Izpeljava tipov

Primer

$$\lambda f.\, \lambda x.\, f(fx) \xrightarrow[\text{kako ugotovimo?}]{} (\alpha \to \alpha) \to \alpha \to \alpha$$

1.faza: določimo tipe in nastavimo enačbe med parametri → spremenljivke za tipe

Hindley-Milnerjev algoritem

$$\dfrac{\dfrac{\dfrac{}{\Gamma \vdash f : \beta} \qquad \dfrac{\dfrac{}{\Gamma \vdash f : \beta} \quad \dfrac{}{\Gamma \vdash x : \gamma}}{\Gamma \vdash f\,x : \delta}}{\underbrace{f:\beta,\ x:\gamma}_{\Gamma} \vdash f(fx) : \varepsilon}}{\dfrac{f : \beta \vdash \lambda x.\, f(fx) : \gamma \to \varepsilon}{\emptyset \vdash \lambda f.\, \lambda x.\, f(fx) : \beta \to \gamma \to \varepsilon}}$$

$$\beta = \gamma \to \delta$$
$$\beta = \delta \to \varepsilon$$

$$\dfrac{\dfrac{\dfrac{}{x:\alpha \vdash x:\alpha}}{\emptyset \vdash \lambda x.\, x : \alpha \to \alpha} \qquad \dfrac{}{\emptyset \vdash 1 : int}}{\emptyset \vdash (\lambda x.\, x)\, 1 : \beta}$$

$$\alpha \to \alpha = int \to \beta$$

2.faza: rešimo enačbe

$$\dfrac{\beta = \gamma \to \delta \qquad \beta = \delta \to \varepsilon}{} \rightsquigarrow \dfrac{\gamma \to \delta = \delta \to \varepsilon}{\beta \mapsto \gamma \to \delta} \rightsquigarrow \dfrac{\gamma = \delta \qquad \delta = \varepsilon}{\beta \mapsto \gamma \to \delta}$$

$$\rightsquigarrow \dfrac{\delta = \varepsilon}{\beta \mapsto \delta \to \delta \qquad \gamma \mapsto \delta} \rightsquigarrow \dfrac{}{\beta \mapsto \varepsilon \to \varepsilon \qquad \gamma \mapsto \varepsilon \qquad \delta \mapsto \varepsilon}$$

Če dobljeno substitucijo uporabimo na tipu $\beta \to \gamma \to \varepsilon$ iz 1.faze, dobimo $(\varepsilon \to \varepsilon) \to \varepsilon \to \varepsilon$, kar je to, kar smo hoteli.

Nastavljanje enačb

$$A ::= bool \mid int \mid A \to B \mid \alpha$$
$$\mathcal{E} ::= \emptyset \mid A_1 = A_2,\ \mathcal{E}$$

izrazi $M, N ::= x \mid \lambda x. M \mid M N$
$\mid \underline{m} \mid M+N \mid M*N \mid -M$
$\mid M=N \mid M<N \mid M>N$
$\mid true \mid false \mid \text{if } M \text{ then } N_1 \text{ else } N_2$
$\mid \text{rec } f x. M$

$$\boxed{\Gamma \vdash M : A \mid \mathcal{E}}$$

$$\frac{(x:A) \in \Gamma}{\Gamma \vdash x : A \mid \emptyset}$$

$$\frac{\Gamma, x:\alpha \vdash M : A \mid \mathcal{E} \quad (\alpha \text{ svež})}{\Gamma \vdash \lambda x. M : \alpha \to A \mid \mathcal{E}}$$

$$\frac{\Gamma \vdash M : A_1 \mid \mathcal{E}_1 \qquad \Gamma \vdash N : A_2 \mid \mathcal{E}_2 \quad (\alpha \text{ svež})}{\Gamma \vdash M N : \alpha \mid A_1 = A_2 \to \alpha, \mathcal{E}_1, \mathcal{E}_2}$$

$$\frac{}{\Gamma \vdash \underline{m} : int \mid \emptyset}$$

$$\frac{\Gamma \vdash M : A_1 \mid \mathcal{E}_1 \qquad \Gamma \vdash N : A_2 \mid \mathcal{E}_2}{\Gamma \vdash M+N : int \mid A_1 = int, A_2 = int, \mathcal{E}_1, \mathcal{E}_2}$$

ostalo doma

$$\frac{\Gamma \vdash M : A \mid \mathcal{E} \qquad \Gamma \vdash N_1 : A_1 \mid \mathcal{E}_1 \qquad \Gamma \vdash N_2 : A_2 \mid \mathcal{E}_2 \quad (\alpha \text{ svež})}{\Gamma \vdash \text{if } M \text{ then } N_1 \text{ else } N_2 : \alpha \mid \alpha = A_1, \alpha = A_2, A = bool, \mathcal{E}, \mathcal{E}_1, \mathcal{E}_2}$$

$$\frac{\Gamma, f:\alpha \to \beta, x:\alpha \vdash M : A \mid \mathcal{E}}{\Gamma \vdash \text{rec } f x. M : \alpha \to A \mid \beta = A, \mathcal{E}}$$

Reševanje enačb

$$\boxed{\mathcal{E} \searrow \sigma}$$

$$\sigma ::= \emptyset \mid \alpha \mapsto A, \sigma$$

$$\boxed{\sigma(A)}$$

$$\sigma(int) = int$$

$$\sigma(bool) = bool$$

$$\sigma(A \to B) = \sigma(A) \to \sigma(B)$$

$$\sigma(\alpha) = \begin{cases} A & (\alpha \mapsto A) \in \sigma \\ \alpha & \text{sicer} \end{cases}$$

$$\boxed{\sigma' \circ \sigma}$$

$$(\sigma' \circ \sigma) = \underbrace{\alpha_i \mapsto \sigma'(A_i)}_{(\alpha_i \mapsto A_i) \in \sigma}, \; \alpha_i' \mapsto A_i' \quad \begin{matrix} \alpha_i' \notin \sigma \\ (\alpha_i' \mapsto A_i') \in \sigma' \end{matrix}$$

Trditev

$$(\sigma' \circ \sigma)(A) = \sigma'(\sigma(A))$$

$$\frac{}{\emptyset \searrow \emptyset}$$

$fp(\mathrm{int}) = \emptyset$
$fp(\mathrm{bool}) = \emptyset$
$fp(A \to B) = fp(A) \cup fp(B)$
$fp(\alpha) = \{\alpha\}$

	α	int	bool	$A \to B$
α	●	●	●	●
int	●	●	×	×
bool	●	×	●	×
$A \to B$	●	×	×	●

$$\frac{\mathcal{E} \searrow \sigma}{A = A, \mathcal{E} \searrow \sigma} \qquad \frac{(\alpha \mapsto A)(\mathcal{E}) \searrow \sigma \qquad \alpha \notin fp(A)}{\alpha = A, \mathcal{E} \searrow \alpha \mapsto \sigma(A), \sigma}$$

$$\frac{A_1 = A_2, B_1 = B_2, \mathcal{E} \searrow \sigma}{A_1 \to B_1 = A_2 \to B_2, \mathcal{E} \searrow \sigma} \qquad \frac{(\alpha \mapsto A)(\mathcal{E}) \searrow \sigma \qquad \alpha \notin fp(A)}{A = \alpha, \mathcal{E} \searrow \alpha \mapsto \sigma(A), \sigma}$$

$$\boxed{\sigma \models \mathcal{E}}$$

$$\frac{}{\sigma \models \emptyset} \qquad \frac{\sigma(A) = \sigma(B) \qquad \sigma \models \mathcal{E}}{\sigma \models (A = B, \mathcal{E})}$$

Trditev

Če velja $\Gamma \vdash M : A \mid \mathcal{E}$ in $\sigma \models \mathcal{E}$
potem velja tudi $\sigma(\Gamma) \vdash M : \sigma(A)$

Dokaz

Z indukcijo na $\Gamma \vdash M : A \mid \mathcal{E}$.
Npr. za aplikacijo dobimo

$$\frac{\Gamma \vdash M : A_1 \mid \mathcal{E}_1 \qquad \Gamma \vdash N : A_2 \mid \mathcal{E}_2}{\Gamma \vdash M\,N : \alpha \mid A_1 = A_2 \to \alpha, \mathcal{E}_1, \mathcal{E}_2}$$

Ker velja $\sigma \models A_1 = A_2 \to \alpha, \mathcal{E}_1, \mathcal{E}_2$ velja $\sigma(A_1) = \sigma(A_2) \to \sigma(\alpha)$ ter $\sigma \models \mathcal{E}_1$ in $\sigma \models \mathcal{E}_2$
Tedaj po I.P. velja $\sigma(\Gamma) \vdash M : \sigma(A_1)$ in $\sigma(\Gamma) \vdash N : \sigma(A_2)$.
Tedaj velja

$$\frac{\sigma(\Gamma) \vdash M : \underbrace{\sigma(A_2) \to \sigma(\alpha)}_{= \sigma(A_1)} \qquad \sigma(\Gamma) \vdash N : \sigma(A_2)}{\sigma(\Gamma) \vdash M\,N : \sigma(\alpha)}$$

$\blacksquare$

Trditev Če $\mathcal{E} \searrow \sigma$, potem $\sigma \models \mathcal{E}$.

Dokaz Z indukcijo na $\mathcal{E} \searrow \sigma$.

Lema Če obstaja σ, da velja $\sigma \models \mathcal{E}$, potem obstaja σ', da velja $\mathcal{E} \searrow \sigma'$.

Dokaz

Glavno vprašanje je, ali se naš algoritem sploh konča. Tega ne moremo pokazati z indukcijo na $\mathcal{E}$, saj se med posameznimi koraki dolžina lahko tudi poveča. Namesto tega uporabimo leksikografsko ureditev na

$$(|fp(\mathcal{E})|, |\mathcal{E}|)$$

$$|\mathcal{E}| := \sum_{A=B \in \mathcal{E}} |A| + |B|$$

$$|int| = |bool| = |\alpha| = 1$$

$$|A \to B| = |A| + |B| + 1$$

$$\frac{}{\emptyset \searrow \emptyset} \qquad \frac{\mathcal{E} \searrow \sigma}{A = A, \mathcal{E} \searrow \sigma} \qquad \frac{A_1 = A_2, B_1 = B_2, \mathcal{E} \searrow \sigma}{A_1 \to B_1 = A_2 \to B_2, \mathcal{E} \searrow \sigma}$$

$$\frac{(\alpha \mapsto A)(\mathcal{E}) \searrow \sigma \qquad \alpha \notin fp(A)}{\alpha = A, \mathcal{E} \searrow \alpha \mapsto \sigma(A), \sigma} \qquad \frac{(\alpha \mapsto A)(\mathcal{E}) \searrow \sigma \qquad \alpha \notin fp(A)}{A = \alpha, \mathcal{E} \searrow \alpha \mapsto \sigma(A), \sigma}$$

- Če je $\mathcal{E} = \emptyset$, potem $\emptyset \searrow \emptyset$
- Če je $\mathcal{E} = A = B, \mathcal{E}'$, potem pa velja $\sigma(A) = \sigma(B)$. Kdaj se to lahko zgodi?
  - Če je že $A = B$, po i.p. obstaja $\mathcal{E}' \searrow \sigma'$, zato $\mathcal{E} \searrow \sigma'$.
  - sicer sta A in B kompatibilne oblike, torej eden parameter ali oba funkcijska tipa.
    + Če je $A = A_1 \to A_2$ in $B = B_1 \to B_2$, tedaj po i.p. obstaja σ', da velja $A_1 = B_1, A_2 = B_2, \mathcal{E}' \searrow \sigma'$, zato $A = B, \mathcal{E}' \searrow \sigma'$.
    + Če je $A = \alpha$, potem po i.p. velja $(\alpha \mapsto B)(\mathcal{E}) \searrow \sigma'$, saj iz $\sigma \models \alpha = B, \mathcal{E}$ sledi tudi $\sigma \models (\alpha \mapsto B)(\mathcal{E})$. Pokazati moramo še $\alpha \notin fv(B)$. Vemo, $\sigma(\alpha) = \sigma(B)$, zato je edina možnost $\alpha = B$, kar smo že obravnavali.
    + Če je $B = \alpha$, ravnomo podobno.

Trditev Če velja $\sigma \models \mathcal{E}$ in $\mathcal{E} \searrow \sigma'$, potem obstaja σ'', da velja $\sigma = \sigma'' \circ \sigma'$.

Dokaz Z indukcijo na $\mathcal{E} \searrow \sigma'$. $\blacksquare$

$\mathcal{E}$ $\alpha \to \mathrm{int} = \beta \to \gamma$

σ $\alpha \mapsto \mathrm{int}, \beta \mapsto \mathrm{int}, \gamma \mapsto \mathrm{int}$

σ' $\alpha \mapsto \beta, \gamma \mapsto \mathrm{int}$

σ'' $\beta \mapsto \mathrm{int}$

Trditev Če velja $\Gamma \vdash M : A$, potem obstaja $A', \mathcal{E}$, da velja $\Gamma \vdash M : A' \mid \mathcal{E}$, in obstaja $\sigma \models \mathcal{E}$, da je $\sigma(A') = A$.

Dokaz TAPL.