Final Exam review sheet: Math 451, Fall 2014

Principle of Mathematical Induction: Let $(P_n)_{n\in\mathbb{N}}$ be a sequence of mathematical statements indexed by the natural numbers $\mathbb{N}$. If P_1 is true and P_{n+1} is true whenever P_n is true, then P_n is true for all $n \in \mathbb{N}$.

Rational Zeros Theorem: Suppose that $c_0, c_1, \dots c_{n-1}, c_n$ are integers such that $c_0 \neq 0$ and $c_n \neq 0$ and r is a rational solution of

$$c_n x^n + c_{n_1} x^{n-1} + \cdots c_1 x + c_0 = 0.$$

If $r = \frac{c}{d}$ where c and d are integers with no common factors, then c divides c_0 and d divides c_n.
Definitions: Let S be a non-empty subset of $\mathbb{R}$. If there exists $s_0 \in S$, so that $s \leq s_0$ whenever $s \in S$, then s_0 is the **maximum** of S, and we write $s_0 = \max S$. If there exists $s_1 \in S$ so that $s \geq s_1$ for all $s \in S$, then s_1 is called the **minimum** of S and we write $s_1 = \min S$.

If there exists $M \in \mathbb{R}$ so that $s \leq M$ for all $s \in S$, then M is an **upper bound** for S. If there exists $m \in \mathbb{R}$, so that $s \geq m$ for all $s \in S$, then m is a **lower bound** for S. S is **bounded** if it has both an upper bound and a lower bound. Equivalently, S is bounded if there exist $R \in \mathbb{R}$ so that if $s \in S$, then $|s| \leq R$.
Completeness Axiom: If S is a non-empty subset of $\mathbb{R}$ which has an upper bound, then S has a least upper bound, called the **supremum** and written $\sup S$.
Corollary: If S is a non-empty subset of $\mathbb{R}$ which has a lower bound, then S has a greatest lower bound, called the **infimum** and written $\inf S$.
Conventions: If S has no upper bound, we write $\sup S = +\infty$ and if S has no lower bound, we write $\inf S = -\infty$.
Archimidean Property: If $a > 0$ and $b > 0$, then there exists a natural number $n \in \mathbb{N}$ so that $na > b$.
Density of Rationals: If $a, b \in \mathbb{R}$ and $a < b$, then there exists a rational number $r \in \mathbb{Q}$ so that $a < r < b$.
Corollary: If $a, b \in \mathbb{R}$ and $a < b$, then there exists an irrational number x so that $a < x < b$.

Sequences

Definition: A sequence (s_n) converges to a real number s if for any $\epsilon > 0$, there exists N so that if $n > N$, then $|s_n - s| < \epsilon$. In this case, we write $\lim s_n = s$.

(Our sequences will typically be indexed by the natural numbers $\mathbb{N}$, but may also be indexed by any subset of the form $\{n \in \mathbb{Z} \mid n \geq m\}$ for some $m \in \mathbb{Z}$, in which case we write $(s_n)_{n=m}^{\infty}$.)
Basic examples: a) If $p > 0$, then $\lim \frac{1}{n^p} = 0$. b) If $|a| < 1$, then $\lim a^n = 0$. c) $\lim n^{\frac{1}{n}} = 1$.
d) If $a > 0$, then $\lim a^{\frac{1}{n}} = 1$.
Fact: If (s_n) is a convergent sequence it is bounded, i.e the set $\{s_n \mid n \in \mathbb{N}\}$ is a bounded subset of $\mathbb{R}$.
Limit laws: a) If $k \in \mathbb{R}$ and (s_n) converges to s, then (ks_n) converges to ks.
b) If (s_n) and (t_n) are convergent sequences, then $(s_n + t_n)$ is convergent and $\lim s_n + t_n = \lim s_n + \lim t_n$.
c) If (s_n) and (t_n) are convergent sequences, then $(s_n t_n)$ is convergent and $\lim s_n t_n = (\lim s_n)(\lim t_n)$.
d) If (s_n) and (t_n) are convergent sequences, $\lim t_n \neq 0$ and t_n is non-zero for all $n \in \mathbb{N}$, then then $(\frac{s_n}{t_n})$ is convergent and $\lim \frac{s_n}{t_n} = \frac{\lim s_n}{\lim t_n}$.
Comparison laws: Let (a_n), (b_n) and (s_n) be sequences.
a) (s_n) converges to 0 if and only if $(|s_n|)$ converges to 0.
b) If (s_n) is convergent and $s_n \geq a$ for all but finitely many values of n, then $\lim s_n \geq a$.
c) If (s_n) is convergent and $s_n \leq b$ for all but finitely many values of n, then $\lim s_n \leq b$.
d) If (a_n) and (b_n) are convergent and $a_n \leq b_n$ for all but finitely many values of n, then $\lim a_n \leq \lim b_n$.
e) (Squeeze principle) If $a_n \leq s_n \leq b_n$ for all $n \in \mathbb{N}$ and $\lim a_n = \lim b_n$, then (s_n) converges and $\lim s_n = \lim a_n$.

Definitions: A sequence (s_n) is **decreasing** if, for all $n \in \mathbb{N}$, $s_n \geq s_{n+1}$. A sequence (s_n) is **increasing** if $s_n \leq s_{n+1}$ for all $n \in \mathbb{N}$. A sequence is **monotone** if it is either increasing or decreasing.
Theorem: Every bounded monotone sequence is convergent.
Definition: A sequence $(s_{n_k})_{k \in N}$ is a *subsequence* of (s_n) if each $n_k \in \mathbb{N}$ and $n_1 < n_2 < n_3 < \cdots < n_k < n_{k+1} < \cdots$.
Fact: If (s_n) converges to s and (s_{n_k}) is a subsequence of (s_n), then (s_{n_k}) also converges to s.
Theorem: Every sequence has a monotone subsequence.
Bolzano-Weierstrass Theorem: Every bounded sequence has a convergent subsequence.

Definition: A sequence (s_n) is **Cauchy** if for any $\epsilon > 0$, there exists N so that if $n, m > N$, then $|s_n - s_m| < \epsilon$.
Theorem: A sequence (s_n) is a Cauchy sequence if and only if it is convergent.
Definitions: A sequence (s_n) **diverges to** $+\infty$, written $\lim s_n = +\infty$ if for all $M > 0$ there exists N such that if $n > N$, then $s_n > M$. A sequence (s_n) **diverges to** $-\infty$, written $\lim s_n = -\infty$, if for all $M < 0$ there exists N such that if $n > N$, then $s_n < M$.
Facts: a) (Sample Limit Law) If (s_n) and (t_n) are sequences, $\lim s_n = +\infty$ and (t_n) converges to a positive real number or diverges to $+\infty$, then $\lim s_n t_n = +\infty$.
b) (Sample comparison law) If (s_n) and (t_n) are sequences, $\lim s_n = +\infty$ and $t_n \geq s_n$ for all $n \in \mathbb{N}$, then $\lim t_n = +\infty$.
c) If (s_n) is unbounded and non-decreasing, then $\lim s_n = +\infty$.
d) If (s_n) is unbounded and non-increasing, then $\lim s_n = -\infty$.
Definitions: If (s_n) is a sequence, we define

$$\limsup s_n = \lim_{N \to \infty} \sup\{s_n \mid n > N\}, \text{ and}$$

$$\liminf s_n = \lim_{N \to \infty} \inf\{s_n \mid n > N\}.$$

If (s_n) is not bounded above, then we say $\limsup s_n = +\infty$ and if (s_n) is not bounded below then we define $\liminf s_n = -\infty$. With these conventions lim sup and lim inf s_n are always defined.
Theorem: If (s_n) is a sequence, then $\lim s_n$ exists if and only if $\liminf s_n = \limsup s_n$. Moreover, if $\lim s_n$ exists, then $\lim s_n = \liminf s_n = \limsup s_n$.

Infinite series

Definitions: If $\sum_{n=1}^{\infty} a_n$ is an infinite series, we consider the partial sum $s_n = a_1 + \cdots + a_n$. We say that the series **converges** to s if $\lim s_n = s$. We then write $\sum_{n=1}^{\infty} a_n = s$. If $\lim s_n = +\infty$, then we say that $\sum_{n=1}^{\infty} a_n$ diverges to $+\infty$ and we write $\sum_{n=1}^{\infty} a_n = +\infty$. Similarly, if $\lim s_n = -\infty$, then we say that $\sum_{n=1}^{\infty} a_n$ diverges to $-\infty$ and we write $\sum_{n=1}^{\infty} a_n = -\infty$.

We say that $\sum_{n=1}^{\infty} a_n$ satisfies the **Cauchy criterion** if there exists N so that if $n, m > N$, then $|s_n - s_m| < \epsilon$.

We say that $\sum_{n=1}^{\infty} a_n$ **converges absolutely** if $\sum_{n=1}^{\infty} |a_n|$ converges.
Facts: a) An infinite series is convergent if and only if it satisfies the Cauchy criterion.
b) If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim a_n = 0$.
Comparison Tests: Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be infinite series.
a) If $\sum_{n=1}^{\infty} a_n$ converges and $|b_n| \leq a_n$ for all n, then $\sum_{n=1}^{\infty} b_n$ converges.
b) If $\sum_{n=1}^{\infty} a_n = +\infty$ and $b_n \geq a_n$ for all n, then $\sum_{n=1}^{\infty} b_n = +\infty$.
c) If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then it converges.
Root and Ratio Tests: Let $\sum_{n=1}^{\infty} a_n$ be an infinite series.
a) If $\limsup \frac{|a_{n+1}|}{|a_n|} < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.
b) If $\liminf \frac{|a_{n+1}|}{|a_n|} > 1$, then $\sum_{n=1}^{\infty} a_n$ does not converge.
c) If $\limsup |a_n|^{\frac{1}{n}} < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.
d) If $\limsup |a_n|^{\frac{1}{n}} > 1$, then $\sum_{n=1}^{\infty} a_n$ does not converge.

Examples: a) $\sum_{n=1}^{\infty} a^n$ is convergent if and only if $|a| < 1$.
b) $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if and only if $p > 1$.
Alternating Series Test: If (a_n) is a non-increasing sequence, $\lim a_n = 0$ and $a_n \geq 0$ for all $n \in \mathbb{N}$, then $\sum_{n=1}^{\infty}(-1)^n a_n$ converges.

Continuity

Definition: A function f is **continuous at** $x_0 \in dom(f)$ if for every sequence (x_n) in $dom(f)$ which converges to x_0, we have $\lim f(x_n) = f(x_0)$. The function f is said to be **continuous** if it is continuous at every point in $dom(f)$.
Theorem: A function f is continuous at $x_0 \in dom(f)$ if and only if for all $\epsilon > 0$ there exists $\delta > 0$ so that if $x \in dom(f)$ and $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \epsilon$.
Basic facts: Suppose that f and g are functions which are continuous at $x_0 \in \mathbb{R}$, then
a) If $k \in \mathbb{R}$, then kf and $|f|$ are continuous at x_0.
b) $f + g$, $f - g$ and fg are continuous at x_0.
c) If $g(x_0) \neq 0$, then $\frac{f}{g}$ is continuous at x_0.
d) If h is continuous at $f(x_0)$, then $h \circ f$ is continuous at x_0.
e) Polynomials and rational functions are continuous. Moreover, $r(x) = \sqrt{x}$ is continuous.
Theorem: If f is continuous on the closed interval $[a, b]$, then f is bounded and it achieves its maximum and its minimum on $[a, b]$, i.e. there exists $x_1, x_2 \in [a, b]$ so that $f(x_1) \leq f(x) \leq f(x_2)$ for all $x \in [a, b]$.
Intermediate Value Theorem: If f is continuous on a closed interval $[a, b]$ and y lies between $f(a)$ and $f(b)$, then there exists $c \in (a, b)$ so that $f(c) = y$.
Corollary: If f is continuous on an interval I, then $f(I)$ is an interval or a single point.
Definition: A function f is **uniformly continuous** on $S \subset dom(f)$ if for all $\epsilon > 0$ there exists $\delta > 0$ so that if $x, y \in S$ and $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.
Theorem: If f is continuous on a closed interval $[a, b]$, then it is uniformly continuous on $[a, b]$.
Fact: If f is uniformly continuous on S and (x_n) is a Cauchy sequence in S, then $f(x_n)$ is a Cauchy sequence.

Differentiability

Definition: If $S \subset \mathbb{R}$ and $a \in \mathbb{R}$ (or $a = \pm\infty$), there exists a sequence (x_n) in S which converges to a and f is a function defined on S, then $\lim_{x\to a^S} f(x) = L$ (where $L \in R$ or $L = \pm\infty$) if for any sequence (x_n) in S such that $\lim x_n = a$, we have $\lim f(x_n) = L$.
In the case, that I is an open interval about a and $S = I - \{a\}$, then we write $\lim_{x\to a^S} f(x)$ as simply $\lim_{x\to a} f(x)$.
In the case, that $S = (a, b)$, we write $\lim_{x\to a^S} f(x)$ as simply $\lim_{x\to a^+} f(x)$.
In the case, that $S = (c, a)$, we write $\lim_{x\to a^S} f(x)$ as simply $\lim_{x\to a^-} f(x)$.
Fact: If f is defined on an open interval about $a \in \mathbb{R}$, then f is continuous at a if and only if $\lim_{x\to a} f(x) = f(a)$.
Basic facts: Suppose that f and g are functions defined on S and $\lim_{x\to a^S} f(x) = L_1$ and $\lim_{x\to a^S} g(x) = L_2$ both exist and are finite, then
a) If $k \in \mathbb{R}$, then $\lim_{x\to a^S} kf(x)$ exists and $\lim_{x\to a^S} kf(x) = kL_1$,
b) $\lim_{x\to a^S}(f+g)(x)$, $\lim_{x\to a^S}(f-g)(x)$, and $\lim_{x\to a^S}(fg)(x)$ all exist, and $\lim_{x\to a^S}(f+g)(x) = L_1 + L_2$, $\lim_{x\to a^S}(f-g)(x) = L_1 - L_2$,and $\lim_{x\to a^S}(fg)(x) = L_1L_2$.
c) If $L_2 \neq 0$, then $\lim_{x\to a^S} \frac{f}{g}(x)$ exists and $\lim_{x\to a^S} \frac{f}{g}(x) = \frac{L_1}{L_2}$.
d) If h is a function defined on $f(S) \cup \{L_1\}$ and continuous at L_1, then $\lim_{x\to a^S}(h \circ f)(x)$ exists and equals $h(L_1)$.
e) If I is an open interval about a and f is defined on $I - \{a\}$, then $\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^+} f(x) = L$ and $\lim_{x\to a^-} f(x) = L$
Theorem: If f is a function defined on S, a is the limit of some sequence in S and $L \in \mathbb{R}$, then $\lim_{x\to a^S} f(x) = L$ if and only if for all $\epsilon > 0$ there exists $\delta > 0$ such that if $|x - a| < \delta$ and $x \in S$, then $|f(x) - L| < \epsilon$.

Definition: Suppose that f is defined on an open interval containing a. We say that f is **differentiable** at a if $\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$ exists and is a real number. In this, case we write $f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}$.

Facts: a) If f is differentiable at a, then f is continuous at a.
b) If $f(x) = c$ is a constant function and $x_0 \in \mathbb{R}$, then f is differentiable at x_0 and $f'(x_0) = 0$.
c) If $n \in \mathbb{N}$, $x_0 \in \mathbb{R}$ and $f(x) = x^n$, then f is differentiable at x_0 and $f'(x_0) = nx_0^{n-1}$.

Differentiation rules: Suppose that f and g are functions which are differentiable at $x_0 \in \mathbb{R}$.
a) If $k \in \mathbb{R}$, then kf is differentiable at x_0 and $(kf)'(x_0) = kf'(x_0)$.
b) $f+g$, $f-g$, and fg are differentiable at x_0 and $(f+g)'(x_0) = f'(x_0)+g'(x_0)$, $(f-g)'(x_0) = f'(x_0) - g'(x_0)$, and $(fg)'(x_0) = f(x_0)g'(x_0) + f'(x_0)g(x_0)$.
c) If $g(x_0) \neq 0$, then $\frac{f}{g}$ is differentiable at x_0 and $(\frac{f}{g})'(x_0) = \frac{f'(x_0)g(x_0)-f(x_0)g'(x_0)}{(g(x_0))^2}$
d) If h is differentiable at $f(x_0)$, then $h\circ f$ is differentiable at x_0 and $(h\circ f)'(x_0) = h'(f(x_0))f'(x_0)$.
e) Polynomials and rational functions are differentiable on their domains.

Theorem: Suppose that f is defined on an open interval I containing x_0 and assumes its maximum or minimum at x_0. If f is differentiable at x_0, then $f'(x_0) = 0$.

Rolle's Theorem: If f is continuous on $[a, b]$, differentiable on (a, b) and $f(a) = f(b)$, then there exists $x_0 \in (a, b)$ so that $f'(x_0) = 0$.

Mean Value Theorem: If f is continuous on $[a, b]$ and differentiable on (a, b), then there exists $x_0 \in (a, b)$ so that $f'(x_0) = \frac{f(b)-f(a)}{b-a}$.

Corollaries: Suppose that f and g are differentiable on (a, b).
a) If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on (a, b).
b) If $f'(x) = g'(x)$ for all $x \in (a, b)$, then there exists $c \in \mathbb{R}$ so that $f(x) = g(x) + c$ for all $x \in (a, b)$.
c) If $f'(x) > 0$ for all $x \in (a, b)$, then f is strictly increasing on (a, b) (i.e. if $x, y \in (a, b)$ and $y > x$, then $f(y) > f(x)$).
d) If $f'(x) \geq 0$ for all $x \in (a, b)$, then f is increasing on (a, b) (i.e. if $x, y \in (a, b)$ and $y > x$, then $f(y) \geq f(x)$).
e) If $f'(x) < 0$ for all $x \in (a, b)$, then f is strictly decreasing on (a, b).
f) If $f'(x) \leq 0$ for all $x \in (a, b)$, then f is decreasing on (a, b).

Intermediate Value Theorem for Derivatives: If f is differentiable on (a, b), $x_1, x_2 \in (a, b)$, $x_1 < x_2$ and c lies between $f'(x_1)$ and $f'(x_2)$, then there exists $x_0 \in (x_1, x_2)$ so that $f'(x_0) = c$.

Theorem: If f is continuous and one-to-one on an open interval I, $J = f(I)$, f is differentiable at $x_0 \in I$ and $f'(x_0) \neq 0$, then f^{-1} is differentiable at $y_0 = f(x_0)$ and $(f^{-1})'(y_0) = \frac{1}{f'(x_0)}$.

Power series and uniform convergence

Theorem: If $\sum_{n=0}^{\infty} a_n x^n$ is a power series. Let $\beta = \limsup |a_n|^{\frac{1}{n}}$ and let $R = \frac{1}{\beta}$. (If $\beta = 0$, let $R = +\infty$ and if $\beta = +\infty$, let $R = 0$.) Then
(1) $\sum_{n=0}^{\infty} a_n x^n$ converges if $|x| < R$, and
(2) $\sum_{n=0}^{\infty} a_n x^n$ diverges if $|x| > R$.

Addendum: If $\lim\left|\frac{a_{n+1}}{a_n}\right|$ exists, then $\beta = \lim\left|\frac{a_{n+1}}{a_n}\right|$.

Notation: R is called the **radius of convergence** and the set of values where $\sum a_n x^n$ converges is called the **interval of convergence**.

Definition: If (f_n) is a sequence of functions, all of which are defined on S, then (f_n) **converges pointwise on** S to a function f defined on S if for all $x \in S$, $\lim f_n(x) = f(x)$.

The sequence (f_n) **converges uniformly on** S to f if for all $\epsilon > 0$, there exists N so that if $x \in S$ and $n > N$, then $|f_n(x) - f(x)| < \epsilon$.

Theorem: Suppose that (f_n) is a sequence of functions which converges uniformly on S to f. If f_n is continuous at x_0 for all $n \in \mathbb{N}$, then f is continuous at x_0.

Definition: A sequence (f_n) of functions, all defined on S, is **uniformly Cauchy on S** if for all $\epsilon > 0$, there exists N such that if $x \in S$ and $n, m > N$, then $|f_n(x) - f_m(x)| < \epsilon$.
Theorem: If a sequence (f_n) of functions is uniformly Cauchy on S, then there exists a function f defined on S so that (f_n) converges uniformly to f on S.

Theorem: If the power series $\sum_{k=0}^{\infty} a_k x^k$ has radius of convergence R and $R_1 \in (0, R)$, then $\left(\sum_{k=0}^{n} a_k x^k\right)$ converges uniformly on $[-R_1, R_1]$.
Corollary: If the power series $\sum_{k=0}^{\infty} a_k x^k$ has radius of convergence R, then the series converges to a continuous function on $(-R, R)$.

Integration

Definitions: Suppose that f is a bounded function on $[a, b]$. If $S \subset [a, b]$, then let

$$M(f, S) = \sup\{f(x) \mid x \in S\} \quad \text{and} \quad m(f, S) = \inf\{f(x) \mid x \in S\}.$$

A **partition** P of $[a, b]$ is a finite collection of points $P = \{a = t_0 < t_1 < \cdots t_{n-1} < t_n = b\}$.
The **Upper Darboux sum** of f with respect to a partition P is

$$U(f, P) = \sum_{k=1}^{n} M(f, [t_{k-1}, t_k])(t_k - t_{k-1})$$

and the **Lower Darboux sum** of f with respect to P is defined to be

$$L(f, P) = \sum_{k=1}^{n} m(f, [t_{k-1}, t_k])(t_k - t_{k-1}).$$

The **Upper Darboux integral** of f on $[a, b]$ is defined to be

$$U(f) = \inf\{U(f, P) \mid P \text{ is a partition of } [a, b]\},$$

while the **Lower Darboux integral** of f on $[a, b]$ is defined to be

$$L(f) = \sup\{L(f, P) \mid P \text{ is a partition of } [a, b]\},$$

The function f is said to be **integrable** on $[a, b]$ if $L(f) = U(f)$ and its integral is

$$\int_a^b f = U(f) = L(f).$$

Basic properties: If f is a bounded function on $[a, b]$, P and Q are partitions of $[a, b]$ and $P \subset Q$, then

(1) $L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P)$.
(2) $L(f) \leq U(f)$.

Cauchy criterion for integrability: A bounded function f on $[a, b]$ is integrable if and only if for all $\epsilon > 0$ there exists a partition P of $[a, b]$ so that $U(f, P) - L(f, P) < \epsilon$.

Theorem: If f is continuous on $[a, b]$, then f is integrable on $[a, b]$.

Theorem: If f is monotonic on $[a.b]$, then f is integrable on $[a, b]$.
Theorem: If f and g are integrable on $[a, b]$ and $f(x) \leq g(x)$ for all $x \in [a, b]$, then $\int_a^b f \leq \int_a^b g$.

Intermediate Value Theorem for Integrals: If f is continuous on $[a, b]$, then there exists $x \in (a, b)$ so that

$$f(x) = \frac{1}{b - a} \int_a^b f.$$

Facts: (1) If $a < c < b$ and f is integrable on $[a, c]$ and on $[c, b]$, then f is integrable on $[a, b]$ and

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

(2) If f is integrable on $[a, b]$ and $c \in (a, b)$, then f is integrable on $[a, c]$.

(3) If f is integrable on $[a, b]$ and $c \in \mathbb{R}$, then cf is integrable on $[a, b]$ and $\int_a^b cf = c \int_a^b f$.

(4) If f and g are integrable on $[a, b]$, then $f + g$ and $f - g$ are integrable on $[a, b]$, $\int_a^b f + g = \int_a^b f + \int_a^b g$ and $\int_a^b f - g = \int_a^b f - \int_a^b g$.

(5) If g is a continuous function on $[a, b]$, $g(x) \geq 0$ for all $x \in [a, b]$, and $\int_a^b g = 0$, then $g(x) = 0$ for all $x \in [a, b]$.

Fundamental Theorem of Calculus I: If g is a continuous function on $[a, b]$ such that g is differentiable on (a, b) and g' is integrable on $[a, b]$, then

$$\int_a^b g' = g(b) - g(a).$$

Fundamental Theorem of Calculus II: If f is an integrable function on $[a, b]$, let

$$F(x) = \int_a^x f(t)dt.$$

Then F is continuous on $[a, b]$. Moreover, if f is continuous at $x_0 \in (a, b)$, then F is differentiable at x_0 and $F'(x_0) = f(x_0)$.

Integration by Parts: If u and v are continuous functions on $[a, b]$ which are differentiable on (a, b) and u' and v' are integrable on $[a, b]$, then

$$\int_a^b uv' + \int_a^b u'v = u(b)v(b) - u(a)v(a).$$