Sample Questions

1. State and prove the classical Peano existence theorem for

$$y' = f(t,y),\ y(t_0) = y_0,$$

where f is continuous.

2. State and prove the classical Cauchy existence and uniqueness theorem for

$$y' = f(t,y),\ y(t_0) = y_0,$$

where f and $\partial f/\partial y$ are continuous.

3. Verify that the system

$$\begin{aligned} x' &= cos(xy) - x, \\ y' &= -y + x^2 + 1 \end{aligned}$$

has an equilbrium point at

$$(x,y) \doteq (0.632639, 1.4003)$$

and write down the linearization of the system around this equilibrium solution. Analyze the behavior of the linearization and relate it to the behavior of non-linear system near the equilibrium point.

4. Consider the Runge-Kutta numerical scheme

$$\begin{aligned} k_1 &= hf(x_n, y_n), \\ k_2 &= hf(x_n + h/2, y_n + k_1/2), \\ k_3 &= hf(x_n + h/2, y_n + k_2/2), \\ k_4 &= hf(x_n + h, y_n + k_3), \\ y_{n+1} &= y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4). \end{aligned}$$

(a)

Compute the local truncation error.

(b)

Count the number of function evaluations required for a single step of this scheme.

(c)

Make appropriate assumptions on the function f and estimate the global error involved in using this scheme to solve the IVP

$$y' = f(t,y),\ y(0) = y_0$$

on the interval $[0, L]$.

(d)

Perform a stability analysis of the scheme.

5.

Consider the linear multistep method

$$y_{n+1} = y_n + \frac{h}{12}(23y'_n - 16y'_{n-1} + 5y'_{n-2}).$$

(a)

Compute the local truncation error.

(b)

Count the number of function evaluations required for a single step of this scheme.

(c)

Make appropriate assumptions on the function f and estimate the global error involved in using this scheme to solve the IVP

$$y' = f(t,y),\ y(t_0) = y_0$$

on the interval $[0, L]$.

6.

Give an example of a stiff system of ODE. Explain why a general numerical scheme is unlikely to provide an accurate solution to a stiff system.