Energy dissipation

The rate of energy dissipation during vibration is another measure of the level of damping present. If there is force excitation $x(t)$ at a single point and if the velocity of this point in the direction of the force is $v(t)$, then the instantaneous power $\Pi(t)$ which enters the system is

$$\Pi(t) = x(t)v(t). \tag{3.51}$$

For steady-state harmonic excitation such that

$$x(t) = x_0 \mathrm{e}^{\mathrm{i}\omega t} \tag{3.52}$$

where x_0 is real, the corresponding velocity response can be written as

$$v(t) = V(\mathrm{i}\omega)\, x_0 \mathrm{e}^{\mathrm{i}\omega t} \tag{3.53}$$

where $V(\mathrm{i}\omega)$ is the direct mobility of the vibrating system for excitation at the point in question. We can now substitute from (3.52) and (3.53) into (3.51). Before doing so we must take account of the meaning implied by (3.52) and (3.53) that the l.h.s. is equal to the real part (or the imaginary part) of the r.h.s. in each case. We must identify which meaning we want, and in this case we shall choose (arbitrarily) 'the imaginary part'. Then (3.51) becomes

$$\begin{aligned}\Pi(t) &= \mathrm{Im}\{x_0\, \mathrm{e}^{\mathrm{i}\omega t}\}\, \mathrm{Im}\{V(\mathrm{i}\omega)\, x_0\, \mathrm{e}^{\mathrm{i}\omega t}\} \\ &= (x_0 \sin \omega t)(\mathrm{Re}\{V(\mathrm{i}\omega)\}x_0 \sin \omega t + \mathrm{Im}\{V(\mathrm{i}\omega)\}x_0 \cos \omega t) \\ &= x_0^2\, \mathrm{Re}\{V(\mathrm{i}\omega)\} \sin^2 \omega t + x_0^2\, \mathrm{Im}\{V(\mathrm{i}\omega)\} \sin \omega t \cos \omega t.\end{aligned} \tag{3.54}$$

The mean power flow, when averaged over a full cycle, is given by

$$\langle \Pi(t) \rangle = \frac{1}{T}\int_0^T \Pi(t)\, \mathrm{d}t \tag{3.55}$$

where $T = 2\pi/\omega$. On substituting for $\Pi(t)$ from (3.54) and evaluating the integral we find that

$$\langle \Pi(t) \rangle = \tfrac{1}{2}\, x_0^2\, \mathrm{Re}\{V(\mathrm{i}\omega)\} \tag{3.56}$$

since

$$\frac{\omega}{2\pi}\int_0^{2\pi/\omega} \sin^2 \omega t\, \mathrm{d}t = \frac{1}{2} \tag{3.57}$$

and

$$\frac{\omega}{2\pi}\int_0^{2\pi/\omega} \sin\omega t\cos\omega t \, \mathrm{d}t = 0. \tag{3.58}$$

Consider this calculation for a lightly damped linear system when the excitation frequency ω is close to a natural frequency, for example ω_j. Then one mode dominates the response and we assume, from (2.57), that the receptance is given by

$$H(\mathrm{i}\omega) \simeq \frac{\beta_j(\mathrm{i}\omega) + \alpha_j}{(\mathrm{i}\omega)^2 + 2\zeta_j\omega_j(\mathrm{i}\omega) + \omega_j^2}. \tag{3.59}$$

The corresponding mobility is, from (2.9),

$$V(\mathrm{i}\omega) = \mathrm{i}\omega H(\mathrm{i}\omega) \simeq \frac{\beta_j(\mathrm{i}\omega)^2 + \alpha_j(\mathrm{i}\omega)}{(\mathrm{i}\omega)^2 + 2\zeta_j\omega_j(\mathrm{i}\omega) + \omega_j^2}. \tag{3.60}$$

For small damping we have seen that ζ_j and β_j are small and so, near resonance,

$$\underset{\omega=\omega_j}{V(\mathrm{i}\omega)} \simeq \frac{\alpha_j(\mathrm{i}\omega_j)}{2\zeta_j\omega_j(\mathrm{i}\omega_j)} = \frac{\alpha_j}{2\zeta_j\omega_j}. \tag{3.61}$$

Then, on substituting this result into (3.56), the average power flow into the mode is

$$\langle \Pi(t) \rangle \simeq \frac{x_0^2\alpha_j}{4\zeta_j\omega_j} \tag{3.62}$$

where x_0 is the amplitude of the excitation (units of force), α_j is a real coefficient (units of mass^{-1}), ω_j is the natural frequency (units of time^{-1}) and ζ_j is the modal damping ratio (dimensionless). The modal damping ratio for a lightly damped mode of a linear system can be calculated from (3.62) if the average power flow into the system at resonance can be measured.