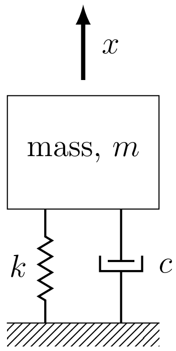
Documentation of the Spring Mass Damper solution using the RK4 ODE solver. Reference: http://www.ahmedmogahed.me/tutorials/mass-spring-damper/

Spring Mass Damper - IC Problem

This solution is an initial condition problem with no forcing function or controller.

System Diagram:



Equation of Motion:

Simple spring - mass - damper equation of motion:

$$m * \ddot{x} + c * \dot{x} + k * x = 0$$

where:

$$\begin{aligned} x &= \text{displacement - } m \\ m &= \text{mass - } kg \\ c &= \text{damping factor - } \frac{N}{m/s} \\ k &= \text{spring stiffness - } \frac{N}{m} \end{aligned}$$

Reformulated for only single order equations for use with the RK4 solver:

$$
\begin{aligned}
x &= x_1 \\
\dot{x_1} &= x_2 \\
\dot{x_2} &= -\frac{c}{m} * x_2 - \frac{k}{m} * x_1
\end{aligned}
$$

Initial Conditions:

$$
\begin{aligned}
x_1(t_0) &= x_0 \\
x_2(t_0) &= 0
\end{aligned}
$$

State Vector:

$$
State = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
$$

Solution Process

1. Define State Equations
   (a) Initialize array containing equations
2. Define initial conditions
   (a) This is the initial state vector for the system
3. Define solution time vector
4. Solution loop: While ($t <= t_{end}$)
   (a) Solve for new state vector using RK4
   (b) Store data

Exact Solution:

Three different exact solutions are possible for this system based on the system parameters.

Underdamped Solution: $\zeta < 1$

$$x(t) = Ae^{-\zeta\omega_n t} sin(\omega_d t + \phi)$$

where:

$$\omega_n = \sqrt{\frac{k}{m}}$$
$$\zeta = \frac{c}{2\omega_n m}$$
$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$
$$A = \frac{\sqrt{(x_0\omega_d)^2 + (v_0 + x_0\zeta\omega_n)^2}}{\omega_d}$$
$$\phi = arctan\frac{x_0\omega_d}{v_0 + x_0\zeta\omega_n}$$

Critically Damped Solution: $\zeta = 1$

$$x(t) = c_1 e^{-\zeta\omega_n t} + c_2 t e^{-\zeta\omega_n t}$$

where:

$$c_1 = x_0$$
$$c_2 = v_0 + x_0\zeta\omega_n$$

Overdamped Solution $\zeta > 1$

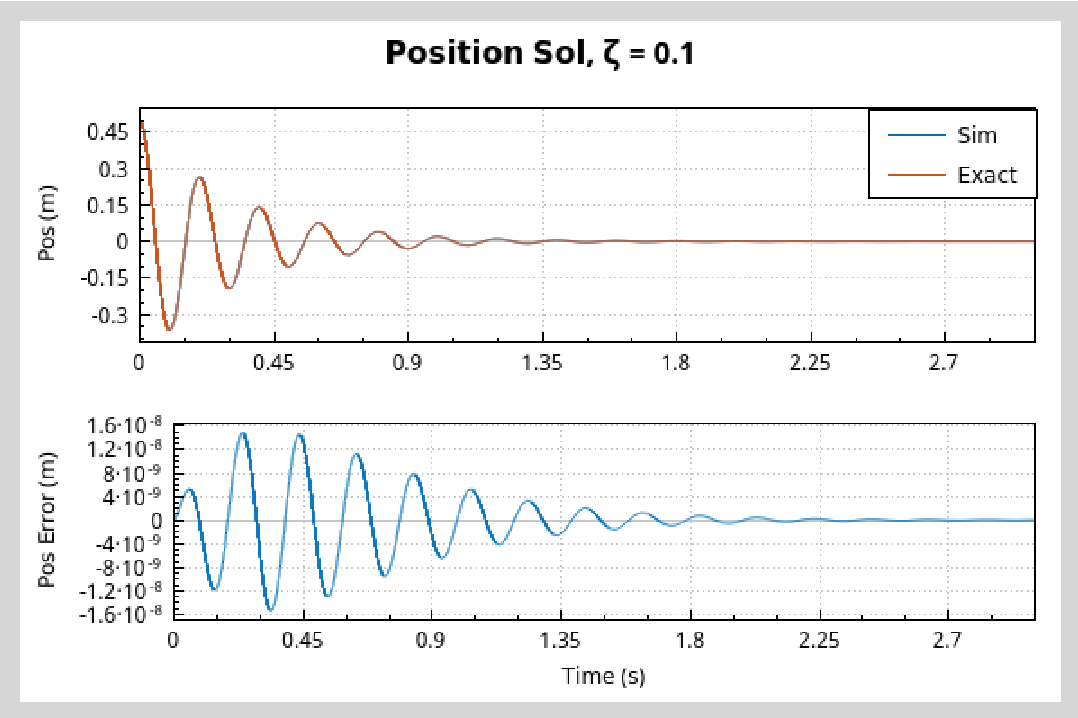
$$x(t) = e^{-\zeta\omega_n t}\left(c_1 e^{\omega_n\sqrt{\zeta^2-1}t} + c_2 e^{-\omega_n\sqrt{\zeta^2-1}t}\right)$$

where:

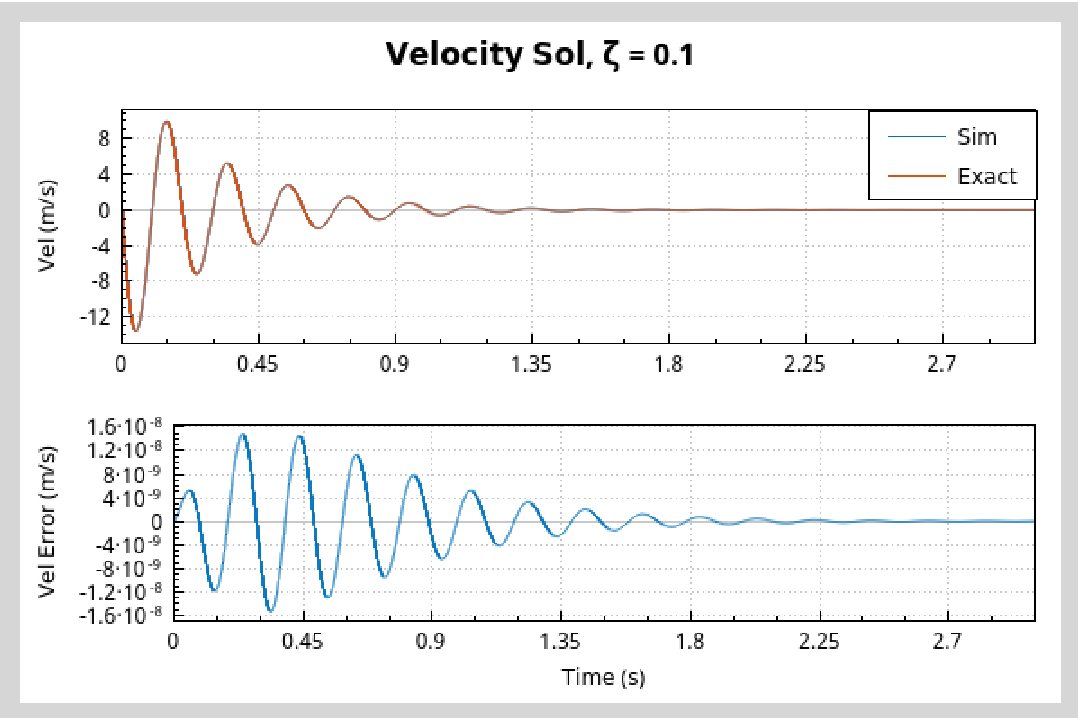
$$c_1 = \frac{x_0\omega_n(\sqrt{\zeta^2-1}+\zeta) + v_0}{2\omega_n\sqrt{\zeta^2-1}}$$
$$c_2 = \frac{x_0\omega_n(\sqrt{\zeta^2-1}-\zeta) - v_0}{2\omega_n\sqrt{\zeta^2-1}}$$

Simulation results:

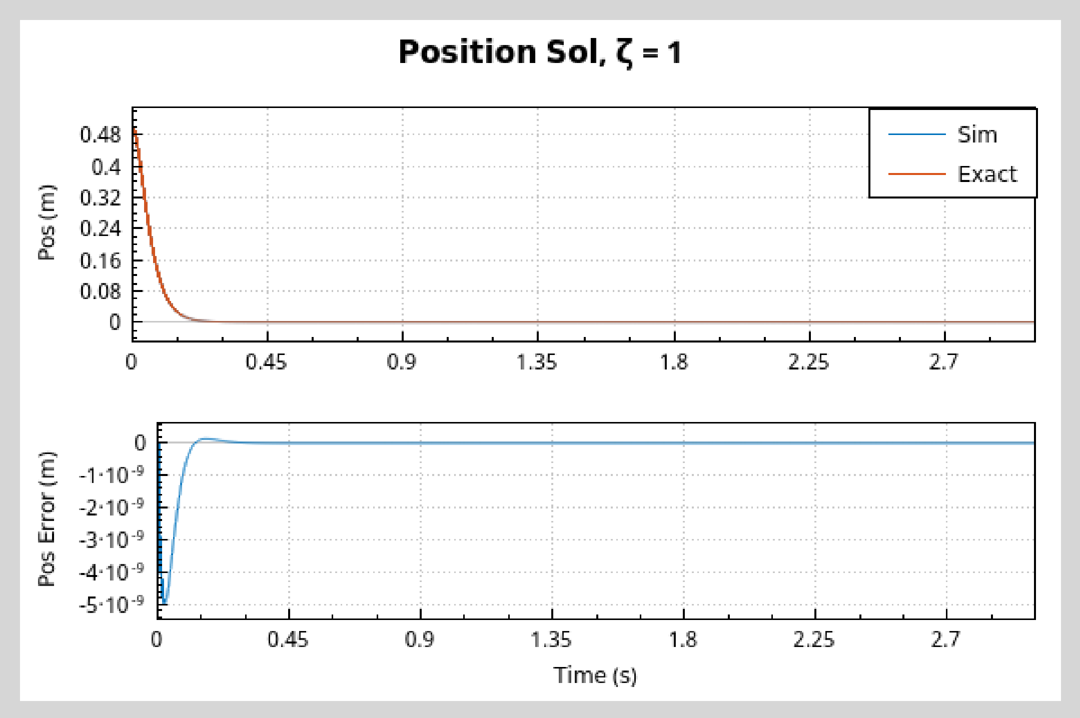
Using $m = 1kg$ and $k = 1000\frac{N}{m}$ the natural frequency of the system is ≈ 5 Hz. Three different values of c are chosen to exercise each results case.

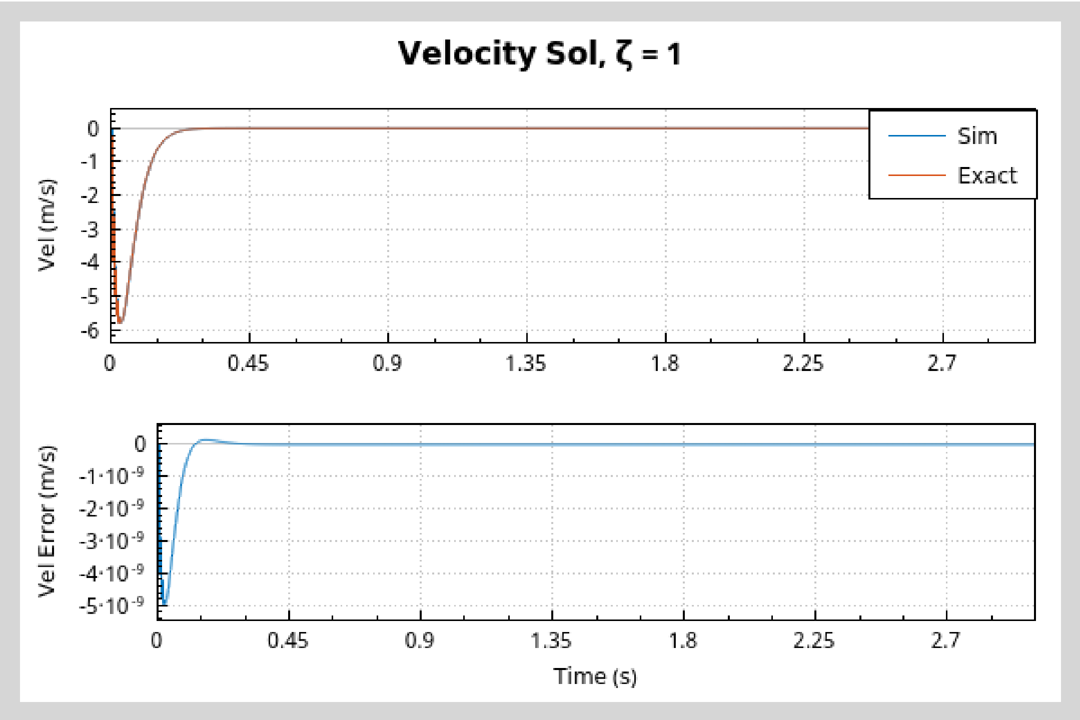


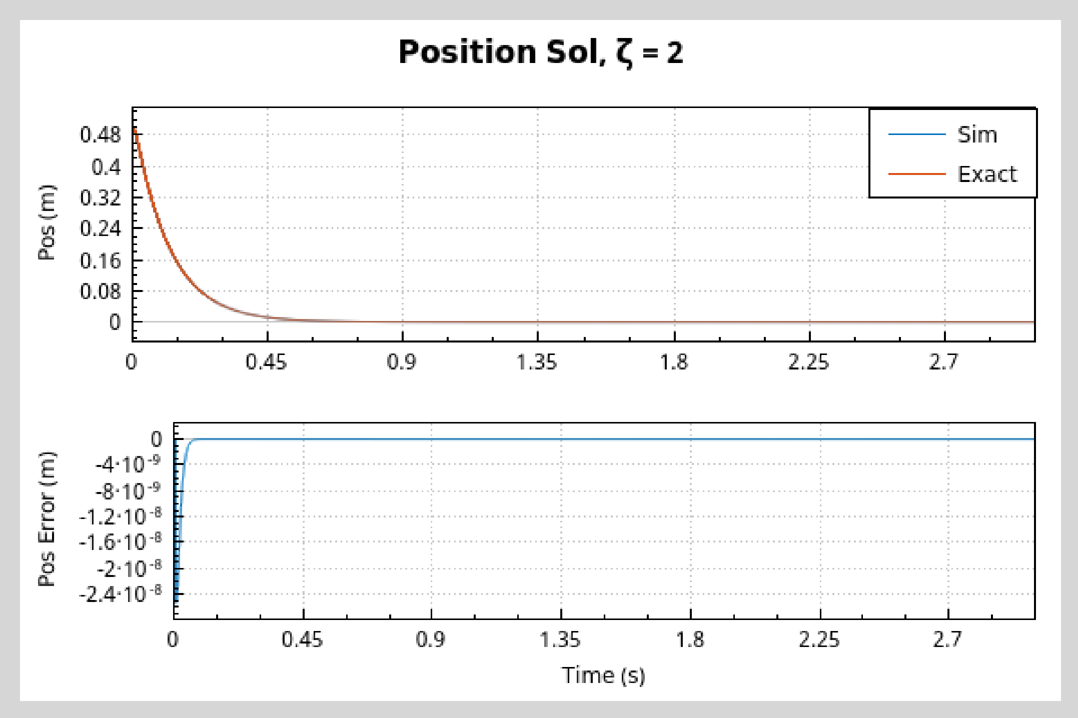
Position Sol, ζ = 0.1
Sim
Exact
Pos (m)
0.45
0.3
0.15
0
-0.15
-0.3
0
0.45
0.9
1.35
1.8
2.25
2.7
Pos Error (m)
1.6·10⁻⁸
1.2·10⁻⁸
8·10⁻⁹
4·10⁻⁹
0
-4·10⁻⁹
-8·10⁻⁹
-1.2·10⁻⁸
-1.6·10⁻⁸
Time (s)

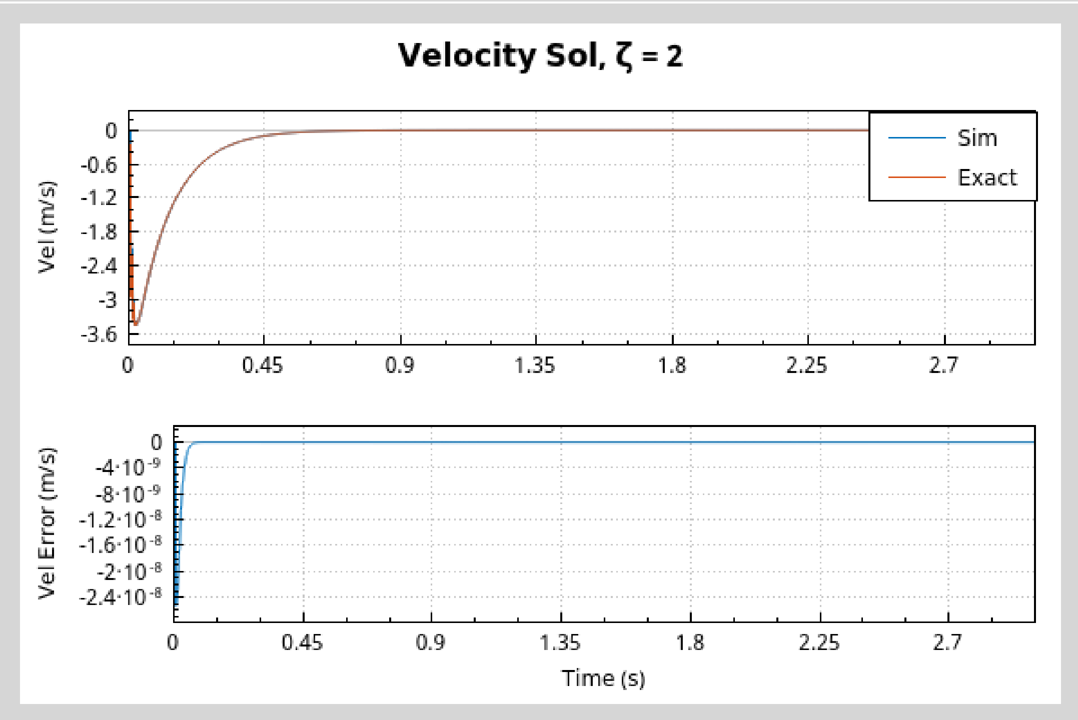


Velocity Sol, ζ = 0.1
Sim
Exact
Vel (m/s)
8
4
0
-4
-8
-12
0
0.45
0.9
1.35
1.8
2.25
2.7
Vel Error (m/s)
1.6·10⁻⁸
1.2·10⁻⁸
8·10⁻⁹
4·10⁻⁹
0
-4·10⁻⁹
-8·10⁻⁹
-1.2·10⁻⁸
-1.6·10⁻⁸
Time (s)


Position Sol, ζ = 1
Sim
Exact
Pos (m)
0.48
0.4
0.32
0.24
0.16
0.08
0
0
0.45
0.9
1.35
1.8
2.25
2.7
Pos Error (m)
0
-1·10⁻⁹
-2·10⁻⁹
-3·10⁻⁹
-4·10⁻⁹
-5·10⁻⁹
Time (s)


Velocity Sol, ζ = 1
Sim
Exact
Vel (m/s)
0
-1
-2
-3
-4
-5
-6
0
0.45
0.9
1.35
1.8
2.25
2.7
Vel Error (m/s)
0
-1·10⁻⁹
-2·10⁻⁹
-3·10⁻⁹
-4·10⁻⁹
-5·10⁻⁹
Time (s)

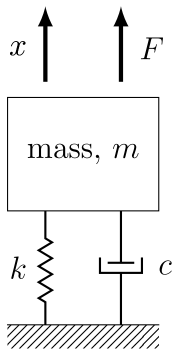

Position Sol, ζ = 2
Pos (m)
Sim
Exact
Pos Error (m)
Time (s)


Velocity Sol, ζ = 2
Vel (m/s)
Sim
Exact
Vel Error (m/s)
Time (s)

Spring Mass Damper - Forced Input

This section covers the solution of the forced problem. The input force is a swept sine signal.

System Diagram:



Equations of Motion:

Forced spring - mass - damper equation of motion:

$$m * \ddot{x} + c * \dot{x} + k * x = F$$

State equation form:

$$\begin{aligned} x &= x_1 \\ \dot{x_1} &= x_2 \\ \dot{x_2} &= -\frac{c}{m} * x_2 - \frac{k}{m} * x_1 + \frac{F}{m} \end{aligned}$$

Linear swept sine wave algorithm (reference: http://www.vibrationdata.com/tutorials/sweep.pdf)

$$F(t) = amp * sin(\frac{\pi * f_2 * t^2}{t_2})$$

This assumes the starting time and frequency are zero and:

$$\begin{aligned} f_2 &= \text{End Frequency - } Hz \\ t_2 &= \text{End Time - } s \\ t &= \text{Current Time - } s \\ amp &= \text{Amplitude - } N \end{aligned}$$

Solution Process

1. Define State Equations
   (a) Initialize array containing equations
2. Define initial conditions
   (a) Initial state vector for the system
   (b) Initial force on the system
3. Define solution time vector
4. Solution loop: While ($t <= t_{end}$)
   (a) Calculate Force
   (b) Store data
   (c) Solve for new state vector using RK4

Simulation Results

The results shown below are based on input parameters specifically chosen to set the natural frequency of the system to $\approx 5\ Hz$. The frequency rate for the sine sweep will be set such that this frequency will be hit at $\approx 15\ s$ of a 30 s sweep.

$$
\begin{aligned}
m &= 1\ kg \\
c &= 1\ \frac{N}{m/s} \\
k &= 1000\ \frac{N}{m} \\
t_2 &= 30\ Hz \\
f_2 &= 10\ s
\end{aligned}
$$