The Serre Spectral Sequence

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Introduction

The long exact sequence for a pair (X, A) is an invaluable tool for computing homology, so much so that some might argue that it is what makes homology even remotely tractable. A natural question to entertain is whether this tool can be pushed further, for instance to obtain some kind of homological relationship from a nested sequence of spaces $X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n$ of arbitrary length. Perhaps a better question is whether such a thing, should it exist, actually offers any additional information. It turns out that the answer to both questions is, at least in some capacity, "yes."

Possibly more surprising, this question gives rise to a rich homological theory of objects called *spectral sequences* (if one manages to wade through an ocean of indices, anyway), which enable one to dissect the homology of a chain complex by scattering its data across an infinite number of "pages." Succinctly, each page of a spectral sequence consists of a collection of $\mathbb{Z}$-bigraded modules (or, more generally, objects of an abelian category), connected by morphisms which compose to zero, such that the following page is given by taking homology. Spectral sequences emerge across a myriad of disciplines, and can be used in a wide variety of applications: they appear in algebraic geometry as a means to compute sheaf cohomology, and in commutative algebra to extract information about the derived functors Tor and Ext.

In addition to introducing the general theory of spectral sequences, the focus of this paper will be on a particular sequence called the *Serre spectral sequence*, named after Jean-Pierre Serre, which is used to analyze the homology of topological spaces that enjoy a certain homotopy lifting property. We will outline how this technology is constructed, and then explore a handful of ways it can be harnessed, concluding with a brief list of additional topics that build on the content of this paper.

Before we proceed, we will make a few remarks about notation. We will denote by fg the composition of the map f with the map g, that is, "f after g." Additionally, we will write (C, ∂) for a chain complex with terms C_n and boundary maps $\partial_n : C_n \to C_{n-1}$, and given a map $f : C \to D$ of chain complexes we let $f_* : H_n(C) \to H_n(D)$ be the induced map on homology.

In their infinite wisdom, the elders of homological algebra omitted as many indices as possible without leading to ambiguity; in keeping with tradition we will adopt this practice as well, leaving out indices whenever possible to (hopefully) improve readibility. We have also quarantined a few technical proofs and lemmata in the appendix, as they are orthogonal to the purpose of the paper and some may find their excessive homological content offensive.

Finally, throughout the paper we fix a ring R, and work over the category of R-modules. However, we note that all of the content of section 1 can be generalized over almost any abelian category $\mathcal{A}$ (for certain results about spectral sequences of filtrations, one may also need to require that $\mathcal{A}$ be (co)complete and its monomorphisms and epimorphisms satisfy some mild technical property). For further details, see [1].

1 Filtered Complexes and Spectral Sequences

We begin by recalling a few basic definitions:

Definition 1.1. Given an R-module M, a *filtration* of M is a $\mathbb{Z}$-indexed family $\{F_n M\}$ of nested submodules

$$\cdots \subseteq F_p M \subseteq F_{p+1} M \subseteq \cdots \subseteq M.$$

Similarly, a *filtration* for a chain complex (C, ∂) is given by a filtration $\{F_n C_i\}$ for each term C_i of C, such that these filtrations are compatible with the boundary map ∂; that is, such that

$$\partial(F_p C_i) \subseteq F_p C_{i-1}$$

for each n and i. The compatibility of each filtration with ∂ turns $\{F_n C_i : i \in \mathbb{Z}\}$ into a chain complex $(F_n C, \partial)$, with boundary maps given by restricting ∂.

We then have the following:

Proposition 1.2. A filtration $\{F_n C\}$ of a complex (C, ∂) induces a filtration $\{F_n H_i(C)\}$ on the homology $H_i(C)$, with

$$F_p H_i(C) := \iota_*(H_i(F_p C))$$

where ι is the inclusion map $F_p C \hookrightarrow C$.

Proof. This follows immediately from the definitions. □

Definition 1.3. We say that a filtration $\{F_n C\}$ is *bounded* if, for each $n \in \mathbb{Z}$, there exist integers $s < t$ such that $F_s C_n = 0$ and $F_t C_n = C_n$. In this case the filtration for C in degree n is given by

$$0 = F_s C_n \subseteq F_{s+1} C_n \subseteq \cdots \subseteq F_{t-1} C_n \subseteq F_t C_n = C_n$$

so there are only finitely-many nontrivial terms in the filtration of each term of C.

Definition 1.4. Given a filtration $\{F_n C\}$ of a complex C and fixed $p \in \mathbb{Z}$, we define the *associated graded complex (in degree p)* to be the quotient complex

$$G_p C := \frac{F_p C}{F_{p-1} C}.$$

By the proposition above, the filtration on C induces a filtration on its homology, so we similarly define the *associated graded homology (in degree p)* by

$$G_p H_i(C) := \frac{F_p H_i(C)}{F_{p-1} H_i(C)}.$$

Note that since $G_p C$ is itself a complex, we can take its homology $H_i(G_p C)$. A natural question to consider is what the relation between $H_i(G_p C)$ and $G_p H_i(C)$ is. The rest of this section is devoted to answering this question by constructing a mechanism called a *spectral sequence* from a given filtration of C. We will see shortly that this allows us to "approximate" $G_p H_i(C)$ with $H_i(G_p C)$, and under favorable circumstances we can recover $H_i(C)$ from its associated graded components, allowing us to determine the homology of C from the spectral sequence of a filtration.

In practice, we can will choose a suitably nice filtration whose homology can be easily computed, making the corresponding spectral sequence (to be assembled momentarily) easy to unpack. This is reminiscent to the process of computing the homology of a space X by selecting a convenient subspace $A \subseteq X$ such that $H_n(A)$ and $H_n(X, A)$ are already understood, and then applying the long exact sequence for relative homology. The similarity is no coincidence, and in fact we will soon see that this technique is simply a special case of the theory of spectral sequences.

Definition 1.5. A *spectral sequence* (E^r, d^r) is a family of R-modules E^r_{pq}, where the set $\{E^r_{pq} : p, q \in \mathbb{Z}\}$ is called the r^{th} *page*, for all $p, q, r \in \mathbb{Z}$ with $r \geq 0$ (note: some authors allow r to start at any fixed $a \in \mathbb{Z}$, but in the interest of simplicity we insist that $a = 0$) together with maps $d^r = d^r_{pq} : E^r_{pq} \to E^r_{p-r,q+r-1}$, called *differentials*, such that $d^r d^r = 0$ and E^{r+1}_{pq} is (isomorphic to) the homology $H_{pq}(E^r, d^r)$ of E^r_{pq}:

$$E^{r+1}_{pq} \cong \frac{\ker(d^r : d_r : E^r_{pq} \to E^r_{p-r,q+r-1})}{\operatorname{im}(d^r : E^r_{p+r,q-r+1} \to E^r_{pq})}.$$

A spectral sequence (E^r, d^r) is said to be *bounded* if, for each $r \geq 0$ there are only finitely-many nonzero terms of total degree n in the r^{th} page (i.e. only finitely many nonzero E^r_{pq} with $p + q = n$) for each $n \in \mathbb{Z}$. Since E^{r+1}_{pq} is a subquotient of E^r_{pq}, by induction it is equivalent to only require that the spectral sequence be bounded on the 0^{th} page.

Note that if (E^r, d^r) is a bounded spectral sequence then given $p, q \in \mathbb{Z}$, $n = p+q$, there exists some $m \geq 0$ such that $E^0_{st} = 0$ for all $s, t \in \mathbb{Z}$ with $s+t = n-1$ and either $s < -m$ or $t > m$. It follows that $E^r_{st} = 0$ for all such s and t as well, so for r sufficiently large we see that $E^r_{p-r,q+r-1} = E^r_{p+r,q-r+1} = 0$, and hence the differentials d^r mapping into and out of $E^r pq$ vanish for r large. Thus the homology stabilizes, that is, for some $r_0 \geq 0$ and each $r \geq r_0$ we have $E^r_{pq} = E^{r_0}_{pq}$.

Definition 1.6. We denote by E^∞_{pq} the stable term $E^{r_0}_{pq}$ described above, and call the set $\{E^\infty_{pq} : p, q \in \mathbb{Z}\}$ the *limit page* of (E^r, d^r).

The utility of a spectral sequence manifests itself in the form of convergence:

Definition 1.7. Let $\{A_n\}$ be a family of R-modules indexed by $\mathbb{Z}$, such that each A_n admits a finite filtration $\{F_p A_n\}$:

$$0 = F_s A_n \subseteq F_{s+1} A_n \subseteq \cdots \subseteq F_{t-1} A_n \subseteq F_t A_n = A_n.$$

We say that a bounded spectral sequence (E^r, d^r) *converges* to $\{A_n\}$, and write $E^0_{pq} \Rightarrow A_{p+q}$, if

$$E^\infty_{pq} \cong \frac{F_p A_{p+q}}{F_{p-1} A_{p+q}} = G_p A_{p+q}$$

for each p, q.

We now arrive at the much-anticipated spectral sequence of a filtration. A sketch of the proof is deferred to the appendix, where it has been exiled for its length and inelegance.

Theorem 1.8. Let (C, ∂) be a complex with a bounded filtration $\{F_n C\}$. Then there exists a bounded spectral sequence (E^r, d^r) with

$$E^1_{pq} = H_{p+q}(G_p C)$$

and

$$\begin{aligned} d^1 : H_{p+q}(G_p C) &\to H_{p+q-1}(G_{p-1} C) \\ [x + F_{p-1} C] &\mapsto [\partial(x) + F_{p-2} C] \end{aligned}$$

where $x \in F_p C$ with $\partial(x) \in F_{p-1} C$. Moreover $E^0_{pq} \Rightarrow H_{p+q}(C)$, that is,

$$E^\infty_{pq} \cong G_p H_{p+q}(C)$$

for each p, q.

In the following sections we will explore a particular spectral sequence that arises from the construction above, which is widely used in the context of algebraic topology. First, however, we will use the machinery above to recover a fundamental result in homological algebra.

Corollary 1.9. Given a short exact sequence of chain complexes

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

there is a long exact sequence

$$\cdots \xrightarrow{g_*} C_{n+1} \xrightarrow{d} A_n \xrightarrow{f_*} B_n \xrightarrow{g_*} C_n \xrightarrow{d} A_{n-1} \xrightarrow{g_*} \cdots$$

Proof. Consider the filtration on B given by $F_pB_i = B_i$ for each $p > 0$, $F_0B_i = f_i(A_i)$, and $F_pB_i = 0$ for $i < 0$. Then the graded components of the filtration are given by $G_1B \cong C$, $G_0B \cong A$, and $G_pB = 0$ for all $p \neq 0, 1$. Since this filtration is clearly bounded, by Theorem 1.8 there exists a spectral sequence (E^r, d^r) with $E^0_{pq} \Rightarrow H_{p+q}(B)$ with

$$E^1_{pq} = H_{p+q}(G_pB) = \begin{cases} H_q(A) & \text{if } p = 0 \\ H_{q+1}(C) & \text{if } p = 1 \\ 0 & \text{otherwise} \end{cases}$$

and differentials $d^1_q : H_{q+1}(C) \to H_q(A)$. Taking the homology, we find that

$$E^2_{pq} = \begin{cases} \ker d^1_q & \text{if } p = 1 \\ \operatorname{coker} d^1_q & \text{if } p = 0 \\ 0 & \text{otherwise} \end{cases}$$

and since the E^2 page is concentrated in the columns $p = 0, 1$ we find that the differentials d^2 are all zero, so $E^\infty = E^2$.

$$\begin{array}{c|cc} 2 & H_2(A) \longleftarrow & H_3(C) \\ 1 & H_1(A) \longleftarrow & H_2(C) \\ 0 & H_0(A) \longleftarrow & H_1(C) \\ \hline & 0 & 1 \end{array} \qquad\qquad \begin{array}{c|cc} 2 & \operatorname{coker} d^1_2 & \ker d^1_2 \\ 1 & \operatorname{coker} d^1_1 & \ker d^1_1 \\ 0 & \operatorname{coker} d^1_0 & \ker d^1_0 \\ \hline & 0 & 1 \end{array}$$

E^1 $\qquad\qquad$ E^2

It follows from the convergence of the spectral sequence that

$$\ker d^1_q = E^\infty_{1,q} \cong \frac{F_1H_{q+1}(B)}{F_0H_{q+1}(B)} = \frac{H_{q+1}(B)}{\operatorname{im} f_*}$$

and

$$\operatorname{coker} d^1_q = E^\infty_{0,q} \cong \frac{F_0H_q(B)}{F_{-1}H_q(B)} = \operatorname{im} f_*$$

giving us a short exact sequence

$$0 \to \operatorname{coker} d^1_q \to H_{q+1}(B) \to \ker d^1_q \to 0$$

for each q. One checks directly that the following triangles commute

$$
\begin{tikzcd}
 & \ker d^1_q \arrow[d] \\
H_{q+1}(B) \arrow[ur] \arrow[r, "g_*"'] & H_{q+1}(C)
\end{tikzcd}
\qquad
\begin{tikzcd}
\operatorname{coker} d^1_q \arrow[dr] & \\
H_q(A) \arrow[u] \arrow[r, "f_*"'] & H_q(B)
\end{tikzcd}
$$

where the maps into and out of the homology of B are the maps in the short exact sequence above. Thus we obtain the desired long exact sequence via the Splicing Lemma (see Lemma 6.1 in the appendix) by precomposing each map $\operatorname{coker} d^1_{q-1} \to H_q(B)$ with the quotient $H_q(A) \to \operatorname{coker} d^1_q$ and postcomposing each map $H_{q+1}(B) \to \ker d^1_q$ with the inclusion $\ker d^1_q \to H_{q+1}(C)$. $\square$

There are far more direct proofs for the corollary which do not make any mention of spectral sequences (indeed, it is one of the first results any student of homological algebra learns, and is arguably where the subject of "homological algebra" truly begins). However, while this approach is certainly overkill, it illuminates an important connection between the theory developed above and some existing techniques in algebraic topology. In particular, this offers some justification for our claim that the spectral sequence of a filtration generalizes relative homology. Namely, given a pair of spaces (X, A) and taking the short exact sequence of complexes above to be

$$0 \to C(A) \to C(X) \to C(X, A) \to 0,$$

we are able to recover the long exact sequence on relative homology from Corollary 1.9.

2 Fibrations

Before we can begin our discussion of the Serre spectral sequence, we must first make a brief detour into the world of homotopical topology to understand *Serre fibrations*. To avoid repetition, we fix topological spaces E and B, and a continuous map $\pi : E \to B$.

Definition 2.1. We say that $E \xrightarrow{\pi} B$ has the *homotopy lifting property* with respect to a space X if, given a homotopy $h_t : X \times I \to B$ and a lift $\widetilde{h}_0 : X \to E$ of h_t for $t = 0$, there exists a homotopy $\widetilde{h}_t : X \times I \to E$ lifting h_t and which agrees with the lift $\widetilde{h}_0$ for $t = 0$. Diagrammatically, $\widetilde{h}_t$ makes the following diagram commute (wherein the left vertical map is the inclusion of $X \cong X \times \{0\}$ into $X \times I$):

$$
\begin{tikzcd}
X \arrow[r, "\widetilde{h}_0"] \arrow[d, hook] & E \arrow[d, "\pi"] \\
X \times I \arrow[r, "h_t"'] \arrow[ur, dashed, "\widetilde{h}_t"] & B
\end{tikzcd}
$$

For instance, covering spaces satisfy the homotopy lifting property with respect to all spaces; in fact, in this case the induced homotopy is unique, which accounts for much of the rich theory of covering spaces. Dropping the requirement that the lift be unique leads us to the following definition:

Definition 2.2. $E \xrightarrow{\pi} X$ is a *fibration* (also called a *Hurewicz fibration*) if it satisfies the homotopy lifting property with respect to all spaces.

Fibrations provide a generalization of the notion of *fiber bundles*, which in turn generalize the projection map $B \times F \to B$. A fiber bundle is a surjection $E \xrightarrow{\pi} B$ which behaves locally like the projection map from the product space $B \times F$, and enjoy the homotopy lifting property with respect to suitably "nice" spaces (in particular, CW complexes). These provide us a first example of the following definition:

Definition 2.3. $E \xrightarrow{\pi} X$ is a *Serre fibration* if it satisfies the homotopy lifting property with respect to all CW complexes. We say that E is the *total space* of the fibration, and that B is the *base space*.

An equivalent definition requires only that Serre fibrations have the homotopy lifting property only with respect to all closed disks D^n. One can then lift along each cell of a CW complex to obtain the more general lifting condition against all CW complexes.

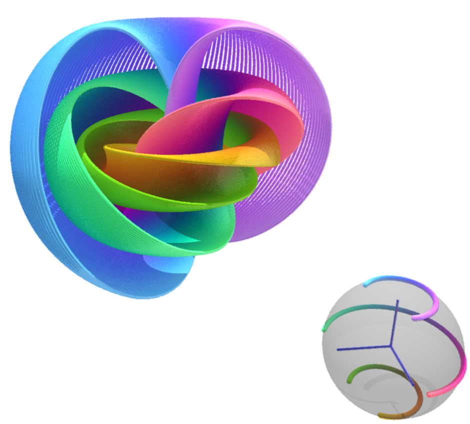
We often care not only about the base space and total space of a (Serre) fibration, but also about the fiber $F = \pi^{-1}(x)$ of a basepoint $x \in B$. Throughout the subsequent sections we will assume that a particular fiber F is given, together with an inclusion map $i : F \to E$, and write

$$F \xrightarrow{i} E \xrightarrow{\pi} B$$

for the complete data of the fibration. For our purposes we will only ever care about the homology of F, so the following proposition (which we state without proof) helps justify our blasé attitude towards the basepoint.

Proposition 2.4. Suppose $\pi : E \to B$ is a Serre fibration with B path-connected, and let $x, y \in B$. Then the fibers $F_x = \pi^{-1}(x)$ and $F_y = \pi^{-1}(y)$ are weakly homotopy equivalent, so in particular $H_q(F_x) \cong H_q(F_y)$ for all q.

Arguably the most (in?)famous example of a Serre fibration is the *Hopf fibration*, given by a map $S^3 \to S^2$ with fiber S^1. Explicitly, consider the quotient map $\pi : \mathbb{C}^2 - \{0\} \to \mathbb{CP}^1 \cong S^2$ which identifies each point (z, w) with $(\lambda z, \lambda w)$ for all nonzero $\lambda \in \mathbb{C}$. Since the unit circle in $\mathbb{C}^2$ is homeomorphic to S^3 there is an inclusion map $\iota : S^3 \to \mathbb{C}^2 - \{0\}$, and it can be verified that the restriction $\pi\iota$ of π along ι is the desired Serre fibration $S^3 \to S^2$ (in fact it is a fiber bundle), fiber S^1. More generally, if $n \neq 0$ then there is a fiber bundle (and hence a Serre fibration) $S^1 \to S^{2n+1} \to \mathbb{CP}^n$ which is also sometimes called a Hopf fibration (although the use of the definite article usually implies that $n = 1$).



A depiction of the Hopf fibration, created by Niles Johnson using SageMath [2]

Given some based space (B, b_0), one may wish to consider the *path space* PB over B, consisting of all paths $I \to B$ starting at the basepoint b_0 and equipped with the compact-open topology (i.e. generated by the subbasis of subsets U^K of paths which carry a given compact subset $K \subseteq I$ to an open set $U \subseteq B$). We note that PB is contractible, for each path $\gamma \in PB$ can be homotoped to the constant path at b_0 by pulling γ back along itself (leaving $\gamma(0)$ fixed).

Similarly, one can consider only those paths in B which both start and end at b_0, which forms the *loop space* ΩB over B (equipped with the subspace topology induced by PB). There is a natural map $\mathrm{ev}_1 : PB \to B$ given by evaluating a path $\gamma \in PB$ at the endpoint 1, that is, $\mathrm{ev}_1(\gamma) = \gamma(1)$, and in fact it is not hard to check that this yields a (Serre) fibration

$$\Omega B \to PB \xrightarrow{\mathrm{ev}_1} B.$$

At first glance loop spaces may appear geometrically impenetrable. After all, how could one go about visualizing every loop in B as a space in its own right? How might we go about understanding such a space? In the following section we will develop a tool which will enable us to compute the homology of ΩB (when B is sufficiently amiable, anyway), along with that of many other spaces which fit into a Serre fibration.

3 The Serre Spectral Sequence

Finally, we are ready to construct the Serre Spectral Sequence. As with Theorem 1.8 we will only outline the main ideas of the proof, but more detailed arguments can be found in [3] and [4].

Theorem 3.1 (Serre Spectral Sequence)**.** Let $F \to E \xrightarrow{\pi} B$ be a Serre fibration such that B is simply-connected. Then there exists a spectral sequence (E^r, d^r) with

$$E^2_{pq} = H_p(B; H_q(F))$$

which converges to $G_pH_{p+q}(E)$ for some filtration on the homology of E.

For simplicity, the argument we present makes a few assumptions: firstly, we assume that B is a CW complex; to prove the general case, one approximates B with a CW complex B', and then uses the hypothesis that $\pi_1(B)$ is trivial (in fact, one only needs the weaker assumption that $\pi_1(B)$ acts trivially on the homology of F), along with some additional machinery such as the Hurewicz Theorem – see [4]. Furthermore, we assume that B is finite-dimensional so that the filtration defined in the proof is bounded. For B an arbitrary CW complex one passes to the direct limit over the filtered diagram of the skeleta, as noted in [5].

Proof of Theorem 3.1. Suppose B is a finite-dimensional CW complex, and let B^p denote its p-skeleton. Then we define a filtration on the chain complex $C(E)$ associated to E by

$$F_pC_i(E) := C_i(\pi^{-1}(B^p)),$$

noting that this filtration is bounded as $B^p = E$ for all sufficiently large p. Theorem 1.8 then produces a spectral sequence (E^r, d^r) with $E^1_{pq} = H_{p+q}(G_pC(E))$ which converges to $G_pH_{p+q}(C(E)) = G_pH_{p+q}(E)$. Thus we have

$$G_pC_i(E) = \frac{F_pC_i(E)}{F_{p-1}C_i(E)} = \frac{C_i(\pi^{-1}(B^p))}{C_i(\pi^{-1}(B^{-1}))} = C_i(\pi^{-1}(B^p), \pi^{-1}(B^{p-1})),$$

so

$$E^1_{pq} = H_{p+q}(\pi^{-1}(B^p), \pi^{-1}(B^{p-1})).$$

We aim to show that $E^1_{pq} \cong C^{CW}_p(B; H_q(F))$, the cellular chain group associated to B with coefficients in $H_q(F)$. For the closure e_α of each p-cell in B^p (i.e. the image of each characteristic map) we have an inclusion of pairs

$$i_\alpha : (\pi^{-1}(e_\alpha), \pi^{-1}(\partial e_\alpha)) \hookrightarrow (\pi^{-1}(B^p), \pi^{-1}(B^{p-1}))$$

so the universal property of the coproduct induces a map

$$i : \coprod_\alpha (\pi^{-1}(e_\alpha), \pi^{-1}(\partial e_\alpha)) \to (\pi^{-1})B^p), \pi^{-1}(B^{p-1})),$$

where ∂e_α denotes the boundary of the cell. We will show that the induced map

$$i_* : \bigoplus_\alpha H_{p+1}(\pi^{-1}(e_\alpha), \pi^{-1}(\partial e_\alpha)) \to H_{p+q}(\pi^{-1})B^p), \pi^{-1}(B^{p-1})) = E^1_{pq}$$

is an isomorphism, using a proof similar to that of Lemma 9.2.2 in [6]. For each e_α, let e'_α be a closed p-disk contained in the interior of e_α. It is easy to see that the inclusion maps

$$(B^p, B^{p-1}) \hookrightarrow (B^p, B^p - \cup_\alpha \operatorname{int}(e'_\alpha))$$
$$(e_\alpha, \partial e_\alpha) \hookrightarrow (e_\alpha, \operatorname{int}(e'_\alpha))$$

are homotopy equivalences, where $\operatorname{int}(e'_\alpha)$ is the interior of e'_α – this is evident from the fact that a (closed) cell with a disk removed from its interior clearly deformation retracts to its boundary. Pulling back along π, we find that the corresponding inclusions

$$(\pi^{-1}(B^p), \pi^{-1}(B^{p-1})) \hookrightarrow (\pi^{-1}(B^p), \pi^{-1}(B^p) - \cup_\alpha \pi^{-1}(\operatorname{int} e'_\alpha))$$
$$(\pi^{-1}(e_\alpha), \pi^{-1}(\partial e_\alpha)) \hookrightarrow (\pi^{-1}(e_\alpha), \pi^{-1}(\operatorname{int}(e'_\alpha)))$$

are likewise homotopy equivalences, since π is a Serre fibration (so we can lift each homotopy along π). Thus, passing to homology we obtain the following commutative diagram, wherein each map is induced by the corresponding inclusion (or sum of inclusions, in the case of the left-hand column):

$$\begin{tikzcd}
\bigoplus_\alpha H_{p+q}(\pi^{-1}(e_\alpha), \pi^{-1}(\partial e_\alpha) \arrow[r, "i_*"] \arrow[d] & H_{p+q}(\pi^{-1}(B^p), \pi^{-1}(B^{p-1})) \arrow[d] \\
\bigoplus_\alpha H_{p+q}(\pi^{-1}(e_\alpha), \pi^{-1}(e_\alpha - \operatorname{int}(e'_\alpha))) \arrow[r] & H_{p+q}(\pi^{-1}(B^p), \pi^{-1}(B^p) - \cup_\alpha \pi^{-1}(\operatorname{int} e'_\alpha)) \\
\bigoplus_\alpha H_{p+q}(\pi^{-1}(e'_\alpha), \pi^{-1}(\partial e'_\alpha)) \arrow[r] \arrow[u] & H_{p+q}(\cup_\alpha \pi^{-1}(e'_\alpha), \cup_\alpha \pi^{-1}(\partial e'_\alpha)) \arrow[u]
\end{tikzcd}$$

The top vertical maps are isomorphisms, since as noted above they are induced by homotopy equivalences. Additionally, the bottom vertical maps are isomorphisms by excision, and the bottom horizontal map is an isomorphism since each e'_α is disjoint (and thus so are their pullbacks). Therefore by the commutativity of the diagram i_* is an isomorphism, as desired.

We further observe that there is an isomorphism

$$\bigoplus_\alpha H_q(F) \cong H_q(F) \otimes \bigoplus_\alpha \mathbb{Z} \cong H_q(F) \otimes H_p(B^p, B^{p-1})$$

as $H_p(B^p, B^{p-1})$ is the direct sum of copies of $\mathbb{Z}$, one for each p-cell of B. Finally, we claim that there is an isomorphism

$$\bigoplus_\alpha H_{p+q}(\pi^{-1}(e_\alpha), \pi^{-1}(\partial e_\alpha)) \cong \bigoplus_\alpha H_q(F).$$

This follows from a technical argument, expanded on in [4], wherein one uses the hypothesis that $\pi_1(B)$ is trivial to ensure that the isomorphism between the homology groups of each fiber do not depend on a choice of path.

Combining the arguments above, we have isomorphisms

$$E^1_{pq} \cong \bigoplus_\alpha H_{p+q}(\pi^{-1}(e_\alpha), \pi^{-1}(e'_\alpha)) \cong \bigoplus_\alpha H_q(F) \cong H_q(F) \otimes H_p(B^p, B^{p-1}) = H_p(B^p, B^{p-1}; H_q(F)),$$

where the rightmost term is precisely the p^{th} cellular chain group of B with coefficients in $H_q(F)$. One further checks that the map d^1 is the cellular boundary map, from which it follows that

$$E^2_{pq} \cong H^{CW}_p(B; H_q(F)).$$

Since $E^0_{pq} \Rightarrow H_{p+q}(E)$ by Theorem 1.8, this completes the proof. $\square$

This spectral sequence is more than a mere curiosity, and we will see shortly how it allows one to extract a great deal of information about the spaces F, E, and B in a Serre fibration.

4 Computations and Corollaries

As a first application of the Serre spectral sequence, we will see how the mere existence of a fibration encodes some information about the spaces involved.

Corollary 4.1. Let $F \to E \xrightarrow{\pi} B$ be a Serre fibration. If B is simply-connected then E is path-connected if and only if F is path-connected.

Proof. Let (E^r, d^r) be the Serre spectral sequence associated to the given fibration. We note that $E^2_{pq} = 0$ when at least one of p or q is less than 0, since then we have

$$E^2_{pq} \cong H_p(B; H_q(F)),$$

and $H_q(F) = 0$ when $q < 0$, $H_p(B; G) = 0$ when $p < 0$ for any group G. Thus the nonnegative terms of E^2_{pq} lie in the first quadrant, so we have

$$E^2_{0,0} = E^\infty_{0,0} \cong G_0 H_0(E).$$

Moreover, $G_p H_0(E) = 0$ for $p \neq 0$, for

$$G_p H_0(E) = G_p H_{p-p}(E) = E^\infty_{p,-p}$$

and if $p \neq 0$ then one of p and $-p$ is less than 0, in which case $E^\infty_{p,-p} = 0$. Therefore the filtration on $H_0(E)$ consists of only two distinct terms, namely, 0 and $H_0(E)$, and we have

$$H_0(B; H_0(F)) \cong E^2_{0,0} \cong H_0(E).$$

Now, suppose F is path-connected. Then we have $H_0(F) \cong \mathbb{Z}$, and since B is also path-connected it follows that $H_0(E) \cong H_0(B) \cong \mathbb{Z}$, so E has a single path-component. Conversely, if E is path-connected then $H_0(B; H_0(F)) \cong H_0(E) \cong \mathbb{Z}$, and since B is path-connected $H_0(B; H_0(F)) \cong H_0(F)$, so we conclude that $H_0(F) \cong \mathbb{Z}$ (i.e. F has a single path-component). $\square$

Next, we will derive a few homological results about fibrations involving spheres. In particular, the Serre spectral sequence gives rise to long exact "transfer" sequences, which allow one to compare the various homology groups of a given space. If the homology in low degrees is easy to understand, this will allow us to determine the higher homology groups via induction (which is precisely how we will later compute the homology of the loop space ΩS^n). The following proof expands on arguments found in Chapter 5 of [1].

Corollary 4.2 (Wang Sequence)**.** Let $F \to E \xrightarrow{\pi} S^n$ be a Serre fibration whose base space is an n-sphere (with $n \neq 0, 1$). Then there is a long exact sequence

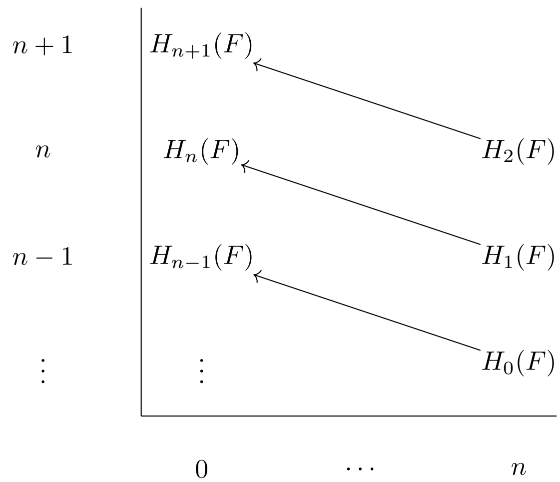
$$\cdots \to H_q(F) \xrightarrow{i_*} H_q(E) \to H_{q-n}(F) \xrightarrow{d^n} H_{q-1}(F) \xrightarrow{i_*} H_{q-1}(E) \to \cdots$$

where $i : F \to E$ is the inclusion map (for F a fixed fiber). In particular, if $q \leq n-2$ then $H_q(F) \cong H_q(E)$.

Proof. By construction, the Serre spectral sequence (E^r, d^r) associated to the fibration π has E^2 page

$$E^2_{pq} = H_p(S^n; H_q(F)) \cong \begin{cases} H_q(F) & \text{if } p = 0, n \\ 0 & \text{otherwise} \end{cases}$$

so we see that the spectral sequence collapses on the second page to the columns $p = 0, n$ (that is, all nonzero terms are contained in these columns). It follows that all terms on subsequent pages likewise vanish off of these columns, since E^{r+1}_{pq} is a subquotient of E^r_{pq}, and thus d^r vanishes for $r \geq 2$ unless $r = n$ (for otherwise either the domain or codomain of d^r is 0). It follows that $E^2_{pq} = E^n_{pq}$ and $E^{n+1}_{pq} = E^\infty_{pq}$.



The E^n page of the spectral sequence.

Now, since E^{n+1} is the homology of E^n, and the only nonzero differentials are given by $H_q(F) = E^n_{n,q} \to E^n_{0,q+n-1} = H_{q+n-1}(F)$, we have

$$\begin{aligned} E^\infty_{n,q} &= \ker(d^n : H_q(F) \to H_{q+n-1}(F)) \\ E^\infty_{0,q+n-1} &= \operatorname{coker}(d^n : H_q(F) \to H_{q+n-1}(F)) \end{aligned}$$

which gives us an exact sequence

$$0 \to E^\infty_{n,q} \to H_q(F) \xrightarrow{d^n} H_{q+n-1}(F) \to E^\infty_{0,q+n-1} \to 0.$$

On the other hand, we have $E^\infty_{pq} \cong G_p H_{p+q}(E)$ for some filtration on $H_{p+q}(E)$, and for a given total degree $p + q = s$ we have $E^\infty_{pq} = 0$ unless $p = 0$, $q = s$ or $p = n$, $q = s - n$. Thus the filtration of $H_s(E)$ has at most a single intermediate term $F_0 H_s(E)$ lying between 0 and $H_s(E)$, and we have

$$\begin{aligned} E^\infty_{0,s} &\cong G_0 H_s(E) = \frac{F_0 H_s(E)}{F_{-1} H_s(E)} = \frac{F_0 H_s(E)}{0} = F_0 H_s(E) \\ E^\infty_{n,s-n} &\cong G_n H_s(E) = \frac{F_n H_s(E)}{F_{n-1} H_s(E)} = \frac{H_s(E)}{F_0 H_s(E)} \end{aligned}$$

giving us a short exact sequence

$$0 \to E^\infty_{0,s} \to H_s(E) \to E^\infty_{n,s-n} \to 0.$$

Applying the Splicing Lemma (stated in the appendix as Lemma 6.1) to this exact sequence and the one above then yields the desired long exact sequence. The second claim follows immediately, for if $s \leq n - 1$ then the terms $H_{s-n}(F)$ vanish, whereby the Wang sequence has the form

$$0 = H_{s-n}(F) \xrightarrow{d^n} H_q(F) \xrightarrow{i_*} H_q(E) \to H_{s-n-1}(F) = 0$$

for $q = s - 1 \leq n - 2$. $\square$

We obtain a similar result via a virtually identical proof, except the spectral sequence instead collapses to the rows $q = 0, n$ on the E^2 page (rather than the columns $p = 0, n$):

Corollary 4.3 (Gysin Sequence)**.** Let $E \xrightarrow{\pi} B$ be a Serre fibration with fiber S^n, $n \neq 0$, such that B is simply-connected. Then there is a long exact sequence

$$\cdots \to H_{p-n}(B) \to H_p(E) \xrightarrow{\pi_*} H_p(B) \xrightarrow{d^{n+1}} H_{p-n-1}(B) \to H_{p-1}(E) \to \cdots$$

We conclude this section by computing the homology of the loop space of S^n for $n \geq 2$, using the Serre fibration $\Omega S^n \to PS^n \xrightarrow{\mathrm{ev}_1} S^n$ described in section 2 along with the preceding corollaries:

Proposition 4.4. The homology of the loop space ΩS^n (with $n \geq 2$) is given by

$$H_p(\Omega S^n) \cong \begin{cases} \mathbb{Z} & \text{if } n-1 \text{ divides } p \\ 0 & \text{otherwise} \end{cases}$$

Proof. First, we observe that since the path space PS^n is contractible (see section 2) it is in particular path-connected, so by Corollary 4.1 the fiber ΩS^n is as well. It follows that $H_0(\Omega S^n) \cong \mathbb{Z}$. Now, let $q > 0$ and suppose by induction that we have proven the claim for all $p < q$. Then the Wang sequence (see Corollary 4.2) yields an exact sequence

$$H_{q+1}(PS^n) \to H_{q-n+1}(\Omega S^n) \to H_q(\Omega S^n) \to H_q(PS^n)$$

and since PS^n is contractible and $q > 0$ the outermost terms vanish, so $H_{q-n+1}(\Omega S^n) \cong H_q(\Omega S^n)$. But q is divisible by $n-1$ if and only if $q - n + 1 = q - (n-1)$ is, so by induction we have

$$H_q(\Omega S^n) \cong H_{q-n+1}(\Omega S^n) \cong \begin{cases} \mathbb{Z} & \text{if } n-1 \text{ divides } q \\ 0 & \text{otherwise} \end{cases}$$

proving the claim by induction. $\square$

5 Further Topics

Like all good things in life, there is a dual version of the Serre spectral sequence, or more generally, for cohomological spectral sequences associated to filtrations. Given a cochain complex with a (decreasing) filtration one can construct a cohomological spectral sequence (E_r, d_r) with $d_r^{pq} : E_r^{pq} \to E_r^{p+r,q-r+1}$, which will converge to the graded components of the cohomology of the complex when the filtration is bounded. As discussed in [5], this offers the advantage of a bilinear product on the complex inducing a product on the spectral sequence, which in turn descends to the product on cohomology when one passes to the limit page. In particular, in the case of the Serre spectral sequence for cohomology, one can recover the cohomology ring of the total space of a Serre fibration. To see this theory developed in detail, see [4].

Another generalization of the Serre spectral sequence enables one to drop the condition that the base space of a fibration act trivially on the homology of the fibers. In this case, the sequence E^0_{pq} associated to the fibrations $F \to E \xrightarrow{\pi} B$ converges to $H_p(B; \{H_q(F_x)\})$, the p^{th} homology group of B with *local coefficients* given by the local system of homology groups for the fiber $F_x = \pi^{-1}(x)$ as x ranges over B. This is expanded upon in [3], wherein the Serre spectral sequence is developed in this more general context, and the version we presented is given as a special case. One can further generalize by considering $H_q(F; G)$ for G some abelian group (rather than fixing $G = \mathbb{Z}$), although this minor change does not affect any of the theory presented in this paper.

Finally, one can extend the theory of spectral sequences further in the homological direction. For instance, double complexes (which can be viewed as "two-dimensional" chain complexes) are an important object of study in homological algebra, but are trickier to get a solid grasp of than regular complexes. In particular, one often cares about the *total complex* of a double complex, which is a chain complex whose terms are given by the direct sum along the antidiagonals of the double complex (i.e. of lattice points with the same total degree). It is possible to adapt the theory of spectral sequences associated to a filtration to study the homology of the total complex of a double complex, and from here one can generalize derived functors using *hyperhomology.* So-called "hyper-derived functors" can be understood using spectral sequences, as elaborated on in [1].

6 Appendix

Throughout the paper, we have banished several results and proofs to the appendix; the reader who is easily disturbed by excessive index-pushing or homological algebra should exercise caution as they proceed through this section. Depending on one's ideology, the appendix can be skipped entirely and its contents taken on faith.

Several proofs in previous sections required that we assemble an exact sequence by gluing together shorter ones. In the interest of being entirely above board, we now justify why this seemingly harmless act is indeed as humane as it appears. The proof is adapted from a similar argument in [3].

Lemma 6.1 (Splicing Lemma)**.** Suppose $A \to B \xrightarrow{f} C$ and $D \xrightarrow{g} E \to F$ are exact sequences of R-modules, and suppose further that there is an isomorphism $\varphi : \operatorname{coker} f \cong \ker g$. Then there is an exact sequence

$$A \to B \xrightarrow{f} C \xrightarrow{\psi} D \xrightarrow{g} E \to F \tag{1}$$

where $\psi : C \to D$ is given by $x \mapsto \varphi([x])$, with $[x]$ the image of x in $\operatorname{coker} f$.

Proof. Let $x \in \ker \psi$. Then $\varphi([x]) = 0$, so $x \in \operatorname{im} f$ (as φ is injective). On the other hand, clearly $\operatorname{im} f$ is contained in $\ker \psi$, so we conclude that $\ker \psi = \operatorname{im} f$, proving the exactness of (1) at C. Similarly, suppose $y \in \ker g$. Then there exists a unique $[x] \in \operatorname{coker} f$ such that $\varphi([x]) = y$, as φ is surjective, and thus we find that $\psi(x) = y$, so $\ker g \subseteq \operatorname{im} \psi$. The reverse inclusion is clear, for $g\psi(x) = g\varphi([x]) = 0$ as the image of φ is $\ker g$. Therefore (1) is exact at D; exactness at the remaining terms follows from the hypotheses. $\square$

Finally, we provide a sketch of the construction of the spectral sequence associated to a filtration below, as promised. Since the proof consists primarily of diagram chasing and verifying a number of small technical details, we content ourselves by merely outlining the main ideas; a more thorough treatment can be found in [3].

Proof of Theorem 1.8. We begin by making the following definitions:

$$Z^r_{pq} := F_p C_{p+q} \cap \partial^{-1}(F_{p-r} C_{p+q-1})$$
$$B^r_{pq} := F_p C_{p+q} \cap \partial(F_{p+r} C_{p+q+1})$$

for all $r, p, q \in \mathbb{Z}$ with $r \geq 0$. In words, these modules define subsets of $F_p C_{p+q}$, with Z^r_{pq} consisting of those elements with boundary in $F_{p-r} C_{p+q-1}$ and B^r_{pq} of those elements which are the boundary of an element of $F_{p+r} C_{p+q+1}$. It is easily verified that

$$B^0_{pq} \subseteq B^1_{pq} \subseteq \cdots \subseteq B^\infty_{pq} \subseteq Z^\infty_{pq} \subseteq \cdots \subseteq Z^1_{pq} \subseteq Z^0_{pq}$$

and

$$\partial(Z^r_{p+r,q-r+1}) = B^r_{pq},$$

and from the definitions we see that Z^r_{pq} contains $Z^{r-1}_{p-1,q+1}$ and B^{r-1}_{pq}, so we define

$$E^r_{pq} := \frac{Z^r_{pq}}{Z^{r-1}_{p-1,q+1} + B^{r-1}_{pq}}$$

for each $r > 0$. We first aim to construct differentials d^r which make (E^r, d^r) a spectral sequence. Let $\eta^r = \eta^r_{pq} : Z^r_{pq} \to E^r_{pq}$ be the quotient map. One then checks that

$$\partial(Z^{r-1}_{p-1,q+1} + B^{r-1}_{pq}) \subseteq Z^{r-1}_{p-r-1,q+r} + B^{r-1}_{p-r,q+r-1},$$

that is, ∂ maps the kernel of η^r_{pq} into the kernel of $\eta^r_{p-r,q+r-1}$, so by the universal property of the quotient there exists a unique map $d^r : E^r_{pq} \to E^r_{p-r,q+r-1}$ making the following diagram (with exact rows) commute:

$$\begin{tikzcd}
0 \arrow[r] & T^r_{pq} \arrow[r] \arrow[d, "\partial"'] & Z^r_{pq} \arrow[r, "\eta^r"] \arrow[d, "\partial"'] & E^r_{pq} \arrow[r] \arrow[d, dashed, "d^r"] & 0 \\
0 \arrow[r] & T^r_{p-r,q+r-1} \arrow[r] & Z^r_{p-r,q+r-1} \arrow[r, "\eta^r"'] & E^r_{p-r,q+r-1} \arrow[r] & 0
\end{tikzcd}$$

where $T^r_{pq} = Z^{r-1}_{p-1,q+1} + B^{r-1}_{pq}$. Consequently we see that $d^r d^r = 0$ by attaching another copy of this diagram (with p and q replaced with $p-r$ and $q+r-1$, respectively) with its top row identified with the bottom row above. Precomposing with η^r_{pq} and using the commutativity of the right-hand square twice we obtain

$$d^r d^r \eta^r = d^r \eta^r \partial = \eta^r \partial\partial = 0,$$

and since η^r is an epimorphism it follows that $d^r d^r = 0$.

It remains to be shown that each E^{r+1}_{pq} is the homology $H_{pq}(E^r, d^r)$ of E^r_{pq}. A diagram chase reveals that

$$\begin{tikzcd}
T^r_{pq} \arrow[r, hook] & Z^{r+1}_{pq} \arrow[r, hook] \arrow[d, dashed, "\mu"'] \arrow[dd, bend right=60, "\gamma"'] & Z^r_{pq} \arrow[r, "\partial"] \arrow[d, "\eta^r"'] & Z^r_{p-r,q+r-1} \arrow[d, "\eta^r"] \\
 & \ker d^r \arrow[r, hook] \arrow[d, two heads, "\pi"'] & E^r_{pq} \arrow[r, "d^r"] & E^r_{p-r,q+r-1} \\
 & H_{pq}(E^r, d^r) & &
\end{tikzcd}$$

Figure 3

commutes, where π is the quotient map $\ker d^r \to H_{pq}(E^r, d^r)$, and μ is induced by the universal property of the kernel. Explicitly,

$$\partial(Z^{r+1}_{pq}) \subseteq F_{p-r-1}C_{p+q-1} \subseteq F_{p-r+1}C_{p+q-1}$$

by definition, and since $\partial\partial = 0$ this in fact shows that

$$\partial(Z^{r+1}_{pq}) \subseteq Z^{r+1}_{p-r+1,q+r} \subseteq Z^{r-1}_{p-r+1,q+r} \subseteq T^r_{p-r,q+r-1},$$

so we see that

$$d^r \eta^r (Z^{r+1}_{pq}) = \eta^r \partial (Z^{r+1}_{pq}) = 0$$

by the commutativity of the right-hand square, and thus the restriction of η^r to Z^{r+1}_{pq} factors uniquely through $\ker d^r$ via the map μ.

Now, let $\gamma = \pi\mu$, the curved arrow in Figure 3. Additional fiddling with the definitions shows that

$$T^{r+1}_{pq} = Z^r_{p-1,q+1} + B^r_{pq} = Z^{r+1}_{pq} \cap (\eta^r)^{-1}(\operatorname{im} d^r) \tag{2}$$

and that

$$\ker \gamma = Z^{r+1}_{pq} \cap (\eta^r)^{-1}(\operatorname{im} d^r)$$

so putting this together we have $\ker\gamma = T_{pq}^{r+1}$. Moreover, a short diagram chase shows that μ is surjective, and thus so is γ (as π is clearly surjective). It then follows from the First Isomorphism Theorem that

$$H_{pq}^r(E^r, d^r) \cong \frac{Z_{pq}^{r+1}}{\ker\gamma} = \frac{Z_{pq}^{r+1}}{Z_{p-1,q+1}^r + B_{pq}^r} = E_{pq}^{r+1}$$

for $r > 0$.

For $r = 0$, we define

$$\begin{aligned} Z_{pq}^{-1} &:= F_{p-1}C_{p+q} \\ B_{pq}^{-1} &:= \partial(F_{p-1}C_{p+q+1}) \end{aligned}$$

and define E_{pq}^0 using the same formula used to define the remaining pages above. Then

$$E_{pq}^0 = \frac{F_pC_{p+q} \cap \partial^{-1}(F_pC_{p+q-1})}{F_{p-1}C_{p+q} + \partial(F_{p-1}C_{p+q+1}} = \frac{F_pC_{p+q}}{F_{p-1}C_{p+q}} = G_pC_{p+q},$$

as $\partial(\partial(F_{p-1}C_{p+q+1}) \subseteq F_{p-1}C_{p+q}$ by hypothesis, so we can define d^0 to be the induced boundary map on the quotient complex G_pC, that is, such that

$$d^0(x + F_{p-1}C_{p+q}) = \partial(x) + F_{p-1}C_{p+q-1}.$$

The argument above for $r > 0$ still holds for $r = 0$, so that

$$E_{pq}^1 \cong H_{pq}(E^0, d^0) = H_{p+q}(G_pC),$$

and unwinding the definitions we see that d^1 corresponds to the map which carries the class $[x+F_{p-1}C]$ with $x \in F_pC_{p+q}$, $\partial(x) \in F_{p-1}C_{p+q-1}$ to $[\partial(x) + F_{p-2}C]$, as desired.

To complete the proof, we must show that (E^r, d^r) converges to $G_pH_{p+q}(C)$. Since the filtration of C is bounded, we find that for all r sufficiently large $F_{p-r}C_{p+q-1} = 0$ and $F_{p+r}C_{p+q+1} = C_{p+q+1}$, so

$$\begin{aligned} Z_{pq}^r &= F_pC_{p+q} \cap \partial^{-1}(0) = F_pC_{p+q} \cap \ker(\partial : C_{p+q} \to C_{p+q-1}) \\ B_{pq}^r &= F_pC_{p+q} \cap \partial(F_{p+r}C_{p+q+1}) = F_pC_{p+q} \cap \operatorname{im}(\partial : C_{p+q+1} \to C_{p+q}). \end{aligned}$$

Thus

$$E_{pq}^\infty = \frac{F_pC_{p+q} \cap \ker\partial}{F_{p-1}C_{p+q} \cap \ker\partial + F_pC_{p+q} \cap \operatorname{im}\partial},$$

and if $\lambda : \ker\partial \to H_{p+q}(C)$ is the quotient map then a simple computation shows that

$$\lambda(F_pC_{p+q} \cap \ker\partial) = \iota_*(H_{p+q}(F_pC)) = F_pH_{p+q}(C)$$

where the equality on the right-hand side is the definition of the filtration on homology given in Proposition 1.2. Moreover, one checks that

$$\lambda(F_{p-1}C_{p+q} \cap \ker\partial + F_pC_{p+q} \cap \operatorname{im}\partial) = F_{p-1}H_{p+q}(C),$$

so if η^∞ is the quotient map $F_pC_{p+q} \cap \ker\partial \to E_{pq}^\infty$ then the universal property of the quotient induces a unique map

$$\rho : E^\infty \to \frac{F_pH_{p+q}(C)}{F_{p-1}H_{p+q}(C)}$$

making the following square commute:

$$\begin{tikzcd}
F_pC_{p+q} \cap \ker \partial \arrow[r, "\eta^\infty"] \arrow[d, "\lambda"'] & E^\infty_{pq} \arrow[d, dashed, "\rho"] \\
F_pH_{p+q}(C) \arrow[r] & \frac{F_pH_{p+q}(C)}{F_{p-1}H_{p+q}(C)}
\end{tikzcd}$$

But the restriction of λ to $F_pC_{p+q} \cap \ker \partial$ is a surjection, as noted above, as is the quotient map along the bottom horizontal, so ρ is likewise surjective. Finally, routine verification shows that ρ is injective, and hence yields an isomorphism

$$E^\infty_{pq} \cong \frac{F_pH_{p+q}(C)}{F_{p-1}H_{p+q}(C)} = G_pH_{p+q}(C)$$

completing the proof. □

References

[1] C. A. Weibel, *An Introduction to Homological Algebra.* Cambridge University Press, 1994. While the writing is fairly dry and the text is often more terse than perhaps desirable for a first treatment, chapter 5 covers the general theory of spectral sequences very thoroughly. Many of the applications of the Serre spectral sequence we discussed are presented in section 5.3, while sections 5.1 through 5.5 cover the spectral sequence of a filtration.

[2] N. Johnson, "A visualization of the hopf fibration." `https://nilesjohnson.net/hopf.html`, 2011.

[3] M. Holmberg-Péroux, "The serre spectral sequence." `http://homepages.math.uic.edu/~mholmb2/serre.pdf`, 2013. A self-contained and fairly concise treatment of the Serre spectral sequence, which the main results presented here in greater detail.

[4] A. Hatcher, "Spectral sequences." `https://pi.math.cornell.edu/~hatcher/AT/ATch5.pdf`. Hatcher's arguments are notoriously difficult to follow, and although chapter 5 (which covers spectral sequences) is no exception it is a useful reference for many of the minor technicalities assumed in the proof of Theorem 3.1. We believe that this text is at its best when supplemented with another text (such as Spanier or Fomenko and Fuchs).

[5] M. Hutchings, "Introduction to spectral sequences." `https://math.berkeley.edu/~hutching/teach/215b-2011/ss.pdf`, 2011. A helpful summary of many of the ideas presented above. Very few complete proofs are presented, but Hutchings provides an overview of the general theory which may be useful as a prelude to some of the other materials we reference.

[6] E. Spanier, *Algebraic Topology.* Springer, 1966. This text is very dense and abstract, which is either an asset or a liability, depending on one's perspective. We believe it serves as a useful foil to Hatcher, although with respect to spectral sequences it was only referenced for the proof of Theorem 3.1.

[7] A. Fomenko and D. Fuchs, *Homotopical Topology.* Springer, 2016. Similar to Spanier in function, but with a more modern treatment and possibly slightly easier to read. Chapter 3 covers spectral sequences of various shapes, sizes, and colors in great detail, and is a good alternative to chapter 5 of Hatcher.