Brezplačni izreki!

tip	izrek
$\vdash h : \alpha\ list \to \alpha$	$[\![h]\!] \circ map\ f = f \circ [\![h]\!]$
$\vdash r : \alpha\ list \to \alpha\ list$	$[\![r]\!] \circ map\ f = map\ f \circ [\![r]\!]$
$\vdash s : \alpha\ list$	$[\![s]\!] = []$
$\vdash i : \alpha \to \alpha$	$[\![i]\!] = id$
$\vdash z : \alpha \times \beta \to \beta \times \alpha$	$[\![z]\!](x,y) = (y,x)$
$m : (\alpha \to \beta) \to \alpha\ list \to \beta\ list$	$[\![m]\!] f = [\![m]\!]\, id \circ map\ f = map\ f \circ [\![m]\!]\, id$

$f : X \to X'$
$g : Y \to Y'$

Ideja

Vemo že, da iz $\vdash M : A$ sledi $[\![M]\!] \in [\![A]\!]$

Če razširimo

$$A ::= \dots \mid \alpha$$

potem lahko za poljubna prirejanje $\vec{X}$, ki prostim parametrom $\alpha_1, \dots, \alpha_n$ priredi množice $X_1, \dots, X_n$ definiramo $[\![A]\!]_{\vec{X}}$

$$[\![bool]\!]_{\vec{X}} = \mathbb{B} \quad [\![int]\!]_{\vec{X}} = \mathbb{Z} \quad [\![A \to B]\!]_{\vec{X}} = [\![B]\!]_{\vec{X}}^{[\![A]\!]_{\vec{X}}} \quad [\![\alpha_i]\!]_{X_1,\dots,X_n} = X_i$$

Npr.

$$[\![\alpha \to \beta \to bool]\!]_{\mathbb{R}, \mathbb{C}}^{\alpha\ \ \beta} = \left(\mathbb{B}^{\mathbb{C}}\right)^{\mathbb{R}} \qquad [\![\alpha \to \beta \to \alpha]\!]_{\mathbb{B},\mathbb{R}} = \left(\mathbb{B}^{\mathbb{R}}\right)^{\mathbb{B}} \qquad [\![\alpha \to \beta \to \alpha]\!]_{\mathbb{B},\mathbb{H}} = \left(\mathbb{B}^{\mathbb{H}}\right)^{\mathbb{B}}$$

Podobno lahko za $\vdash M : A$ definiramo $[\![M]\!]_{\vec{X}} \in [\![A]\!]_{\vec{X}}$

Še več:

Vsak tip bomo interpretirali z relacijo.
Za vsak parameter α_i določimo množici X_i, X_i' in relacijo $R_i \subseteq X_i \times X_i'$

$$[\![M]\!]_{\vec{X}} \in [\![A]\!]_{\vec{X}} \qquad [\![M]\!]_{\vec{X}'} \in [\![A]\!]_{\vec{X}'}$$

Če A interpretiramo z ustrezno relacijo $[\![A]\!]_{\vec{R}} \subseteq [\![A]\!]_{\vec{X}} \times [\![A]\!]_{\vec{X}'}$ velja

$$[\![M]\!]_{\vec{X}} \ [\![A]\!]_{\vec{R}} \ [\![M]\!]_{\vec{X}'}$$

Primer

$\vec{X} = \mathbb{B}, \mathbb{R}$

$\vec{X}' = \mathbb{B}, \mathbb{H}$

$\vec{R} = \emptyset, \mathcal{G}_\iota$ $\quad \iota : \mathbb{R} \to \mathbb{H} \quad \iota(x) = x + 0\cdot i + 0\cdot j + 0\cdot k$

$\mathcal{G}_{f : X \to Y} = \{(x,y) \in X \times Y \mid y = f\,x\} = \{(x, f\,x) \mid x \in X\}$

$M = \lambda x.\, \lambda y.\, x$

$[\![M]\!]_{\vec X} = b \in \mathbb{B} \mapsto x \in \mathbb{R} \mapsto b$

$[\![M]\!]_{\vec X'} = b \in \mathbb{B} \mapsto q \in \mathbb{H} \mapsto b$

Ideja je, da sta funkciji v relaciji, kadar argumente v relaciji na domeni slikata v rezultate v relaciji na kodomeni.

Vidimo, da sta $[\![M]\!]_{\vec X}$ in $[\![M]\!]_{\vec X'}$ v relaciji neodvisno od $\vec R$.

Kako definiramo $[\![\![A]\!]\!]_{\vec X, \vec X', \vec R}$?

$$[\![\![\alpha_i]\!]\!]_{R_1,\dots,R_n} = R_i \subseteq [\![\alpha_i]\!]_{X_1,\dots,X_n} \times [\![\alpha_i]\!]_{X'_1,\dots,X'_n} = X_i \times X'_i$$

$$[\![\![\mathrm{bool}]\!]\!]_{\vec R} = (=_{\mathbb B}) = \{(b,b) \mid b \in \mathbb B\} \subseteq [\![\mathrm{bool}]\!]_{\vec X} \times [\![\mathrm{bool}]\!]_{\vec X'} = \mathbb B \times \mathbb B$$

$$[\![\![\mathrm{int}]\!]\!]_{\vec R} = (=_{\mathbb Z}) \subseteq \mathbb Z \times \mathbb Z$$

$$[\![\![A \to B]\!]\!]_{\vec R} = [\![\![A]\!]\!]_{\vec R} \longrightarrow [\![\![B]\!]\!]_{\vec R}$$

$$[\![\![A]\!]\!]_{\vec R} \subseteq [\![A]\!]_{\vec X} \times [\![A]\!]_{\vec X'} \qquad [\![\![B]\!]\!]_{\vec R} \subseteq [\![B]\!]_{\vec X} \times [\![B]\!]_{\vec X'}$$

kjer je za $R \subseteq X \times X'$ in $S \subseteq Y \times Y'$ relacija $R \to S \subseteq Y^X \times Y'^{X'}$ podana z

$$\underset{X \to Y}{f}\ (R \to S)\ \underset{X' \to Y'}{g} \iff \forall x \in X, x' \in X'.\ x R x' \Rightarrow (f\,x) S (g\,x')$$

Podobno bi lahko definirali

$[\![\![A \times B]\!]\!]_{\vec R} = [\![\![A]\!]\!]_{\vec R} \times [\![\![B]\!]\!]_{\vec R}$, kjer je $(x,y)\ R \times S\ (x',y') \iff x R x' \land y S y'$

$[\![\![A + B]\!]\!]_{\vec R} = [\![\![A]\!]\!]_{\vec R} + [\![\![B]\!]\!]_{\vec R}$, kjer je $\iota_1(x)\ (R+S)\ \iota_1(x')$ če $x R x'$

ali $\iota_2(y)\ (R+S)\ \iota_2(y')$ če $y R y'$

$[\![\![\mathrm{list}\ A]\!]\!]_{\vec R} = \mathrm{list}\ [\![\![A]\!]\!]$, kjer je $xs\ (\mathrm{list}\ R)\ xs'$, če $|xs| = |xs'|$ in

$x_1 R x_1', \dots, x_n R x_n'$.

<u>Trditev</u> Za $f : X \to Y$ velja
$$\text{list}\ \mathcal{G}_f = \mathcal{G}_{\text{map}\, f}$$

<u>Trditev</u> Za $f : X \to X'$, $g : Y \to Y'$, $h : X \to Y$, $h' : X' \to Y'$

$$h\ (\mathcal{G}_f \to \mathcal{G}_g)\ h' \iff h' \circ f = g \circ h$$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"'] & X' \arrow[d, "h'"] \\ Y \arrow[r, "g"'] & Y' \end{tikzcd}$$

<u>Izrek</u> Naj bo:
- $\Gamma \vdash M : A$
- $\alpha_1, \dots, \alpha_n$ prosti parametri v Γ in A $\quad fp(\Gamma) \cup fp(A)$
- $\underbrace{X_1, \dots, X_n}_{\vec{X}}$, $X_1', \dots, X_n'$ poljubne množice
- $R_1 \subseteq X_1 \times X_1', \dots, R_n \subseteq X_n \times X_n'$ poljubne relacije

Tedaj je
$$[\![\Gamma \vdash M : A]\!]_{\vec{X}}\ \left([\![\![\Gamma]\!]\!]_{\vec{R}} \to [\![\![A]\!]\!]_{\vec{R}} \right)\ [\![\Gamma \vdash M : A]\!]_{\vec{X}'}$$

<u>Dokaz</u> Z indukcijo na $\Gamma \vdash M : A$.

<u>Posledica</u> Za ustrezne $\vec{X}, \vec{X}', \vec{R}$ velja
$$[\![\vdash M : A]\!]_{\vec{X}}\ [\![\![A]\!]\!]_{\vec{R}}\ [\![\vdash M : A]\!]_{\vec{X}'}$$

<u>Posledica</u> Naj bo $\vdash h : \alpha\,\text{list} \to \alpha$ in $f : X \to Y$. Tedaj velja
$$[\![h]\!]_Y \circ \text{map}\, f = f \circ [\![h]\!]_X$$

<u>Dokaz</u> Vemo, da za poljuben $X, Y, R \subseteq X \times Y$ velja
$$[\![h]\!]_X\ [\![\![\alpha\,\text{list} \to \alpha]\!]\!]_R\ [\![h]\!]_Y$$
$$\iff [\![h]\!]_X\ (R\,\text{list} \to R)\ [\![h]\!]_Y$$

Če vzamemo $R = \mathcal{G}_f$ je $R\,\text{list} = \mathcal{G}_f\ \text{list} = \mathcal{G}_{\text{map}\, f}$

Torej $[\![h]\!]_X \, (\mathcal{C}_{\mathrm{map}\, f} \to \mathcal{C}_f) \, [\![h]\!]_Y$.

$$\iff \quad f \circ [\![h]\!]_X = [\![h]\!]_Y \circ \mathrm{map}\, f$$

$$\begin{tikzcd} \mathrm{list}\, X \arrow[r, "\mathrm{map}\, f"] \arrow[d, "{[\![h]\!]_X}"'] & \mathrm{list}\, Y \arrow[d, "{[\![h]\!]_Y}"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

<u>Trditev</u> Za $\vdash z : \alpha \times \beta \to \beta \times \alpha$ in poljubni X, Y velja

$$[\![z]\!]_{X,Y}(x,y) = (y,x) \quad \text{za vse } x \in X, y \in Y$$

<u>Dokaz</u> Za poljubne X, X', Y, Y', $R \subseteq X \times X'$, $S \subseteq Y \times Y'$ velja

$$[\![z]\!]_{X,Y} \, (R \times S \to S \times R) \, [\![z]\!]_{X',Y'}$$

Torej za $x R x'$ in $y S y'$ velja $[\![z]\!]_{X,Y}(x,y) \, (S \times R) \, [\![z]\!](x',y')$

Vzamemo $X = X'$, $Y = Y'$ in za vsak $x \in X, y \in Y$ vzamemo

$R_x = \{(x,x)\}$ in $S_y = \{(y,y)\}$

Tedaj velja $x R_x x$ in $y R_y y$, zato je

$$[\![z]\!]_{X,Y}(x,y) \, (S_y \times R_x) \, [\![z]\!]_{X,Y}(x,y)$$

torej $[\![z]\!]_{X,Y}(x,y) = (y,x)$.