Name: ______________

Show an appropriate amount of work.

1. Compute magnitude and argument for all possible values of

$z = \left(\frac{i}{\sqrt{1+i}}\right)^3$

abs(z_1) = $\boxed{2^{-3/4}}$ abs(z_2) = $\boxed{9\pi/8}$

arg(z_1) = $\boxed{2^{-3/4}}$ arg(z_2) = $\boxed{-15\pi/8}$

2. The roots of $z^2 + 2\,i\,z + 5 = 0$ are

$z_1 = \boxed{0} + i\,\boxed{-1+\sqrt{6}}$

$z_2 = \boxed{0} + i\,\boxed{-1-\sqrt{6}}$

3. $i^{-i} = \boxed{E^{\pi/2}} + i\,\boxed{0}$

$= \boxed{E^{\pi/2}}\, e^{\boxed{I\,0}}$

work on A)

4. Sketch Points

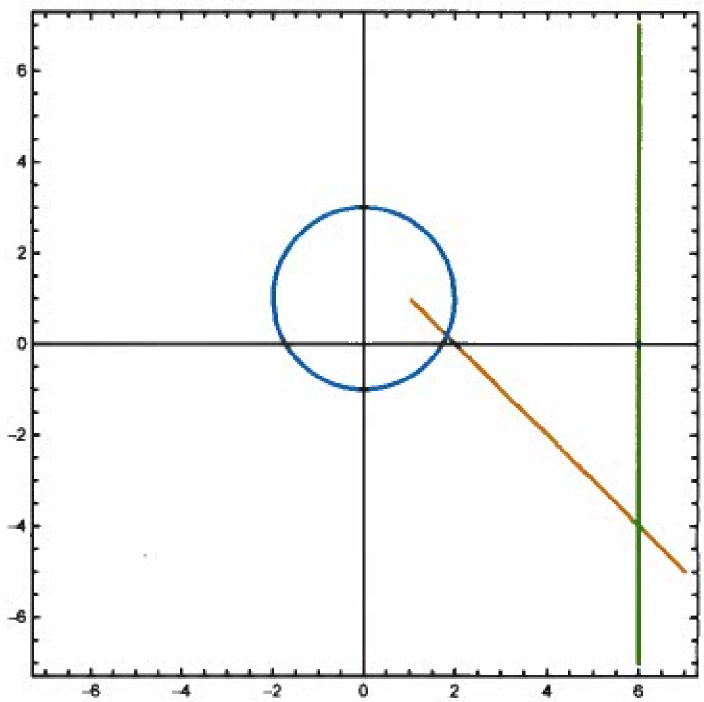
4.1. Sketch the set of points C_1 that satisfy $| z - i | < 2$. Label the set C_1

4.2. Sketch the set of points C_2 that satisfy $\mathrm{re}(z - 2 + i) = 4$. Label the set C_2

4.3. Sketch the set of points C_3 that satisfy $\arg(z - i - 1) = -\pi/4$. Label the set C_3

In[12]:=
```
ComplexContourPlot[{
  Abs[z - i] == 2,
  Arg[z - i - 1] == -π / 4,
  Re[z - 2 + i] == 4
 }, {z, 7},
 Axes → True]
```

Out[12]=



1) $\left(\frac{I}{\sqrt{1+I}}\right)^3 = \left(\frac{E^{I\pi/2}}{\sqrt{\sqrt{2}\, E^{I\pi/4}}}\right)^3$ A)

$$= \left(\frac{E^{I\pi/2}}{\sqrt{\sqrt{2}}\, E^{I\pi/8}}\right)^3 \text{ or } \left(\frac{E^{I\pi/2}}{\sqrt{\sqrt{2}}\, E^{I(\pi/8+\pi)}}\right)^3$$

$$= 2^{-3/4}\left(E^{I(\pi/2-\pi/8)}\right)^3 \text{ or } 2^{-3/4}\left(E^{I(\pi/2-\pi/8-\pi)}\right)^3$$

$$= 2^{-3/4} E^{I\, 9\pi/8} \text{ or } 2^{-3/4} E^{-I\, 15\pi/8}$$

---

2) $(z+I)^2 + 1 + 5 = 0$

$$(z+I)^2 = -6$$

$$z + I = \pm\sqrt{6}\, I$$

$$z = (-1 \pm \sqrt{6})\, I$$

---

3) $I^{-I} = \left(E^{I\pi/2}\right)^{-I} = E^{-I^2\pi/2} = E^{\pi/2} + 0I$

5. For $f(z) = u(x, y) + i\, v(x, y)$ and write down the CR equations

$\boxed{u_x = v_y}$ and $\boxed{u_y = -v_x}$

6. For $2x + i x^3 + 2 i y - 3x^2 y - 3 i x y^2 + y^3$ complete the following

$u(x, y) = \boxed{2x - 3x^2 y + y^3}$ and $v(x, y) = \boxed{x^3 + 2y - xy^2}$

$\partial_x u = \boxed{2 - 6xy}$ and $\partial_x v = \boxed{3x^2 - 2xy}$

$\partial_y u = \boxed{}$ and $\partial_y v = \boxed{2 - 6xy}$

$2xy - 3x^2$

Analytic Yes or No $\boxed{\text{yes}}$.

If yes $f(z) = \boxed{I z^3 + 2z}$

7. Is $\sin(x - i\, y)$ analytic? Yes or No $\boxed{\text{NO}}$. Show any needed work.

8. Label poles with $P_1, P_2, \ldots$ Label zeros with $Z_1, Z_2, \ldots$

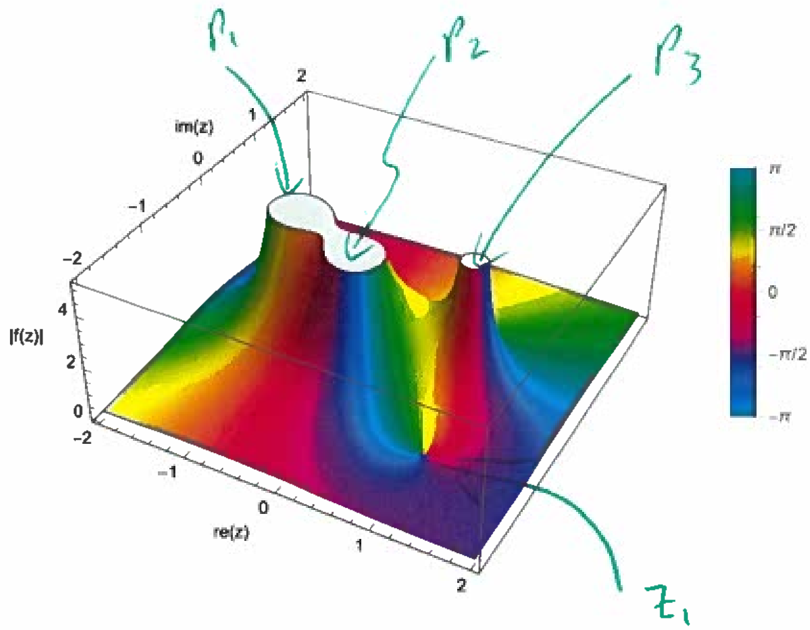
Res(f, P_1) = $\boxed{-\frac{3}{2}\,\frac{(-2+I)}{-2(-1+I)}}$ Res(f, P_2) = $\boxed{-\frac{3}{2}\,\frac{(I/2-1)}{(-I/2-1)(-I/2+1)}}$

Res(f, P_3) = $\boxed{-\frac{3}{2}\,\frac{I}{2(1+I/2)}}$

In[ ]:= $f[z\_] := 3\,\dfrac{z - 1 + i}{(1 - z^2)\,(i + 2\,z)}$ $= -\frac{3}{2}\left[\dfrac{z-1+I}{(z-1)(z+1)(z+I/2)}\right]$

```
ComplexPlot3D[ f[z],
 {z, 2}, PlotLegends → Automatic,
 AxesLabel → {"re(z)", "im(z)", "|f(z)|"}]
```

Out[ ]=



$P_1 = -1$

$P_2 = -I/2$

$P_3 = 1$

$Res_1 = -\frac{3}{2}\,\dfrac{-1-1+I}{(-1-1)(-1+2I/2)}$

$Res_2 = -\frac{3}{2}\,\dfrac{-I/2-1+I}{(-I/2-1)(-I/2+1)}$

$Res_3 = -\frac{3}{2}\left[\dfrac{1-1+I}{2\,(1+I/2)}\right]$

9. Complete the following. The TS

$\log(i+z)=\sum_{k=0}^{\infty} a_k z^k$ where $a_k = \boxed{I^{K+1}/K+1 \quad K \neq 0}$ $\quad a_0 = \log(I)$

converges for $\boxed{|z| < 1}$. Justify your

convergence statement using a Calculus II theorem. Give a simpler complex analytic justification below.

$$\frac{1}{I+z} = \frac{-I}{1-Iz} = -I\left(\frac{1}{1-Iz}\right) = -I\sum_{K=0}^{\infty}(Iz)^K$$

$$\log(I+z) = \log(I) + \int_0^z \frac{1}{1-I\xi}\,d\xi = \log(I) - \sum_{K=0}^{\infty}\frac{(Iz)^{K+1}}{K+1}$$

---

or $\quad f(z) = \log(I+z)$

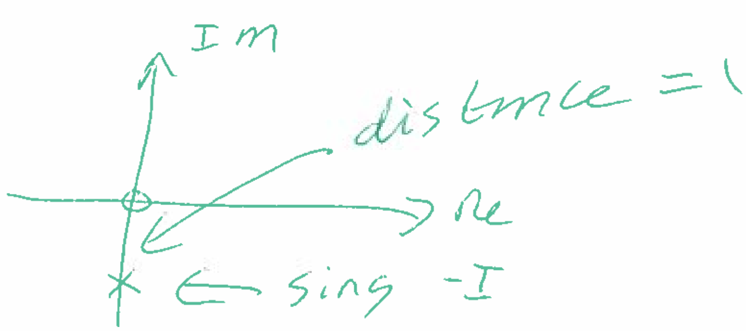
$f'(z) = \frac{1}{I+z} = (I+z)^{-1}$ $\qquad$ etc.

$f''(z) = -(1+z)^{-2}$

$f'''(z) = 2(1+z)^{-3}$

$\vdots$

$\lim \frac{K+1}{K} = 1 \qquad R = \frac{1}{1} = 1$



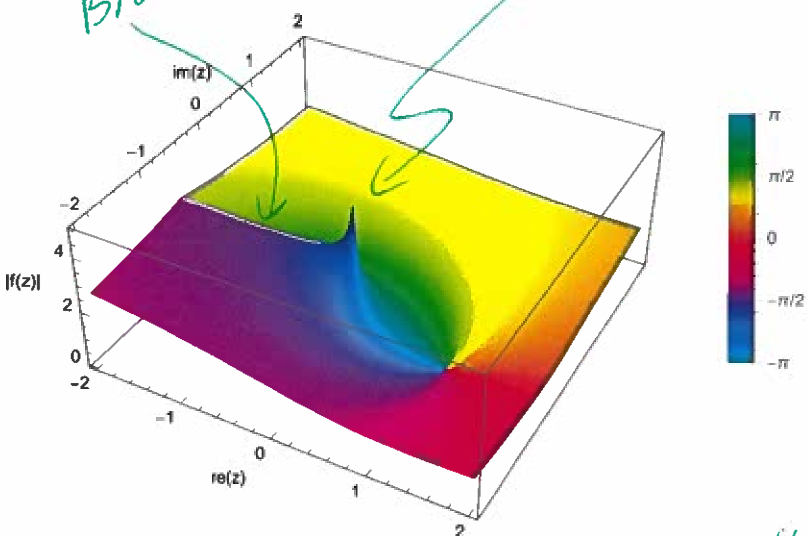
10. Label and explain the singularity and color discontinuity visible for $\log(i/2+z)$ in the plot below using appropriate language. The radius of convergence for the TS $f(z)=\sum_{k=0}^{\infty} a_k (z-z_0)^k$ with $z_0 == 1$ is $R = \boxed{\sqrt{5/4}}$.

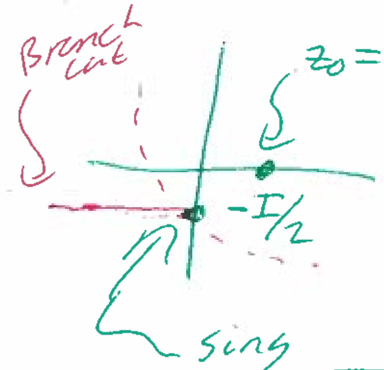
Explain how you know and explain what the TS converges to around the color discontinuity.

```
In[ ]:= f[z_] := Log[i / 2 + z]
        ComplexPlot3D[f[z], {z, 2}, PlotLegends → Automatic,
         AxesLabel → {"re(z)", "im(z)", "|f(z)|"}]
Out[ ]=
```

Branch cut

sing





$dist = \sqrt{1^2 + (1/2)^2}$

$= \sqrt{5/4}$

TS converges to f up to sing. Disc does not contact the branch cut!

11. For the closed CCW contour Γ shown below

$p_1(t) = \boxed{5\,E^{\pi I + \pi I t}}$ for $0 \le t < 1$

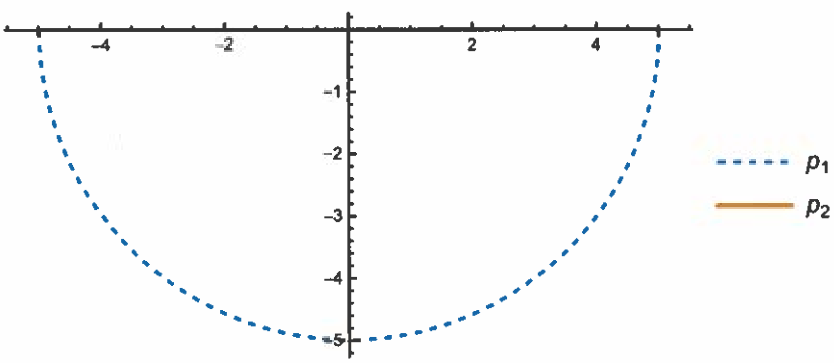
$p_2(t) = \boxed{5 - 10t}$ for $0 \le t < 1$

$\int_\Gamma f(z)\,dz = \int_0^1 \boxed{f(p_1(t))\,p_1'(t)}\,dt + \int_0^1 \boxed{f(p_2(t))\,p_2'(t)}\,dt.$

For $f(z) = z^2$ evaluate $\int_\Gamma f(z)\,dz = \boxed{0}$ and

$\int_{\Gamma_2} f(z)\,dz = \boxed{-250/3}$. Show appropriate work below.

Out[ ]=



no sing inside Γ

$$\int_\Gamma z^2\,dz = 0$$

$$\int_{\Gamma_1} f(z)\,dz + \int_{\Gamma_2} f(z)\,dz = 0$$

$$\int_{\Gamma_2} f(z)\,dz = \int_5^{-5} z^2\,dz = -\int_{-5}^{5} z^2\,dz$$

$$= -\left[\left[z^3/3 \right]_{-5}^{5} \right]$$

$$= -\left[\frac{125}{3} + 125/3 \right]$$

$$= -250/3$$

12. Compute $\int_\Gamma \frac{e^z}{(z^2+2z+1)(z+i)}\,dz = \boxed{0}$ for the CCW contour Γ. Show appropriate work.

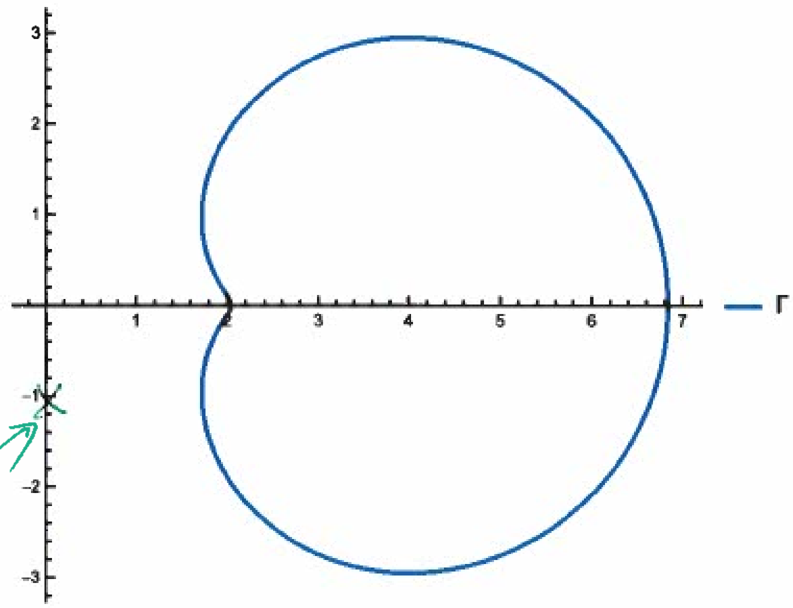
poles

$z^2+2z+1=0$
$(z+1)^2=0$
$z=-1$

$z+I=0$
$z=-I$

In[ ]:=
```
p1[t_] := 2 + (E^(2 π i t) + 1.2)^2
ParametricPlot[ReIm[p1[t]], {t, 0, 1}, PlotLegends → {"Γ"},
 AxesOrigin → {0, 0}]
```

Out[ ]=



poles

duble pole at -1
pole at $z=-I$
no poles in Γ