Προηγμένα Θέματα Αλγορίθμων
Αναφορά για Semidefinite Programming

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Περίληψη

Το περιεχόμενο της παρούσας αναφοράς πραγματεύεται τον τομέα του Semidefinite Programming (SDP). Αρχικά, ορίζουμε θεωρητικά αυτή την κλάση προβλημάτων κυρτής βελτιστοποίησης και, ακολούθως, ασχολούμαστε με αλγοριθμικές εφαρμογές του SDP, δίνοντας έμφαση σε προσεγγιστικούς αλγορίθμους.

ΕΘΝΙΚΟ ΜΕΤΣΟΒΙΟ ΠΟΛΥΤΕΧΝΕΙΟ

Part 1 - Introduction

Semidefinite programming (SDP) is another problem class for convex optimization. SDP is closely related to linear programming (LP) and is equivalent to another convex optimization class called vector programming (VP). All linear programs can be expressed as SDP's. In a set-theoretic point of view, we can think of convex optimization classes in the following way : Convex Programs $\supset$ SDP $\equiv$ Vector Programs $\supset$ LP $\supset$ ILP.
SDP has applications in fields like convex constrained optimization, combinatorial optimization (graph problems) and control theory. It can be used both in proofs and in the design of approximation algorithms for NP-hard problems using the schema of SDP relaxation and Rounding. Besides the applications, another answer on the question why use SDP could be that quadratic approximations are better than linear approximations. We will study in detail the application of SDP on MAXCUT problem, giving an approximation algorithm with optimal approximation ratio if UGC is true.

Part 2 - Review of Convex Programming

Part 2.1 - LP Review

The standard form of an LP program is :

$$LP : min \sum_{i=1}^{n} c_i x_i = min \vec{c}^T \bullet \vec{x}$$
$$s.t. : \vec{a_j} \bullet \vec{x} = \vec{b_j}, \quad j \in \{1, 2, ..., k\} = [k]$$
$$\vec{x} \in \mathbb{R}^n_{\geq 0}$$

As we can see the objective function is linear subject to linear (in)equality constraints generating a convex polytope as feasible region. But linear is a function that is at the same time convex and concave. So, we can see that LP is a special case of Convex Programming, where we require convexity both to the objective function and to the constraints.

Part 2.2 - On Convex Programming

Let $\mathbb{X}$ be a real-valued vector space and let $\mathcal{S} \subset \mathbb{X}$ be a convex set. We consider a set of convex functions $f_i : \mathcal{S} \to \mathbb{R}, i \in \{0\} \cup [k]$,
i.e., satisfy :

$$f_i(\alpha \mathbf{x} + \beta \mathbf{y}) \leq \alpha f_i(\mathbf{x}) + \beta f_i(\mathbf{y}) \quad \forall \mathbf{x}, \mathbf{y} \in \mathcal{S}, \alpha, \beta \in \mathbb{R}_{\geq 0} \quad with\ \alpha + \beta = 1$$

Then a convex optimization problem is one of the form :

$$CP : min \ \ f_0(\mathbf{x})$$
$$s.t. : f_i(\mathbf{x}) \leq \mathbf{b_i}, \quad i \in [k]$$
$$\mathbf{x} \in \mathcal{S}$$

Why do we care about convexity? Because finding a local minimum, it directly has to be a global minimum too. Up to that point, we have seen that LP is a special case of Convex Programming (CP) and that in CP the objective function and the constraints have to be convex (in LP,they are both convex and concave, and so they are linear). So what about SDP?

Part 2.3 - Connecting the dots

So, we have seen that $LP \subset CP$. In the middle of these two problem classifications, the SDP arises.

$$LP \subset SDP \subset CP$$

As we will see, the SDP looks remarkably similar to LP. In SDP, we have a linear objective function too and the crucial difference is found in the constraints. SDP is a generalization of LP and so we can think that we broaden our workspace from variables to vectors relaxing our constraints as we will see. We relax a LP to a Vector program. In LP, our vector $\mathbf{x} = (x_1, ..., x_n)$ must lie in $R^n_{\geq 0}$ and so it must consist of n non-negative variables. In SDP, we replace the nonnegativity constraints on the n real variables $x_1, ..., x_n$ of LP with the positive semidefiniteness constraints on n vector variables that consist an array $X = (\mathbf{x_1}, \mathbf{x_2}, ..., \mathbf{x_n})$.

In fact, we broaden our workspace from n variables (or one vector) to n vectors of n variables (or one square matrix). This idea of augmenting dimension is seen in a lot of mathematics fields : Complex numbers (proof of Fundamental Theorem of Algebra), Stochastic processes, Geometry, etc.

It is helpful to think our matrix variable X in SDP as a vector of n^2 variables s.t. $(x_{11}, x_{12}, ..., x_{ij}, ..., x_{nn}) \in \mathbb{R}^{n^2}$. The constraint in LP that $\mathbf{x}$ lies in $\mathbb{R}^n_{\geq 0}$ is now replaced with the constraint that X lies in the cone of positive semidefinite matrices $\mathcal{S}^n_+$, i.e., the set of all symmetric positive semidefinite matrices of dimension $n \times n$. So the fact that "$\mathbf{x} \geq \mathbf{0}$" (meaning that each of the n variables must be nonnegative), we can think that in SDP, we have the constraint "$X \succeq 0$" stating that X must be pos. semidefinite (each of the n eigenvalues of X must be nonnegative). So we broaden our space from the nonnegative orthant $R^n_{\geq 0}$ to the $\mathcal{S}^n_+$ by replacing nonnegativity of variables to nonnegativity of eigenvalues. The space remains convex since the cone $\mathcal{S}^n_+$ is a proper cone (i.e. closed, convex, pointed and solid). Now we have to think of a linear objective function of the matrix X.

Part 2.4 - Some Linear Algebra

SDP is closely connected to linear algebra term 'semidefinite matrix'.
Positive Semidefinite Matrix (PSD) : A symmetric $n \times n$ matrix A is called PSD if $\mathbf{x}^T A\mathbf{x} \geq 0 \ \ \forall \mathbf{x} \in \mathbb{R}^n$. The set of PSD matrices is denoted by $\mathcal{S}^n_+$.

PSD equivalences : The following statements are equivalent :

- The symmetric matrix A is PSD ($A \succeq 0$).
- All eigenvalues of A are nonnegative.
- There exists U s.t. $A = UU^T$ (Cholesky decomposition).

In part 2.3, we spoke about the cone of positive semidefinite matrices $\mathcal{S}_+^n$:

What is a cone? Let C be a subset of a vector space $\mathbb{V}$. We say that C is a (linear) cone if $\forall \mathbf{x} \in C$ and $\alpha > 0$, the product $\alpha\mathbf{x} \in C$. The cone $\mathcal{S}_+^n = \{A \in Sym(\mathbb{R}_{n\times n}) : A \succeq 0\}$ is a closed convex cone in $\mathbb{R}^{n^2}$.

Before we define SDP, we have to define an appropriate inner-product for our workspace.
Let $\mathbb{S}^n = \{X_{n\times n} : X^T = X, x_{ij} \in \mathbb{R}\}$ the set of $n \times n$ symmetric matrices. We now make $\mathbb{S}^n$ an inner-product space $(\mathbb{S}^n, <, >)$ by defining :

$$< A, B >=< A, B >_{\mathbb{S}^n} := tr(A^T B) = \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}B_{ij}$$

We can think of the inner product $<, >_{\mathbb{S}^n}$ as a matrix inner product by compressing the columns of A and the rows of B and getting the normal vector inner product.
Working with that inner product space, we can define an SDP by expressing the linear objective function of X as $< C, X >$. Note that if X is a symmetric matrix, there is no loss of generality in assuming that the matrix C is also symmetric. One very useful property is that, letting $X \in \mathbb{S}^n$, then $X \succeq 0$ iff $< A, X >\geq 0$, $\forall A \succeq 0$.

Part 3 - Semidefinite Programming

Part 3.1 - Definition of SDP

Let $X \in \mathbb{S}^n$. Then a SDP is an optimization of the following form :

$$\begin{gathered} SDP : min < C, X > \\ s.t. :< A_i, X >= b_i, \quad i \in [k] \\ X \succeq 0 \end{gathered}$$

The matrix variable has to lie in the closed convex cone $\mathcal{S}_+^n$. The data of the SDP consists of the symmetric matrix C and the finite collection $\{(A_i, b_i)\}$ that defines the k linear equations. $C, A_i \in \mathbb{S}^n, \forall i \in [k]$.

This is how we model a SDP optimization problem. It looks really close to a LP representation with the exceptions that now our variable is a symmetric matrix and we have semi-positive definiteness constraints.

Part 3.2 - Why $LP \subset SDP$

We have seen in the introduction that $LP \subset SDP$. The reduction follows :

Consider a LP problem :

$$\begin{gathered} LP : min \ \ \mathbf{c}^T \bullet \mathbf{x} \\ s.t. : \mathbf{a_i} \bullet \mathbf{x} = \mathbf{b_i}, \quad i \in [k] \\ \mathbf{x} \in \mathbb{R}^n_{\geq 0} \end{gathered}$$

Define : $A_i = \begin{bmatrix} a_{i1} & & \\ & \ddots & \\ & & a_{in} \end{bmatrix} = Diag(a_{i1}, ..., a_{in}), i \in [k]$ and $C = Diag(c_1, ..., c_n)$, then the LP can be written as a SDP problem of the form :

$$SDP_{LP} : min < C, X >$$
$$s.t. :< A_i, X >= b_i, \quad i \in [k]$$
$$X_{ij} = 0, \quad i \in [n], j > i$$
$$X(= Diag(x_1, ..., x_n)) \succeq 0$$

Part 3.3 - What about SDP Duality ?

Exactly like working on LP duality, we can easily define the SDP dual as follows :
The dual problem of SDP is :

$$SDD : max \sum_{i=1}^{k} b_i y_i$$
$$s.t. : \sum_{i=1}^{k} A_i y_i + S = C$$
$$S \succeq 0$$

or equivalently : $S \succeq 0 \Rightarrow C - \sum_{i=1}^{k} A_i y_i \succeq 0$, with $S, A_i, C \in \mathbb{S}^n$

Part 3.4 - Feasible solutions

A matrix in $\mathbb{R}_{n \times n}$ satisfying all the constraints of SDP is a feasible solution of SDP. Since a convex combination of positive semidefinite matrices is positive semidefinite, it is easy to see that the set of feasible solutions is convex, i.e., if $A, B \in \mathbb{R}_{n \times n}$ are feasible solutions then so is any convex combination of these solutions. One difference we can notice on SDP, that does not extend from LP, is that there is no direct analog of a "basic feasible solution" for SDP. There is no finite algorithm for solving SDP.

Part 3.5 - On SDP Weak Duality

Given a feasible solution X of SDP and a feasible solution (y,S) of SDD, the duality gap is $< C, X > - \sum_{i=1}^{k} b_i =< S, X >\geq 0$. If $< S, X >= 0$, then X and (y, S) are each optimal solutions of SDP and SDD, respectively, and furthermore, $SX = 0$.

Consider the SDP $p^* = min_{X \in \mathcal{S}_+^n} < C, X > s.t. < A_i, X >= b_i, i \in [m]$ and its dual $d^* = max_{\mathbf{y}} \mathbf{y}^T \mathbf{b} s.t. \sum_{i=1}^{m} y_i A_i \succeq C$. The following holds :
1)Duality is symmetric, in the sense that the dual of the dual is the primal.

2)Weak duality always holds : $< C, X > \geq \sum_{i=1}^{k} b_i y_i$.
3)If the primal (resp. dual) problem is bounded below (resp. above), and strictly feasible, then $p^* = d^*$ and the dual (resp. primal) is attained.
4)If both problems are strictly feasible, then $p^* = d^*$ and both problems are attained.

Another propertiy of LP that does not extend to SDP is that there may be a finite or infinite duality gap. The SDP and/or SDD may or may not attain their optima. Both SDP and SDD will attain their common optimum if both programs have feasible solutions in the interior of the semidefinite cone. Contrarily to linear optimization problems, SDPs can fail to have a zero duality gap, even when they are feasible. Consider the example : $p^* = min_{\mathbf{x}} x_2$ s.t. $\begin{pmatrix} x_2+1 & 0 & 0 \\ 0 & x_1 & x_2 \\ 0 & x_2 & 0 \end{pmatrix} \succeq 0$.
We have $p^* = 0$.
The dual is $d^* = max_{Y \in \mathcal{S}_+^3}(-Y_{11}) : Y \succeq 0, Y_{22} = 0, 1 - Y_{11} - 2Y_{23} = 0$ Any dual feasible Y satisfies $Y_{23} = 0$ (since $Y_{22} = 0$), thus $Y_{11} = -1 = d^*$.

Part 4 - Applications of SDP

In this part, we use the following shortcuts : QP stands for Quadratic Programming and VP for Vector Programming, where a quadratic program is defined as follows :

$$QP : min \frac{1}{2}\mathbf{x}^T Q \mathbf{x} + \mathbf{c}^T \mathbf{x}$$
$$s.t. : A\mathbf{x} \leq \mathbf{b}$$
$$Q \in \mathbb{S}^n$$

We have already seen a framework for creating approximation algorithms :
$ILP \rightarrow LP$ and then Rounding (f.e. Set Cover and $x_i > \frac{1}{f}$)

By formulating an optimization problem in terms of SDP, we can create approx. algorithms in the same way. For example, MAXCUT is NP-hard. We formulate this problem as ILP/IQP and we approximate using the scheme :
$QP \rightarrow SDP(\equiv VP)$ and then Randomized Rounding.

We have not yet defined anything about VP, because we prefer to examine VP through an application.

Part 4.1 - Simple MAXCUT definition

The MAXCUT problem is defined as follows : Given the undirected unweighted graph $G = (V, E)$, find $max_{\mathcal{U} \subseteq V} |E(\mathcal{U}, V \setminus \mathcal{U})|$. This version is based on an undirected unweighted graph.

We can define the ILP version of MAXCUT as following :
Let $X_u = \mathbb{1}[u \in \mathcal{U}] = \begin{cases} 1, & if \ \ u \in \mathcal{U} \\ 0, & otherwise \end{cases}$.

Then :

$$ILP : \frac{1}{2} max \sum_{(u,v)\in E} M_{uv}$$
$$s.t. : M_{uv} \leq X_u + X_v, \quad \forall u, v$$
$$M_{uv} \leq (1 - X_u) + (1 - X_v), \quad \forall u, v$$
$$M_{uv}, X_u \in \{0, 1\} \forall u, v$$

M_{uv} captures whether edge (u,v) is in cut.

Part 4.2 - MAXCUT on weighted G

Definition : Given the undirected graph $G = (V, E)$, with edge weights $w : E \to Q^+$, find a partition $(S, \bar{S})$ of V, so as to maximize the total weight of edges in this cut.

We know that MAXCUT is NP-hard. So, we want to find an approximation algorithm for MAXCUT. But first let's define an appropriate program for MAXCUT. We can easily represent MAXCUT using QP as follows :

$$MAXCUT\_QP : max \frac{1}{2} \sum_{(i,j)\in E} w_{ij}(1 - y_i y_j)$$
$$s.t. : y_i \in \{-1, +1\}, \quad i \in [n]$$

This Quadratic Program ($y_i^2 = 1, y_i \in \mathbb{Z}$) models the MAXCUT. Consider the cut $U = \{i : y_i = -1\}$ and $\bar{U} = \{i : y_i = +1\}$. If an edge (i, j) is in this cut, then $y_i y_j = -1$, otherwise $+1$. So the objective function gives the total weight of the cut. Goal : Find $U, \bar{U}$ that maximize the total weight.

Part 4.3 - Thinking on MAXCUT_QP's relaxation

In order to obtain a relaxation, we will allow the variables to be in a higher dimension space. We want to relax the constraints $y_i^2 = 1, y_i \in \mathbb{Z}$. As we saw in SDP, we go from real-valued variables to real vector variables. We call this a Vector Program. So :

$$MAXCUT\_VP : max \frac{1}{2} \sum_{(i,j)\in E} w_{ij}(1 - \mathbf{v}_i \mathbf{v}_j)$$
$$s.t. : \mathbf{v}_i \bullet \mathbf{v}_i = +1, \quad i \in [n]$$
$$\mathbf{v}_i \in \mathbb{R}_n, \quad i \in [n]$$

This program is a relaxation since we can take any feasible solution y of MAXCUT_QP and generate a solution of MAXCUT_VP by setting $\mathbf{v}_i = (y_i, 0, 0, ..., 0)$. This is a VP problem. Thus, the main idea of a vector program can be thought of as being obtained from a linear program by replacing each variable with an inner product of a pair of these vectors.

More formally, a vector program is defined over n vector variables in $\mathbb{R}^n$, say $\{\mathbf{v}_{1\leq i\leq n}\}$ and is the problem of optimizing a linear function of the inner products $\mathbf{v}_i \bullet \mathbf{v}_j$, $1 \leq i \leq j \leq n$, subject to linear constraints on these inner products.

The most positive consequence is that we can now solve this problem in polynomial time. So, we would like to round the solution to obtain a near-optimal cut.

But, before we proceed to the algorithm, we have to prove that VP is equivalent to semidefinite program SDP.

We will show that corresponding to each feasible solution to VP, there is a feasible solution to SDP of the same objective function value, and vice versa :
$\Rightarrow$ Let $\{\mathbf{a}_i\}_{i=1}^n$ be a SOL(VP). Construct the matrix $U = (\mathbf{a}_1|...|\mathbf{a}_n)$ whose columns are the n vectors. Then the matrix $A = U^T U$ is a SOL(SDP) with the same objective function value.
$\Leftarrow$ Let matrix A be a SOL(SDP). Then $A \succeq 0$ and, by Cholesky decomposition, $A = U^T U$. Let $\{\mathbf{a}_i\}_{i=1}^n$ be the columns of U. These columns are a SOL(VP) with the same objective function value.

Part 4.4 - MAXCUT in SDP form

So, a SDP relaxation for MAXCUT is :

$$MAXCUT\_SDP : max \frac{1}{2} \sum_{1\leq i\leq j\leq n} w_{ij}(1 - y_i y_j)$$
$$s.t. : y_i^2 = 1, \quad i \in [n], v_i \in V$$
$$Y \succeq 0, Y \in \mathbb{S}^n$$

Semidefinite programs, and consequently vector programs, can be solved within an additive error of ϵ, for any $\epsilon > 0$, in time polynomial in n and $log(1/\epsilon)$, using the ellipsoid algorithm. The ellipsoid algorithm will be seen in Part 5. Now, we have to use a rounding algorithm for getting a MAXCUT approximation.

Part 4.5 - Goemans-Willimason Theorem for MAXCUT

The idea behind constructing a randomized rounding algorithm for MAXCUT follows :

- Let us assume that we have OPT(MAXCUT_VP). Let $\{\mathbf{a}_i\}_{i=1}^n$ be an optimal solution with value OPT_{VP}.
- These vectors lie on the $n-$ dimensional unit sphere S_{n-1}. Let θ_{ij} be the angle between vectors $\mathbf{a}_i$ and $\mathbf{a}_j$.
- This pair's contribution to OPT_{VP} is $\frac{w_{ij}}{2}(1 - cos\theta_{ij})$.
- The closer the angle is to π, the larger the contribution will be. So, we want the graph vertices v_i and v_j to be separated in the cut $(S, \bar{S})$ if θ_{ij} is large.

Idea : Pick $\mathbf{r}$ to be a uniformly distributed vector on S_{n-1} and let $S = \{v_i : \mathbf{a}_i \bullet \mathbf{r} \geq 0\}$.

Let's first consider the following lemma :

Lemma (Cutting Probability) : $\mathbb{P}[v_i \in S, v_j \in \bar{S}] = \frac{\theta_{ij}}{\pi}$.

To prove the lemma, we can project $\mathbf{r}$ onto the plane containing $\mathbf{a}_i$ and $\mathbf{a}_j$. If the projection lies in one of the two arcs of angle θ_{ij}, and the two vertices will be separated and since, $\mathbf{r}$ is picked randomly from the S_{n-1}, its projection will be random too.

How can we pick a random vector from the sphere S_{n-1}?
Pick independently $x_1, ..., x_n$ from the distribution $\mathcal{N}(0, 1)$. Then, if we normalize each x_i with the vector's $\mathbf{x} = (x_1, ..., x_n)$ $\mathcal{L}_2$ norm, we get a random unit vector on S_{n-1}, where $\mathcal{L}_2(\mathbf{x}) = \sqrt{\sum x_i^2}$.

This idea generates the following algorithm :
MAXCUT - GW Algorithm
1. Solve VP/SDP. Let $\{\mathbf{a}_i\}_{i=1}^n$ be an OPT(VP) solution.
2. Pick $\mathbf{r}$ to be a uniformly distributed vector on S_{n-1}.
3. Let $S = \{v_i : \mathbf{a}_i \bullet \mathbf{r} \geq 0\}$.
4. Output $Weight(E(S, \bar{S})) = W$.

The output W is a random variable that corresponds the weight of edges in the cut. Define the Goemans-Williamson constant :

$$\alpha_{GW} = \frac{2}{\pi} min_{0 \leq \theta \leq \pi} \frac{\theta}{1 - cos\theta} > 0.87856.$$

Then we get the following lemma : Lemma $\mathbb{E}[W] \geq \alpha_{GW} OPT_{VP}$.

The proof follows : $\forall \theta \in [0, \pi]$, $\frac{\theta}{\pi} \geq \alpha_{GW} \frac{1-cos\theta}{2}$. (From definition of the constant)
Now, we have $\mathbb{E}[W] = \sum_{1 \leq i \leq j \leq n} w_{ij} \mathbb{P}[v_i, v_j \ separated]$.
From Cutting Probability Lemma : $\mathbb{E}[W] = \sum_{1 \leq i \leq j \leq n} w_{ij} \frac{\theta_{ij}}{\pi} \geq \alpha_{GW} \sum w_{ij} \frac{1-cos\theta}{2} = \alpha_{GW} OPT_{VP}$.

Theorem [0.87856 - approximation for MAXCUT]
There is a randomized approximation algorithm for MAXCUT achieving an approximation factor of 0.87856.

Proof :

- Let $T = \sum_{e \in E} w(e)$ and define $\beta : \mathbb{E}[W] = \beta T$.
- Let $p = \mathbb{P}[W < (1 - \epsilon)\beta T]$, for some constant $\epsilon > 0$.
- Since the random variable W is upper bounded by T, $\mathbb{E}[W] = \beta T \leq p(1 - \epsilon)\beta T + (1 - p)T$.
- So, $p \leq \frac{1-\beta}{1-\beta+\beta\epsilon}$.
- By the previous lemma and because of VP relaxation, $T \geq \beta T = \mathbb{E}[W] \geq \alpha_{GW} OPT_{VP} \geq \alpha_{GW} OPT \geq \frac{\alpha T}{2}$.
- So, $\frac{\alpha_{GW}}{2} \leq \beta \leq 1$ and we get $p \leq 1 - \frac{\epsilon \alpha_{GW}/2}{1+\epsilon-\alpha_{GW}/2} = 1 - c$.
- Run our MAXCUT rounding algorithm $1/c$ times, and output the max weight cut founded in these runs. Then :

- $\mathbb{P}[maxW \geq (1-\epsilon)\beta T] \geq 1-(1-c)^{1/c} \geq 1-\frac{1}{e}$.
- Since $\beta T \geq \alpha_{GW} OPT > 0.87856 OPT$, we can choose $\epsilon > 0$ s.t. $(1-\epsilon)\beta T \geq 0.87856 OPT$.

Part 4.6 - Other applications

- Eigenvalue optimization :
  Given symmetric $n \times n$ matrices $B, \{A_i\}_{i=1}^k$, choose weights $w_1, ..., w_k$ to create a new matrix $S := B - \sum_{i=1}^k w_i A_i$ s.t. : $\lambda_{max}(S) - \lambda_{min}(S)$ is minimized.
- 3-colorabilility :
  We want to color a 3-colorable graph. We can color it with $\mathcal{O}(\sqrt{n})$ colors. But can we do better ?

  We will define $\tilde{\mathcal{O}}$ as follows : A function $g(n) = \tilde{\mathcal{O}}(f(n))$ if there exists some constant $c \geq 0$ and some n_0 such that for all $n \geq n_0$, $g(n) = \mathcal{O}(f(n) log^c n)$.

  Using semidefinite programming, we can obtain an algorithm that uses $\tilde{\mathcal{O}}(n^{0.387})$ colors. The best known algorithm does not use much fewer than $\tilde{\mathcal{O}}(n^{0.387})$ colors
- Satisfiability : There is a polynomial time approximation algorithm for MAX-3-SAT with an approximation ratio of 7/8.
- Discrepancy and number theory (See [3])

Part 5 - Algorithms for SDP

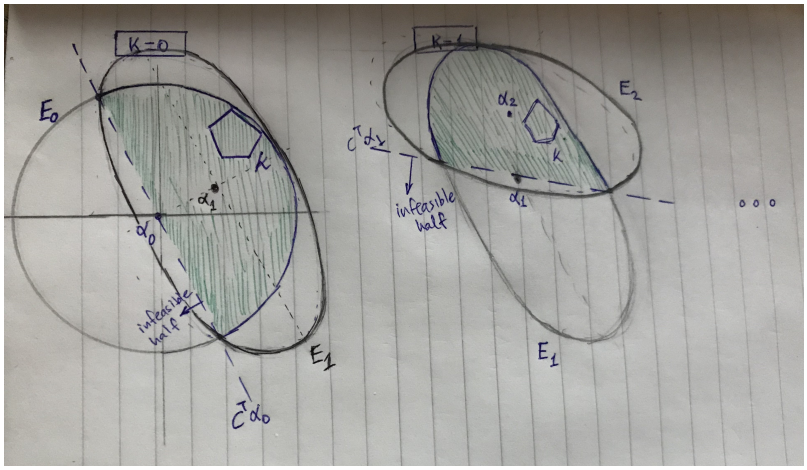
The main algorithm that we use to solve SDP/VP problems is Ellipsoid Algorithm.

An ellipsoid is defined as follows : Given a center α and matrix $A \succ 0$, the ellipsoid $E(\alpha, A)$ is defined as $\{x \in \mathbb{R}^n : (x-\alpha)^T A^{-1}(x-\alpha) \leq 1\}$. As we can see it is strongly connected to positive definiteness.

The problem considered by Ellipsoid algorithm can be stated as : "Given a bounded convex set $\mathcal{K} \in \mathbb{R}^n$, find $x \in \mathcal{K}$."

The algorithm works as follows :
1) Let E_0 be an ellipsoid containing $\mathcal{K}$
2) While center a_k of E_k is not in $\mathcal{K}$:
2a)Let $c^T x \leq c^T a_k$ be s.t. $\{x : c^T x \leq c^T a_k\} \supseteq \mathcal{K}$
(to chop off infeasible half-ellipsoid).
2b)Let E_{k+1} be the minimum volume ellipsoid that contains the
feasible half-ellipsoid $E_k \cap \{x : c^T x \leq c^T a_k\}$
2c) $k++$;
2d) Jump to (2)

It can be proved that $\frac{Vol(E_{k+1})}{Vol(E_k)} < e^{-\frac{1}{2(n+1)}}$ and so the ellipsoids constructed shrink in volume.



As we saw ellipsoid algorithm, finds a feasible solution. How do we connect feasibility to optimization ? We will focus on the most important situation in combinatorial optimization when we are given a set $S \subseteq \{0,1\}^n$ and $\mathcal{K} = convex(S)$.

Let $\mathbf{c}^T\mathbf{x}$ be the objective function we want to minimize over the convex region $\mathcal{K}, \mathbf{c} \in \mathbb{R}^n$. Assume that $\mathbf{c} \in \mathbb{Z}^n$. Instead of optimizing, we can check the non-emptiness of $\mathcal{K}' = \mathcal{K} \cap \{\mathbf{x} : \mathbf{c}^T\mathbf{x} \leq l + \frac{1}{2}\}$ for $l \in \mathbb{Z}$ and our optimum value corresponds to $minl$. Since $S \subseteq \{0,1\}^n$, $-nmax_i c_i \leq l \leq nmax_i c_i$. To find l use binary search with total cost $(\mathcal{O}(log(n) + log(max_i c_i)))$.

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