Topoi and Internal Logic in Algebraic Geometry

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1 Introduction

By design, scheme theory shares close ties with (commutative) algebra, and one of the most powerful tools available in algebraic geometry is the ability to formally exploit this analogy. Many results in algebraic geometry essentially consist of some argument to reduce a problem to the affine case, and then use some formal correspondence between affine geometry and commutative algebra to proceed in a completely algebraic fashion.

Topoi, an especially nice class of categories which enjoy most of the same properties as the category of sets, allow us to carry the analogy between schemes theory and commutative algebra further. Within every topos there exists a kind of "internal logic," which mirrors the first-order logic (up to a small caveat, which we elaborate on in Remark 3.20) upon which most of classical mathematics is built. In this paper, we will explore how the internal logic of sheaf topoi, and especially those associated to schemes, can be used to translate between algebraic geometry and commutative algebra, and does away with much of the sophistication of working directly with sheaves.

After briefly outlining a few notational conventions in Section 2, we will describe the general machinery of elementary topoi (Section 3.1), their internal first-order languages (Section 3.2), and the semantic system that bridges syntax and content (Section 3.3). (The reader disinterested in the general categorical technology underpinning the internal logic of sheaves discussed in subsequent sections can comfortably skip Section 3, and simply take Theorem 4.5 and Theorem 4.6 as definitions as opposed to theorems.) We then go on to describe how the mechanisms of Section 3 can be interpreted in the "Little Zariski Topos," the case most relevant to us. Finally, in Section 5, we will apply the internal logic of the Little Zariski Topos to establish a dictionary between concepts in scheme theory and their algebraic siblings, and then use this to prove a handful of familiar results using topos-theoretic techniques.

2 Notation

We will write **Set** to denote the category of sets. For any category $\mathcal{C}$, we will denote by よ the (covariant) Yoneda embedding $\mathcal{C} \hookrightarrow \mathbf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set})$, and we will write $\mathcal{C}(X, Y)$ for the set[1] of morphisms $X \to Y$ in $\mathcal{C}$.

Additionally, given a topological space X, we will write $\mathbf{Sh}(X)$ for the category of sheaves on X, and $\mathrm{Open}(X)$ for the poset category of open subsets of X, ordered by inclusion. For sheaves $\mathscr{F}, \mathscr{G} \in \mathbf{Sh}(X)$, we will use $\mathscr{H}\!om(\mathscr{F}, \mathscr{G})$ to denote the internal hom from $\mathscr{F}$ to $\mathscr{G}$; that is, given an open set $U \subseteq X$, we have

$$\mathscr{H}\!om(\mathscr{F}, \mathscr{G})(U) := \mathbf{Sh}(U)(\mathscr{F}|_U, \mathscr{G}|_U).$$

Finally, we denote the sheafification functor by $(-)^{\mathrm{sh}}$.

3 Topoi and Internal Logic

In this section we lay the foundation for studying the internal logic of topoi. After defining elementary topoi in Section 3.1, we will construct their first-order language in Section 3.2, and conclude by establishing the semantic system that governs it in Section 3.3.

3.1 Elementary Topoi

Definition 3.1. Let $\mathcal{C}$ be a finitely complete category with terminal object $*$. A *subobject classifier* is a monomorphism $\top : * \hookrightarrow \Omega$ such that for each monomorphism $U \hookrightarrow X$ there exists a unique morphism $\chi_U : X \to \Omega$ making the square

$$\begin{tikzcd} U \arrow[r] \arrow[d, hook] & * \arrow[d, hook, "\top"] \\ X \arrow[r, "\chi_U"'] & \Omega \end{tikzcd}$$

commutative and cartesian (that is, such that it is a pullback square in $\mathcal{C}$). We will frequently abuse notation by referring to the codomain Ω as the subobject classifier, when this does not lead to ambiguity.

The notation for the induced map χ_U should remind the reader of the characteristic function of a subset $U \subseteq X$. This is no coincidence; indeed, if one takes $\mathcal{C}$ to be the category of sets, then the map $\top : * \hookrightarrow \{0, 1\}$ which sends the point to 1 is a subobject classifier. In this case, given a subset $U \subseteq X$, the map χ_U is precisely the characteristic function of U.

Proposition 3.2. Let $\mathcal{C}$ be a category with a subobject classifier Ω and pullbacks, and let X be an object of $\mathcal{C}$. Then there is a natural bijection

$$\mathrm{Sub}_{\mathcal{C}}(X) \cong \mathcal{C}(X, \Omega)$$

between the set of subobjects of X in $\mathcal{C}$ and the set of morphisms $X \to \Omega$ in $\mathcal{C}$. Since $\mathrm{Sub}_{\mathcal{C}}(X)$ is naturally endowed with a partial ordering, this identification makes $\mathcal{C}(X, \Omega)$ into a poset.

Proof. First, let $\iota : U \hookrightarrow X$ be a monomorphism. Then by definition there exists a unique map $\chi_U : X \to \Omega$ such that

$$\begin{tikzcd} U \arrow[r] \arrow[d, hook, "\iota"'] & * \arrow[d, hook, "\top"] \\ X \arrow[r, dashed, "\chi_U"'] & \Omega \end{tikzcd}$$

is a pullback square. Additionally, given another monomorphism $\kappa : V \hookrightarrow X$ and an isomorphism $U \cong V$ commuting with ι and κ, we find that $\chi_U = \chi_V$. This follows from the fact that the three squares (two inner and one outer) of

$$\begin{tikzcd} V \arrow[r, equal, "\sim"] \arrow[d, hook, "\kappa"'] & U \arrow[r] \arrow[d, hook, "\iota"'] & * \arrow[d, hook, "\top"] \\ X \arrow[r, equal] \arrow[rr, dashed, bend right, "\chi_V"'] & X \arrow[r, dashed, "\chi_U"] & \Omega \end{tikzcd}$$

[1] All of our categories will be locally small, so there are no size issues to be wary of.

commute, so $\chi_V\kappa$ is equal to the outer top right composite by the commutativity of the outer square, which is equal to $\chi_U\kappa$ by the fact that the left and right squares commute. Since κ is a monomorphism, the claim follows.

The above shows that the map $\mathrm{Sub}_{\mathcal{C}}(X) \to \mathcal{C}(X,\Omega)$ given by sending the subobject represented by ι to χ_U is well-defined. Conversely, given $f : X \to \Omega$, the pullback along $\top$ yields a monomorphism ρ_f into X, and it is straightforward to verify that the map $\mathcal{C}(X,\Omega) \to \mathrm{Sub}_{\mathcal{C}}(X)$ which sends f to the subobject represented by ρ_f defines an inverse to the preceding map. □

Definition 3.3. An *elementary topos* is a category $\mathcal{E}$ with the following properties:

(1) $\mathcal{E}$ is finitely complete.

(2) $\mathcal{E}$ is cartesian closed.

(3) $\mathcal{E}$ has a subobject classifier.

We will subsequently refer to elementary topoi merely as "topoi," since for our purposes the notion of an elementary topos will suffice. In other contexts, one often also speaks of "Grothendieck topoi," which are a strengthening of the definition above, so in the literature it is necessary to make a distinction between the two.

Example 3.4. The category of sets is a topos. More generally, if X is a topological space, then the category $\mathbf{Sh}(X)$ of sheaves on X forms a topos. It is easy to show that $\mathbf{Sh}(X)$ satisfies the first two properties above, while the subobject classifier is the sheaf Ω which carries an open subset $U \subseteq X$ to the set of open subsets of U (equipped with the subspace topology).

Throughout the remainder of this section we fix a topos $\mathcal{E}$, with internal hom $[-,-]$, terminal object $*$, and subobject classifier $\top : * \to \Omega$. The remaining results within this subsection will assist us in Section 3.2 in building the first-order language of topoi.

Lemma 3.5. Given an object X of $\mathcal{E}$, there is a canonical map $\delta_X : X \times X \to \Omega$.

Proof. We first observe that the diagonal map $\Delta : X \to X \times X$ is a monomorphism, for given a pair of maps $f, g : Y \to X$ with $\Delta f = \Delta g$, we have

$$f = \pi_1 \Delta f = \pi_1 \Delta g = g,$$

where $\pi_1 : X \times X \to X$ is the projection onto the first factor. Thus, we obtain the desired map δ_X as the characteristic morphism of Δ:

$$\begin{tikzcd} X \arrow[r] \arrow[d, hook, "\Delta"'] & * \arrow[d, "\top"] \\ X \times X \arrow[r, dashed, "\delta_X"'] & \Omega \end{tikzcd}$$

□

The map δ_X can be thought of as identifying those pairs (x, y) in $X \times X$ with $x = y$; that is, it picks out the diagonal of $X \times X$.

The structure of the subobject classifier Ω is central to the internal logic of a topos. In particular, Ω is equipped with maps that behave like the connectives of propositional logic:

Proposition 3.6. The subobject classifier[2] Ω of $\mathcal{E}$ has the structure of an *internal Heyting algebra* (see section IV.8 of [1]). In particular, there exist morphisms

$$\wedge, \vee, \Rightarrow : \Omega \times \Omega \to \Omega$$
$$\top, \bot : * \to \Omega$$

(with $\top$ given by the subobject classifier) such that $\wedge$ and $\vee$ satisfy the axioms of a lattice (as the *meet* and *join*, respectively), and making the diagrams below commute:

$$\begin{tikzcd} \Omega \times * \arrow[r, equal, "\sim"] \arrow[d, "\mathrm{id}_\Omega \times \top"'] & \Omega \arrow[d, equal] & \Omega \times * \arrow[l, equal, "\sim"'] \arrow[d, "\mathrm{id}_\Omega \times \bot"] \\ \Omega \times \Omega \arrow[r, "\wedge"'] & \Omega & \Omega \times \Omega \arrow[l, "\vee"] \end{tikzcd} \qquad \begin{tikzcd} \Omega \times \Omega \times \Omega \arrow[r, "\mathrm{id}_\Omega \times \wedge"] \arrow[d, "\mathrm{id}_\Omega \times \tau \times \mathrm{id}_\Omega"'] & \Omega \times \Omega \arrow[r, "\Rightarrow"] & \Omega \\ \Omega \times \Omega \times \Omega \times \Omega \arrow[r, "\mathrm{id}_\Omega \times \tau \times \mathrm{id}_\Omega"'] & \Omega \times \Omega \times \Omega \times \Omega \arrow[r, "\Rightarrow \times \Rightarrow"'] & \Omega \times \Omega \arrow[u, "\wedge"'] \end{tikzcd}$$

[2]More precisely, the underlying object of the subobject classifier $\top : * \hookrightarrow \Omega$.

$$
\begin{tikzcd}
\Omega \arrow[r] \arrow[d, "\Delta"'] & * \arrow[d, "\top"] \\
\Omega \times \Omega \arrow[r, "\Rightarrow"'] & \Omega
\end{tikzcd}
\qquad
\begin{tikzcd}
\Omega \times \Omega \arrow[r, "\wedge"] \arrow[d, "\Delta \times \mathrm{id}_\Omega"'] & \Omega \\
\Omega \times \Omega \times \Omega \arrow[r, "\mathrm{id}_\Omega \times \Rightarrow"'] & \Omega \times \Omega \arrow[u, "\wedge"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\Omega \times \Omega \arrow[r, "\pi_2"] \arrow[d, "\mathrm{id}_\Omega \times \Delta"'] & \Omega \\
\Omega \times \Omega \times \Omega \arrow[r, "\mathrm{id}_\Omega \times \Rightarrow"'] & \Omega \times \Omega \arrow[u, "\wedge"']
\end{tikzcd}
$$

where π_2 is the projection onto the second factor and τ is the transposition map. We additionally define an endomorphism $\neg : \Omega \to \Omega$ to be the composite

$$\Omega \cong \Omega \times * \xrightarrow{\mathrm{id}_\Omega \times \perp} \Omega \times \Omega \xrightarrow{\Rightarrow} \Omega,$$

which makes

$$
\begin{tikzcd}
* \arrow[r, "\top"] \arrow[dr, "\perp"'] & \Omega \arrow[d, "\neg"] \\
& \Omega
\end{tikzcd}
\qquad
\begin{tikzcd}
\Omega & \\
\Omega \arrow[u, "\neg"] & * \arrow[l, "\perp"] \arrow[ul, "\top"']
\end{tikzcd}
$$

commute.

Proof. The construction of the morphisms above can be found in the proof of Theorem 1 (Internal) of section IV.8 of [1]. □

The Heyting algebra structure of Ω will be used in the sequel to define logical connectives, mirroring those of classical logic. To complete our first-order language, described in Section 3.2, we will need some suitable notion of quantifiers as well, which we lay the foundation for below.

We recall that in **Set** the subobject classifier $\Omega = \{0, 1\}$ represents the contravariant power set functor $\mathcal{P}(-) : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Set}$, that is, for a set X there is a natural bijection $\mathcal{P}(X) \cong \mathbf{Set}(X, \Omega)$. Viewing this as an internal hom object in the category of sets makes it clear how to organically extend this definition to any topos:

Definition 3.7. The *power object* functor $\mathcal{P} : \mathcal{E}^{\mathrm{op}} \to \mathcal{E}$ is given by

$$\mathcal{P}(-) := [-, \Omega].$$

Definition 3.8. Let X and Y be objects of $\mathcal{E}$, and let $\varphi : \mathcal{P}X \to \mathcal{P}Y$ and $\psi : \mathcal{P}Y \to \mathcal{P}X$ be morphisms in $\mathcal{E}$. Then φ is *internally left adjoint* to ψ (and conversely, ψ is *internally right adjoint* to φ) if, for each object $A \in \mathcal{E}$, the induced functors of poset categories

$$
\begin{tikzcd}
\mathcal{E}(A, \mathcal{P}X) \arrow[r, shift left, "\varphi_*"] & \mathcal{E}(A, \mathcal{P}Y) \arrow[l, shift left, "\psi_*"]
\end{tikzcd}
$$

form an adjoint pair. Here the partial ordering on the external hom sets of $\mathcal{E}$ is given by Proposition 3.2.

Proposition 3.9. Let $f : X \to Y$ be a morphism in $\mathcal{E}$. Then the map $\mathcal{P}f : \mathcal{P}Y \to \mathcal{P}X$ has internal left and right adjoints $\exists_f, \forall_f : \mathcal{P}X \to \mathcal{P}Y$, respectively.

Proof. See the proof of Theorem 1.5 in [2]. □

3.2 The Mitchell-Bénabou Language

We have now established the necessary groundwork to define the internal language of a topos, called the *Mitchell-Bénabou language*. Throughout the remainder of this section we fix a topos $\mathcal{E}$, with internal hom $[-, -]$, terminal object $*$, and subobject classifier $\top : * \to \Omega$.

Definition 3.10. A *type* in the this language is an object of $\mathcal{E}$.

A type serves as a domain over which "terms" can be quantified. Terms, defined below, will play an analogous role as they do in first-order logic – namely, a term will stand in for an "element" in a formula (although the exact meaning of "element" must be adapted since we are no longer working in the category of sets). Each term has an interpretation as a morphism in $\mathcal{E}$, and the codomain of this morphism, viewed as a term, is its *type*.

Definition 3.11. A *term* in this language is defined recursively:

(1) A *variable* of type X is a term whose interpretation is the identity map $\mathrm{id}_X : X \to X$.

(2) Suppose given terms $\sigma : U \to X$ and $\tau : V \to Y$ of types X and Y, respectively. Then there is a term $\langle \sigma, \tau \rangle$ of type $X \times Y$, with interpretation $\sigma \times \tau : U \times V \to X \times Y$.

(3) Suppose given terms $\sigma : U \to X$ and $\theta : W \to X$ of the same type X. Then there is a term $\sigma = \theta$ of type Σ, with interpretation given by the composite

$$(\sigma = \theta) : U \times V \xrightarrow{\sigma \times \theta} X \times X \xrightarrow{\delta_X} \Omega,$$

where δ_X is the morphism of Lemma 3.5

(4) Suppose given a term $\sigma : U \to X$ of type X, together with a morphism $f : X \to Y$. Then there is a term $f\sigma$ of type Y with interpretation given by the composite

$$f\sigma : U \xrightarrow{\sigma} X \xrightarrow{f} Y.$$

(5) Suppose given terms $\sigma : U \to X$ and $\omega : T \to [X, Y]$ of types X and $[X, Y]$, respectively. Then there is a term $\omega(\sigma)$ of type Y with interpretation given by the composite

$$\omega(\sigma) : T \times U \xrightarrow{\omega \times \sigma} [X, Y] \times X \xrightarrow{\mathrm{ev}} Y,$$

where $\mathrm{ev} : [X, Y] \times X \to Y$ is the transpose of the identity map on $[X, Y]$. When $Y = \Omega$ we may also denote the term $\omega(\sigma)$ by $\sigma \in \omega$.

(6) Suppose given a variable x of type X and a term $\mu : X \times S \to Z$ of type Z. Then there is a term $\lambda x.\mu(x)$ of type $[X, Z]$ with interpretation given by the transpose $S \to [X, Z]$ of μ.

Definition 3.12. A term of type Ω is a *formula.*

Using the internal Heyting algebra structure of Ω that we exhibited in Proposition 3.6, we can define the ordinary logical connectives on formulas in this language. In particular, suppose given formulas $\varphi : U \to \Omega$ and $\psi : V \to \Omega$. Then we define formulas

$$\begin{aligned} \varphi \wedge \psi : U \times V &\xrightarrow{\varphi \times \psi} \Omega \times \Omega \xrightarrow{\wedge} \Omega \\ \varphi \vee \psi : U \times V &\xrightarrow{\varphi \times \psi} \Omega \times \Omega \xrightarrow{\vee} \Omega \\ \varphi \Rightarrow \psi : U \times V &\xrightarrow{\varphi \times \psi} \Omega \times \Omega \xrightarrow{\Rightarrow} \Omega \\ \neg \varphi : U &\xrightarrow{\varphi} \Omega \xrightarrow{\neg} \Omega \end{aligned}$$

To complete our language, we need to define quantifiers. These should take as input a formula $\varphi(x)$ with a free variable x of type X, and yield a new formula with all the same free variables at φ except x. For this, we recall the internal adjunctions

$$\begin{tikzcd} \mathcal{E}(-, \mathcal{P}X) \arrow[r, shift left=2, "(\exists_p)_*"] \arrow[r, shift right=2, "(\forall_p)_*"'] & \mathcal{E}(-, \mathcal{P}*) \arrow[l, "\mathcal{P}p"' description] \end{tikzcd}$$

of Proposition 3.9, with f taken to be the unique morphism $p : X \to *$.

Definition 3.13. Given a formula $\varphi(x) : X \times U \to \Omega$ as above, we define the following formulas:

(1) $(\forall x : X)\varphi(x)$ with interpretation

$$U \xrightarrow{\lambda x.\varphi(x)} [X, \Omega] \xrightarrow{\forall_p} [*, \Omega] \cong \Omega.$$

(2) $(\exists x : X)\varphi(x)$ with interpretation

$$U \xrightarrow{\lambda x.\varphi(x)} [X, \Omega] \xrightarrow{\exists_p} [*, \Omega] \cong \Omega.$$

When the type of x is clear or irrelevant, we will often abbreviate $\forall x : X$ by $\forall x$ (and similarly for $\exists$). We will also occasionally abuse notation and write $\forall x, y : X$ instead of $\forall x : X \forall y : X$. In this case we must exercise caution, as the order in which we quantify over variables will matter in certain cases.

3.3 Joyal-Kripke Semantics

Finally, we conclude this section by assigning meaning to the syntactic system built in Section 3.2. These are called the *Joyal-Kripke semantics*, and provide some general notion of when an "element" can be said to "satisfy" a formula. The exact meaning of the terms "element" and "satisfy" will be made explicit shortly.

Definition 3.14. Let φ be a formula with free variables $x_1, \ldots, x_n$, with x_i of type X_i. Then we define the object

$$\{(x_1, \ldots, x_n) : \varphi(x_1, \ldots, x_n)\} \in \mathcal{E}$$

to be the pullback

$$\begin{tikzcd}
\{(x_1, \ldots, x_n) : \varphi(x_1, \ldots, x_n)\} \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner"] & * \arrow[d, hook, "\top"] \\
X_1 \times \cdots \times X_n \arrow[r, "\varphi(x_1, \ldots, x_n)"'] & \Omega .
\end{tikzcd}$$

As the notation suggests, the object $\{x : \varphi(x)\}$ can be thought of as the subset of X consisting of those "elements" which "satisfy" $\varphi(x)$. In the category of sets, this is exactly the case when we identify an element of a set S with a morphism $* \to S$ from the one-point set. In a general topos, however, morphisms out of the terminal object are not in general sufficient to probe their entire codomain. Instead, we adopt the only sensible generalization of "elements" available to us as category theorists, taking inspiration from the Yoneda lemma, by viewing *all* morphisms into an object as "elements" of that object:

Definition 3.15. Let U and X be objects of $\mathcal{E}$. A *generalized element* of X (with shape U) is a morphism $\alpha : U \to X$ in $\mathcal{E}$.

Definition 3.16. Given a generalized element $\alpha : U \to X$ of X with shape U, together with a formula $\varphi(x)$ with a free variable x of type X, we say that U *forces* $\varphi(\alpha)$, denoted $U \Vdash \varphi(\alpha)$, if α factors through $\{x : \varphi(x)\}$:

$$\begin{tikzcd}
 & \{x : \varphi(x)\} \arrow[r] \arrow[d, hook] & * \arrow[d, hook, "\top"] \\
U \arrow[ur, dashed] \arrow[r, "\alpha"'] & X \arrow[r, "\varphi(x)"'] & \Omega .
\end{tikzcd}$$

More generally, if $\varphi(x_1, \ldots, x_n)$ has free variables $x_1, \ldots, x_n$ with x_i of type X_i, then given generalized elements $\alpha_i : U \to X_i$ for $i = 1, \ldots, n$ we say that U forces $\varphi(\alpha_1, \ldots, \alpha_n)$, written $U \Vdash \varphi(\alpha_1, \ldots, \alpha_n)$, if the induced map $\langle \alpha_1, \ldots, \alpha_n \rangle : U \to X_1 \times \cdots \times X_n$ factors through $\{(x_1, \ldots, x_n) : \varphi(x_1, \ldots, x_n)\}$.

Note that when $\top : * \hookrightarrow \Omega$ is interpreted as the inclusion of "true" into Ω, the condition above can be read as approximating the statement that $\varphi(\alpha)$ is true.

Proposition 3.17 (Monotonicity)**.** Let $\alpha : U \to X$ be a generalized element, let $\varphi(x)$ a formula with a free variable x of type X such that $U \Vdash \varphi(\alpha)$, and let $f : V \to U$ be a morphism in $\mathcal{E}$. Then $V \Vdash \varphi(\alpha f)$.

Proof. By hypothesis, we have a commuting solid diagram

$$\begin{tikzcd}
 & & \{x : \varphi(x)\} \arrow[r] \arrow[d, hook] & * \arrow[d, hook, "\top"] \\
V \arrow[urr, dashed] \arrow[r, "f"'] & U \arrow[ur] \arrow[r, "\alpha"'] & X \arrow[r, "\varphi(x)"'] & \Omega,
\end{tikzcd}$$

and seek to construct the dashed arrow $V \to \{x : \varphi(x)\}$ such that αf factors through it. Since the square on the right is a pullback, it will suffice to show that the bottom horizontal composite is equal to the composite of the unique map $V \to *$ and the monomorphism $\top : * \to \Omega$. But the former is equal to the composite

$$V \xrightarrow{f} U \to \{x : \varphi(x)\} \hookrightarrow X \xrightarrow{\varphi(x)},$$

which is the same as the composite

$$V \xrightarrow{f} U \to \{x : \varphi(x)\} \to * \xrightarrow{\top} \Omega.$$

But $*$ is terminal, so this is the same as the map $V \to * \xrightarrow{\top} \Omega$, as desired, and thus the dashed map above exists by the universal property of the pullback. $\square$

Proposition 3.18 (Locality)**.** Let $\alpha : U \to X$ be a generalized element, let $f : V \twoheadrightarrow U$ be an epimorphism, and let $\varphi(x)$ a formula with a free variable x of type X such that $V \Vdash \varphi(\alpha f)$. Then $U \Vdash \varphi(\alpha)$.

Proof. This follows from an argument similar to Proposition 3.17. □

We would like a way to verify when an object forces a formula on some generalized element which reflects our intuition for how to manipulate formulas classically. In particular, we would like some means of performing induction on formulas, by stripping away each quantifier or connective one-by-one. The following result, which we will later adapt to the particular context of schemes (and modules over them), will give us the means to do this. We state the proposition without proof, although one can be found in section VI.6 of [1]:

Proposition 3.19. Let $\alpha : U \to X$ be a generalized element, and let $\varphi(x)$ and $\psi(x)$ be formulas with a free variable x of type X. Then

(1) $U \Vdash \varphi(\alpha) \wedge \psi(\alpha)$ if and only if $U \Vdash \varphi(\alpha)$ and $U \Vdash \psi(\alpha)$.

(2) $U \Vdash \varphi(\alpha) \vee \psi(\alpha)$ if and only if there exist morphisms $p : V \to U$ and $q : W \to U$ such that $p + q : V \sqcup W \twoheadrightarrow U$ is an epimorphism and both $V \Vdash \varphi(\alpha p)$ and $W \Vdash \psi(\alpha q)$.[3]

(3) $U \Vdash \varphi(\alpha) \Rightarrow \psi(\alpha)$ if and only if for any morphism $p : V \to U$ such that $V \Vdash \varphi(\alpha p)$, we have $V \Vdash \psi(\alpha p)$.

(4) $U \Vdash \neg\varphi(\alpha)$ if and only if for any morphism $p : V \to U$, if $V \Vdash \varphi(\alpha p)$ then $V \cong \varnothing$, the initial object of $\mathcal{E}$.[4]

If $\theta(x, y)$ is a formula with an additional free variable y of type Y, then

(1) $U \Vdash \exists y \theta(\alpha, y)$ if and only if there exists an epimorphism $p : V \twoheadrightarrow U$ and a generalized element $\beta : V \to Y$ such that $V \Vdash \theta(\alpha p, \beta)$.

(2) $U \Vdash \forall y \theta(\alpha, y)$ if and only if for every object V, each morphism $p : V \to U$, and each generalized element $\beta : V \to Y$, we have $V \Vdash \theta(\alpha p, \beta)$.

Remark 3.20. It is at this point we must pause to note a subtle but significant distinction between the Joyal-Kripke semantics and classical first-order logic. Our semantic system is what logicians refer to as "constructive," or more precisely, "intuitionistic." This means, for instance, that we cannot appeal to the law of the excluded middle or double negation elimination; that is, neither of the following must hold in general:

$$\varphi \vee \neg\varphi \qquad \neg\neg\varphi \Rightarrow \varphi.$$

Consequently, we must take care that all of our internal arguments (i.e. formal manipulations of the Mitchell-Bénabou language using Joyal-Kripke semantics) are intuitionistically valid; externally (i.e. reasoning outside of the internal language of a topos), we still allow ourselves to reason classically. See Example 4.8 for an illustration of how these perspectives differ.

4 The Little Zariski Topos

In this section, we will provide an overview of how to interpret (in a slightly more concrete way) the machinery of Section 3 in categories of sheaves on topological spaces (see Example 3.4), or more precisely, in the *Little Zariski Topos* – that is, the topos given by $\mathbf{Sh}(X)$, where X is the underlying space of a scheme. Throughout this section we fix an arbitrary topological space X.

4.1 Sheaf Semantics

Our objective is to apply the internal logic of topoi to algebraic geometry, so naturally we are most concerned with those topoi which occur most often in the context of algebraic geomety: sheaves. To that end, we have the following adaptations of the semantics of forcing defined in Section 3.3, which will become useful when we begin translating first-order statements into the context of scheme theory. For a complete derivation of this result, see Theorem 1 of section VI.7 in [1]; we note that the authors work in the more general context of Grothendieck topologies and sheaves on sites, as opposed to spaces.

[3] It is a theorem that every topos is finitely cocomplete, so the use of the coproduct here is not as reckless as it may seem at first glance.

[4] Again, $\mathcal{E}$ has all finite limits, so an initial object $\varnothing$ is guaranteed to exist.

Definition 4.1. Let $U \subseteq X$ be an open set, and let $\mathscr{F}$ be a sheaf on X. Additionally, suppose given a generalized element $\alpha : \text{よ}(U)^{\mathrm{sh}} \to \mathscr{F}$ and a formula $\varphi(x)$ with a free variable x of type $\mathscr{F}$. We say that U *forces* $\varphi(\alpha)$, denoted $U \Vdash \varphi(\alpha)$, if $\text{よ}(U)^{\mathrm{sh}}$ forces $\varphi(\alpha)$ under Definition 3.16.

Remark 4.2. The main point of restricting the shape of our generalized elements is that we are, in effect, limiting ourselves to quantifying over sections of sheaves. This is made explicit by the Yoneda lemma: by adjunction, a morphism of sheaves $\alpha : \text{よ}(U)^{\mathrm{sh}} \to \mathscr{F}$ is the same as a morphism of presheaves $\text{よ}(U) \to \mathscr{F}$, that is, a natural transformation, and the Yoneda lemma tells us that this is precisely an element of $\mathscr{F}(U)$. In light of this, we will subsequently denote a generalized element α as above by $\alpha \in \mathscr{F}(U)$ (or, more often, by $s \in \mathscr{F}(U)$, to reinforce that we should now be thinking in terms of sections).

We will soon see why limiting generalized elements in this way to ensure that they look like sections of sheaves will help us use sheaf topoi to study algebraic geometry. First, however, we interpret some of the definitions and results of Section 3 in this new framework.

Proposition 4.3. Given a formula φ with interpretation $\mathscr{F} \to \Omega$, the sheaf $\{f : \varphi(f)\}$ is represented by the union of all open subsets $U \subseteq X$ such that $U \Vdash \varphi$. Conversely, given an open subset $U \subseteq X$, we can define a formula φ_U such that for any open $V \subseteq X$ we have $V \Vdash \varphi_U$ if and only if $V \subseteq U$.

Proof. The first claim follows directly from the definitions. For the second, we take $\varphi_U := \chi_U$ in the diagram below:

$$\begin{tikzcd} \text{よ}(U) \arrow[r, "!"] \arrow[d, hook] & * \arrow[d, hook, "\top"] \\ \text{よ}(X) \arrow[r, dashed, "\chi_U"'] & \Omega \end{tikzcd} .$$

If $V \subseteq U$ then clearly the map $\text{よ}(V) \hookrightarrow \text{よ}(X)$ factors through $\text{よ}(U)$. Conversely, if $V \subseteq X$ is any open set with $V \Vdash \varphi_U$, then by definition we have a map $\text{よ}(V) \to \text{よ}(U)$, so by the Yoneda Lemma we obtain an inclusion $V \subseteq U$. $\square$

Under the interpretation of Definition 4.1, the monotonicity result Proposition 3.17 can be rewritten as the following, which shows that forcing is preserved under restriction:

Proposition 4.4 (Sheaf Monotonicity)**.** Let $V \subseteq U \subseteq X$ be open sets. Additionally, let $\varphi(x_1, \ldots, x_n)$ be a formula with free variables $x_1, \ldots, x_n$, where x_i has type $\mathscr{F}_i \in \mathbf{Sh}(X)$. Additionally, let s_i be a section in $\mathscr{F}_i(U)$ for each i. If $U \Vdash \varphi(s_1, \ldots, s_n)$ then $V \Vdash \varphi(s_1|_V, \ldots, s_n|_V)$.

We can also restate the forcing rules of Proposition 3.19, and in fact extend them slightly:

Theorem 4.5. Let $U \subseteq X$ be an open set, let $\mathscr{F}$ be a sheaf on X, and let φ and ψ be formulas. Then

(1) $U \Vdash \varphi \wedge \psi$ if and only if $U \Vdash \varphi$ and $U \Vdash \psi$.

(2) $U \Vdash \varphi \vee \varphi$ if and only if there exists an open cover $U = \bigcup_{i \in I} U_i$ such that for each $i \in I$ we have $U_i \Vdash \varphi$ or $U_i \Vdash \psi$.

(3) $U \Vdash \varphi \Rightarrow \psi$ if and only if for all open $V \subseteq U$, if $V \Vdash \varphi$ then $V \Vdash \psi$.

(4) $U \Vdash \neg\varphi$ if and only if for all open $V \subseteq U$, if $V \Vdash \varphi$ then $V = \varnothing$.

If $\theta(x)$ is a formula with a free variable of type $\mathscr{F} \in \mathbf{Sh}(X)$, then

(1) $U \Vdash (\exists x : \mathscr{F})\theta(x)$ if and only if there exists an open cover $U = \bigcup_{i \in I} U_i$ and sections $s_i \in \mathscr{F}(U_i)$ such that $U_i \Vdash \theta(s_i)$ for each i.

(2) $U \Vdash (\forall x : \mathscr{F})\theta(x)$ if and only if for each open subset $V \subseteq U$ and each section $s \in \mathscr{F}(V)$ we have $V \Vdash \theta(s)$.

We can further extend these rules to dissect atomic formulas, and to handle infinite conjunctions/disjunctions:

Theorem 4.6. Let $U \subseteq X$ be an open set, and let $\mathscr{F}$ be a sheaf on X. Additionally, let s and t be sections in $\mathscr{F}(U)$, and let $\mathscr{G}$ be a subsheaf of $\mathscr{F}$. Then

(1) $U \Vdash (s = t)$ if and only if $s = t$ as elements of the set $\mathscr{F}(U)$.

(2) $U \Vdash (s \in \mathscr{G})$ if and only if s is contained in the set of sections $\mathscr{G}(U)$.

(3) $U \Vdash \top$ is always satisfied.

(4) $U \Vdash \bot$ if and only if U is empty.

Additionally, given a set J and formulas φ_j for each $j \in J$, we have

(1) $U \Vdash \bigwedge_{j\in J}$ if and only if $U \Vdash \varphi_j$ for each $j \in J$.

(2) $U \Vdash \bigvee_{j\in J}$ if and only if there exists some open cover $U = \bigcup_{i\in I} U_i$ such that for each i we have $U_i \Vdash \varphi_j$ for some $j \in J$.

From the above, we can prove our first concrete result about sheaves: that monomorphisms (resp. epimorphisms) in $\mathbf{Sh}(X)$ are precisely the same thing as injective (resp. surjective) maps, defined internally. While this proposition is not tremendously useful to us, it will illustrate how one can use Kripke-Joyal semantics to formulate properties of sheaves in more familiar algebraic terms. We will see much more of this idea in Section 5.

Corollary 4.7. Let $\alpha : \mathscr{F} \to \mathscr{G}$ be a morphism in $\mathbf{Sh}(X)$. Then

$$X \Vdash (\forall x, y : \mathscr{F})(\alpha(x) = \alpha(y) \Rightarrow x = y)$$

if and only if α is a monomorphism, and

$$X \Vdash (\forall y : \mathscr{G} \exists x : \mathscr{F})(\alpha(x) = y)$$

if and only if α is an epimorphism.

Proof. We will prove the first claim, noting that the second follows from a similar, albeit slightly more annoying (owing to the cumbersome nature of epimorphisms in $\mathbf{Sh}(X)$) argument. By Theorem 4.5, we have that

$$X \Vdash (\forall x, y : \mathscr{F})(\alpha(x) = \alpha(y) \Rightarrow x = y)$$

if and only if, for each open $V \subseteq U \subseteq X$ and each pair of sections $s \in \mathscr{F}(U)$ and $t \in \mathscr{F}(V)$, we have

$$V \Vdash \alpha(s|_V) = \alpha(t) \Rightarrow s|_V = t.$$

Applying Theorem 4.5 again, this is equivalent to $W \Vdash s|_W = t|_W$ whenever $W \Vdash \alpha(s|_W) = \alpha(t|_W)$ for each open $W \subseteq V$. Finally, this is equivalent to $s|_W = t|_W$ whenever $\alpha(s|_W) = \alpha(t|_W)$, again by Theorem 4.5.

Now, α is a monomorphism if and only if each component of α is injective. This clearly implies that if $\alpha(s|_W) = \alpha(t|_W)$ then $s|_W = t|_W$. Conversely, if this holds for all U, V, and W, given some open $T \subseteq X$, to show that the T-component of α is injective it suffices to take $U = V = W = T$. □

From Theorem 4.5 and Theorem 4.6 we can also demonstrate a concrete example of how the internal intuitionistic framework we are working within differs from external, classical logic.

Example 4.8. Suppose φ is some formula such that $X \Vdash \neg\varphi$. If X is nonempty, then intuitionistically this is stronger than the statement that X does not force φ. By Theorem 4.5, the former is equivalent to the statement that the only open subset of X which forces φ is the empty set, that is, that φ holds nowhere in X, not just that it does not hold globally (which is what it means for X to fail to force φ).

For instance, if φ were the statement that some global section f of a sheaf $\mathscr{F} \in \mathbf{Sh}(X)$ were invertible, then $X \Vdash \neg\varphi$ implies not only that f is not invertible over X (which is all that would be implied by $X \nVdash \varphi$), but also that it is noninvertible on every open subset of X.

Finally, interpreting Proposition 3.18, we obtain the following, which elucidates the use of the term "locality":

Proposition 4.9 (Sheaf Locality)**.** Let $U \subseteq X$ be an open set, and let $U = \bigcup_{i\in I} U_i$ be an open cover. Additionally, let φ be a formula. Then $U \Vdash \varphi$ if and only if $U_i \Vdash \varphi$ for each $i \in I$.

Note that the forward direction of this result comes from Proposition 4.4, as each U_i is contained in U.

4.2 Geometric Formulas

Stalks play an important role in the study of sheaves: we often care about the behavior of a sheaf "near a point." To that end, we introduce the following definition:

Definition 4.10. Let φ be a formula with terms $x_1, \ldots, x_n$, with x_i of type $\mathscr{F}_i$ for each i. Additionally, let $p \in X$ be a point. Then φ *holds at* p if the first-order formula φ_p obtained by replacing each x_i with its stalk as a term of type $(\mathscr{F}_i)_p$, holds in the usual first-order language.

Example 4.11. To illustrate this definition concretely, given a morphism $\alpha : \mathscr{F} \to \mathscr{G}$ in $\mathbf{Sh}(X)$, consider the formula

$$\varphi = (\forall y : \mathscr{G} \exists x : \mathscr{F})(\alpha(x) = y),$$

which states that α is surjective, or equivalently, an epimorphism (see Corollary 4.7). For φ to hold at $p \in X$ is to say that

$$\varphi_p = (\forall y : \mathscr{G}_p \exists x : \mathscr{F}_p)(\alpha_p(x) = y)$$

holds, that is, that the induced map $\alpha_p : \mathscr{F}_p \to \mathscr{F}_p$ is surjective.

Of particular importance when studying local behavior is a subclass of formulas called *geometric formulas*[5].

Definition 4.12. A formula φ is *geometric* if it only consists of the connectives

$$\exists, \wedge, \vee, \bigvee, \top, \bot, \in, =,$$

but does not contain $\forall$, $\bigwedge$, and $\Rightarrow$. A *geometric implication* is a formula of the form

$$\forall x_1, \ldots, x_n(\psi(x_1, \ldots, x_n) \Rightarrow \theta(x_1, \ldots, x_n))$$

where ψ and θ are geometric.

These formulas enjoy the particularly nice property that they allow for translation between stalks and local sections:

Proposition 4.13. Let φ be a geometric formula, and let $p \in X$ be a point. Then φ holds at p if and only if $U \Vdash \varphi$ for some open neighborhood $U \subseteq X$ of p.

Proof. This can be verified by induction on φ. We will content ourselves with outline the case $\varphi = \psi \wedge \theta$ with ψ and θ geometric, noting that the remaining cases follow via similar arguments. By definition φ_p holds if and only if ψ_p and θ_p both hold, so by induction there exist open neighborhoods U and V of p such that $U \Vdash \psi$ and $V \Vdash \theta$. It follows from Proposition 4.4 that $U \cap V$ forces both ψ and θ, and thus $U \cap V \Vdash \varphi$ by Theorem 4.5. $\square$

This yields the following corollary, which tells us that geometric implications can be checked pointwise:

Corollary 4.14. Let $U \subseteq X$ be an open set, and let φ be a geometric implication. Then $U \Vdash \varphi$ if and only if φ holds at each point $p \in U$.

Proof. In the interest of brevity, we assume without loss of generality that φ is of the form

$$(\forall x : \mathscr{F})(\psi(x) \Rightarrow \theta(x))$$

with ψ and θ geometric. By Theorem 4.5, if $U \Vdash \varphi$ then for each open $V \subseteq U$ and each section $s \in \mathscr{F}(V)$ we have

$$V \Vdash (\psi(s) \Rightarrow \theta(s)),$$

which is equivalent to the statement that $V \Vdash \theta(s)$ whenever $V \Vdash \psi(s)$[6]. Fix $s \in \mathscr{F}(U)$, and let $p \in U$. Then by Proposition 4.13 $\psi_p(s_p)$ (resp. $\theta_p(s_p)$) holds if and only if $V \Vdash \psi(s)$ (resp .$V \Vdash \theta(s)$). Thus we have that

$$\psi_p(s_p) \Rightarrow \theta_p(s_p)$$

for all sections $s \in \mathscr{F}(U)$, and since the map $\mathscr{F}(U) \to \mathscr{F}_p$ is surjective it follows that

$$(\forall x : \mathscr{F}_p)(\psi_p(x) \Rightarrow \theta_p(x))$$

holds. This is precisely φ_p, proving the forward direction.

Conversely, suppose φ_p holds for each $p \in U$. Then by Theorem 4.5 and Proposition 4.13, for each $p \in U$, there exists an open neighborhood $U_p \subseteq U$ of p such that $U_p \Vdash \varphi$. Therefore by Proposition 4.9 we conclude that $U \Vdash \varphi$, proving the backward direction. $\square$

[5] So-named because they are preserved by pulling back along *geometric morphisms*, which are adjoint functor pairs whose left adjoint is finitely continuous.

[6] We do not need to consider open subsets of V here, as V is already an arbitrary open subset of U.

5 The Internal Logic of Schemes

Finally, we have developed enough machinery to begin applying topos theory to schemes. In this section we will use the internal logic of sheaves to establish close analogies between commutative algebra and algebraic geometry, and translating classical proofs in the context of the former to prove familiar results about schemes.

We begin by recalling the axioms of a commutative ring with underlying set R:

(1) R is equipped with operations $+, \cdot : R \times R \to R$ and $- : R \to R$ and elements $0, 1 \in R$.

(2) $+$ and $\cdot$ are associative and commutative.

(3) 0 is a unit for $+$, and 1 is a unit for $\cdot$.

(4) For each $r \in R$ we have $r + (-r) = 0$.

(5) $\cdot$ distributes over $+$.

We can reformulate these axioms using the internal language above, taking as our topos the category of sets. The complete formula capturing each axiom is inordinately long so we omit it, but for illustrative purposes the following would be the axiomatization for $+$, $-$, and 0 satisfying (4):

$$\exists + : [R \times R, R] \exists - : [R, R] \exists 0 : R \forall r : R(+(r, -(r)) = 0).$$

Having translated the ring axioms into the Mitchell-Bénabou language of Section 3.2, we can formulate "rings" internal to any topos. In particular:

Proposition 5.1. Let X be a topological space. Then a ring internal to the topos $\mathbf{Sh}(X)$ of set-valued sheaves on X is precisely the same thing as a ring-valued sheaf.

Furthermore, we have the following characterization of invertible sections/stalks:

Lemma 5.2. Let X be a topological space, let $U \subseteq X$ be an open set, and let $\mathscr{F}$ be a sheaf of rings on X. Then for a global section $s \in \mathscr{F}(X)$, the following are equivalent:

(1) s is invertible in the ring $\mathscr{F}(X)$.

(2) s_x is invertible in the stalk $\mathscr{F}_x$ for each $x \in X$.

(3) s is invertible in $\mathscr{F}$, viewed as a ring internal to $\mathbf{Sh}(X)$. That is, we have

$$X \Vdash \exists t : \mathscr{F}(s \cdot t = 1).$$

Proof. First, we observe that the formula $\varphi(s)$ in (3) is geometric. It follows from Proposition 4.13 that $X \Vdash \varphi(s)$ if and only if $\varphi_x(s_x)$ holds for each $x \in X$. But given x, we have

$$\varphi_x(s_x) = \exists t : \mathscr{F}_x(s_x \cdot t_x = 1),$$

which is precisely the statement that s_x is invertible in the stalk $\mathscr{F}_x$. Thus (2) and (3) are equivalent.

Additionally, it is clear that (1) implies (2). On the other hand, suppose $X \Vdash \varphi(s)$. Then by Theorem 4.5 we have an open cover $X = \bigcup_{i \in I} U_i$ and sections $t_i \in \mathscr{F}(U_i)$ such that

$$U_i \Vdash s|_{U_i} \cdot t_i = 1$$

for each $i \in I$. Moreover, the usual proof of the uniqueness of inverses holds intuitionistically (i.e. in the Joyal-Kripke semantics of Section 3.3), so $t_i|_{U_i \cap U_j} = t_j|_{U_i \cap U_j}$. It follows that the t_i glue to form a global section $t \in \mathscr{F}(U)$, and since $s \cdot t$ restricts to 1 on the open cover $\{U_i\}_{i \in I}$ we conclude that $s \cdot t = 1$, completing the proof by showing that (3) implies (1). $\square$

We now arrive at a juncture, caused by the bifurcation of classical and intuitionistic logic. We must take care to choose our definitions carefully from here on out, as formulations which are classically equivalent may fail to be the same in our intuitionistic language. For instance:

Definition 5.3. A *local ring* is a ring R such that

$$\neg(1=0) \wedge \forall x, y : R(\mathrm{inv}(x+y) \Rightarrow \mathrm{inv}(x) \vee \mathrm{inv}(y))$$

holds, where $\mathrm{inv}(r)$ is the formula

$$\exists s : R(s \cdot r = 1),$$

which says that r is (internally) invertible.

We adopt this definition rather than making some statement about maximal ideals because we cannot quantify over all ideals of R using our first-order language, and of the handful of possible first-order formulations of locality (equivalent and nonequivalent) this one will suite our purposes the best. In particular, it allows us to easily characterize schemes in the internal language of sheaves.

Proposition 5.4. Let X be the underlying space of a scheme with structure sheaf $\mathcal{O}_X$. Then $\mathcal{O}_X$ is a local ring in the internal language of $\mathbf{Sh}(X)$.

Proof. Recall that for a formula φ, we have

$$\neg\varphi = \varphi \Rightarrow \perp$$

by definition (see Proposition 3.6 and Definition 3.12), so if φ is a geometric formula then $\neg\varphi$ is a geometric implication. The formula above in Definition 5.3 is thus the conjunction of a pair of geometric implications, so by Theorem 4.5 and Corollary 4.14 it suffices to check the condition at each point $x \in X$. But at a given point x this says precisely that the stalk $\mathcal{O}_{X,x}$ is a local ring, that is, that $(X, \mathcal{O}_X)$ is a scheme. □

Remark 5.5. This clarifies why we only demand that the *stalks* of a scheme be local rings, as opposed to naïvely demanding each ring of sections to be local. Locality of the stalks is exactly the condition necessary for schemes to look like local rings from the internal perspective.

We can also reformulate many familiar properties of schemes using the internal language. For instance, we have

Proposition 5.6. A scheme $(X, \mathcal{O}_X)$ is reduced if and only if $\mathcal{O}_X$ is a reduced ring in the internal language of $\mathbf{Sh}(X)$.

Proof. We can express the condition for a ring R to be reduced in the internal language as

$$\varphi := \forall r : R\left(\left(\bigvee_{n \geq 0} r^n = 0\right) \Rightarrow r = 0\right),$$

noting that this is a geometric implication. Thus, for $\mathcal{O}_X$ to be (internally) reduced is equivalent to each stalk $\mathcal{O}_{X,x}$ with $x \in X$ being a reduced ring by Corollary 4.14. This is precisely what it means for $(X, \mathcal{O}_X)$ to be a reduced scheme. □

Next, we turn our attention to integral schemes. First, we must identify what it means for a ring (internal to some topos) to be "integral." It turns out that we have two competing notions, which are equivalent in the classical framework, but distinct in our intuitionistic one.

Definition 5.7. A ring R is a *weak integral domain* if

$$\neg(0=1) \wedge \forall x, y : R(x \cdot y = 0 \Rightarrow (x = 0 \vee y = 0))$$

holds, and is a *strong integral domain* if

$$\neg(0=1) \wedge \forall x : R(x = 0 \vee \mathrm{reg}(x))$$

holds, where

$$\mathrm{reg}(r) := \forall s : R(r \cdot s = 0 \Rightarrow s = 0).$$

Proposition 5.8. Let $(X, \mathcal{O}_X)$ be a ringed space. Then X is integral at all points, that is, each stalk of $\mathcal{O}_X$ is an integral ring, if and only if $\mathcal{O}_X$ is a weak integral domain in the internal language of $\mathbf{Sh}(X)$.

Proof. The condition for being a weak integral domain is the conjunction of geometric implications, so the claim follows from Theorem 4.5 and Corollary 4.14. □

In the case that $(X, \mathcal{O}_X)$ is a locally Noetherian scheme, we can do even better. First, we need a small lemma, which is easily verified using ordinary scheme-theoretic methods, or can be proven using the internal language (by way of some more sophisticated machinery that we have omitted; see Proposition 9.4 of [3]).

Lemma 5.9. Let $(X, \mathcal{O}_X)$ be a locally Noetherian scheme, let $U \subseteq X$ be an open set, and let $s \in \Gamma(U; \mathcal{O}_X)$ be a section. Then for a given point $x \in U$ the following are equivalent:

(1) $\mathrm{reg}(s_x)$ holds with $R = \mathcal{O}_{X,x}$.

(2) There exists some open $V \subseteq U$ such that $\mathrm{reg}(s_y)$ holds with $R = \mathcal{O}_{X,y}$ for each $y \in V$.

Proposition 5.10. Let $(X, \mathcal{O}_X)$ be a locally Noetherian scheme. Then $\mathcal{O}_X$ is a weak integral domain if and only if it is a strong integral domain (internally to $\mathbf{Sh}(X)$).

Proof. We can assume that $0 \neq 1$ in $\mathcal{O}_X$, since that condition is present in both internal manifestations of integral rings. By Theorem 4.5, we find that $\mathcal{O}_X$ is a strong integral domain if and only if for each open $U \subseteq X$ and each section $s \in \Gamma(U; \mathcal{O}_X)$ we have

$$U \Vdash (s = 0 \vee \mathrm{reg}(s)),$$

which is equivalent to there existing an open cover $U = \bigcup_{i \in I} U_i$ such that, for each $i \in I$, either $s|_{U_i} = 0$ or $\mathrm{reg}(s|_{U_i})$. But $\mathrm{reg}(r)$ is a geometric implication, so by Corollary 4.14 it holds on an open neighborhood if and only if it holds at each point of that neighborhood, and thus $\mathrm{reg}(s|_{U_i})$ holds if and only if $\mathrm{reg}(s_x)$ holds for each $x \in U_i$. Since being a weak and strong integral domain are equivalent for ordinary rings, the desired result follows from Proposition 5.8, as this shows that each stalk of $\mathcal{O}_X$ must be an integral domain. □

Lastly, we will establish a dictionary between several basic properties of modules over a scheme and their analogs in commutative algebra. Throughout the rest of this section, we fix a scheme $(X, \mathcal{O}_X)$. Our first order of business is to state what should at this point be expected:

Proposition 5.11. An $\mathcal{O}_X$-module is precisely a module over the ring $\mathcal{O}_X$, in the internal language of $\mathbf{Sh}(X)$.

The proof is essentially the same method as one would use to show Proposition 5.1; the forcing rules, locality, and monotonicity for sheaf semantics handle the restricting and gluing of sections, which is the only place where one might encounter difficulties translating between algebra and geometry.

Proposition 5.12. An $\mathcal{O}_X$-module $\mathscr{F}$ is finite locally free if and only if $\mathscr{F}$ is a finite free module over the ring $\mathcal{O}_X$, in the internal language of $\mathbf{Sh}(X)$, or more precisely,

$$X \Vdash \bigvee_{n \geq 0} \exists e_1, \ldots, e_n : \mathscr{F}\, \forall f : \mathscr{F}\, \exists! a_1, \ldots, a_n : \mathcal{O}_X \Big(f = \sum_{k=1}^{n} a_k \cdot e_k \Big).$$

Proof. Interpreting the internal formula above using Theorem 4.6, it is equivalent to the existence of an open cover $X = \bigcup_{i \in I} U_i$ and integers $n_i \geq 0$ for each $i \in I$ such that

$$U_i \Vdash \exists e_1, \ldots, e_{n_i} : \mathscr{F}\, \forall f : \mathscr{F}\, \exists! a_1, \ldots, a_{n_i} : \mathcal{O}_X \Big(f = \sum_{k=1}^{n_i} a_k \cdot e_k \Big).$$

Unwinding the quantifiers, we find that this is the same as the existence of an open cover $U_i = \bigcup_{j \in J_i} V_j^i$ so that for each $j \in J_i$, each open $W \subseteq V_j^i$, and each section $f \in \mathscr{F}(W)$, there exist unique sections $a_1, \ldots, a_{n_i} \in \Gamma(W; \mathcal{O}_X))$ so that

$$f = \sum_{k=1}^{n_i} a_k \cdot e_k.$$

Taking $W = V_j^i$ shows that $\mathscr{F}$ is locally finite free, and conversely if $\mathscr{F}$ is locally finite free then the cover $X = \bigcup_{i,j} V_j^i$ satisfies the desired conditions (with U_i taken to be the union of all the V_j^i). □

Remark 5.13. A similar argument proves that a $\mathcal{O}_X$-module is of finite type if and only if it is finitely generated from the internal perspective.

As we have seen, we can reformulate many properties of schemes in more familiar, algebraic terms using the internal language of sheaf topoi. We conclude this paper by studying a few examples which illustrate the power of this idea, by proving a handful of scheme-theoretic results using the internal language and techniques from ordinary commutative algebra.

Proposition 5.14. Suppose given a short exact sequence of $\mathcal{O}_X$-modules

$$0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0.$$

If $\mathscr{F}$ and $\mathscr{H}$ are of finite type (resp. finite locally free) then $\mathscr{G}$ is of finite type (resp. finite locally free).

Proof. Using the dictionary established by Proposition 5.12 and Remark 5.13, it will suffice to supply an intuitionistically valid proof of the corresponding result for finite free or finitely generated modules. The usual proofs in commutative algebra of these facts are indeed valid intuitionistically, as one can easily check. □

We can also prove more sophisticated results, like the one that follows, by reducing a scheme-theoretic problem to finding an intuitionistically valid proof of a simple result in commutative algebra.

Proposition 5.15. Let $\mathcal{L}$ be a line bundle on X, and suppose given global sections $f_1, \ldots, f_n \in \Gamma(X; \mathcal{L})$. The following are equivalent:

(1) $f_1, \ldots, f_n$ globally generate $\mathcal{L}$.

(2) We have

$$X \Vdash \bigvee_{i=1}^{n} \exists \alpha : [\mathcal{L}, \mathcal{O}_X](\mathrm{inv}(\alpha(f_i)) \wedge \mathrm{iso}(\alpha)),$$

where inv is defined in Definition 5.3 and for some $\beta : \mathscr{H}\!om(\mathscr{F}, \mathscr{G})$ we have

$$\mathrm{iso}(\beta) := \exists \gamma : \mathscr{H}\!om(\mathscr{G}, \mathscr{F})(\beta \circ \gamma = \mathrm{id}_{\mathscr{G}} \wedge \gamma \circ \beta = \mathrm{id}_{\mathscr{F}}).[7]$$

In other words, for some i there exists an isomorphism $L \cong \mathcal{O}_X$ under which the image of f_i is invertible.

Proof. By Proposition 5.4 and Proposition 5.12, we seek an intuitionistically valid proof that given a local ring R and a free R-module L of rank 1 with elements $f_1, \ldots, f_n \in L$, these elements generate L if and only if there exists an isomorphism $L \cong R$ such that the image of f_i is invertible for some i. Clearly given such an isomorphism α the element f_i generates L; for if $r \in R$ is the inverse of $\alpha(f_i)$ then for any $g \in L$ we have

$$\alpha(g) = \alpha(g) \cdot (r \cdot \alpha(f_i)),$$

so

$$g = \alpha^{-1}(\alpha(g) \cdot (r \cdot \alpha(f_i))) = (\alpha(g) \cdot r) \cdot f_i.$$

Conversely, if $f_1, \ldots, f_n$ generate L, then fix an isomorphism $\alpha : L \cong R$; such an isomorphism exists, since L has rank 1. Then there exist $r_1, \ldots, r_n \in R$ such that

$$1 = \sum_{i=1}^{n} r_i \cdot \alpha(f_i),$$

and since R is local one of the summands $r_i \cdot \alpha(f_i)$ is invertible, and therefore so is $\alpha(f_i)$. □

We conclude with a few results about relationships between properties of the space X underlying a scheme and the internal language of $\mathbf{Sh}(X)$, starting with quasicompactness. Although it is not possible to describe quasicompactness completely within the internal language of $\mathbf{Sh}(X)$, we can characterize it using a "metaproperty" on the forcing relation of X.

Proposition 5.16. The space X is quasicompact if and only if, for each directed set[8] $(I, \preceq)$ and each *monotone* family of formulas $(\varphi_i)_{i \in I}$ such that if $i \preceq j$ then $X \Vdash \varphi_i \Rightarrow \varphi_j$, if

$$X \Vdash \bigvee_{i \in I} \varphi_i$$

then $X \Vdash \varphi_i$ for some $i \in I$.

[7] Here $\circ$ and id are defined internally, via the cartesian closed structure of $\mathbf{Sh}(X)$.
[8] That is, a poset such that each pair of elements has a common upper bound.

Proof. First, suppose X is quasicompact, and let $(\varphi_i)_{i\in I}$ be a given monotone family of formulas. For each $i \in I$, let U_i be the union of all open sets which force φ_i, that is, the largest open subset of X with $U_i \Vdash \varphi_i$. Suppose $i \preceq j$. Then $X \Vdash \varphi_i \Rightarrow \varphi_j$, so since $U_i \Vdash \varphi_i$ by construction it follows from Theorem 4.5 that $U_i \Vdash \varphi_j$. Therefore $U_i \subseteq U_j$, so $i \mapsto U_i$ defines a monotone map of posets $(I, \preceq) \to (\mathrm{Open}(X), \subseteq)$.

Now, if

$$X \Vdash \bigvee_{i\in I} \varphi_i$$

then by Theorem 4.6 there exists an open cover $X = \bigcup_{j\in J} V_j$ such that for each $j \in J$ we have $V_j \Vdash \varphi_i$ for some $i \in I$. Thus, for each j, we have $V_j \subseteq U_i$ for some i, so

$$X = \bigcup_{j\in J} V_j \subseteq \bigcup_{i\in I} U_i \subseteq X,$$

and hence $X = \bigcup_{i\in I} U_i$. But X is quasicompact, so X is covered by some finite subcover $U_{i_1}, \ldots, U_{i_n}$. Since I is directed, there exists a common upper bound k for each of $i_1, \ldots, i_n$. Thus by the monotonicity of $(\varphi_i)_i$ and of $(U_i)_{i\in I}$, we conclude that

$$X = \bigcup_{j=1}^{n} U_{i_j} \subseteq U_k \subseteq X,$$

so $U_k = X$ and we have $X \Vdash \varphi_k$.

Conversely, suppose given a monotone family of open subsets $(U_i)_{i\in I}$ indexed by I. For each $i \in I$, let φ_i be the formula φ_{U_i} of Proposition 4.3. Then by Theorem 4.6, Proposition 4.3, and the argument above, we find that

$$X \Vdash \bigvee_{i\in I} \varphi_i$$

if and only if $X = \bigcup_{i\in I} U_i$, and that $X \Vdash \varphi_i$ if and only if $X = U_i$. The remaining direction of the proof then follows immediately. $\square$

We can also characterize the irreducibility of the space X using a metaproperty of the internal language of $\mathbf{Sh}(X)$:

Proposition 5.17. The space X is irreducible if and only if, given a pair of formulas φ and ψ, if $X \Vdash \neg(\varphi \wedge \psi)$ then $X \Vdash \neg\varphi$ or $X \Vdash \neg\psi$.

Proof. Unwinding $X \Vdash \neg(\varphi \wedge \psi)$ with Theorem 4.5 and Theorem 4.6, we find that it is equivalent to the statement that if $U, V \subseteq X$ are open sets with $U \Vdash \varphi$ and $V \Vdash \psi$ then $U \cap V = \varnothing$. Similarly, using the same theorems, we find that $X \Vdash \neg\varphi$ (resp. ψ) if and only if, for each open $W \subseteq X$, if $W \Vdash \varphi$ (resp. ψ) then $W = \varnothing$.

Suppose X is irreducible, and that $X \Vdash \neg(\varphi \wedge \psi)$. Suppose further that there exist nonempty open subsets $U \Vdash \varphi$ and $V \Vdash \psi$. Then since X is irreducible $U \cap V$ is nonempty, which is impossible. Therefore we must have either $X \Vdash \neg\varphi$ or $X \Vdash \neg\psi$.

Conversely, if U and V are nonempty open subsets of X then we consider the formulas φ_U and φ_V of Proposition 4.3. Suppose towards contradiction that $U \cap V = \varnothing$. By construction, we have $U \Vdash \varphi_U$ and $V \Vdash \varphi_V$, so it follows from Proposition 4.3 and the above that $X \Vdash \neg(\varphi_U \wedge \varphi_V)$. Thus by hypothesis $X \Vdash \neg\varphi_U$ or $X \Vdash \neg\varphi_V$, so we must have $U = \varnothing$ or $V = \varnothing$, both of which are contradictions. Hence $U \cap V$ is nonempty, so X is irreducible. $\square$

Finally, we close by using this result, along with a simple lemma, to re-derive the following classical scheme-theoretic result, using the machinery of the internal language of $\mathbf{Sh}(X)$.

Lemma 5.18. For any scheme $(X, \mathcal{O}_X)$, we have

$$X \Vdash \forall s : \mathcal{O}_X(\neg\mathrm{inv}(s) \Rightarrow \mathrm{nil}(s)),$$

where inv is defined in Definition 5.3 and

$$\mathrm{nil}(r) := \bigvee_{n\geq 0} r^n = 0.$$

Informally, this lemma states that noninvertible sections are nilpotent. That is, viewed as internal rings, schemes are very nearly fields, modulo their nilradical.

Proof. By Proposition 4.9 and the fact that $(X, \mathcal{O}_X)$ is a scheme, it will suffice to prove the claim for $X = \operatorname{Spec} A$ for some ring A. Let $a \in \Gamma(X; \mathcal{O}_X) = A$, noting that we do not need to pass to an open subset of X since we could restrict to a distinguished open and then replace A with the appropriate localization, and suppose

$$X \Vdash \neg\mathrm{inv}(a).$$

Then if $a|_U$ is invertible for $U \subseteq X$, we must have $U = \varnothing$. Thus, if $\mathfrak{p} \subseteq A$ is a prime ideal then the restriction of a to the complement of the closed set $V(\mathfrak{p})$, which is the image of a in the localization $A_{\mathfrak{p}}$, is noninvertible, so $a \in \mathfrak{p}$. It follows that a is in the intersection of every prime ideal, which is the nilradical of A, and therefore $X \Vdash \mathrm{nil}(a)$. □

It follows immediately that *reduced* schemes are precisely internal fields in $\mathbf{Sh}(X)$:

Corollary 5.19. If $(X, \mathcal{O}_X)$ is a reduced scheme then $\mathcal{O}_X$ is a field from the internal perspective of $\mathbf{Sh}(X)$, in the sense that

$$X \Vdash \forall s : \mathcal{O}_X(\neg\mathrm{inv}(s) \Rightarrow s = 0).$$

From Corollary 5.19 we can at last convince ourselves, using the internal language of $\mathbf{Sh}(X)$, that irreducible, reduced schemes are integral:

Proposition 5.20. Suppose $(X, \mathcal{O}_X)$ is an irreducible, reduced scheme. Then the stalks of $\mathcal{O}_X$ are integral domains.

Proof. We apply Proposition 5.4 and Proposition 5.6 to reduce the claim to showing intuitionistically that a reduced local ring R satisfying the property that non-invertible elements are zero is a weak integral domain (see Definition 5.7). Let r, s be of type R, and suppose $r \cdot s = 0$. Then we have

$$R \Vdash \neg\mathrm{inv}(r \cdot s),$$

since $R \Vdash \neg(0 = 1)$ as R is local, so 0 cannot be invertible. It follows that

$$R \Vdash \neg(\mathrm{inv}(r) \wedge \mathrm{inv}(s)),$$

for if both r and s are invertible so is their product. But X is irreducible, so by the metaproperty of Proposition 5.17 we find that $R \Vdash \neg\mathrm{inv}(r)$ or $R \Vdash \neg\mathrm{inv}(s)$. By Corollary 5.19 it follows that $r = 0$ or $s = 0$, completing the proof. □

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