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**David Taylor Research Center**

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Ship Materials Engineering Department

Research and Development Report

The Relationship of Traditional Damping Measures for Materials with High Damping Capacity    DTRC-SME-91/05

# The Relationship of Traditional Damping Measures for Materials with High Damping Capacity

by

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## ABSTRACT

In the field of material damping a number of measures are used to express the level of damping which a material possesses. Such measures are required when evaluating material and system responses to dynamic loading conditions. The most widely used measures of damping capacity include the tangent of the phase lag, tan $\phi$, damping ratio, $\zeta$, specific damping capacity, $\psi$, loss factor, $\eta$, inverse quality factor, $Q^{-1}$, and log decrement, $\delta$. Each of these damping constants are defined in relation to the method used to measure them. Thus it can sometimes be difficult to compare the damping capacity of one material to another. By their inherent definitions, the measures of damping capacity listed above can be simply interrelated when damping levels are within the range $0 < \tan\phi < 0.14$ (i.e. $Q^{-1} = \eta = \tan\phi = E'/E'' = \delta/\pi = \psi/2\pi = 2\zeta$). However, these widely used interrelationships are actually approximations based on two simple anelastic models and an assumption of low damping. This assumption simplifies otherwise complicated nonlinear conversions. When higher levels of damping are of interest ($0.14 < \tan\phi$) the simple linear relationships given above can produce up to 40% error when converting from one damping measure to another. In this paper the basic measures of damping will be reviewed and the exact formulas which relate the various damping measures will be presented. Both the exact formulas and the approximations will be given in the discussion and in a table contained in the summary section. The nonlinear relationships will be useful in cases involving high damping, especially when damping levels are near $\tan\phi = 1$.

## ADMINISTRATIVE INFORMATION

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## INTRODUCTION

Simply stated, damping is the dissipation of energy in vibrating systems which results in either the control of the amplitude of oscillations or their eventual decay. In mechanical systems damping can be classified as passive or active. Active damping uses externally applied feedback control forces to limit the deformation and motion of large structural configurations. Passive

damping makes use of properties which are inherent to the system and which arise from energy dissipation taking place within stressed elements of a vibrating system or from energy being imparted to a surrounding dissipative medium. The passive damping which is of interest in this discussion involves mechanically stressed materials or structural elements and the energy dissipation associated with cyclic strain or stress.

The science of passive damping is separated into two areas: material damping and system damping. In material damping the energy dissipation inherent to the material is of interest. Some mechanisms which produce material damping in metals include movement of point or line defects and relative movement of domain walls. In polymeric materials the damping is often due to rotation or sliding of long monomer chains. In either case these dissipative effects are often broadly referred to as "internal friction," and this is a term which is widely used in the literature. In mechanical and structural assemblies the overall energy dissipation of the system is of interest and this can result from the addition of extra dissipative devices or materials (e.g. riveted joints with friction, viscous dampers, plastically deforming elements, viscoelastic layers) and/or damping due to interaction of the structure with its environment.

In material damping two primary classes of materials are studied: metal alloys and polymers. Each class has advantages and disadvantages as independent materials. Overall metals have good stiffness, strength, and creep characteristics. However, metals possess damping capacities which are generally much lower than that of polymers especially when service temperatures are near the polymer glass transition temperature. Generally speaking, polymers have excellent intrinsic damping capacities but suffer from poor stiffness, strength, and creep characteristics and also suffer from strongly temperature dependent material properties. To treat these independent problems designers have developed schemes that implement polymer layers or devices into metal structures thus improving the overall system damping. Also, researchers are attempting to develop new ways to increase the intrinsic damping in metals [1].

The discussion in this paper centers around the various measures of damping which are used to determine the dissipative capability of homogeneous linear materials. A variety of

damping measures are commonly used and the choice of one single measure depends on application, test method or both. The measures of damping which are most commonly used arise from an assumption of some mechanical model of physical behavior. The mechanical model can be chosen for the sake of simplicity or for the sake of modeling a more complex physical behavior. The latter choice is preferable although it is often impractical due to mathematical nonlinearities. For this reason simple models of damping are used in the majority of cases.

The primary measures of damping used in analysis and experimentation can be derived from two simple models of anelastic behavior namely the Kelvin-Voigt model and the complex modulus model. Following the definition in [2], *anelastic material behavior* requires that the following conditions be imposed on stress, strain, and equilibrium in a material:

1) For each value of stress in a material there must be an equilibrium value of strain (as a corollary, this condition requires a complete recovery of strain upon unloading to zero stress).
2) The equilibrium response is arrived at following some sufficient time delay (self adjustment or relaxation).
3) A linear stress-strain relationship is required.

This definition differs from that of an ideally elastic material only in the condition imposed by Item 2. For an ideally elastic material the equilibrium response is instantaneous and thus the difference between an ideally elastic material and an anelastic material is based only on the condition of instantaneity [2].

The condition of linearity is of utmost practical importance to both the derivation and meaning of the measures of damping. Fortunately many materials under low to moderate stresses (i.e. stresses much lower than the yield point) satisfy the conditions of linearity [2]. Linearity is embodied in the principle of superposition which states that when a sequence of stresses are applied to a material at different times the newly applied stress contributes to the resulting strain as though it were acting alone. In more specific terms this means that if there

exists a stress history $\sigma_1(t)$ which produces a strain history $\varepsilon_1(t)$ and a separate stress history $\sigma_2(t)$ which produces a strain history of $\varepsilon_2(t)$ then the sum $\sigma_1(t) + \sigma_2(t)$ will produce a strain of $\varepsilon_1(t) + \varepsilon_2(t)$.

The purpose of this paper is to present the models upon which the different measures of damping capacity are based and, through analytical methods, present the inter-relationships among them. It is shown that the nonlinear conversion of one damping measure to another produces a significantly different result when compared to the widely used linear conversions and that some of these conversions are valid for a only a limited range of values. These nonlinear relationships will be helpful in the correlation of material damping data.

## DISCUSSION

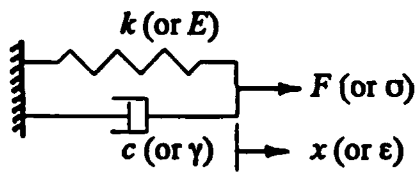
The measures of damping which are most commonly used in dynamic analyses and reporting of experimental data are the following: tangent of the phase lag (tan $\phi$), the damping ratio ($\zeta$), the specific damping capacity ($\psi$), the loss factor ($\eta$), the inverse quality factor ($Q^{-1}$), and the log decrement ($\delta$). These measures are used because they can derived from two simple models of mechanical behavior: 1) the Kelvin-Voigt (KV) model and 2) the complex spring (CS) (or Kimball-Lovell complex modulus [3]) model. Both these models represent simple anelastic behavior [2,4]. In the discussion we will sometimes refer to the KV and CS models as damped material elements.

In deriving the various damping measures we will consider two idealized cases involving both the KV and CS models. The first case will be that of an inertialess system to which a harmonic force is applied. This system is an idealized representation of a damping material under the influence of an applied harmonic force. The second case will be that of a one-dimensional system with inertia. Inertia will be included by the addition of a single rigid external mass to the damped material element. For these two cases we will show that the steady state dynamic response characteristics are quite different for the KV and CS elements; especially when the energy dissipation levels are large.

Note that when considering a single degree of freedom (SDOF) system that contains inertia we will restrict the inertia to be a rigid and external mass. In many cases of actual materials and test samples distributed inertia is important. However, all cases involving distributed inertia can be shown to be mathematically equivalent to the SDOF oscillator [2]. As a major consequence of this it follows that for a given frequency of vibration the damping exhibited by a homogeneous linear material is independent of the test specimen design and of the loading configuration [2,5].

### SDOF MATERIAL REPRESENTATIONS

Let us first give the mathematical equations and mechanical analogies of the two models without inertia. First, consider the Kelvin-Voigt model which consists of a perfectly elastic spring arranged in parallel with a dashpot that behaves as a linearly viscous fluid. The model representation and associated equations are as follows:




$$c\dot{x} + kx = F \tag{1}$$

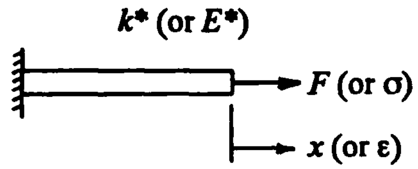
$$\gamma\dot{\varepsilon} + E\varepsilon = \sigma \tag{1a}$$

In Eq. (1) $F$ and $x$ are generalized force and deformation variables where $F$ represents the force applied across the entire element, $x$ represents the deformation of the model away from the undeformed geometry of the spring, $k$ is the elastic spring constant, and $c$ the viscosity coefficient of the dashpot. $F$, $x$, $k$, and $c$ are the counterparts of stress, strain, elastic modulus, and Newtonian viscosity, respectively, of the material. For example Eq. (1a) is the material counterpart of Eq. (1) for the case of uniaxial stressing where $\sigma$ is the stress, $\varepsilon$ is the strain, $E$ is the elastic modulus, and $\gamma$ is the Newtonian viscosity coefficient. Similar material counterparts could be written for the cases of shear loading and bulk response.

In the KV model the spring is an elastic element which provides a purely elastic restoring force and the dashpot provides linearly viscous damping. Becau . the spring and dashpot are

arranged in parallel, anelastic behavior is produced in simple step loading and unloading conditions [4]. For fully cyclic loading conditions the dashpot removes a portion of the energy which is input to the system and thus giving rise to damping.

Next consider the CS model. This model consists of a single spring which is characterized by a complex number as follows:




$$k^* x = F \tag{2}$$

$$E^* \varepsilon = \sigma \tag{2a}$$

where

$$k^* = k_1 + ik_2 \qquad E^* = E_1 + iE_2 \ , \qquad i = \sqrt{-1}$$

In Eq. (2) the real part of $k^*$ ($k_1$) is frequently called the storage modulus and the imaginary part of $k^*$ ($k_2$) is called the loss modulus. Eq. (2a) is the material counterpart to Eq. (2) for the case of uniaxial stressing. It is important to note that this model is truly valid only for the case of harmonic input (for an explanation regarding this restriction see [2]). Also, because the behavior is linear, the stiffness and damping constants of the material can be simply related to the stiffness and damping constants of a specimen [5].

Recall that mass has not been included in either of the simple representations given thus far. Such representations are considered to be simple models of material behavior where inertia terms are negligible compared to the stiffness and damping terms. Thus, in the discussion of the cases which include inertia we will sometimes refer to the KV and CS models as damped material elements.

Now let us take each model and solve for its response to the case of sinusoidal force input in the steady state. Specifically let $F = F_0 \sin \omega t$ where $F_0$ is the peak force and $\omega$ is the frequency of harmonic input. Note that for the two models to have equivalent stiffness we are specifying that $k$ in the KV model is equal to $k_1$ in the CS model, i.e. $k = k_1$. The steady state

solution for the case of the KV model is straightforward [6,7] and the details will not be repeated here. With some knowledge of the complex variable theory [7] the steady state solution for the CS model is also straightforward. It turns out that the steady state solution for both models can be written as follows:

$$F = F_0 \sin \omega t \tag{3}$$

$$x = X_0 \sin (\omega t\text{-}\phi) \tag{4}$$

where

$$X_0 = \frac{F_0}{k\sqrt{1 + \tan^2\phi}} \tag{5}$$

$$\tan \phi = \frac{k_2}{k_1} \qquad \text{for the CS model} \tag{6a}$$

$$\tan \phi = \frac{c\omega}{k} \qquad \text{for the KV model} \tag{6b}$$

Here $X_0$ is the amplitude of the steady state response and $\phi$ is a phase angle by which the response lags the input.

### TANGENT OF THE PHASE LAG: TAN $\phi$

Because the material response, $x$, lags the force input by the phase angle, $\phi$, (also called phase shift and loss angle) a number of commercially developed testing devices (e.g. the Polymer Laboratory DMTA and the Dupont DMA) have been designed to experimentally determine a material's damping capacity by measuring the phase shift. Therefore we have just defined one of the measures of damping: namely tan $\phi$. In the case of the CS model tan $\phi$ is determined by material properties alone, as seen in Eq. (6a). Also, because tan $\phi$ is a constant, the overall amplitude $X_0$ is determined only by material properties and the amplitude of the force input. Thus the overall response of the CS element is independent of the applied frequency. Specifically, tan $\phi$ is dependent upon the loss modulus $k_2$. However, due to the linearly viscous nature of the damping in the KV model tan $\phi$ is dependent upon the product of the viscosity

coefficient and the frequency (see Eq. (6b)) and thus the response of the KV element to sinusoidal input is dependent upon the frequency. Therefore when using the CS model one assumes that the effect of damping on the material response is present even at very low frequencies and that this damping is constant, while for the KV model the response is greatly affected by the rate of sinusoidal input.

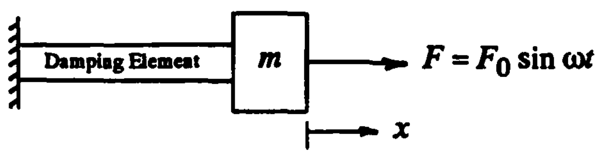
When the force, Eq. (3), input is plotted against the response Eq. (4) the result is an elliptical hysteresis loop as shown in Figure 1. More specifically the plot is that of an ellipse which is inclined with respect to the $x$ axis. Two straight lines which show the inclination of this axis are given in the plot. First note the solid line which is called the storage modulus line; this line represents the stiffness of the spring ($k$) in the KV model and the storage modulus ($k_1$) of the CS model. Also, note that this line connects the origin to the point of maximum deformation. The second straight line shown is dashed and represents the major axis of the ellipse. As the damping decreases the major axis rotates toward the storage modulus line and the hysteresis loop becomes thinner, ultimately collapsing onto the storage modulus line in the limit of zero damping. This progression is shown in Figure 2 for three different values of tan $\phi$: tan $\phi$ = 1, .6, and .2.

Recall from Eq. (6b) that $\tan \phi = c\omega/k$ for the KV element. Thus in order to increase tan $\phi$ in the KV model it is only necessary to increase the frequency $\omega$ of the harmonic input. Since no material constants are changed in the process, Figure 2 clearly shows one feature of the frequency dependent nature of the KV model. Namely, for an increase in the frequency of harmonic force input, the hysteresis loop widens, the major axis of the ellipse rotates away from the storage modulus line, and the position of peak deformation decreases. Such behavior will not be exhibited in the response of the complex spring. Recall from Eq. (6a) that $\tan \phi = k_2/k_1$. Also, by examining Eq. (5) it is clear that the frequency of loading does not affect the amplitude of the response. Thus the complex spring, in an of itself is a rate independent damping element. Note however that the KV model response will be equivalent to the CS model response if the damping coefficient in the KV model has the property that $c=k_2/\omega$. This equivalence is valid

only for the case where $\omega$ is the constant frequency of a single frequency harmonic input. Thus the CS model can be considered to be a special case of the KV model, but only when the loading is harmonic with a single frequency of $\omega$ and also when $k_2 = c\omega$.

### SDOF Systems with Inertia

Before proceeding to other measures of damping let us consider the second idealized case wherein a single rigid mass is externally attached to each of the damped material element. When the mass is acted upon by a harmonic force we have what is known as the *simple harmonic oscillator*. This is depicted below:




For the purposes of the discussion here the damping element is represented by either the CS or KV model. Also note that $x$ is the displacement of the mass away from a reference configuration wherein the damped element is undeformed and is without any residual stresses. The response of the harmonic oscillator will lead to a number of quantities which are used as measures of material damping.

The equations which govern the harmonic oscillator are as follows for the two models under consideration:

$$m\ddot{x} + k^{*}x = F_0 \sin \omega t \qquad \text{(CS Model)}$$

$$m\ddot{x} + c\dot{x} + kx = F_0 \sin \omega t \qquad \text{(KV Model)}$$

The steady state solution of either of the above differential equations is as follows [2,7]:

$$x = X_0 \sin (\omega t\text{-}\theta) \tag{7}$$

where $$X_0 = \frac{F_0/k}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \tan^2\phi}} \tag{8}$$

$$\tan\theta = \frac{\tan\phi}{1-\left(\frac{\omega}{\omega_n}\right)^2} \tag{9}$$

$$\omega_n = \sqrt{\frac{k}{m}}$$ is the natural frequency

and as before

$$\tan\phi = \frac{k_2}{k_1}$$ for the CS model

$$\tan\phi = \frac{c\omega}{k} = 2\zeta\frac{\omega}{\omega_n}$$ for the KV model

Here $\theta$ is the phase lag of the displacement of the mass with respect to the force applied to the mass whereas $\phi$ is the phase lag of strain with respect to stress in the material. Also, $\zeta$ is the damping ratio in the KV-mass model (defined as $\zeta = c/c_c$ where $c_c$ is the critical value of damping: $c_c = 2\sqrt{km}$). Note that for the case of quasi-static loading (where $\omega/\omega_n \ll 1$) the solution reduces to that of the inertialess system, Eqs. (4)-(6), with $\theta \approx \phi$.

## SPECIFIC DAMPING CAPACITY: $\psi$

With this background now in place let us now proceed to the evaluation of other measures of damping. Presently let us consider the measure of damping known as the specific damping capacity. The specific damping capacity (denoted as $\psi$) is defined by the following expression:

$$\psi = \frac{\Delta W}{W} \tag{10}$$

Here $\Delta W$ represents the energy absorbed per cycle of deformation in the steady state and the quantity $W$ in the denominator is a measure of stored energy. $\Delta W$ is also called the absolute

damping energy of the specimen [5] and therefore $\psi$ is a relative damping measure. For the KV and CS elements without mass $\Delta W$ can be found directly by using Eqs. (3) and (4) in conjunction with the following integral:

$$\Delta W = \int_{\text{cycle}} F\,dx = \int_{0}^{2\pi/\omega} F\dot{x}\ dt$$

which yields

$$\Delta W = \pi F_0 X_0 \sin\phi$$

A variety of definitions for stored energy can be used in the expression for $\psi$. For example, in [5] $W$ is defined to be the total strain energy for the entire specimen at maximum deformation. Other authors (e.g. [8]) take $W$ to be the peak potential energy solely in the elastic component of the model. Still others have defined $W$ to be the instantaneous kinetic plus potential energies of vibrating systems with inertia [9]. Thus the choice of $W$ which is made greatly influences the value of $\psi$. In the following paragraphs we will individually consider these three specific definitions of the energy storage term, $W$, applied to both the inertialess material response and to the system response with inertia.

First let us consider the case of the inertialess material response where $W$ is a strain energy quantity; specifically let $W_1$ be the strain energy stored in the material upon loading from zero deformation to maximum deformation i.e.

$$W_1 = \int_{0}^{X_0} F\,dx = \int_{\frac{\phi}{\omega}}^{\frac{1}{\omega}\left(\phi+\frac{\pi}{2}\right)} F\dot{x}\ dt$$

The strain energy represented by $W_1$ is shown as the cross-hatched region of Figure 3. Using Eqs. (3) and (4) in the above integral produces the following result:

$$W_1 = \frac{F_0 X_0}{2} \cos\phi \left[1 + \frac{\pi}{2}\tan\phi\right]$$

Following a similar process let us next examine the slightly different case where $W$ is the strain energy stored in the inertialess material upon loading from zero force to maximum force, i.e.

$$W_2 = \int_{-X_0 \sin\phi}^{X_0 \cos\phi} F\,dx = \int_0^{\frac{1}{\omega}\frac{\pi}{2}} F\dot{x}\;dt$$

This strain energy is shown as the cross-hatched region of Figure 4. Interestingly, the result of integration is identical to that of the previous case

$$W_2 = \frac{F_0 X_0}{2} \cos\phi \left[1 + \frac{\pi}{2}\tan\phi\right] = W_1$$

Thus the strain energy stored when loading from zero force to maximum force is identical to the strain energy stored when loading from zero deformation to maximum deformation. This result is not intuitively expected because the shape of the stress-strain curve follows an elliptical loop rather than a loading path which passes through the origin. It should be pointed out that the equivalence of $W_1$ and $W_2$ is a consequence of linearity; for nonlinear materials (i.e. materials with non-elliptical hysteresis loops) $W_1 \neq W_2$.

Next, consider yet another definition of stored energy for the inertialess material response. In this third case let $W$ be defined as the energy stored in the purely elastic component of each damped material element and let this energy be computed for the case of maximum deformation in the cyclic response. The elastic component of the KV and CS models is a purely elastic spring and is represented by the stiffness constant $k$. The shaded portion of Figure 5 represents the maximum potential energy stored in the spring, which is

$$W_3 = \frac{1}{2} kX_0^2 = \frac{F_0 X_0}{2} \cos \phi$$

Therefore, at maximum deformation, the energy stored in the elastic component of the material model is a quantity which has a simpler form than strain energies $W_1$ and $W_2$.

According to Eq. (10), for the three separate definitions of stored energy $W_1$, $W_2$, and $W_3$ which apply to the inertialess material response, the specific damping capacity becomes $\psi_1$, $\psi_2$, and $\psi_3$ as follows for both the KV and CS models:

$$\psi_1 = \psi_2 = 2\pi \frac{\tan \phi}{1 + \frac{\pi}{2} \tan \phi} \quad , \quad \psi_3 = 2\pi \tan \phi$$

Clearly $\psi_3$ gives the simplest relationship between $\tan \phi$ and $\psi$. In fact, $\psi_3$ defines a relationship which is widely used.

Next we will consider the computation of $\psi$ for the oscillating system with either KV or CS type damping this time adding mass. Even though the vibrational response is now different because of the inertia which is present, we will be able to arrive at a definition of $\psi$ which is identical to $\psi_3$. However in order to obtain this identity it will again be necessary to make a careful selection of the stored energy term $W$.

The solution for the response of the harmonic oscillator will afford us two very important analyses; the first analysis involves the computation of the specific damping capacity and the second analysis involves the computation of the inverse quality factor. Because the energy dissipation is associated with the steady state response of the damped spring-mass system (Eq. (7)) we will specially denote the dissipated energy as $\Delta W_{sys}$:

$$\Delta W_{sys} = \int_{cycle} F \, dx = \int_0^{2\pi/\omega} F\dot{x} \, dt$$

In this case the energy dissipation is quite different than that of the previous case because of the added mass and thus the result is strongly dependent upon the frequency $\omega$

$$\Delta W_{sys} = \pi F_0 X_0 \sin\theta = \pi F_0 X_0 \frac{\tan\phi}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \tan^2\phi}}$$

The energy storage quantities that were used for the inertialess response ($W_1$, $W_2$, and $W_3$) may also be used for the simple harmonic oscillator. However the most convenient expressions for $\psi$ are produced by using the third energy definition. Following the methodology used in [9] the stored energy $W$ is specified as a simple energy storage term, as was done in the case of $W_3$. Thus let us consider $W_4$ to represent the energy stored in the elastic spring at maximum deflection in the cyclic oscillation. No restriction is placed on frequency and the result is as follows:

$$W_4 = \frac{1}{2} k X_0^2$$

Using Eq. (8) and rearranging $W_4$ gives

$$W_4 = \frac{1}{2} F_0 X_0 \frac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \tan^2\phi}}$$

and therefore

$$\psi_4 = \frac{\Delta W_{sys}}{W_4} = 2\pi \tan\phi$$

Alternatively, W could have been specified as the total energy stored in the damped spring-mass oscillator; i.e. the instantaneous potential energy of the spring plus the instantaneous kinetic energy of the mass:

$$W_5 = \frac{1}{2}kx^2 + \frac{1}{2}m\dot{x}^2$$

Plugging the response, Eq. (7), into $W_5$ produces the following result:

$$W_5 = \left[1-\left[\frac{\omega}{\omega_n}\right]^2\right]\frac{1}{2}kx^2 + \left[\frac{\omega}{\omega_n}\right]^2\frac{1}{2}kX_0^2$$

Note that the energy stored in the damped system is dependent upon both frequency and on the instantaneous position $x$. However, when the system is oscillating at resonance the energy stored in the system becomes constant for every position in the cycle:

$$W_5\Big|_{\omega=\omega_n} = \frac{1}{2}kX_{0Res}^2 = \frac{1}{2}F_0X_{0Res}\frac{1}{\tan\phi} \quad , \qquad X_{0Res} = X_0\Big|_{\omega=\omega_n}$$

By computing the specific damping capacity of the system at resonance the following result is produced which is identical to $\psi_3$:

$$\psi_5 = \frac{\Delta W_{sys}}{W_5}\Bigg|_{\omega=\omega_n} = 2\pi\ \tan\phi$$

The specific damping capacities which pertain to the harmonic oscillator are exhibited in $\psi_4$ and $\psi_5$. Thus when considering an oscillating spring-mass system with damping of either the Kelvin-Voigt or the complex spring type, the tangent of the phase lag can be determined by computing either of the following: $(\Delta W_{sys}/W_4$ or $(\Delta W_{sys}/W_5)\big|_{\omega=\omega_n})$ where $\Delta W_{sys}$ is the damping energy dissipated per cycle, $W_4$ is the energy stored in the spring at maximum deflection $X_0$, and $W_5$ is the total energy of the oscillating spring-mass system.

## LOSS FACTOR: $\eta$

The next measure of damping considered is the loss factor. By using proper energy storage definitions the loss factor allows for a simple and direct inter-relationship between actual energy absorbed and phase lag. The symbol denoting loss factor is $\eta$ and is defined as follows:

$$\eta = \frac{\psi}{2\pi} = \frac{1}{2\pi}\frac{\Delta W}{W} \qquad (11)$$

Using this definition, and referring back to our recent definitions of $W$ we have the following results:

$$\eta_1 = \eta_2 = \frac{\tan\phi}{1 + \frac{\pi}{2}\tan\phi}$$

$$\eta_3 = \tan\phi \qquad \text{and also} \qquad \eta_4 = \eta_5 = \tan\phi$$

To the best of the authors' knowledge the definitions given by $\eta_1$ and $\eta_2$ have not been reported nor are they of primary interest. However the definitions given by $\eta_3$, $\eta_4$, and $\eta_5$ are of primary interest because of their simple form and because the conversion $\eta = \tan\phi$ is widely used in damping work. It is important to note, however, that the form of this widely used conversion is due entirely to the definition of $W$ in the denominator of Eq. (11). Therefore workers in the field of damping who report measured values of the loss factor should precisely specify what they are using as their measure of stored energy or, more simply, report the specific damping capacity.

It is important to reiterate that $\eta = \tan\phi$ only in the three specific cases for which $\psi = 2\pi \tan\phi$. The first case pertains to the system without inertia. The loss factor $\eta$ in this case is defined by $1/2\pi$ times the energy dissipated during one full cycle of deformation for an inertialess linear material which can be modeled by either a complex spring or a Kelvin-Voigt element, divided by the energy stored an the purely elastic component of the material at maximum deformation. The second and third cases pertain to the vibrating linear SDOF system (i.e. the system with inertia) with the element attached to the mass being either of the KV or CS type. In the second case the loss factor is defined as the product of $1/2\pi$ and the energy dissipated by the system in one full cycle of steady state resonant deformation divided by the total energy of the system at resonance (kinetic energy of the mass plus potential energy of the spring). Finally in the third case the loss factor is defined by $1/2\pi$ times the energy dissipated by the system in one full cycle of steady state deformation (at any frequency) divided by the energy stored an the purely elastic component of the damping element at maximum deformation. *Only*

*by defining the loss factor precisely in these terms is* $\eta = \tan\phi$. Thus for these special definitions $\eta$ represents a constant material damping property in the case of a complex spring and a frequency dependent damping property in the case of the Kelvin-Voigt model (see Eqs. (6a) and (6b)).

### INVERSE QUALITY FACTOR: $Q^{-1}$

The next measure of damping which we will consider is the inverse quality factor ($Q^{-1}$). This measure is based on the frequency response of the damped spring-mass system under harmonic steady state conditions. Also called the resonance curve breadth factor, $Q^{-1}$ is determined by the half power bandwidth of the response amplitude vs. frequency plot and it is convenient because in conditions of low damping it can be simply related to the previously discussed measures of damping.

The frequency dependent response amplitude, $X_0$, of the damped spring-mass system was given earlier in Eq. (8). By dividing $X_0$ by $F_0/k$ we now obtain the magnification ratio, $M$. Since tan $\phi$ is different for the KV and CS systems the magnification ratio must be considered separately for these systems.

$$M = M(\omega) = \frac{X_0}{F_0/k} \quad , \quad M_{CS} = \frac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \tan^2\phi}} \tag{12a}$$

$$M_{KV} = \frac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta\frac{\omega}{\omega_n}\right]^2}} \tag{12b}$$

The dimensionless magnification ratio is so named because it magnifies a static deflection of $F_0/k$ to the dynamic response amplitude $X_0$. In other words by multiplying $M$ with the static deflection produced in a spring of stiffness k by steady force $F_0$, we obtain the amplitude of steady state vibration that results when the same force acts harmonically dynamic with frequency $\omega$.

A concise graphical interpretation of the response can be made by plotting $M$ against $\omega/\omega_n$ (called the frequency ratio) as shown in Figure 6 and 7 for the KV and CS models respectively. The curves given in these figures are known as frequency response curves and clearly show that the magnitude of the steady state response is strongly dependent upon both the frequency and the amount of damping. The two sets of frequency response curves are similar in that the damping level ($\zeta$ in the KV-mass model and tan $\phi$ in the CS model) controls the height of the response peak near the region of resonance. Indeed the resonant peak is sharply defined for low levels of damping and becomes infinite in the limit of zero damping. However, as damping is increased the peaks of both response curves are lowered dramatically.

There are a few notable differences in the separate response curves, Figures 6 and 7, which should be pointed out. The first difference is the position of the peak response on the frequency scale. In both figures the peak is located at the position of resonance ($\omega/\omega_n = 1$) when damping levels are small enough so as to approach zero. However as the amount of damping increases the peak of the KV response shifts away from the resonant frequency to lower frequencies whereas the position of peak response of the CS model remains located at $\omega_n$. This is due to the fact that the KV element by itself acts in a frequency dependent manner while the CS element does not. Also, as $\omega/\omega_n$ approaches zero for the KV-mass system, $M$=1 regardless of the amount of damping. This is because the viscous damping force vanishes as the rate of loading approaches zero. Such behavior is not exhibited in the CS element. In the CS element energy is dissipated independently of the rate of loading and a damping force is produced even in the limit of quasi-static loading, i.e. at $\omega/\omega_n$=0. Thus in the quasi-static condition $M$ is reduced away from an upper bound of $M$=1 as tan $\phi$ is increased in the CS element. In addition, the response associated with the CS model has a dynamic response peak located at $\omega/\omega_n$=1 for all levels of damping whereas the peak response associated with the KV model becomes barely distinguishable from $M$=1 as damping is increased past $\zeta$ = 0.5. Indeed the peak of the resonance curve of the KV-mass system becomes $M$=1 and is located at $\omega/\omega_n$=0 for adequately high levels of damping (i.e. as $\zeta$ approaches 1).

The definition of the inverse quality factor used in the damping analyses of mechanically vibrating systems was actually borrowed from the theory of electrical circuits. Therefore it is important that we give a brief background of the quality factor as it applies to the electrical theory. This is useful because resistor-inductor-capacitor (*RLC* where the symbols *R*, *L*, and *C* represent resistance, inductance, and capacitance, respectively) circuits are analogous to spring-mass-damper (*kmc*) systems. Indeed, the following quantities are analogous to one another: applied force and impressed voltage, charge and displacement, resistance and damping (in fact resistance is a dissipative mechanism), inductance and mass, and elastance ($1/C$) and stiffness.

In our discussion of the quality factor of *electrical* systems we will use the specially superscripted symbol $Q^e$. The quality factor, $Q^e$, was originally developed to express a figure of merit (or quality) of highly underdamped resonant electrical circuits or filter networks which, for certain important applications, need to be as free from damping as possible. By definition, for any resonant circuit, the sharpness of the resonant frequency response is determined by the amount of energy that *can be stored in the circuit*, compared with the energy that is lost during one complete period of resonant oscillation [10]. Note that this quantity is analogous to our previous definition of $\psi$ and $\eta$ of mechanical damping. Indeed, for electric circuits the general definition of quality factor states that $Q^e$ is equal to the product of $2\pi$ and the ratio of stored electromagnetic energy to energy lost in the total network [11].

However, when considering the condition of electrical resonance in either parallel or series *RLC* circuits the quality factor can be shown to reduce to the following expression [10]:

$$Q_r^e = \frac{\omega_n}{\omega_2 - \omega_1} \qquad (3)$$

Here the subscript $r$ denotes the resonant condition and $\omega_n$ is the resonant frequency of the circuit. The frequencies $\omega_2$ and $\omega_1$ define two points on the response curve that have a value of $1/\sqrt{2}$ times the response at resonance. And because the response squared is proportional to the power dissipated during cyclic loading these frequencies are called the half-power point

frequencies. In this context the difference $\omega_2$-$\omega_1$ defines the so-called half-power bandwidth of the frequency response and we specify that $\omega_2>\omega_1$. Although this relationship is derived specifically for parallel and series resonant *RLC* circuits it is commonly used for all electrical networks whose magnitude plots exhibit frequency selectivity (i.e. resonance) [11].

It is the relationship for $Q^e$ as given by Eq. (13) which is used for mechanical damping. Specifically, as it applies to the response amplitude vs. frequency plots of mechanical systems, the inverse relationship of Eq. (13) is used as a measure of mechanical damping [2,6,12]. Thus the inverse quality factor of mechanical systems is defined as:

$$Q^{-1} = \frac{\omega_2 - \omega_1}{\omega_n} \tag{14}$$

Here, $\omega_n$, $\omega_1$ and $\omega_2$ all have the same meaning as in the electrical analogy; $\omega_1$ and $\omega_2$ define the half power points of the magnification ratio vs. frequency ratio plots of mechanically vibrating systems and $\omega_n$ is the natural frequency. Specifically $\omega_1$ and $\omega_2$ are the response frequencies which satisfy $X_0(\omega) = \frac{1}{\sqrt{2}}\left[X_0\big|_{\omega=\omega_n}\right]$.

Let us consider $Q^{-1}$ first for the CS model and then for the KV model. The response amplitude at resonance is denoted as $X_{0_{Res}}$ and, according to Eq. (8), is equal to $F_0/(k \tan\phi)$. The frequencies $\omega_1$ and $\omega_2$ are found by equating Eq. (8) with $(1/\sqrt{2})X_{0_{Res}}$, squaring both sides, rearranging, and solving the following equation for $\omega$:

$$\left[1-\left[\frac{\omega}{\omega_n}\right]^2\right]^2 + \tan^2\phi = 2\tan^2\phi$$

The roots of this equation are as follows: $\left[\frac{\omega}{\omega_n}\right]^2_{(1,2)} = 1 \pm \tan\phi$

and the result for the inverse quality factor is:

$$Q^{-1} = \frac{\omega_2 - \omega_1}{\omega_n} = \sqrt{1+\tan\phi} - \sqrt{1-\tan\phi} \tag{15}$$

This equation is analogously derived in [8].

Before proceeding further we wish to make an observation regarding the magnification ratio. Note that if Eq. (12a) is evaluated for $\omega = \omega_n$ a simple relationship between the magnification ratio and tan $\phi$ is produced, namely $M(\omega_n) = 1/\tan\phi$. This relationship can be used to determine damping experimentally rather than $(\omega_2 - \omega_1)/\omega_n$. Indeed, it would appear that the height of the peak of $M$ would be easy to measure from a given set of experimental data. However, in actual experimental conditions only the height of the response peak is measured and this height, by itself, does not convey an adequate amount of information to determine the damping. In such case the inverse quality factor must be used.

Even though the response amplitude is real-valued and defined for all values of tan $\phi$ it is important to note that $Q^{-1}$ is not. In fact $Q^{-1}$ is real valued only when the arguments under each square root are positive; thus $Q^{-1}$ is defined only in the range of $-1 \leq \tan\phi \leq 1$. However actual materials cannot possess negative damping and must have $\tan\phi \geq 0$ since $\tan\phi = 0$ represents a perfectly elastic material. Therefore we further restrict Eq. (15) to the range: $0 \leq \tan\phi \leq 1$. The need for this restriction can be seen visually in Figure 8 where the square of the magnification ratio is plotted against the frequency ratio. Careful examination of this plot reveals that when $\tan\phi \leq 1$ there will always exist two specific half power frequencies, $\omega_1$ and $\omega_2$, corresponding to points on either side of $\omega_n$ where $M^2_{CS}(\omega) = M^2_{CS}(\omega_n)/2$. Note that as tan $\phi$ increases and approaches the upper limit of $\tan\phi = 1$ the frequency $\omega_1$ shifts to the left. At $\tan\phi = 1$ the half power frequencies are $\omega_1 = 0$ and $\omega_2 = 1.41\omega_n$. However, as the damping increases past this upper limit the only half power frequency which can be determined from the plot is $\omega_2$ and this frequency lays to the right of $\omega_n$. To the left of $\omega_n$ it is clear that $M^2_{CS}(\omega) > M^2_{CS}(\omega_n)/2$ and $\omega_1$ is indeterminate. Thus Eq. (15) can only be used for $0 \leq \tan\phi \leq 1$.

When damping levels are very low we can simplify the expression for $Q^{-1}$ by using the series expansion for the square root quantities [13] in Eq. (15); doing this Eq. (15) becomes

$$Q^{-1} = \tan\phi \left[ 1 + \frac{1}{8}\tan^2\phi + \frac{5}{64}\tan^4\phi + \cdots \right]$$

When considering only small values of tan $\phi$ (i.e. tan $\phi \ll 1$) any terms in the series with an order of two or higher are negligible with respect to unity and the following approximation can be used:

$$Q^{-1} \approx \tan\phi \qquad \text{for} \qquad \tan\phi \ll 1 \tag{16}$$

The exact and the approximate forms of $Q^{-1}$ are plotted vs. tan $\phi$ in Figure 9. Notice that the exact form, Eq. (15), deviates significantly from the approximate form, Eq. (16), especially in the region of tan $\phi$=1. A plot of the error between the approximate and exact values of $Q^{-1}$ is shown in Figure 10; at tan $\phi$ = 1 the error is significant and has a value of 29%. At lower values the error is less; at tan $\phi$ = .5 the error is significantly reduced to a value of 3.4%. At tan $\phi$ = .1 the error is only .13%. Thus $Q^{-1} \approx \tan\phi$ is accurate for small levels of damping. In fact, the approximation is accurate to within 1% error in the range $0 \leq \tan\phi \leq .28$ but not outside.

We can make a similar analysis of the KV-mass system for $Q^{-1}$. In this case we will use Eq. (12b) and square both sides. By solving $M^2_{KV}(\omega) = M^2_{KV}(\omega_n)/2$ for the half power point frequencies and by substituting the results into Eq. (14) the following inverse quality factor is obtained for the KV-mass system:

$$Q^{-1}_{KV} = \sqrt{1 - 2\zeta^2 + 2\zeta\sqrt{1 + \zeta^2}} - \sqrt{1 - 2\zeta^2 - 2\zeta\sqrt{1 + \zeta^2}} \tag{17}$$

The first term in Eq. (17) is $\omega_2/\omega_n$ and the second term is $\omega_1/\omega_n$. By setting $\omega_1/\omega_n = 0$ it is possible to determine the upper limit on $\zeta$, i.e. the limit for which $Q^{-1}$ remains real valued. Thus Eq. (17) is valid only when $\zeta$ is in the following range: $0 \leq \zeta \leq .353$.

When the damping is low, it is possible to simplify Eq. (17) as was done previously for the inverse quality factor of the CS-mass system. For cases of very low damping we can say that $\zeta \ll 1$ and assume that $\zeta^2$ is negligible with respect to unity. Thus

$$Q_{KV}^{-1} \approx \sqrt{1+2\zeta} - \sqrt{1-2\zeta} \qquad \text{for} \qquad \zeta \ll 1$$

By applying the proper series expansion [13] to the square root quantities above and dropping terms of order $\zeta^2$ or higher we obtain the following simplified approximation:

$$Q_{KV}^{-1} \approx 2\zeta \qquad \text{for} \qquad \zeta \ll 1 \tag{18}$$

The exact and approximate inverse quality factors of the KV-mass system are compared in Figure 11. This figure is similar to that shown earlier for the CS-mass system (Figure 9). As $\zeta$ approaches the upper limit for its use in Eq. (17) we again note that the exact values of the inverse quality factor are significantly higher than those of the approximation. The error between the exact and approximate inverse quality factors is plotted in Figure 12. At the upper limit of $\zeta = .353$ the error is 42%. At lower values of $\zeta$ the error is reduced; at a value of $\zeta = .177$ the error is 6.5% and at $\zeta = .0353$ the error is 0.25%. For accuracy to within 1% error the damping ratio is limited to the range of $0 \leq \zeta \leq .0705$.

Thus the inverse quality factor, $Q^{-1}$, is related to two important measures of damping namely tan $\phi$ and $\zeta$. The way in which $Q^{-1}$ is related to these measures can be either simple and linear or more complicated and nonlinear depending upon the amount of damping. If damping is very small then one may use the simple approximation (given by either Eq. (16) or Eq. (18)) to convert from $Q^{-1}$ to either tan $\phi$ or $\zeta$. However, if damping levels are higher then the exact form of $Q^{-1}$ (given by either Eq. (15) or (17)) is recommended.

**LOG DECREMENT: $\delta$**

The next measure of damping considered is the log decrement which is denoted as $\delta$. This quantity gives a measure of the free decay of oscillations in damped systems possessing mass. In many simple systems with low damping levels the free response is recorded experimentally and the decay envelope is observed to be exponential. Then by taking the natural logarithm of any two successive amplitudes of oscillation a constant is obtained which can be used as a convenient measure of damping; this constant is called the logarithmic decrement. In

order to determine the simplest interrelationship between $\delta$ and material properties we need only to solve the equations which govern the free decay of the KV and CS models with mass. These equations are deducible from our earlier differential equations governing systems with inertia and are as follows [2,7]:

$$\text{CS:} \qquad \ddot{x} + \omega_n^2 (1 + i \tan \phi) x = 0 \tag{19a}$$

$$\text{KV:} \qquad \ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = 0 \tag{19b}$$

Let us briefly discuss Eq. (19a) as it applies to the free decay problem. First, note that the complex modulus is contained in the coefficient multiplying $x$. As we noted earlier, the use of complex modulus is truly valid only for the case where both the response $x$ and the force $F$ are purely harmonic in time. In the present case we are considering the free decay of vibration in systems which, after an initial excitation, are isolated from external forces. Thus because a harmonic input is not present the free response cannot be a simple harmonic unless internal friction is absent. But, since we are including internal friction (in the form of a simple anelastic model) the free response resulting form the solution of Eq. (19a) must have a decaying amplitude and thus does not possess a simple harmonic form. Therefore the use of Eq. (19a) is not strictly correct for the case of free vibrations. However, for cases of adequately small damping the free response closely approximates simple harmonic behavior and for these cases the results involving log decrement can be useful.

The solution of the freely decaying CS-mass system, Eq. (19a), can be found [2] as:

$$x = x_0 e^{i\omega^* t} \tag{20}$$

where $\omega^* = \omega_n \sqrt{1 + i \tan \phi}$ and $x_0$ is an arbitrary initial displacement. The square root of the complex number contained in $\omega^*$ can be simplified by following the rules of complex variable theory [7] and employing appropriate trigonometric relations. The result is as follows:

$$\omega^* = \omega_0 (1 + ib)$$

where
$$\omega_0 = \omega_n \frac{1}{\sqrt{1 - \tan^2 \frac{\phi}{2}}} \quad , \quad 0 \le \phi < \frac{\pi}{2}$$

$$b = \tan \frac{\phi}{2}$$

The frequency $\omega_0$ is the damped natural frequency and $b\omega_0$ is the rate of free decay; both these quantities are directly affected by the magnitude of the damping. Plugging $\omega^*$ back into Eq. (20) produces the following result:

$$x = x_0 \, e^{-b\omega_0 t} \, e^{i\omega_0 t} \tag{21}$$

As $b$ (or $\phi$) decreases to very small numbers the value of the damped natural frequency $\omega_0$ approaches that of its undamped counterpart $\omega_n$. Thus for adequately small levels of damping the response given by Eq. (21) approaches that of simple harmonic behavior.

By computing the ratio of the response at time t to the response one free decay cycle later, and then taking the natural logarithm of this ratio we are making use of the definition of the log decrement.

$$\delta = \ln \frac{x(t)}{x\left(t + \frac{2\pi}{\omega_0}\right)} = 2\pi b \qquad \text{for} \qquad 0 \le \phi < \frac{\pi}{2} \tag{22}$$

Using the definition of $b$, which contains the loss angle, we obtain the following result:

$$\delta = 2\pi \tan \frac{\phi}{2} \tag{23}$$

Thus for this simple model of free decay the log decrement is conveniently related to the loss angle $\phi$.

If the damping is very small such that $\tan(\phi/2) \approx \phi/2$ then the following approximation may be formed via Eq. (23):

$$\delta \approx \pi\phi \qquad \text{for} \qquad \frac{\phi}{2} \ll 1 \tag{24}$$

Many researchers prefer to work with $\tan\phi$ rather than $\phi$. Therefore if the loss angle is also small with respect to unity such that $\tan\phi \approx \phi$ then the following approximation can be used:

$$\delta \approx \pi \tan\phi \qquad \text{for} \qquad \phi \ll 1 \tag{25}$$

Both approximations given by Eqs. (24) and (25) have been reported [2] and are widely used.

Recall that in our earlier discussion we noted that the use of Eq. (19a) was not strictly correct because the complex modulus is contained in the coefficient of $x$ and the subsequent free response was not a simple harmonic. Indeed by inspection of Eq. (21) it is seen that as the damping increases (i.e. as $b$ increases) the rate of free decay will also increase and thus the response will further depart from simple harmonic behavior. For $\omega_0$ to be within 1% of the natural frequency, the damping must be in the range of $0 \leq \tan\phi \leq .286$ (or $0 \leq \delta \leq .281\pi$). At the upper limit of $\tan\phi = .286$ the error incurred by using Eq. (24) is .6% while the error incurred by using Eq. (25) is 2%. With respect to Eq. (23), Eqs. (24) and (25) become less accurate as damping increases. Figure 13 shows that Eq. (24) actually underestimates the exact value of $\delta$ and Eq. (25) overestimates it. The error between the estimated and exact values of $\delta$ increase significantly as $\tan\phi$ approaches unity. This is shown in Figure 14. One should keep in mind, however, that the accuracy of the CS model in governing free decay decreases as damping increases due to the assumption simple harmonic motion. Ultimately it is up to the judgement of the engineer conducting free decay analyses as to whether or not the CS model is useful for their purposes, and for what levels of damping it may apply.

According to Nowick and Berry [2], when the damping problem is to be solved exactly one must utilize a specific differential equation to relate the stress and strain in the material of

interest. When this is done the relationship between $\delta$ and $\phi$ will not be as direct as it is in Eq. (23).

To illustrate this point we can use the free decay of the KV-mass system which is found by solving Eq. (19b) for the initial conditions of $x(0) = x_0$ and $\dot{x}(0) = v_0$. The result is obtained in a straightforward manner [6,7] and is as follows:

$$x = X e^{-\zeta\omega_n t} \sin\left(\sqrt{1-\zeta^2}\,\omega_n t + \xi\right)$$

where $$X = \frac{x_0}{\sin \xi}$$

$$\tan \xi = \frac{v_0 + \zeta\omega_n x_0}{\sqrt{1-\zeta^2}\,\omega_n x_0}$$

Taking the log decrement according to its definition produces the following result:

$$\delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \quad , \quad 0 \le \zeta < 1 \tag{26}$$

An approximate form can be used when $\zeta$ is small, i.e.

$$\delta \approx 2\pi\zeta \quad , \quad \zeta \ll 1 \tag{27}$$

and this approximation is accurate to within 1% of Eq. (26) in the range of $0 \le \zeta \le .141$ (or $0 \le \delta \le .886$).

Eqs. (26) and (27) are simple and useful relationships. However recall that the definition of $\zeta$ (given following Eq. (9)) involves not only the damping coefficient $c$ but also the stiffness $k$ and mass $m$. Thus $\delta$ for the KV-mass system is tied not only to the damping and stiffness of the material but also to the mass. This result is in contrast to the previous case in that $\delta$ of the CS-mass system was directly related to $\phi$, which is determined only by the damping and stiffness of the material.

A similar point is made by Nowick and Berry [2] in regard to the "standard anelastic solid" (i.e. a spring arranged in series with a KV element). The "standard anelastic solid" is a model in which both the response amplitude and the loss angle $\phi$ are frequency dependent and the dependence is such that the $\phi$ vs. $\omega$ curve displays a peak. By using this model to represent system behavior an important inference can be made between the log decrement and the loss angle. The important result, originally given by Zener [14], states that the peak value $\phi_m$ of the loss angle is related to the peak value $\delta_m$ of the log decrement as follows:

$$\tan \phi_m \approx \phi_m \approx \frac{\delta_m}{\pi}\left[1 - \frac{\delta_m}{2\pi} + \cdots\right]$$

Apparently, even in the case of low damping materials the approximations given by Eqs. (24) and (25) are only first corrections if the material of interest simulates the behavior of the standard elastic solid.

To further elaborate on this point let us compare the first and second order corrections of $\delta_m$ for the special value of $\tan \phi_m = 0.1$

1st order $\qquad \delta_m \approx .1\pi$

2nd order $\qquad \delta_m\left[1 - \dfrac{\delta_m}{2\pi}\right] \approx .1\pi \qquad \Rightarrow \qquad \delta_m \approx .106\pi$

Assuming that the second order correction is more accurate than the first the error incurred by using the first order approximation is 5.7%, which is an appreciable amount. As pointed out by Nowick and Berry [2], this result is significant in that an observed exponential decay of oscillations does not automatically guarantee the validity of Eqs. (23)-(25).

It should be noted that the exponential decay of oscillations in damped systems is another direct consequence of the linearity built into the models used to mimic material or system behavior. Furthermore, if the amplitude envelope of an observed set of decaying oscillations is not exponential in time, then the material or system being considered is not anelastic and the

damping is a function of strain amplitude [2]. Such materials and systems are termed "nonlinear" because they cannot be represented by any of the linear models of anelasticity.

## SUMMARY

In this paper, the exact relationships which allow for conversion from one damping measure to another are derived and presented. All theoretical work was based on two simple models of linear anelasticity: the model of Kelvin and Voigt and the complex modulus model. These relationships show that when the storage energy term of the specific damping capacity is suitably defined, the relationship between it and other measures of damping is significantly simplified. Also, the approximate conversion between the inverse quality factor and tan $\phi$ is shown to underestimate the correct value by approximately 30% or less. A similar error was produced by using approximate conversions involving the logarithmic decrement.

A summary of results is presented in Tables 1-3. Table 1 gives the basic definitions of the various damping measures; Table 2 gives the exact relationships between the various damping measures and the loss angle $\phi$, and finally, Table 3 presents the widely used conversions between damping constants along with an associated set of applicability ranges.

**Table 1. Damping definitions.**

| DAMPING DEFINITIONS | REMARKS |
|---|---|
| **Tangent of the Phase Lag**<br><br>$\tan \phi = \frac{k_2}{k_1}$ Complex Spring Eq. (6a)<br><br>$\tan \phi = \frac{c\omega}{k} = 2\zeta \frac{\omega}{\omega_n}$ Kelvin-Voigt Model, Eq. (6b) | $0 \leq \phi < \frac{\pi}{2}$<br>$k_1$: elastic (or storage) modulus<br>$k_2$: loss modulus<br><br>$c$: damping coefficient<br>$k$: elastic stiffness<br>$\omega$: frequency of harmonic input<br>$\zeta$: damping ratio<br>$\omega_n$: natural frequency |
| **Specific Damping Capacity**<br><br>$\psi = \frac{\Delta W}{W}$ Eq. (10) | $\Delta W$: the energy dissipated per cycle of harmonic steady state oscillation.<br><br>$W$: a quantity representing energy storage during the oscillation (e.g. strain energy, elastic potential energy, kinetic energy, etc.) |
| **Loss factor**<br><br>$\eta = \frac{\Delta W}{2\pi W}$ Eq. (11) | $\Delta W$, $W$: are defined the same as above. |
| **Inverse Quality Factor**<br><br>$Q^{-1} = \frac{\omega_2 - \omega_1}{\omega_n}$ Eq. (14) | $\omega_2 - \omega_1$: the half-power bandwith of the frequency response peak.<br><br>$\omega_n$: the natural frequency. |
| **Logarithmic De ment**<br><br>$\delta = \ln \frac{X_n}{X_{n+1}}$ Eq. (22) | $X_n$: The amplitude of a free decaying response after $n$ cycles. |

**Table 2. Exact conversions.**

| EXACT CONVERSIONS | | REMARKS |
|---|---|---|
| $\tan\phi = \frac{k_2}{k_1}$<br>$\tan\phi = \frac{c\omega}{k} = 2\zeta\frac{\omega}{\omega_n}$ | CS<br>KV | $0 \le \phi < \frac{\pi}{2}$ |
| $\psi = 2\pi\tan\phi$ | KV & CS | $0 \le \phi < \frac{\pi}{2}$<br><u>Relationship holds when</u><br>$W = \frac{1}{2}kX_0^2$<br>where $X_0$ is the response amplitude of the oscillating steady state system either with or without inertia.<br><u>or when</u><br>$W = \left[\frac{1}{2}kx^2 + \frac{1}{2}m\dot{x}^2\right]\Big\vert_{\omega=\omega_n}$<br>where $x$ and $\dot{x}$ are the instantaneous position and velocity of the mass. |
| $\eta = \frac{\psi}{2\pi} = \tan\phi$ | KV & CS | $0 \le \phi < \frac{\pi}{2}$<br>Relationship holds for $W$ defined as above |
| $Q_{CS}^{-1} = \sqrt{1+\tan\phi} - \sqrt{1-\tan\phi}$<br>$Q_{KV}^{-1} = \sqrt{1-2\zeta^2+2\zeta\sqrt{1+\zeta^2}} - \sqrt{1-2\zeta^2-2\zeta\sqrt{1+\zeta^2}}$ | | $0 \le \tan\phi \le 1$ or $0 \le Q^{-1} \le \sqrt{2}$<br>$0 \le \zeta \le .3535$ or $0 \le Q^{-1} \le \sqrt{1.5}$ |
| $\delta = 2\pi\tan\frac{\phi}{2}$<br>$\delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$ | CS<br>KV | $0 \le \phi \le \frac{\pi}{2}$ or $0 \le \delta \le 2\pi$<br>$0 \le \zeta \le 1$ or $0 \le \delta \le \infty$ |

**Table 3. Approximate conversions.**

| DAMPING APPROXIMATION | | RANGE SO ERROR OF APPROX. ≤ 1% | | |
|---|---|---|---|---|
| $Q^{-1} \approx \tan\phi$ | CS | $0 \leq \tan\phi \leq .28$ | | |
| $Q^{-1} \approx 2\zeta$ | KV | $0 \leq \zeta \leq .070$ | or | $0 \leq Q^{-1} \leq .14$ |
| $\delta \approx \pi\phi$ | CS | $0 \leq \phi \leq .30$ | or | $0 \leq \delta \leq .94$ |
| $\delta \approx \pi \tan\phi$ | CS | $0 \leq \tan\phi \leq .19$ | or | $0 \leq \delta \leq .60$ |
| $\delta \approx 2\pi\zeta$ | KV | $0 \leq \zeta \leq .14$ | or | $0 \leq \delta \leq .88$ |

| GENERAL APPROXIMATIONS VALID TO WITHIN 1% OF EXACT VALUE: | |
|---|---|
| CS: $\tan\phi = \eta = \frac{\psi}{2\pi} \approx Q^{-1} \approx \frac{\delta}{\pi}$ | $0 \leq \tan\phi \leq .19$ |
| KV: $\zeta = \frac{\eta}{2\frac{\omega}{\omega_n}} = \frac{\psi}{4\pi\frac{\omega}{\omega_n}} \approx \frac{Q^{-1}}{2} \approx \frac{\delta}{2\pi}$ | $0 \leq \zeta \leq .070$ |

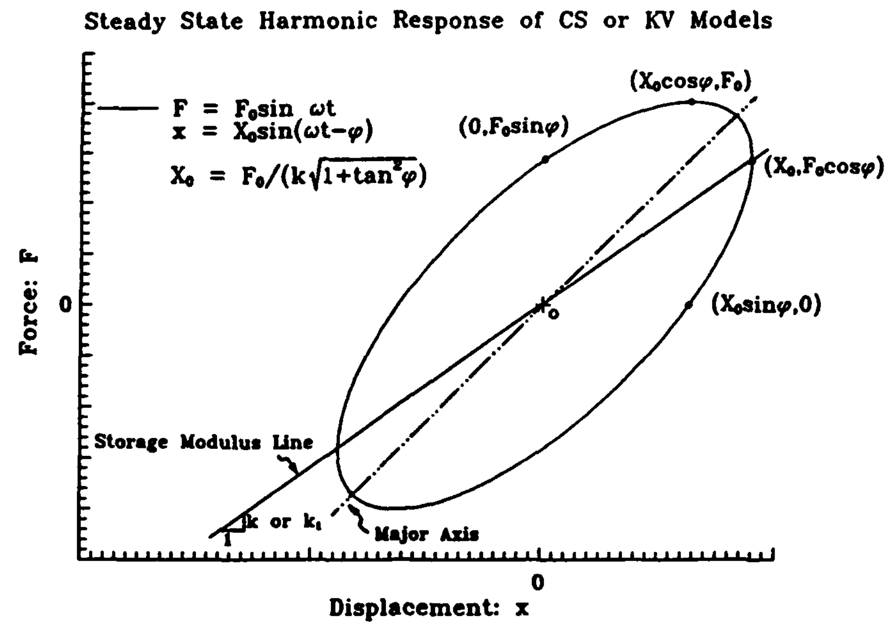



**Fig. 1.** Characteristics of linear anelastic steady state hysteresis.

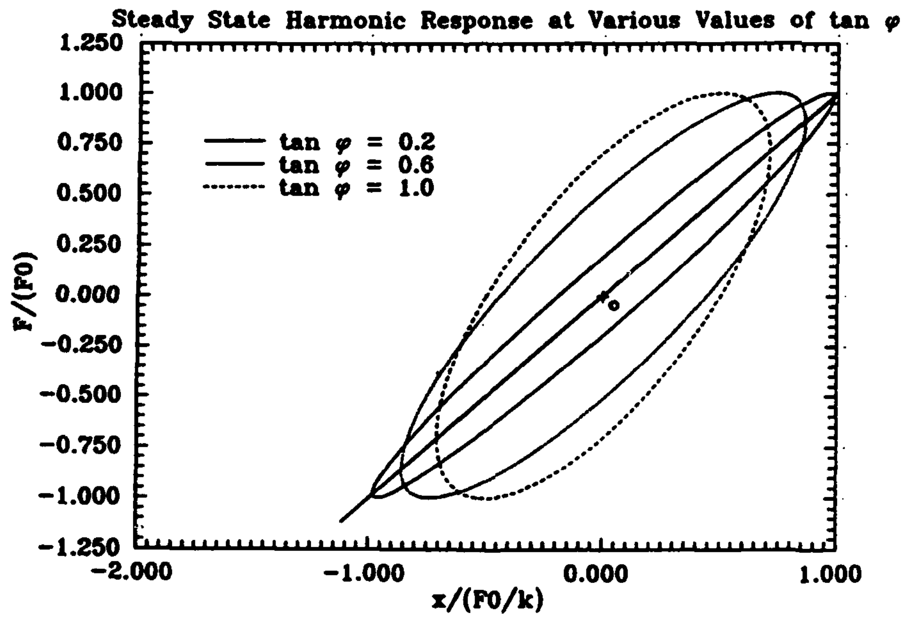



**Fig. 2.** Elliptical hysteresis for varying levels of damping.

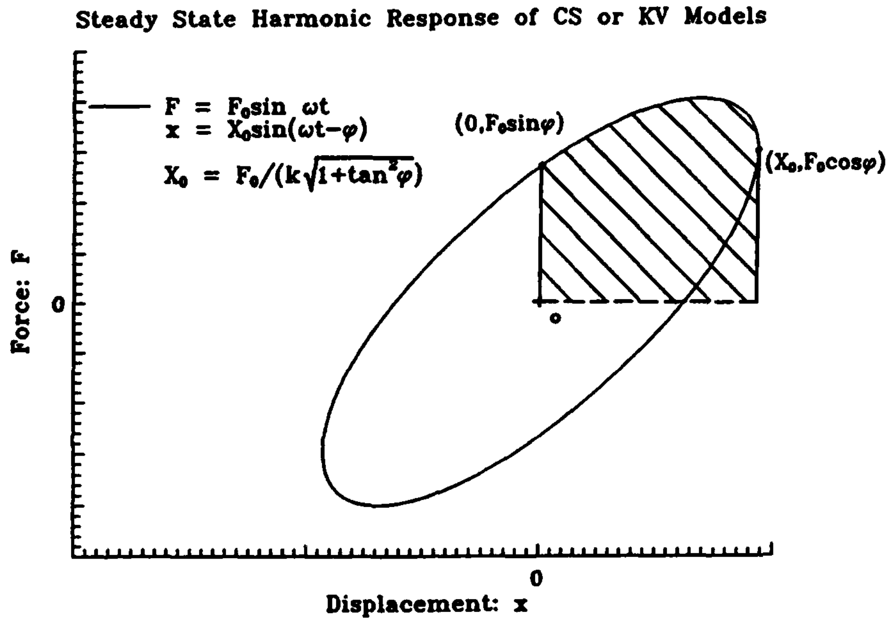



**Fig. 3.** Strain energy developed when loading from zero displacement (or strain) to maximum displacement.

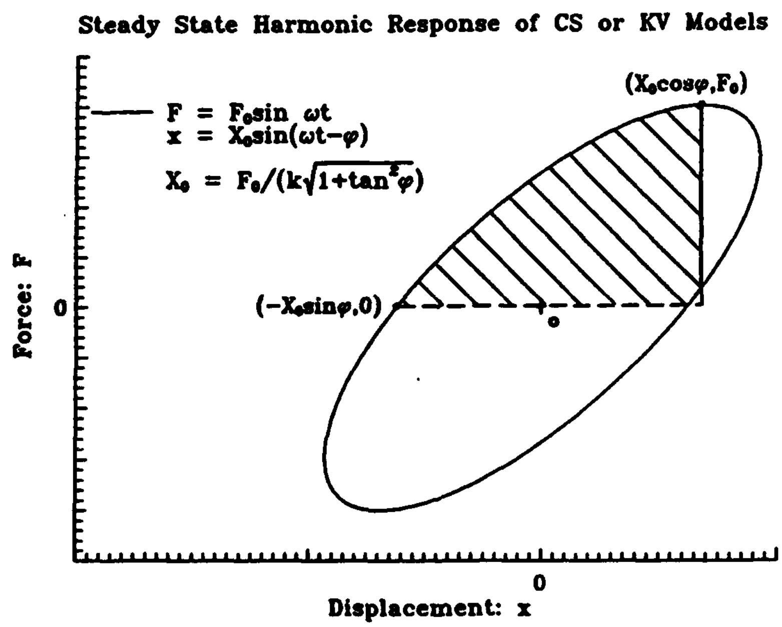



**Fig. 4.** Strain energy developed when loading from zero force (or stress) to maximum force.

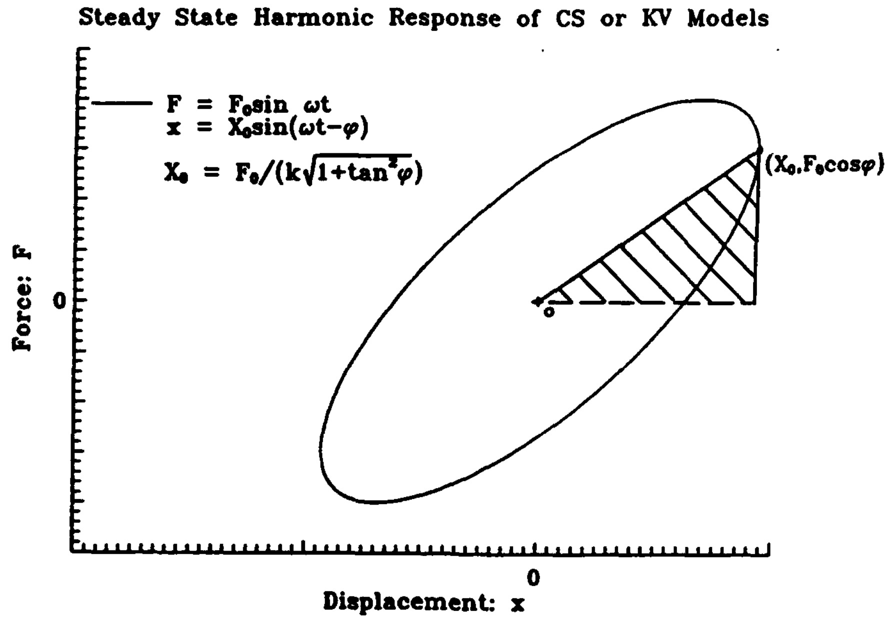



**Fig. 5.** Energy stored in the elastic component of KV or CS models (= .5 $kX_0^2$).

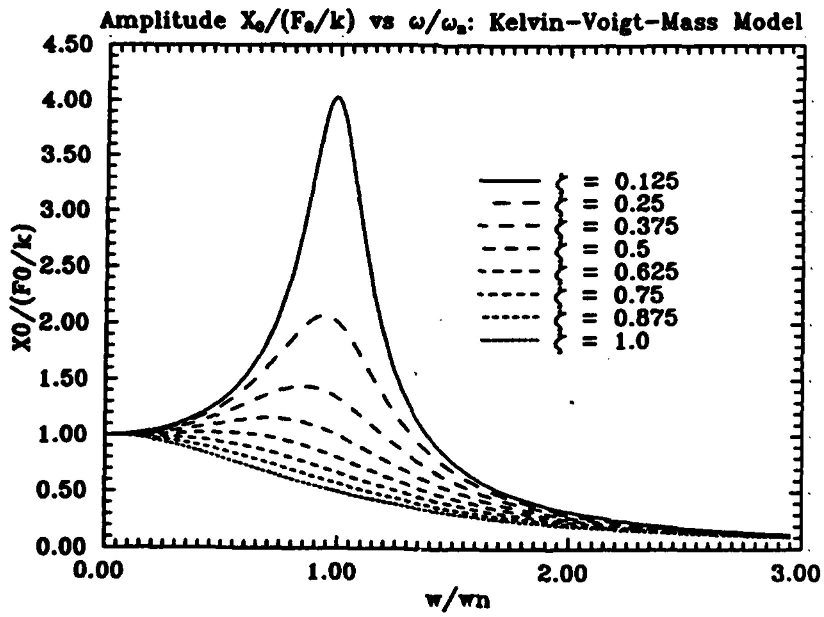



**Fig. 6.** Plot of magnification ratio of KV model for various levels of ζ.

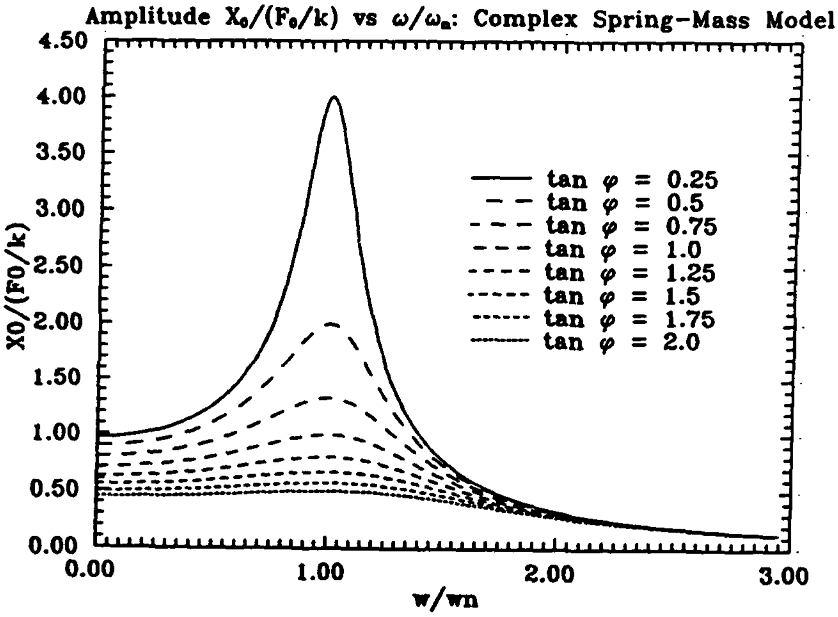



**Fig. 7.** Plot of magnification ratio of CS model for various levels of tan $\phi$.

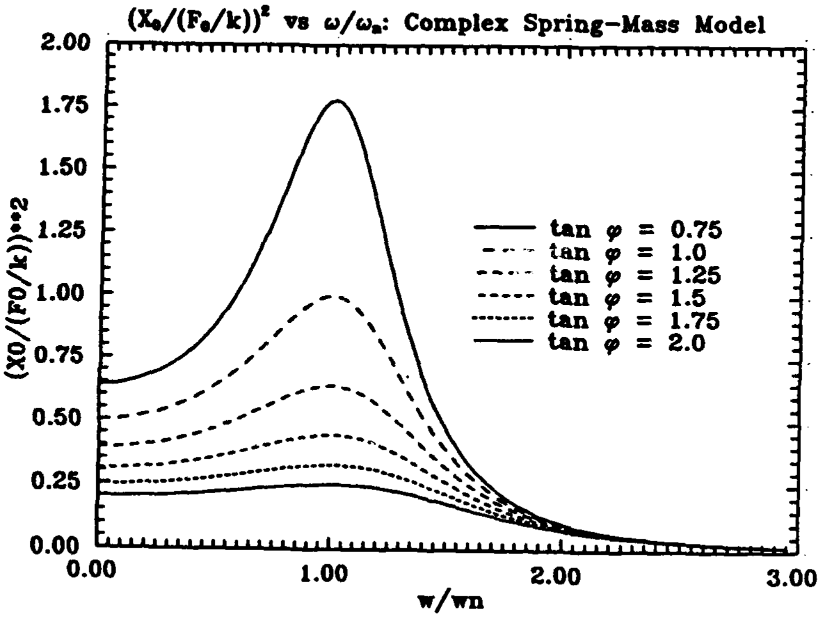



**Fig. 8.** Square of magnification ratio of CS model.

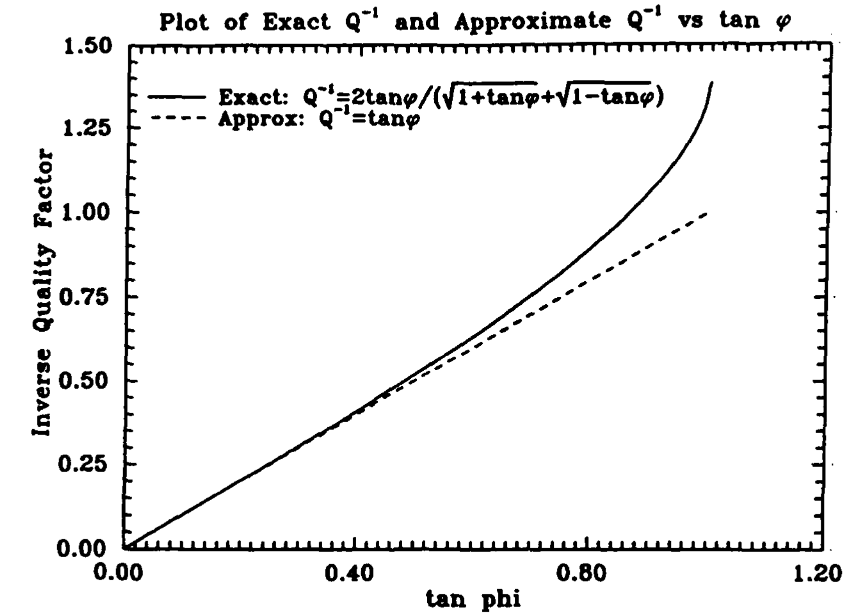



**Fig. 9.** Exact and approximate curves of $Q^{-1}$ for the CS model.

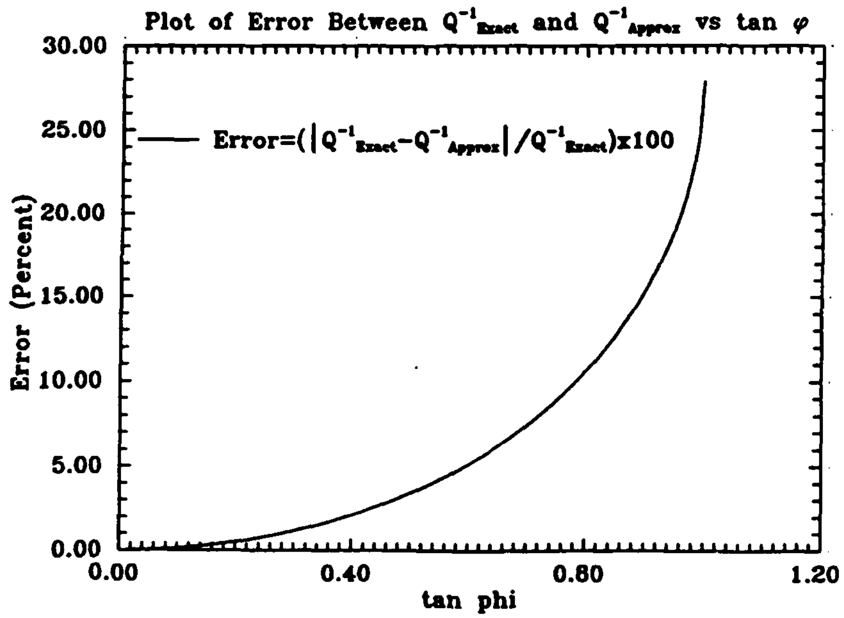



**Fig. 10.** Plot of error incurred by using approximate $Q^{-1}$ for the CS model.

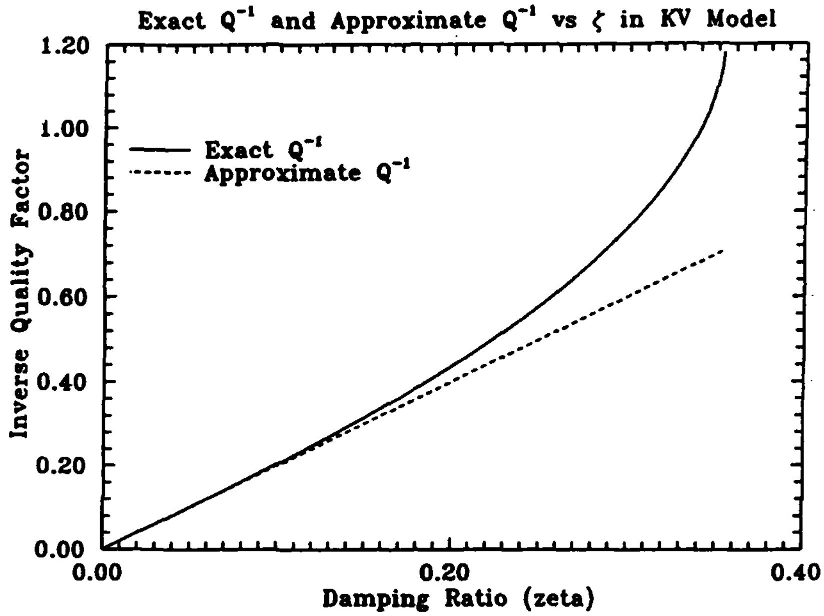



**Fig. 11.** Exact and approximate curves of $Q^{-1}$ for the KV model.

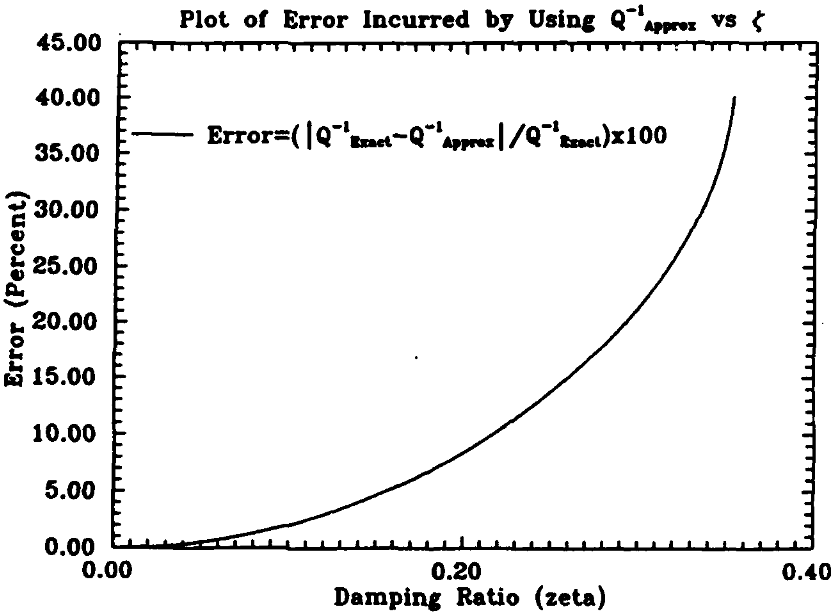



**Fig. 12.** Plot of error incurred by using approximate $Q^{-1}$ for the KV model.

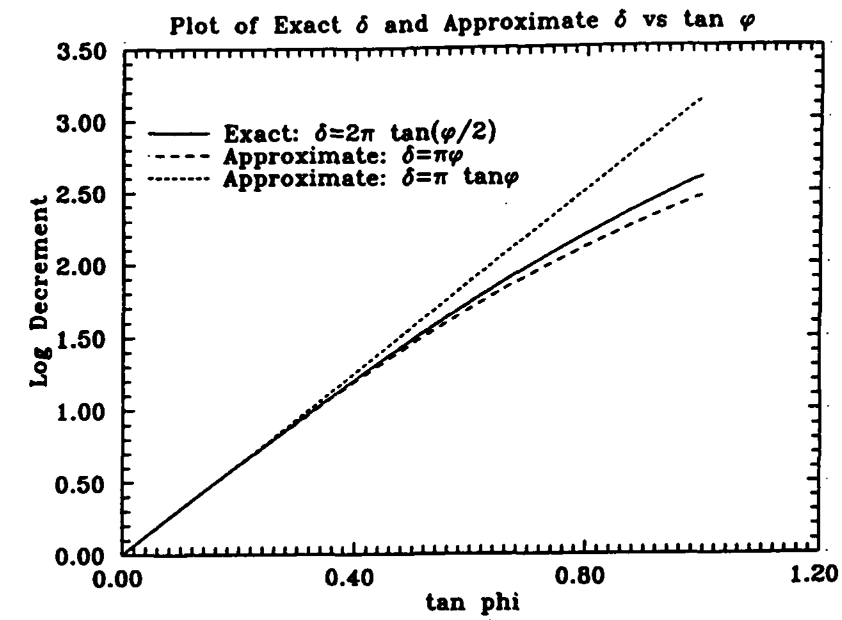



**Fig. 13.** Exact and approximate curves of log decrement for the CS model.

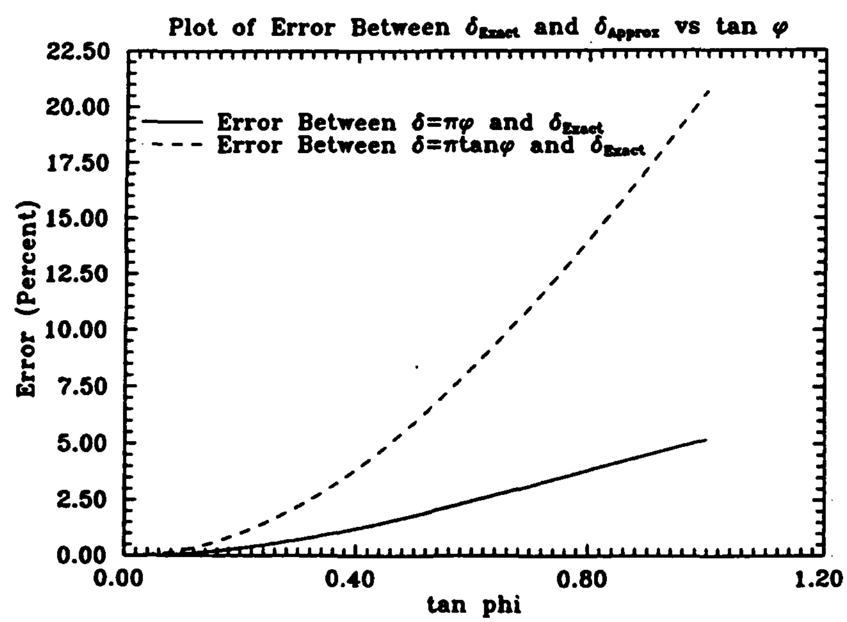



**Fig. 14.** Plot of error incurred by using approximate equation for δ in the CS model.

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13. ABSTRACT *(Maximum 200 words)*

In the field of material damping a number of measures are used to express the level of damping which a material possesses. Such measures are required when evaluating material and system responses to dynamic loading conditions. The most widely used measures of damping capacity include the tangent of the phase lag, tan $\phi$, damping ratio, $\zeta$, specific damping capacity, $\psi$, loss factor, $\eta$, inverse quality factor, $Q^{-1}$, and log decrement, $\delta$. Each of these damping constants are defined in relation to the method used to measure them. Thus it can sometimes be difficult to compare the damping capacity of one material to another. By their inherent definitions, the measures of damping capacity listed above can be simply interrelated when damping levels are within the range $0 < \tan\phi < 0.14$ (i.e. $Q^{-1} = \eta = \tan\phi = E'/E'' = \delta/\pi = \psi/2\pi = 2\zeta$). However, these widely used interrelationships are actually approximations based on two simple anelastic models and an assumption of low damping. This assumption simplifies otherwise complicated nonlinear conversions. When higher levels of damping are of interest ($0.14 < \tan\phi$) the simple linear relationships given above can produce up to 40% error when converting from one damping measure to another. In this paper the basic measures of damping will be reviewed and the exact formulas which relate the various damping measures will be presented. Both the exact formulas and the approximations will be given in the discussion and in a table contained in the summary section. The nonlinear relationships will be useful in cases involving high damping, especially when damping levels are near $\tan\phi = 1$.

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