Elementary Matheatics I - Numbers

Pre-requisites

None.

Contents

1 Integers: An Intuitive Definition

We begin with an instuitive definition of the integers. We start by introducing the positive numbers as those numbers most familiar to us and move progressively into the abstract integers.

The most familiar numbers are those used in everyday life

$$1, 2, 3, 4, \ldots$$

called the **positive numbers**. Another familiar number is

$$0.$$

Taking together the positives numbers and zero we have what are called the **natural numbers**

$$0, 1, 2, 3, \ldots$$

These numbers represent only symbols that mean nothing to us until we give them meaning. We do this by relating each number to the other on a number-line:

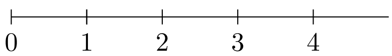

Figure 1: Natural Numbers

Each number is placed at equal intervals apart along the line; the distance from $0 \to 1$ is the same as the distance from $1 \to 2$ and so forth. We call zero the **origin** and each number represents its distance from the origin; 4 is that number which is four intervals away from the origin.

If we extend the number-line to the left we get the **negative integers**[1]

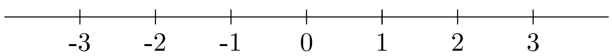

Figure 2: Integers

and when we take all of these togethers (positive integers, natural numbers, negative integers) we get the set[2] of numbers called the **integers**.

[1] It is worth noting here that 'negative' and 'minus' are often used interchangeably but, following Serge Lange's distate for this mistake, I will not be using them to mean the same thing. See Q4.

[2] A technical term representing a concept we are not yet prepared to deal with, but for now can be taken to mean 'collection', 'class' or 'species'

1.1 Addition and Algebra Introduction

Now we move onto performing operations on the integers which enable us to both manipulate the integers and discovers its rules.

Addition is the most fundamental operation we can perform on the integers which involves moving along the number-line (in any direction) according to a certain amount, we denote the operation with the + symbol.

Let us begin by adding zero to any number:

$$5 + 0 = 5.$$

We don't move left or right and remain precisely where we started. This leads to our first axiom of the integers.

Axiom I. Additive Identity of the Integers

Zero added to any number results in that same number.

Notice here that we stated this rule using words[3] which makes our rule excessively verbose and muddy. It would be convenient to introduce a notation for speaking about any numbers of a set. Let us do that by restating our axiom in this new notation

Axiom I. Additive Identity of the Integers

For any integer a, we have

$$0 + a = a.$$

We introduce the letter a to represent any integer; we do not care if its 0, 1, -1442 or any integer you can think of, so long as *it is is an integer* you can replace a with anything and the axiom applies. We will use this notation going forward.

1.2 Rules ofAdding Integers

[3] This was actually common before the creation of algebraic notation.*The Compendious Book on Calculation by Completion and Balancing* is a 9^{th} century algebra text written entirely in prose and is the source of our word 'algebra'.

2 Socratic Questions

Q1: What are these symbols we call numbers?

The symbols themselves $(0, 1, 2,)$ *etc are a conventional notation used to represent a understood distance away from the origin. This notation is Arabic, but other notation systems exist such as Roman Numerals , Egyptian, etc. Our numeric system is base-*10 *and positional which means we use ten digits* (0 *through* 9) *and the position of a digit in a number represents its place-value.*

Q2: What does ... mean?

This is an ellipsis that is shorthand notation for "and so on" or "so forth". It indicates that continuation of a sequence beyond what is written explicitly. Likewise the line extending past the final number on the number-line plays a similar role

Q3: What is an axiom?

Axiom is mathematical jargon for a rule that is asserted to be true but not proven. It is an assumption we make from which we then extrapolate new rules. The axioms governing integers for everyday mathematicians are the Peano Axioms, by Giuseppe Peano. Axioms are not empirical claims, but premises and, as such, cannot be disproven in the oridnary sense but can be rejected. A familiar axiom to you that has since been rejected is Euclids axiom that two parallel lines never meet - this is no longer accepted wholesale, such as in the field of non-Euclidean geometry.

Q4: Why 'minus' and not 'negative'

I use 'minus' for $-a$ *wherever the sign*[4] *of the value represented by a is ambiguous. Given an integer a the phrase 'negative a' carries with it the implication that the value represented by a is negative, that is it exists to the left of zero on the number-line. The phrase 'minus a' carries with it no such implication and instead refers to the operation.*

This may sound like a nit-pick but clear nomenclature makes the busy work of mathematics less cumbersome later on when performing large operations and multi-step deductions where such simple mistakes are common.

[4]'sign' refers to the positivity or negativity of a number

References

[1] Serge Lang. *Basic mathematics.* eng. Repr., corr. 3. print. New York Berlin Heidelberg: Springer, 1998. ISBN: 978-0-387-96787-5.