Probability

STAT 1F92 ✵ Component 03 ✵ October 04 – October 31, 2023[1]

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We will cover probability rules, the addition rule and complements, independence and the multiplication rule and conditional probability and the general multiplication rule, permutations and combinations.

✵ October 05, 2023 ✵

3 Probability

In this component, we will go over typical probability rules, addition rule and complements, independence and the multiplication rule, conditional probability, the general multiplication rule and counting techniques (permutation and combinations).

3.1 Probability Rules

We achieve the 'true' probability of an objective when we repeat the experiment many times. This is referred to as *the law of large numbers*. In probability, an *experiment* is any task with uncertain results that can be repeated. For instance, flipping a fair coin. It could land on heads (50%) or tails (50%) but we are uncertain with which face it will land on. Of course, this task can be repeated as many times as we want.

Definition 3.1: Sample space in probability (S)

The sample space, denoted by S, is the collection of all possible outcomes.

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In other words, the sample space is a list of all possible results of a probability experiment. We must get the result from that list and we cannot obtain a result that is *not* found on the list.

Definition 3.2: An event in probability

An *event*, which consists of one or more outcomes, is any collection of outcomes from a probability experiment. A *simple event* is denoted by e_i (the ith simple event) and in general, events are denoted using capital letters such as E.

An outcome is the result of one trial of a probability experiment.

Example 3.1: Events and the Sample Space of a Probability Experiment

- **Identify the outcomes of rolling a fair die**
  We have six different outcomes (*i.e.*, the sample space has a list of six items) and can be labelled using the notation e_i, as follows:

$$e_1 = \text{rolling one } (i.e., 1), \quad e_2 = \text{rolling two } (i.e., 2), \quad e_3 = \text{rolling three } (i.e., 3),$$
$$e_4 = \text{rolling four } (i.e., 4), \quad e_5 = \text{rolling five } (i.e., 5), \quad e_6 = \text{rolling six } (i.e., 6).$$

- **What is the sample space?**
  The sample space, denoted by S, is the possible outcomes, which is:

$$S = \{e_1, e_2, e_3, e_4, e_5, e_6\} = \{1, 2, 3, 4, 5, 6\}$$

- **What is the event of rolling an even number?**
  Let $E = \{\text{rolling an even number}\}$, which will contain the outcomes $E = \{2, 4, 6\}$
- **What is the event of rolling an odd number larger than one?**
  Let $E = \{\text{rolling an odd number larger than one}\}$, which will contain the outcomes $E = \{3, 5\}$

Probability will always be between 0 and 1 (both inclusive), where a probability of 0 means the event will never occur (*i.e.*, impossible) and a probability of 1 means the event will always occur (*i.e.*, certain). An *unusual* event is an event that has a low probability, less than 0.05 (or less than 5%) of occurring. Typical cut-off points that statisticians use are 0.01, 0.05 or 0.10.

Definition 3.3: What is $P(E)$

We use the notation $P(E)$ to denote "the probability that event E occurs."

For instance, let the event E denote rolling a 2, then $P(E) = \frac{1}{6}$, as there are six possible outcomes and rolling a two is a single outcome out of the six. Letting the event E denoting rolling an odd number gives the probability $P(E) = \frac{3}{6} = \frac{1}{2}$, as there are three odd numbers we can get out of the six total outcomes.

Definition 3.4: Rules of probability

The two rules *must* be satisfied:

- Any event E has a probability of $P(E)$ of occurring, which must be between zero and one (*i.e.*, $0 \leqslant P(E) \leqslant 1$)
- Suppose we have n outcomes and our sample space is $S = \{e_1, e_2, \ldots, e_n\}$, then the sum of the probabilities of all outcomes must equal 1:[a]

$$P(e_1) + P(e_2) + \cdots + P(e_n) = 1$$

This means that we cannot have negative probabilities nor probabilities that are larger than 1.

[a] In realistic situations, it might occur that the probabilities add up to 0.9999999... but we can assume this is 1

For instance, suppose we have a fair die, then we have six outcomes $S = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ and each outcome has a probability of $\frac{1}{6}$ of occurring, hence, the total probability is:

$$P(e_1) + P(e_2) + P(e_3) + P(e_4) + P(e_5) + P(e_6) = 1$$

$$\frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = 1$$

A *probability model* is a table containing a list of events with their outcomes. For example, suppose that we have asked 100 people about their favourite colours and here are the results:

Colour	Probability	Percentage
Red	0.09	09%
Green	0.15	15%
Blue	0.33	33%
Magenta	0.25	25%
Orange	0.18	18%
Sum →	1.00	100%

Table 3.1: The probability model of favourite colours generated by asking 100 people

It is well known that the probability of obtaining heads is 50% and tails 50% using a fair coin. This is because there are two sides and they are equal, hence, it is fifty-fifty. This is referred to as the *classical method*. Suppose we wanted to perform the same concept using the data in Table 3.1. We have five options and they are *not* equally likely options. How can we, in general, conclude the probabilities of choosing the colours? The data has been conducted once and these are the values we are dealing with. It is fair to assume that the green colour must always be near 15%? Not necessarily in general. Instead, we would have to perform the experiment multiple times and we will *eventually* reach the 'true' probabilities. This is referred to as the *empirical method*. In Table 3.1, the experiment was repeated 100 times (*i.e.*, a single trial is asking one person).

Definition 3.5: Approximating probabilities using the empirical approach

The probability of an event E occurring is approximately the number of times event E is observed divided by the number of repetitions of the experiment, given as:

$$P(E) \approx \text{relative frequency of E} = \frac{\text{frequency of } E}{\text{number of trials of experiment}}.$$

That is, the probability of the event E is approximated through the number of trials. Suppose we flip a fair coin 100 times and we are interested in getting tails. Then we might get tails 48 times out of the 100. We repeat the experiment again and we might get tails 51 times out of 100. We might perform the experiment yet another time and might get tails 50 times out of 100. We see that the probability of receiving tails will be approximated as we perform the experiment many times (remember the law of large numbers).

Example 3.2: Using relative frequencies to approximate probabilities

Suppose that we have 250 students last year in STAT 1F92 and 60 of them completed the course with a grade of A (*i.e.*, received a grade between 80% and 100%). We can *approximate* the total number of students that will receive a grade of A to be about

$$\frac{60}{250} \approx 0.24.$$

This means that for this year, we expect to receive *about* a quarter of the students to receive a grade of A in the course.

Example 3.3: Building a probability model from survey data

Suppose people were asked about their favourite social media platform and the data are recorded as follows:

Platform	Frequency	Relative frequency
Snapchat	129	0.07104
Reddit	492	0.27093
Instagram	247	0.13601
TikTok	482	0.26542
Facebook	92	0.05066
LinkedIn	50	0.02753
Twitter	263	0.14482
WhatsApp	61	0.03359
Total →	1816	1

There are 1816 individuals who can be seen as the trials of the probability experiment. The experiment is to choose one of the eight platforms and this will be repeated 1816 times. The relative frequency (percentage) column is helpful because we can find the relative frequency of, say, TikTok, which happens to be

$$\frac{482}{1816} \approx 0.26542 \tag{1}$$

We can approximate what happens if we surveyed 18160 (ten times the original number). For instance, LinkedIn would have 500 as the frequency. This is an approximation, *not* a guarantee.

Definition 3.6: Computing probability using the classical method

In case an experiment has n equally likely outcomes and the number of ways that an event E can occur is m, then the probability of E, denoted as $P(E)$, is

$$P(E) = \frac{\text{number of ways that } E \text{ can occur}}{\text{number of possible outcomes}} = \frac{m}{n}$$

Put differently, suppose S is the sample space of the experiment, then

$$P(E) = \frac{N(E)}{N(S)},$$

where $N(E)$ is the number of outcomes in E, and $N(S)$ is the number of outcomes in the sample space. Think of $N(...)$ as the 'count' of outcomes, which is `=COUNT(...)` in Excel.

Example 3.4: Computing probability using the classical method

Suppose we have two dice to role. The sample space will contain the outcomes found in Table 3.2.

$(1, 1) = 2$	$(1, 2) = 3$	$(1, 3) = 4$	$(1, 4) = 5$	$(1, 5) = 6$	$(1, 6) = 7$
$(2, 1) = 3$	$(2, 2) = 4$	$(2, 3) = 5$	$(2, 4) = 6$	$(2, 5) = 7$	$(2, 6) = 8$
$(3, 1) = 4$	$(3, 2) = 5$	$(3, 3) = 6$	$(3, 4) = 7$	$(3, 5) = 8$	$(3, 6) = 9$
$(4, 1) = 5$	$(4, 2) = 6$	$(4, 3) = 7$	$(4, 4) = 8$	$(4, 5) = 9$	$(4, 6) = 10$
$(5, 1) = 6$	$(5, 2) = 7$	$(5, 3) = 8$	$(5, 4) = 9$	$(5, 5) = 10$	$(5, 6) = 11$
$(6, 1) = 7$	$(6, 2) = 8$	$(6, 3) = 9$	$(6, 4) = 10$	$(6, 5) = 11$	$(6, 6) = 12$

Table 3.2: The possible dice roles with their sum

Since there are 36 possible outcomes, the count of our sample space is $N(S) = 36$.

Let E be the event of 'rolling a two on the die'. The count of event E is column two and row two from Table 3.2, which is 11 (don't double count!). Hence, $N(E) = 11$. The total probability of achieving event E is $\frac{N(E)}{N(S)} = \frac{11}{36} = 0.30556$.

Let F be the event of 'rolling the sum of two'. There is a single count for this which is $(1, 1)$. Hence, $N(F) = 1$. The total probability of achieving event F is $\frac{N(F)}{N(S)} = \frac{1}{36} = 0.02778$.

Let G be the event of 'rolling the sum of six'. There are five counts for this: $(5,1),(4,2),(3,3),(2,4),(1,5)$. Hence, $N(G)=5$. The total probability of achieving event G is $\frac{N(G)}{N(S)}=\frac{5}{36}=0.13889$.

Here is a different yet similar concept to keep in mind. How can we find the amount/probability of something that is a part of multiple counts? For instance, suppose we have males and females in a room. What is the amount of males in the room?

Example 3.5: Finding the probability of an event from multiple counts

Suppose there are 13 males and 20 females in a room. What is the probability/percentage of females? We know that the *total* number of people is

$$\begin{aligned} N(\text{total}) &= N(\text{males}) + N(\text{females}) \\ &= 13 + 20 \\ &= 33 \end{aligned} \tag{2}$$

The probability of females will be

$$\begin{aligned} \text{probability} &= \frac{N(\text{females})}{N(\text{total})} \\ &= \frac{N(\text{females})}{N(\text{males}) + N(\text{females})} \\ &= \frac{20}{13+20} \\ &= 0.60606 \end{aligned} \tag{3}$$

3.1.1 Tree Diagram

A *tree diagram* is a way to visualize all of the possible outcomes of outputs that could equally occur. For instance, what are the different outcomes when having three children that could either be boy or a girl, with the assumption each has 50% of occurring? Mathematically, there will be eight different outcomes. This is because the first child has two options (either boy or girl), the second child also has two options and the third child has two options as well. It will be $2^3 = 8$, or in general:

$$(\#\text{ of options})^n, \tag{4}$$

where # of options is the count of possible options that are available (it is two in our case as the options could be either boy or girl) and n is the number of iterations. The n value will be three because we are interested in three children. This can be visualized using a tree diagram, as shown in Figure 3.1.

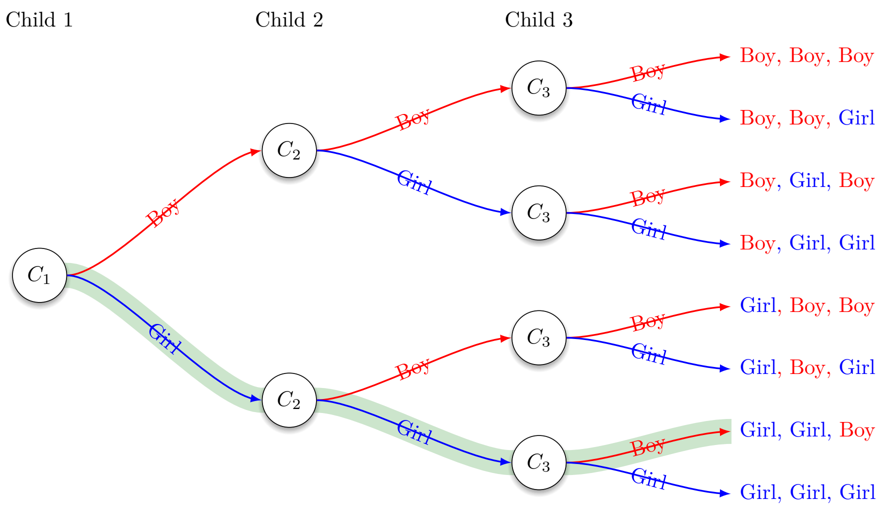


Figure 3.1: A tree diagram of the possible outcomes of having three children where each child could either by a boy or girl. The path of getting Girl, Girl, Boy has been highlighted. The labels C_1, C_2 and C_3 refer to the first, second and third child, respectively.

The *decimal* number system is what we use every day with digits ranging from 0 to 9. Since we are dealing with two values, we can map our data into binary representation. Binary representation is a number system that only uses 0's and 1's (*i.e.*, only two options available). This is what computers understand! All numbers are mapped to a binary representation and mathematical operations such as addition, subtraction, etc. are applied on binary values. Suppose we have three blanks □□□ to fill with binary values, what are the possible outcomes? It is similar to what we did above with the Boy/Girl example. We can map to the digit 0 to Boy and the digit 1 to Girl. This is shown in Table 3.3.

Decimal	Binary	Boy/Girl
0	000	Boy, Boy, Boy
1	001	Boy, Boy, Girl
2	010	Boy, Girl, Boy
3	011	Boy, Girl, Girl
4	100	Girl, Boy, Boy
5	101	Girl, Boy, Girl
6	110	Girl, Girl, Boy
7	111	Girl, Girl, Girl

Table 3.3: The decimal representation along with the binary representation and the equivalent of Boy/Girl outcomes.

Now, suppose we have five blanks □□□□□ to fill with three values. What is the number of total outcomes (oh, don't try to list them)? It will be $3^5 = 243$ different outcomes. How about having five blanks but now we have ten values to choose from? It will be $10^5 = 100000$ different outcomes.

October 17, 2023

3.2 The Addition Rule, Complements and Venn Diagrams

In general, probability events could be mixed with one another or separate. For instance, suppose we have a six-sided die with the numbers coloured as such:

1 2 3 4 5 6

Let us say that we have two events to place our elements in, which are red and blue events. It would make sense to place the values 1, 3 and 5 in the red event and place 2, 4 and 6 in the blue event. Visually, it would look like:

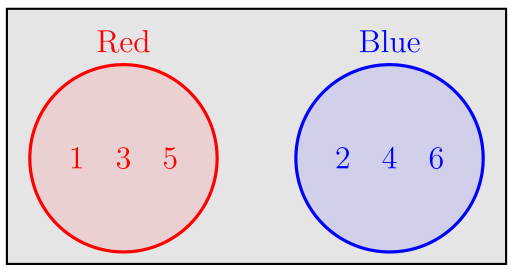


Figure 3.2: Two separate/disjoint events

Now, suppose we have a six-sided die that has the numbers coloured as such:

1 2 3 4 5 6

Suppose we have two events to place our elements in, which are red-coloured values and even values. We cannot have our events to be separate because we have common elements. For example, the value 2 is red *and* even. Of course, we cannot just duplicate the number and place it twice. It must be added to the diagram once and only once. Our two events will be combined, like so:

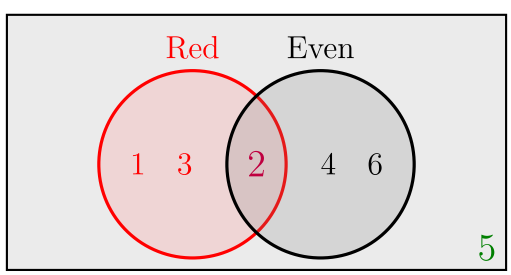


Figure 3.3: Two joint/combined events

We see that the value 2 is placed between the two events and the value 5 *outside* of both events because it is *not* a part of either. These visualizations are referred to as a *Venn* *diagram*. The important point about Venn diagrams is that they will include all of the elements in the sample space, and each data point will be shown once and only once within the specified rectangle somewhere.

Definition 3.7: Disjoint Events

Two events are disjoint (or *mutually exclusive*) if they have no outcomes/elements in common.

In general, we could have zero or more events/circles in our Venn diagram.

Definition 3.8: Addition Rule for Disjoint Events

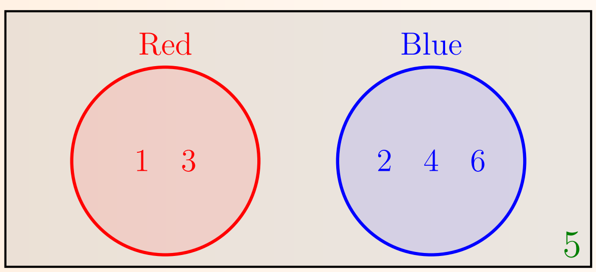
Suppose E and F are disjoint (or mutually exclusive) events, then

$$P(E \text{ or } F) = P(E) + P(F). \tag{5}$$

That is, if we have the events $E_1, E_2, \ldots, E_n$ which are all disjoint from each other, then the probability of getting E_1 or E_2 or $\cdots$ or E_n is

$$P(E_1 \text{ or } E_2 \text{ or } \cdots \text{ or } E_n) = P(E_1) + P(E_2) + \cdots + P(E_n). \tag{6}$$

Example 3.6: Finding the probabilities from a Venn diagram (disjoint events)



Let us use the above Venn diagram and find the probabilities of each event occurring. We know that the count of sample space is $N(S) = 6$. The count of the Red event is $N(\text{Red}) = 2$ and $N(\text{Blue}) = 3$. For instance, the probability of getting the Red event is

$$P(\text{Red}) = \frac{N(\text{Red})}{N(S)} = \frac{2}{6} = 0.33333. \tag{7}$$

Furthermore, we can find the probability of getting either a Red or Blue event. In this case, it would be adding the count of the two events, denoted as $N(\text{Red or Blue})$:

$$N(\text{Red or Blue}) = N(\text{Red}) + N(\text{Blue}) = 2 + 3 = 5. \tag{8}$$

Hence, the probability of getting either the Red event or Blue event, denote by $P(\text{Red or Blue})$, is

$$P(\text{Red or Blue}) = \frac{N(\text{Red} + \text{Blue})}{N(S)} = \frac{N(\text{Red}) + N(\text{Blue})}{N(S)} = \frac{5}{6} = 0.83333. \tag{9}$$

We were able to add the probabilities in the previous example because the events are disjoint (*i.e.*,mutually exclusive). Suppose that two events are joint (*i.e.*, mutually inclusive), how can we find their probabilities? It would be similar to adding both probabilities of the two events but now we have to also subtract the intersection of the two events as we would double count the intersection.

Example 3.7: Finding the probabilities from a Venn diagram (joint events)

Let us use the Venn diagram from Figure 3.3 and find the probabilities of each event occurring. We know that the count of sample space is $N(S) = 6$. The count of the Red event is $N(\text{Red}) = 3$ and $N(\text{Even}) = 3$. What is the probability of getting either a Red or an Even event? Previously, we said to add the probabilities. Now, we cannot just 'add' because our count would be 6, the entire space which is not correct. Instead, we have to add as well as subtract the common elements. So, the count of Red and Even is 1 (the data value 2). This is denoted by $P(\text{Red and Even})$. Hence,

$$P(\text{Red and Even}) = 1. \tag{10}$$

To find the correct probability of getting either a Red or an Even, we will use the following formula:

$$\begin{aligned} P(\text{Red or Even}) &= P(\text{Red}) + P(\text{Even}) - P(\text{Red and Even}) \\ P(\text{Red or Even}) &= \frac{N(\text{Red})}{N(S)} + \frac{N(\text{Even})}{N(S)} - \frac{P(\text{Red and Even})}{N(S)} \\ &= \frac{3}{6} + \frac{3}{6} - \frac{1}{6} \\ &= 0.83333 \end{aligned} \tag{11}$$

Definition 3.9: The General Addition Rule

Suppose E and F are joint or disjoint events, then

$$P(E \text{ or } F) = P(E) + P(F) - P(E \text{ and } F). \tag{12}$$

3.2.1 Probabilities Using a Deck of Cards

Let us use the deck of cards that are found in Table 3.4 for the next example to determine the probabilities of the questions found in Example 3.8.

Suit	Ace									Jack	Queen	King
Clubs	A♣	2♣	3♣	4♣	5♣	6♣	7♣	8♣	9♣	10♣	J♣	Q♣	K♣
Spades	A♠	2♠	3♠	4♠	5♠	6♠	7♠	8♠	9♠	10♠	J♠	Q♠	K♠
Diamonds	A♦	2♦	3♦	4♦	5♦	6♦	7♦	8♦	9♦	10♦	J♦	Q♦	K♦
Hearts	A♥	2♥	3♥	4♥	5♥	6♥	7♥	8♥	9♥	10♥	J♥	Q♥	K♥

Table 3.4: A standard deck of cards with 52 different cards and the typical four suits

Example 3.8: Probability of drawing from a deck

What is the probability of drawing

- A heart ♥?
  - **Answer:** $P(\heartsuit) = \frac{13}{52} = 0.25$, because there are 13 cards of hearts out of the entire 52 cards
- A heart ♥ or spade ♠?
  - **Answer:** $P(\heartsuit \text{ or } \spadesuit) = \underbrace{\frac{13}{52}}_{\heartsuit} + \underbrace{\frac{13}{52}}_{\spadesuit} = 0.5$, because there are 13 cards of hearts and another 13 cards of spades
- A card with a value of 5?
  - **Answer:** $P(\text{value 5 card}) = \frac{4}{52} = 0.07692$, because there are four cards with the value 5 in the deck
- A black-coloured card?
  - **Answer:** $P(\text{black card}) = \frac{26}{52} = 0.5$, because there are 26 cards that are coloured black, they are the clubs ♣ and spades ♠ suits
- A black 7?
  - **Answer:** $P(\text{black 7}) = \frac{2}{52} = 0.03846$, because, although there are four cards with value 7, only two are black: 7♣ and 7♠
- A queen or a king?
  - **Answer:** $P(\text{queen or king}) = \frac{4}{52} + \frac{4}{52} = 0.15385$, because there are four queens and four kings
- An black queen or a red king?
  - **Answer:** $P(\text{black queen or red king}) = \frac{2}{52} + \frac{2}{52} = 0.07692$, because there are two black queens (Q♣ and Q♠) and two red kings (K♦ and K♥)
- A black 2 or a card from the suit of heats ♥?
  - **Answer:** $P(\text{black 2 or } \heartsuit) = \frac{2}{52} + \frac{13}{52} = 0.28846$, because there are two black 2's (2♣ and 2♠), and 13 cards of hearts
- A 10 or a card from the suit of heats ♥?
  - **Answer:** $P(\text{value 10 card or } \heartsuit) = \frac{4}{52} + \frac{13}{52} - \underbrace{\frac{1}{52}}_{10\heartsuit} = 0.30769$, because there are four 10's and 13 hearts, but we double counted 10♥, so we need to subtract it

  Note: focus on the two last examples, they are similar but drastically different! Refer to Table 3.4 to visually see why.

In general, when dealing with two events, if both events are found on horizontal rows, or both event are found on vertical columns, double counting will not occur and the two event will be independent. In the case where one event spans complete/partial row(s) and the other spans complete/partial column(s), then double counting might occur. In case one spans a complete row(s) and the other a complete column(s), double counting must occur. For instance, the second last example had one event to span vertically and the other horizontally, but double counting didn't occur because the vertical span was partial (*i.e.*, we got lucky). However, in the last example, there was a vertical partial span, but we were unlucky and had to remove the double counted card.

3.2.2 Complement of an Event

Suppose S is our sample space and E is an event. The *complement* of the event E, denoted as E^c, is all outcomes in the sample space S that are not in the event E. This is because the events E and E^c are mutually *exclusive*. That is, an outcome could be in only E or only E^c, but not both. Assuming there is a single event E in the experiment, then $P(E)+P(E^c) = 1$.

Definition 3.10: Complement Rule

Given we have a single event E, then E^c is the event represents the complement (invert) of E, then

$$P(E^c) = 1 - P(E). \tag{13}$$

Visually, the outcomes of the complement of an event are found outside the event's circle.

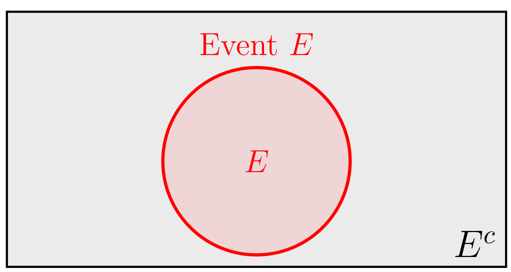


Figure 3.4: The event E and its complement E^c

3.2.3 Contingency Tables

A *contingency table* (or two-way table) relates two categories of data in a single table. We are able to calculate the probabilities based on the properties (*i.e.*, colours) and the two categories that are associated with these properties. Suppose we have asked males and females about their favourite colour and recorded the results:

Colour	Males	Females
Red	23	52
Green	18	39
Blue	4	10
Magenta	62	40
Orange	41	21
Sum →	148	162
Total sum	310	

Table 3.5: A contingency table of favourite colours generated of males and females (usually, the sum/total sum is not a part of the table but not wrong to include)

There are multiple questions/answers in Example 3.9 about different events occurring based on the data in contingency table in Table 3.5.

Example 3.9: Probability of selecting people based on their gender and favourite colour

Out of the entire population (which is 310), what is the probability of:

- Males who favour magenta?
  - **Answer:** $P(\text{Male favours magenta}) = \dfrac{62}{310} = 0.2$, because out of the entire experiment, the males who favour magenta is 62
- Females who favour magenta?
  - **Answer:** $P(\text{Female favours magenta}) = \dfrac{40}{310} = 0.12903$, because out of the entire experiment, the females who favour magenta is 40
- People who favour magenta?
  - **Answer:** $P(\text{magenta}) = \underbrace{\dfrac{62}{310}}_{\text{Males}} + \underbrace{\dfrac{40}{310}}_{\text{Females}} = 0.32903$, we need to add the portion of the males who favour magenta as well as the females
- Males who favour green or blue?
  - **Answer:** $P(\text{Male favours green or blue}) = \dfrac{18}{310} + \dfrac{4}{310} = 0.07097$
- Females who favour red or green or magenta or orange?
  - **Answer:** Two ways to solve this. The is first is summing up all of the four events with respect to the sum of females (pay attention to the denominators):

$$\begin{aligned} P(\text{Female favours red or green or magenta or orange}) &= \frac{52}{310} + \frac{39}{310} + \frac{40}{310} + \frac{21}{310} \\ &= 0.49032 \end{aligned} \tag{14}$$

The second (quicker and reduces the chances of making mistakes) is to focus on the event/colour that is not included in the question (*i.e.*, blue) and then subtract it from the total number of females (*i.e.*, $\frac{162}{300}$). We assume everything is valid except the event that is not included:

$$\frac{162}{310} - \frac{10}{310} = 0.49032$$

- People who favour blue or orange?
  - **Answer:**

$$P(\text{blue or orange}) = \frac{\overbrace{4}^{\text{Males}} + \overbrace{10}^{\text{Females}}}{310} + \frac{\overbrace{41}^{\text{Males}} + \overbrace{21}^{\text{Females}}}{310} = \frac{14}{310} + \frac{62}{310} = 0.24516$$

- A male or someone who favour green?
  - **Answer:** This is tricky because we will double count. The male event and green event are not mutually exclusive (*i.e.*, they are combined). We have to
    i. Count the total males
    ii. Count the total people who favour green
    iii. Add the two total above and then subtract the intersection, which is the males who favour green

$$\begin{aligned} P(\text{Male or green}) &= \underbrace{\frac{\overbrace{23}^{\text{red}} + \boxed{\overbrace{18}^{\text{green}}} + \overbrace{4}^{\text{blue}} + \overbrace{62}^{\text{magenta}} + \overbrace{41}^{\text{orange}}}{310}}_{\text{Total males}} + \underbrace{\frac{\boxed{\overbrace{18}^{\text{Males}}} + \overbrace{39}^{\text{Females}}}{310}}_{\text{green}} - \underbrace{\frac{18}{310}}_{\text{Male and green}} \\ &= \frac{148}{310} + \frac{57}{310} - \frac{18}{310} \\ &= \frac{148 + 57 - 18}{310} \\ &= 0.60323 \end{aligned} \quad (15)$$

October 19, 2023

3.3 Independence and the Multiplication Rule

In the previous section, we talked about how we can find the probabilities of two events where one or the other could occur. It required to add the probabilities of the two events (and subtract the intersection to not double count). Now, we will examine how to find the probability of *both* events occurring.

Definition 3.11: Dependence vs Independence

Suppose we have two events, then they are

- **Independent** if the occurrence of an event **doesn't** change the probability of another event occurring
- **Dependent** if the occurrence of an event change the probability of another event occurring

In other words, they are independent if they are not related to one another, otherwise, dependent.

In our daily lives, everything we practically do is dependent one another. For instance, the event of having a low GPA average will yield a low probability of the event of getting accepted into a top-tier university. However, the higher the average the better the probability of the second event of occurring. The event of driving faster than the speed limit generates a higher probability for the event of a collision of occurring. The event of spending your money wisely will increase the probability of owning a house.

Furthermore, there are instances where events are independent. For example, the winner of the World Cup will not affect your salary. The number of watches you own doesn't affect the number of siblings you have. The last meal you ate doesn't influence the fact that Canada is a country.

Definition 3.12: Disjoint Events versus Independent Events

Disjoint events and independent events are **different** concepts.

- **Disjoint events:** two events don't have anything common between them (if one event occurs, then we are certain that the other didn't)
- **Independence events:** if one event occurs, then it doesn't change the probability of the other event occurring
- In a nutshell:
  - Disjoint $\to$ no common elements between events
  - Independence $\to$ event probabilities not affected by other events

For example, rolling a fair die. Suppose event E is rolling an even number and event F is rolling an odd number (*i.e.*, $P(E) = 0.5$ and $P(F) = 0.5$):

$$E = \{2, 4, 6\}, \qquad F = \{1, 3, 5\} \tag{16}$$

We see that they are disjoint. Are they independent? They are not independent (*i.e.*, dependent). This is because if we knew that we will roll an even, the probability will be $P(E) = 1$ and $P(F) = 0$, not $P(E) = 0.5$ and $P(F) = 0.5$.

Now, suppose we want to roll a fair die twice. Let the event $E = \{1, 2, 3, 4, 5, 6\}$ be rolling the first time and $F = \{1, 2, 3, 4, 5, 6\}$ be rolling the second time. Of course, they are independent events as there are no common outputs (even though they have the same digits, colour matters here). Suppose we get a 3 on the first roll, does that affect (*i.e.*, change the probability of) the value we will get in the second roll? No! not at all. Now, suppose we get a 1 on the first roll, does that influence the second roll? Unsurprisingly, no! Hence, this is an example where we have two disjoint *and* independent events.

Definition 3.13: Multiplication rule for independent events

Suppose E and F are independent events, then

$$P(E \text{ and } F) = P(E) \cdot P(F) \tag{17}$$

That is, if we have the events $E_1, E_2, \ldots, E_n$ which are all independent from each other, then the probability of getting E_1 and E_2 and $\cdots$ and E_n is

$$P(E_1 \text{ and } E_2 \text{ and } \cdots \text{ and } E_n) = P(E_1) \cdot P(E_2) \cdot \ldots \cdot P(E_n). \tag{18}$$

Example 3.10: Finding the probabilities of independent events

Suppose we will flip a fair coin twice and could get either heads (H) or tails (T). The sample space will be

$$S = \{HH, HT, TH, TT\}$$

What is the probability of getting heads on both flips? There are multiple ways of solving this. The first is looking at the sample space (when available) and counting the number of times that we get two heads. This happens to be just one. Hence,

$$P(\text{first flip } H \text{ and second flip } H) = \frac{N(HH)}{N(S)} = \frac{1}{4}. \tag{19}$$

The second approach is finding the probability of each flip individually:

- During each flip, the coin could either land on heads (50%) or tails (50%)
- Let event E be the *first* flip of heads, which has a probability of $P(E) = \frac{1}{2}$
- Let event F be the *second* flip of heads, which has a probability of $P(F) = \frac{1}{2}$
- Since we want to find the probability of *both* flips occurring, we would have to multiply their probability, as given in Definition 3.13:

$$P(E \text{ and } F) = P(E) \cdot P(F) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}. \tag{20}$$

Using a fair coin or a fair die, or both (if both, then flipping the coin as well as rolling a die is considered as a single 'turn'), what are the probability of (tip: focus on the number of iteration along with if it is an **OR** versus **AND**)

- Rolling a die once to get 2 or 5: $\frac{2}{6} = 0.33333$
- Rolling a die once to get 2 and 5: $\frac{0}{6} = 0$, you cannot get both numbers using a single roll
- Rolling a die twice to get 2 or 5: $\frac{20}{36} = \frac{5}{9}$. We assume this is saying that you have to get a (2,5), or (5,2), or (2,2) or (5,5), or (2,1), (2,3), etc. From Table 3.2, whenever we see a 2 or 5 in $(\square, \square)$, it is a part of the event. They are 20 in total.
- Rolling a die twice and getting a (2 first and then a 5 second) or (5 first and then a 2 second): $\frac{1}{36} + \frac{1}{36} = \frac{2}{36} = \frac{1}{18}$, we could either get $(2, 5)$ or $(5, 2)$
- Getting heads or rolling a 3?: $\frac{1}{2} + \frac{1}{6} = \frac{2}{3}$
- Getting heads and rolling a 3?: $\frac{1}{2} \cdot \frac{1}{6} = \frac{1}{12}$
- Getting heads twice or rolling (in that order) 1 and 6?: $\frac{1}{4} + \frac{1}{36} = \frac{5}{18}$
- Getting heads twice and rolling (in that order) 1 and 6?: $\frac{1}{4} \cdot \frac{1}{36} = \frac{1}{144}$
- Getting heads once or rolling 1 or 6 (rolling performed twice)?: $\frac{1}{2} + \frac{1}{6} + \frac{1}{6} = \frac{5}{6}$, here, we assume we evaluate the first roll on the spot and then worry about the second roll later
- Getting heads once or rolling (in that order) 1 and 6 (rolling performed twice)?: $\frac{1}{2} + \left(\frac{1}{6} \cdot \frac{1}{6}\right) = \frac{19}{36}$. There is 50% chance of getting heads and $\frac{1}{36}$ of getting 1 and then 6.
- Getting heads once and (rolling 1 or 6 (rolling performed twice))?: $\frac{1}{2} \cdot \left(\frac{1}{6} + \frac{1}{6}\right) = \frac{1}{6}$
- Getting heads once and (rolling 1 and 6 (rolling performed twice))?: $\frac{1}{2} \cdot \left(\frac{1}{6} \cdot \frac{1}{6}\right) = \frac{1}{72}$.

3.3.1 Computing At-Least Probabilities

Suppose we roll a fair die, then what is the probability that we get at least[2] 2? This is a fancy way of saying we need to add the events that match the condition. That is, we will have to add the probability of getting a 2, getting a 3, getting a 4, getting a 5 along with

[2] Always include the event specified when the word at least is used, in our case, it is the event of getting the value 2, 3, 4, 5 or 6

getting a 6. It is:

$$\begin{aligned} P(\text{at least } 2) &= P(2) + P(3) + P(4) + P(5) + P(6) \\ P(2 \text{ or } 3 \text{ or } 4 \text{ or } 5 \text{ or } 6) &= P(2) + P(3) + P(4) + P(5) + P(6) \\ &= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} \\ &= \frac{5}{6} \end{aligned} \tag{21}$$

What we could have also done is find the probability of getting less than 2 and then subtract that from the total 1:

$$\begin{aligned} P(\text{at least } 2) &= 1 - P(\text{less than } 2) \\ &= 1 - P(1) \\ &= 1 - \frac{1}{6} \\ &= \frac{6}{6} - \frac{1}{6} \\ &= \frac{5}{6} \end{aligned} \tag{22}$$

So, we could either

- Add the probabilities of the events that satisfy the condition, or,
- Add the probabilities of the events that **don't** satisfy the events and then subtract them from the total, which is one (*i.e.*, $1-$ (probabilities of invalid events))

Here are some exercises of finding the probability that:

- Rolling at least a 1 using a fair die? **Answer:** 1 or 100% (you could either roll a 1, 2, 3, 4, 5 or 6, and at least 1 means all possibility). $P(\text{at least } 1) = \frac{6}{6} = 1$.
- Rolling at least a 6 using a fair die? **Answer:** At least 6 means 6, 7, 8, etc. Since a die has the values from 1 to 6, it is $\frac{1}{6}$, rolling the value six. $P(\text{at least } 6) = \frac{1}{6}$.
- Rolling at least a 4 using a fair die? **Answer:** At least 4 means 4, 5, 6, etc. $P(\text{at least } 4) = \frac{3}{6} = 0.5$.
- Suppose we have 200 males, (males $= \{m_1, m_2, \ldots, m_{200}\}$), and each male has a probability of 0.9953 of passing the course (mutually exclusive), find the probability of:
  - At least one male doesn't pass (*i.e.*, at least one male failed)
    i. This means we could have:
    ii. One male failed, or
    iii. Two males failed, or
    iv. Three males failed, or
    v. Four males failed, or,
    vi. $\vdots$
    vii. 198 males failed, or
    viii. 199 males failed, or
    ix. 200 males (all) failed
    x. Any of the above points is a part of "at least one male doesn't pass", we need to find the probabilities of all of these cases, add them together to find the total

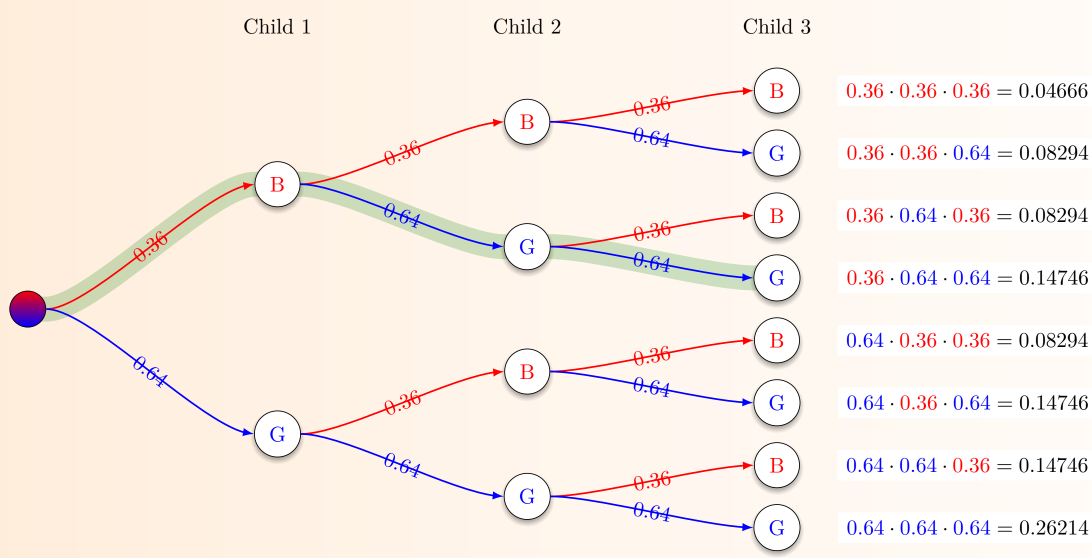
probability of at least one male doesn't pass

Alternatively, we could find the reverse of the question. Instead of finding at least one male not passing (*i.e.*, one male failed, or two males failed, or more etc.), the reverse is finding the probability of all of the males that passed, then subtracting that result from one.

$$\begin{aligned} P(\text{at least one failed}) &= 1 - P(\text{zero failed}) \\ &= 1 - P(\text{all passed}) \\ &= 1 - P(m_1 \text{ passed and } m_2 \text{ passed and } \dots \text{ and } m_{200} \text{ passed}) \\ &= 1 - \underbrace{P(m_1 \text{ passed}) \cdot P(m_2 \text{ passed}) \cdot \dots \cdot P(m_{200} \text{ passed})}_{200 \text{ times}} \\ &= 1 - \underbrace{(0.9953) \cdot (0.9953) \cdot (0.9953) \cdot \dots \cdot (0.9953)}_{200 \text{ times}} \\ &= 1 - (0.9953)^{200} \\ &= 0.61024 \end{aligned} \tag{23}$$

Example 3.11: Probability tree

Suppose we have 36% probability of getting a boy (B) and 64% probability of getting a girl (G). draw the probability tree[a] of getting three children. Find the probability of getting Boy, Girl, Girl.



Probability tree with eight outputs with a total probability of 1

The probability of getting Boy, Girl and Girl is $0.36 \cdot 0.64 \cdot 0.64 = 0.14746$.

[a]This is a probability tree, not a tree diagram, they are different.

3.4 Conditional Probability

Suppose we have a fair die and wanted to know the probability of getting the value two. It will be $\frac{1}{6}$ for obvious reasons. Now, suppose we want to find the probability of getting the value two *given* that it will be even. Well, since we know that it must be even, the even values are Even $= \{2, 4, 6\}$. Since we want to find the probability of getting the value two, then it will be $\frac{1}{3}$, where the numerator is getting two and the denominator is the count of the even values.

Another instance is finding the probability of rolling a die and getting a value less than five. This means we could either get $1, 2, 3$ or 4, which is $\frac{4}{6}$. Now, suppose we what to know the probability of getting a value less than five *given* that it is even. Again, the possible values that are less than five are $1, 2, 3$ or 4, but because we know we will get an even value, then we need to include the even values only, which are 2 and 4. Hence, there are three even values on a die and we want to get an even value that is less than 5, then the probability will be $\frac{2}{3}$, where the numerator is the values two and four, and the denominator is the total of even values on a die.

Let us use formal notations to see how conditional probability works. The first example could be written as:
Let E be the event of getting the value two and F be the event of getting an even value. Then, the conditional probability of getting the value two given it is even is written as

$$P(\text{two} \mid \text{even}) = \frac{P(\text{two and even})}{P(\text{even})} = \frac{N(\text{two and even})}{N(\text{even})}. \tag{24}$$

The event E will be $E = \{2\}$ and event F will be $F = \{2, 4, 6\}$. We can see that the count of the event E is 1, as this includes the value two only. The count of even is three. Hence, the intersection of both events is the total, which is $\frac{1}{3}$.

The second example could be translated to:
Let E be the event of getting a value less than five and F be the event of getting an even value. Then, the conditional probability of getting the value two given it is even is written as

$$P(\text{less than 5} \mid \text{even}) = \frac{P(\text{less than 5 and even})}{P(\text{even})} = \frac{N(\text{less than 5 and even})}{N(\text{even})}. \tag{25}$$

The event E will be $E = \{1, 2, 3, 4\}$ and event F will be $F = \{2, 4, 6\}$. Now, the intersection between both events are the values two and four, which has a count of 2. Event F has a count of 3, hence, the probability is $\frac{2}{3}$. Let us visualize this using a Venn diagram. We see that since we have event F to occur, we focus on event F, as well as focus on how much of event E is in F (*i.e.*, the intersection between E and F).

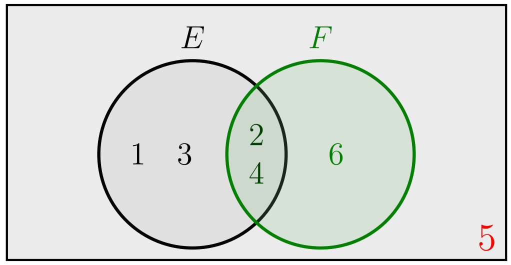


Figure 3.5: Two joint/combined events where we are trying to understand $P(E \mid F)$. The idea is to have the denominator as the count of the elements in the green circle and the numerator to be the intersection between the two events $P(E \text{ and } F)$, which contains the values 2 and 4, meaning that $N(E \text{ and } F) = 2$. The total probability is $P(E \mid F) = \frac{2}{3}$.

Definition 3.14: Conditional probability rule

Suppose events E and F are any two events, then

$$P(E \mid F) = \frac{P(E \text{ and } F)}{P(F)} = \frac{N(E \text{ and } F)}{N(F)} \tag{26}$$

The notation $P(E \mid F)$ is read as "the probability of event E given event F". It is the probability that the event E occurs, given that the event F has occurred. Note that $P(F) \neq 0$ as we cannot divide by zero.

Refer to the Table 3.5 to answer the following questions:

Colour	**Males**	**Females**
Red	23	52
Green	18	39
Blue	4	10
Magenta	62	40
Orange	41	21
Sum $\rightarrow$	148	162
Total sum	310	

Table 3.6: A contingency table of favourite colours generated of males and females (usually, the sum/total sum is not a part of the table but not wrong to include)

- $P(\text{red}) = \dfrac{23 + 52}{310}$
- $P(\text{blue}) = \dfrac{4 + 10}{310}$
- $P(\text{males}) = \dfrac{148}{310}$

- $P(\text{females}) = \dfrac{162}{310}$
- $P(\text{magenta} \mid \text{male}) = \dfrac{62}{148}$
- $P(\text{orange} \mid \text{female}) = \dfrac{21}{162}$
- $P(\text{green} \mid (\text{male or female})) = \dfrac{18+39}{148+162} = \dfrac{18+39}{310}$
- $P((\text{red or blue}) \mid (\text{male or female})) = \dfrac{23+52+4+10}{148+162}$

3.5 Permutations

Let us introduce the factorial notation.

Definition 3.15: Factorial notation

For any integer value $n \geqslant 0$, the *factorial* notation, $n!$, is a compact form of multiplying all of the values from 1 to n, formally defined as

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1 \tag{27}$$

For the value zero, zero factorial is defined as 1, *i.e.*,

$$0! = 1 \tag{28}$$

Here are some examples:

$$\begin{aligned}
0! \quad &= 1 && = 1 \\
1! \quad &= 1 && = 1 \\
2! \quad &= 2 \cdot 1 && = 2 \\
3! \quad &= 3 \cdot 2 \cdot 1 && = 6 \\
4! \quad &= 4 \cdot 3 \cdot 2 \cdot 1 && = 24 \\
5! \quad &= 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 && = 120 \\
6! \quad &= 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 && = 720 \\
7! \quad &= 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 && = 5040
\end{aligned} \tag{29}$$

Let us introduce the idea of arranging objects where order matters. Suppose we have five people and three different chairs. How many different seating arrangements can we have? Let us label the five people A, B, C, D and E, and the chairs as squares □□□. The chair on the left will have five people to choose from. Suppose we chose one, then the middle chair will have four people to choose from, as we have seated one person on the left chair. The chair on the right will have three people to choose from, as we have sat down two people. In total, we will have $5 \cdot 4 \cdot 3 = 60$ different seating arrangements to seat five different people on three different chairs. There are some assumptions here: once we seat one person, then they will stay seated and we don't have duplicates of the objects. For instance, if we had the five people A, B, B, C and D, and we chose B to sit, then *which* B are we talking about? The left one or right one? This is not valid in our situation. We have to have distinct/unique objects (or people in this case).

In the above example, we have concluded that given five people and three chairs, there are $5 \cdot 4 \cdot 3 = 60$ possible arrangements to seat the five people. Now, suppose we have five people and five chairs, how many possible arrangements do we have? It is a continuation of the previous example, we start with:

- # of people = 5, # of chairs = 5, choose one person to sit on one chair, to be left with
- # of people = 4, # of chairs = 4, choose one person to sit on one chair, to be left with
- # of people = 3, # of chairs = 3, choose one person to sit on one chair, to be left with
- # of people = 2, # of chairs = 2, choose one person to sit on one chair, to be left with
- # of people = 1, # of chairs = 1, choose one person to sit on one chair, to be left with zero people and zero chairs

In other words, it will be $5! = 120$ possible arrangements.

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Definition 3.16: Permutations formula

The number of arrangements of r objects from n objects is

$$ {}_{n}P_{r} = \frac{n!}{(n-r)!}, \tag{30}$$

where:

- The n objects are distinct (*i.e.*, unique)
- Repetition of objects is not allowed (*i.e.*, if we use the element Z, then we cannot use it again)
- Order matters (*i.e.*, ABC is different than CBA)
- Other notations include nPr, $P(n,r)$, ${}^{n}P_{r}$

Order matters in the sense that each of the r positions are 'ranked' in some way. For instance, suppose Alex finished the race first, then Bob second and then Cameron third. It is a must that Alex gets the gold medal, then Bob gets the silver and Cameron the bronze medal. Consider the five people and three chairs example, can we come up with a formula that deals with factorials? We have $5 \cdot 4 \cdot 3 = 60$, let us rewrite it in the form of a factorial. We are missing the two numbers $2 \cdot 1$. We want to insert $2 \cdot 1$ but also get the value 60. We can add them to the numerator (which will be 5!) but also place them in the denominator, so that we multiple by $2 \cdot 1$ and divide by $2 \cdot 1$, to get 60:

$$\text{Total number of arrangements} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1} = 60 \tag{31}$$

Can we write the denominator as a form of factorial? Let us visually see the division better:

$$\text{Total number of arrangements} = \frac{\overbrace{\underbrace{5 \cdot 4 \cdot 3}_{\substack{\text{used} \\ r=3}} \cdot \underbrace{2 \cdot 1}_{\substack{\text{missing} \\ n-r}}}^{\text{n=5}}}{\underbrace{2 \cdot 1}_{\substack{\text{missing} \\ n-r \\ 5-3}}} = 60 \tag{32}$$

There are five elements, which is n, and three are used, refer to the number of used elements as r. The denominator needs to be the "leftover", which is $n - r$. That should make sense, if we use r elements from the total n, then there are $n - r$ elements left. Hence, if we have five objects ($n = 5$), three are used ($r = 3$), then the missing number of elements is $n - r = 5 - 3 = 2$. In the denominator, we need to have 2!, which is $2 \cdot 1$. To conclude, we can represent the denominator not as $n - r$, but as $n - r$ factorial, *i.e.*, $(n - r)!$. As for the numerator, it will be the definition of a factorial:

$$\text{Total number of arrangements} = {}_5P_3 = \frac{5!}{(5-3)!} = \frac{5!}{(2)!} = \frac{5!}{2!} = 60 \tag{33}$$

Example 3.12: Permutations of a committee

Suppose we ten people and would like to form a committee that contains a president, vice president, manager and an employee. How many possible arrangements are there to form such a committee?

It is always favourable to extract the values of n, r and $n - r$ from the question:

$$n = 10, \quad r = 4, \quad n - r = 10 - 4 = 6. \tag{34}$$

The total of used positions will be four, as $r = 4$, which must start at n, then decrement three more values (count the numbers because it is deceiving). They will be:

$$10 \cdot 9 \cdot 8 \cdot 7 \tag{35}$$

Typically, the number of numbers from 10 to 7 is 3, but here, we must include 10 *as well as* 7, which happens to be 4. The missing values are $n - r = 6$:

$$6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \tag{36}$$

Thus, the answer so far is:

$$\text{Total arrangements} = {}_{10}P_4 = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} \tag{37}$$

We can rewrite the numerator as $n! = 10!$ and the denominator as $(n - r)! = (10 - 4)!$, which becomes:

$$\text{Total arrangements} = {}_{10}P_4 = \frac{10!}{(10-4)!} = \frac{10!}{6!} = 5040. \tag{38}$$

Therefore, there are 5040 possible permutations to arrangement ten people into four ordered positions.

Example 3.13: Permutations of finishing a race

What is the number of permutations are there for 50 runners to complete a race consisting of 1st, 2nd, 3rd and 4th?

Let us gather the data from the question. There are 50 runners, which means $n = 50$, we are interested in four positions, which means $r = 4$, and the leftover is $n - r = 50 - 4 = 46$:

$$n = 50, \quad r = 4, \quad n - r = 46. \tag{39}$$

The total permutations is given by

$$\begin{aligned} {}_{n}P_{r} &= \frac{n!}{(n-r)!} \\ {}_{50}P_{4} &= \frac{50!}{(50-4)!} \\ {}_{50}P_{4} &= \frac{50!}{(46)!} \\ {}_{50}P_{4} &= 5527200 \end{aligned} \tag{40}$$

Therefore, there are 5527200 different ways to arrange the winners.

October 31, 2023

3.6 Combinations

In the previous section, we found how to calculate the different arrangements when order *matters*. Suppose we have five people named A, B, C, D and E. We want to find the number of permutation of selecting three people out of the five. The total different permutations is ${}_{5}P_{3} = 60$. The possible outcomes are found in Table 3.7.

#01	ABC	ACB	BAC	BCA	CAB	CBA
#02	ABD	ADB	BAD	BDA	DAB	DBA
#03	ABE	AEB	BAE	BEA	EAB	EBA
#04	ACD	ADC	CAD	CDA	DAC	DCA
#05	ACE	AEC	CAE	CEA	EAC	ECA
#06	ADE	AED	DAE	DEA	EAD	EDA
#07	BCD	BDC	CBD	CDB	DBC	DCB
#08	BCE	BEC	CBE	CEB	EBC	ECB
#09	BDE	BED	DBE	DEB	EBD	EDB
#10	CDE	CED	DCE	DEC	ECD	EDC

Table 3.7: The different permutations of selecting three people from a total of five people.

We can see all of the 60 possible permutations divided into ten different groups. Now, our objective is to find the possible *combinations* where order *doesn't* matter. Look at the first row, it contains the possible arrangements of people A, B and C. Suppose that we are interested in the fact that the arrangements contain A, B and C, *not* the order. In other words, let us consider all of the permutations in the first row as a *single* combination. Similarly, let us consider the permutations in the second row as a single combination, and we will do the same for the remaining rows. Hence, we can conclude that the different possible number of combinations of choosing three people out of five people where order *doesn't* matter is 10, and given as:

#01	#02	#03	#04	#05	#06	#07	#08	#09	#10
ABC	ABD	ABE	ACD	ACE	ADE	BCD	BCE	BDE	CDE

Mathematically, we have divided the total of each row by six, to get 1, then summed all of these ten 1's to get a total of 10. The question is *why* six? Let us focus only on the first row. It could be formed by asking what is the different permutations of arranging three objects into three ordered positions? This a fancy way of saying it will be 3!. The first position has three choices, the second position has two choices and the last position has a single choice. The same occurs for the second row and other remaining rows. We can summarize it as get the total permutations, which is 60, and then divide by 6 to get rid of extra rearranged elements. Let us write it as multiplying the total by the fraction $\frac{1}{6}$:

$$\begin{aligned}\text{Total combinations} &= 10\\ &= \frac{1}{6}\cdot 60\\ &= \frac{1}{3\cdot 2\cdot 1}\cdot 60\\ &= \frac{1}{3!}\cdot {}_5P_3\\ &= \frac{1}{3!}\cdot\frac{5!}{(5-3)!}\\ &= \frac{5!}{3!\cdot(5-3)!}\end{aligned} \tag{41}$$

Since $n = 5$, $r = 3$ and $n - r = 2$, we can generate the formula of combinations to be:

$$\text{Total combinations} = \frac{n!}{r!\cdot(n-r)!}. \tag{42}$$

Definition 3.17: Combinations formula

The number of arrangements, *where order doesn't matter*, of r objects from n objects is

$${}_nC_r = \frac{n!}{r!\cdot(n-r)!}, \tag{43}$$

where:

- The n objects are distinct (*i.e.*, unique)
- Repetition of objects is not allowed (*i.e.*, if we use the element Z, then we cannot use it again)
- Order *doesn't* matters (*i.e.*, ABC is seen the same as CBA)
- Other notations include nCr, $C(n, r)$, ${}^nC_r$

Example 3.14: Selection of people

Suppose we have 15 people and would like to select seven people where the order doesn't matter. How many combinations are possible? It is always favourable to extract the values of n, r and $n - r$ from the question:

$$n = 15, \quad r = 7, \quad n - r = 15 - 7 = 8. \tag{44}$$

The total of used positions will be seven, as $r = 7$, which must start at n, then decrement six more values (count the numbers because it is deceiving). They will be:

$$15\cdot 14\cdot 13\cdot 12\cdot 11\cdot 10\cdot 9 \tag{45}$$

Typically, the number of numbers from 15 to 9 is 6, but here, we must include 15 *as well as* 9, which happens to be 7. The missing values are $n - r = 8$:

$$8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1 \tag{46}$$

Thus, the answer so far is:

$$\text{Total arrangements} = {}_{15}P_7 = \frac{15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} \quad (47)$$

We can rewrite the numerator as $n! = 15!$ and the denominator as $(n-r)! = (15-7)!$, which becomes:

$$\text{Total arrangements} = {}_{15}P_7 = \frac{15!}{(15-7)!} = \frac{15!}{8!}. \quad (48)$$

We now need to divide the numerator by $r!$ to get rid of the different permutations for each distinct combination:

$$\text{Total arrangements} = {}_{15}C_7 = \frac{n!}{r! \cdot (n-r)!} = \frac{15!}{7! \cdot (15-7)!} = \frac{15!}{7! \cdot 8!} = 6435. \quad (49)$$

Therefore, there are 6435 possible combinations to for choosing seven people from a group of 15 people where order doesn't matter.

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Example 3.15: Choosing from distinct and non-distinct objects

A continuation of combinations. A DNA chain consists of four possible letters A, C, G and T. How many different chains could be generated by using three As, four Cs, two Gs, and one T?

In total, we will need to have three + four + two + one, which is ten characters. Thus, $n = 10$, we have ten different spots to fill. Let us visualize this:

- At the beginning, we have 10 spots open, let us choose three to fill the As, there ${}_{10}C_3$ ways

A	A	A							

- Now, we have 7 spots free as we inserted three As previously, let us choose four spots to place the Cs, ${}_7C_4$

A	A	A	C	C	C	C			

- Now, we have 3 spots free as we inserted three As and four Cs, let us choose two spots to insert the two Gs, ${}_3C_2$

A	A	A	C	C	C	C	G	G	

- Lastly, we have one spot that is left to insert the T, ${}_1C_1$

A	A	A	C	C	C	C	G	G	T

Hence, the total number of combinations is $\underbrace{{}_{10}C_3}_{\text{Three As}} \cdot \underbrace{{}_{7}C_4}_{\text{Four Cs}} \cdot \underbrace{{}_{3}C_2}_{\text{Two Gs}} \cdot \underbrace{{}_{1}C_1}_{\text{One T}} = 12600$

We are not required to fill in the spots in order. For instance, we could have chosen any of the empty spots to place the three As. Regardless of which spot is filled, we will have the same result. Furthermore, we are not obligated to fill in the As first. We could have filled in the letters in the following order: G, C, T and A, which would have generated the following combinations:

$$\underbrace{{}_{10}C_2}_{\text{Two Gs}} \cdot \underbrace{{}_{8}C_4}_{\text{Four Cs}} \cdot \underbrace{{}_{4}C_1}_{\text{One T}} \cdot \underbrace{{}_{3}C_3}_{\text{Three As}} = 12600 \tag{50}$$

3.7 Permutations of Non-distinct Objects

Suppose we have ten tiles with five blue, four red and one white. None of the tiles are numbered, which mean all the blue tiles is identical to each other, all of the red are identical to each other and the white one is identical to itself. How many ways can we permute them? Let us look at Figure 3.6 to visualize the task.

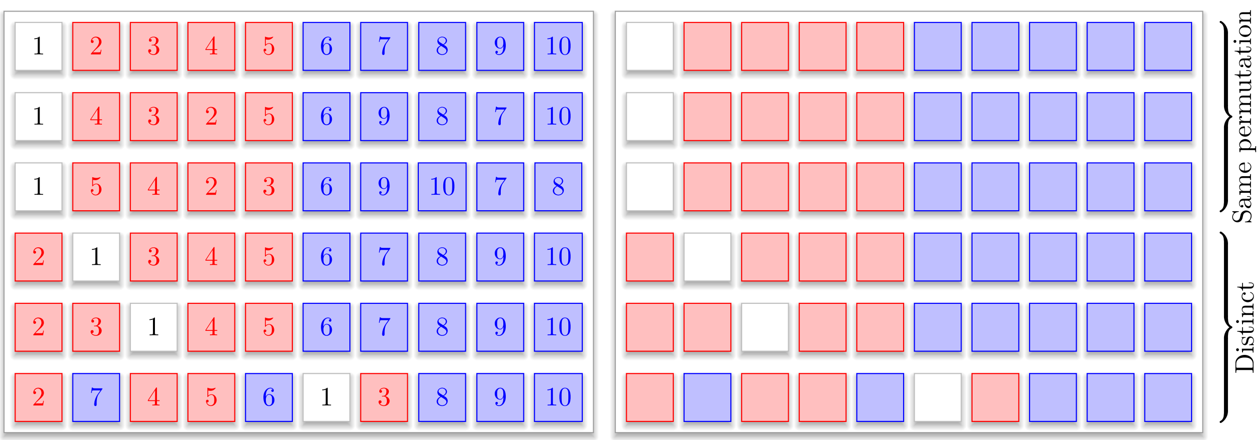

Figure 3.6: The left diagram numbers the objects to show rearranging elements in the first three rows produces a different permutation. However, without the numberings, as shown on the right, the first three rows have the same permutation, *i.e.*, duplicated. The bottom three rows are three different permutations, whether they are labelled or not. Since in reality, there are no labels, we look at the right diagram and conclude there exist four distinct permutations. The first three rows are seen as one permutation and the last three are three different permutations.

Suppose there are 10 distinct objects, then there are $10! = 3628800$ ways to arrange them. In our case, we don't have distinct objects, so the total number of permutation will be less than $10!$. For each of the distinct permutation, we need to divide by $x!$, where x is the number of the objects of one type. For instance, look at the top-left row in Figure 3.6. How many different permutations possible to keep the order the same (*i.e.*, one white, four red and five blue) but change the permutation of the red tiles? There are 4! different permutation of having the same order of colours but rearranging the red tiles among themselves. However, all of these rearrangements are considered equal (*i.e.*, counts toward a single arrangement). Hence, we need to divide by 4! to ensure the different number of red permutation is not counted. The same argument could be made for the blue tiles, which means we need to divide by 5!. The same thought applies on the white tile, but dividing by $1! = 1$ doesn't make a difference. We can see the first three rows in the left diagram are considered the same. Furthermore, the fourth row is different than the above ones, and the last two are also different. So, we have four different orderings on the left. To calculate the total permutations in our case, we will have to divide the total, which is 10! by $5! \cdot 4! \cdot 1$:

$$\text{Total permutations} = \frac{10!}{5! \cdot 4! \cdot 1!} = 1260 \tag{51}$$

From 3.6 million permutations to 1260, quite a significant difference!

Definition 3.18: Permutations of non-distinct objects

The number of permutations of n objects of which x_1 are of one type, x_2 are of a second type, ..., and x_s are of the sth kind, is given by

$$\frac{n!}{x_1! \cdot x_2! \cdot x_3! \cdot \ldots \cdot x_s!}, \tag{52}$$

where $x_1 + x_2 + x_3 + \cdots + x_s = n$.

In the above example, we had $x_1 = 5$ (blue tiles), $x_2 = 4$ (red tiles) and $x_3 = 1$ (white tile). The following is also valid: $x_1 = 1$ (white tile), $x_2 = 4$ (red tiles) and $x_3 = 5$ (blue tiles), which generate the following equivalent equation:

$$\text{Total permutations} = \frac{10!}{1! \cdot 4! \cdot 5!} = 1260 \tag{53}$$