Ordinary Differential Equations Comprehensive Exam

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1. Consider the initial value problem

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0.$$

(a) State the classical Peano Existence Theorem.

(b) State the classical Cauchy Existence and Uniqueness Theorem.

(c) Prove any one of the two theorems.

2. Find all equilibrium points of the system

$$\frac{dx}{dt} = y - 4x$$
$$\frac{dy}{dt} = x^2 - y^2 + 60.$$

Linearize the system around each equilibrium point, analyze the behavior of the obtained linear system near each equilibrium point and relate it to the behavior of the nonlinear system (near the corresponding point). Show several representative solutions in the phase plane, including any separatrices.

3. State the Lyapunov Stability Theorem. Using the Lyapunov function $V(x, y) = x^2 + y^2$, investigate whether the system

$$\begin{aligned}\frac{dx}{dt} &= -x + y + xy \\ \frac{dy}{dt} &= x - y - x^2 - y^3\end{aligned}$$

has an asymptotically stable equilibrium point at the origin.

4. What is a limit cycle? Show that the system

$$\frac{dx}{dt} = x - y - x^3$$
$$\frac{dy}{dt} = x + y - y^3$$

has a limit cycle.

5. Write down a second order explicit Runge-Kutta method for $y' = f(t, y)$. Write both the Butcher tableau for the method, and the explicit derivation of y_{n+1} from y_n and the stages.

   (a) Define local truncation error.

   (b) Verify that your scheme is second order accurate (globally) and state all required assumptions on f.

   (c) Explain how you could combine your scheme with Euler's method to create an adaptive scheme. Count the function evaluations required for a single step of your adaptive scheme. Describe the step-size control you would implement.

6. Introduce an A-stable two-step second-order accurate multistep method for solving $y' = f(t, y), y(t_0) = y_0$. Show that your scheme is consistent, convergent and A-stable. How would you create a third-order accurate A-stable multi-step method? Explain (in detail) the approach you would use, or explain why this would not be possible.