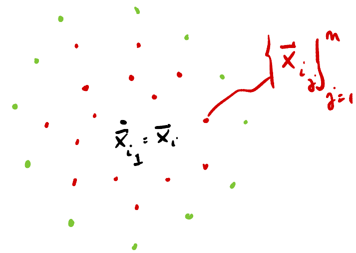
The goal is to analyse the Differentiation Matrix properties



Given a large set of nodes $\{\vec{x}_i\}_{i=1}^{N}$ located in $\mathbb{R}^d$.

1. Infinitely smooth Radial function $\phi(r)$

A local interpolant takes the form $s(\vec{x}) = \sum_{j=1}^{m} \lambda_j \phi(\|\vec{x} - \vec{x}_{i_j}\|)$

where $\vec{x}_{i_1}$ is the central node and $\{x_{i_j}\}_{j=1}^{m}$ are its m neighbours

We can impose the interpolation condition at each neighbouring node.

Let's denote $\{f(\vec{x}_{i_j})\}_{j=1}^{m}$ by $\vec{f}$

Define the collocation matrix $A_{jk} = [\phi(\|\vec{x}_{i_j} - \vec{x}_{i_k}\|)]_{j,k=1}^{m}$, s.t. $A\vec{\lambda} = \vec{f}$ (1)

We wish to find the weights $\{w_j\}_{j=1}^{m}$ s.t. $\sum_{j=1}^{m} w_j f(\vec{x}_j) = L_x g(\vec{x})\Big|_{\vec{x}=\vec{x}_i}$

Now denote $\{w_j\}_{j=1}^{m}$ by $\vec{w}$, and $\{L_x g(\vec{x})|_{\vec{x}=\vec{x}_{i_j}}\}_{j=1}^{m}$ by $\vec{g}$, so, in matrix form, we seek $\vec{w}$ s.t. $\vec{f}^T\vec{w} = g_1$ (2)

We can apply the differential operator L to the local interpolant.

$L_{\vec{x}} s(\vec{x}) = \sum_{j=1}^{m} \lambda_i L_{\vec{x}} \phi(\|\vec{x} - \vec{x}_{i_j}\|)$ and evaluate it at the central point $\vec{x}_i = \vec{x}_{i_1}$.

$$L_x g(\vec{x}_i) \approx L_{\vec{x}} s(\vec{x})\Big|_{\vec{x}=\vec{x}_i} = \sum_{j=1}^{m} \lambda_i L_{\vec{x}} \phi(\|\vec{x} - \vec{x}_{i_j}\|)\Big|_{\vec{x}=\vec{x}_i} = \left(L_{\vec{x}} \phi(\|\vec{x} - \vec{x}_{i_1}\|)\Big|_{\vec{x}=\vec{x}_i} \quad L_{\vec{x}} \phi(\|\vec{x} - \vec{x}_{i_2}\|)\Big|_{\vec{x}=\vec{x}_i} \quad \cdots \quad L_{\vec{x}} \phi(\|\vec{x} - \vec{x}_{i_m}\|)\Big|_{\vec{x}=\vec{x}_i} \right) \vec{\lambda}$$

$$= B_i \vec{\lambda}$$

From (1) (since A is invertible) $g_1 = B_i A^{-1} \vec{f}$

from (2), $\vec{w} = (B_i A^{-1})^T = (A^{-1})^T B_i^T = (A^T)^{-1} B_i^T$

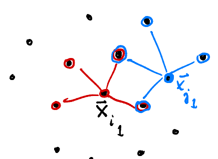
So $\boxed{A^T \vec{w} = B_i^T}$

Note: A is a local collocation matrix, associated to its cluster center $\vec{x}_i$

B_i is L applied to the local interpolant and evaluated at $\vec{x}_i$.

Global Representation:

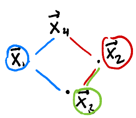
Consider a graph of each center with its m neighbours



A set of adjacency matrix can be built.

For instance:

$$E = \begin{pmatrix} 1 & 0 & 1 & 1 & \cdots \\ 0 & 1 & 1 & 1 & \cdots \\ 0 & 1 & 1 & 0 & \cdots \\ & & \cdots & & \end{pmatrix}$$



Properties:
(1) E is $N \times N$ (N is the total number of nodes)
(2) $\sum_{i=1}^{N} E_{ji} = m, \forall j$ (m is fixed: the number of neighbours per cluster)
(3) E is not necessarily symmetric
(4) $E_{ii} = 1, \forall i$

Let A be the global collocation matrix, with all the nodes. Similarly, let B be the global differentiated interpolant evaluated at each node.

Let A_i be the local collocation matrix associated to the cluster of which the central node is $\vec{x}_i$. Similarly, B_i is the local interpolant differentiated and evaluated at the central node (recall it is a row vector)

The neighbours of $\vec{x}_i$ are $\{\vec{x}_{i_1}, \vec{x}_{i_2}, \dots, \vec{x}_{i_m}\}$

The indices of the neighbours are the positions of the non-zero entries of $E_{i,:}$.

Let P_i be the $m\times N$ reduction matrix, so $\boxed{\underset{m\times m}{A_i} = P_i \underset{N\times N}{A} P_i^T}$

For instance, $E_{i,:} = [0\ 1\ 0\ 0\ 1\ 1]$, $P_i = \underset{3\times 6}{\begin{bmatrix} 0&1&0&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{bmatrix}}$

$N=6$, $m=3$

$$A_i = P_i A P_i^T = \begin{bmatrix} 0&1&0&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{bmatrix} \begin{bmatrix} \phi_{11} & \phi_{12} & \phi_{13} & \cdots & \phi_{16}\\ \phi_{21} & \phi_{22} & & & \vert\\ \vdots & & & & \vert\\ \phi_{61} & - & - & - & \phi_{66} \end{bmatrix} \begin{bmatrix} 0&0&0\\ 1&0&0\\ 0&0&0\\ 0&0&0\\ 0&1&0\\ 0&0&1 \end{bmatrix} = \begin{bmatrix} \phi_{22} & \phi_{25} & \phi_{26}\\ \phi_{52} & \phi_{55} & \phi_{56}\\ \phi_{62} & \phi_{65} & \phi_{66} \end{bmatrix}$$

$$\text{Similarly,}\quad B_i = P_i(\overset{\text{central node index}}{c},:)\, B\, P_i^T = [0\ 0\ 0\ 0\ 1\ 0] \begin{bmatrix} L\phi_{11} & L\phi_{12} & \cdots\\ \vdots & \ddots & \vdots\\ L\phi_{61} & \cdots & L\phi_{66} \end{bmatrix} \begin{bmatrix} 0&0&0\\ 1&0&0\\ 0&0&0\\ 0&0&0\\ 0&1&0\\ 0&0&1 \end{bmatrix} = [L\phi_{52}\ L\phi_{55}\ L\phi_{56}]$$

So the local cluster differentiation weights satisfy

$$A_i^T \vec{w}_i = B_i^T$$

Here, A_i is symmetric, so $A_i \vec{w}_i = B_i^T$

$$P_i A P_i^T \vec{w}_i = P_i B^T P_i(c,:)^T$$

$$\vec{w}_i = (P_i A P_i^T)^{-1}(P_i B^T P_i^T(:,c))$$

Building the Differentiation Matrix:

$\vec{w}_i$ contains the entries of the i^{th} row, along the non-zero columns of $E_{i,:}$

So $\underset{N\times 1}{D_i} = P_i^T(:,c)\ \underset{1\times m}{\vec{w}_i^{\,T}}\ \underset{m\times N}{P_i}$

$$\vec{w}_i = (P_i A P_i^T)^{-1}(P_i B^T P_i^T(:,c))$$

So $$D = \sum_{i=1}^{N} P_i^T(:,c)\, \vec{w}_i^{\,T} P_i$$

$$= \sum_{i=1}^{N} P_i^T(:,c) \underbrace{\left(\overbrace{P_i(c,:) B P_i^T}^{i^{th}\text{ row of }B}\right)\left(P_i A^T P_i^T\right)^{-1}}_{\vec{w}_i^{\,T}} P_i$$

zero matrix except 1 at (i,i)

Note: For the symmetric version of D:

$$\underset{m\times m}{W_i} = (P_i A P_i^T)^{-1}(\underset{m\times n}{P_i} B \underset{n\times m}{P_i^T})$$

$$D = \sum_{i=1}^{N} P_i^T (P_i A P_i^T)^{-1}(P_i B P_i^T) P_i$$