The Homotopy Hypothesis via Kan Complexes

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1 Introduction

Simplicial sets, or more precisely quasi-categories, offer a convenient and reasonably tame model for ∞-categories. A particularly nice feature of simplicial sets is that they are specified by purely combinatorial data, while still bearing a passing resemblance to topological spaces. One often thinks of a simplicial set not as a collection of sets and maps joining them, but rather as simplices "glued" together, much like how one constructs a simplicial or CW complex. In fact, this analogy is true in some stronger formal sense, which we aim to make precise in this paper.

The *homotopy hypothesis*, which informally states that ∞-groupoids are equivalent to topological spaces, makes the comparison between ∞-categorical structures and spaces concrete. There are a number of ways in which this statement can be formally realized, each corresponding to a different model of ∞-categories. In the case of quasi-categories, it states that equipping the category of simplicial sets with a particular model structure yields a Quillen equivalence with the category of topological spaces (with its usual model structure). Moreover, this model structure on simplicial sets identifies Kan complexes, which we take as our model for ∞-groupoids, as the bifibrant (fibrant and cofibrant) objects, and hence this Quillen equivalence allows one to translate between the homotopy theory of ∞-groupoids and that of topological spaces.

We omit the ∞-categorical statement of the homotopy hypothesis, and instead content ourselves by establishing a 1-categorical equivalence between homotopy categories. After establishing some notational conventions in section 2, we begin by reviewing the main definitions and fundamental results of the theory of model categories in sections 3 and 4, before describing a model structure on simplicial sets to facilitate this Quillen equivalence in section 5. Finally, we conclude by proving the homotopy hypothesis in section 6.

2 Notation

We write **Set** and **Top** for the category of sets and the category of topological spaces, respectively. Given a category $\mathcal{C}$, we use $\mathrm{PSh}(\mathcal{C})$ to denote the category of (set-valued) presheaves on $\mathcal{C}$, that is, the category of (contravariant) functors $\mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$, and write $よ : \mathcal{C} \hookrightarrow \mathrm{PSh}(\mathcal{C})$ for the (covariant) Yoneda embedding. Additionally, we write Δ for the simplex category, whose objects are the sets $[n] := \{0, 1, \ldots, n\}$ for each

$n \geq 0$, and whose morphisms are nondecreasing functions between these sets. We then form the category of simplicial sets, $\mathbf{sSet} := \mathrm{PSh}(\Delta)$.

Unless otherwise specified, we denote by $\varnothing$ the initial object of a category, and by $*$ the terminal object. In **Top**, we write Δ^n_{Top} for the standard topological n-simplex, I for the compact interval $[0,1]$, and S^n for the n-sphere. In **sSet**, we use Δ^n to denote the standard n-simplex, that is, the simplicial set $\mathrm{Hom}_\Delta(-,[n]) = よ([n])$, and $\partial\Delta^n$ for the boundary of this simplex (whose m-simplices are the nonsurjective maps $[m] \to [n]$ in Δ). Finally, we write Λ^n_k for the standard simplicial (n,k)-horn (whose m-simplices are those maps $[m] \to [n]$ in Δ whose image excludes some $j \neq k$); informally, this is the simplicial subset of $\partial\Delta^n$ obtained by deleting the k^{th} face.

3 Model Categories

We begin by recalling some basic definitions and results from the theory of model categories.

Definition 3.1. A *model category* is a bicomplete category $\mathcal{C}$ equipped with three collections of morphisms, $\mathrm{Cof}(\mathcal{C})$, $\mathrm{Fib}(\mathcal{C})$, and $\mathrm{W}(\mathcal{C})$, called *cofibrations*, *fibrations*, and *weak equivalences*, respectively; we say that a morphism is a *trivial cofibration* if it belongs to both $\mathrm{Cof}(\mathcal{C})$ and $\mathrm{W}(\mathcal{C})$, and is a *trivial fibration* if it belongs to both $\mathrm{Fib}(\mathcal{C})$ and $\mathrm{W}(\mathcal{C})$. These classes are subject to the following conditions:

(1) The class $\mathrm{W}(\mathcal{C})$ has the 2-of-3 property, that is, given composable morphisms f and g, if at least two of f, g, and gf are in $\mathrm{W}(\mathcal{C})$ then so is the third.

(2) Each distinguished class of morphisms, $\mathrm{Cof}(\mathcal{C})$, $\mathrm{Fib}(\mathcal{C})$, and $\mathrm{W}(\mathcal{C})$, is closed under retracts (in the arrow category $\mathrm{Arr}(\mathcal{C})$ of $\mathcal{C}$).

(3) Every cofibration has the left lifting property against trivial fibrations, and every fibration has the right lifting property against trivial cofibrations. Diagrammatically, given a solid commuting square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "g"'] \arrow[ur, dashed, "h"] & Y \end{tikzcd}$$

there exists a diagonal lift if i is a cofibration (resp. trivial cofibration) and p is a trivial cofibration (resp. fibration).

(4) Every morphism f can be factored (functorially) in two ways: $f = \widetilde{p}i$ with i a cofibration and $\widetilde{p}$ a trivial fibration; and $f = p\widetilde{i}$ with $\widetilde{i}$ a trivial cofibration and p a fibration.

We note that a few stronger conditions have been assumed here, as in [1], such as the functoriality of the factorization systems and bicompleteness. In some of the literature these hypotheses are weakened slightly, but in nature the model categories one encounters are typically well-behaved, as is the case for those we consider in this paper.

For a first example of a model category, we make the following definition:

Definition 3.2. The *Serre model structure* on **Top** is specified by the following:

(1) The class of cofibrations is given by all retracts of CW inclusions.

(2) The class of fibrations is given by all Serre fibrations.

(3) The class of weak equivalences is given by all weak homotopy equivalences.

The lifting properties of the cofibrations and fibrations in this model structure can be extracted from their corresponding homotopy lifting properties, and the factorization systems are given by factoring a map f through its mapping cylinder or mapping path space.

It turns out that the data specifying a model structure can be weakened slightly. In particular, if one understands trivial cofibrations (resp. trivial fibrations) then the class of fibrations (resp. cofibrations) can be recovered. This is made explicit by the following proposition, which says that each class of morphism is determined by its lifting property.

Proposition 3.3. Let f be a morphism in a model category.

(1) If f has the left lifting property against all trivial fibrations then f is a cofibration.

(2) If f has the right lifting property against all trivial cofibrations then f is a fibration.

(3) If f has the left lifting property against all fibrations then f is a trivial cofibration.

(4) If f has the right lifting property against all cofibrations then f is a trivial fibration.

Proof. We prove only the first claim, as the rest follow by an identical argument. Suppose $f : X \to Y$ has the left lifting property against all trivial fibrations. Then we can factor $f = pi$, with p a trivial fibration and i a cofibration, and thus we obtain a lift g as depicted below:

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & A \arrow[d, "p"] \\ Y \arrow[r, equal] \arrow[ur, dashed, "g"] & B \end{tikzcd} .$$

From this, we obtain a retract diagram

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "f"'] & X \arrow[r, equal] \arrow[d, "i"] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & A \arrow[r, "p"'] & Y \end{tikzcd}$$

so f is a retract of the cofibration i, and is consequently a cofibration. □

In the case of the model structure on **Top** described above, this conveniently allows us to forget the cofibrations, which are much more difficult to pin down than the fibrations.

When working with classical homotopy theory, one quickly discovers that certain spaces are much more manageable than others. There are, however, nicer subclasses of spaces. One particularly nice subclass is that of CW complexes, which behave better than the average space in the context of homotopy theory, and which are sufficiently "dense" in **Top** in the sense that very little homotopical data is lost by limiting one's attention to them. When studying model categories, one often prefers to restrict to a smaller class of "homotopically well-mannered" objects.

Definition 3.4. An object $X \in \mathcal{C}$ is said to be *cofibrant* if the unique map $\varnothing \to X$ is a cofibration, and dually, is said to be *fibrant* if the unique map $X \to *$ is a fibration. An object which is both fibrant and cofibrant is called *bifibrant*.

Definition 3.5. Let $\mathcal{C}$ be a model category.

(1) We denote by $\mathcal{C}_{\mathrm{c}}$ the full subcategory of $\mathcal{C}$ spanned by the cofibrant objects.

(2) We denote by $\mathcal{C}_{\mathrm{f}}$ the full subcategory of $\mathcal{C}$ spanned by the fibrant objects.

(3) We denote by $\mathcal{C}_{\mathrm{cf}}$ the full subcategory of $\mathcal{C}$ spanned by the bifibrant objects.

In **Top**, the cofibrant objects are (retracts of) CW complexes, while all objects are fibrant (the constant map to the point is always a Serre fibration).

From the factorization systems of a model category, we note that for any X we can factor the unique map $\varnothing \to X$ as $\varnothing \to QX \xrightarrow{q_X} X$, where the map $\varnothing \to QX$ is a cofibration and q_X is a trivial fibration (so in particular is a weak equivalence). We say that QX is a *cofibrant replacement* of X, and by the functoriality of the factorization this defines the action of an endofunctor $Q : \mathcal{C} \to \mathcal{C}$ on objects. Moreover, the q_X assemble to form a natural weak equivalence $q : Q \Rightarrow \mathrm{id}_{\mathcal{C}}$, and the pair (Q, q) defines a *left deformation* of $\mathcal{C}$ (see [2]). Dually, we can factor the map $X \to *$ through a fibrant object RX via a weak equivalence $r_X : X \to RX$, which defines the functor $R : \mathcal{C} \to \mathcal{C}$, called *fibrant replacement* and a natural weak equivalence $r : \mathrm{id}_{\mathcal{C}} \Rightarrow R$ making (R, r) a *right deformation* of $\mathcal{C}$.

Lemma 3.6 (Ken Brown's Lemma)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be model categories, and suppose $F : \mathcal{C} \to \mathcal{D}$ is a functor which sends trivial cofibrations between cofibrant objects to weak equivalences. Then F preserves all weak equivalences between cofibrant objects.

There is, of course, a dual result for fibrations and fibrant objects, but we will not need this result in the sequel.

Proof. Suppose given a weak equivalence $f : X \to Y$ of cofibrant objects in $\mathcal{C}$. There is an induced map $f \amalg \mathrm{id}_Y : X \amalg Y \to Y$, which we can factor as the composite pj with p a trivial fibration and j a cofibration. Since X and Y are cofibrant, and one easily verifies that cofibrations are stable under pushouts (this follows from an easy diagram chase), we find that the inclusion maps $\iota_X : X \to X \amalg Y$ and $\iota_Y : Y \to X \amalg Y$ are cofibrations (as they are given by the pushouts of the maps from the initial object $\varnothing$ to Y and X, respectively).

Now, p is a weak equivalence, and we have $p(j\iota_X) = (f \amalg \mathrm{id}_Y)\iota_X = f$ and $p(j\iota_Y) = (f\amalg)\iota_Y = \mathrm{id}_Y$, both of which are weak equivalences, so by the 2-of-3 property $j\iota_X$ and $j\iota_Y$ are trivial cofibrations. By hypothesis, it follows that their images under F are weak equivalences in $\mathcal{D}$. Moreover,

$$Fp \circ F(j\iota_Y) = F(p(j\iota_Y)) = F(\mathrm{id}_Y) = \mathrm{id}_{FY}$$

is a weak equivalence, so by the 2-of-3 property Fp is a weak equivalence. Finally,

$$Fp \circ F(j\iota_X) = F(p(j\iota_X)) = Ff,$$

and both terms on the left are weak equivalences, so by the 2-of-3 property so is Ff, as desired. □

The intuition captured by a model category structure (which one should view as a particular presentation of the underlying category, rather than being intrinsic to it – a category can be equipped with many different model structures) is that it admits some kind of homotopy theory. We conclude this section by making this idea precise by associating to a model category a "homotopy category," wherein we formally identify objects which are related by a weak equivalence.

Definition 3.7. Let $\mathcal{C}$ be a model category. The *homotopy category* of $\mathcal{C}$ is the localization $\mathrm{Ho}\,\mathcal{C} := \mathcal{C}[\mathrm{W}(\mathcal{C})^{-1}]$.

More concretely, it is a category $\mathrm{Ho}\,\mathcal{C}$ equipped with a functor $\gamma : \mathcal{C} \to \mathrm{Ho}\,\mathcal{C}$ such that the image of each weak equivalence under γ is an isomorphism in $\mathrm{Ho}\,\mathcal{C}$, and moreover is the initial such category over $\mathcal{C}$ – that is, if $F : \mathcal{C} \to \mathcal{D}$ is a functor which inverts each weak equivalence in $\mathcal{C}$, then F factors uniquely through a functor $\mathrm{Ho}\,\mathcal{C} \to \mathcal{D}$:

$$\begin{tikzcd} \mathcal{C} \arrow[d, "\gamma"'] \arrow[dr, "F"] & \\ \mathrm{Ho}\,\mathcal{C} \arrow[r, dashed] & \mathcal{D} \end{tikzcd} .$$

While the construction of the homotopy category may appear unwieldy, and indeed in general the morphisms in localizations behave quite badly, it turns out that the model structure gives rise to a much more tractable presentation. For this, one needs to define a suitable notion of a homotopy between morphisms in a model category, which is a somewhat technical ordeal. In particular, there is a handedness to homotopies in a general model category (i.e. one defines both left and right homotopies), and then proves that these agree and define an equivalence relation on morphisms with a cofibrant domain and fibrant codomain. We refer to [1] for the details of these constructions, and instead present the culmination of them in the form of what some authors (for instance [1], [3]) refer to as "the fundamental theorem of model categories."

Theorem 3.8 (Fundamental Theorem of Model Categories)**.** Let $\mathcal{C}$ be a model category, and let $\gamma : \mathcal{C} \to \mathrm{Ho}\,\mathcal{C}$ be the localization functor.

(1) Let $\mathcal{C}_{\mathrm{cf}}/\sim$ be the category with the same objects as $\mathcal{C}_{\mathrm{cf}}$, whose morphisms given by homotopy classes of morphisms in $\mathcal{C}$. Then the inclusion $\mathcal{C}_{\mathrm{cf}} \hookrightarrow \mathcal{C}$ induces an equivalence of categories $\mathcal{C}_{\mathrm{cf}}/\sim \simeq \mathrm{Ho}\,\mathcal{C}$.

(2) If $f : X \to Y$ is a morphism in $\mathcal{C}$ such that γf is an isomorphism in $\mathrm{Ho}\,\mathcal{C}$, then f is a weak equivalence.

In classical homotopy theory, one often prefers to replace the localization of **Top** wherein all weak homotopy equivalences are inverted with the category of CW complexes modulo homotopy. The equivalence of these two categories demonstrates a first application of the fundamental theorem of model categories, since, as noted above, the bifibrant objects in **Top** are (retracts of) CW complexes.

The second part of the theorem above is of particular interest to us because it allows us to pin down weak equivalences using the homotopy category, which is essential to the machinery we develop in the following section.

4 Derived Functors and Quillen Adjunctions

Having defined model categories, a natural question one may ask is how they relate to each other. In particular, how might one define a notion of "morphisms between model categories"? The most useful answer, it turns out, takes the form not of a single functor between model categories (as one might expect), but rather is given by an adjunction compatible with the model structures of the two categories it bridges.

Definition 4.1. Let $\mathcal{C}$ and $\mathcal{D}$ be model categories. A *Quillen adjunction* is an adjunction $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$ such that one of the following (equivalent) conditions are satisfied:

(1) F preserves cofibrations and trivial cofibrations.

(2) G preserves fibrations and trivial fibrations.

(3) F preserves cofibrations and G preserves fibrations.

(4) F preserves trivial cofibrations and G preserves trivial fibrations.

In this case we say that F is a *left Quillen functor* and that G is a *right Quillen functor*.

One views a Quillen adjunction as above as a "morphism of model categories" $\mathcal{C} \to \mathcal{D}$, with the directionality favoring that of the left Quillen functor (although this convention is arbitrary).

Lemma 4.2. Left Quillen functors preserve weak equivalences between cofibrant objects. Dually, right Quillen functors preserve weak equivalences between fibrant objects.

Proof. This follows immediately from Ken Brown's lemma 3.6. □

An important property of Quillen functors is that they descend to functors between homotopy categories[1]. This allows us to understand model categories at the level of their homotopy categories, discarding the pathological objects (i.e. those which are not fibrant or cofibrant, for instance) by inverting weak equivalences. In classical homotopy theory we do this often, preferring to work with CW complexes instead of general spaces; by Whitehead's theorems, this sacrifice is minimal, and exhibits the more general paradigm of model categories merely serving as a vehicle for their homotopy categories.

Definition 4.3. Suppose $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$ is a Quillen adjunction of categories.

(1) The *total left derived functor* $\mathbb{L}F : \operatorname{Ho}\mathcal{C} \to \operatorname{Ho}\mathcal{D}$ of the left Quillen functor F is the composite $\operatorname{Ho}F \circ \operatorname{Ho}Q$, where $\operatorname{Ho}Q$ and $\operatorname{Ho}F$ are the induced maps

$$\begin{tikzcd} \mathcal{C} \arrow[r, "Q"] \arrow[d] & \mathcal{C}_c \arrow[r, "F"] \arrow[d] & \mathcal{D} \arrow[d] \\ \operatorname{Ho}\mathcal{C} \arrow[r, dashed, "\operatorname{Ho}Q"'] & \operatorname{Ho}\mathcal{C}_c \arrow[r, dashed, "\operatorname{Ho}F"'] & \operatorname{Ho}\mathcal{D} \end{tikzcd}.$$

[1] This is a special case of a more general phenomenon. Given any homotopical category, which is some weakening of model categories, one can form a homotopy category. One can then take derived functors via Kan extensions. For further details, see chapter 2 of [2].

(2) Dually, the *total right derived functor* $\mathbb{R}G : \operatorname{Ho}\mathcal{D} \to \operatorname{Ho}\mathcal{C}$ of the right Quillen functor G is the composite $\operatorname{Ho}G \operatorname{Ho}R$, where $\operatorname{Ho}R$ and $\operatorname{Ho}G$ are the induced maps

$$\begin{tikzcd} \mathcal{D} \arrow[r, "R"] \arrow[d] & \mathcal{D}_{\mathrm{f}} \arrow[r, "G"] \arrow[d] & \mathcal{C} \arrow[d] \\ \operatorname{Ho}\mathcal{D} \arrow[r, dashed, "\operatorname{Ho}R"'] & \operatorname{Ho}\mathcal{D}_{\mathrm{f}} \arrow[r, dashed, "\operatorname{Ho}G"'] & \operatorname{Ho}\mathcal{C}\ . \end{tikzcd}$$

The induced maps described above exist by the universal property of the corresponding homotopy categories: $\operatorname{Ho}Q$ is induced by the composite of Q and the localization functor $\mathcal{C}_{\mathrm{c}} \to \operatorname{Ho}\mathcal{C}_{\mathrm{c}}$, since Q sends weak equivalences to weak equivalences by the 2-of-3 property and the naturality square of q, and the localization functor inverts weak equivalences; similarly, $\operatorname{Ho}F$ exists since left Quillen functors send weak equivalences between cofibrant objects to weak equivalences by Ken Brown's lemma 3.6, which are then inverted by the functor $\mathcal{D} \to \operatorname{Ho}\mathcal{D}$. The existence of $\mathbb{R}G$ follows by a dual argument.

Proposition 4.4. Suppose $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$ is a Quillen adjunction, with unit η and counit ε, between model categories. Then $\mathbb{L}F \dashv \mathbb{R}G$, where the unit is the image of $GrFQ \circ \eta Q$ in $\operatorname{Ho}\mathcal{C}$ and the counit is the image of $\varepsilon R \circ FqGR$ in $\operatorname{Ho}\mathcal{D}$.

To view a pair of model categories as "equivalent," it turns out that demanding a categorical equivalence is in some ways too strong. In particular, the model structure encodes a kind of homotopy theory, so we are often only interested in the underlying homotopy category of a model category. For instance, in classical homotopy theory one rarely distinguishes spaces up to homeomorphism, and instead identifies them up to homotopy equivalence. In keeping with this paradigm, we loosen the conditions for a pair of model categories to be "the same" by requiring only that a Quillen adjunction between them descend to an equivalence on the underlying homotopy categories.

Definition 4.5. Let $F : \mathcal{C} \to \mathcal{D} : G$ be a Quillen adjunction of model categories. We say that this defines a *Quillen equivalence* if the total derived functors $\mathbb{L}F : \operatorname{Ho}\mathcal{C} \leftrightarrows \operatorname{Ho}\mathcal{D} : \mathbb{R}G$ form an adjoint equivalence.

While this definition is conceptually useful, in practice it is cumbersome and difficult to work with. Fortunately, there are more explicit conditions for verifying that a Quillen adjunction is an equivalence, which we now outline.

Proposition 4.6. Suppose $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$ is a Quillen adjunction of model categories. Then the following are equivalent:

(1) The pair $F \dashv G$ form a Quillen equivalence.

(2) For all cofibrant X in $\mathcal{C}$ and fibrant Y in $\mathcal{D}$, a morphism $FX \to Y$ is a weak equivalence if and only if its transpose $X \to GY$ is a weak equivalence.

The proof of this proposition can be found in [1] (as proposition 1.3.13); we omit it here, as it is technical and not especially enlightening. Of more significance is the following characterization of Quillen equivalences, which we will later make use of in proving the main result:

Corollary 4.7. Suppose $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$ is a Quillen adjunction of model categories. Then the following are equivalent:

(1) The pair $F \dashv G$ forms a Quillen equivalence.

(2) F reflects weak equivalences between cofibrant objects, and the map $FQGY \to Y$ given by the transpose of $q_{GY} : QGY \to GY$ is a weak equivalence for all fibrant Y in $\mathcal{D}$.

Proof. First, suppose $F \dashv G$ is a Quillen equivalence, and let $Y \in \mathcal{D}$ be a fibrant object. Then since $q_{GY} : QGY \to GY$ is a weak equivalence and QGY is cofibrant, we conclude by proposition 4.6 that its transpose is likewise a weak equivalence. Now, suppose $f : A \to B$ is a morphism between cofibrant objects in $\mathcal{C}$ such that Ff is a weak equivalence. Then applying F to the naturality square

$$\begin{tikzcd} QA \arrow[r, "Qf"] \arrow[d, "q_A"'] & QB \arrow[d, "q_B"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

we conclude from the 2-of-3 property of weak equivalences, together with lemma 4.2 that FQf is a weak equivalence. Passing to the homotopy category of $\mathcal{D}$, we find that $\mathbb{L}Ff$ is an isomorphism, and since $\mathbb{L}F$ is an equivalence we find that f is an isomorphism in $\operatorname{Ho}\mathcal{C}$. By theorem 3.8 it follows that f is a weak equivalence in $\mathcal{C}$.

Conversely, suppose condition (2) holds, and let η and ε denote the unit and counit of the derived adjunction $\mathbb{L}F \dashv \mathbb{R}G$ (respectively). Then ε_X is the image in $\operatorname{Ho}\mathcal{D}$ of the transpose of q_{GRX} for each $X \in \mathcal{D}$, and since this transpose is a weak equivalence by hypothesis (as RX is fibrant) we conclude that ε is a natural isomorphism. To complete the proof, we must verify that η is an isomorphism.

By the triangle identities, we know that $\mathbb{L}F\eta$ is a right inverse of $\varepsilon\mathbb{L}F$, and is consequently an isomorphism as well. But η is the image of $h = GrFQ \circ \widetilde{\eta}Q$ by proposition 4.4, where $\widetilde{\eta} : \mathrm{id}_{\mathcal{C}} \Rightarrow GF$ is the unit of $F \dashv G$. Moreover, $\mathbb{L}F\eta$ is the image of FQh in $\operatorname{Ho}\mathcal{C}$, so by theorem 3.8 FQh is a weak equivalence. It follows that Qh is a weak equivalence, since F reflects weak equivalences between cofibrant objects by assumption, and Q reflects weak equivalences by the 2-of-3 property (using the naturality square of the weak equivalence $q : Q \Rightarrow \mathrm{id}_{\mathcal{D}}$). Thus h is a weak equivalence, so η is an isomorphism. □

5 A Model Structure for Simplicial Sets

Next we review some of the basic theory of simplicial sets. First, we recall the construction of the geometric realization and singular simplicial set functors:

Definition 5.1. We define a functor $|-| : \Delta \to \mathbf{Top}$ by $[n] \mapsto \Delta^n_{\mathrm{Top}}$ (on morphisms we define the obvious map on vertices, then extend linearly). The *geometric realization* functor $|-| : \mathbf{sSet} \to \mathbf{Top}$ is then defined to be the left Kan extension of $|-|$ along the Yoneda embedding $よ : \Delta \hookrightarrow \mathbf{sSet}$:

$$\begin{tikzcd} \Delta \arrow[r, "|-|"] \arrow[d, hook, "よ"'] & \mathbf{Top} \\ \mathbf{sSet} \arrow[ur, dashed, "|-|"'] & \end{tikzcd}.$$

Since $\mathbf{Top}$ is cocomplete and Δ is small, this is a pointwise Kan extension, so in particular for any $X \in \mathbf{sSet}$ we have a formula

$$|X| \cong \operatorname{colim} |\Delta^n|$$

where the colimit is taken over all maps $\Delta^n \to X$, that is, all n-simplices of X (via the Yoneda lemma). Informally, $|X|$ is given by converting each n-simplex of X into a topological n-simplex, then using the face and degeneracy maps of X as gluing data to identify the faces of each n-simplex with the corresponding lower-dimensional simplices.

We will later need the following property of geometric realization (which we state without proof, although one can be found in [4]):

Lemma 5.2. The geometric realization functor $|-|$ commutes with finite products.

Now, by a basic result in ordinary category theory, any left Kan extension $\operatorname{Lan}_{よ} F : \operatorname{PSh}(\mathcal{C}) \to \mathcal{D}$ of a functor $F : \mathcal{C} \to \mathcal{D}$ along $よ$ with $\mathcal{C}$ small and $\mathcal{D}$ cocomplete has a right adjoint $R : \mathcal{D} \to \operatorname{PSh}(\mathcal{C})$, defined by

$$R(d) := \mathcal{D}(\operatorname{Lan}_{よ} F(-), d).$$

In the particular case when $F = |-| : \Delta \to \mathbf{Top}$, this yields the following definition:

Definition 5.3. The *singular simplicial set* functor $\mathrm{Sing} : \mathbf{Top} \to \mathbf{sSet}$, defined on objects by

$$\mathrm{Sing}(T)_n := \mathbf{Top}(\Delta^n_{\mathrm{Top}}, T)$$

for a topological space T, is the right adjoint of the geometric realization functor $|-|$.

Our goal is to impose a suitable model structure on **sSet** such that this adjunction defines a Quillen equivalence with **Top**, equipped with the Serre model structure described in 3.2.

Definition 5.4. A morphism $\varphi : X \to Y$ of simplicial sets is a *Kan fibration* if it has the right lifting property against all horn inclusions $\Lambda^n_k \hookrightarrow \Delta^n$ (with $n \geq 0$ and $0 \leq k \leq n$):

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & X \arrow[d, "\varphi"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} .$$

Definition 5.5. The *Kan-Quillen model structure* on **sSet** is specified by the following:

(1) The class of cofibrations is given by all monomorphisms.

(2) The class of fibrations is given by all Kan fibrations.

(3) The class of weak equivalences consists of all maps φ such that the geometric realization $|\varphi|$ is a weak homotopy equivalence in **Top**.

One can check (with some combination of blood, sweat, and tears) that this defines a model structure on the category of simplicial sets; we refer the morbidly curious to [1], [3], or [5] for details. We note that all simplicial sets are cofibrant, since the inclusion $\varnothing \hookrightarrow X$ is monic for all $X \in \mathbf{sSet}$, so the left deformation given by cofibrant replacement is the identity.

We devote the remainder of this section to outlining the homotopy theory of simplicial sets, and in particular its relation to classical homotopy theory, for this will aid us in proving the main result.

Definition 5.6. A simplicial set X is a Kan complex if, for each n and each $0 \leq k \leq n$, every map $\Lambda^n_k \to X$ has a lift to a map $\Delta^n \to X$:

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd} .$$

Kan complexes provide a model for ∞-groupoids: the inner-horn lifting property turns them into quasi-categories, so they can be viewed as ∞-categories (with composition of the given faces of the horn witnessed by a lift filling that horn), and outer-horn lifting yields inverse maps. One can verify that, for a given space $T \in \mathbf{Top}$, the singular simplicial set $\mathrm{Sing}\, T$ is a Kan complex. It can also easily be checked that the fibrant objects (and thus also the bifibrant objects) in **sSet** are precisely Kan complexes.

Definition 5.7. Let X and Y be Kan complexes. A pair of morphisms $f, g : X \to Y$ in **sSet** are *homotopic* if there is a morphism $h : X \times \Delta^1 \to Y$, called a *simplicial homotopy* from f to g, which satisfies $h|_{X\times\{0\}} = f$ and $h|_{X\times\{1\}} = g$.

One verifies that, for Kan complexes X and Y, pointed simplicial homotopy defines an equivalence relation on the set of pointed morphisms $X \to Y$.

Definition 5.8. Let X and Y be Kan complexes. We write $[X, Y]$ for the set of homotopy classes of morphisms $X \to Y$.

As one might hope, simplicial homotopies correspond to topological homotopies under our adjunction.

Proposition 5.9. Let T be a topological space, and let X be a simplicial set. Under the adjunction $|-| : \mathbf{sSet} \leftrightarrows \mathbf{Top} : \mathrm{Sing}$, simplicial homotopies of morphisms $X \to \mathrm{Sing}\, T$ correspond to homotopies of maps $|X| \to T$.

Proof. A pointed homotopy is a morphism $X \times \Delta^1 \to \operatorname{Sing} T$, whose transpose is a map $|X \times \Delta^1| \to T$. But $|-|$ commutes with finite products by lemma 5.2, and $|\Delta^1| \cong I$, so this is the same as a map $|X| \times I \to T$. $\square$

Definition 5.10. Let X be a Kan complex. Then the *set of path components* of X is the set

$$\pi_0(X) := [\Delta^0, X].$$

When X is the singular simplicial set of a topological space T, it turns out that this agrees with the set of path components $\pi_0(T)$.

Proposition 5.11. Let $T \in \mathbf{Top}$. There is a canonical bijection

$$\pi_0(\operatorname{Sing} T) \cong \pi_0(T)$$

which is given by the $|-| \dashv \operatorname{Sing}$ adjunction.

Proof. This follows immediately from proposition 5.9. $\square$

6 The Homotopy Hypothesis

We are finally ready to attack the main result, namely, that the adjunction $|-| : \mathbf{sSet} \leftrightarrows \mathbf{Top} : \operatorname{Sing}$ defines a Quillen equivalence on the model structures described in 3.2 and 5.5. We begin by establishing that Sing is a right Quillen functor, by showing that it preserves (and reflects) fibrations and trivial fibrations.

Proposition 6.1. The following are equivalent for a morphism f in $\mathbf{Top}$:

(1) f is a Serre fibration.

(2) $\operatorname{Sing} f$ is a Kan fibration.

Proof. There are evident homeomorphisms $|\Delta^n| \cong \Delta^n_{\mathrm{Top}} \cong I^n$, whose composite restricts to a homeomorphism $|\Lambda^n_k| \cong I^{n-1} \times \{0\}$. That is, we have a commuting square

$$\begin{tikzcd}
|\Lambda^n_k| \arrow[r, "\sim"] \arrow[d, hook] & I^{n-1} \times \{0\} \arrow[d, hook] \\
|\Delta^n| \arrow[r, "\sim"] & I^n
\end{tikzcd}$$

so f is a Serre fibration if and only if f has the right lifting property against the geometric realization of all horn inclusions. Passing over the $|-| \dashv \operatorname{Sing}$ adjunction, we find that this is equivalent to $\operatorname{Sing} f$ having the right lifting property against all horn inclusions, so we conclude that f is a Serre fibration if and only if $\operatorname{Sing} f$ is a Kan fibration. $\square$

We would like to show that the same holds for trivial fibrations, from which it will follow that the singular simplicial set functor $\operatorname{Sing} : \mathbf{Top} \to \mathbf{sSet}$ defines a right Quillen functor. However, to prove this, we need the following auxiliary results, which simplify the conditions for being a trivial fibration in $\mathbf{sSet}$ and in $\mathbf{Top}$.

Lemma 6.2. A morphism $\varphi : X \to Y$ in $\mathbf{sSet}$ is a trivial fibration if and only if it has the right lifting property against the inclusion $\partial\Delta^n \hookrightarrow \Delta^n$ for all n.

Proof. By 3.3, we know that $\varphi : X \to Y$ is a trivial fibration if and only if it has the right lifting property against all cofibrations, which are precisely the monomorphisms in $\mathbf{sSet}$. Thus, one direction of the claim is automatic. For the other direction, suppose φ has the right lifting property against all boundary inclusions for the standard n-simplices, and suppose further that we are given a monomorphism (i.e. a cofibration) $A \hookrightarrow B$ and a commuting square

$$\begin{tikzcd}
A \arrow[r] \arrow[d, hook] & X \arrow[d, "\varphi"] \\
B \arrow[r] & Y
\end{tikzcd} .$$

We know that A is the filtered colimit of its n-skeleta $\mathrm{sk}^n(A)$ (and similarly for B), so it will suffice to lift φ against the inclusions $\mathrm{sk}^n(A) \hookrightarrow \mathrm{sk}^n(B)$ inductively. For the base case we take $\mathrm{sk}^{-1}(A) = \mathrm{sk}^{-1}(B)$ to be the initial object of **sSet**, so the lifting is trivial. Now, for the inductive step let $n \geq 0$, and suppose we have already found a lift against $\mathrm{sk}^{n-1}(A) \hookrightarrow \mathrm{sk}^{n-1}(B)$. Then by hypothesis each composite

$$\partial\Delta^n \to \mathrm{sk}^{n-1}(X) \to X$$

extends to a map $\Delta^n \to X$, and thus by the universal property of the coproduct we obtain an extension

$$\begin{tikzcd}
\coprod \partial\Delta^n \arrow[r] \arrow[d, hook] & \mathrm{sk}^{n-1}(A) \arrow[r] & X \\
\coprod \Delta^n \arrow[urr, dashed] & &
\end{tikzcd}.$$

Moreover, recall that the n-skeleton of B is a pushout

$$\begin{tikzcd}
\coprod \partial\Delta^n \arrow[r] \arrow[d, hook] & \mathrm{sk}^{n-1}(B) \arrow[d, hook] \\
\coprod \Delta^n \arrow[r] & \mathrm{sk}^n(B) \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

where the coproducts are indexed by the nondegenerate n-simplices of B (see [4] for details). Since we already have a lift $\mathrm{sk}^{n-1}(B) \to X$ of φ by induction, compatible with the map $\coprod \Delta^n \to X$ just constructed, we obtain an induced map $\mathrm{sk}^n(B) \to X$ by the universal property of the pushout which solves the desired lifting problem. ☐

Lemma 6.3. Let $f : T \to S$ be a morphism in **Top**. Then the following are equivalent:

(1) f is a weak homotopy equivalence and a Serre fibration, that is, f is a trivial fibration in **Top**.

(2) f has the right lifting property against the inclusion $\partial I^n \hookrightarrow I^n$ for all n.

Proof. If f is a trivial fibration then since $\partial I^n \hookrightarrow I^n$ is a CW inclusion f automatically lifts against it on the right. Conversely, suppose (2) holds. First, we show that f is a weak homotopy equivalence. The boundary case $n = 0$ is clear, since the lifting condition becomes path lifting, so we assume $n \geq 1$. Fix a basepoint $t \in T$, and let $s = f(t) \in S$. Given a class $\alpha \in \pi_n(S, s)$, fix a representative $a : (I^n, \partial I^n) \to (S, s)$. Then the lifting property of f yields a map $b : (I^n, \partial I^n) \to (T, t)$, and thus an element $\beta = [b] \in \pi_n(T, t)$, satisfying

$$f_*(\beta) = [fb] = [a] = \alpha,$$

so $f_* : \pi_n(T, t) \to \pi_n(S, s)$ is surjective. Now, suppose given $\gamma \in \pi_n(T, t)$, represented by some $c : (I^n, \partial I^n) \to (T, t)$, which is contained in the kernel of f_*. Then there is a (based) homotopy $h_t : I^{n+1} \to S$ from fc to the constant map at s. Thus, we have a solid commuting square

$$\begin{tikzcd}
\partial I^{n+1} \arrow[r, "g"] \arrow[d, hook] & T \arrow[d, "f"] \\
I^{n+1} \arrow[r, "h_t"'] \arrow[ur, dashed, "\widetilde{h}_t"] & S
\end{tikzcd}$$

where g is given by c on $I^n \times \{0\}$ and is the constant map at t on all other faces. By the right lifting property of f, we obtain a homotopy lift $\widetilde{h}_t : I^{n+1} \to T$ as depicted above, which yields a (based) homotopy from c to the constant map at t. Thus γ is trivial in $\pi_n(T, t)$, so f_* is injective. It follows that f is a weak equivalence.

We conclude by checking that f is a Serre fibration. It will suffice to show that it has the right lifting property against all inclusions $I^n \times \{0\} \hookrightarrow I^{n+1}$, or equivalently, against all inclusions $I^n \times \{0\} \cup \partial I^n \times I \hookrightarrow$

I^{n+1}; we denote by $\partial^1 I^{n+1}$ the domain of this inclusion, that is, the boundary of the cube with the top face deleted. Suppose given a commuting square

$$\begin{tikzcd} \partial^1 I^{n+1} \arrow[r, "a"] \arrow[d, hook] & T \arrow[d, "f"] \\ I^{n+1} \arrow[r, "b"'] & S . \end{tikzcd}$$

Then the restriction a' of a to the boundary of the top face $\partial I^n \times \{1\} \cong \partial I^n$ and the restriction b' of b to the top face $I^n \times \{1\} \cong I^n$ yields a commuting square

$$\begin{tikzcd} \partial I^n \arrow[r, "a'"] \arrow[d, hook] & T \arrow[d, "f"] \\ I^n \arrow[r, "b'"'] \arrow[ur, dashed, "c'"] & S \end{tikzcd}$$

and thus a lift $c' : I^n \to T$. Since c' is compatible with a, this lets us define a map $c : \partial I^{n+1} \to T$, defined by c' on the top face $I^n \times \{1\}$ and by a on the remaining faces $\partial^1 I^{n+1}$, making the following solid diagram commute

$$\begin{tikzcd} \partial I^{n+1} \arrow[r, "c"] \arrow[d, hook] & T \arrow[d, "f"] \\ I^{n+1} \arrow[r, "b"'] \arrow[ur, dashed] & S . \end{tikzcd}$$

Thus we obtain a lift $I^{n+1} \to T$ which restricts to a on $\partial^1 I^{n+1}$, and hence f is a Serre fibration. $\square$

Proposition 6.4. The following are equivalent for a morphism f in **Top**:

(1) f is a trivial fibration in **Top**.

(2) $\operatorname{Sing} f$ is a trivial fibration in **sSet**.

Proof. By lemma 6.2 we know that $\operatorname{Sing} f$ is a trivial fibration if and only if it has the right lifting property against all inclusions $\partial\Delta^n \hookrightarrow \Delta^n$, and passing over the $|-| \dashv \operatorname{Sing}$ adjunction this holds if and only if f has the right lifting property against the inclusion $|\partial\Delta^n| \hookrightarrow |\Delta^n|$. But there is clearly a homeomorphism $|\Delta^n| \cong I^n$ which restricts to $|\partial\Delta^n| \cong \partial I^n$, and by lemma 6.3 f has the right lifting property against the inclusion $\partial I^n \hookrightarrow I^n$ for all n if and only if f is a trivial fibration in **Top**, completing the proof. $\square$

Corollary 6.5. The adjunction $|-| : \mathbf{sSet} \leftrightarrows \mathbf{Top} : \operatorname{Sing}$ forms a Quillen adjunction.

Proof. By proposition 6.1 and proposition 6.4 we know that Sing is a right Quillen functor, and hence the adjunction is a Quillen adjunction. $\square$

We conclude by proving that the $|-| \dashv \operatorname{Sing}$ adjunction in fact defines a Quillen equivalence.

Theorem 6.6 (Homotopy Hypothesis)**.** The Quillen adjunction $|-| : \mathbf{sSet} \leftrightarrows \mathbf{Top} : \operatorname{Sing}$ forms a Quillen equivalence, and thus the homotopy categories for **sSet** (with the Kan-Quillen model structure) and **Top** (with the Serre model structure) are equivalent.

Proof. By corollary 4.7, it will suffice to show that $|-|$ reflects weak equivalences between cofibrant simplicial sets, and that the transpose of $q_{\operatorname{Sing} T}$ is a weak equivalence for all fibrant $T \in \mathbf{Top}$. Since we defined weak equivalences in **sSet** to be those morphisms carried to weak homotopy equivalences by $|-|$, the first claim is immediate. Moreover, as noted in section 5, the cofibrant replacement functor in **sSet** is trivial, so $q_{\operatorname{Sing} T}$ is the identity on $\operatorname{Sing} T$, and thus its transpose is the counit $\varepsilon_T : |\operatorname{Sing} T| \to T$.

We aim to prove that ε_T is a weak homotopy equivalence for all spaces T, using induction on n to show that the induced map on π_n is an isomorphism. For the base case, we first note that we have a bijection $\pi_0(\operatorname{Sing} T) \cong \pi_0(T)$ under the $|-| \dashv \operatorname{Sing}$ adjunction by proposition 5.11; since $\operatorname{Sing} T$ is the coproduct of its connected components and $|-|$ preserves colimits as a left adjoint, it follows that we have a canonical bijection

$$\pi_0(|\operatorname{Sing} T|) \cong \pi_0(\operatorname{Sing} T) \cong \pi_0(T),$$

and tracing through the definitions shows that this is induced by ε_T. Now, let $n > 0$, and suppose by induction that we have shown that ε_S induces an isomorphism on π_{n-1} for all spaces S. Consider the pathspace fibration

$$\Omega T \to PT \xrightarrow{p} T$$

for X. By proposition 6.1 $\operatorname{Sing} p$ is a Kan fibration, so $|\operatorname{Sing} p|$ is a Serre fibration by the main result in [6], and we have the following commutative diagram

$$\begin{tikzcd}
|\operatorname{Sing} \Omega T| \arrow[r] \arrow[d, "\varepsilon_{\Omega T}"'] & |\operatorname{Sing} PT| \arrow[r, "|\operatorname{Sing} p|"] \arrow[d, "\varepsilon_{PT}"] & |\operatorname{Sing} T| \arrow[d, "\varepsilon_T"] \\
\Omega T \arrow[r] & PT \arrow[r, "p"'] & T
\end{tikzcd}$$

where each row is a Serre fibration. Moreover, PT is homotopy equivalent to a point, and since homotopies are preserved by the $|-| \dashv \operatorname{Sing}$ adjunction via proposition 5.9 we find that $|\operatorname{Sing} PT|$ is likewise contractible. Thus, applying the long exact sequence for a Serre fibration to the diagram above, we obtain a commuting square

$$\begin{tikzcd}
\pi_n(|\operatorname{Sing} T|, s) \arrow[r, "\sim"] \arrow[d, "(\varepsilon_T)_*"'] & \pi_{n-1}(|\operatorname{Sing} \Omega T|, s') \arrow[d, "(\varepsilon_{\Omega T})_*"] \\
\pi_n(T, t) \arrow[r, "\sim"] & \pi_{n-1}(\Omega T, t')
\end{tikzcd}$$

where $t \in T$ is a chosen basepoint, and the remaining basepoints are chosen to be compatible with t. The horizontal maps are isomorphisms by exactness, since the homotopy groups of $|\operatorname{Sing} PT|$ and of PT vanish, and by induction the right vertical map, induced by $\varepsilon_{\Omega T}$ on π_{n-1}, is an isomorphism. Therefore the map induced by ε_T on π_n is an isomorphism, so ε_T is a weak homotopy equivalence, completing the proof. $\square$

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