_3$ tudi zvezna preslikava.

Dokaz

- $x \leq_1 y \Rightarrow fx \leq_2 fy$

  $(g \circ f)\, x = g(f x) \leq_3 g(f y) = (g \circ f)\, y$

- $$(g \circ f)\left(\bigvee_i x_i\right) = g\left(f\left(\bigvee_i x_i\right)\right) = g\left(\bigvee_i f x_i\right) = \bigvee_i g(f x_i) = \bigvee_i (g \circ f)\, x_i$$

■

Izrek (Knasper-Tarski / Kleene)

Naj bo $(D, \leq)$ domena in $f: D \to D$ zvezna preslikava. Tedaj ima f najmanjšo fiksno točko $\mathrm{fix}\, f$.

Dokaz

$$x_0 = \bot$$
$$x_{i+1} = f x_i$$

Ker je f monotona in $\bot$ najmanjši element, je $\{x_i\}_i$ veriga.

$$x_0 = \bot \leq f\bot = x_1$$
$$x_1 = f x_0 \leq f x_1 = x_2$$
$$\vdots$$

Definirajmo $\mathrm{fix}\, f = \bigvee_i x_i = \bigvee_i f^i \bot$.

Pokažimo, da je $\mathrm{fix}\, f$ fiksna točka.

$$f(\mathrm{fix}\, f) = f\left(\bigvee_i f^i \bot\right) = \bigvee_i f^{i+1} \bot = \mathrm{fix}\, f.$$

Pokažimo, da je najmanjša. Naj bo $y = fy$.

$$x_0 = \bot \leq y \qquad x_1 = f\bot \leq fy = y \quad \dots \quad x_{i+1} = f x_i \leq fy = y$$

Torej je y zgornja meja verige $\{x_i\}_i$ in $\mathrm{fix}\, f = \bigvee_i x_i \leq y$

Interpretacija tipa

Vsak tip A bomo interpretirali z domeno

$$[\![\mathrm{int}]\!] = \mathbb{Z}_\perp$$

$$[\![\mathrm{bool}]\!] = \mathbb{B}_\perp$$

sicer denotacijska semantika ne bi bila zadostna.

$$[\![A \to B]\!] = \big[[\![A]\!] \to_\perp [\![B]\!]\big]_\perp$$

D.N.

kjer je $[D_1 \to D_2]$ domena vseh zveznih funkcij $D_1 \to D_2$ opremljena z relacijo $\leq$

$$f \leq g \iff \forall x \in D_1.\ f\,x \leq g\,x.$$

$$[\mathbb{S} \to \mathbb{S}] \qquad \begin{array}{c} k_\top \\ | \\ \mathrm{id} \\ | \\ k_\perp \end{array}$$

$[D_1 \to_\perp D_2]$ pa domena vseh strogih zveznih funkcij, torej takih, da je $f\,\perp_1 = \perp_2$

Interpretacija izrazov

$\Gamma \vdash M : A$ bomo interpretirali z zvezno preslikavo

$[\![\Gamma \vdash M : A]\!] : [\![\Gamma]\!] \to [\![A]\!]$, kjer je

$$[\![x_1 : A_1, \dots, x_n : A_n]\!] = [\![A_1]\!] \times \dots \times [\![A_n]\!]$$

izrazi $M, N ::= x \mid \lambda x.M \mid MN$
$\mid \underline{m} \mid M+N \mid M*N \mid -M$
$\mid M=N \mid M<N \mid M>N$
$\mid \text{true} \mid \text{false} \mid \text{if } M \text{ then } N_1 \text{ else } N_2$
$\mid \text{rec}\, f x . M$

$$\llbracket x_1 : A_1, \dots, x_n : A_n \vdash x_i : A_i \rrbracket (a_1, \dots, a_n) = a_i$$
$$\llbracket \Gamma \vdash \lambda x.M : A \to B \rrbracket (\vec{a}) = y \in \llbracket A \rrbracket \longmapsto \llbracket \Gamma, x : A \vdash M : B \rrbracket (\vec{a}, y)$$
$$\llbracket \Gamma \vdash MN : B \rrbracket (\vec{a}) = \underset{\llbracket B \rrbracket^{\llbracket A \rrbracket}}{\llbracket \Gamma \vdash M : A \to B \rrbracket (\vec{a})} \Big(\overset{\llbracket A \rrbracket}{\llbracket \Gamma \vdash N : A \rrbracket (\vec{a})} \Big)$$
$$\llbracket \Gamma \vdash \underline{m} : \text{int} \rrbracket (\vec{a}) = m$$
$$\llbracket \Gamma \vdash M+N : \text{int} \rrbracket (\vec{a}) = \llbracket \Gamma \vdash M : \text{int} \rrbracket (\vec{a}) + \llbracket \Gamma \vdash N : \text{int} \rrbracket (\vec{a})$$
$$\llbracket \Gamma \vdash M * N : \text{int} \rrbracket (\vec{a}) = \llbracket M \rrbracket (\vec{a}) \cdot \llbracket N \rrbracket (\vec{a})$$
$$\vdots$$
$$\llbracket \Gamma \vdash \text{true} : \text{bool} \rrbracket (\vec{a}) = tt$$
$$\vdots$$
$$\llbracket \Gamma \vdash \text{if } M \text{ then } N_1 \text{ else } N_2 : A \rrbracket (\vec{a}) = \begin{cases} \llbracket N_1 \rrbracket (\vec{a}) ; & \llbracket M \rrbracket (\vec{a}) = tt \\ \llbracket N_2 \rrbracket (\vec{a}) ; & \llbracket M \rrbracket (\vec{a}) = ff \end{cases}$$

$$\llbracket x_1 : A_1, \dots, x_n : A_n \underset{\pi_i}{\overset{}{\vdash}} x_i : A_i \rrbracket (a_1, \dots, a_n) = a_i$$

Pokažimo, da je preslikava zvezna

$$\pi_i \Big(\bigvee_j (a_{1j}, \dots, a_{nj}) \Big) = \pi_i \Big(\bigvee_j a_{1j}, \dots, \bigvee_j a_{nj} \Big)$$
$$= \bigvee_j a_{ij} = \bigvee_j \pi_i (a_{1j}, \dots, a_{nj})$$

$$\llbracket \Gamma \vdash M+N : \text{int} \rrbracket (\vec{a}) = \llbracket M \rrbracket (\vec{a}) +_\perp \llbracket N \rrbracket (\vec{a})$$

kjer je $m +_\perp n = m + n$

podobno za ostale

$$\perp +_\perp n = m +_\perp \perp = \perp +_\perp \perp = \perp$$

$$\llbracket \Gamma \vdash \lambda x . M : A \to B \rrbracket (\vec{a}) = y \in \llbracket A \rrbracket \longmapsto \begin{cases} \perp & \text{če je } y = \perp \\ \llbracket M \rrbracket (\vec{a}, y) & \text{sicer} \end{cases}$$

da dobimo stroge funkcije

$$\llbracket M N \rrbracket (\vec{a}) = \begin{cases} \perp & \llbracket M \rrbracket (\vec{a}) = \perp \\ \llbracket M \rrbracket (\vec{a}) \big(\llbracket N \rrbracket (\vec{a}) \big) & \end{cases}$$

$$\llbracket \Gamma \vdash \text{rec}\, f x . M : A \to B \rrbracket (\vec{a}) = \text{fix} \Big(g \in \overset{[A \to_\perp B]_\perp}{\overset{\|}{\llbracket A \to B \rrbracket}} \mapsto$$
$$y \in \llbracket A \rrbracket \longmapsto \llbracket M \rrbracket (\vec{a}, g, y) \Big)$$

Trditev $\mathrm{fix} : [D \to D] \to D$ je zvezna.