Separation of Variables for Laplace's Equation

Mixed Boundary Conditions on a Square

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Problem Statement

We solve Laplace's equation

$$\Delta u = 0 \quad \text{on } \Omega = (0, \pi) \times (0, \pi),$$

with boundary conditions:

- Neumann:

$$\partial_x u(0, y) = \partial_x u(\pi, y) = 0$$

- Dirichlet:

$$u(x, 0) = 0, \qquad u(x, \pi) = g(x).$$

Goal: express the solution explicitly using separation of variables.

Strategy: Separation of Variables

We seek solutions of the form

$$u(x, y) = X(x)Y(y).$$

Substituting into $\Delta u = 0$:

$$X''(x)Y(y) + X(x)Y''(y) = 0.$$

Divide by $X(x)Y(y)$:

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda.$$

This yields two ODEs:

$$\begin{cases} X'' + \lambda X = 0, \\ Y'' - \lambda Y = 0. \end{cases}$$

The x-Problem (Neumann BCs)

The x-equation:

$$X'' + \lambda X = 0,$$

with

$$X'(0) = X'(\pi) = 0.$$

Eigenvalues and eigenfunctions:

$$\lambda_n = n^2, \qquad X_n(x) = \cos(nx), \quad n = 0, 1, 2, \ldots$$

- Neumann BCs select *cosine* modes
- These form an orthogonal basis in $L^2(0, \pi)$

The y-Problem

For each $n \geq 1$:

$$Y_n'' - n^2 Y_n = 0.$$

General solution:

$$Y_n(y) = A_n \sinh(ny) + B_n \cosh(ny).$$

Apply the boundary condition:

$$u(x,0) = 0 \quad \Rightarrow \quad Y_n(0) = 0,$$

which gives

$$B_n = 0.$$

Thus,

$$Y_n(y) = A_n \sinh(ny).$$

Separated Solutions

For each $n \geq 1$, we obtain:

$$u_n(x, y) = A_n \cos(nx) \sinh(ny).$$

By linearity, the general solution is:

$$u(x, y) = \sum_{n=1}^{\infty} A_n \cos(nx) \sinh(ny).$$

(If g has nonzero mean, a separate $n = 0$ mode may be handled explicitly.)

Imposing the Top Boundary Condition

We impose:

$$u(x,\pi) = g(x).$$

Substitute the series:

$$g(x) = \sum_{n=1}^{\infty} A_n \cos(nx) \sinh(n\pi).$$

Thus g must be expanded in a cosine series:

$$g(x) = \sum_{n=1}^{\infty} \hat{g}_n \cos(nx),$$

where

$$\hat{g}_n = \frac{2}{\pi} \int_0^{\pi} g(x) \cos(nx)\, dx.$$

Solving for the Coefficients

Matching coefficients gives:

$$A_n = \frac{\hat{g}_n}{\sinh(n\pi)}.$$

Therefore, the solution is

$$\boxed{u(x, y) = \sum_{n=1}^{\infty} \frac{\hat{g}_n}{\sinh(n\pi)} \cos(nx) \sinh(ny)}$$

This satisfies:

- $\Delta u = 0$
- Neumann BCs in x
- Dirichlet BCs in y

Interpretation and Remarks

- Cosine modes arise from Neumann conditions
- Hyperbolic sine enforces zero value at $y = 0$
- $\sinh(n\pi)$ strongly damps high-frequency modes
- The solution is smooth for smooth g

This construction is a prototype for:

- Poisson problems
- Heat equation with mixed BCs
- Spectral methods in rectangles

Summary

1. Separate variables
2. Solve Sturm–Liouville problem in x
3. Solve ODE in y
4. Expand boundary data in eigenfunctions
5. Match coefficients

$$\boxed{u(x,y) = \sum_{n=1}^{\infty} \frac{2}{\pi} \frac{\cos(nx)\sinh(ny)}{\sinh(n\pi)} \int_0^{\pi} g(s)\cos(ns)\,ds}$$