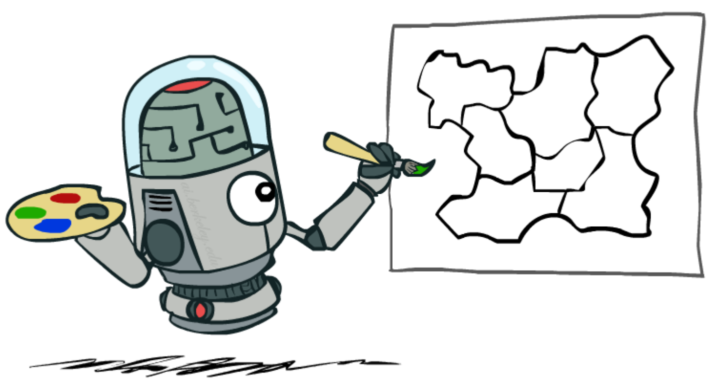
Introduction to Artificial Intelligence

Lecture 4: Constraint satisfaction problems

LIÈGE université

Today

- Constraint satisfaction problems:
  - Exploiting the representation of a state to accelerate search.
  - Backtracking.
  - Generic heuristics.
- Logical agents
  - Propositional logic for reasoning about the world.
  - ... and its connection with CSPs.



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Constraint satisfaction problems

Motivation

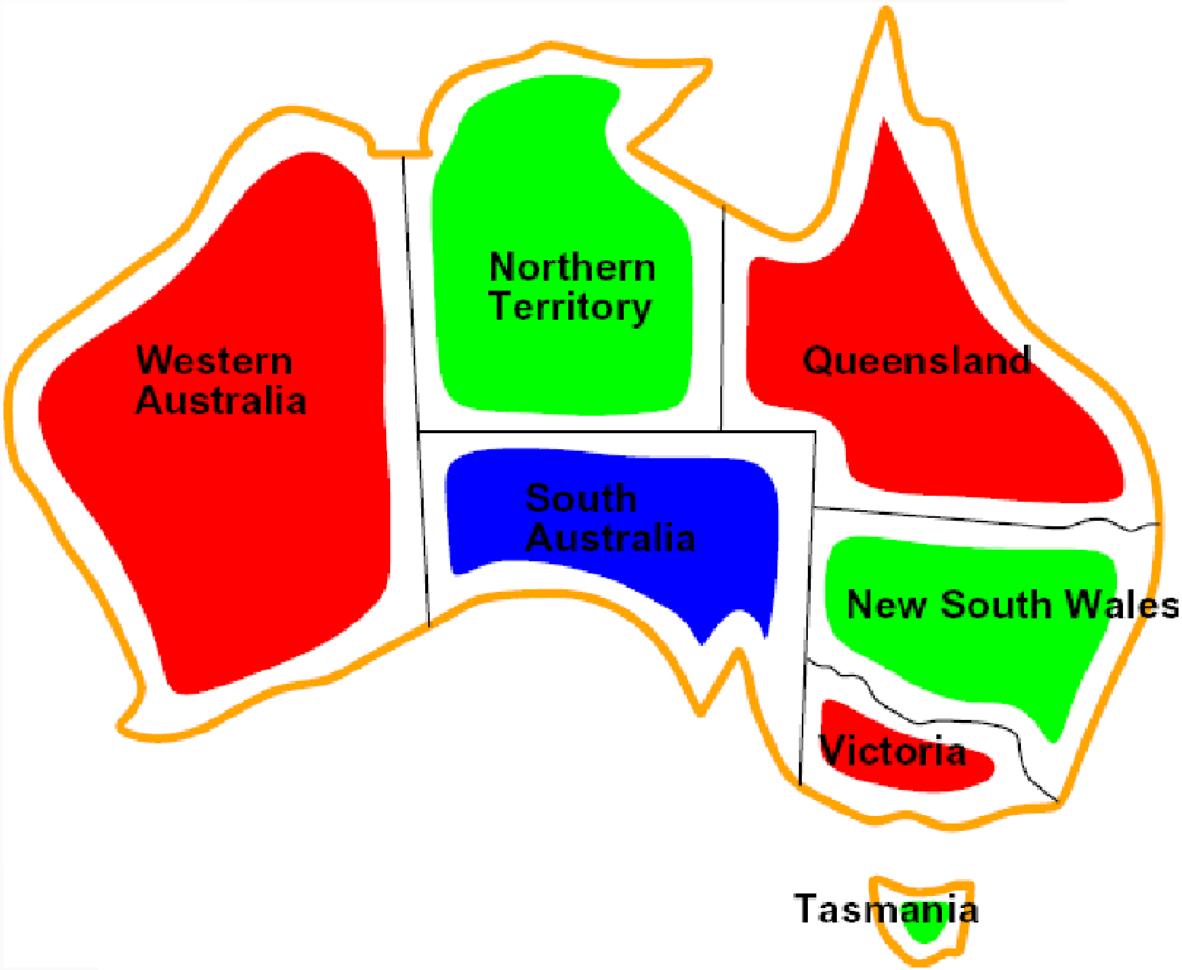
- In standard search problems:
  - States are evaluated by domain-specific heuristics.
  - States are tested by a domain-specific function to determine if the goal is achieved.
  - From the point of view of the search algorithms however, states are atomic.
    - A state is a black box.
- Instead, if states have a factored representation, then the structure of states can be exploited to improve the efficiency of the search.
- Constraint satisfaction problem algorithms take advantage of this structure and use general-purpose heuristics to solve complex problems.
- Main idea: eliminate large portions of the search space all at once, by identifying combinations of variable/value that violate constraints.

Constraint satisfaction problems

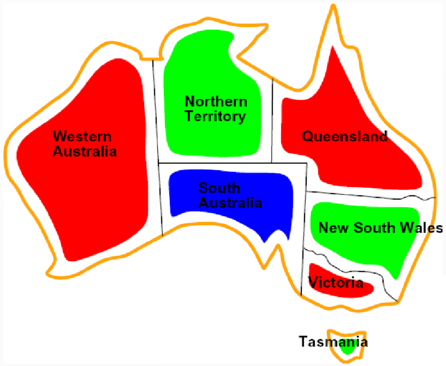
Formally, a constraint satisfaction problem (CSP) consists of three components X, D and C:

- X is a set of variables, $\{X_1, ..., X_n\}$,
- D is a set of domains, $\{D_1, ..., D_n\}$, one for each variable,
- C is a set of constraints that specify allowable combinations of values.

Example: Map coloring

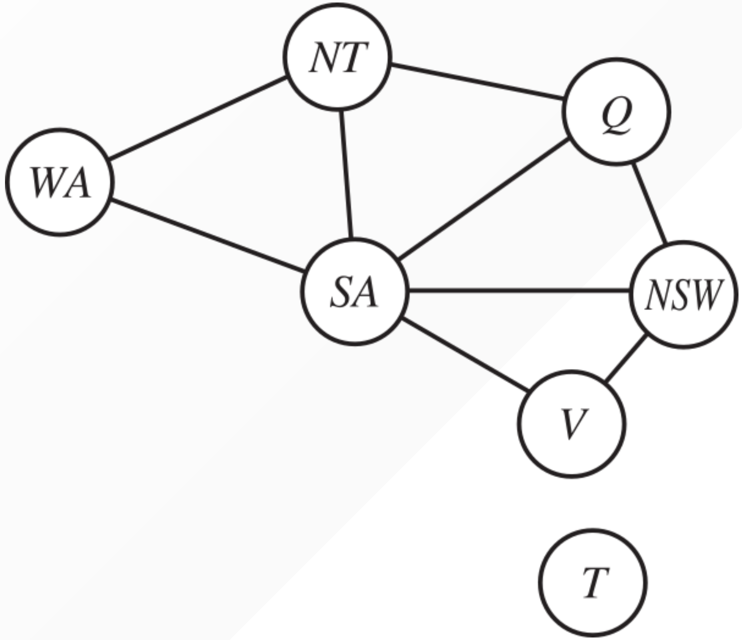


Example: Map coloring



- Variables: $X = \{WA, NT, Q, NSW, V, SA, T\}$
- Domains: $D_i = \{red, green, blue\}$ for each variable.
- Constraints: $C = \{SA \neq WA, SA \neq NT, SA \neq Q, \ldots\}$
  - Implicit: $WA \neq NT$
  - Explicit: $(WA, NT) \in \{\{red, green\}, \{red, blue\}, \ldots\}$
- Solutions are assignments of values to the variables such that constraints are all satisfied.
  - e.g., $\{WA = red, NT = green, Q = red, SA = blue,$
    $NSW = green, V = red, T = green\}$

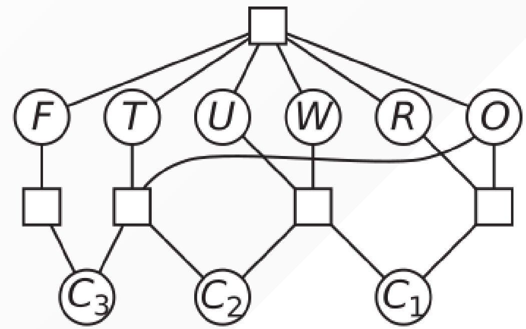
Constraint graph



- Nodes = variables of the problems
- Edges = constraints in the problem involving the variables associated to the end nodes.
- General purpose CSP algorithms use the graph structure to speedup search.
    - e.g., Tasmania is an independent subproblem.

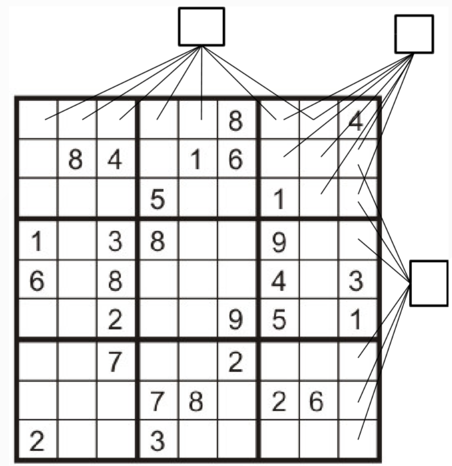
Example: Cryptarithmetic

$$\begin{array}{r} T\ W\ O \\ +\ T\ W\ O \\ \hline F\ O\ U\ R \end{array}$$



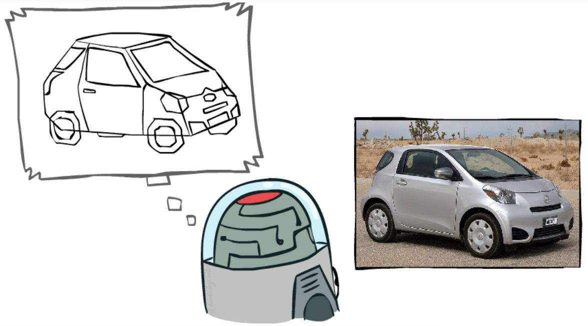
- Variables: $\{T, W, O, F, U, R, C_1, C_2, C_3\}$
- Domains: $D_i = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$
- Constraints:
  - $\text{alldiff}(T, W, O, F, U, R)$
  - $O + O = R + 10 \times C_1$
  - $C_1 + W + W = U + 10 \times C_2$
  - ...

Example: Sudoku



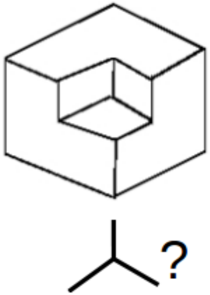
- Variables: each (open) square
- Domains: $D_i = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$
- Constraints:
  - 9-way $\mathrm{alldiff}$ for each column
  - 9-way $\mathrm{alldiff}$ for each row
  - 9-way $\mathrm{alldiff}$ for each region

Example: The Waltz algorithm



The Waltz algorithm is a procedure for interpreting 2D line drawings of solid polyhedra as 3D objects. Early example of an AI computation posed as a CSP.

CSP formulation:



- Each intersection is a variable.
- Adjacent intersections impose constraints on each other.
- Solutions are physically realizable 3D objects.

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Variations on the CSP formalism

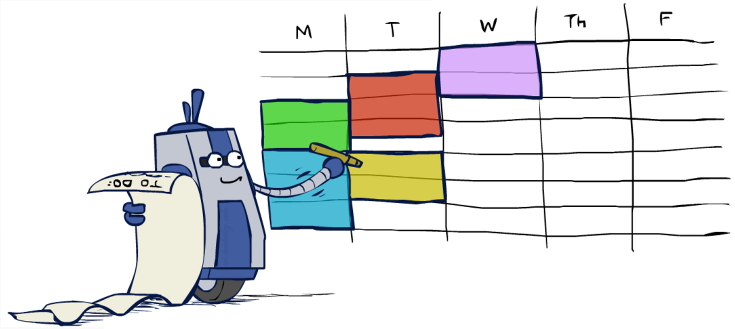
- Discrete variables
  - Finite domains
    - Size d means $O(d^n)$ complete assignments.
    - e.g., boolean CSPs, including the SAT boolean satisfiability problem (NP-complete).
  - Infinite domains
    - e.g., job scheduling, variables are start/end days for for each job.
    - need a constraint language, e.g. $start_1 + 5 \leq start_2$.
    - Solvable for linear constraints, undecidable otherwise.
- Continuous variables
  - e.g., precise start/end times of experiments.
  - Linear constraints solvable in polynomial time by LP methods.

Variations on the CSP formalism

- Varieties of constraints:
    - Unary constraint involve a single variable.
        - Equivalent to reducing the domain, e.g. $SA \neq green$.
    - Binary constraints involve pairs of variables, e.g. $SA \neq WA$.
    - Higher-order constraints involve 3 or more variables.
- Preferences (soft constraints)
    - e.g., red is better than green.
    - Often representable by a cost for each variable assignment.
    - Results in constraint optimization problems.
    - (We will ignore those for now.)

Real-world examples

- Assignment problems
  - e.g., who teaches what class
- Timetabling problems
  - e.g., which class is offered when and where?
- Hardware configuration
- Spreadsheets
- Transportation scheduling
- Factory scheduling
- Circuit layout
- ... and many more



Notice that many real-world problems involve real-valued variables.

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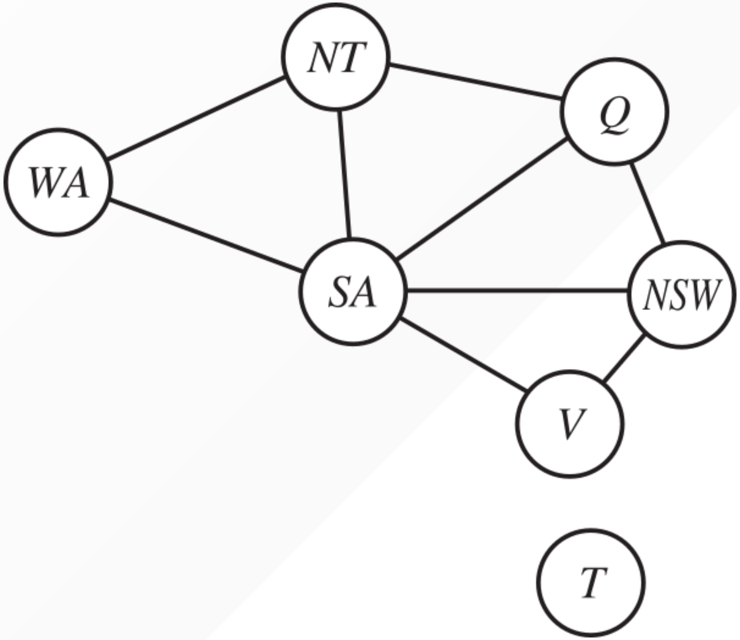
Solving CSPs

Standard search formulation

- CSPs can be cast as standard search problems.
  - For which we have solvers, including DFS, BFS or A*.
- States are partial assignments:
  - The initial state is the empty assignment $\{\}$.
  - Actions: assign a value to an unassigned variable.
  - Goal test: the current assignment is complete and satisfies all constraints.
- This algorithm is the same for all CSPs!

Search methods

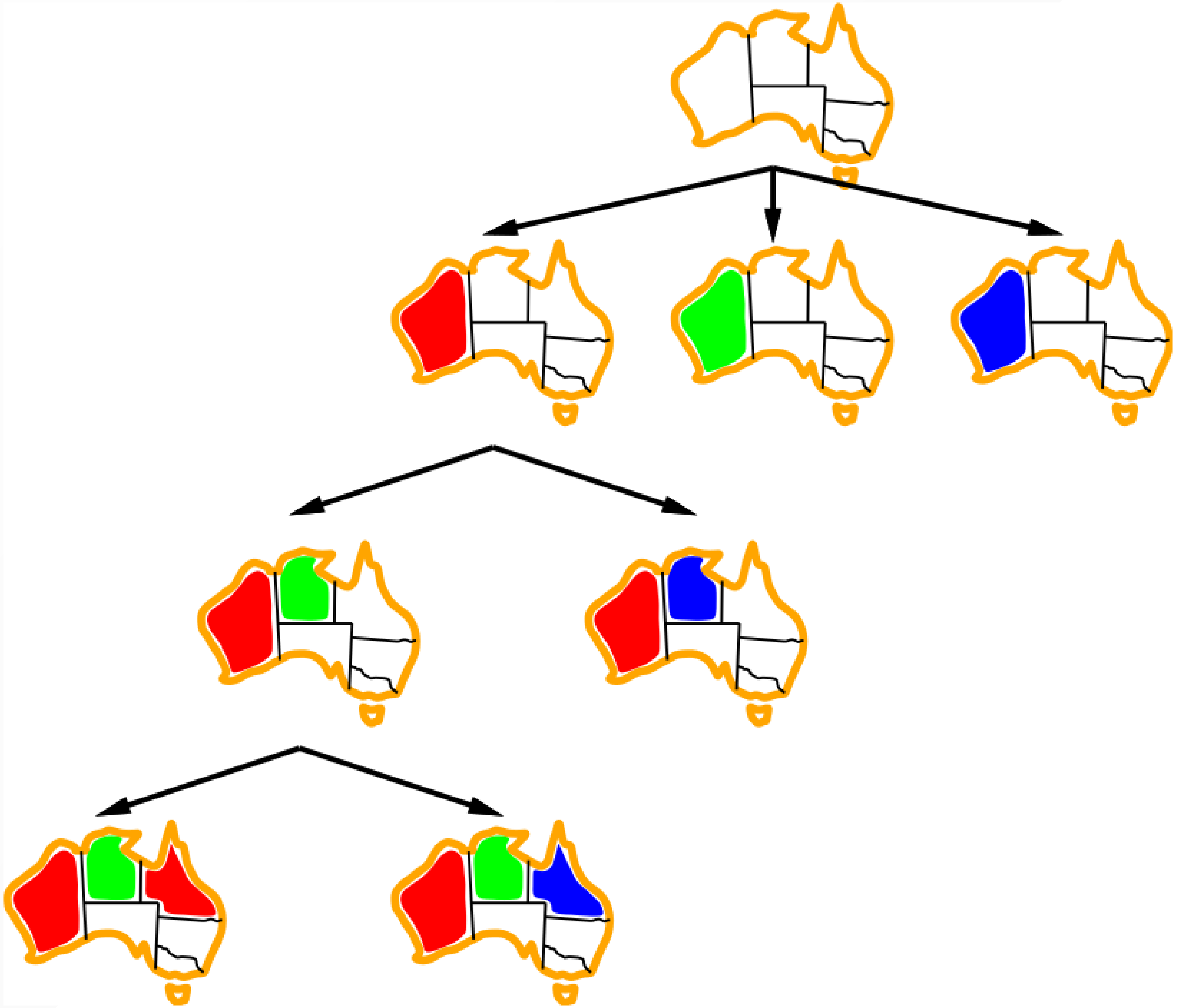


- What would BFS or DF do? What problems does naive search have?
- For n variables of domain size $d, b = (n - l)d$ at depth l.
  - We generate a tree with $n!d^n$ leaves even if there are only d^n possible assignments!

Backtracking search

- Backtracking search is the basic uninformed algorithm for solving CSPs.
- Idea 1: One variable at a time:
    - The naive application of search algorithms ignore a crucial property: variable assignments are commutative. Therefore, fix the ordering.
        - $WA = red$ then $NT = green$ is the same as $NT = green$ then $WA = red$.
    - One only needs to consider assignments to a single variable at each step.
        - $b = d$ and there are d^n leaves.
- Idea 2: Check constraints as you go:
    - Consider only values which do not conflict with current partial assignment.
    - Incremental goal test.

Backtracking example



Backtracking search

```
function BACKTRACKING-SEARCH(csp) returns solution/failure
    return RECURSIVE-BACKTRACKING({ }, csp)

function RECURSIVE-BACKTRACKING(assignment, csp) returns soln/failure
    if assignment is complete then return assignment
    var ← SELECT-UNASSIGNED-VARIABLE(VARIABLES[csp], assignment, csp)
    for each value in ORDER-DOMAIN-VALUES(var, assignment, csp) do
        if value is consistent with assignment given CONSTRAINTS[csp] then
            add {var = value} to assignment
            result ← RECURSIVE-BACKTRACKING(assignment, csp)
            if result ≠ failure then return result
            remove {var = value} from assignment
    return failure
```

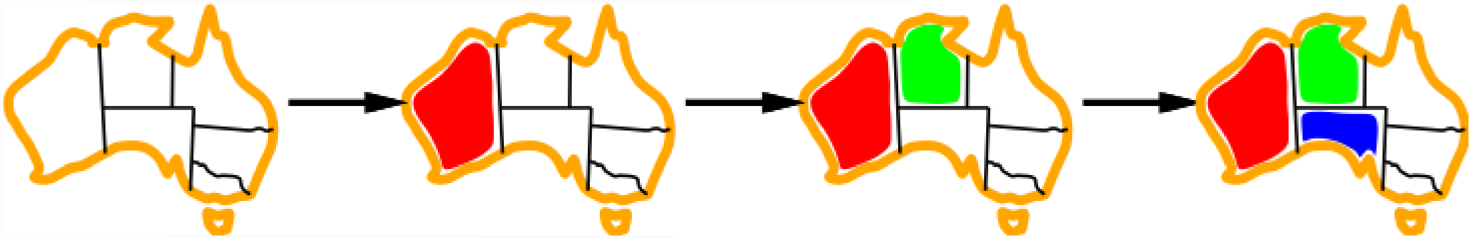
- Backtracking = DFS + variable-ordering + fail-on-violation
- What are the choice points?

Improving backtracking

- Can we improve backtracking using general-purpose ideas, without domain-specific knowledge?
- Ordering:
  - Which variable should be assigned next?
  - In what order should its values be tried?
- Filtering: can we detect inevitable failure early?
- Structure: can we exploit the problem structure?

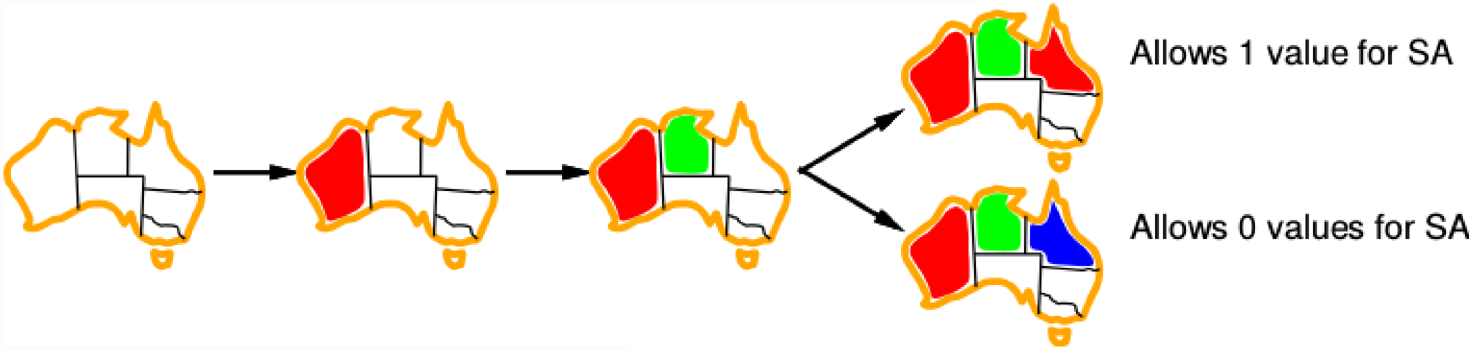
Variable ordering

- Minimum remaining values: Choose the variable with the fewest legal values left in its domain.
- Also known as the fail-first heuristic.
    - Detecting failures quickly is equivalent to pruning large parts of the search tree.



Value ordering

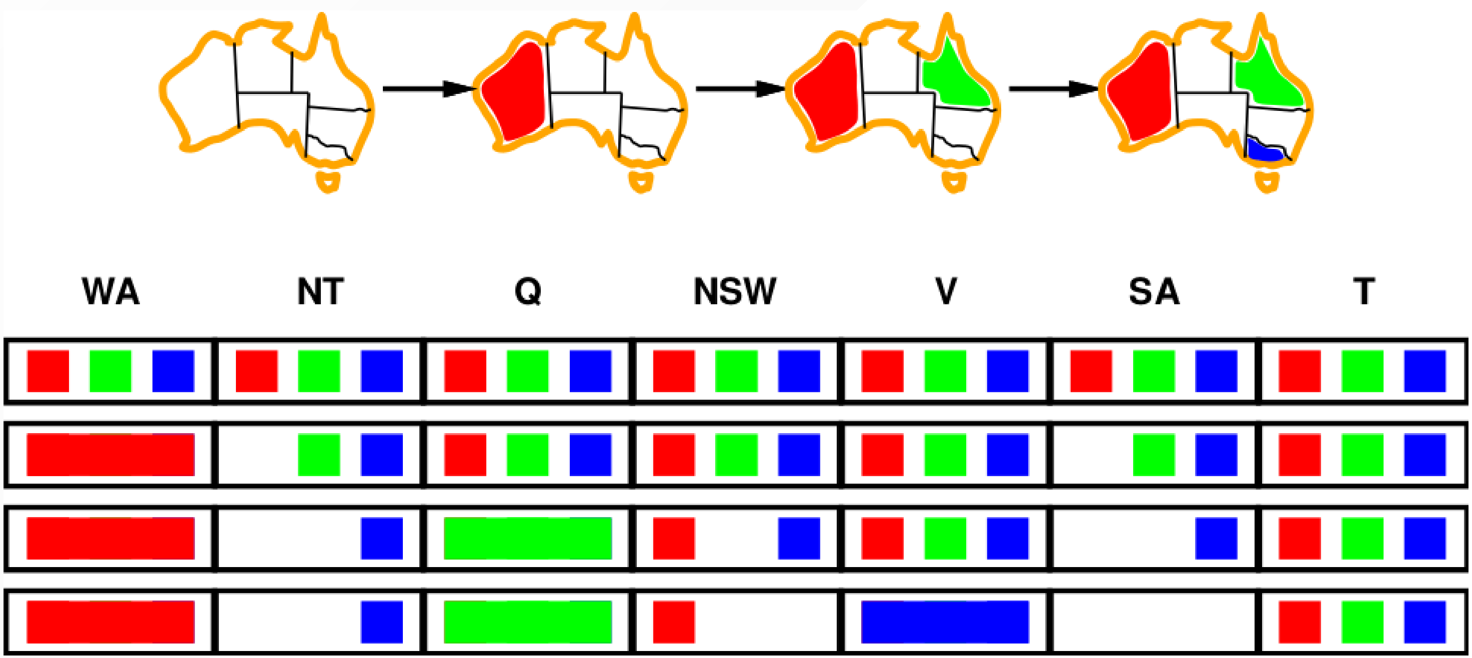
- Least constraining value: Given a choice of variable, choose the least constraining value.
- i.e., the value that rules out the fewest values in the remaining variables.



[Q] Why should variable selection be fail-first but value selection be fail-last?

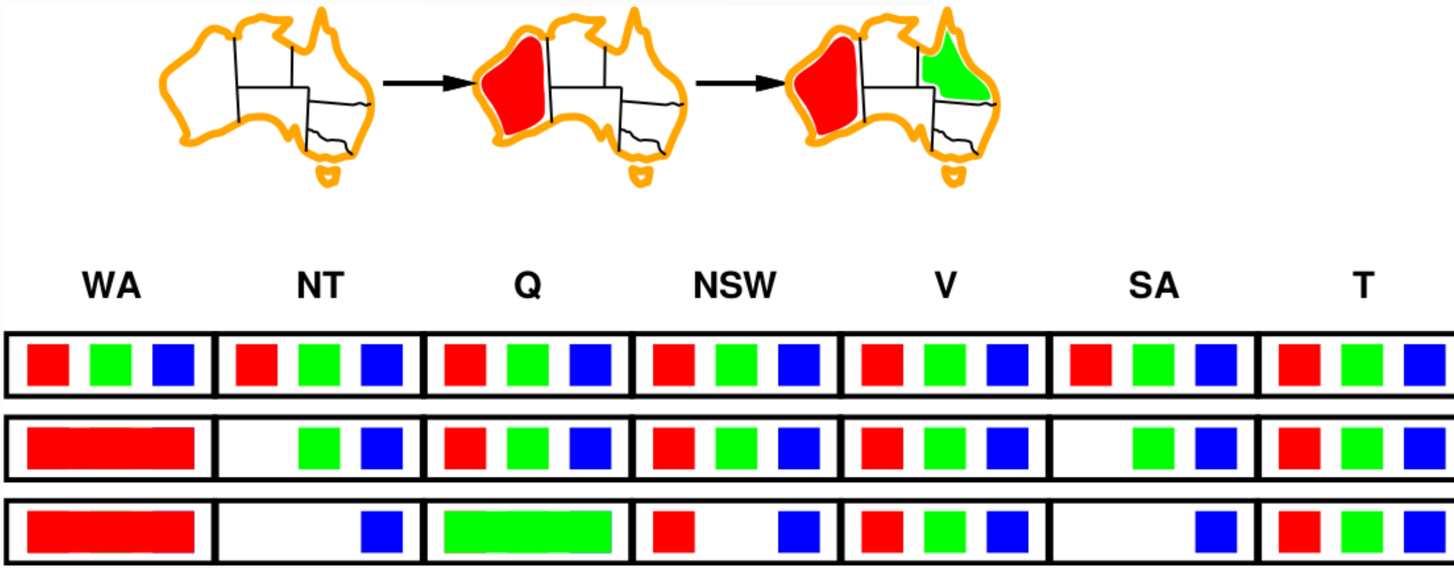
Filtering: Forward checking

- Keep track of remaining legal values for unassigned variables.
  - Whenever a variable X is assigned, and for each unassigned variable Y that is connected to X by a constraint, delete from Y's domain any value that is inconsistent.
- Terminate search when any variable has no legal value left.



Filtering: Constraint propagation

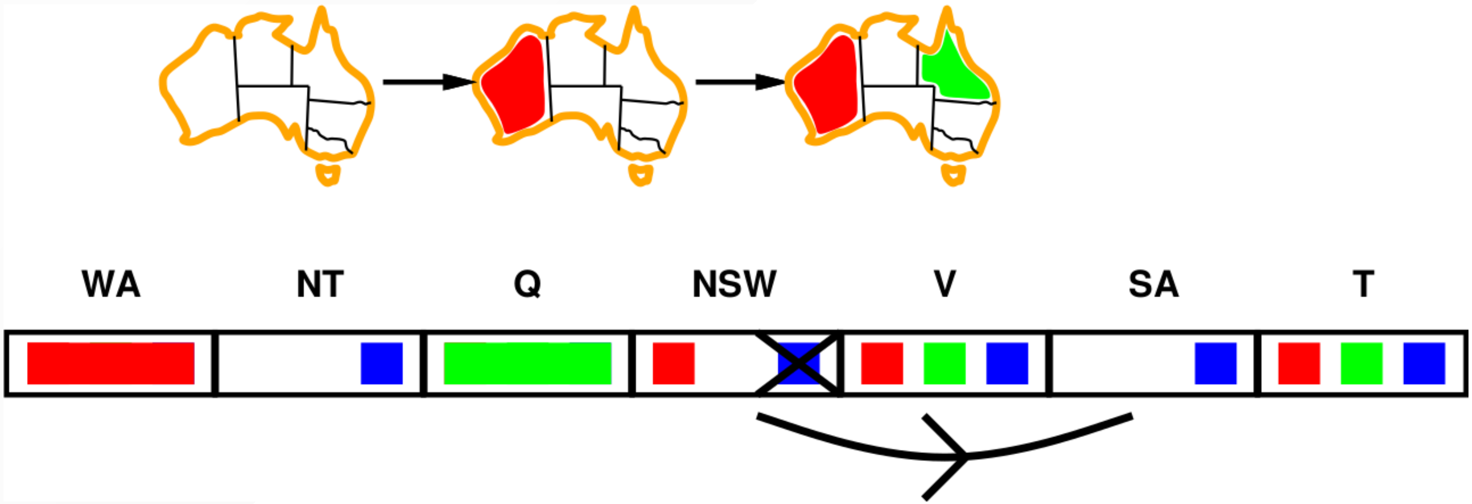
Forward checking propagates information assigned to unassigned variables, but does not provide early deteciton for all failures:



- NT and SA cannot both be blue!
- Constraint propagation repeatedly enforces constraints locally.

Arc consistency

- An arc $X \rightarrow Y$ is consistent if and only if for every value x in the domain of X there is some value y in the domain of Y that satisfies the associated binary constraint.
- Forward checking $\Leftrightarrow$ enforcing consistency of arcs pointing to each new assignment.
- This principle can be generalized to enforce consistency for all arcs.



Arc consistency algorithm

function AC-3(*csp*) **returns** false if an inconsistency is found and true otherwise
  inputs: *csp*, a binary CSP with components $(X,\ D,\ C)$
  local variables: *queue*, a queue of arcs, initially all the arcs in *csp*

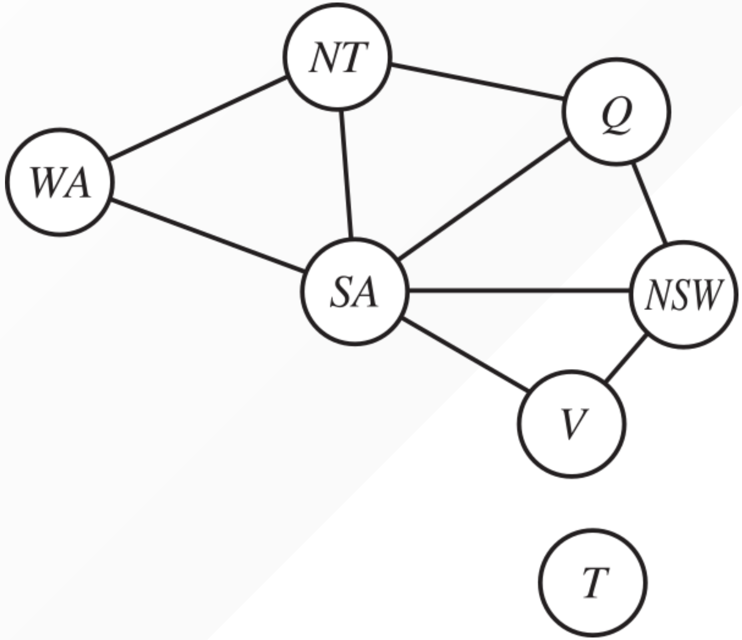
  while *queue* is not empty **do**
    $(X_i,\ X_j) \leftarrow$ REMOVE-FIRST(*queue*)
    if REVISE(*csp*, $X_i,\ X_j$) **then**
      if size of $D_i = 0$ **then return** *false*
      for each X_k **in** X_i.NEIGHBORS - $\{X_j\}$ **do**
        add $(X_k,\ X_i)$ to *queue*
  return *true*

---

function REVISE(*csp*, $X_i,\ X_j$) **returns** true iff we revise the domain of X_i
  revised $\leftarrow$ *false*
  for each x **in** D_i **do**
    if no value y in D_j allows (x,y) to satisfy the constraint between X_i and X_j **then**
      delete x from D_i
      revised $\leftarrow$ *true*
  return *revised*

[Q] When in backtracking shall this procedure be called?

Structure (1)

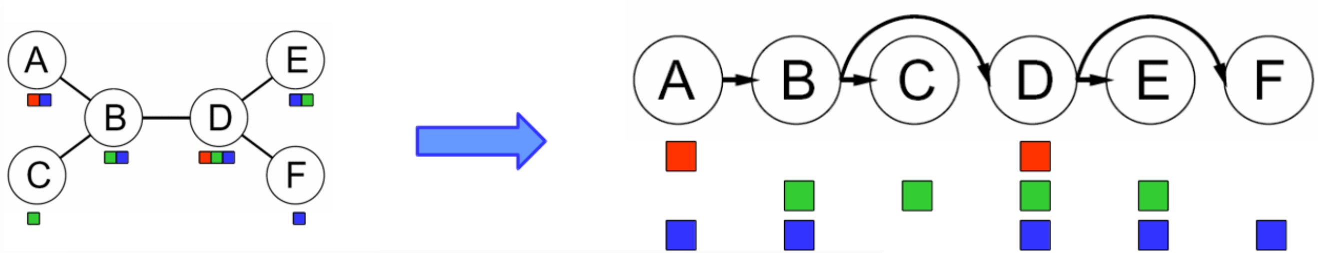


- Tasmania and mainland are independent subproblems.
    - Any solution for the mainland combined with any solution for Tasmania yields a solution for the whole map.
- Independence can be ascertained by finding connected components of the constraint graph.

Structure (2)

- Time complexity: Assume each subproblem has c variables out of n in total. Then $O(\frac{n}{c}d^c)$.
  - E.g., $n = 80, d = 2, c = 20$.
  - 2^{80} = 4 billion years at 10 million nodes/sec.
  - 4×2^{20} = 0.4 seconds at 10 million nodes/sec.

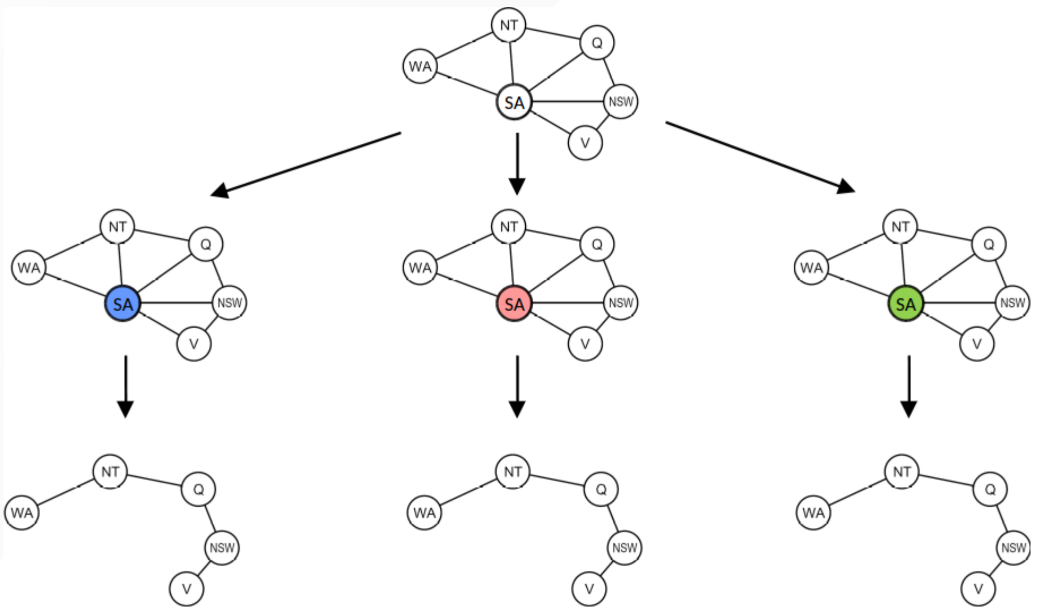
Tree-structured CSPs



- Algorithm for tree-structured CSPs:
  - Order: choose a root variable, order variables so that parents precede children (topological sort).
  - Remove backward:
    - for $i = n$ down to 2, enforce arc consistency of $parent(X_i) \to X_i$.
  - Assign forward:
    - for $i = 1$ to n, assign X_i consistently with its $parent(X_i)$.
- Time complexity: $O(nd^2)$
  - Compare to general CSPs, where worst-case time is $O(d^n)$.

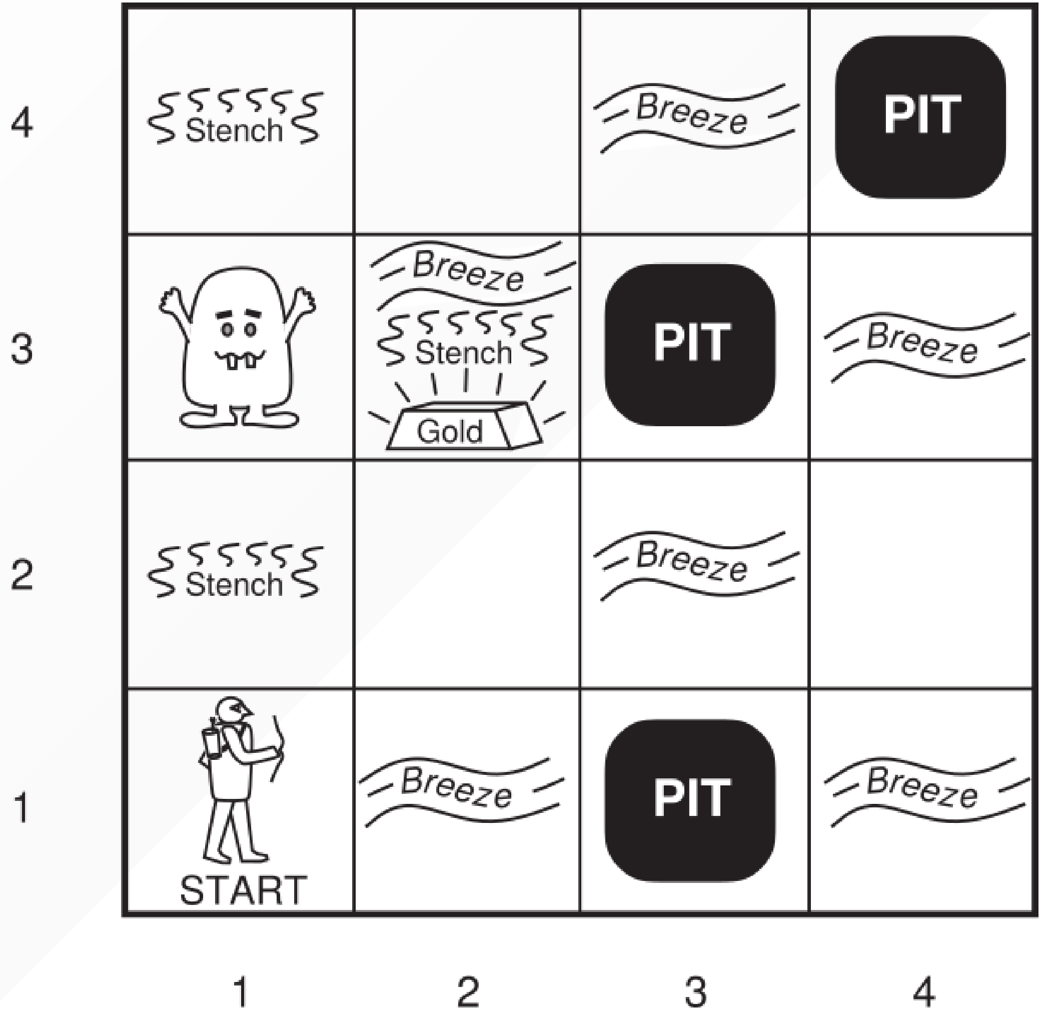
Nearly tree-structured CSPs

- Conditioning: instantiate a variable, prune its neighbors' domains.
- Cutset conditioning:
  - Assign (in all ways) a set S of variables such that the remaining constraint graph is a tree.
  - Solve the residual CSPs (tree-structured).
  - If the residual CSP has a solution, return it together with the assignment for S.



Logical agents

The Wumpus world



PEAS description

- Performance measure:
  - +1000 for climbing out of the cave with gold;
  - -1000 for falling into a pit or being eaten by the wumpus;
  - -1 per step.
- Environment:
  - 4×4 grid of rooms;
  - The agent starts in the lower left square labeled $[1, 1]$, facing right;
  - Locations for gold, the wumpus and pits are chosen randomly from squares other than the start square.
- Actuators:
  - Forward, Turn left by $90°$ or Turn right by $90°$.
- Sensors:
  - Squares adjacent to wumpus are smelly;
  - Squares adjacent to pit are breezy;
  - Glitter if gold is in the same square;
    - Gold is picked up by reflex, and cannot be dropped.
  - You bump if you walk into a wall.
  - The agent program with receives the percept $[Stench, Breeze, Glitter, Bump]$.

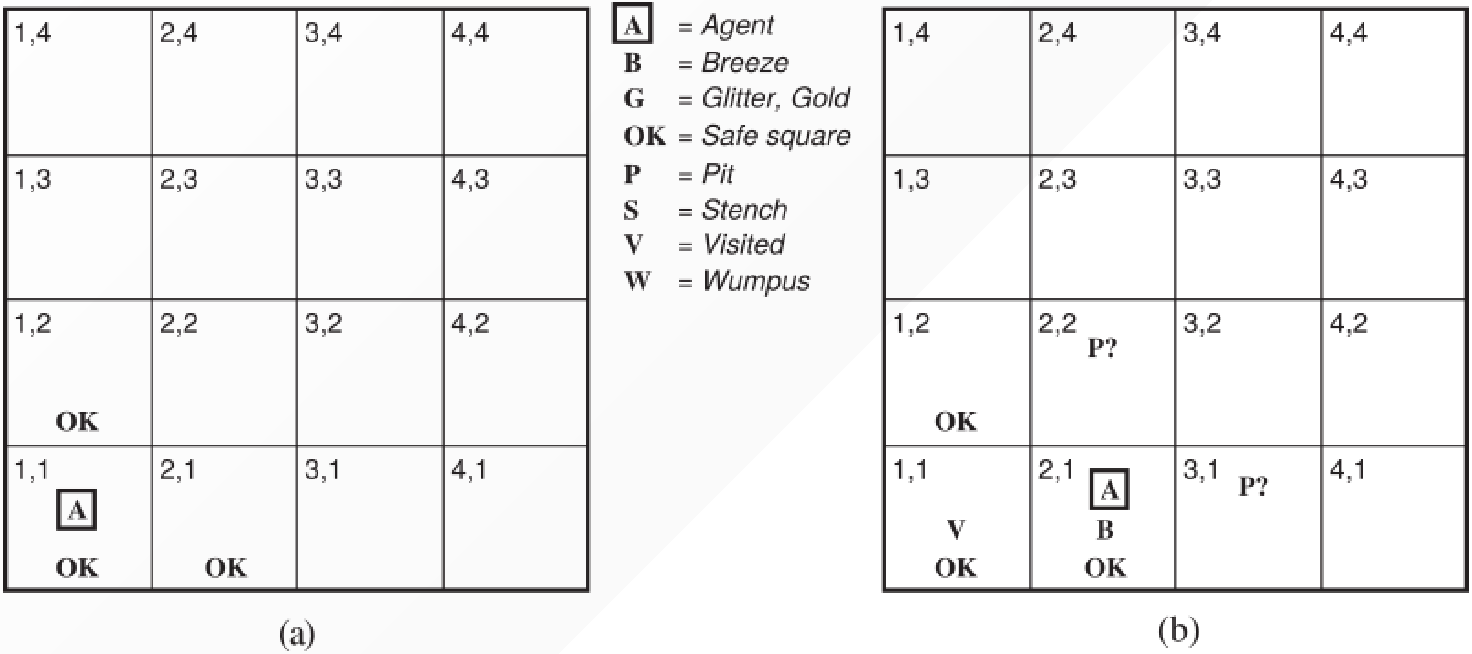
Wumpus world characterization

- Deterministic: Yes, outcomes are exactly specified.
- Static: Yes, Wumpus and pits dot not move.
- Discrete: Yes.
- Single-agent: Yes, Wumpus is essential a natural feature.
- Fully observable: No, only local perception.
- Episodic: No, what was observed before is very useful.

The agent need to maintain a model of the world and to update this model upon percepts.

We will use logical reasoning to overcome the initial ignorance of the agent.

Exploring the Wumpus world (1)



(a) Percept = $[None, None, None, None]$

(b) Percept = $[None, Breeze, None, None]$

Exploring the Wumpus world (2)

(a) (b)

(a) Percept = $[Stench, None, None, None]$

(b) Percept = $[Stench, Breeze, Glitter, None]$

Logical agents

- Most useful in non-episodic, partially observable environments.
- Logic (knowledge-based) agents combine:
  - A knowledge base (KB): a list of facts that are known to the agent.
  - Current percepts.
- Hidden aspects of the current state are inferred using rules of inference.
- Logic provides a good formal language for both
  - Facts encoded as axioms.
  - Rules of inference.

Propositional logic: Syntax

The syntax of propositional logic defines allowable sentences.

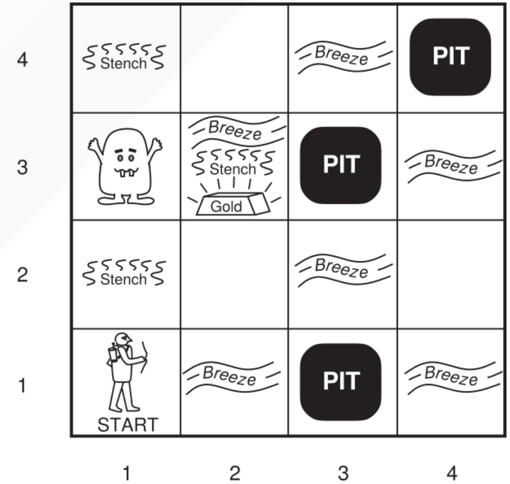
$$
\begin{array}{rcl}
\mathit{Sentence} & \rightarrow & \mathit{AtomicSentence} \mid \mathit{ComplexSentence} \\
\mathit{AtomicSentence} & \rightarrow & \mathit{True} \mid \mathit{False} \mid P \mid Q \mid R \mid \ldots \\
\mathit{ComplexSentence} & \rightarrow & (\ \mathit{Sentence}\ ) \mid [\ \mathit{Sentence}\ ] \\
 & \mid & \neg\ \mathit{Sentence} \\
 & \mid & \mathit{Sentence} \land \mathit{Sentence} \\
 & \mid & \mathit{Sentence} \lor \mathit{Sentence} \\
 & \mid & \mathit{Sentence} \ \Rightarrow\ \mathit{Sentence} \\
 & \mid & \mathit{Sentence} \ \Leftrightarrow\ \mathit{Sentence} \\
 & & \\
\text{OPERATOR PRECEDENCE} & : & \neg, \land, \lor, \Rightarrow, \Leftrightarrow
\end{array}
$$

Propositional logic: Semantics

- In propositional logic, a model is an assignment of truth values for every proposition symbol.
  - E.g., if the sentences of the knowledge base make use of the symbols P_1, P_2 and P_3, then one possible model is $m = \{P_1 = false, P_2 = true, P_3 = true\}$.
- The semantics for propositional logic specifies how to (recursively) evaluate the truth value of any complex sentence, with respect to a model m, as follows:
  - The truth value of a proposition symbol is specified in m.
  - $\neg P$ is true iff P is false;
  - $P \wedge Q$ is true iff P and Q are true;
  - $P \vee Q$ is true iff either P or Q is true;
  - $P \Rightarrow Q$ is true unless P is true and Q is false;
  - $P \Leftrightarrow Q$ is true iff P and Q are both true of both false.

Wumpus world sentences

- Let $P_{i,j}$ be true if there is a pit in $[i, j]$.
- Let $B_{i,j}$ be true if there is a breeze in $[i, j]$.



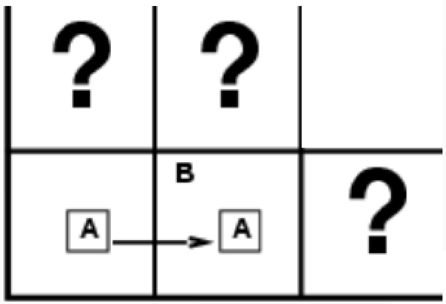
Examples:

- Start: $\neg P_{1,1}, \neg B_{1,1}, B_{2,1}$
- Pits cause breezes in adjacent squares:
  - $B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})$
  - $B_{2,1} \Leftrightarrow (P_{1,1} \vee P_{2,2} \vee P_{3,1})$

Entailment

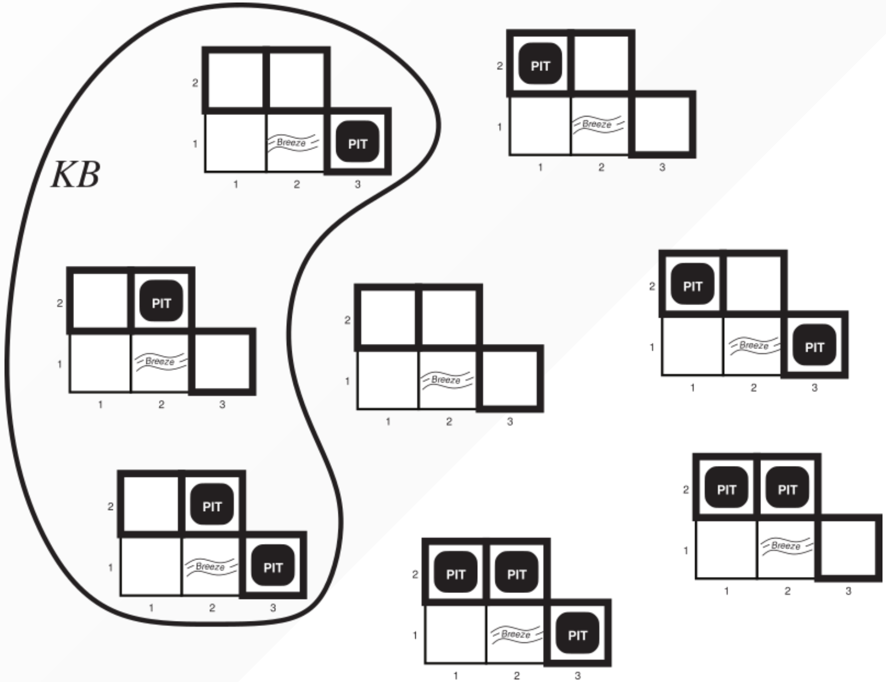
- We say a model m satisfies a sentence α if α is true in m.
  - $M(\alpha)$ is the set of all models of α.
- $\alpha \models \beta$ iff $M(\alpha) \subseteq M(\beta)$.
  - We say that the sentence α entails the sentence β.
  - β is true in all models where α is true.
  - That is, β follows logically from α.

Wumpus models (1)



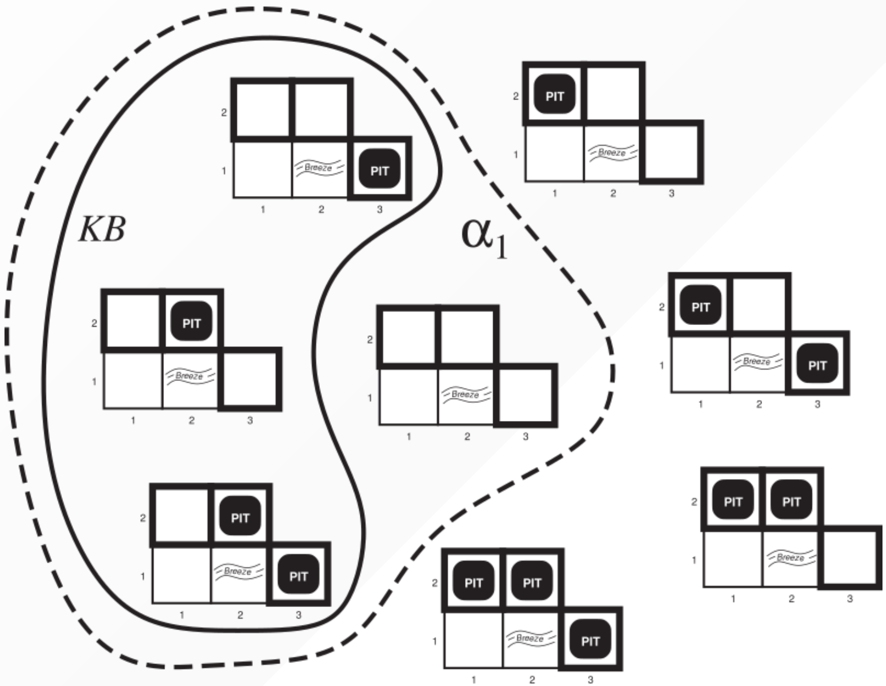
- Let consider possible models for KB assuming only pits and a reduced Wumpus world with only 5 squares and pits.
- Situation after:
  - detecting nothing in $[1, 1]$,
  - moving right, breeze in $[2, 1]$.

Wumpus models (2)



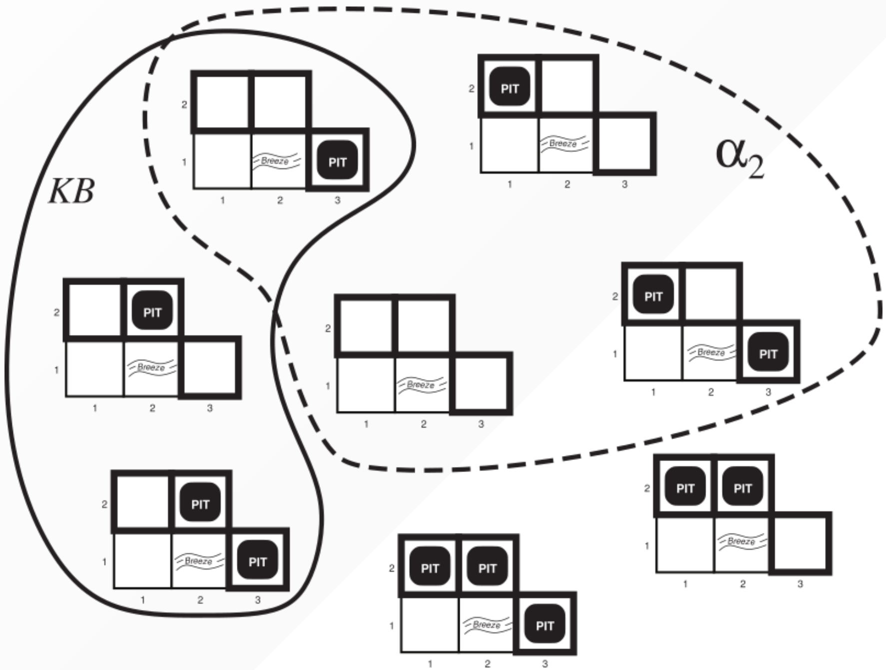
- All 8 possible models in the reduced Wumpus world.
- The knowledge base KB contains all possible Wumpus worlds consistent with the observations and the physics of the world.

Entailments (1)



- α_1 = "$[1, 2]$ is safe". Does KB entails α_1?
- $KB \vDash \alpha_1$ since $M(KB) \subseteq M(\alpha_1)$.
  - This proof is called model checking because it enumerates all possible models to check whether α_1 is true in all models where KB is true.
- Entailment can be used to carry out logical inference.

Entailments (2)



- α_2 = "$[2,2]$ is safe". Does KB entails α_2?
- $KB \not\models \alpha_2$ since $M(KB) \not\subseteq M(\alpha_2)$.
- We cannot conclude whether $[2,2]$ is safe (it may or may not).

Unsatisfiability theorem

$$\alpha \models \beta \text{ iff } (\alpha \land \neg\beta) \text{ is unsatisfiable}$$

- α is unsatisfiable iff $M(\alpha) = \{\}$.
    - i.e., there is no assignment of truth values such that α is true.
- Proving $\alpha \models \beta$ by checking the unsatisfiability of $\alpha \land \neg\beta$ corresponds to the proof technique of reductio ad absurdum.
- Checking the satisfiability of a sentence α can be cast as CSP!
    - More efficient than enumerating all models.
    - But remains NP-complete.
    - See also SAT solvers, tailored for this specific problem.

Summary

- Constraint satisfaction problems:
  - States are represented by a set of variable/value pairs.
  - Backtracking, a form of depth-first search, is commonly used for solving CSPs.
  - The complexity of solving a CSP is strongly related to the structure of its constraint graph.
- Logical agents:
  - Intelligent agents need knowledge about the world in order to reach good decisions.
  - Logical inference can be used as tool to reason about the world.
    - The inference problem can be cast as the problem of determining the unsatisfiability of a formula.
    - This in turn can be cast as a CSP.