Ordinary Differential Equations
Comprehensive Exam

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Complete any 5 questions. Start each question on a new page. Label the pages and problems clearly. Indicate clearly which five questions you want to have graded. Calculators are NOT ALLOWED.

1. State and prove the classical Picard iterate based existence and uniqueness theorem for Initial Value Problems. Make sure you clearly define all terms and hypotheses.

2. Consider the ODE system

$$\begin{aligned}\tfrac{dx}{dt} &= -y + x(x^2+y^2)\sin\left(\tfrac{1}{\sqrt{x^2+y^2}}\right)\\ \tfrac{dx}{dt} &= x + y(x^2+y^2)\sin\left(\tfrac{1}{\sqrt{x^2+y^2}}\right)\end{aligned}$$

   for $x^2+y^2 \neq 0$, with $\frac{dx}{dt} = \frac{dy}{dt} = 0$ at $(0,0)$. Find all the limit cycles Γ_n and show that they accumulate at the origin. Discuss the stability of these limit cycles.

3. **a) Use polar coordinates** to determine whether the origin is a center, a stable focus or an unstable focus of the following system:

$$\begin{aligned}\dot{x} &= -y + xy^2\\ \dot{y} &= x + y^3.\end{aligned}$$

b) Determine the nature of the critical points of the following system. Be as specific as possible.

$$\begin{aligned} \dot{x} &= -4y + 2xy - 8 \\ \dot{y} &= 4y^2 - x^2. \end{aligned}$$

4. Consider the following system of ODEs for the unknown functions $x(t), y(t)$:

$$\begin{aligned} \dot{x} + 2x &= xy, \\ 2\dot{y} + y &= x^2. \end{aligned}$$

a) Find all equilibrium points.

b) Find the linearization of this system near the origin.

c) Sketch as accurately as possible the phase portrait for the linearization from part **b**.

5. Consider the following method for solving $y'(t) = f(t, y(t))$:

$$y_{n+1} = y_n + \frac{1}{2}hf(t_n, y_n) + \frac{1}{2}hf(t_{n+1}, y_n + hf(t_n, y_n)).$$

(i) Find the expression for the stability function $R(z) = |\frac{y_{n+1}}{y_n}|$ for this method. Plotting the stability region is not required.

(ii) Is this method A-stable? Justify your answer.

(iii) What is the order of accuracy of this method?

6. Write down a second order Runge-Kutta scheme for $x'(t) = f(x(t))$. Make sure you define all your terms.

   (a) Define local truncation order.

   (b) Verify that your scheme is second order.

   (c) Explain how you could combine your scheme with Euler's method to create an adaptive scheme. Count the function evaluations required for a single step of your adaptive scheme. Describe the stepsize control you would implement.