The damped harmonic oscillator

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Goals:

- Understand the behaviour of this paradigm exactly solvable physics model that appears in numerous applications.
- Understand the connection between the response to a sinusoidal driving force and intrinsic oscillator properties.
- Understand the connection between the Q factor, width of this response and energy dissipation.

The damped harmonic oscillator

1. A damped harmonic oscillator is displaced by a distance x_0 and released at time $t = 0$. Show that the subsequent motion is described by the differential equation

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + m\gamma\frac{\mathrm{d}x}{\mathrm{d}t} + m\omega_0^2 x = 0\,,$$

or equivalently

$$m\ddot{x} + m\gamma\dot{x} + m\omega_0^2 x = 0\,,$$

with $x = x_0$ and $\dot{x} = 0$ at $t = 0$, explaining the physical meaning of the parameters m, γ and ω_0.

- Find and sketch solutions for (i) overdamping, (ii) critical damping, and (iii) underdamping. (iv) What happens for $\gamma = 0$?
- For a lightly damped oscillator the quality factor, or Q-factor, is defined as

$$Q = \frac{\text{energy stored}}{\text{energy lost per radian of oscillation}}\,.$$

  Show that $Q = \omega_0/\gamma$.

We now add a driving force $F\cos(\omega t)$ to the harmonic oscillator so that its equation becomes

$$m\ddot{x} + m\gamma\dot{x} + m\omega_0^2 x = F\cos(\omega t)\,.$$

- Explain what is meant by the steady state solution of this equation, and calculate the steady state solution for the displacement $x(t)$ and the velocity $\dot{x}(t)$.
- Sketch the amplitude and phase of $x(t)$ and $\dot{x}(t)$ as a function of ω.
- Determine the resonant frequency for both the displacement and the velocity.
- Defining $\Delta\omega$ as the full width at half maximum of the resonance peak calculate $\Delta\omega/\omega_0$ to leading order in γ/ω_0.
- For a lightly damped, driven oscillator near resonance, calculate the energy stored and the power supplied to the system. Hence confirm that $Q = \omega_0/\gamma$ as shown above. How is Q related to the width of the resonance peak?

Solution: The forces on the mass m are $F_s = -kx = -m\omega_0^2 x$ due to the spring and $F_f = -m\gamma\dot{x}$ due to friction γ. The equation follows from Newton's law $m\ddot{x} = F_s + F_f$.

The characteristic polynomial for ansatz $x(t) = \mathrm{e}^{\lambda t}$ is $\lambda^2 + \gamma\lambda + \omega_0^2 = 0$ leading to eigenfrequencies

$$\lambda_{1,2} = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \omega_0^2}\,.$$

[1]These problems are based on problem sets by Prof J. Yeomans.

We get (i) overdamping when $\gamma > 2\omega_0$ and hence solutions do not oscillate, (ii) critical damping for $\gamma = 2\omega_0$ and (iii) underdamping for $\gamma < 2\omega_0$. Different solutions are shown in Fig. 1. The general solution is given by

$$x(t) = \Re\left\{A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t}\right\} .$$

and can be simplified for the different situations (writing $\alpha = \sqrt{|\omega_0^2 - \gamma^2/4|}$) for the three cases

(i) $x(t) = e^{-\gamma t/2}[A\cosh(\alpha t) + B\sinh(\alpha t)]$ or equivalently $x(t) = e^{-\gamma t/2}(Ce^{\alpha t} + De^{-\alpha t})$

(ii) $x(t) = e^{-\gamma t/2}(A + Bt)$

(iii) $x(t) = e^{-\gamma t/2}[A\cos(\alpha t) + B\sin(\alpha t)]$

using the standard procedure for degenerate roots of the characteristic polynomial in (ii).

By matching the initial conditions we find for the different cases

(i) $A = x_0$ and $B = x_0\gamma/(2\alpha)$ or equivalently $C = x_0(\alpha + \gamma/2)/(2\alpha)$ and $D = x_0(\alpha - \gamma/2)/(2\alpha)$

(ii) $A = x_0$ and $B = x_0\gamma/2$

(iii) $A = x_0$ and $B = x_0\gamma/(2\alpha)$

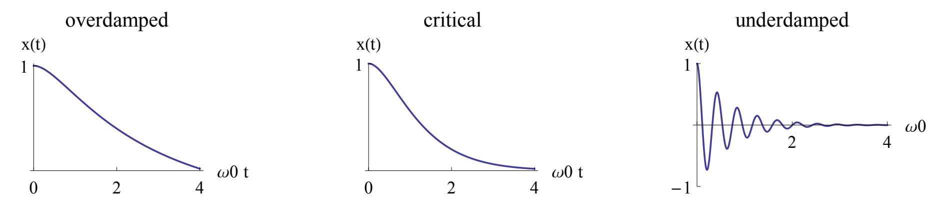

Figure 1: Oscillator displacement for different dampings.

The energy stored in the harmonic oscillator is the sum of kinetic and elastic energy

$$E(t) = \frac{m\dot{x}(t)^2}{2} + \frac{m\omega_0^2 x(t)^2}{2} .$$

In order to proceed for the lightly damped case it is easiest to write $x(t) = A\cos(\alpha t - \phi)e^{-\gamma t/2}$ and thus $\dot{x}(t) = -A\alpha\sin(\alpha t - \phi)e^{-\gamma t/2} - \gamma x(t)/2$. Since lightly damped means $\gamma \ll \omega_0$ we may neglect the second term in $\dot{x}(t)$ and approximate $\alpha \approx \omega_0$. Then the expression for the energy simplifies to

$$E(t) = \frac{m\omega_0^2}{2} A^2 e^{-\gamma t} .$$

A radian corresponds to the time difference $\tau = 1/\omega_0$ and so we find the energy lost per radian

$$E_L = E(0) - E(1/\omega_0) = \frac{m\omega_0^2}{2} A^2(1 - e^{-\gamma/\omega_0}) \approx \frac{m\omega_0\gamma}{2} A^2 .$$

by expanding $e^{-\gamma/\omega_0} \approx 1 - \gamma/\omega_0$ for $\gamma \ll \omega_0$. Hence the result $Q = E(0)/E_L = \omega_0/\gamma$ follows as required.

We now turn to the forced damped harmonic oscillator. The solutions to the homogeneous equation will damp out on a time scale $1/\gamma$. At times $t \gg 1/\gamma$ only terms arising from the particular solution will remain. These terms describe the stationary state[2]. We work out a particular solution using the ansatz $x(t) = \Re\left\{\mathcal{A}(\omega)e^{i\omega t}\right\}$ and find

$$\mathcal{A}(\omega) = \frac{F}{m(\omega_0^2 - \omega^2 + i\gamma\omega)} = |\mathcal{A}(\omega)|e^{-i\varphi} ,$$

where

$$|\mathcal{A}(\omega)| = \frac{F}{m\sqrt{((\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2)}} \quad \text{and} \quad \varphi = \begin{cases} \arctan\left(\frac{\gamma\omega}{\omega_0^2 - \omega^2}\right) & \text{for} \quad \omega \le \omega_0 \\ \pi + \arctan\left(\frac{\gamma\omega}{\omega_0^2 - \omega^2}\right) & \text{for} \quad \omega > \omega \end{cases} .$$

[2]Since we can always move a term from the homogeneous solution to the particular solution it is strictly speaking not accurate to say that the particular solution is the stationary state.

Magnitude $|\mathcal{A}(\omega)|$ and phase φ are shown in Fig. 2 as a function of ω. The velocity is given by $\dot{x}(t) = \Re\left\{i|\mathcal{A}(\omega)|\omega e^{i\omega t}\right\} = -|\mathcal{A}(\omega)|\omega\cos(\omega t - (\varphi + \pi/2))$, i.e. there is an additional shift of $\pi/2$ compared to the displacement. The additional factor of ω shifts the maximum amplitude of $\dot{x}(t)$ compared to that of $x(t)$. Amplitude and phase of $\dot{x}(t)$ are shown in Fig. 2.

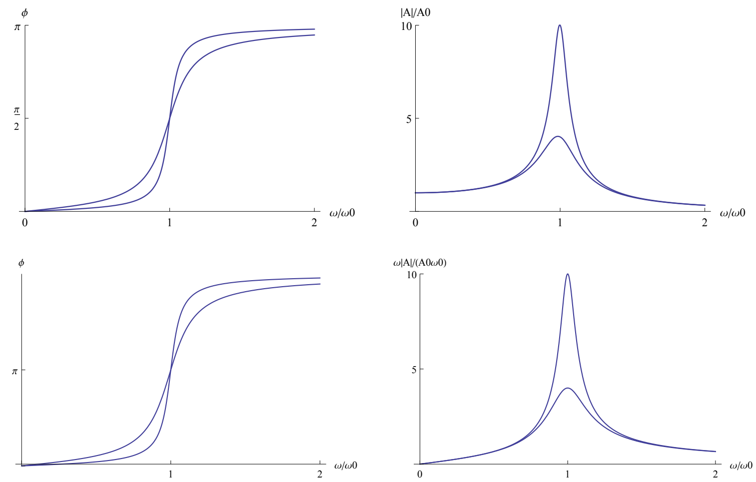

Figure 2: Displacement and velocity response to periodic driving for $\gamma = \omega_0/10$ and $\gamma = \omega_0/4$.

The maximum of the displacement amplitude is found by solving $\mathrm{d}|\mathcal{A}(\omega)|/\mathrm{d}\omega = 0$ giving a resonance frequency $\omega_x^2 = \omega_0^2 - \gamma^2/2$. For the maximum velocity amplitude we solve $\mathrm{d}|\omega\mathcal{A}(\omega)|/\mathrm{d}\omega = 0$ and find the resonance frequency $\omega_{\dot{x}} = \omega_0$.

We write the full width half maximum as $\Delta\omega = \omega_2 - \omega_1$ with $\mathcal{A}(\omega_{i=1,2}) = \mathcal{A}(\omega_x)/2$. We take the square of this expression and find

$$\frac{1}{(\omega_0^2 - \omega_i^2)^2 + \gamma^2\omega_i^2} = \frac{1}{4}\frac{1}{(\omega_0^2 - \omega_x^2)^2 + \gamma^2\omega_x^2}$$

This can be re-written as $\gamma^4 + \gamma^2(\omega_i^2 - 4\omega_0^2) + (\omega_0^2 - \omega_i^2)^2 = 0$. This can in principle be solved for ω_i but since we have assumed the oscillator to be lightly damped and have worked out quantities like Q only to lowest order in γ/ω_0 we instead only look for a solution valid to this order. We thus substitute $\omega_i \approx \omega_0(1 + \beta\gamma)$ obtaining $\gamma^4 + \gamma^2(\beta\gamma\omega_0 - 3\omega_0^2) + \beta^2\gamma^2\omega_0^2 = 0$. We now ignore any terms of $\mathcal{O}(\gamma^3)$ and $\mathcal{O}(\gamma^4)$ and thus get the approximate solution $\beta = \pm\sqrt{3}$ and thus $\omega_0^2 - \omega_i^2 \approx \pm\sqrt{3}\gamma\omega_0$ to lowest order. A Taylor series expansion in γ/ω_0 yields

$$\Delta\omega = \omega_2 - \omega_1 = \omega_0\left(\sqrt{1 + \sqrt{3}\gamma/\omega_0} - \sqrt{1 - \sqrt{3}\gamma/\omega_0}\right) \approx \sqrt{3}\gamma\,.$$

Hence $\Delta\omega/\omega_0 = \sqrt{3}\gamma/\omega_0$.

Near resonance $\omega \approx \omega_0$ and we thus find for the energy of the oscillator

$$E = \frac{m}{2}\dot{x}(t)^2 + \frac{m\omega_0^2}{2}x(t)^2 \approx \frac{F^2}{2m\gamma^2}\,.$$

The average supplied power is given by

$$P = \overline{F\cos(\omega t)\dot{x}(t)} = -F|\mathcal{A}(\omega)|\omega\overline{\cos(\omega t)\sin(\omega t - \varphi)} = -F|\mathcal{A}(\omega)|\omega\overline{\cos(\omega t)(\sin(\omega t)\cos(\varphi) - \cos(\omega t)\sin(\varphi))}\,.$$

Near resonance we have $\varphi \approx \pi/2$ and $\omega \approx \omega_0$ so that

$$P = \frac{F|\mathcal{A}(\omega_0)|\omega_0}{2} = \frac{F^2}{2m\gamma}\,.$$

In the steady state the energy dissipated per radian must be equal to the energy supplied by the external force per radian $E_L = P\tau = P/\omega_0$. Thus

$$Q = \frac{E}{E_L} = \frac{\omega_0}{\gamma} \quad \text{and} \quad \frac{\Delta\omega}{\omega_0} = \frac{\sqrt{3}}{Q}\,.$$