GRADED MONOIDAL CATEGORIES AND INTERNALIZATION

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Abstract. In stable homotopy theory there are a variety of different ways to construct a point-set level smash product of spectra, each with its own advantages and limitations. Several of these constructions involve "internalizing" some kind of external product. In this paper, we aim to formalize this process which allows a graded, external product to be molded into an internal one. After introducing the general theory of internalization and investigating some of its categorical properties, we outline how our objects of study can be realized as algebras over an operad, and conclude by describing how both Day convolution and the smash product of S-modules fit into this framework.

Contents

1. Introduction

Unsurprisingly, in homotopy theory one is often concerned only with objects of interest up to homotopy. However, when possible, one may wish to lift a homotopical phenomenon to the point-set level, and study things prior to passing to the homotopy category. In the particular case of spectra, one might desire a well-behaved smash product which lifts the canonical one dwelling within the homotopy category of spectra. However, in [6], it is shown that no such smash product can exist, without surrendering at least one property desirable of this product.

In fact, many such weaker smash products of spectra exist, depending on which properties are relinquished and what point-set model of spectra is favored. For instance, one can form a smash product of symmetric or orthogonal spectra using Day convolution and the results of [7]. Alternatively, if one wishes to use the coordinate-free spectra of [1] then restricting to S-modules yields another smash product. Both of these constructions entail some "external," graded product which is flattened into a single degree to yield the desired "internal" smash product. In this paper, we propose a general categorical axiomatization of this procedure of "internalizing" a graded product.

We begin by briefly outlining the general notational conventions used throughout this paper in Section 2. In Section 3 we introduce the basic definitions and results underpinning internalization: Section 3.1 is devoted to describing and providing a handful of examples of graded monoidal categories, while Section 3.2 goes on to define internalizations of these categories, before proving some fundamental results about internalization. In Section 4 we form the category of graded monoidal categories with internalizations by defining functors between them, and investigate some of the general categorical properties of internalization. The focus of Section 5 is the construction of an operad whose algebras are internalized categories. (Both Section 4 and Section 5 can be skipped by a reader interested only in the application of internalization to smash products of spectra.) Finally, we conclude with an overview of the two main examples of internalization in Section 6: Day convolution is reinterpreted through the lens of internaliZation in Section 6.1, and in Section 6.2 we reconstruct the smash product of S-modules as the internalization of the external product of spectra.

2. Notation

Before we begin assembling the machinery of internalization, we make a few brief notes on the notation used throughout this paper.

We denote by $\mathbb{1}$ the terminal category, consisting of a unique object and its identity morphism. Given a category $\mathcal{C}$ and objects $x, y \in \mathcal{C}$, we write $\mathcal{C}(x, y)$ for the class of morphisms with domain x and codomain y; if $\mathcal{C}$ is enriched over some monoidal category $\mathcal{V}$, then we write $\underline{\mathcal{C}}(x, y)$ for the hom-object in $\mathcal{V}$ of morphisms from x to y.

Additionally, we will write $\mathbb{N}$ for the set of natural numbers (starting at 0). Given $m \in \mathbb{N}$ we denote by $\mathcal{C}^m$ the m^{th} power of $\mathcal{C}$, that is, the iterated product

$$\mathcal{C}^m := \prod_{i=1}^{m} \mathcal{C}.$$

Note that $\mathcal{C}^0 = \mathbb{1}$. If $f : \mathcal{C}^m \to \mathcal{D}$ is a functor and $x_1, \ldots, x_m \in \mathcal{C}$, we will occasionally write

$$f(x_i)_{i=1}^m := f(x_1, \ldots, x_m)$$

to conserve (an admittedly minimal amount of) horizontal space.

Finally, much of the content of the following sections involves objects which are $\mathbb{N}$-graded. Consequently, a great deal of indices are required to keep track of degrees. When possible, we omit these indices, provided that the degree of the objects we are concerned with are either clear from context or unimportant. In particular, our diagrams will (mercifully) almost always be free of indices.

3. External and Internal Products

Our aim is to formalize the procedure of coercing a kind of external, graded product into an ordinary monoidal product. To achieve this, we must first make sense of what it means for a (graded) category to be equipped with such an external product. We define the notion of a "graded monoidal product" in Section 3.1, and introduce generalizations of a few common properties enjoyed by ordinary monoidal categories (namely, symmetry and closure). We then go on to make rigorous the structure which enables internalizing a graded product, and establish some basic results regarding internalization, in Section 3.2.

3.1. **Graded Monoidal Categories.** We begin by making the following definition:

Definition 3.1. A *graded monoidal category* $(\mathcal{C}_\bullet, \mu, 1)$ consists of the following data:

(1) For each $m \in \mathbb{N}$ a category $\mathcal{C}_m$, called the *degree m component* of $\mathcal{C}_\bullet$. (We will also occasionally write $\mathcal{C}$ for the coproduct of the components of $\mathcal{C}_\bullet$.)
(2) A bifunctor $\mu_{m,n} : \mathcal{C}_m \times \mathcal{C}_n \to \mathcal{C}_{m+n}$ for each pair $m, n \in \mathbb{N}$. We will often abuse notation by writing μ for the gestalt of these functors. (We will often omit degree indices; since μ is degree-additive, the degree of $\mu(x, y)$ is uniquely determined by the degrees of x and y, so this is unambiguous.)
(3) An object $1 \in \mathcal{C}_0$, called the *unit* of μ.
(4) For each $m, n, r \in \mathbb{N}$ a natural isomorphism

$$\alpha_{m,n,r} : \mu_{m,n+r}(\mathrm{id}_{\mathcal{C}_m}, \mu_{n,r}) \cong \mu_{m+n,r}(\mu_{m,n}, \mathrm{id}_{\mathcal{C}_r}),$$

called the *associators* of μ.
(5) Natural isomorphisms

$$\lambda_m : \mu_{0,m}(1, \mathrm{id}_{\mathcal{C}_m}) \cong \mathrm{id}_{\mathcal{C}_m}$$

and

$$\rho_m : \mu_{m,0}(\mathrm{id}_{\mathcal{C}_m}, 1) \cong \mathrm{id}_{\mathcal{C}_m}$$

for each $m \in \mathbb{N}$, called the *left unitors* and *right unitors* of μ, respectively.

This data is further subject to the same coherence conditions as ordinary monoidal categories[1]; that is, for each $x, y, z, w \in \mathcal{C}$ the following diagrams commute:

$$\begin{tikzcd} & \mu(x,y) & \\ \mu(x,\mu(1,y)) \arrow[ur, "\mu(x,\lambda_y)"] \arrow[rr, "\alpha_{x,1,y}"'] & & \mu(\mu(x,1),y) \arrow[ul, "\mu(\rho_x,y)"'] \end{tikzcd}$$

$$\begin{tikzcd} \mu(x,\mu(y,\mu(z,w))) \arrow[rr, "\alpha_{x,y,\mu(z,w)}"] \arrow[d, "\mu(x,\alpha_{y,z,w})"'] & & \mu(\mu(x,y),\mu(z,w)) \arrow[d, "\alpha_{\mu(x,y),z,w}"] \\ \mu(x,\mu(\mu(y,z),w)) \arrow[dr, "\alpha_{x,\mu(y,z),w}"'] & & \mu(\mu(\mu(x,y),z),w) \\ & \mu(\mu(x,\mu(y,z)),w) \arrow[ur, "\mu(\alpha_{x,y,z},w)"'] & \end{tikzcd}$$

We think of the collection of categories $\mathcal{C}_\bullet$ as a single monoidal category, with the additional structure of a grading. Indeed, μ defines an ordinary monoidal structure on $\mathcal{C} := \coprod_{m\in\mathbb{N}} \mathcal{C}_m$, since each of this category's morphisms are contained within its independent graded components (so the naturality and coherence of of the structure maps can be checked degree-wise).

Examples of graded monoidal categories are abundant in nature:

[1] If we were taking care to track degrees everywhere, the coherence conditions for graded monoidal categories would differ only in that they would be decorated with a cumbersome quantity of indices. In an effort to minimize the reader's vertigo, we will omit the degree indices from all diagrams throughout this paper.

Example 3.2. Any monoidal category $\mathcal{C}$ can be viewed as a graded monoidal category in a number of ways. One can form the *trivial graded category* $\mathcal{C}(0)_\bullet$, which is given by $\mathcal{C}$ in degree 0, and which is the empty category in all higher degrees. (One could equally take $\mathcal{C}(0)_m$ to be the terminal category $\mathbb{1}$ with one object and one morphism for each $m > 0$.) The monoidal product on $\mathcal{C}$ then naturally extends to a graded monoidal product on $\mathcal{C}(0)_\bullet$.

The ordinary monoidal category $\mathcal{C}$ also gives rise to the *constant graded category* $\overline{C}_\bullet$, defined by $\overline{\mathcal{C}}_m := \mathcal{C}$ for each $m \in \mathbb{N}$. Again, it is clear that the monoidal product on $\mathcal{C}$ extends to a graded monoidal product on $\overline{\mathcal{C}}_\bullet$.

Finally, one can produce a *free graded monoidal category* $\mathrm{Free}(\mathcal{C})_\bullet$ on any given category $\mathcal{C}$, monoidal or not. We let $\mathrm{Free}(\mathcal{C})_m := \mathcal{C}^m$ for each $m \in \mathbb{N}$, where $\mathcal{C}^0 = \mathbb{1}$, and let $\mathrm{Free}(\mathcal{C})_m \times \mathrm{Free}(\mathcal{C})_n \to \mathrm{Free}(\mathcal{C})_{m+n}$ be given by concatenation. This construction enjoys a handful of nice properties when $\mathcal{C}$ is monoidal, which are outlined in Section 4 (see Proposition 4.8 and Proposition 4.9).

Example 3.3. Every graded integral domain $S = \bigoplus_{d \geq 0} S_d$ forms a *preadditive graded monoidal category* $\mathcal{S}_\bullet$ – that is, a graded monoidal category such that $\mathcal{S}_m$ is enriched over abelian groups for each $m \in \mathbb{N}$. In each degree $\mathcal{S}_\bullet$ has only one object, while the morphisms of $\mathcal{S}_m$ are those elements of S with degree m (i.e. which have nonzero projections to S_m, and trivial projections to all higher degrees). The addition of morphisms within the same component is defined using the addition of S. In fact, we can further define the sum of morphisms in different components this way, since the degree of an element of S is well-defined, so $\mathcal{S} := \bigsqcup_{m \in \mathbb{N}} \mathcal{S}_m$ is itself a preadditive category.

We equip $\mathcal{S}_\bullet$ with the graded product $\mathcal{S}_m \times \mathcal{S}_n \to \mathcal{S}_{m+n}$ which acts on morphisms via the multiplication of S. Since S is a graded ring, and the product of elements of S of degrees m and n is an element of degree $m+n$ (hence the requirement that S be an integral domain), this defines a graded monoidal product. We note that the set of morphisms in $\mathcal{S}$ corresponds precisely to the underlying set of S.

Example 3.4. The category **FinMan** of finite-dimensional (topological, n-differentiable, smooth, etc.) connected manifolds can be turned into a graded category $\mathbf{FinMan}_\bullet$, where $\mathbf{FinMan}_m$ is the full subcategory of m-dimensional manifolds. The categorical product of **FinMan** then forms a graded monoidal product on $\mathbf{FinMan}_\bullet$, since it is dimension-additive.

Note that we have lost the data of morphisms between manifolds of distinct dimension in forming $\mathbf{FinMan}_\bullet$. As we will see in the following sections, the purpose of a graded monoidal category will often be to study the degree 1 component in particular. In practice, we will typically build the higher degrees of a graded monoidal category from degree 1, as opposed to stripping a category of enough data to reduce it to a graded monoidal category.

As a kind of specialization of monoidal categories, we would like to extend some of the properties desired of an ordinary monoidal product to the graded setting. We start by describing what it means for a graded monoidal product to be symmetric. Throughout the remainder of this section, we fix a graded monoidal category $(\mathcal{C}_\bullet, \mu, 1)$, using the same notation for the associators and unitors as above.

Definition 3.5. For each $m, n \in \mathbb{N}$, we write $\tau_{m,n}$ for the *transposition functor* $\mathcal{C}_m \times \mathcal{C}_n \to \mathcal{C}_n \times \mathcal{C}_m$. Explicitly, this is defined via the universal property of the product to be the unique functor making

$$\begin{tikzcd} & \mathcal{C}_m \times \mathcal{C}_n \arrow[dl] \arrow[d, dashed, "\tau_{m,n}"] \arrow[dr] & \\ \mathcal{C}_n & \mathcal{C}_n \times \mathcal{C}_m \arrow[l] \arrow[r] & \mathcal{C}_m \end{tikzcd}$$

commute, where the unlabeled arrows are the appropriate projection functors.

Definition 3.6. $\mathcal{C}_\bullet$ is *symmetric* if, for each $m, n \in \mathbb{N}$, there exist natural isomorphisms $\sigma_{m,n} : \mu_{m,n} \cong \mu_{n,m}\tau_{m,n}$, called the *braidings* of μ, making the following diagrams commute for each $x, y, z \in \mathcal{C}$:

$$\begin{tikzcd} & x & \\ \mu(1,x) \arrow[ur, "\lambda_x"] \arrow[rr, "\sigma_{1,x}"'] & & \mu(x,1) \arrow[ul, "\rho_x"'] \end{tikzcd}$$

$$\begin{tikzcd} \mu(x,\mu(y,z)) \arrow[r, "\mu(x,\sigma_{y,z})"] \arrow[d, "\alpha_{x,y,z}"'] & \mu(x,\mu(z,y)) \arrow[d, "\alpha_{x,z,y}"] \\ \mu(\mu(x,y),z) \arrow[d, "\sigma_{\mu(x,y),z}"'] & \mu(\mu(x,z),y) \arrow[d, "\mu(\sigma_{x,z},y)"] \\ \mu(z,\mu(x,y)) \arrow[r, "\alpha_{z,x,y}"'] & \mu(\mu(z,x),y) \end{tikzcd}$$

$$\begin{tikzcd} \mu(x,y) \arrow[rr, equal] \arrow[dr, "\sigma_{x,y}"'] & & \mu(x,y) \\ & \mu(y,x) \arrow[ur, "\sigma_{y,x}"'] & \end{tikzcd}$$

As in Definition 3.1, this definition directly generalizes the classical case. The coherence conditions for graded symmetric monoidal categories are precisely the same as the usual ones, with the exception that the sufficiently motivated reader could insert degree indices in each diagram above. We can likewise stratify the notion of a closed monoidal category by grading the internal hom:

Definition 3.7. $\mathcal{C}_\bullet$ is *closed* if, for each $m, n \in \mathbb{N}$, there exists a bifunctor $\mathscr{F}_{m,n} : \mathcal{C}_n^{\mathrm{op}} \times \mathcal{C}_{m+n} \to \mathcal{C}_m$ such that, given $x \in \mathcal{C}_m, y \in \mathcal{C}_n$, and $z \in \mathcal{C}_{m+n}$, there is a natural isomorphism

$$\mathcal{C}_{m+n}(\mu_{m,n}(x,y), z) \cong \mathcal{C}_m(x, \mathscr{F}_{m,n}(y,z)).$$

The collection of bifunctors $\mathscr{F}$ are called the *internal hom* of $\mathcal{C}_\bullet$.

Remark 3.8. In the case that μ is symmetric, $\mathscr{F}$ defines a two-variable adjunction with μ.

Remark 3.9 (Coherence)**.** As noted previously, everything defined above is a direct extension of ordinary monoidal categories. All of the coherence conditions are exactly the same as usual, ignoring the minor technicality of tracking degrees. Consequently, it should not be surprising that Mac Lane's Coherence Theorem for Monoidal Categories will still hold in the slightly more general setting of graded monoidal categories.

The main idea is that the usual proof of the theorem can be adapted by simply formally inserting degree indices where necessary, when constructing the preorder category of words as in VII.2 of [5]. At the beginning of the proof one must exercise slightly more caution when assembling each set of valid words inductively, but ultimately this does not cause any issues, and the rest of the argument is identical. We will not need the coherence theorem in the sequel, but it is worth mentioning, if only to emphasize that graded monoidal categories are not so far a departure from their ungraded cousins.

3.2. **Internalization.** In the preceding section, we laid the foundations for studying graded monoidal products. In this section, we describe the additional structure required to drag the external product of a graded monoidal category into the degree one component[2], and explore some of the basic behavior of this structure.

Definition 3.10. Let $(\mathcal{C}_\bullet, \mu, 1)$ be a graded monoidal category. An *internalization of* μ consists of:

(1) For each $m \in \mathbb{N}$ a functor $\Phi_m : \mathcal{C}_m \to \mathcal{C}_1$, with a natural isomorphism $\iota : \Phi_1 \cong \mathrm{id}_{\mathcal{C}_1}$. (As usual, we will write Φ to denote the collection of all the Φ_m).
(2) For each $m, n \in \mathbb{N}$, natural isomorphisms

$$\mathcal{L}_{m,n} : \Phi_{n+1}\mu_{1,n}(\Phi_m, \mathrm{id}_{\mathcal{C}_n}) \cong \Phi_{m+n}\mu_{m,n}$$
$$\mathcal{R}_{m,n} : \Phi_{m+1}\mu_{m,1}(\mathrm{id}_{\mathcal{C}_m}, \Phi_n) \cong \Phi_{m+n}\mu_{m,n}$$

called the *left absorbers* and *right absorbers* of Φ, respectively.[3]

This data is further subject to the coherence conditions below – given $x, y, z \in \mathcal{C}$, the following diagrams commute:

$$\begin{tikzcd}
\Phi\mu(\Phi x, \Phi y) \arrow[r, "\mathcal{L}_{x,\Phi y}"] \arrow[d, "\mathcal{R}_{\Phi x, y}"'] & \Phi\mu(x, \Phi y) \arrow[d, "\mathcal{R}_{x,y}"] \\
\Phi\mu(\Phi x, y) \arrow[r, "\mathcal{L}_{x,y}"'] & \Phi\mu(x, y),
\end{tikzcd}$$

$$\begin{tikzcd}
\Phi\mu(x, \Phi\mu(y, \Phi z)) \arrow[r, "\mathcal{R}_{x,\mu(y,\Phi z)}"] \arrow[d, "\Phi\mu(x, \mathcal{R}_{y,z})"'] & \Phi\mu(x, \mu(y, \Phi z)) \arrow[r, "\Phi\alpha_{x,y,\Phi z}"] & \Phi\mu(\mu(x,y), \Phi z) \arrow[d, "\mathcal{R}_{\mu(x,y),z}"] \\
\Phi\mu(x, \Phi\mu(y, z)) \arrow[r, "\mathcal{R}_{x,\mu(y,z)}"'] & \Phi\mu(x, \mu(y, z)) \arrow[r, "\Phi\alpha_{x,y,z}"'] & \Phi\mu(\mu(x,y), z),
\end{tikzcd}$$

$$\begin{tikzcd}
\Phi\mu(\Phi\mu(\Phi x, y), z) \arrow[r, "\mathcal{L}_{\mu(\Phi x,y),z}"] \arrow[d, "\Phi\mu(\mathcal{L}_{x,y}, z)"'] & \Phi\mu(\mu(\Phi x, y), z) \arrow[r, "\Phi\alpha^{-1}_{\Phi x,y,z}"] & \Phi\mu(\Phi x, \mu(y,z)) \arrow[d, "\mathcal{L}_{x,\mu(y,z)}"] \\
\Phi\mu(\Phi\mu(x, y), z) \arrow[r, "\mathcal{L}_{\mu(x,y),z}"'] & \Phi\mu(\mu(x, y), z) \arrow[r, "\Phi\alpha_{x,y,z}"'] & \Phi\mu(x, \mu(y, z)),
\end{tikzcd}$$

$$\begin{tikzcd}
\Phi\mu(x, \Phi\mu(y, z)) \arrow[r, "\mathcal{R}_{x,\mu(y,z)}"] \arrow[d, "\Phi\mu(x, \mathcal{L}^{-1}_{y,z})"'] & \Phi\mu(x, \mu(y, z)) \arrow[r, "\Phi\alpha_{x,y,z}"] & \Phi\mu(\mu(x,y), z) \arrow[r, "\mathcal{L}^{-1}_{\mu(x,y),z}"] & \Phi\mu(\Phi\mu(x,y), z) \arrow[d, "\Phi\mu(\mathcal{R}^{-1}_{x,y}, z)"] \\
\Phi\mu(x, \Phi\mu(\Phi y, z)) \arrow[r, "\mathcal{R}_{x,\mu(\Phi y,z)}"'] & \Phi\mu(x, \mu(\Phi y, z)) \arrow[r, "\Phi\alpha_{x,\Phi y,z}"'] & \Phi\mu(\mu(x, \Phi y), z) \arrow[r, "\mathcal{L}^{-1}_{\mu(x,\Phi y),z}"'] & \Phi\mu(\Phi\mu(x, \Phi y), z),
\end{tikzcd}$$

[2]There is nothing exceptional about degree one, and in theory one could define an internalization into any degree. However, in practice we are most concerned with the degree one component, so we elect to bake our bias towards this component into our definitions for simplicity.

[3]The absorbers are so-named because they allow the outermost factor of Φ to "absorb" all inner ones. We make this explicit in Corollary 5.17.

and lastly that

$$\mathcal{R}_{m,1} = \Phi_{m+1}\mu(\mathrm{id}, \iota)$$
$$\mathcal{L}_{1,m} = \Phi_{m+1}\mu(\iota, \mathrm{id}).$$

If $(\mathcal{C}_\bullet, \mu, 1)$ is equipped with an internalization Φ as above, we say that $\mathcal{C}_\bullet$ is an *internalized monoidal category* (we omit the term "graded" for brevity), and write $(\mathcal{C}_\bullet, \Phi)$ when Φ is not clear from context, leaving the absorbers implicit.

As suggested by the name, the purpose of an internalization Φ on a graded monoidal category $\mathcal{C}_\bullet$ is to "internalize" its external product μ. The functors Φ allow us to normalize the degree of μ, while the absorbers enable us to extract the ordinary monoidal coherence axioms from the graded ones.

Theorem 3.11. *Let $(\mathcal{C}_\bullet, \mu, 1)$ be a graded monoidal category equipped with an internalization $(\Phi, \mathcal{L}, \mathcal{R})$. Then the bifunctor $\otimes : \mathcal{C}_1 \times \mathcal{C}_1 \to \mathcal{C}_1$ defined by*

$$x \otimes y := \Phi_2\mu_{1,1}(x, y)$$

defines a monoidal product on $\mathcal{C}_1$, with:

(1) unit given by $I := \Phi_0(1)$,
(2) associator given by

$$a_{x,y,z} := (\mathcal{L}^{-1}_{2,1})_{\mu(x,y),z} \circ \Phi_3(\alpha_{1,1,1})_{x,y,z} \circ (\mathcal{R}_{1,2})_{x,\mu(y,z)},$$

(3) left and right unitors given (respectively) by

$$l_x := \iota_x \circ \Phi_1(\lambda_1)_x \circ (\mathcal{L}_{0,1})_{1,x}$$
$$r_x := \iota_x \circ \Phi_1(\rho_2)_x \circ (\mathcal{R}_{1,0})_{x,1},$$

where $x, y, z \in \mathcal{C}_1$. We say that $\otimes$ is the internalization of μ with respect to Φ.

Proof. First, we observe that $a, l,$ and r have the correct types. In particular, given $x, y, z \in \mathcal{C}_1$, $a_{x,y,z}$ is the composite

$$x \otimes (y \otimes z) \xrightarrow{\mathcal{R}_{x,\mu(y,z)}} \Phi\mu(x, \mu(y, z)) \xrightarrow{\Phi\alpha_{x,y,z}} \Phi\mu(\mu(x, y), z) \xrightarrow{\mathcal{L}^{-1}_{\mu(x,y),z}} (x \otimes y) \otimes z,$$

l_x is the composite

$$I \otimes x \xrightarrow{\mathcal{L}_{1,x}} \Phi\mu(1, x) \xrightarrow{\Phi\lambda_x} \Phi(x) \xrightarrow{\iota_x} x,$$

and r_x is the composite

$$x \otimes I \xrightarrow{\mathcal{R}_{x,1}} \Phi\mu(x, 1) \xrightarrow{\Phi\rho_x} \Phi(x) \xrightarrow{\iota_x} x.$$

We must verify that the specified natural transformations satisfy the coherence conditions of a monoidal product, namely, that the following diagrams commute for all $x, y, z, w \in \mathcal{C}_1$:

(3.12)
$$\begin{tikzcd} & x \otimes y & \\ x \otimes (1 \otimes y) \arrow[ur, "x\otimes l_y"] \arrow[rr, "a_{x,1,y}"'] & & (x \otimes 1) \otimes y \arrow[ul, "r_y \otimes y"'] \end{tikzcd}$$

(3.13)
$$
\begin{tikzcd}
x \otimes (y \otimes (z \otimes w)) \arrow[rr, "a_{x,y,z\otimes w}"] \arrow[d, "x\otimes a_{y,z,w}"'] & & (x \otimes y) \otimes (z \otimes w) \arrow[d, "a_{x\otimes y,z,w}"] \\
x \otimes ((y \otimes z) \otimes w) \arrow[dr, "a_{x,y\otimes z,w}"'] & & ((x \otimes y) \otimes z) \otimes w \\
& (x \otimes (y \otimes z)) \otimes w \arrow[ur, "a_{x,y,z}\otimes w"'] &
\end{tikzcd}
$$

First, we will show that (3.12) commutes. By the definitions of a, l, and r, we can decompose the diagram into the outermost diagram below:

$$
\begin{tikzcd}
& \Phi\mu(x,\Phi y) \arrow[r, "\mathcal{R}_{x,y}"] & \Phi\mu(x,y) & \Phi\mu(\Phi x,y) \arrow[l, "\mathcal{L}_{x,y}"'] \\
\Phi\mu(x,\Phi\mu(1,y)) \arrow[ur, "\Phi\mu(x,\Phi\lambda_y)"] \arrow[r, "\mathcal{R}_{x,\mu(1,y)}"'] & \Phi\mu(x,\mu(1,y)) \arrow[ur, "\Phi\mu(x,\lambda_y)"'] \arrow[r, "\Phi\alpha_{x,1,y}"'] & \Phi\mu(\mu(x,1),y) \arrow[u, "\Phi\mu(\rho_x,y)"] & \Phi\mu(\Phi\mu(x,1),y) \arrow[l, "\mathcal{L}_{\mu(x,1),y}"] \arrow[u, "\Phi\mu(\Phi\rho_x,y)"'] \\
\Phi\mu(x,\Phi\mu(I,y)) \arrow[u, "\Phi\mu(x,\mathcal{L}_{1,y})"] \arrow[r, "\mathcal{R}_{x,\mu(I,y)}"'] & \Phi\mu(x,\mu(I,y)) \arrow[r, "\Phi\alpha_{x,I,y}"'] & \Phi\mu(\mu(x,I),y) \arrow[r, "\mathcal{L}^{-1}_{\mu(x,I),y}"'] & \Phi\mu(\Phi\mu(x,I),y) \arrow[u, "\Phi\mu(\mathcal{R}_{x,1},y)"'] \ .
\end{tikzcd}
$$

But each inner subdiagram above commutes: the top left and right squares commute by the naturality of $\mathcal{L}$ and $\mathcal{R}$; the top middle triangle is one of the coherence conditions for μ; and the bottom rectangle is one of the coherence conditions for Φ. Therefore the outermost subdiagram commutes, and $\otimes$ satisfies the first coherence condition.

It remains to be shown that (3.13) commutes. Again, we expand the diagram to form the one below:

$$
\begin{tikzcd}
\Phi\mu(x,\Phi\mu(y,\Phi\mu(z,w))) \arrow[r, "\mathcal{R}_{x,\mu(y,\Phi\mu(z,w))}"] \arrow[d, "\Phi\mu(x,\mathcal{R}_{y,\mu(z,w)})"'] & \Phi\mu(x,\mu(y,\Phi\mu(z,w))) \arrow[r, "\Phi\alpha_{x,y,\Phi\mu(z,w)}"] & \Phi\mu(\mu(x,y),\Phi\mu(z,w)) \arrow[r, "\mathcal{L}^{-1}_{\mu(x,y),\Phi\mu(z,w)}"] \arrow[d, "\mathcal{R}_{\mu(x,y),\mu(z,w)}"'] & \Phi\mu(\Phi\mu(x,y),\Phi\mu(z,w)) \arrow[d, "\mathcal{R}_{\Phi\mu(x,y),\mu(z,w)}"] & \\
\Phi\mu(x,\Phi\mu(y,\mu(z,w))) \arrow[r, "\mathcal{R}_{x,\mu(y,\mu(z,w))}"'] \arrow[dd, "\Phi\mu(x,\Phi\alpha_{y,z,w})"'] & \Phi\mu(x,\mu(y,\mu(z,w))) \arrow[r, "\Phi\alpha_{x,y,\mu(z,w)}"'] \arrow[dd, "\Phi\mu(x,\alpha_{y,z,w})"'] & \Phi\mu(\mu(x,y),\mu(z,w)) \arrow[r, "\mathcal{L}^{-1}_{\mu(x,y),\mu(z,w)}"'] \arrow[d, "\Phi\alpha_{\mu(x,y),z,w}"'] & \Phi\mu(\Phi\mu(x,y),\mu(z,w)) \arrow[r, "\Phi\alpha_{\Phi\mu(x,y),z,w}"'] & \Phi\mu(\mu(\Phi\mu(x,y),z),w) \arrow[d, "\mathcal{L}^{-1}_{\mu(\Phi\mu(x,y),z),w}"] \\
& & \Phi\mu(\mu(\mu(x,y),z),w) \arrow[r, "\mathcal{L}^{-1}_{\mu(\mu(x,y),z),w}"'] & \Phi\mu(\Phi\mu(\mu(x,y),z),w) \arrow[r, "\Phi\mu(\mathcal{L}^{-1}_{\mu(x,y),z}{,}w)"'] & \Phi\mu(\Phi\mu(\Phi\mu(x,y),z),w) \\
\Phi\mu(x,\Phi\mu(\mu(y,z),w)) \arrow[r, "\mathcal{R}_{x,\mu(\mu(y,z),w)}"'] \arrow[d, "\Phi\mu(x,\mathcal{L}^{-1}_{\mu(y,z),w})"'] & \Phi\mu(x,\mu(\mu(y,z),w)) \arrow[r, "\Phi\alpha_{x,\mu(y,z),w}"'] & \Phi\mu(\mu(x,\mu(y,z)),w) \arrow[r, "\mathcal{L}^{-1}_{\mu(x,\mu(y,z)),w}"'] \arrow[u, "\Phi\mu(\alpha_{x,y,z}{,}w)"] & \Phi\mu(x,\mu(\Phi\mu(y,z),w)) \arrow[r, "\Phi\mu(\Phi\alpha_{x,y,z}{,}w)"'] & \Phi\mu(\Phi\mu(\mu(x,y),z),w) \arrow[u, "\Phi\mu(\mathcal{L}^{-1}_{\mu(x,y),z}{,}w)"'] \\
\Phi\mu(x,\Phi\mu(\Phi\mu(y,z),w)) \arrow[r, "\mathcal{R}_{x,\mu(\Phi\mu(y,z),w)}"'] & \Phi\mu(x,\mu(\Phi\mu(y,z),w)) \arrow[r, "\Phi\alpha_{x,\Phi\mu(y,z),w}"'] & \Phi\mu(\mu(x,\Phi\mu(y,z)),w) \arrow[r, "\mathcal{L}^{-1}_{\mu(x,\Phi\mu(y,z)),w}"'] & \Phi\mu(\Phi\mu(x,\Phi\mu(y,z)),w) \arrow[u, "\Phi\mu(\mathcal{R}_{x,\mu(y,z)}{,}w)"'] \ . &
\end{tikzcd}
$$

Once again, we will find that each inner subdiagram commutes. All three of the hexagons, the top-right square, and the bottommost octagon commute by the coherence conditions of Definition 3.10. Additionally, the middle-left square is precisely the naturality of $\mathcal{R}$. Finally, the middle pentagon is the image of pentagonal coherence condition for α under Φ, and thus also commutes. Therefore the outermost subdiagram commutes, completing the proof. $\square$

Since we have generalized the notions of symmetric and closed monoidal products to the graded setting, it would be impolite not to do the same for internalizations:

Definition 3.14. Let $(\mathcal{C}_\bullet, \mu, 1)$ be a symmetric graded monoidal category with braiding σ, equipped with an internalization $(\Phi, \mathcal{L}, \mathcal{R})$. Then Φ is a *symmetric internalization* if, for all $x, y \in \mathcal{C}$, the diagram

$$\begin{tikzcd}
\Phi\mu(\Phi x, y) \arrow[r, "\Phi\sigma_{\Phi x, y}"] \arrow[d, "\mathcal{L}_{x,y}"'] & \Phi\mu(y, \Phi x) \arrow[d, "\mathcal{R}_{y,x}"] \\
\Phi\mu(x, y) \arrow[r, "\Phi\sigma_{x,y}"'] & \Phi\mu(y, x)
\end{tikzcd}$$

commutes.

Corollary 3.15. *Let $(\mathcal{C}_\bullet, \mu, 1)$ be a symmetric graded monoidal category with braiding σ, equipped with a symmetric internalization $(\Phi, \mathcal{L}, \mathcal{R})$. Then the natural isomorphism*

$$s_{x,y} := \Phi_2(\sigma_{1,1})_{x,y}$$

defines a braiding on the internalization $\otimes$ of μ with respect to Φ, making $(\mathcal{C}_1, \otimes, I)$ into a symmetric monoidal category.

Proof. We must verify that the natural isomorphism s satisfies the three coherence conditions for a symmetric monoidal category, namely, that the following diagrams commute for all $x, y, z \in \mathcal{C}_1$:

$$\begin{tikzcd}
 & x & \\
I \otimes x \arrow[ur, "l_x"] \arrow[rr, "s_{1,x}"'] & & x \otimes I \arrow[ul, "r_x"']
\end{tikzcd} \tag{3.16}$$

$$\begin{tikzcd}
x \otimes (y \otimes z) \arrow[r, "x \otimes s_{y,z}"] \arrow[d, "a_{x,y,z}"'] & x \otimes (z \otimes y) \arrow[d, "a_{x,z,y}"] \\
(x \otimes y) \otimes z \arrow[d, "s_{x \otimes y, z}"'] & (x \otimes z) \otimes y \arrow[d, "s_{x,z} \otimes y"] \\
z \otimes (x \otimes y) \arrow[r, "a_{z,x,y}"'] & (z \otimes x) \otimes y
\end{tikzcd} \tag{3.17}$$

$$\begin{tikzcd}
x \otimes y \arrow[rr, equal] \arrow[dr, "s_{x,y}"'] & & x \otimes y \\
 & y \otimes x, \arrow[ur, "s_{y,x}"'] &
\end{tikzcd} \tag{3.18}$$

where a, l, and r are defined in Theorem 3.11. First, we observe that (3.16) commutes by the commutativity of

$$\begin{tikzcd}
 & \Phi x & \\
\Phi\mu(1,x) \arrow[ur, "\Phi\lambda_x"] \arrow[rr, "\Phi\sigma_{1,x}"] & & \Phi\mu(x,1) \arrow[ul, "\Phi\rho_x"'] \\
\Phi\mu(\Phi(1),x) \arrow[u, "\mathcal{L}_{1,x}"] \arrow[rr, "\Phi\sigma_{\Phi(1),x}"'] & & \Phi\mu(x,\Phi(1)) \arrow[u, "\mathcal{R}_{x,1}"']
\end{tikzcd}.$$

Each inner subdiagram commutes – the triangle is image of one of the coherence conditions on σ under Φ, and the square commutes by hypothesis – and, post-composition with ι_x yields (3.16).

Next, we will show that (3.17) commutes. Rewriting the diagram, we find that (3.17) is the outermost subdiagram of

$$\begin{tikzcd}
 & \Phi\mu(x,\Phi\mu(y,z)) \arrow[r, "\Phi\mu(x,\Phi\sigma_{y,z})"] \arrow[d, "\mathcal{R}_{x,\mu(y,z)}"'] & \Phi\mu(x,\Phi\mu(z,y)) \arrow[d, "\mathcal{R}_{x,\mu(z,y)}"] & \\
 & \Phi\mu(x,\mu(y,z)) \arrow[r, "\Phi\mu(x,\sigma_{y,z})"'] \arrow[d, "\Phi\alpha_{x,y,z}"'] & \Phi\mu(x,\mu(z,y)) \arrow[d, "\Phi\alpha_{x,z,y}"] & \\
\Phi\mu(\Phi\mu(x,y),z) \arrow[d, "\Phi\sigma_{\Phi\mu(x,y),z}"'] & \Phi\mu(\mu(x,y),z) \arrow[l, "\mathcal{L}^{-1}_{\mu(x,y),z}"'] \arrow[d, "\Phi\sigma_{\mu(x,y),z}"'] & \Phi\mu(\mu(x,z),y) \arrow[r, "\mathcal{L}^{-1}_{\mu(x,z),y}"] \arrow[d, "\Phi\mu(\sigma_{x,z},y)"] & \Phi\mu(\Phi\mu(x,z),y) \arrow[d, "\Phi\mu(\Phi\sigma_{x,z},y)"] \\
\Phi\mu(z,\Phi\mu(x,y)) \arrow[r, "\mathcal{R}_{z,\mu(x,y)}"'] & \Phi\mu(z,\mu(x,y)) \arrow[r, "\Phi\alpha_{z,x,y}"'] & \Phi\mu(\mu(z,x),y) \arrow[r, "\mathcal{L}^{-1}_{\mu(z,x),y}"'] & \Phi\mu(\Phi\mu(z,x),y)
\end{tikzcd}.$$

This diagram commutes, since the topmost square commutes by the naturality of $\mathcal{R}$, the bottom-left square commutes by hypothesis, the bottom-right square commutes by the naturality of $\mathcal{L}$, and and middle hexagon commutes since it is the image of one of the coherence conditions on σ under Φ. Thus, (3.17) commutes.

Finally, it is immediate that (3.18) commutes, since it is simply the image of a coherence condition on σ under Φ. Therefore s defines a braiding for $\otimes$ which makes $(\mathcal{C}_1, \otimes, I)$ into a symmetric monoidal category. $\square$

Definition 3.19. Let $(\mathcal{C}_\bullet, \mu, 1)$ be a closed graded monoidal category with internal hom $\mathscr{F}$, equipped with an internalization $(\Phi, \mathcal{L}, \mathcal{R})$. Then Φ is a *closed internalization* if, for each $m \in \mathbb{N}$, there exists a right adjoint $\Psi_m : \mathcal{C}_1 \to \mathcal{C}_m$ to Φ_m. For brevity, we write $\Phi \dashv \Psi$ to denote the adjunction.

Corollary 3.20. *Let $(\mathcal{C}_\bullet, \mu, 1)$ be a closed graded monoidal category with internal hom $\mathscr{F}$, equipped with a closed internalization $(\Phi, \mathcal{L}, \mathcal{R})$, with $\Phi \dashv \Psi$. Then $(\mathcal{C}_1, \otimes, I)$ is a closed monoidal category with internal hom*

$$F(x, y) := \mathscr{F}_{1,1}(x, \Psi_2 y)$$

for each $x, y \in \mathcal{C}_1$.

Proof. Let $x, y, z \in \mathcal{C}_1$. Then the claim follows immediately by composing the natural isomorphisms

$$\begin{aligned}\mathcal{C}_1(x \otimes y, z) &= \mathcal{C}_1(\Phi_2 \mu_{1,1}(x, y), z) \\ &\cong \mathcal{C}_2(\mu_{1,1}(x, y), \Psi_2 z) \\ &\cong \mathcal{C}_1(x, \mathscr{F}_{1,1}(y, \Psi_2 z)) \\ &= \mathcal{C}_1(x, F(y, z)).\end{aligned}$$

□

In Section 6, we'll use the results above to recover two constructions of point-set smash products of spectra, Day convolution and the smash product of S-modules, and show that these are automatically closed and symmetric.

Remark 3.21. All of the definitions and constructions in Section 3.1 and Section 3.2 can be interpreted more generally in the enriched setting. In particular, we could fix a finitely-complete[4] monoidal category $(\mathcal{V}, \otimes_{\mathcal{V}}, I_{\mathcal{V}})$ and consider external products (as $\mathcal{V}$-functors) on graded $\mathcal{V}$-categories. We omit a technical description of this generalization, as the interpretations of the arguments above would be at best cumbersome, and at worst entirely unreadable. In Section 6.1 the internalized categories we are interested in will be enriched, so we note that this is possible only so that the reader is not completely blindsided by the sudden use of enriched categories.

4. Categorical Properties of Internalization

Internalization enjoys a handful of convenient formal properties. In this section, we will focus on showing that the procedure in Theorem 3.11 for turning an internalized category into an ordinary monoidal category is functorial (Proposition 4.5), and has a left adjoint section (Proposition 4.8 and Proposition 4.9).

To simplify notation, throughout this section we fix graded monoidal categories $(\mathcal{C}_\bullet, \mu^{\mathcal{C}}, 1^{\mathcal{C}})$ and $(\mathcal{D}_\bullet, \mu^{\mathcal{D}}, 1^{\mathcal{D}})$.

Definition 4.1. A *graded monoidal functor* $f : \mathcal{C}_\bullet \to \mathcal{D}_\bullet$ consists of:

(1) a functor $f_m : \mathcal{C}_m \to \mathcal{D}_m$ for each $m \in \mathbb{N}$,
(2) a natural isomorphism

$$f_{m+n} \mu^{\mathcal{C}}_{m,n} \cong \mu^{\mathcal{D}}_{m,n}(f_m, f_n)$$

for each $m, n \in \mathbb{N}$,
(3) and an isomorphism $f_0(1^{\mathcal{C}}) \cong 1^{\mathcal{D}}$.

Remark 4.2. Our "monoidal functors" are taken to be strong, as opposed to lax. This choice was arbitrary, and made primarily for notational convenience. If the isomorphisms of Definition 4.1 were weakened to natural transformations, that is, if we instead considered *lax graded monoidal functors*, then all of the subsequent results in this section remain true.

[4]The completeness hypothesis is necessary to form the binary product of $\mathcal{V}$-categories.

Remark 4.3. Clearly the identity functor on a graded monoidal category is itself a graded monoidal functor, and by composing the natural transformations of graded monoidal functors $\mathcal{C}_\bullet \to \mathcal{D}_\bullet \to \mathcal{E}_\bullet$ we find that their composite is again graded monoidal. In other words, graded monoidal categories[5] and graded monoidal functors form a category.[6]

Definition 4.4. Suppose $\mathcal{C}_\bullet$ and $\mathcal{D}_\bullet$ are equipped with internalizations $\Phi^{\mathcal{C}}$ and $\Phi^{\mathcal{D}}$, respectively. An *internalized monoidal functor* $f : (\mathcal{C}_\bullet, \Phi^{\mathcal{C}}) \to (\mathcal{D}_\bullet, \Phi^{\mathcal{D}})$ is a graded monoidal functor $\mathcal{C}_\bullet \to \mathcal{D}_\bullet$, together with natural isomorphisms

$$f_1 \Phi^{\mathcal{C}}_m \cong \Phi^{\mathcal{D}}_m f_m$$

for each $m \in \mathbb{N}$.

If $g : (\mathcal{D}_\bullet, \Phi^{\mathcal{D}}) \to (\mathcal{E}_\bullet, \Phi^{\mathcal{E}})$ is another internalized monoidal functor then the natural isomorphisms

$$g_1 f_1 \Phi^{\mathcal{C}}_m \cong g_1 \Phi^{\mathcal{D}}_m f_m \cong \Phi^{\mathcal{E}}_m g_m f_m$$

witness that the composite gf is again an internalized monoidal functor. Since the identity functor on a given internalized category is clearly an internalized monoidal functor, we can form the category **IntMonCat** of internalized categories and internalized monoidal functors between them.

Our first result confirms that the internalization procedure of Theorem 3.11 is canonical, or more precisely, is functorial.

Proposition 4.5. *Internalization defines a functor* $\mathrm{Int} : \mathbf{IntMonCat} \to \mathbf{MonCat}$.

Proof. The action of Int on objects is clear: an internalized monoidal category $(\mathcal{C}_\bullet, \Phi^{\mathcal{C}})$ is carried to the monoidal category $(\mathcal{C}_1, \otimes^{\mathcal{C}}, I^{\mathcal{C}})$ of Theorem 3.11. Given an internalized monoidal functor $f : (\mathcal{C}_\bullet, \Phi^{\mathcal{C}}) \to (\mathcal{D}_\bullet, \Phi^{\mathcal{D}})$, we define $\mathrm{Int}(f) := f_1$. We must verify that f_1 is monoidal, and that the action of Int on morphisms is functorial.

Let λ_m be the natural isomorphism $f_1 \Phi^{\mathcal{C}}_m \cong \Phi^{\mathcal{D}}_m f_m$, let $\nu_{m,n}$ be the natural isomorphism $f_{m+n}\mu^{\mathcal{C}}_{m,n} \cong \mu^{\mathcal{C}}_{m,n}(f_m, f_n)$, and let η be the isomorphism $f_0(1^{\mathcal{C}}) \cong 1^{\mathcal{D}}$. Then the composite

$$f_1 \Phi^{\mathcal{C}}_2 \mu^{\mathcal{C}}_{1,1} \xrightarrow{\lambda_2 \mu^{\mathcal{C}}_{1,1}} \Phi^{\mathcal{D}}_2 f_2 \mu^{\mathcal{C}}_{1,1} \xrightarrow{\Phi^{\mathcal{D}}_2 \nu_2} \Phi^{\mathcal{D}}_2 \mu^{\mathcal{D}}_{1,1}(f_1, f_1) \tag{4.6}$$

witnesses that f_1 is preserves the internalized products $\otimes^{\mathcal{C}}$ and $\otimes^{\mathcal{D}}$, while the composite

$$f_1 \Phi^{\mathcal{C}}_0(1^{\mathcal{C}}) \xrightarrow{(\lambda_0)_{1^{\mathcal{C}}}} \Phi^{\mathcal{D}}_0 f_0(1^{\mathcal{C}}) \xrightarrow{\Phi^{\mathcal{D}}_0 \eta} \Phi^{\mathcal{D}}_0(1^{\mathcal{D}}) \tag{4.7}$$

witnesses that f_1 preserves the internalized units $I^{\mathcal{C}}$ and $I^{\mathcal{D}}$. Thus f_1 is indeed monoidal.

Now, suppose given an internalized monoidal category $(\mathcal{E}_\bullet, \Phi^{\mathcal{E}})$, along with an internalized monoidal functor $g : (\mathcal{D}_\bullet, \Phi^{\mathcal{D}}) \to (\mathcal{E}_\bullet, \Phi^{\mathcal{E}})$. Then

$$\mathrm{Int}(gf) = (gf)_1 = g_1 f_1 = \mathrm{Int}(g)\,\mathrm{Int}(f)$$

and

$$\mathrm{Int}(\mathrm{id}_{\mathcal{C}_\bullet}) = \mathrm{id}_{\mathcal{C}_1},$$

so Int preserves composition and identities at the level of the underlying functors of internalized monoidal functors. One also verifies that the composites of (4.6) and (4.7), and the corresponding isomorphisms for $\mathrm{Int}(g)$, compose to form the isomorphisms for $\mathrm{Int}(gf)$,

[5] As usual when referring to categories of categories, we must fix an upper bound on the size of the categories in question to avoid size issues.

[6] In the same way that the category of monoidal categories forms a 2-category with the addition of monoidal transformations, the category of graded monoidal categories can be elevated to a 2-category.

although we omit the details as they are not especially enlightening. Therefore Int is functorial, as desired. □

The Int functor is especially well-behaved, as it admits a right inverse. In particular, every ordinary monoidal category is the internalization of an internalized category. Given a monoidal category, we construct the corresponding *free internalized category* in the following proposition. We further show that this construction is functorial, and that the internalization of the free internalized category is precisely the monoidal category we started with.

Proposition 4.8. *There is a functor* $\mathrm{Free} : \mathbf{MonCat} \to \mathbf{IntMonCat}$ *which defines a section for* Int*, that is, such that* $\mathrm{Int} \circ \mathrm{Free} = \mathrm{id}_{\mathbf{MonCat}}$.

Proof. Let $(\mathcal{C}, \otimes, I)$ be a monoidal category. We begin by defining a graded category $\mathrm{Free}(\mathcal{C})_\bullet$ by

$$\mathrm{Free}(\mathcal{C})_m := \mathcal{C}^m,$$

the product of m-copies of $\mathcal{C}$ (where $\mathcal{C}^0 = \mathbb{1}$ is the terminal category). This graded category is naturally equipped with an external monoidal structure $(\mu^{\mathrm{Free}}, 1^{\mathrm{Free}})$, namely,

$$\begin{aligned} \mu^{\mathrm{Free}}_{m,n} &: \mathrm{Free}(\mathcal{C})_m \times \mathrm{Free}(\mathcal{C})_n \to \mathrm{Free}(\mathcal{C})_{m+n} \\ &((x_1, \ldots, x_m), (y_1, \ldots, y_n)) \mapsto (x_1, \ldots, x_m, y_1, \ldots, y_n) \end{aligned}$$

with 1^{Free} given by the unique object of $\mathbb{1}$. In other words, μ^{Free} is simply concatenation of finite strings in $\mathcal{C}$, and 1^{Free} is the empty string. This is clearly associative and unital up to (coherent) natural isomorphism.

We note that the construction of $(\mathrm{Free}(\mathcal{C})_\bullet, \mu^{\mathrm{Free}}, 1^{\mathrm{Free}})$ did not require $\mathcal{C}$ to be monoidal. The monoidal structure on $\mathcal{C}$ is instead necessary to define a (nearly) canonical internalization of $\mathrm{Free}(\mathcal{C})_\bullet$, which we now describe. We take $\Phi^{\mathrm{Free}}_0 : \mathrm{Free}(\mathcal{C})_0 \to \mathrm{Free}(\mathcal{C})_1$ to be the functor $\mathbb{1} \to \mathcal{C}$ which identifies the monoidal unit I, and for $m > 0$, we define

$$\begin{aligned} \Phi^{\mathrm{Free}}_m &: \mathrm{Free}(\mathcal{C})_m \to \mathrm{Free}(\mathcal{C})_1 \\ (x_1, \ldots, x_m) &\mapsto \bigotimes_{i=1}^{m} x_i. \end{aligned}$$

(Clearly Φ_1 is the identity, so u is trivial.) If $\otimes$ is not strict then Φ^{Free}_m is only well-defined up to natural isomorphism, since we must choose a particular way to paranthesize the iterated monoidal product. The associator of $\otimes$, together with Mac Lane's Coherence Theorem, establish a canonical dictionary between each possible construction of Φ^{Free}_m, so there are no significant ramifications of this technicality.

The left and right absorbers of $\Phi^{\mathrm{Free}}_\bullet$ are simply induced by the monoidal structure of $\otimes$. If $m, n > 0$ then $\mathcal{L}_{m,n}$ is simply the composite of the associators of $\otimes$ which witness the isomorphism

$$\bigotimes_{i=1}^{m} x_i \otimes \bigotimes_{j=m+1}^{m+n} x_j \cong \bigotimes_{i=1}^{m+n} x_i.$$

Similarly, if $m = 0$ then $\mathcal{L}_{0,n}$ is induced simply the left unitor

$$I \otimes \bigotimes_{j=1}^{n} y_j \cong \bigotimes_{i=1}^{n} y_j.$$

The right absorbers are defined analogously, and by Mac Lane's Coherence Theorem our candidate absorbers satisfy the axioms of Definition 3.10, making $(\mathrm{Free}(\mathcal{C})_\bullet, \Phi^{\mathrm{Free}}_\bullet)$ an internalized category.

Next, we must check that the construction $\mathrm{Free}(-)_\bullet$ is functorial. Given a monoidal functor $f : (\mathcal{C}, \otimes_\mathcal{C}, 1_\mathcal{C}) \to (\mathcal{D}, \otimes_\mathcal{D}, 1_\mathcal{D})$, we define $\mathrm{Free}(f)_\bullet : \mathrm{Free}(\mathcal{C})_\bullet \to \mathrm{Free}(\mathcal{D})_\bullet$ by

$$\mathrm{Free}(f)_m(x_1, \dots, x_m) := (f(x_1), \dots, f(x_m)).$$

This definition is clearly compatible with the graded monoidal structures of $\mathrm{Free}(\mathcal{C})_\bullet$ and $\mathrm{Free}(\mathcal{D})_\bullet$, and thus $\mathrm{Free}(f)_\bullet$ defines a graded monoidal functor. Moreover, we have

$$\begin{aligned}\mathrm{Free}(f)_1 \Phi^{\mathrm{Free}(\mathcal{C})}_m (x_i)_{i=1}^m &= f(x_1 \otimes_\mathcal{C} \cdots \otimes_\mathcal{C} x_m) \\ &\cong f(x_1) \otimes_\mathcal{D} \cdots \otimes_\mathcal{D} f(x_m) \\ &= \Phi^{\mathrm{Free}(\mathcal{D})}_m \mathrm{Free}(f)_m (x_i)_{i=1}^m\end{aligned}$$

since f is monoidal. Thus $\mathrm{Free}(f)_\bullet$ is an internalized monoidal functor, defining the action of Free on morphisms. It is not difficult to verify that Free preserves composition and identities, and thus defines a functor $\mathrm{Free} : \mathbf{MonCat} \to \mathbf{IntMonCat}$.

Finally, we will show that Free defines a section for Int. Given a monoidal category $(\mathcal{C}, \otimes, I)$, we must verify that the internalization of $(\mathrm{Free}(\mathcal{C})_\bullet, \Phi^{\mathrm{Free}}_\bullet)$ – see Theorem 3.11 – is $(\mathcal{C}, \otimes, I)$. Since $\mathrm{Free}(\mathcal{C})_1 = \mathcal{C}$ is the underlying category of this internalization, we need only verify that the monoidal structures agree. By construction, we have

$$\Phi^{\mathrm{Free}}(1^{\mathrm{Free}}) = I$$

and

$$\Phi^{\mathrm{Free}} \mu^{\mathrm{Free}}(x, y) = x \otimes y$$

for each $c, d \in \mathcal{C}$, so we need only check that the associator and unitors induced by their graded analogs α, λ, ρ are the associator a and unitors l, r of $(\mathcal{C}, \otimes, I)$. For $x, y, z \in \mathcal{C}$, the internalized associator

$$(\mathcal{L}^{-1} \circ \Phi\alpha \circ \mathcal{R})_{x,y,z}$$

is a word in $x, y, z, \otimes$, and a, so since it has the same domain and codomain as $a_{c,d,e}$ they must be equal by Mac Lane's Coherence Theorem. Similarly,

$$(u \circ \Phi\lambda \circ \mathcal{L})_x$$

and

$$(u \circ \Phi\rho \circ \mathcal{R})_x$$

are words in $x, \otimes, l$, and r, so they are equal to l_c and r_c by Mac Lane's Coherence Theorem, completing the proof. □

Our last main result of this section justifies the name of the functor Free, by showing that it is left adjoint to Int. In other words, the free internalized category associated to a monoidal category is the "most efficient" way to assign a grading, external product, and internalization to that monoidal category.

Proposition 4.9. *The free functor* Free *of Proposition 4.8 is left adjoint to* Int*:*

$$\begin{tikzcd} \mathbf{MonCat} \arrow[r, bend left, "\mathrm{Free}"] \arrow[r, phantom, "\perp"] & \mathbf{IntMonCat} \arrow[l, bend left, "\mathrm{Int}"] \end{tikzcd} .$$

Proof. Given a monoidal category $(\mathcal{C}, \otimes_{\mathcal{C}}, I_{\mathcal{C}})$ and an internalized monoidal category $(\mathcal{D}_\bullet, \Phi^{\mathcal{D}}_\bullet)$, we will establish a bijection

$$(-)^\flat : \mathbf{IntMonCat}(\mathrm{Free}(\mathcal{C})_\bullet, \mathcal{D}_\bullet) \cong \mathbf{MonCat}(\mathcal{C}, \mathrm{Int}(\mathcal{D}_\bullet)) : (-)^\sharp. \tag{4.10}$$

First, suppose given an internalized monoidal functor $f : \mathrm{Free}(\mathcal{C})_\bullet \to \mathcal{D}_\bullet$. We define the $f^\flat : \mathcal{C} \to \mathrm{Int}(\mathcal{D}_\bullet)$ to be the functor f_1, which has the desired type, since $\mathrm{Free}(\mathcal{C})_1 = \mathcal{C}$ and the underlying category of $\mathrm{Int}(\mathcal{D}_\bullet)$ is $\mathcal{D}_1$. To see that f_1 is monoidal, we first observe that for $x, y \in \mathcal{C}$, we have

$$x \otimes_{\mathcal{C}} y = \Phi_2^{\mathrm{Free}(\mathcal{C})} \mu_{1,1}^{\mathrm{Free}(\mathcal{C})}(x, y)$$

by Proposition 4.8. It follows that

$$f^\flat(x \otimes_{\mathcal{C}} y) = f_1 \Phi_2^{\mathrm{Free}(\mathcal{C})} \mu_{1,1}^{\mathrm{Free}(\mathcal{C})}(x, y),$$

and since f is internalized monoidal we have isomorphisms

$$f_1 \Phi_2^{\mathrm{Free}(\mathcal{C})} \mu_{1,1}^{\mathrm{Free}(\mathcal{C})}(x, y) \cong \Phi_2^{\mathcal{D}} f_2 \mu_{1,1}^{\mathrm{Free}(\mathcal{C})}(x, y) \cong \Phi_2^{\mathcal{D}} \mu_{1,1}^{\mathcal{D}}(f_1 x, f_1 y).$$

By definition the right-hand side is precisely the monoidal product of x and y in $\mathrm{Int}(\mathcal{D}_\bullet)$. Similarly, by Proposition 4.8 we have

$$f^\flat(I_{\mathcal{C}}) = f_1 \Phi_0^{\mathrm{Free}(\mathcal{C})}(1^{\mathrm{Free}(\mathcal{C})}) \cong \Phi_0^{\mathcal{D}} f_0(1^{\mathrm{Free}(\mathcal{C})}) \cong \Phi_0^{\mathcal{D}}(1^{\mathcal{D}}),$$

which is the identity of $\mathrm{Int}(\mathcal{D}_\bullet)$. Thus $f^\flat$ is indeed a monoidal functor.

Conversely, suppose given a monoidal functor $g : \mathcal{C} \to \mathrm{Int}(\mathcal{D}_\bullet)$. We define $g^\sharp : \mathrm{Free}(\mathcal{C})_\bullet \to D_\bullet$ by taking $g_0^\sharp : \mathbb{1} \to \mathcal{D}_0$ to be the functor which identifies $1^{\mathcal{D}} \in \mathcal{D}_0$, and defining

$$\begin{aligned} g_m^\sharp &: \mathrm{Free}(\mathcal{C})_m \to \mathcal{D}_m \\ (x_1, \ldots, x_m) &\mapsto \mathop{\boldsymbol{\mu}}_{i=1}^{m} g(x_i) \end{aligned}$$

for $m > 0$. As in the proof of Proposition 4.8, we must select a bias for $\mu^{\mathcal{D}}$ to define the iterated product.[7] To be concrete, we assume that all parantheses are concentrated on the left, that is, that

$$\mathop{\boldsymbol{\mu}}_{i=1}^{m} d_i = \mu^{\mathcal{D}}(\mu^{\mathcal{D}}(\cdots \mu^{\mathcal{D}}(d_1, d_2), d_3), \cdots, d_m).$$

Since the external monoidal product of $\mathrm{Free}(\mathcal{C})_\bullet$ is given by concatenation, $g^\sharp$ is automatically graded monoidal. To see that it is an internalized monoidal functor, we first observe that

$$g_1^\sharp \Phi_0^{\mathrm{Free}(\mathcal{C})}(1^{\mathrm{Free}(\mathcal{C})}) = g(I_{\mathcal{C}}) \cong \Phi_0^{\mathcal{D}}(1^{\mathcal{D}}) = \Phi_0^{\mathcal{D}} g_0^\sharp(1^{\mathrm{Free}(\mathcal{C})})$$

as g is monoidal. Additionally, using the isomorphism $\mathrm{id}_{\mathcal{D}_1} \cong \Phi_1^{\mathcal{D}}$, we have

$$g_1^\sharp \Phi_1^{\mathrm{Free}(\mathcal{C})}(x) = g_1^\sharp(x) \cong \Phi_1^{\mathcal{D}} g_1^\sharp(x)$$

for each $x \in \mathcal{C}$. Now, suppose by induction that for a given $k \geq 1$ we have a natural isomorphism

$$g_1^\sharp \Phi_k^{\mathrm{Free}(\mathcal{C})} \cong \Phi_k^{\mathcal{D}} g_k^\sharp.$$

Let $x_1, \ldots, x_k, x_{k+1} \in \mathcal{C}$. Then we have

$$g_1^\sharp \Phi_{k+1}^{\mathrm{Free}(\mathcal{C})}(x_1, \ldots, x_k, x_{k+1}) = g\left(\bigotimes_{i=1}^{k} x_i \otimes x_{k+1}\right) \cong \Phi_2^{\mathcal{D}} \mu_{1,1}(g\left(\bigotimes_{i=1}^{k} x_i\right), g(x_{k+1}))$$

[7]In fact, this implies that the "adjunction" is really a 2-adjunction, that is, an equivalence of the hom-categories in (4.10). We will only be concerned with functors up to natural isomorphism, and consequently we opt to sweep this technical non-issue under the rug and never think about it again.

as g is monoidal. But

$$g\left(\bigotimes_{i=1}^{k} x_i\right) = g_1^\sharp \Phi_k^{\mathrm{Free}(\mathcal{C})}(x_1, \dots, x_k),$$

so by the induction hypothesis

$$g_1^\sharp \Phi_{k+1}^{\mathrm{Free}(\mathcal{C})}(x_1, \dots, x_k, x_{k+1}) \cong \Phi_2^{\mathcal{D}} \mu_{1,1}(\Phi_k^{\mathcal{D}} g_k^\sharp(x_1, \dots, x_k), g(x_{k+1})).$$

Composing with the left absorber $\mathcal{L}_{k,1}$ for $\Phi_\bullet^{\mathcal{D}}$, this is isomorphic to

$$\Phi_{k+1}^{\mathcal{D}} \mu_{k,1}(g_k^\sharp(x_1, \dots, x_k), g(x_{k+1})) = \Phi_{k+1}^{\mathcal{D}} \mathop{\mu}_{i=1}^{k+1} g(x_i) = \Phi_{k+1}^{\mathcal{D}} g_{k+1}^\sharp(x_1, \dots, x_{k+1}).$$

Thus, by induction we conclude that

$$g_1^\sharp \Phi_m^{\mathrm{Free}(\mathcal{C})} \cong \Phi_m^{\mathcal{D}} g_m^\sharp,$$

so $g^\sharp$ is an internalized monoidal functor.[8]

Finally, to verify that $(-)^\flat$ and $(-)^\sharp$ are inverses, let f and g be as above, and let $c, x_1, \dots, x_m \in \mathcal{C}$. Then

$$(f^\flat)_m^\sharp(x_1, \dots, x_m) = \mathop{\mu}_{i=1}^{m} f^\flat(x_i) = \mathop{\mu}_{i=1}^{m} f_1(x_i).$$

But f is a graded monoidal functor, so

$$\mathop{\mu}_{i=1}^{m} f_1(x_i) \cong f_m\left(\mathop{\mu}_{i=1}^{m} x_i\right) = f_m(x_1, \dots, x_m),$$

by the definition of $\mu^{\mathrm{Free}(\mathcal{C})}$, and hence $(f^\flat)^\sharp \cong f$. On the other hand,

$$(g^\sharp)^\flat(c) = g_1^\sharp(c) = g(c),$$

so $(g^\sharp)^\flat = g$, completing the proof. □

As an immediate consequence of Proposition 4.9, we have:

Corollary 4.11. *Internalization preserves limits.*

5. The Operadic Point of View

In this section we will construct a $\mathbb{N}$-colored (non-symmetric) operad of categories which encodes the structure of graded monoidal categories with internalizations. More precisely, we will show that algebras over this operad are exactly internalized categories.

We write $\mathbb{N}^{<\infty}$ for the set of finite sequences of natural numbers. Given elements $s := (s_1, \dots, s_m)$ and $t := (t_1, \dots, t_n)$ of $\mathbb{N}^{<\infty}$, we denote by $s \cdot t$ their concatenation

$$s \cdot t := (s_1, \dots, s_m, t_1, \dots, t_n).$$

Additionally, we write $\varnothing$ for the empty string.

Definition 5.1. A *fork* is a word w in the letters $\{1, (-)_m, \mu_{m,n}, \Phi_m\}_{m,n\in\mathbb{N}}$ with a particular form, equipped with a *domain* $d(w) \in \mathbb{N}^{<\infty}$ and a *codomain* $c(w) \in \mathbb{N}$, defined recursively as follows:

(1) $w = 1$, with $d(w) = \varnothing$ and $c(w) := 0$.
(2) $w = (-)_m$, with $d(w) := (m)$ and $c(w) := m$.

[8] The induction used here can also be circumvented using Corollary 5.17, by extracting all but the outermost application of $\Phi^{\mathcal{D}}$ and then using the associators to obtain the desired isomorphism $g_1^\sharp \Phi_m^{\mathrm{Free}} \cong \Phi_m^{\mathcal{D}} g_m^\sharp$.

(3) $w = \mu_{m,n}(u, v)$, where u and v are forks satisfying $c(u) = m$ and $c(v) = n$, with $d(w) := d(u) \cdot d(v)$ and $c(w) := m + n$.
(4) $w = \Phi_m(u)$, where u is a fork satisfying $c(u) = m$, with $d(w) := d(u)$ and $c(w) := 1$.

Example 5.2. The following are all examples of well-defined forks:
- $\mu_{m,n+r}((-)_m, \mu_{n,r}((-)_n, (-)_r))$, with domain (m, n, r) and codomain $m + n + r$.
- $\Phi_{m+1}\mu_{m,1}((-)_m, \Phi_0(1))$, with domain (m) and codomain 1.
- $\mu_{1,1}(\Phi_m((-)_m), \Phi_m((-)_m))$, with domain (m, m) and codomain 2.
- $\mu_{0,0}(1, 1)$, with domain $\varnothing$ and codomain 0.

The idea is that the set of forks captures precisely the structure of those objects which can be formed in an internalized category. The internalized category operad we aim to construct will have forks as its operations. To distinguish forks by their type, we make the following definition:

Definition 5.3. The *signature* $\sigma(w)$ of a fork w is the totality of its domain and codomain: if $d(w) = (m_1, \dots, m_r)$ and $c(w) = n$ then

$$\sigma(w) := (m_1, \dots, m_r; n).$$

We say that w is a $\sigma(w)$-fork.

We seek an operad of categories, not merely of sets, so we must now define morphisms of forks. When realizing internalized categories as algebras over this operad, these morphisms will become exactly the natural isomorphisms that define the structure of the graded monoidal category and its internalization – namely, the associators, unitors, and absorbers.

Definition 5.4. Let u and v be forks with the same signature σ. An *atomic morphism of σ-forks* $\psi : u \to v$ is a formal expression of one of the following forms:
(1) $\psi = \mathrm{id}_\sigma$, where $u = v$.
(2) $\psi = \iota$, where $c(v) = 1$ and $u = \Phi_1(v)$.
(3) $\psi = \alpha_{m,n,r}$, where x, y, z are forks with codomains m, n, and r (respectively) such that $u = \mu_{m,n+r}(x, \mu_{n,r}(y, z))$ and $v = \mu_{m+n,r}(\mu_{m,n}(x, y), z)$.
(4) $\psi = \lambda_m$, where x is a fork with codomain m such that $u = \mu_{0,m}(1, x)$ and $v = x$.
(5) $\psi = \rho_m$, where x is a fork with codomain m such that $u = \mu_{m,0}(x, 1)$ and $v = x$.
(6) $\psi = \mathcal{L}_{m,n}$, where x, y are forks with codomains m and n (respectively) such that $u = \Phi_{n+1}\mu_{1,n}(\Phi_m x, y)$ and $v = \Phi_{m+n}\mu_{m,n}(x, y)$.
(7) $\psi = \mathcal{R}_{m,n}$, where x, y are forks with codomains m and n (respectively) such that $u = \Phi_{m+1}\mu_{m,1}(x, \Phi_n y)$ and $v = \Phi_{m+n}\mu_{m,n}(x, y)$.

A *morphism of* σ-forks $\psi : u \to v$ is defined recursively:
(1) ψ is atomic.
(2) $\psi = \mu_{m,n}(\beta, \gamma)$, where $\beta : u' \to v'$ and $\gamma : u'' \to v''$ are morphisms of forks (with $c(u') = c(v') = m$, $c(u'') = c(v'') = n$), $u = \mu_{m,n}(u', u'')$, and $v = \mu_{m,n}(v', v'')$.
(3) $\psi = \Phi_m(\beta)$, where $\beta : u' \to v'$ is a morphism of forks (with $c(u') = c(v') = m$), and $u = \Phi_m(u')$.
(4) $\psi = \beta^{-1}$, where $\beta : v \to u$ is a morphism of forks.
(5) $\psi = \gamma \circ \beta$, where $\beta : u \to w$ and $\gamma : w \to v$ are morphisms of forks.

We have now defined the "free" category of forks, wherein morphisms are formal expressions of the form above and composition is merely concatenation. To shape this into a more useful category, we must impose relations on these expressions to endow them with the desired behaviors.

Definition 5.5. Let u and v be forks with the same signature σ, and let $M_\sigma(u,v)$ be the set of morphisms of σ-forks $u \to v$. We define an equivalence relation $E_\sigma(u,v)$ on $M_\sigma(u,v)$ so that

(1) Formal composition is associative and has id_σ as its unit, up to equivalence.
(2) Formal inverses are inverses up to equivalence.
(3) Φ_m and $\mu_{m,n}$ are functorial, up to equivalence.
(4) $\alpha_{m,n,r}$, λ_m, ρ_m, ι, $\mathcal{L}_{m,n}$, and $\mathcal{R}_{m,n}$ are natural.
(5) $\alpha_{m,n,r}$, λ_m, and ρ_m satisfy the coherence axioms of Definition 3.1, up to equivalence.
(6) $\mathcal{L}_{m,n}$ and $\mathcal{R}_{m,n}$ satisfy the coherence axioms of Definition 3.10, up to equivalence.

The generators of this equivalence relation can be made explicit; for instance, if $\beta : u \to w$, $\gamma : w \to w'$, and $\delta : w' \to v$, then the formal composites $\delta \circ (\gamma \circ \beta)$ and $(\delta \circ \gamma) \circ \beta$ are equivalent. We omit the full description of each generator.

Definition 5.6. Given a signature $\sigma = (m_1, \ldots, m_k; n)$, the *category of σ-forks*, denoted $\mathcal{P}_\sigma = \mathcal{P}(m_1, \ldots, m_k; n)$ has as its objects the set of σ-forks, and where the set of morphisms

$$\mathcal{P}_\sigma(u,v) := \frac{M_\sigma(u,v)}{E_\sigma(u,v)}.$$

Writing $[\psi]$ for the equivalence class of $\psi \in M_\sigma(u,v)$, we define composition by

$$\begin{aligned} \circ : \mathcal{P}_\sigma(v,w) \times \mathcal{P}_\sigma(u,v) &\to \mathcal{P}_\sigma(u,w) \\ (\gamma, \beta) &\mapsto [\gamma \circ \beta]. \end{aligned}$$

By Definition 5.5 this indeed defines a category, where the identity on the fork u is given by $[\mathrm{id}_\sigma]$.

We will subsequently suppress the brackets and refer only to the category $\mathcal{P}_\sigma$, interpreting the above notion of equivalence as equality.

Since forks are intended to serve as the operations of our operad, they should themselves be composable as n-morphisms. The composition of forks will also give rise to a composition of fork morphisms, both of which we now define:

Definition 5.7. Let $u_1, \ldots, u_r, v$ be forks such that

$$d(v) = (c(u_1), \ldots, c(u_r)).$$

Then we define the *composite fork* $v \diamond (u_1, \ldots, u_n)$ recursively:

(1) If $v = 1$ then $r = 0$, so $\{u_1, \ldots, u_r\}$ is empty. We define this vacuous composite to be v.
(2) If $v = (-)_m$ then $r = 1$, and we define
$$v \diamond u_1 := u_1.$$
(3) If $v = \mu_{m,n}(w_1, w_2)$ then we have $d(v) = d(w_1) \cdot d(w_2)$, and hence $r = m + n$, so we define
$$v \diamond (u_1, \ldots, u_r) := \mu_{m,n}(w_1 \diamond (u_1 \ldots u_m), w_2 \diamond (u_{m+1}, \ldots, u_r)).$$
(4) If $v = \Phi_m(w)$ then we have $d(v) = d(w_1)$, so we define
$$v \diamond (u_1, \ldots, u_r) := \Phi_m(w \diamond (u_1, \ldots, u_r)).$$

Given fork morphisms $\psi_i : u_i \to u_i'$ (with the same signatures as above), we define $v(\psi_1, \ldots, \psi_r) : v \diamond (u_1, \ldots, u_r) \to v \diamond (u_1', \ldots, u_r')$ again using recursion:

(1) If $v = 1$ then (vacuously)
$$v(\psi_1, \dots, \psi_r) := \mathrm{id}\,.$$
(2) If $v = (-)_m$ then $r = 1$ and
$$v(\psi_1) := \psi_1.$$
(3) If $v = \mu_{m,n}(w_1, w_2)$ then we define
$$v(\psi_1, \dots, \psi_r) := \mu_{m,n}(w_1(\psi_1, \dots, \psi_m), w_2(\psi_{m+1}, \dots, \psi_r)).$$
(4) If $v = \Phi_m(w)$ then we define
$$v(\psi_1, \dots, \psi_r) := \Phi_m(w(\psi_1, \dots, \psi_r)).$$

Finally, if $\psi : v \to v'$ then we define $\psi(u_1, \dots, u_r) : v \diamond (u_1, \dots, u_r) \to v' \diamond (u_1, \dots, u_r)$ recursively as follows:

(1) If ψ is atomic then we simply take $\psi(u_1, \dots, u_r)$ to be the corresponding atomic morphism $v \diamond (u_1, \dots, u_r) \to v' \diamond (u_1, \dots, u_r)$.
(2) If $\psi = \mu_{m,n}(\beta, \gamma)$ then
$$\psi(u_1, \dots, u_r) := \mu_{m,n}(\beta(u_1, \dots, u_m), \gamma(u_{m+1}, \dots, u_r)).$$
(3) If $\psi = \Phi_m(\beta)$ then
$$\psi(u_1, \dots, u_r) := \Phi_m(\beta(u_1, \dots, u_r)).$$
(4) If $\psi = \beta^{-1}$ then
$$\psi(u_1, \dots, u_r) := \beta(u_1, \dots, u_r)^{-1}.$$
(5) If $\psi = \gamma \circ \beta$ then
$$\psi(u_1, \dots, u_r) := \gamma(u_1, \dots, u_r) \circ \beta(u_1, \dots, u_r).$$

Finally, we define the *vertical composite* $\psi \diamond (\psi_1, \dots, \psi_r) : v \diamond (u_1, \dots, u_r) \to v' \diamond (u'_1, \dots, u'_r)$ by
$$\psi \diamond (\psi_1, \dots, \psi_r) := \psi(u'_1, \dots, u'_r) \circ v(\psi_1, \dots, \psi_r).$$

As one might hope, vertical composition is "functorial," in that it behaves well with respect to composition and identity morphisms. It also satisfies a natural "interchange law." This behavior is captured by the following lemmata, which are the last ingredients we need to complete the assembly of our internalized category operad.

Lemma 5.8. *Let ψ and ψ_i be as in Definition 5.7, and let $\theta : v' \to v''$ and $\theta_i : u'_i \to u''_i$ for $i = 1, \dots, r$ be morphisms of forks. Then*
$$v(\theta_1 \circ \psi_1, \dots, \theta_r \circ \psi_r) = v(\theta_1, \dots, \theta_r) \circ v(\psi_1, \dots, \psi_r) \tag{5.9}$$
and
$$(\theta \circ \psi)(u_1, \dots, u_r) = \theta(u_1, \dots, u_r) \circ \psi(u_1, \dots, u_r). \tag{5.10}$$

Proof. We verify the equality (5.9) by induction on v. In the vacuous case $v = 1$ both the left and right sides are id.

If $v = (-)_m$ then $r = 1$, and we have
$$v(\theta_1 \circ \psi_1) = \theta_1 \circ \psi_1 = v(\theta_1) \circ v(\psi_1).$$

If $v = \mu_{m,n}(x, y)$ then $r = m + n$ and we have
$$\begin{aligned} v(\theta_1 \circ \psi_1, \dots, \theta_r \circ \psi_r) &= \mu_{m,n}(x(\theta_i \circ \psi_i)_{i=1}^m, y(\theta_j \circ \psi_j)_{j=m+1}^r) \\ &= \mu_{m,n}(x(\theta_i)_{i=1}^m \circ x(\psi_i)_{i=1}^m, y(\theta_j)_{j=m+1}^r \circ y(\psi_j)_{j=m+1}^r), \end{aligned}$$

where the second equality follows from induction. By the functoriality of $\mu_{m,n}$, this is equal to

$$\mu_{m,n}(x(\theta_i)_{i=1}^m, y(\theta_j)_{j=m+1}^r) \circ \mu_{m,n}(x(\psi_i)_{i=1}^m, y(\psi_j)_{j=m+1}^r),$$

which is precisely

$$v(\theta_1, \ldots, \theta_r) \circ v(\psi_1, \ldots, \psi_r).$$

Finally, if $v = \Phi_m(x)$ then

$$\begin{aligned} v(\theta_1 \circ \psi_1, \ldots, \theta_r \circ \psi_r) &= \Phi_m(x(\theta_1 \circ \psi_1, \ldots, \theta_r \circ \psi_r)) \\ &= \Phi_m(x(\theta_1, \ldots, \theta_r) \circ x(\psi_1, \ldots, \psi_r)), \end{aligned}$$

where the second equality follows by induction. Since Φ_m is functorial this is equal to

$$\Phi_m(x(\theta_1, \ldots, \theta_r)) \circ \Phi_m(x(\psi_1, \ldots, \psi_r)) = v(\theta_1, \ldots, \theta_r) \circ v(\psi_1, \ldots, \psi_r),$$

completing the verification of (5.9).

To finish the proof, we observe that (5.10) follows immediately from Definition 5.7. □

Lemma 5.11. *Given forks v and $u_1, \ldots, u_r$ with $\sigma := \sigma(v)$ and $\sigma_i := \sigma(u_i)$, we have*

$$v(\mathrm{id}_{\sigma_1}, \ldots, \mathrm{id}_{\sigma_r}) = \mathrm{id}_{\sigma'} = \mathrm{id}_\sigma(u_1, \ldots, u_r). \tag{5.12}$$

Proof. To prove the left equality, we proceed by induction on v. If $v = 1$ or $v = (-)_m$ then this follows immediately. Suppose $v = \mu_{m,n}(x, y)$. Then by induction we have

$$\begin{aligned} v(\mathrm{id}_{\sigma_1}, \ldots, \mathrm{id}_{\sigma_r}) = \mu_{m,n}(x(\mathrm{id}_{\sigma_1}, \ldots, \mathrm{id}_{\sigma_m}), y(\mathrm{id}_{\sigma_{m+1}}, \ldots, \mathrm{id}_{\sigma_r})) \\ = \mu_{m,n}(\mathrm{id}_{\sigma(x)}, \mathrm{id}_{\sigma(y)}), \end{aligned}$$

so using the functoriality of $\mu_{m,n}$ we conclude that

$$v(\mathrm{id}_{\sigma_1}, \ldots, \mathrm{id}_{\sigma_r}) = \mathrm{id}_{\sigma'}.$$

Finally, if $v = \Phi_m(x)$ then

$$v(\mathrm{id}_{\sigma_1}, \ldots, \mathrm{id}_{\sigma_r}) = \Phi_m(x(\mathrm{id}_{\sigma_1}, \ldots, \mathrm{id}_{\sigma_r})) = \Phi_m(\mathrm{id}_{\sigma(x)}) = \mathrm{id}_{\sigma'},$$

where the second equality follows from induction and the third from the functoriality of Φ_m. This proves the left-hand equality in (5.12).

The right-hand equality in (5.12) follows directly from the fact that id_σ is atomic, and from Definition 5.7. □

Lemma 5.13. *If ψ_i and ψ are as in Definition 5.7 then*

$$\psi \diamond (\psi_1, \ldots, \psi_r) = v'(\psi_1, \ldots, \psi_r) \circ \psi(u_1, \ldots, u_r). \tag{5.14}$$

Proof. We will show that (5.14) holds using induction on v. In the vacuous case $v = 1$ we must have that $v'(\psi_1, \ldots, \psi_r) = v'(\varnothing) = \mathrm{id}$ (this can be checked using the fact that v' must be a word in 1 and $\mu_{0,0}$ to have the same signature as v, along with induction), and thus we have

$$\psi \diamond (\psi_1, \ldots, \psi_r) = \psi(\varnothing) \circ \mathrm{id} = \psi(\varnothing)$$

and

$$v'(\psi_1, \ldots, \psi_r) \circ \psi(u_1, \ldots, u_r) = \mathrm{id} \circ \psi(\varnothing) = \psi(\varnothing).$$

Next, suppose $v = (-)_m$. We proceed by induction on ψ. If ψ is atomic then we must have that $\psi = \mathrm{id}$, so $v' = (-)_m$ and $r = 1$. It follows that

$$\psi(u_1') \circ v(\psi_1) = \mathrm{id} \circ \psi_1 = \psi_1$$

and

$$v'(\psi_1) \circ \psi(u_1) = \psi_1 \circ \mathrm{id} = \psi_1,$$

as desired. If $\psi = \beta^{-1}$ for some $\beta : v' \to v$ for which the claim holds then we have

$$\beta(u_1', \dots, u_r') \circ v'(\psi_1, \dots, \psi_r) = v(\psi_1, \dots, \psi_r) \circ \beta(u_1, \dots, u_r).$$

Then composing with $\psi(u_1', \dots, u_r')$ on the left and $\psi(u_1, \dots, u_r)$ on the right we obtain

$$v'(\psi_1, \dots, \psi_r) \circ \psi(u_1, \dots, u_r) = \psi(u_1', \dots, u_r') \circ v(\psi_1, \dots, \psi_r).$$

Finally, if $\psi = \gamma \circ \beta$ for some pair of morphisms $\beta : v \to w$, $\gamma : w \to v'$ satisfying the induction hypothesis then

$$\begin{aligned}
\psi(u_1', \dots, u_r') \circ v(\psi_1, \dots, \psi_r) &= \gamma(u_i')_{i=1}^r \circ \beta(u_i')_{i=1}^r \circ v(\psi_i)_{i=1}^r \\
&= \gamma(u_i')_{i=1}^r \circ w(\psi_i)_{i=1}^r \circ \beta(u_i)_{i=1}^r \\
&= v'(\psi_i)_{i=1}^r \circ \gamma(u_i)_{i=1}^r \circ \beta(u_i)_{i=1}^r \\
&= v'(\psi_1, \dots, \psi_r) \circ \psi(u_1, \dots, u_r).
\end{aligned}$$

We cannot have that $\psi = \mu_{m,n}(\beta, \gamma)$ or $\psi = \Phi_m(\beta)$, since then the domain of ψ could not be $(-)_m$, so this verifies the equality in (5.14) when $v = (-)_m$.

Now, suppose $v = \mu_{m,n}(x, y)$. Again, we proceed by induction on ψ. If ψ is atomic then to have the right domain we must have $\psi \in \{\alpha, \lambda, \rho, \mathrm{id}\}$, in which case (5.14) holds by naturality. If $\psi = \beta^{-1}$ or $\psi = \gamma \circ \beta$ for morphisms β, γ already satisfying the induction hypothesis then the same argument as the case when $v = (-)_m$ shows that (5.14) holds. If $\psi = \mu_{m,n}(\beta, \gamma)$ with $\beta : x \to x'$ and $\gamma : y \to y'$ satisfying the induction hypothesis then $v' = \mu_{m,n}(x', y')$, and we have

$$\begin{aligned}
\psi(u_i')_{i=1}^r \circ v(\psi_i)_{i=1}^r &= \mu_{m,n}(\beta(u_i')_{i=1}^m, \gamma(u_j')_{j=m+1}^r) \circ \mu_{m,n}(x(\psi_i)_{i=1}^m, y(\psi_j)_{j=m+1}^r) \\
&= \mu_{m,n}(\beta(u_i')_{i=1}^m \circ x(\psi_i)_{i=1}^r, \gamma(u_j')_{j=m+1}^r \circ y(\psi_j)_{j=m+1}^r) \\
&= \mu_{m,n}(x'(\psi_i)_{i=1}^m \circ \beta(u_i)_{i=1}^m, y'(\psi_j)_{j=m+1}^r \circ \gamma(u_j)_{j=m+1}^r) \\
&= v'(\psi_i)_{i=1}^r \circ \psi(u_i)_{i=1}^r.
\end{aligned}$$

Lastly, ψ cannot be of the form $\Phi_m(\beta)$, since then its domain could not be $v = \mu_{m,n}(x, y)$. Thus (5.14) holds when $v = \mu_{m,n}(x, y)$.

Finally, suppose $v = \Phi_m(x)$. As above, we proceed by induction on ψ. If ψ is atomic then we must have $\psi \in \{\iota, \mathcal{L}, \mathcal{R}, \mathrm{id}\}$, so (5.14) follows from naturality. If $\psi = \beta^{-1}$ or $\psi = \gamma \circ \beta$ then by the same argument as the case $v = (-)_m$ shows that (5.14) holds. If $\psi = \Phi_m(\beta)$ for some $\beta : x \to x'$ for which the induction hypothesis holds then we have $v' = \Phi_m(x')$, and

$$\begin{aligned}
\psi(u_i')_{i=1}^r \circ v(\psi_i)_{i=1}^r &= \Phi_m(\beta(u_i')_{i=1}^r) \circ \Phi_m(x(\psi_i)_{i=1}^r) \\
&= \Phi_m(\beta(u_i')_{i=1}^r \circ x(\psi_i)_{i=1}^r) \\
&= \Phi_m(x'(\psi_i)_{i=1}^r \circ \beta(u_i)_{i=1}^r) \\
&= v'(\psi_i)_{i=1}^r \circ \beta(u_i)_{i=1}^r.
\end{aligned}$$

Lastly, we cannot have $\psi = \mu_{m,n}(\beta, \gamma)$, since then v could not be its domain. Thus (5.14) holds when $v = \Phi_m(x)$, completing the proof. □

We have finally established all the technology necessary to define the internalized category operad.

Proposition 5.15. *The categories $\mathcal{P}_\sigma$ assemble to form a non-symmetric $\mathbb{N}$-colored operad of categories $\mathcal{P}$. Given signatures $\sigma_j = (m_1^j, \dots, m_{k_j}^j; n_j)$ for $j = 1, \dots, r$, $\sigma = (n_1, \dots, n_r; n)$, and $\sigma' = (m_1^1, \dots, m_{k_r}^r; n)$, the structure map γ is the functor*

$$\gamma : \mathcal{P}_\sigma \times \mathcal{P}_{\sigma_1} \times \cdots \times \mathcal{P}_{\sigma_r} \to \mathcal{P}_{\sigma'}$$

defined by

$$\gamma(v; u_1, \ldots, u_r) := v \diamond (u_1, \ldots, u_r),$$

(and similarly for morphisms; see Definition 5.7) while the identity map $e_m \in \mathcal{P}(m; m)$ *is the object* $(-)_m$*.*

Proof. We must first check that γ is functorial. Suppose given fork morphisms

$$v \xrightarrow{\psi} v' \xrightarrow{\theta} v''$$

and

$$u_i \xrightarrow{\psi_i} u_i' \xrightarrow{\theta_i} u_i''$$

for $i = 1, \ldots, r$, where $v \in \mathcal{P}_\sigma$ and $u_i \in \mathcal{P}_{\sigma_i}$. Then we have

$$\begin{aligned}\gamma((\theta; \theta_1, \ldots, \theta_r) \circ (\psi; \psi_1, \ldots, \psi_r)) &= \gamma(\theta \circ \psi; \theta_i \circ \psi_i)_{i=1}^r \\ &= (\theta \circ \psi) \diamond (\theta_i \circ \psi_i)_{i=1}^r.\end{aligned}$$

But by Lemma 5.8 and Lemma 5.13 we have

$$\begin{aligned}(\theta \circ \psi)(u_i'')_{i=1}^r \circ v(\theta_i \circ \psi_i)_{i=1}^r &= \theta(u_i'')_{i=1}^r \circ \psi(u_i'')_{i=1}^r \circ v(\theta_i)_{i=1}^r \circ v(\psi_i)_{i=1}^r \\ &= \theta(u_i'')_{i=1}^r \circ v'(\theta_i)_{i=1}^r \circ \psi(u_i')_{i=1}^r \circ v(\psi_i)_{i=1}^r \\ &= (\theta \diamond (\theta_i)_{i=1}^r) \circ (\psi \diamond (\psi_i)_{i=1}^r),\end{aligned}$$

which is precisely

$$\gamma(\theta; \theta_1, \ldots, \theta_r) \circ \gamma(\psi; \psi_1, \ldots, \psi_r),$$

so γ preserves composition. Additionally, by Lemma 5.11 we have

$$\gamma(\mathrm{id}_\sigma; \mathrm{id}_{\sigma_1, \ldots, \sigma_r}) = \mathrm{id}_\sigma \diamond (\mathrm{id}_{\sigma_1}, \ldots, \mathrm{id}_{\sigma_r}) = \mathrm{id}_{\sigma'},$$

and thus γ is indeed a functor.

It remains to be shown that $\mathcal{P}$ satisfies the axioms of a colored operad. We verify only the unitality condition, since associativity follows from a similar (and significantly more tedious) argument. First, we must verify that

$$\gamma((-)_m; w) = w$$

for each $w \in \mathcal{P}(s_1, \ldots, s_p; m)$. This follows immediately from Definition 5.7, for we have

$$\gamma((-)_m; w) = (-)_m \diamond w = w.$$

Additionally, we must check that

$$\gamma(w'; (-)_{m_1}, \ldots, (-)_{m_q}) = w'$$

for each $w' \in \mathcal{P}(m_1, \ldots, m_q; t)$. We proceed by induction on w'. In the vacuous case $w' = 1$ this is trivial. If $w' = (-)_m$ then $q = 1$ and $t = m_1$, and we have

$$\gamma(w'; (-)_{m_1}) = (-)_m \diamond (-)_{m_1} = (-)_{m_1} = w'.$$

Now, suppose $w' = \mu_{a,b}(x, y)$. Then $q = a + b$, and by induction we have

$$\begin{aligned}\gamma(w'; (-)_{m_1}, \ldots, (-)_{m_q}) &= w' \diamond ((-)_{m_1}, \ldots, (-)_{m_q}) \\ &= \mu_{a,b}(x \diamond ((-)_{m_1}, \ldots, (-)_{m_a}), y \diamond ((-)_{m_{a+1}}, \ldots, (-)_{m_q})) \\ &= \mu_{a,b}(x, y),\end{aligned}$$

which is precisely w'.

Finally, if $w' = \Phi_a(x)$ then by induction we have

$$\begin{aligned} \gamma(w'; (-)_{m_1}, \ldots, (-)_{m_q}) &= w' \diamond ((-)_{m_1}, \ldots, (-)_{m_q}) \\ &= \Phi_a(x \diamond ((-)_{m_1}, \ldots, (-)_{m_q})) \\ &= \Phi_a(x), \end{aligned}$$

which is precisely w', completing the proof. □

Theorem 5.16. *An algebra over $\mathcal{P}$ is precisely a graded monoidal category equipped with an internalization.*

Proof. An algebra over $\mathcal{P}$ consists of a category $\mathcal{C}_m$ for each $m \in \mathbb{N}$, together with functors

$$\Omega_{m_1,\ldots,m_r;n} : \mathcal{P}(m_1, \ldots, m_r; n) \times \mathcal{C}_{m_1} \times \cdots \times \mathcal{C}_{m_r} \to \mathcal{C}_n$$

which satisfies the unitality and associativity conditions. We define

$$\begin{aligned} 1 &:= \Omega_{\varnothing;0}(1) \\ \mu_{m,n}(x, y) &:= \Omega_{m,n;m+n}(\mu_{m,n}((-)_m, (-)_n); x, y) \\ \Phi_m(x) &:= \Omega_{m;1}(\Phi_m((-)_m); x) \\ (\alpha_{m,n,r})_{x,y,z} &:= \Omega_{m,n,r;m+n+r}(\alpha_{m,n,r}; x, y, z) \\ (\lambda_m)_x &:= \Omega_{m;m}(\lambda_m; x) \\ (\rho_m)_x &:= \Omega_{m;m}(\rho_m; x) \\ (\mathcal{L}_{m,n})_{x,y} &:= \Omega_{m,n;1}(\mathcal{L}_{m,n}; x, y) \\ (\mathcal{R}_{m,n})_{x,y} &:= \Omega_{m,n;1}(\mathcal{R}_{m,n}; x, y), \end{aligned}$$

and claim that $(\mathcal{C}_\bullet, \mu, 1)$ forms a graded monoidal category with an internalization $\Phi_\bullet$ (with the appropriate natural transformations). But functoriality of μ and Φ, the naturality of α, λ, ρ, $\mathcal{L}$, and $\mathcal{R}$, and all the coherence relations between them are satisfied in $\mathcal{P}$. Moreover, each natural transformation is a priori an isomorphism in $\mathcal{P}$, and has the required domain and codomain, so by construction this defines a graded monoidal category with equipped with an internalization.

The converse follows from the same argument. Given an internalized category $(\mathcal{C}_\bullet, \Phi_\bullet)$, we can define the functors $\Omega_{m_1,\ldots,m_r;n}$ by reversing the definitions above, and then using recursion to extend this definition to all objects and morphisms. That is, every fork and morphism of forks is assembled from components on which the definition of Ω is already given, which uniquely determines $\Omega_{m_1,\ldots,m_r;n}$ for an arbitrary signature. □

We conclude this section with a consequence of Theorem 5.16, which allows us to compute a given fork inside an internalized category by first computing all external monoidal products, and then internalizing. This offer some justification for the term "absorbers" given to the natural isomorphisms of Definition 3.10, as they allow the last application of the internalization functor Φ to "absorb" all other nested instances of Φ wihtin the fork.

Corollary 5.17 (Absorption Coherence)**.** *Let $(\mathcal{C}_\bullet, \mu, 1)$ be a graded monoidal category equipped with an internalization $\Phi_\bullet$, and let w be an object of $\mathcal{C}_1$ of the form*

$$w = \Phi_{m+n}\mu_{m,n}(u, v).$$

Then w is isomorphic to an object of the form

$$\Phi_{m'+n'}\mu_{m',n'}(u', v'),$$

where Φ does not occur in u' or v'.

Proof. Since $(\mathcal{C}_\bullet, \Phi_\bullet)$ is an algebra over $\mathcal{P}$, it will suffice to show that for any fork

$$w = \Phi_{m+n}\mu_{m,n}(u, v)$$

there exists a fork

$$w' = \Phi_{m'+n'}\mu_{m',n'}(u', v')$$

such that Φ does not occur in u' or v'.

We proceed by induction on u to show that w is isomorphic to a fork of the form

$$\Phi_{m'+n'}\mu_{m',n}(u', v'),$$

where Φ does not occur in u', and such that Φ occurs in v and v' an equal number of times.

For the base case, we observe that if $u = 1$ or if $u = (-)_m$ then this is trivial. Namely, we simply take the identity on w to be our isomorphism.

Now, if $u = \Phi_r(x)$ for some fork x then $\mathcal{L}_{1,n}$ defines an isomorphism of forks

$$w = \Phi_{n+1}\mu_{1,n}(\Phi_m(x), v) \cong \Phi_{r+n}\mu_{r,n}(x, v).$$

But by the induction hypothesis there is an isomorphism

$$\Phi_{r+n}\mu_{r,n}(x, v) \cong \Phi_{r'+n'}\mu_{r',n'}(x', v')$$

such that Φ does not occur in x', and so that Φ occurs in v and v' an equal number of times.

Finally, if $u = \mu_{p,q}(x, y)$ then $\Phi_{p+q+n}\alpha_{p,q,n}$ defines an isomorphism of forks

$$w = \Phi_{p+q+n}\mu_{p+q,n}(\mu_{p+q}(x, y)) \cong \Phi_{p+q+n}\mu_{p,q+n}(x, \mu_{q,n}(y, v)).$$

Again, by the induction hypothesis we have an isomorphism

$$\Phi_{p+q+n}\mu_{p,q+n}(x, \mu_{q,n}(y, v)) \cong \Phi_{p'+n'}\mu_{p',n'}(x', v'),$$

where Φ does not occur in x', and where Φ occurs in $\mu_{q,n}(y, v)$ and v' an equal number of times. Since it also appears in v and $\mu_{q,n}(y, v)$ an equal number of times, this proves the desired intermediate result.

To complete the proof, we note that a symmetric argument shows that

$$w \cong \Phi_{m'+n'}\mu_{m',n'}(u', v'),$$

where Φ does not occur in v', and where Φ occurs in u and u' an equal number of times. But then by the argument above there exists an isomorphism

$$\Phi_{m'+n'}\mu_{m',n'}(u', v') \cong \Phi_{m''+n''}\mu_{m'',n''}(u'', v''),$$

where Φ does not occur in u'', and occurs in v' and v'' an equal number of times, namely, zero. Composing the isomorphisms we find that

$$w \cong \Phi_{m''+n''}\mu_{m'',n''}(u'', v''),$$

and since Φ does not occur in u'' or v'' we are done. □

6. Internal Smash Products of Spectra

We conclude this paper by outlining two applications of internalized categories to reconstruct a pair of monoidal products which occur naturally in stable homotopy theory.

6.1. **Day Convolution.** The smash product of symmetric or orthogonal spectra (or, more generally, of diagram spectra over a given monoid object in the category of $\mathcal{D}$-spaces, where $\mathcal{D}$ is some topological category – see [7]) can be formulated as a special case of Day convolution. In this section we describe how Day convolution fits into the framework of an internalized graded monoidal category, and use this to prove that the Day convolution monoidal product is closed and symmetric.

We first recall the general procedure of Day convolution. We fix a bicomplete, closed, symmetric monoidal category $(\mathcal{V}, \otimes_{\mathcal{V}}, I_{\mathcal{V}})$, and a (small) $\mathcal{V}$-enriched monoidal category $(\mathcal{C}, \otimes_{\mathcal{C}}, I_{\mathcal{C}})$. We additionally denote by $\mathrm{Fun}(\mathcal{C}, \mathcal{V})$ the $\mathcal{V}$-enriched functor category, with hom-object $\mathrm{Fun}(\mathcal{C}, \mathcal{V})(F, G)$ defined by the $\mathcal{V}$-enriched end

$$\mathrm{Fun}(\mathcal{C}, \mathcal{V})(F, G) := \int_{c \in \mathcal{C}} \underline{V}(Fc, Gc)$$

for given $\mathcal{V}$-functors $F, G : \mathcal{C} \to \mathcal{V}$.

Definition 6.1. The *Day convolution product* on $\mathrm{Fun}(\mathcal{C}, \mathcal{V})$ is the bifunctor

$$\otimes_{\mathrm{Day}} : \mathrm{Fun}(\mathcal{C}, \mathcal{V}) \times \mathrm{Fun}(\mathcal{C}, \mathcal{V}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{V})$$

defined by

$$F \otimes_{\mathrm{Day}} G := \mathrm{Lan}_{\otimes_{\mathcal{C}}}(F \otimes_{\mathcal{V}} G)$$

for $\mathcal{V}$-functors $F, G \in \mathrm{Fun}(\mathcal{C}, \mathcal{V})$, where

$$(F \otimes_{\mathcal{V}} G)(c) := Fc \otimes_{\mathcal{V}} Gc.$$

By way of the coend formula for left Kan extensions, we compute the image of $c \in \mathcal{C}$ under $F \otimes_{\mathrm{Day}} G$ as

$$(F \otimes_{\mathrm{Day}} G)(c) = \int^{c_1, c_2 \in \mathcal{C}} \mathcal{C}(c_1 \otimes_{\mathcal{C}} c_2, c) \otimes_V Fc_1 \otimes_V Gc_2.$$

Remark 6.2. The product $\otimes_{\mathrm{Day}}$ is indeed monoidal, with unit object given by よ$(I_{\mathcal{C}}) : \mathcal{C} \to \mathcal{V}$, where よ is the $\mathcal{V}$-enriched (contravariant) Yoneda embedding $\mathcal{C}^{\mathrm{op}} \to \mathrm{Fun}(\mathcal{C}, \mathcal{V})$. One can prove this classically, but this will also follow immediately from Theorem 3.11 after we've shown that $\otimes_{\mathrm{Day}}$ arises as the internalization of a graded monoidal product.

Definition 6.3. We define a $\mathcal{V}$-enriched graded monoidal category $(\mathrm{Fun}_{\bullet}(\mathcal{C}, \mathcal{V}), \mu, 1)$ by

(1) $\mathrm{Fun}_m(\mathcal{C}, \mathcal{V}) := \mathrm{Fun}(\mathcal{C}^m, \mathcal{V})$ for $m \in \mathbb{N}$.
(2) $\mu_{m,n}(F, G) : \mathcal{C}^{m+n} \to \mathcal{V}$ is the $\mathcal{V}$-functor

$$\mu_{m,n}(F, G)(c_1, \ldots, c_{m+n}) := F(c_1, \ldots, c_m) \otimes_V G(c_{m+1}, \ldots, c_{m+n})$$

for $m, n \in \mathbb{N}$, where $F : \mathcal{C}^m \to \mathcal{V}$ and $G : \mathcal{C}^n \to \mathcal{V}$ are $\mathcal{V}$-functors.
(3) $1 : \mathcal{C}^0 \to \mathcal{V}$ is the functor which carries the terminal $\mathcal{V}$-category $\mathcal{C}^0$ to the object $I_{\mathcal{V}} \in \mathcal{V}$.

The natural isomorphisms for μ are defined termwise by the corresponding isomorphisms for $\otimes_{\mathcal{V}}$.

We aim to show that $\otimes_{\mathrm{Day}}$ is an internalization of μ:

Definition 6.4. Given $m \in \mathbb{N}$, let $\Phi_m : \mathrm{Fun}(\mathcal{C}^m, \mathcal{V}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{V})$ be the $\mathcal{V}$-functor

$$\Phi_m(F) := \mathrm{Lan}_{\otimes_{\mathcal{C}}^m}(F),$$

where $\otimes_{\mathcal{C}}^m : \mathcal{C}^m \to \mathcal{C}$ is the[9] $\mathcal{V}$-functor

$$(c_1, \ldots, c_m) \mapsto \bigotimes_{i=1}^{m} c_i.$$

In the special case that $m = 0$, we adopt the convention that $\otimes_{\mathcal{C}}^0$ is the $\mathcal{V}$-functor $\mathcal{C}^0 \to \mathcal{C}$ which identifies the unit $I_{\mathcal{C}}$.

Remark 6.5. We note that since $\mathrm{Fun}_0(\mathcal{C}, \mathcal{V}) = \mathrm{Fun}(\mathcal{C}^0, \mathcal{V})$ is naturally isomorphic to $\mathcal{V}$, Φ_0 can be viewed as a $\mathcal{V}$-functor $\mathcal{V} \to \mathrm{Fun}(\mathcal{C}, \mathcal{V})$. Given $v \in \mathcal{V}$ and $c \in \mathcal{C}$, we have

$$\Phi_0(v)(c) = (\mathrm{Lan}_{\otimes_{\mathcal{C}}^0}(v))(c) = \int^{\mathcal{C}^0} \mathcal{C}(I_{\mathcal{C}}, c) \otimes_{\mathcal{V}} v = よ(I_{\mathcal{C}})(c) \otimes_{\mathcal{V}} v$$

by the coend formula for left Kan extensions.

Theorem 6.6. *Φ defines an internalization on* $(\mathrm{Fun}_\bullet(\mathcal{C}, \mathcal{V}), \mu, 1)$.

Proof. First, we observe that $\Phi_1(F)$ is the left Kan extension of F along $\mathrm{id}_{\mathcal{C}}$, so there is a natural isomorphism $u : \Phi_1 \cong \mathrm{id}_{\mathrm{Fun}(\mathcal{C},\mathcal{V})}$.

Next, we will construct the left absorber for Φ as the composite of a chain of natural isomorphisms; the construction of the right absorber is completely dual.

Fix $m, n \in \mathbb{N}$, and let $F : \mathcal{C}^m \to \mathcal{V}$ and $G : \mathcal{C}^n \to \mathcal{V}$ be $\mathcal{V}$-functors. Additionally, let $c \in \mathcal{C}$. We aim to construct a (natural) isomorphism

$$\mathrm{Lan}_{\otimes_{\mathcal{C}}^{n+1}}(\mathrm{Lan}_{\otimes_{\mathcal{C}}^{m}}(F), G)(c) \cong \mathrm{Lan}_{\otimes_{\mathcal{C}}^{m+n}}(F, G)(c).$$

Unwinding the left-hand side using the coend formula for left Kan extensions, we obtain

$$\int^{d, c_1, \ldots, c_n} \mathcal{C}\Big(d \otimes_{\mathcal{C}} \bigotimes_{i=1}^{n} c_i, c\Big) \otimes_{\mathcal{V}} \left(\int^{d_1, \ldots, d_m} \mathcal{C}\Big(\bigotimes_{j=1}^{m} d_j, d\Big) \otimes_{\mathcal{V}} F(d_j)_{j=1}^m \right) \otimes_{\mathcal{V}} \; G(c_i)_{i=1}^n,$$

where both coends are indexed by $\mathcal{C}$. By the Fubini Theorem, this is naturally isomorphic to

$$\int^{d, d_1, \ldots, d_m, c_1, \ldots, c_n} \mathcal{C}\Big(d \otimes_{\mathcal{C}} \bigotimes_{i=1}^{n} c_i, c\Big) \otimes_{\mathcal{V}} \mathcal{C}\Big(\bigotimes_{j=1}^{m} d_j, d\Big) \otimes_{\mathcal{V}} \; F(d_1, \ldots, d_m) \otimes_{\mathcal{V}} \; G(c_1, \ldots, c_n),$$

which is naturally isomorphic to

$$\int^{d_1, \ldots, d_m, c_1, \ldots, c_n} \mathcal{C}\Big(\bigotimes_{j=1}^{m} d_j \otimes_{\mathcal{C}} \bigotimes_{i=1}^{n} c_i, c\Big) \otimes_{\mathcal{V}} \; F(d_j)_{j=1}^m \otimes_{\mathcal{V}} \; G(c_i)_{i=1}^n$$

via the co-Yoneda Lemma. But using the definition of Φ and the coend formula for left Kan extensions, this is precisely

$$\mathrm{Lan}_{\otimes_{\mathcal{C}}^{m+n}}(F, G)(c),$$

so the desired natural isomorphism is the composite of the isomorphisms above. Note that in the case that one of m or n is 0 the same argument holds, with the corresponding m-fold or n-fold tensor products being replaced by the unit object $I_{\mathcal{C}}$.

In the interest of preserving the reader's (and author's) sanity, we omit the verification of the coherence axioms for the absorbers. □

[9] Here one should be slightly careful in the case that $\otimes_{\mathcal{C}}$ is not strict. One is then forced to choose a particular arrangement of parantheses, that is, a bias for the m-fold monoidal product on $\mathcal{C}$. Each natural isomorphism in the internalization is then composed with some sequence of associators to ensure compatibility, but Mac Lane's Coherence Theorem guarantees that this will not cause problems.

Proposition 6.7. *The monoidal product on* $\mathrm{Fun}(\mathcal{C}, \mathcal{V})$ *induced by* Φ *is the Day convolution product* $\otimes_{\mathrm{Day}}$.

Proof. This follows immediately from the definitions. □

Corollary 6.8. *The Day convolution product is a monoidal product, with unit* $よ(I_{\mathcal{C}})$.

Proof. It follows from Proposition 6.7 and Theorem 3.11 that $\otimes_{\mathrm{Day}}$ is a monoidal product, while the second claim is a consequence of Remark 6.5, since

$$\Phi_0(1)(c) = よ(I_{\mathcal{C}})(c) \otimes_{\mathcal{V}} I_{\mathcal{V}} \cong よ(I_{\mathcal{C}})(c)$$

for each $c \in \mathcal{C}$. □

We have shown that Day convolution is simply a special case of internalization. In fact, we can further recover the properties of symmetry and closure that $\otimes_{\mathrm{Day}}$ enjoys by using the formal behavior of internalized categories.

Lemma 6.9. *The graded monoidal category* $\mathrm{Fun}_\bullet(\mathcal{C}, \mathcal{V})$ *is symmetric, as is its internalization (see Definition 3.6 and Definition 3.14).*

Proof. This follows from the symmetry of $\otimes_{\mathcal{V}}$, and a diagram chase on its braiding and the absorbers of Φ. □

Lemma 6.10. *The graded monoidal category* $\mathrm{Fun}_\bullet(\mathcal{C}, \mathcal{V})$ *is closed (see Definition 3.7).*

Proof. We define the internal hom of μ by the $\mathcal{V}$-functors

$$\mathscr{F}_{m,n} : \mathrm{Fun}(\mathcal{C}^n, \mathcal{V})^{\mathrm{op}} \times \mathrm{Fun}(\mathcal{C}^{m+n}, \mathcal{V}) \to \mathrm{Fun}(\mathcal{C}^m, \mathcal{V})$$

by

$$\mathscr{F}_{m,n}(F, G)(c_1, \ldots, c_m) := \int_{d_1,\ldots,d_n \in \mathcal{C}} \underline{V}(F(d_1, \ldots, d_n), G(c_1, \ldots, c_m, d_1, \ldots, d_n)),$$

where $c_1, \ldots, c_m \in \mathcal{C}$, $F : \mathcal{C}^n \to \mathcal{V}$, and $G : \mathcal{C}^{m+n} \to \mathcal{V}$. To verify that this does indeed define an internal hom of μ, fix $m, n \in \mathbb{N}$ and suppose given $\mathcal{V}$-functors $F : \mathcal{C}^m \to \mathcal{V}$, $G : \mathcal{C}^n \to \mathcal{V}$, and $H : \mathcal{C}^{m+n} \to \mathcal{V}$. Then uwinding the definitions, we have

$$\mathrm{Fun}_{m+n}(\mathcal{C}, \mathcal{V})(\mu(F, G), H) = \int_{c_1,\ldots,c_{m+n} \in \mathcal{C}} \underline{V}(F(c_i)_{i=1}^m \otimes_{\mathcal{V}} G(c_i)_{i=m+1}^{m+n}, H(c_i)_{i=1}^{m+n}),$$

and the right-hand side is isomorphic to

$$\int_{c_1,\ldots,c_{m+n} \in \mathcal{C}} \underline{V}(F(c_1, \ldots, c_m), \underline{V}(G(c_{m+1}, \ldots, c_{m+n}), H(c_1, \ldots, c_{m+n})))$$

using the closed structure of $\mathcal{V}$. Now, using the universal property of ends, this is isomorphic to

$$\int_{c_1,\ldots,c_m \in \mathcal{C}} \underline{V}(F(c_i)_{i=1}^m, \int_{c_{m+1},\ldots,c_{m+n}} \underline{V}(G(c_i)_{i=m+1}^{m+n}, H(c_i)_{i=1}^{m+n})),$$

which is precisely

$$\mathrm{Fun}_m(\mathcal{C}, \mathcal{V})(F, \mathscr{F}(G, H)),$$

and hence μ is closed with internal hom $\mathscr{F}$. □

Lemma 6.11. *The internalization* Φ *of* μ *has a right adjoint, and is closed (see Definition 3.19).*

Proof. Since μ is closed by Lemma 6.10, the second claim follows immediately from the first. Given $m \in \mathbb{N}$ and a $\mathcal{V}$-functor $G : \mathcal{C}^m \to \mathcal{V}$, we define a $\mathcal{V}$-functor $\Psi_m : \mathrm{Fun}(\mathcal{C}, \mathcal{V}) \to \mathrm{Fun}(\mathcal{C}^m, \mathcal{V})$ by

$$\Psi_m(G)(c_1, \ldots, c_m) := G\Big(\bigotimes_{i=1}^{m} c_i\Big)$$

for each $c_1, \ldots, c_m \in \mathcal{C}$. Then since Φ_m is given by left Kan extension along $\otimes_{\mathcal{C}}^m$, and Ψ_m precomposes with $\otimes_{\mathcal{C}}^m$, it is automatic that Φ_m is left adjoint to Ψ_m. □

Proposition 6.12. *The Day convolution product is symmetric and closed.*

Proof. Symmetry follows from Corollary 3.15 and Lemma 6.10, and closure follows from Corollary 3.20, Lemma 6.10, and Lemma 6.11. □

To specialize to the case of spectra (symmetric, orthogonal, etc.) we refer to [7]. Fix a symmetric monoidal category $\mathcal{D}$ enriched over based topological spaces, and recall that a *$\mathcal{D}$-space* is a continuous functor $\mathcal{D} \to \mathbf{Top}_*$. Taking $\mathcal{V} := \mathbf{Top}_*$, with symmetric monoidal product given by the usual smash product $\wedge$, and $\mathcal{C} := \mathcal{D}$, we see that $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is precisely the category $\mathcal{D}(\mathbf{Top}_*)$ of $\mathcal{D}$-spaces. Day convolution then provides a means of turning the external product induced by $\wedge$ into the *internal smash product of $\mathcal{D}$-spaces* – see Definition 21.4 of [7].

A *$\mathcal{D}$-spectrum* (Definition 1.9 in [7]) over a monoid object R in $\mathcal{D}(\mathbf{Top}_*)$ is a $\mathcal{D}$-space equipped with a right action by R. By Theorem 2.2 of [7] the category of $\mathcal{D}$-spectra is equivalent to the category of $\mathcal{D}_R$-spaces, where $\mathcal{D}_R$ is a $\mathbf{Top}_*$-enriched category defined using $\mathcal{D}$ and R (see Construction 2.1 of [7] for a complete definition). Moreover, when R is a commutative monoid $\mathcal{D}_R$ is naturally equipped with a symmetric monoidal product, in which case Day convolution induces a symmetric monoidal product on the category of $\mathcal{D}$-spectra.

Finally, we note that both symmetric spectra and orthogonal spectra can be realized as $\mathcal{D}$-spectra over commutative monoids. In the case of the former (see Example 4.2 of [7]), we have $\mathcal{D} := \Sigma$, the category of (unbased) finite sets $[n] := \{0, \ldots, n\}$ for each $n \in \mathbb{N}$ and their permutations, and $R := S_\Sigma$, the continuous functor $\Sigma \to \mathbf{Top}_*$ which sends $[n]$ to S^n. For the latter (Example 4.4 of [7]) we instead take $\mathcal{D}$ to be the category of (unbased) finite-dimensional real inner product spaces and isometric linear isomorphisms between them, and let R be the functor $\mathcal{D} \to \mathbf{Top}_*$ sending V to the one-point compactification S^V. In both cases, the resulting internal smash product induced by Day convolution is precisely the usual smash product of the corresponding spectra.

6.2. **The Smash Product of S-Modules.** Finally, we turn our attention to S-modules, defined in [1]. After reviewing the relevant definitions, we will show how the smash product of S-modules can be realized as an internalization of the external smash product of coordinate-free spectra.

Recall that, given a real inner product U isomorphic to $\mathbb{R}^\infty$, the direct sum of countably-many copies of $\mathbb{R}$, there exists a category $\mathscr{S}U$ of *coordinate-free spectra over the universe U*. Throughout the remainder of this section, we will refer to these simply as "spectra (over U)."

The category $\mathscr{S}$ enjoys a number of convenient properties. By Proposition I.1.1 of [1] it is both complete and cocomplete. Moreover, given a pair of universes U and U', there is an *external smash product*

$$\wedge : \mathscr{S}U \times \mathscr{S}U' \to \mathscr{S}(U \oplus U')$$

of spectra over U and U', which is associative and symmetric (up to natural isomorphism). Lastly, there is a functor $\Sigma^\infty : \mathbf{Top}_* \to \mathscr{S}U$, called the *suspension spectrum functor*, which identifies a based space with a spectrum over U, and we write S to denote the suspension spectrum $\Sigma^\infty S^0$ of the 0-sphere S^0 by S.

For universes U and U', let $\mathscr{I}(U, U')$ be the set of linear isometries $U \to U'$. Viewing U and U' as topological spaces, whose topologies are induced by their finite dimensional subspaces, we can equip $\mathscr{I}(U, U')$ with the function space topology. Now, given any space A and a continuous map $\alpha : A \to \mathscr{I}(U, U')$, together with a spectrum E over U, there exists a spectrum $A \ltimes E$ over U', called the *half-twisted smash product* of A and E. Moreover, this construction is functorial in E, and satisfies the following properties (see Proposition I.2.2 of [1]):

(1) For each $E \in \mathscr{S}U$ there is a canonical isomorphism
$$\{\mathrm{id}_U\} \ltimes E \cong E.$$
(2) For spaces A, B with maps $\alpha : A \to \mathscr{I}(U, U')$ and $\beta : B \to \mathscr{I}(U', U'')$ and a spectrum $E \in \mathscr{S}U$ there is a canonical isomorphism
$$(B \times A) \ltimes E \cong B \ltimes (A \ltimes E).$$
(3) Let A_1 and A_2 be spaces with maps $\alpha_i : A_i \to \mathscr{I}(U_i, U_i')$. Then given spectra $E_1 \in \mathscr{S}U_1$ and $E_2 \in \mathscr{S}U_2$ there is a canonical isomorphism
$$(A_1 \times A_2) \ltimes (E_1 \wedge E_2) \cong (A_1 \ltimes E_1) \wedge (A_2 \ltimes E_2).$$
(4) For each $E \in \mathscr{S}U$ and each based space X there is a canonical isomorphism
$$A \ltimes (E \wedge X) \cong (A \ltimes E) \wedge X.$$

Now, writing U^j for the direct sum of j copies of a universe U, and taking $\mathscr{L}(j) := \mathscr{I}(U^j, U)$, there is a symmetric operad, called the *linear isometries operad*, equipped with structure maps
$$\gamma : \mathscr{L}(k) \times \mathscr{L}(j_1) \times \cdots \times \mathscr{L}(j_k) \to \mathscr{L}(j_1 + \cdots + j_k),$$
defined by
$$\gamma(g; f_1, \ldots, f_k) := g(f_1, \ldots, f_k).$$
For each j and each spectrum $E \in \mathscr{S}U^j$, the identity map on $\mathscr{L}(j)$ then induces a spectrum $\mathscr{L}(j) \ltimes E \in \mathscr{S}U$. Moreover, the endofunctor
$$\mathscr{L}(1) \ltimes (-) : \mathscr{S}U \to \mathscr{S}U$$
defines a monad, which we denote by $\mathbb{L}$, with multiplication map $\mathbb{L}\mathbb{L}E \to \mathbb{L}E$ induced by the structure map $\gamma : \mathscr{L}(1) \times \mathscr{L}(1) \to \mathscr{L}(1)$, and unit $E \to \mathbb{L}E$ induced by the inclusion $\{0\} = \mathscr{L}(0) \to \mathscr{L}(1)$ which identifies id_U.

We write $\mathscr{S}[\mathbb{L}]$ for the category of $\mathbb{L}$-algebras, which we refer to as $\mathbb{L}$-*spectra*. This category is equipped with a bifunctor $\wedge_{\mathscr{L}} : \mathscr{S}[\mathbb{L}] \times \mathscr{S}[\mathbb{L}] \to \mathscr{S}[\mathbb{L}]$, called the *smash product of* $\mathbb{L}$-*spectra*, defined as follows: let M and N be $\mathbb{L}$-spectra, with actions $\xi_M : \mathbb{L}M \to M$ and $\xi_N : \mathbb{L}N \to N$; we take $M \wedge_{\mathscr{L}} N$ to be the coequalizer of the diagram
$$(\mathscr{L}(2) \times \mathscr{L}(1) \times \mathscr{L}(1)) \ltimes (M \wedge N) \underset{\gamma \ltimes \mathrm{id}}{\overset{\mathrm{id} \ltimes (\xi_M \wedge \xi_N)}{\rightrightarrows}} \mathscr{L}(2) \ltimes (M \wedge N).$$
As shown in Proposition I.5.2 and Theorem I.5.5 of [1] this smash product is symmetric and associative (up to natural isomorphism), and for each $M \in \mathscr{S}[\mathbb{L}]$ there is a canonical map
$$S \wedge_{\mathscr{L}} M \to M,$$

described in Proposition I.8.3 of [1].

An *S-module* is an $\mathbb{L}$-spectrum for which this map is an isomorphism, and we write $\mathscr{M}_S$ for the full subcategory of $\mathscr{S}[\mathbb{L}]$ spanned by the S-modules. For S-modules M and N, we write $M \wedge_S N := M \wedge_{\mathscr{L}} N$, which is again an S-module (see Proposition II.1.2 of [1]), so $\wedge_S$ defines a symmetric monoidal product on $\mathscr{M}_S$.

We aim to show that $\wedge_S$ is realized as the internalization of the external product of spectra $\wedge$. First, we must formulate a suitable graded monoidal category to impose an internalization upon. This is somewhat tricky, primarily because of the non-unitality of $\wedge_{\mathscr{L}}$. We essentially need to define a kind of intermediate, "non-unital" internalized category (i.e. we forget about the degree 0 component, and only require an associative product) which acts like $\mathscr{S}[\mathbb{L}]$ above, and then restrict to those objects which behave like S-modules to recover unitality.

Definition 6.13. Fix $m \geq 1$. Then there is a monad $\mathbb{L}^m$ on $\mathscr{S}U^m$, defined by

$$\mathbb{L}^m(E) := \mathscr{L}(1)^m \ltimes E,$$

where the map $\mathscr{L}(1)^m \to \mathscr{I}(U^m, U^m)$ carries a tuple of isometries $(f_1, \ldots, f_m)$ to the isometry $f_1 \times \cdots \times f_m$. The multiplication map $\mu_m : \mathbb{L}^m\mathbb{L}^m E \to \mathbb{L}^m E$ is induced by m copies of the structure map $\gamma : \mathscr{L}(1) \times \mathscr{L}(1) \to \mathscr{L}(1)$, while the unit $\eta_m : E \to \mathbb{L}^m E$ is induced by the composite

$$E \xrightarrow{\eta} \mathbb{L}E \xrightarrow{\mathbb{L}\eta} \mathbb{L}^2 E \xrightarrow{\mathbb{L}^2\eta} \cdots \xrightarrow{\mathbb{L}^{m-1}\eta} \mathbb{L}^m E,$$

where η is the unit for $\mathbb{L}^1 = \mathbb{L}$, defined above. The category $\mathscr{S}[\mathbb{L}^m]$ of *$\mathbb{L}^m$-spectra* is the category of algebras over $\mathbb{L}^m$.

The categories $\mathscr{S}[\mathbb{L}]_m$ assemble to form a graded category $\mathscr{S}[\mathbb{L}]_\bullet$ (taking $\mathscr{S}[\mathbb{L}]_0$ to be the empty category). The external smash product of spectra then defines an associative graded product on $\mathscr{S}[\mathbb{L}]_\bullet$.

Definition 6.14. For $m, n \geq 1$, there is a bifunctor $\mu : \mathscr{S}[\mathbb{L}]_m \times \mathscr{S}[\mathbb{L}]_n \to \mathscr{S}[\mathbb{L}]_{m+n}$, which, given $(M, \xi_M) \in \mathscr{S}[\mathbb{L}]_m$ and $(N, \xi_N) \in \mathscr{S}[\mathbb{L}]_n$, equips $M \wedge N$ with the structure map $\xi : \mathbb{L}^{m+n}(M \wedge N) \to M \wedge N$ defined by the composite

$$(\mathscr{L}(1)^m \times \mathscr{L}(1)^n) \ltimes (M \wedge N) \cong (\mathscr{L}(1)^m \ltimes M) \wedge (\mathscr{L}(1)^n \ltimes N) \xrightarrow{\xi_M \wedge \xi_N} M \wedge N.$$

We now begin to construct the desired internalization $\Phi_\bullet$. It may seem strange to do this before defining the graded monoidal category that we ultimately aim to internalize, but this category will in fact be defined using our incomplete internalization functors.

Definition 6.15. Given $M \in \mathscr{S}[\mathbb{L}]_m$, we define the underlying object of $\Phi_m(M, \xi_M)$ to be the coequalizer of the diagram

$$(\mathscr{L}(m) \times \mathscr{L}(1)^m) \ltimes M \underset{\gamma \ltimes \mathrm{id}}{\overset{\mathrm{id} \ltimes \xi_M}{\rightrightarrows}} \mathscr{L}(m) \ltimes M.$$

The structure map $\mathbb{L}\Phi_m(M) \to \Phi_m(M)$ is induced by $\gamma : \mathscr{L}(1) \times \mathscr{L}(m) \to \mathscr{L}(m)$.

A priori the internalized product $\wedge_{\mathscr{L}} := \Phi_2\mu_{1,1}$ of $\mathbb{L}$-spectra is associative (see Theorem I.5.5 of [1]). One would hope that S would be the unit of this product, but sadly this is not the case. However, we do in general have a canonical map

$$S \wedge_{\mathscr{L}} M \to M$$

for any $\mathbb{L}$-spectrum M, described in Proposition I.8.3 of [1]. In fact, this generalizes to higher degree spectra and to spaces besides S^0:

Lemma 6.16. *Let (M, ξ_M) be an $\mathbb{L}^m$-spectrum, and let X be a based topological space. Then there is a natural morphism of $\mathbb{L}$-spectra*

$$\Phi_{m+1}\mu(\Phi_0 X, M) \to \Phi_m\mu(X, M).$$

Proof. First, suppose M is a free $\mathbb{L}^m$-algebra $\mathbb{L}^m E = \mathscr{L}(1)^m \ltimes E$, where $E \in \mathscr{S}[U^m]$. Then we seek a natural morphism

$$\mathscr{L}(m+1) \ltimes_{\mathscr{L}(1)^{m+1}} (\Sigma^\infty X \wedge \mathscr{L}(1)^m \ltimes E) \to \mathscr{L}(m) \ltimes_{\mathscr{L}(1)^m} (X \wedge \mathscr{L}(1)^m \ltimes E).$$

First, we observe that γ defines a morphism

$$\mathscr{L}(m+1) \ltimes_{\mathscr{L}(1)^{m+1}} \mathscr{L}(0) \times \mathscr{L}(1)^m \to \mathscr{L}(m),$$

since it coequalizes the pair of morphisms

$$\mathscr{L}(m+1) \times \mathscr{L}(1)^{m+1} \times \mathscr{L}(0) \times \mathscr{L}(1)^m \underset{\gamma\times\mathrm{id}}{\overset{\mathrm{id}\times\gamma^{m+1}}{\rightrightarrows}} \mathscr{L}(m+1) \times \mathscr{L}(0) \times \mathscr{L}(1)^m .$$

This yields a map from

$$\begin{aligned}\mathscr{L}(m+1) \ltimes_{\mathscr{L}(1)^{m+1}} (\Sigma^\infty X \wedge \mathscr{L}(1)^m E) &= \mathscr{L}(m+1) \ltimes_{\mathscr{L}(1)^{m+1}} (\mathscr{L}(0) \ltimes X \wedge \mathscr{L}(1)^m \ltimes E)\\ &\cong (\mathscr{L}(m+1) \times_{\mathscr{L}(1)^{m+1}} \mathscr{L}(0) \times \mathscr{L}(1)^m) \ltimes (X \wedge E)\end{aligned}$$

to

$$\mathscr{L}(m) \ltimes (X \wedge E).$$

But by Lemma 6.20 we have

$$\mathscr{L}(m) \cong \mathscr{L}(m) \times_{\mathscr{L}(1)^m} \mathscr{L}(1)^m,$$

so

$$\begin{aligned}\mathscr{L}(m) \ltimes (X \wedge E) &\cong \mathscr{L}(m) \ltimes_{\mathscr{L}(1)^m} (\mathscr{L}(1)^m \ltimes (X \wedge E))\\ &\cong \mathscr{L}(m) \ltimes_{\mathscr{L}(1)^m} (X \wedge (\mathscr{L}(1)^m \ltimes E)).\end{aligned}$$

Composing the morphisms above yields the desired map.

Now, if M is any $\mathbb{L}^m$-spectrum then there is a split coequalizer diagram

$$\mathbb{L}^m\mathbb{L}^m M \rightrightarrows \mathbb{L}^m M \twoheadrightarrow M .$$

Since split coequalizers are absolute colimits, the canonical morphisms corresponding to the free algebras in the diagram induce a morphism of coequalizer diagrams

$$\begin{tikzcd}
\Phi_{m+1}\mu(\Phi_0 X, \mathbb{L}^m\mathbb{L}^m M) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \Phi_{m+1}\mu(\Phi_0 X, \mathbb{L}^m M) \arrow[r, two heads] \arrow[d] & \Phi_{m+1}\mu(\Phi_0 X, M) \arrow[d, dashed] \\
\Phi_m\mu(X, \mathbb{L}^m\mathbb{L}^m M) \arrow[r, shift left] \arrow[r, shift right] & \Phi_m\mu(X, \mathbb{L}^m M) \arrow[r, two heads] & \Phi_m\mu(X, M)
\end{tikzcd}$$

which induces the desired map for M. □

The map in Lemma 6.16 need not be an isomorphism in general. To obtain the desired internalized category, we restrict our attention to those $\mathbb{L}_m$-spectra for which it is.

Definition 6.17. Given $m \geq 1$, let $\mathscr{M}_m$, *the category of S-modules of degree m*, be the full subcategory of $\mathscr{S}[\mathbb{L}]_m$ spanned by those $\mathbb{L}^m$-algebras M such that the canonical map

$$\Phi_{m+1}\mu(\Phi_0 X, M) \to \Phi_m \mu(X, M)$$

of Lemma 6.16 an isomorphism.

Additionally, let $\mathscr{M}_0 := \mathbf{Top}_*$, the category of based spaces. Then we can naturally extend μ to a graded monoidal product on $\mathscr{M}_\bullet$, with unit S^0.

Remark 6.18. Note that $\mathscr{M}_1 = \mathscr{M}_S$, the category of S-modules (see Definition II.1.1 of [1]). Certainly each spectrum in $\mathscr{M}_1$ is an S-module, by taking $X = S^0$. Conversely, every S-module is a degree 1 S-module by Proposition II.1.4 of [1].

Remark 6.19. If $M \in \mathscr{M}_m$ then $\Phi_m(M) \in \mathscr{M}_1$, so Φ_m restricts to a functor $\mathscr{M}_m \to \mathscr{M}_1$. This is because, given a based space X, we have isomorphisms

$$\Phi_2\mu(\Phi_0 X, \Phi_m M) \cong \Phi_{m+1}\mu(\Phi_0 X, M) \cong \Phi_m\mu(X, M),$$

where the first map is the right absorber, and the second follows from the fact that M a degree m S-module.

To complete the construction of an internalization $\Phi_\bullet$ on $(\mathscr{M}_\bullet, \mu, 1)$, we must define the component $\Phi_0 : \mathscr{M}_0 \to \mathscr{M}_1$. We take $\Phi_0 := \mathscr{L}(0) \ltimes (-) = \Sigma^\infty$, which, by Proposition II.1.2 of [1], defines a functor from based spaces to S-modules, as desired.

We can now verify that the functors $\Phi_\bullet$ indeed define an internalization of μ. For this, we will need a preliminary results, which generalizes Lemma I.5.4 of [1].

Lemma 6.20 (Hopkins)**.** *For integers $k \geq 1$, $j_1, \ldots, j_k \geq 1$, the diagram*

$$\mathscr{L}(k) \times \mathscr{L}(1)^k \times \mathscr{L}(j_1) \times \cdots \times \mathscr{L}(j_k) \underset{\gamma \times \mathrm{id}}{\overset{\mathrm{id} \times \gamma^k}{\rightrightarrows}} \mathscr{L}(k) \times \mathscr{L}(j_1) \times \cdots \times \mathscr{L}(j_k) \xrightarrow{\gamma} \mathscr{L}(j_1 + \cdots + j_k)$$

is a split coequalizer of spaces, and consequently

$$\mathscr{L}(j_1 + \cdots + j_k) \cong \mathscr{L}(k) \times_{\mathscr{L}(1)^k} \mathscr{L}(j_1) \times \cdots \times \mathscr{L}(j_k).$$

Proof. The argument is a direct generalization of the one presented in [1]. Suppose $j_i \geq 1$ for $i = 1, \ldots, k$, and fix isomorphisms $s_i : U^{i_k} \cong U$. We define maps

$$\begin{aligned} h : \mathscr{L}(j_1 + \cdots + j_k) &\to \mathscr{L}(k) \times \mathscr{L}(j_1) \times \cdots \times \mathscr{L}(j_k) \\ f &\mapsto (f\Big(\bigoplus_{i=1}^k s_i\Big)^{-1}; s_1, \ldots, s_k) \end{aligned}$$

and

$$\begin{aligned} r : \mathscr{L}(k) \times \mathscr{L}(j_1) \times \cdots \times \mathscr{L}(j_k) &\to \mathscr{L}(k) \times \mathscr{L}(1)^k \times \mathscr{L}(j_1) \times \cdots \times \mathscr{L}(j_k) \\ (f; g_1, \ldots, g_k) &\mapsto (f; g_1 s_1^{-1}, \ldots, g_k s_k^{-1}; s_1, \ldots, s_k), \end{aligned}$$

and claim that these split the fork above.

First, given $f \in \mathscr{L}(j_1 + \cdots + j_k)$, we observe that

$$\gamma h(f) = f\Big(\bigoplus_{i=1}^k s_i\Big)^{-1}(s_1, \ldots, s_k) = f \bigoplus_{i=1}^k \mathrm{id}_{U^{j_i}} = f,$$

so γh is the identity. Additionally, given $(f; g_1, \ldots, g_k) \in \mathscr{L}(k) \times \mathscr{L}(j_1) \times \cdots \times \mathscr{L}(j_k)$, we have

$$\begin{aligned}(\mathrm{id} \times \gamma^k) r(f; g_1, \ldots, g_k) &= (\mathrm{id} \times \gamma^k)(f; g_1 s_1^{-1}, \ldots, g_k s_k^{-1}; s_1, \ldots, s_k) \\ &= (f; g_1, \ldots, g_k),\end{aligned}$$

so $(\mathrm{id} \times \gamma^k) r$ is the identity as well. Finally,

$$\begin{aligned}(\gamma \times \mathrm{id}) r(f; g_1, \ldots, g_k) &= (\gamma \times \mathrm{id})(f; g_1 s_1^{-1}, \ldots, g_k s_k^{-1}; s_1, \ldots, s_k) \\ &= (f(g_1 s^{-1}, \ldots, g_k s_k^{-1}); s_1, \ldots, s_k),\end{aligned}$$

and

$$\begin{aligned}h\gamma(f; g_1, \ldots, g_k) &= h(f(g_1, \ldots, g_k)) \\ &= (f(g_1, \ldots, g_k)\Big(\bigoplus_{i=1}^{k} s_i\Big)^{-1}; s_1, \ldots, s_k) \\ &= (f(g_1 s^{-1}, \ldots, g_k s_k^{-1}); s_1, \ldots, s_k),\end{aligned}$$

so we conclude that $(\gamma \times \mathrm{id}) r = h\gamma$. ☐

Theorem 6.21. *The functors Φ define an internalization on the graded monoidal category $(\mathscr{M}_\bullet, \mu, S^0)$, and the internalized product on $\mathscr{M}_S$ is precisely $\wedge_S$.*

Proof. We aim to construct absorbers for Φ, as defined in Definition 3.10. We will content ourselves with identifying the left absorbers, as the right absorbers are built through an identical procedure, and omit the verification of the coherence axioms.

Let $M \in \mathscr{M}_m$, $N \in \mathscr{M}_n$. We seek a natural isomorphism

$$\Phi_{n+1}\mu(\Phi_m M, N) \cong \Phi_{m+n}\mu(M, N).$$

First, suppose $m, n > 0$. Since N is an $\mathbb{L}^n$-algebra there is a split coequalizer diagram

$$\mathbb{L}^n\mathbb{L}^n N \rightrightarrows \mathbb{L}^n N \longrightarrow N,$$

so $N \cong \mathscr{L}(1)^n \ltimes_{\mathscr{L}(1)^n} N$. Thus, we have

$$\begin{aligned}\Phi_{n+1}\mu(\Phi_m M, N) &= \mathscr{L}(n+1) \ltimes_{\mathscr{L}(1)^{n+1}} (\mathscr{L}(m) \ltimes_{\mathscr{L}(1)^m} M \wedge N) \\ &\cong \mathscr{L}(n+1) \ltimes_{\mathscr{L}(1)^{n+1}} (\mathscr{L}(m) \ltimes_{\mathscr{L}(1)^m} M \wedge \mathscr{L}(1)^n \ltimes_{\mathscr{L}(1)^n} N) \\ &\cong \mathscr{L}(n+1) \ltimes_{\mathscr{L}(1)^{n+1}} ((\mathscr{L}(m) \times \mathscr{L}(1)^n) \ltimes_{\mathscr{L}(1)^{m+n}} (M \wedge N)).\end{aligned}$$

But by Lemma 6.20 we have

$$\begin{aligned}\mathscr{L}(m+n) \ltimes_{\mathscr{L}(1)^{m+n}} (M \wedge N) &\cong (\mathscr{L}(n+1) \times_{\mathscr{L}(1)^{n+1}} \mathscr{L}(m) \times \mathscr{L}(1)^n) \ltimes_{\mathscr{L}(1)^{m+n}} (M \wedge N) \\ &\cong \mathscr{L}(n+1) \ltimes_{\mathscr{L}(1)^{n+1}} ((\mathscr{L}(m) \times \mathscr{L}(1)^n) \ltimes_{\mathscr{L}(1)^{m+n}} (M \wedge N)),\end{aligned}$$

so we conclude that

$$\Phi_{n+1}\mu(\Phi_m M, N) \cong \mathscr{L}(m+n) \ltimes_{\mathscr{L}(1)^{m+n}} (M \wedge N) = \Phi_{m+n}\mu(M, N),$$

yielding the left absorbers of Φ for positive degrees m and n.

Next, suppose $m = 0$. Then M is a based topological space, and since N is a degree n S-module we have a natural isomorphism

$$\Phi_{n+1}(\Phi_0 M, N) \cong \Phi_n(M, N)$$

by Definition 6.17.

Finally, suppose $n = 0$. Then N is a based topological space, and thus we have natural isomorphisms

$$\begin{aligned}\Phi_1\mu(\Phi_m M, N) &= \mathscr{L}(1) \ltimes_{\mathscr{L}(1)} ((\mathscr{L}(m) \ltimes_{\mathscr{L}(1)^m} M) \wedge N) \\ &\cong (\mathscr{L}(m) \ltimes_{\mathscr{L}(1)^m} M) \wedge N \\ &\cong \mathscr{L}(m) \ltimes_{\mathscr{L}(1)^m} (M \wedge N) \\ &= \Phi_m\mu(M, N),\end{aligned}$$

completing the proof. □

Having shown that the smash product $\wedge_S$ of S-modules is an internalization of $\wedge$, we conclude by proving that it is symmetric, using the formal properties Φ.

Lemma 6.22. *$(\mathscr{M}_\bullet, \mu, S^0)$ is a symmetric graded monoidal category, and Φ is a symmetric internalization.*

Proof. The symmetry of $(\mathscr{M}_\bullet, \mu, S^0)$ comes from that of the external smash product $\wedge$. Naturality of the isomorphisms which define the left and right absorbers in the proof of Theorem 6.21 shows that Φ is symmetric. □

Proposition 6.23. *The monoidal product $\wedge_S$ on the category of S-modules $\mathscr{M}_S$ is symmetric.*

Proof. This follows immediately from Lemma 6.22 and Corollary 3.15. □

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