Polynomial Method(s)

Applications

1. Kakeya-Nikodym problems. (Dvir) '08
2. Erdös distinct problem. (Guth-Katz) 2010
3. Cap set problem (Ellenberg...) (2017)
4. Joint Lines in 3D. (Guth-Katz) '06

Coding Theory - Error-correction.

- Kakeya problem:
  - From Continuous to Finite field
- Main polynomial ingredients.
- 2-distances problem
- Combinatorial Nullstellnsatz

  App 1: Cube covering

  App 2: Sum-sets.

What is special about polynomials Deg $\leq D$?

- $\Theta(D^n)$ degrees of freedom as functions on $\mathbb{F}^n$.
- $\approx D$ deg. of freedom when restricted to a line

  (Vanishing Lemma: if vanishes at $> D$ pts, then vanishes on the whole line.)

Applications

$n=3$
$d=1$

$n=4$

Erdös (1946)

what is the smallest # of distinct distances determined by n points in $\mathbb{R}^2$?

$\sim \frac{n}{\sqrt{\log(n)}}$ (Landau-Ramanujan) for upper bound

$$\lim_{x \to \infty} \frac{\#\{n \leq x : n = SOS_2\}}{x/\sqrt{\ln(x)}} = b \approx 0.76$$

Thm [Guth-Katz '11]

For any n pt set in the plane,

$$\#\text{distinct distances} \geq \frac{c \cdot n}{\log(n)}.$$

$$\frac{cn}{\log n} \leq g(n) \leq \frac{cn}{\sqrt{\log n}}$$

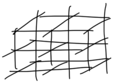
Joints problem

$\mathcal{L}$ set of lines in $\mathbb{R}^3$

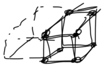
L lines.

A joint of $\mathcal{L}$ is a point which lies in 3 non-coplanar lines.

Example:



grid



$S=2 \to 12$

$S \times S \times S \to$ contains $3S^2 = L$ lines
& contains S^3 joints.

So $\#$ joints $\sim L^{3/2}$

Thm [Guth-Katz]: Any L lines in space determine $\leq 10 L^{3/2}$ joints.

Kakeya

Conj: Slides

[Dvir '09]

Any Kakeya set in $\mathbb{F}_q^n$ contains $\geq c_n q^n$ points.

Ingredients

$\mathbb{F}$ field $\qquad \mathbb{F}[x_1, \dots, x_n]$.

$$\mathrm{Poly}_D(\mathbb{F}^n) = \left\{ \text{poly over } \mathbb{F} \text{ with } \begin{array}{l} \deg \leq D \\ \text{vars} = n \end{array} \right\}$$

$\downarrow$

vector space over $\mathbb{F}$.

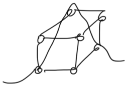
Proposition 1:

The vector space $\mathrm{Poly}_D(\mathbb{F}^n)$ has

$$\dim(\mathrm{Poly}_D(\mathbb{F}^n)) = \binom{D+n}{n} \geq \frac{D^n}{n!}$$

Proof: Basis $\to$ monomial $x_1^{D_1} \dots x_n^{D_n}$ s.t. $\sum D_i \leq D$

$\Rightarrow \#$monomials $= \binom{D+n}{}$



Corollary: [Parameter Counting].

$S \subseteq \mathbb{F}^n$ of size $|S| < \binom{n+D}{D}$

Then, $\exists$ non-zero poly $\hookrightarrow \mathbb{F}[x_1,..,x_n]$ that vanishes on S with degree at most D.

(As long as $\binom{D+n}{n} > |S|$, we have enough params to arrange a non-zero poly that vanishes on S)

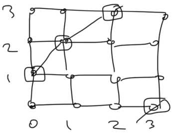
Lemma 1 (Vanishing Lemma).

If L is a line in a vector space

P poly deg $\leq D$

& if P vanishes at $D+1$ points of L

then P vanishes on L.



$v + t\lambda$ $\quad$ $v = (0,1)$, $\lambda = (1,1)$ $\Big\}$ line $= \{(t, 1+t)\}$

$(0,1)$

$(1,2)$

$(2,3)$

$(3,0)$

Lemma 2 (Instead of 1):

Every non-zero poly $f(x_1,..,x_n)$ of deg d over $\mathbb{F}$ with q elements has at most

$d \cdot q^{n-1}$ roots

Proof: $n \geq 2$, $1 \leq d \leq q$, $|\mathbb{F}| = q$.

We reduce to $n=1$:

$f = g + h \rightarrow$ contains only monomials of degree strictly smaller than d.

g: homogeneous

of degree d

Since f non-zero, $g(w) \neq 0$ for some $w \in \mathbb{F}^n, w \neq \vec{0}$.

$\forall u \in \mathbb{F}^n \rightleftarrows L_u = \{u + tw : t \in \mathbb{F}\}$.

$L_u \cap L_v = \phi$

as long as $v \notin L_u$.

Since $w \neq \vec{0}$, each line L_u contains $|L_u| = q$ pts.

Hence, partition $\mathbb{F}^n$ into $\frac{q^n}{q} = q^{n-1}$ lines.

It remains to show that

\# zeroes of f on each of the lines L_u $< d$

$\forall u \in \mathbb{F}^n$: $P_u(t) = f(u + tw) \rightsquigarrow$ poly in t of deg $\leq d$.

Not identically zero

Since the $\underset{\text{in } P_u(t)}{\text{coeff}(t^d)}$ is $g(w) \neq \vec{0}$.

Thus, $P_u(t)$ at most d roots.

$\Rightarrow$ f can vanish on at most d points of L_u.

Hence, f has at most $\underline{\underline{d\, q^{n-1}}}$ roots.

<u>Kakeya</u>:

$f \in \mathbb{F}_q[x_1, .., x_n]$ poly of deg $\leq q-1$.

If f vanishes on a Kakeya set K,

then f is the zero-polynomial.

<u>Proof</u>: Suppose f is non zero.

$f = \sum_{i=0}^{d} \underbrace{f_i}_{\text{homogeneous component}}$, $0 \leq d \leq q-1$

f_d non-zero

Since f vanishes on K, d cannot be zero

$\downarrow$

f_1 is a non zero polynomial

a

Let $v \in \mathbb{F}^n \setminus \{\vec{0}\}$.

Since K Kakeya,

K contains $\{w + tv : t \in \mathbb{F}\}$ for some $w \in \mathbb{F}^n$.

Thus: $\underbrace{f(w+tv)}_{g_{w,v}(t) \text{ in } t \text{ of deg } \leq q-1} = 0 \quad \forall t \in \mathbb{F}$.

& must be the zero poly (all coeffs zero).

$$d\, q^{n-1} \leq (q-1)\, q^{n-1} < q^n$$

$\downarrow$ Lemma 2

f_d must be a zero poly.

Thm: $K \subset \mathbb{F}^n$ Kakeya.

$$|K| \geqslant \binom{|\mathbb{F}|+n-1}{n} \geqslant \frac{1}{n!}\, q^n$$

Nullstellensatz (Alon)

$f \in \mathbb{F}[x_1, .., x_n]$

$\phi \neq S_i \subset \mathbb{F}$.

If $f(x) = 0 \;\; \forall\; x = (x_1, .., x_n) \in \bigtimes_{i=1}^{n} S_i$

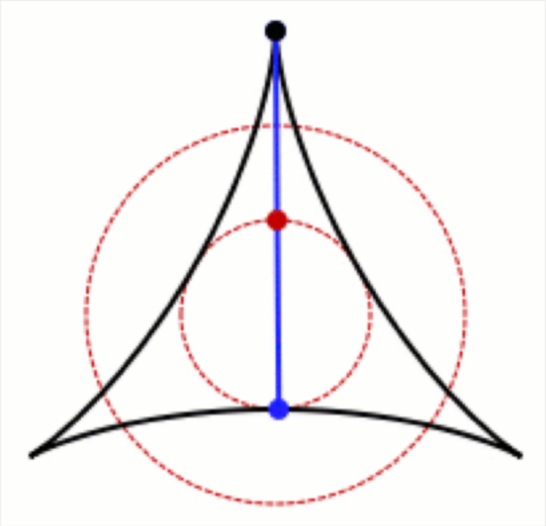
$\Rightarrow$ $\exists$ polys $h_1, .., h_n \in \mathbb{F}[x_1, .., x_n]$ s.t.

$\deg(h_i) \leq \deg(f) - |S_i|$

&

$$f(x_1, .., x_n) = \sum_{i=1}^{n} h_i(x_1, .., x_n) \prod_{s \in S_i} (x_i - s)$$

Covering cube by affine hyperplanes



2-dist problem

what is the max size of a set $P \subset \mathbb{R}^d$ s.t.

dist btwn every 2 pts of P is 1?

→ d-dim simplex

2-dist set → $\exists P: |P| = \binom{d}{2}$.

Thm: Every 2-dist set in $\mathbb{R}^d$ has size at most $\binom{d}{2} + 3d + 2$.

$$f_j(x) = (D(x, P_j)^2 - r^2)(D(x, P_j)^2 - s^2)$$

(1) $\forall q \in P: f_j(q) = 0, q \neq P_j$.

(2) f_j is lin. comb. of

- $\left(\sum_{j=1}^{d} x_j^2\right)^2$
- $x_k \sum_{j=1}^{d} x_j^2$
- x_k
- $x_\ell x_k$
- 1

f_j = vector in $\mathbb{R}^t$, $t = \binom{d}{2} + 3d + 2$

$f_j(x_1, .., x_d) \Leftrightarrow V_j = (v_1^j, \dots, v_t^j)$

(iii) (V_j) lin. indep.

$\Rightarrow \boxed{|P| \le t}$

Comb. Nullstellensatz (Alon).

$f(x_1, .., x_n)$ deg d over $\mathbb{F}$

If coef $(x_1^{t_1} \dots x_n^{t_n}) \neq 0$

$\sum t_i = d$

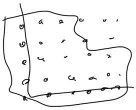
$S_1, \dots, S_n \subset \mathbb{F}$, $|S_i| \geq t_i + 1$

$\Rightarrow \exists \; x \in S_1 \times \dots \times S_n$: $\boxed{f(x) \neq 0}$

$|A+B| \geq \min\{p, |A|+|B|-1\}$.

Assume: $\underline{|A+B| \leq |A|+|B|-2}$

$C \subseteq \mathbb{Z}_p$: $A+B \subseteq C$



& $|C| = \underline{|A|+|B|-2}$.

$$f(x,y) = \prod_{c \in C} (x+y-c)$$

$\forall (a,b) \in A \times B$: $f(a,b) = 0$.

$$\text{coef}\left(x^{|A|-1} y^{|B|-1}\right) = \binom{|A|+|B|-2}{|A|-1} \neq 0$$

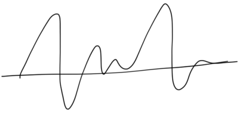
in $\mathbb{Z}_p$, since $|A|+|B|-2 < p$.

By Comb. Nullstellnsatz

$$\left.\begin{array}{l} n=2 \\ S_1 = A \\ S_2 = B \end{array}\right\} \longrightarrow \exists \, (a,b) \in A \times B : f(a,b) \neq 0$$

$\Downarrow$

$$|A+B| \geq \underbrace{|A|+|B|-1}_{\leq p}$$



$(x-2)(x-3)(x-4)$

$\deg f \leq |S| \to f(x) \neq 0$

$T = \{ \ldots \}$.

$|T| \le B$

non-zero low-deg poly that vanishes on T.

$|T| \ge B'$.

the only low-deg poly that vanishes on T is the zero poly.