MATH 323: Probability

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Contents

1 Discrete Random Variables

A random variable is discrete if it only has a finite or countably infinite number of distinct values

3.2-3 Discrete Probability Distribution

- $0 \leq p(y) \leq 1 \quad \forall y$
- $\sum_y p(y) = 1$
- $E(Y) = \sum_y yp(y)$
- $V(Y) = E\left[(Y-\mu)^2\right] = E(Y^2) - \mu^2$
- $E(c) = \sum_y cp(y) = c\sum_y p(y)$
- $E\left[cg(Y)\right] = cE\left[g(Y)\right]$

3.4 Binomial Probability Distribution

Consists of n identical independent trials, resulting in either success (p) or failure (q). Goal is to find number of successes in k trials for some k.

- $p(y) = \binom{n}{y} p^y q^{n-y}$
- $E(Y) = np$
- $\sigma^2 = V(Y) = npq$

3.5 Geometric Probability Distribution

Same constraints as binomial probability distribution, but we are interested in finding the odds that Y is the index of the first success.

- $p(y) = q^{y-1}p$
- $\mu = E(Y) = \frac{1}{p}$
- $\sigma^2 = V(Y) = \frac{1-p}{p^2}$

3.6 Negative Binomial Probability Distribution

Same constraints as binomial probability distribution, but we are interested in finding the position for the k^{th} success, for some k

- $p(y) = \binom{y-1}{r-1} p^r q^{y-r}$
- $\mu = E(Y) = \frac{r}{p}$
- $\sigma^2 = V(Y) = \frac{r(1-p)}{p^2}$

3.8 Poisson Probability Distribution

Used to express probability that a certain number of events occur at a fixed time interval, given λ is the average, granted that the events are independent and identical

- $p(y) = \frac{\lambda^y}{y!} e^{-\lambda}$
- $\mu = E(Y) = \lambda$
- $\sigma^2 = V(Y) = \lambda$

3.9 Moment & Moment-Generating Functions

- ${\mu'}_k = E(Y^k)$
- $\mu_k = E\left[(Y-\mu)^k\right]$
- $m(t) = E(e^{tY})$
- $\frac{d^k m(t)}{dt^k}\Big|_{t=0} = m^{(k)}(0) = {\mu'}_k$

3.11 Tchebysheff's Theorem

If $\mu < \infty$ and $\sigma^2 < \infty$, then $\forall k > 0$:
$P(|Y-\mu| < k\sigma) \geq 1 - \frac{1}{k^2}$ or
$P(|Y-\mu| \geq k\sigma) \leq \frac{1}{k^2}$

Chernoff Inequality

Given a random variable x, moment generating function $M_x(t)$, and $c > 0$:

1. $P(X \geq c) \leq e^{-tc} \cdot M_x(t) \quad \forall t > 0$
2. $P(X \leq c) \leq e^{-tc} \cdot M_x(t) \quad \forall t < 0$

Jenson's Inequality

If $f(x)$ is a convex function, for $0 \leq a \leq 1$ $\forall x_1, x_2 \leq$ Domain of f:
$a \cdot f(x_1) + (1-a) \cdot f(x_2) \geq f(ax_1 + (1-a) \cdot x_2)$

If $E(f(x)) < \infty$, then $E(f(x)) \geq f(E(x))$

Markov Inequality

If $E(X) < \infty$, then $P(X \geq c) \leq \frac{E(X)}{c}$ $\forall c > 0$

Bernoulli Trial

Random experiment for which, given $0 < p < 1$, $P(Y = 1) = p$, $P(Y = 0) = q = 1 - p$

1. Random variable: x
2. Realization: $x = 1$ (success), $x = 0$ (failure)
3. Probability Mass Function: $p_x(x) = p^x(1-p)^{1-x} = p^x q^{1-x} \quad x = 0, 1$
4. Moment Generating Function: $M_x(t) = E(e^{tx}) = \sum_{x=0}^{1} e^{tx} \cdot p^x \cdot (1-p)^{1-x} = q + p \cdot e^t$
5. Expectation $= p$, Variance $= pq$

2 Final Review

Inequality

- Kolmogorov's Axioms
  Last point: If $A_i \cap A_j = \emptyset$, then $P(A_i \cup A_j) = P(A_i) + P(A_j)$
- Boole's Inequality
  For $A_1, A_2, ... \in F$, $P\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} P(A_i)$
  Prove using disjoint settings, using axiom above for summation, and showing that an intersection $A \cap B$ is always less than or equal to A.
- Bonferroni's Inequality
  For $A_1, A_2, ... \in F$, $P\left(\bigcap_{i=1}^{\infty} A_i\right) \geq 1 - \sum_{i=1}^{\infty} P(A_i^C)$
  Prove using DeMorgan's, then Boole's to form the inequality.

Independence

Prove that A is independent from B iff A^C is independent from B

Moment Generation Function

Used to calculate moments (expectation and variance) and showing that distributions are equal.
To find the expectation, derive MGF and set $t = 0$, or derive $log(MGF)$ and set $t = 1$
An MGF is composed of the normalizing constant and the kernel

Transformation

- Jacobian
- CDF

$$\begin{aligned} P(Y \leq y) &= P(g(X) \leq y) \\ &= P(X \leq g^{-1}(y)) \\ &= P_x(g^{-1}(y)) \end{aligned} \tag{1}$$

$$\begin{aligned} f_y(y) &= F_y'(y) \\ &= F_x'(f^{-1}(y))\frac{d}{dy}g^{-1}(y) \end{aligned}$$