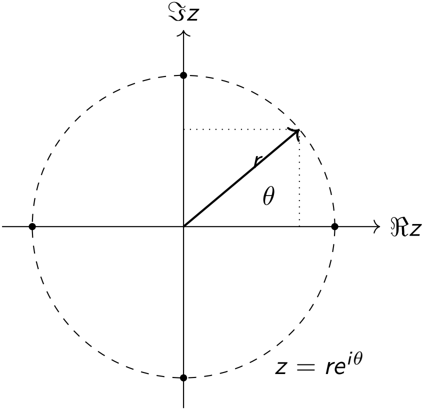
MA4410 — Geometry of Complex Numbers



Geometric Interpretation.

- $|z| = r$ is the distance from the origin.
- $\arg z = \theta$ is the angle from the positive real axis.
- Multiplication adds angles and scales magnitudes.
- nth roots lie evenly spaced on a circle of radius $r^{1/n}$.

MA4410 — Review of Complex Arithmetic

Algebraic Form. A complex number is $z = x + iy$, with

$$\Re z = re(z) = x, \qquad \Im z = im(z) = y, \qquad \bar{z} = x - iy, \qquad |z| = \sqrt{x^2 + y^2}.$$

Polar Form.

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}, \quad r = |z|, \quad \theta = \arg z.$$

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

Addition and Subtraction.

$$(x_1 + iy_1) \pm (x_2 + iy_2) = (x_1 \pm x_2) + i(y_1 \pm y_2).$$

Multiplication.

$$(x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1).$$

In polar form:

$$r_1e^{i\theta_1}\, r_2e^{i\theta_2} = (r_1r_2)e^{i(\theta_1+\theta_2)}.$$

Division.

$$\frac{x_1 + iy_1}{x_2 + iy_2} = \left(\frac{x_1 + iy_1}{x_2 + iy_2}\right)\left(\frac{x_2 - iy_2}{x_2 - iy_2}\right) = \frac{(x_1 + iy_1)(x_2 - iy_2)}{x_2^2 + y_2^2}.$$

Note: $z\bar{z} = |z|^2$
In polar form:

$$\frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}.$$

Integer Powers. If $z = re^{i\theta}$, then the n power is

$$z_n = r^n e^{i(n\theta)}, \qquad k = 0, 1, \ldots, n-1.$$

Integer Roots (De Moivre). If $z = re^{i\theta}$, then the n roots of $w^n = z$ are

$$w_k = r^{1/n} e^{i(\theta + 2\pi k)/n}, \qquad k = 0, 1, \ldots, n-1.$$

MA4410 — Roots of Unity

Definition. The *nth roots of unity* are the solutions of

$$z^n = 1.$$

Writing $1 = e^{i2\pi k}$, the solutions are

$$\omega_k = e^{2\pi ik/n}, \qquad k = 0, 1, \ldots, n-1.$$