23.1 (a)

(a) $\sum n^2 x^n$

Let $a_n = n^2$

By Corollary 12.3, $\beta = \limsup |a_n|^{\frac{1}{n}} = \lim \left|\frac{a_{n+1}}{a_n}\right| = \lim \left|\frac{(n+1)^2}{n^2}\right| = 1$

$\therefore R = 1/\beta = 1$

The radius of convergence is $R=1$

At $x=1$, $\lim n^2 = +\infty$

At $x=-1$, $\lim n^2 \cdot (-1)^n \neq 0$

By Corollary 14.5, at $x=\pm 1$, series does not converge.

$\therefore$ The interval of convergence is $(-1, 1)$.

(b) $\sum \left(\frac{x}{n}\right)^n$

Let $a_n = \left(\frac{1}{n}\right)^n$

By Corollary 12.3, $\beta = \limsup |a_n|^{\frac{1}{n}} = \lim \frac{1}{n} = 0$

$\therefore R = 1/\beta = +\infty$

The radius of convergence is $R = +\infty$.

$\therefore$ The interval of convergence is $(-\infty, +\infty)$

(c) $\sum \left(\frac{2^n}{n^2}\right) x^n$

Let $a_n = \frac{2^n}{n^2}$

By Corollary 12.3, $\beta = \limsup |a_n|^{\frac{1}{n}} = \lim \left|\frac{a_{n+1}}{a_n}\right| = \lim \left|\frac{2^{n+1}}{n^2} \cdot \frac{(n+1)^2}{2^n}\right| = 2$

$\therefore R = 1/\beta = 1/2$

The radius of convergence is $R = 1/2$

At $x = \frac{1}{2}$, $\frac{2^n}{n^2} \cdot \left(\frac{1}{2}\right)^n = \frac{1}{n^2}$. By theorem 15.1, $\sum \frac{1}{n^2}$ converges.

At $x = \frac{1}{2}$, $\frac{2^n}{n^2} \cdot \left(-\frac{1}{2}\right)^n = \frac{1}{n^2}(-1)^n$. By theorem 15.3, $\lim \frac{1}{n^2} = 0$ and $\frac{1}{n^2}$ is monotonicly decreasing. $\sum \frac{1}{n^2}(-1)^n$ converges.

$\therefore$ The interval of convergence is $\left[-\frac{1}{2}, \frac{1}{2}\right]$

(d) $\sum \left(\frac{n^3}{3^n}\right) x^n$

Let $a_n = n^3/3^n$

By Corollary 12.3, $\beta = \limsup |a_n|^{\frac{1}{n}} = \lim \left|\frac{a_{n+1}}{a_n}\right| = \lim \left|\frac{(n+1)^3}{3n^3}\right| = \frac{1}{3}$

$\therefore R = 1/\beta = 3$

The radius of convergence is 3

At $x=3$, $\left(\frac{n^3}{3^n}\right)\cdot 3^n = n^3$. $\lim n^3 \neq 0$.

At $x=-3$, $\left(\frac{n^3}{3^n}\right)\cdot(-3)^n = n^3\cdot(-1)^n$. $\lim n^3\cdot(-1)^n \neq 0$.

By Corollary 14.5, at $x=\pm 3$, series does not converge

$\therefore$ The interval of convergence is $(-3, 3)$.

(e) $\sum \left(\frac{2^n}{n!}\right) x^n$

Let $a_n = 2^n/n!$

By Corollary 12.3, $\beta = \limsup |a_n|^{\frac{1}{n}} = \lim \left|\frac{a_{n+1}}{a_n}\right| = \lim \left|\frac{2}{n+1}\right| = 0$

$\therefore R = 1/\beta = +\infty$

$\therefore$ The interval of convergence is $(-\infty, +\infty)$.

26.2

(a) $\sum_{n=0}^{\infty} \frac{d}{dx}(x^n) = \frac{d}{dx}\left(\frac{1}{1-x}\right) = 1/(1-x)^2$ for $|x|<1$

$\sum_{n=0}^{\infty} n x^{n-1}\cdot x = \sum_{n=0}^{\infty} n x^n = x/(1-x)^2$ for $|x|<1$

$\therefore \sum_{n=0}^{\infty} n x^n = x/(1-x)^2$ for $|x|<1$

(b) From part a),

$\sum_{n=0}^{\infty} \frac{n}{2^n} = \frac{1/2}{(1-\frac{1}{2})^2} = \frac{1/2}{1/4} = 2$

For 14.13 d), $\sum_{n=1}^{\infty} \frac{n-1}{2^{n+1}} = 0 + \frac{1}{2^3} + \frac{2}{2^4} + \frac{3}{2^5} + \cdots = \frac{1}{2}$

$\sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots$

$= 4\cdot\left(\frac{1}{4\cdot 2^1} + \frac{2}{4\cdot 2^2} + \frac{3}{4\cdot 2^3} + \cdots\right)$

$= 4\cdot\frac{1}{2}$

$= 2$

(c) $\sum_{n=1}^{\infty} n/3^n$ and $\sum_{n=1}^{\infty} \frac{(-1)^n n}{3^n}$

From part a),
$$\sum_{n=1}^{\infty} n\left(\tfrac{1}{3}\right)^n = \tfrac{1}{3}\Big/\left(1-\tfrac{1}{3}\right)^2 = \tfrac{1}{3}\Big/\tfrac{4}{9} = 3/4$$

From part a),
$$\sum_{n=1}^{\infty} n\left(\tfrac{-1}{3}\right)^n = \frac{-1/3}{\left(1-\left(-\frac{1}{3}\right)\right)^2} = \frac{\left(-\frac{1}{3}\right)}{16/9} = -\frac{3}{16}$$

26.3

(a) $$\frac{d}{dx}\sum_{n=1}^{\infty} n x^n = \frac{d}{dx}\left[\frac{x}{(1-x)^2}\right] = \frac{(1-x)^2 + 2x(1-x)}{(1-x)^4} = \frac{1+x}{(1-x)^3}$$

$$x\sum_{n=1}^{\infty} n^2 x^{n-1} = \frac{x\cdot(1+x)}{(1-x)^3} = \frac{x+x^2}{(1-x)^3} \text{ for } |x|<1$$

$$\therefore \sum_{n=1}^{\infty} n^2 x^n = \frac{x+x^2}{(1-x)^3} \text{ for } |x|<1$$

(b) $$\sum_{n=1}^{\infty} n^2/2^n = \frac{\frac{1}{2}+\left(\frac{1}{2}\right)^2}{\left(1-\frac{1}{2}\right)^3} = \left(\frac{1}{2}+\frac{1}{4}\right)\cdot\frac{2^3}{1^3} = \frac{3}{4}\cdot 8 = 6$$

$$\sum_{n=1}^{\infty} n^2/3^n = \frac{\frac{1}{3}+\left(\frac{1}{3}\right)^2}{\left(1-\frac{1}{3}\right)^2} = \frac{4}{9}\cdot\frac{27}{8} = 3/2$$

26.4

(a) $$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!},\ \forall x \in \mathbb{R}$$

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n\cdot(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}$$

(b) $$F(x) = \int_0^x e^{-t^2}dt = \int_0^x \left(\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}t^{2n}\right)dt = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_0^x t^{2n}dt$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\cdot\frac{1}{2n+1}\cdot t^{2n+1}\Big|_0^x$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\cdot\frac{1}{2n+1}\cdot\left(x^{2n+1} - 0^{2n+1}\right) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\cdot\frac{x^{2n+1}}{2n+1}$$

$$\therefore \int_0^x e^{-t^2}dt = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\cdot\frac{x^{2n+1}}{2n+1}$$

26.5

Let $f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$ for $x \in \mathbb{R}$

~~Let $a_n = \frac{1}{n!}$ and $R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \left|\frac{1/n!}{1/(n+1)!}\right| = \lim_{n\to\infty} (n+1) = +\infty$~~

The radius of convergence of $f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$ is $R = +\infty$.

Thus, f is differentiable on $(-\infty, +\infty)$, by theorem 26.5.

$f'(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \frac{d}{dx}(x^n) = \sum_{n=1}^{\infty} \frac{1}{n!}(n x^{n-1}) = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} x^{n-1} = \sum_{k=0}^{\infty} \frac{1}{k!} x^k = f(x)$

31.1

$$f^n(x) = \begin{cases} (-1)^{k+1} \sin x & n = 2k+1 \\ (-1)^k \cos x & n = 2k \end{cases} \qquad k = 0, 1, 2, 3, 4$$

At $x = 0$, $f^n(0) = \begin{cases} 0 & n = 2k+1 \\ (-1)^k & n = 2k \end{cases}$

$|f^n(0)|$ is bounded by 1 and all $f^{(n)}$ exists on $(-\infty, +\infty)$

By Corollary 31.4, $\lim_{n\to\infty} R_n(x) = 0$ for $\forall x \in \mathbb{R}$.

~~$R_n(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{n!}(x-c)^k = 0$, for $c \in (-\infty, +\infty)$~~

$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{n!}(x-c)^k$

$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{n!} x^k = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2k!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$ at $c = 0$

Thus, taylor series for $\cos x$ is $\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2k!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$

Let $P_n(x)$ be the partial sum of Maclaurin's Series.

$|\cos x - P_n(x)| \le \left|\frac{f^{(n+1)}(x)}{(n+1)!} x\right|^{n+1} = \left|\frac{d^{n+1}}{dx^{n+1}} \cos x\right| \cdot \frac{|x|^{n+1}}{(n+1)!} \le \frac{|x|^{n+1}}{(n+1)!}$

For fixed $x \in \mathbb{R} \setminus \{-\infty, +\infty\}$, $\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$

$\therefore \lim_{n\to\infty} |\cos x - P_n(x)| \le \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$

Therefore, the taylor series of $\cos x$ always converges to $\cos x$.

31.2

(1) $\sinh x = \frac{1}{2}(e^x - e^{-x})$

The taylor series of e^x is $\sum \frac{x^n}{n!}$

$e^{-x} = \sum \frac{(-x)^n}{n!}$

$\sinh x = \frac{1}{2}\left(\sum \frac{x^n}{n!} - \sum \frac{(-x)^n}{n!}\right)$

For $n=2k$, $\sinh x = \frac{1}{2}\left(\sum \frac{x^{2k}}{(2k)!} - \sum \frac{x^{2k}}{(2k)!}\right) = 0$, $k=0,1,2,3\ldots$

For $n=2k+1$, $\sinh x = \frac{1}{2}\left(\sum \frac{x^{2k+1}}{(2k+1)!} + \sum \frac{x^{2k+1}}{(2k+1)!}\right) = \sum \frac{x^{2k+1}}{(2k+1)!}$, $k=0,1,2,3,\ldots$

Let $a_n = \frac{1}{n!}$, $n=2k$.

By Corollary 12.3, $\beta = \limsup |a_n|^{\frac{1}{n}} = \lim \left|\frac{a_{n+1}}{a_n}\right| = \lim \left|\frac{\frac{1}{(n+1)!}}{\frac{1}{n!}}\right| = \lim \left|\frac{\frac{1}{(2k+2)!}}{\frac{1}{2k!}}\right|$

$= \lim \left|\frac{1}{2k+2}\right| = 0$

$\therefore R = 1/\beta = +\infty$. The interval of convergence is $(-\infty, +\infty)$.

$\therefore$ The taylor series of $\sinh x$ always converges to $\sinh x$.

(2) $\cosh x = \frac{1}{2}(e^x + e^{-x})$

For $n=2k$, $\cosh x = \frac{1}{2}\left(\sum \frac{x^{2k}}{(2k)!} + \sum \frac{(-x)^{2k}}{(2k)!}\right) = \sum \frac{x^{2k}}{(2k)!}$, $k=0,1,2,3,\ldots$

For $n=2k+1$, $\sinh x = \frac{1}{2}\left(\sum \frac{x^{2k+1}}{(2k+1)!} - \sum \frac{(x)^{2k+1}}{(2k+1)!}\right) = 0$

Let $a_n = \frac{1}{n!}$, $n=2k+1$

By Corollary 12.3, $\beta = \limsup |a_n|^{\frac{1}{n}} = \lim \left|\frac{a_{n+1}}{a_n}\right| = \lim \left|\frac{\frac{1}{(n+1)!}}{\frac{1}{n!}}\right|$

$= \lim \left|\frac{\frac{1}{(2k+3)!}}{\frac{1}{(2k+1)!}}\right| = \lim \left|\frac{1}{2k+3}\right| = 0$

$\therefore R = 1/\beta = +\infty$. The interval of convergence is $(-\infty, +\infty)$.

$\therefore$ The taylor series of $\cosh x$ always converges to $\cosh x$.

31.5

(a) From example 3, assume $g^n(x) = e^{-x^{-2}} P(1/x^2)$, with $P_n(x) = a_{3n}x^{3n} + \dots + a_0$

Set $n=0$, $P_0(x) = 1$ and $n=1$, $P_1(x) = 2x^3$

$$g^{n}(x) = e^{-x^{-2}} \sum_{k=0}^{3n} \frac{a_k}{x^k} \text{ for } x \neq 0$$

$$g^{(n+1)}(x) = e^{-x^{-2}} \left[0 - \sum_{k=1}^{3n} \frac{k a_k}{x^{k+1}}\right] + \sum_{k=0}^{3n} \frac{a_k}{x^k} \cdot 2x^{-3} \cdot e^{-x^{-2}}$$

$$= e^{-x^{-2}} \left[2x^{-3} \cdot \sum_{k=0}^{3n} \frac{a_k}{x^k} - \sum_{k=1}^{3n} \frac{k a_k}{x^{k+1}}\right]$$

Then, we have $P_{n+1}(x) = 2\sum_{k=0}^{3n} a_k x^{k+3} - \sum_{k=1}^{3n} k a_k x^{k+1}$

Therefore, $g^n(x) = e^{-x^{-2}} P(1/x^2)$ holds for all n and $x \neq 0$

For $n=1$, $g'(0) = g(0) = 0$;

Assume $g^k(0) = 0$ holds for some $k \geq 0$

$$\lim \frac{g^{(n)}(x) - g^{(n)}(0)}{x - 0} = \lim_{x \to 0} \frac{g^n(x)}{x} = 0$$

$\therefore g^{(n+1)}(x) = \lim_{x \to 0} \frac{g^n(x)}{x} = 0$

$\therefore g^n(0) = 0$ is true for $\forall n \in \mathbb{N}$.

(b) From part a), $g^{(n)}(0) = 0$ for $\forall n \in \mathbb{N}$

With $g(0) = 0$ and $g^{(n)}(x) > 0$ for $\forall n \in \mathbb{N}$, the taylor series of function g only agrees with g at $x = 0$.