Contents

1 Lecture 1 *<2017-09-05 Tue>*

1.1 Algorithm

- Al-Khwarizmi (9th Century)
- Algorismus (Latin)
- Arithmos (Greek)
  - Greek + Latin > Algorithm

A set of step by step instructions

1. Every step simple and precise
2. Produces an answer in finite time (not run forever)

This course will be very rigorous, lots of proofs, but it will take 2-3 months to formally define algorithms, so we'll just have to be satisfied with thsi definition. Formalized in 1930's by Turing and Church.

- Covered in COMP 330

Even though the concept of an algorithm is very simple and intuitive, it's not very obvious to prove things.

- Algorithms are an old concept, have been studied forever. Some examples are really old

1.1.1 Examples of algorithms

- Recipes
- 1600 BC Babylonians (Factorization and square roots)
- Euclid's Algorithm (200 BC)
  - Finding greatest common divisor of 2 numbers

Field of theoretical computer science is much older than first computers.

- Really mature field.
- Computers are just a device that helps us use these things.
- Theoretical computer science is part of math and science and has been studied for milennials

1.2 Teacher's website

`http://www.cs.mcgill.ca/~hatami/`

- He will be following the textbook.

1.3 Stable matching

n men: $m_1, m_2, \ldots, m_n$ n women: $w_1, w_2, \ldots, w_n$

Every man and woman has a ranking of people of other gender.

1.3.1 Ex: $n = 4$

$m_1 : w_3 > w_1 > w_2 > w_4$
$m_2 : w_1 > w_4 > w_2 > w_3$
$m_3 : w_1 > w_2 > w_4 > w_3$
$m_4 : w_2 > w_3 > w_4 > w_1$

$w_1 : m_4 > m_2 > m_1 > m_3$
$w_2 : m_1 > m_2 > m_3 > m_4$
$w_3 : m_2 > m_1 > m_3 > m_4$
$w_4 : m_4 > m_1 > m_2 > m_3$

1. A pairing
    - $m_1 + w_2$
    - $m_2 + w_4$
    - $m_3 + w_1$
    - $m_4 + w_3$

   What is unstable about this? The last pair, m_4 and w_3.
    - m_1 and w_3 prefer each other
2. Unstability: If there is a pair (m, w) such that
    (a) m prefers w to his current partner
    (b) w prefers m to her current partner
    (c) Selfish agents, everyone wants to be with the best possible partner they can find
3. Problem: Can we find a stable matching?

1.4 Stable Matching Algorithm

while $\exists$ a free man $\underline{m}$

- m proposes to the highest-ranked woman $\underline{w}$ that he has not prosed yet
- If $\underline{w}$ is free <u>or</u> prefers m to her current partner, she gets engaged to $\underline{m}$ and her current partner becomes free

else

- She rejects $\underline{m}$ and $\underline{m}$ remains free

End while

1.4.1 For our example:

- m_1 proposes to w_3, accepts $> m_1 + w_3$
- m_2 proposes to w_1, accepts $> m_2 + w_1$
- m_3 proposes to w_1, rejects
  - m_3 proposes to w_2, accepts $> m_3 + w_2$
- m_4 proposes to w_2, rejects
  - m_4 proposes to w_3, rejects
  - m_4 proposes to w_4, accepts $> m_4 + w_4$

Simple example, no one broke up. Let's change the example a bit.

1.4.2 Modified example

$m_1 : w_3 > w_1 > w_2 > w_4$
$m_2 : w_1 > w_4 > w_2 > w_3$
$m_3 : w_1 > w_2 > w_4 > w_3$
$m_4 : w_2 > w_3 > w_4 > w_1$

$w_1 : m_4 > m_2 > m_1 > m_3$
$w_2 : m_1 > m_2 > m_3 > m_4$
$w_3 : m_2 > m_4 > m_3 > m_1$
$w_4 : m_4 > m_1 > m_2 > m_3$

- m_1 proposes to w_3, accepts
- m_2 proposes to w_1, accepts
- m_3 proposes to w_1, rejects
  - m_3 proposes to w_2, accepts
- m_4 proposes to w_2, rejects
  - m_4 proposes to w_3, accepts, breaks up with m_1
- m_1 proposes to w_1, rejects
  - m_1 proposes to w_2, accepts, breaks up with m_3
- m_3 proposes to w_4, accepts

1.4.3 Why isn't this infinite?

$P(t)$: Number of pairs (m, w) such that m has not proposed to w yet at time t (number of iterations of while loop). $P(0) = n^2$ $P(1) = n^2 - 1$ Is it possible that a man proposes to a woman more than once? No.

1. Fact: No man proposes to the same woman more than once.

   Some of these proposals may never happen.

   The quantity P will never go negative.

2. Fact: $P(t)$ decreases by 1 at every iteration.

3. Lemma: The algorithm terminates after at most n^2 iterations. There will be no free men at the end.

4. Fact: Once a woman gets a proposal, she is never free again.

   $\implies$ If a man $\underline{m}$ remains free by the end of the alg it means that at the end all women are engaged. $\implies$ Since there are n men and n women this means that all men are engaged as well.

   $\implies$ At the end every person is engaged.

   - This algorithm gives us a pairing.
     - But we need to show that this is a good pairing, that it's stable.

2 Lecture 2 *<2017-09-07 Thu>*

2.1 Announcements

- Lectures will be recorded.
- Assignment 1 to come out soon, probably early next week.

2.2 Recall:

Stable matching n men n women.

- Not a fundamental problem, but contains many of the elements we'll see later in this course

2.2.1 Ex: $n = 4$

Man	Preference 1	2	3	4
m_1:	$w_3 >$	$w_1 >$	$w_2 >$	w_4
m_2:	$w_1 >$	$w_4 >$	$w_2 >$	w_3
m_3:	$w_1 >$	$w_2 >$	$w_4 >$	w_3
m_4:	$w_2 >$	$w_3 >$	$w_4 >$	w_1

Woman	Pref 1	2	3	4
w_1:	$m_4 >$	$m_2 >$	$m_1 >$	m_3
w_2:	$m_1 >$	$m_2 >$	$m_3 >$	m_4
w_3:	$m_2 >$	$m_4 >$	$m_3 >$	m_1
w_4:	$m_4 >$	$m_1 >$	$m_2 >$	m_3

Same example as last lecture, see matching/use of algorithm in lecture 1. Matching becomes:

m_1	m_2	m_3	m_4
w_2	w_1	w_4	w_3

Top matched with bottom. Does w_1 have a tendancy to break up and go with m_3? No.

2.2.2 Last lecture we proved:

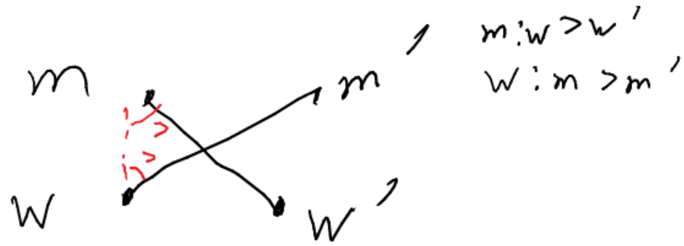
1. The algorithm always terminates.
    - Easy to see from the list of preferences, because we go down the list of the men's preferences, they always go down their list and never go back
2. When the algorithm terminates everybody has a partner.
    - Won't end up with a situation where a man proposes to everyone and gets rejected
    - Women will never be free once they are initially proposed to
    - A man can't be free at the end, because that means all women we're proposed to and all women are married
        - But same amount of women and men

2.3 Stable Matching Algorithm

2.3.1 Does this algorithm produce a stable marriage?

It remains to show that the output is stable.

1. Observation 1
   - Throughout the algorithm every man's partner gets worse and worse
2. Observation 2
   - However, for women it is the opposite
   - They accept the first proposal
   - But every partner gets better and better
3. Theorem: The final matching is stable.
   (a) Proof: Suppose not! Then there exists engaged pairs as in the following:



   But in this case m would have proposed to w before proposing to w', and as a result we know that w would not have ended up with someone worse than m.

2.3.2 Is this algorithm better for men or women?

Let's say (m, w) is valid if there exists some stable matching that pairs m and w.

1. Fact: This algorithm matches every man with their most preferred valid w and every woman with their least preferred valid m.

- For men, they start ambituously and go for their most preferred partner and go down the list if needed
- For women, they start at whatever is first given and only improve if needed
- Formal proof in textbook, won't do it in class as to spend less time on this problem

2.4 Notes on problems

- Formulate the problem as a precise mathematical problem.
  - What is the input?
  - What is the goal?
  - Conditions we want to satisfy?
  - Everything must be precise or else we won't be able to satisfy all these things.
- Design an algorithm
- Analyze the algorithm:
  - It always terminates
    * Show that, no matter the input, it will always stop, no infinite loop
  - It outputs the correct output!
    * In stable marriage, we showed that it is always stable
  - Running time
    * How long does it take to terminate?
      · For stable marriage, we could brute force and try all possible combinations and see if they're stable or not, but that would be $n!$

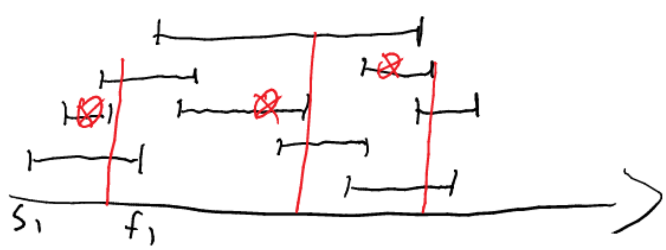
Professor won't do much on first point, about formulating problem as math. Textbook often presents the problem in a bunch of sentences for some real life thing and we need to extract the mathematical problem from there, which the professor isn't a big fan of.

2.5 Some example problems

2.5.1 Interval scheduling

- Let's say you have a room and want to rent it out
- Bunch of offers that say the person wants to use the room from a start time to an end time
- Want to accomodate as many people as possible, but we can't have overlap
  - Maximize number of offers without overlap

1. Input:
   - n requests
   - Starting time $s_1, s_2, \ldots, s_n$
   - Finishing time $f_1, f_2, \ldots, f_n$
   - Such that $s_i < f_i$
2. Problem We want to pick the max number of these tasks s.t. no two overlap. (Maximum bookings, not maximum time, not charging per hour)



   (a) Algorithm?
   - What algorithm is good for this?
   - Pick next available room that finishes the earliest and keep going
   - **Greedy algorithm**

2.5.2 Weighted Interval Scheduling

- Now every offer comes with some value.
- $v_1, \ldots, v_n$
  - where v_i is the value we get from accomodating the i^{th} offer.
- $s_1, \ldots, s_n$
- $f_1, \ldots, f_n$

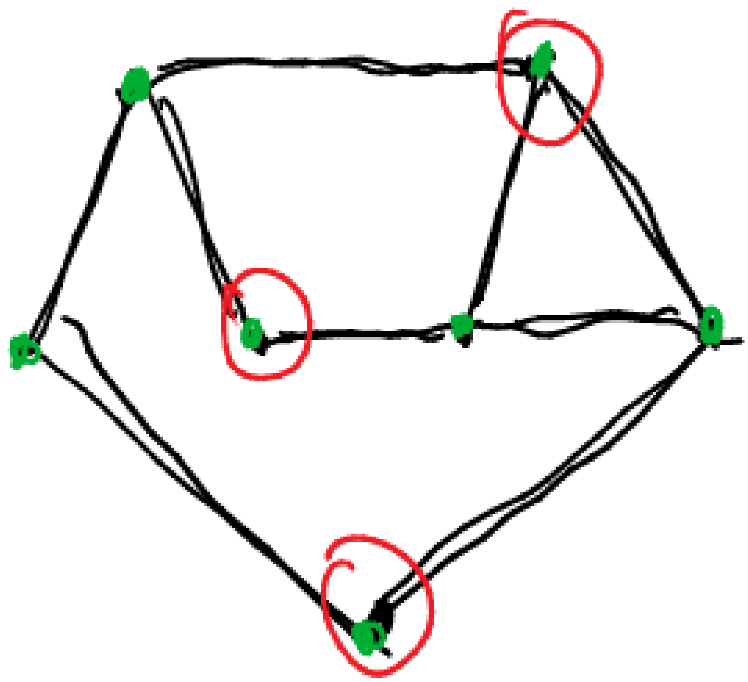
Want higher value, rather than most matchings Why is this harder to solve than the previous problem? Because the previous one is a special case of the first.

- Reduction = reducing this problem to the previous to get an answer
- Setting $v_1 = \ldots = v_n = 1$ solves the previous problem.

We will solve this using **Dynamic Programming**

- Create huge table, keep filling it up as you process input
  - Solve solution for smaller version of problem and keep expanding based on that
- Let's say $A[t]$ = max value if we stop at time t.

1. Independent set problem



Independent set: A set of notes, no two are adjacent. Find the largest independent set.

- Obvious way of doing it without concerning ourselves with time?
  - Brute force
- Without that? You can do some heuristics, but,
  - It is widely believed that every algorithm for this problem is of brute force nature: It is more or less checking all the possible subsets?

(a) P vs NP?

- Most important problem in computer science
- Common belief: $P \neq NP$
- This is an example of a problem which is believed to be NP

So that is essentially a small instruction about the types of problems we'll be seeing in this course. Next lecture we'll be formally going through running time.

3 Lecture 3 *<2017-09-12 Tue>*

3.1 Running Time Analysis

- We will be talking about running time of an algorithm.

3.1.1 Questions

Thinking back without knowledge of running time, what questions can we pose?

- How should we measure the running time of an algorithm?
- How can we compare the efficiency of two algorithms?
- What should we call an <u>efficient</u> algorithm?
  - Brute force isn't efficient for finding a matching.
  - Was our algorithm for stable matchings efficient?
  - We want to understand the concept of efficiency for an algorithm.

1. One option:
   - Call an algorithm efficient if it performs "fast" on <u>"real world"</u> inputs.
     - What is a real world input?
       * Without a good/rigorous definition, then this isn't a good option.
       * Not precise, so this option doesn't work.
2. Option II:
   - Take the set of all inputs of a certain size and take the average <u>running time</u> of our algorithm on them.
     - Maybe the inputs we care about are quite sparse in the set of all inputs.
     - Random inputs might be quite trivial
       * May lead us to think we defined a good algorithm
       * But in reality what we care about is harder
3. Example: Algorithm for prime numbers

- Input: integer n
- Output: Is n a prime number?
  - Alg 1:

    for $i = 2$ to $n - 1$ **do**
      if $n \pmod i = 0$ **then** return False
      end if
    end for
    return true
  - Look at all the numbers between $1, \ldots, N$
  - How many are divisible by $2, 3, 4, 5, 6, 7$? $1 - \frac{1}{2} \times \frac{1}{3} \times \frac{1}{5} \times \frac{1}{7} > 99\%$
  - On average performs well
  - Worst case (prime numbers) does not perform well.
  - While this notion of average time complexity is useful, because the majority of inputs dominate the worst case ones, it is not a very good definition.
- Better to just care about the worst case

4. Worst case time analysis We measure the <u>running time</u> against the worst input of a given <u>size</u>

   - Want to be inddependent of implementation:
     - We will count the number of "simple steps" (e.g. <u>If "$a > b$"</u>, $a := b \times c$)

3.1.2 Efficiency

1. <u>Def:</u> We call an algorithm **efficient** if its running time is bounded by a polynomial $P(n)$ for every input of <u>size</u> (in number of bits) n

   - n efficient
   - n^2 good
   - $n \log n$ good
   - 2^n bad
   - $n \log n < n^2$

   Remember that you need $\log n$ bits to store a number n.

   - Objection: $\underline{n^{100}}$ is considered efficient while it is not practical!

- Answer: Usually the exponents are better. (Rarely see n^{100} if ever)

- Scales well
  - Many interesting algorithms have polynomial time algorithms

2. Alternative Def: Efficient is running time $< n^3$ seems a better def as it overrules cases like n^{100}
   - Let's say you're combining 2 algorithms, say you're running a n^2 algorithm in a n^2 for-loop
     - Suddenly you're stuck with an n^4 algorithm
     - This doesn't allow us to easily stick algorithms in for-loops and the like
   - This is not very robust.
     - The choice of data structure, pseudo-code, ... can change the running time a bit and so this definition is not "robust". Result depends on implementation.

3. Example: Input: An array $A[0 \dots n-1]$

   Goal: Are all elements in $A[]$ distinct?

   for $i = 0$ to $n-2$ **do**
       for $j = i+1$ to $n-1$ **do**
           if $A[i] == A[j]$ **then**
               return "False"
           end if
       end for
   end for
   Return "True"

Step	Iterations
c_1: setting i	$n-1$
c_2: setting j	$\sum_{i=0}^{n-2}\sum_{j=i+1}^{n-1} 1 = \frac{n(n-1)}{2}$
c_3: comparing $A[i] == A[j]$	$\frac{n(n-1)}{2}$
c_4: return False	1
c_5: return True	1

Running time:

$$n - 1 + \frac{n(n-1)}{2} + \frac{n(n-1)}{2} + 1 + 1 = n^2 + 1$$

Efficient

This much accuracy is meaningless: Each one of these commands consist of some more primitive commands and that can depend on your compiler, ...

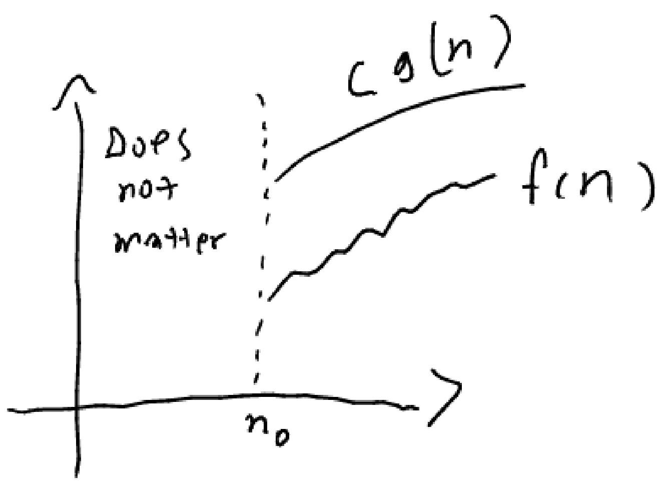
- What matters is that this is quadratic.

3.1.3 Big-O notation

Informally $O(g(n))$ is the set of all functions with smaller or same order of growth.

- You should think of it as a set, not a value.
- $n \in O(n^2)$
- $100n + 5 \in O(n^2)$
- $\frac{1}{2}n(n-1) \in O(n^2)$
- $n^3 \notin O(n^2)$

1. Def: $f(n) \in O(g(n))$ if $\exists n_0, c > 0$ such that $f(n) < cg(n)$ $\forall n > n_0$



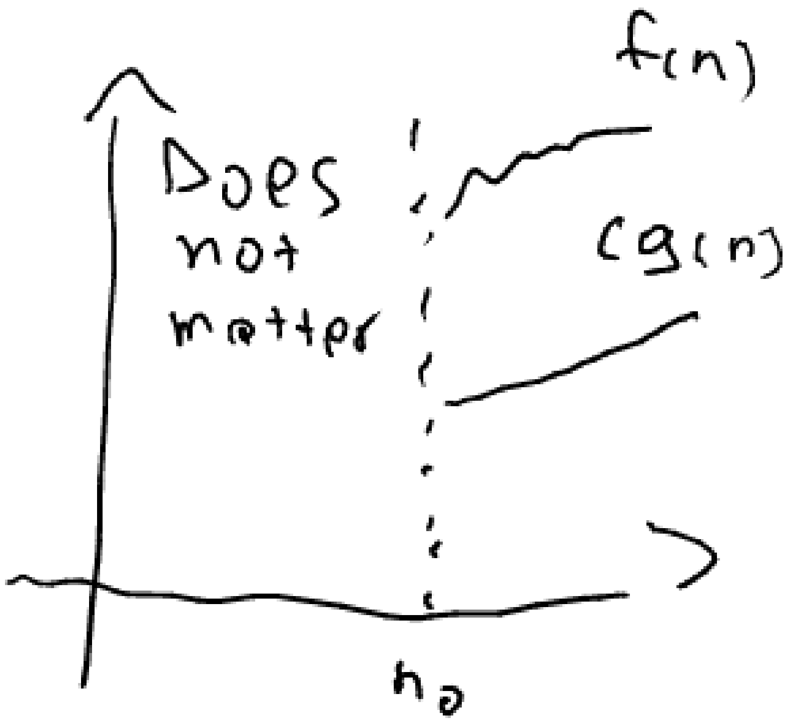
2. Ex: $100n + 5 \in O(n^2)$

   (a) Proof $100n + 5 \overset{n \geq 5, n_0 = 5}{\leq} 100n + n \leq \underbrace{101}_{c=101} n$

3.1.4 Ω-notation:

Informally $f(n) \in \Omega(g(n))$ if $f(n)$ grows faster or the same as $g(n)$

1. Def: $f(n) \in \Omega(g(n))$ if $\exists n_0, c > 0$ such that $f(n) \geq cg(n)\ \forall n \geq n_0$
   (Equivalently $g(n) \in O(f(n))$)



2. Ex: $\frac{n^2}{2} - 5n \in \Omega(n^2)$ $\frac{n^2}{2} - 5n \geq \frac{1}{4}n^2 \implies c = \frac{1}{4}\ \forall n \geq 20 = n_0$

4 Lecture 4 *<2017-09-14 Thu>*

4.1 Recall:

- Big-Oh
- Omega notation

4.2 Examples

Table 2.1 The running times (rounded up) of different algorithms on inputs of increasing size, for a processor performing a million high-level instructions per second. In cases where the running time exceeds 10^{25} years, we simply record the algorithm as taking a very long time.

	n	$n \log_2 n$	n^2	n^3	1.5^n	2^n	$n!$
$n = 10$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	4 sec
$n = 30$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	18 min	10^{25} years
$n = 50$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	11 min	36 years	very long
$n = 100$	< 1 sec	< 1 sec	< 1 sec	1 sec	12,892 years	10^{17} years	very long
$n = 1{,}000$	< 1 sec	< 1 sec	1 sec	18 min	very long	very long	very long
$n = 10{,}000$	< 1 sec	< 1 sec	2 min	12 days	very long	very long	very long
$n = 100{,}000$	< 1 sec	2 sec	3 hours	32 years	very long	very long	very long
$n = 1{,}000{,}000$	1 sec	20 sec	12 days	31,710 years	very long	very long	very long

4.3 Θ-notation:

$f(n) \iff f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$

- Grows at the same rate as $g(n)$

Alternatively:
$\exists n_0, c_1, c_2 \forall n > n_0$, s.t. $c_1 g(n) \leq f(n) \leq c_2 g(n)$

4.3.1 Examples

- $2n^2 + 1 = \Theta(n^2)$
  1. $n^2 - 5n + 10 \leq n^2 \forall n \geq 2$
  2. $n^2 - 5n + 10 \geq \frac{n^2}{2} \forall n \geq 20$

4.4 Theorem

Let $f(n) = a_d n^d + a_{d-1} n^{d-1} + \ldots + a_1 n + a_0, a_d > 0$.
Then $f(n) = \Theta(n^d)$

4.4.1 Proof

- $(f(n) = O(n^d)$
  - $f(n) = a_d n^d + \ldots + a_1 n + a_0 \leq \underbrace{(a_d + |a_{d-1} + \ldots + |a_0|)}_{c} n^d, \forall n \geq 1$
  - E.g. $2n^2 - 5n + 10 \leq (2+5+10)n^2$
- $f(n) = \Omega(n^d)$
  - $a_d n^d + a_{d-1}n^{d-1} + \ldots + a_1 n + a_0 \geq Cn^d$
  - $c = \frac{a_d}{2}$, since a_d is controlling the growth rate of the left hand side.
  - $\frac{a_d}{2} n^d \geq -(a_{d-1}n^{d-1} + a_{d-2}n^{d-2} + \ldots + a_0)$
  - $\frac{a_d}{2} n^d \geq (|a_{d-1}| + \ldots + |a_0|)n^{d-1}$, $\forall n \geq \frac{2(|a_{d-1} + \ldots + |a_0|)}{a_d}$ (by rearranging and isolating n)
  - On the other hand:
    * $(|a_{d-1}| + \ldots + |a_0|)n^{d-1} \geq -(a_{d-1}n^{d-1} + \ldots + a_0)$
  - Note that $|a_r|n^{d-1} \geq -a_r n^r, r \leq d-1$

4.5 Little o and Little omega

- Show strict upper and lower bounds, rather than equalities

$f(n) = o(g(n))$

- $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$
- Little oh implies big-Oh, but not the other way around

$f(n) = \omega(g(n)$

- $\lim_{n\to\infty} \frac{g(n)}{f(n)} = 0$

$n^{1/100}$ vs $\log_2(n)^5$

Claim: $\log_2(n)^5 = o(n^{1/100})$

Proof: $\lim_{n\to\infty} \frac{\log_2(n)^5}{n^{1/100}} = \lim_{n\to\infty} \frac{5\log(n)^4 \frac{\ln(2)}{n}}{\frac{1}{100} n^{\frac{-99}{100}}} = \ldots = 0$ (have to do L'Hopital's 4 more times)

The lesson is that anything in log grows much slower than any polynomial.

4.6 Theorem

- $\forall r > 1, d > 0$
- $n^d = o(r^n)$ (i.e. polynomials grow much slower than exponential functions)

$\underbrace{n^{10000}}_{\text{Better}}$ vs 1.0001^n

4.7 Stable Marriage

Data structures we may use:

- Array $A[0 \dots n-1]$
  - Operation times:
    - Access $A[i] : O(1)$
    - Insert a new entry somewhere in the middle: $O(n)$, need to shift.
    - Delete: $O(n)$
    - Finding an element: $O(n)$ not sorted
      - $O(\log(n))$ sorted
- Linked List
  - Operation times:
    - Access $i - th$ entry: $O(n)$
    - Insert-delete: $O(1)$
    - Finding: $O(n)$

while $\exists$ a free man m **do**
  Let w be the highest-ranked woman m has not proposed to yet.
  if w is free **then**
    (m, w) engaged
  else if w is currently engaged to m' **then**
    if w prefers m to m' **then**
      (m', w) engaged
      m becomes free
    end if
  end if

end while

- Input: Two (men and women) $n \times n$ arrays (rankings)
- Reading input $\Theta(n^2)$ so at best we can hope $\Theta(n^2)$ for the alg.
- The main while loop can repeat $O(n^2)$ times $\implies$ To have total $\Theta(n^2)$ time every iteration must be done in $O(1)$.
- How to implement?
  - When do we know if a man is free?
    - Can have an array of booleans of free men, but then you need for loop to check if there's a free man, which will be $O(n)$
    - Solution 1: Can have a linked list of free men.
      - Delete someone from the list when they get engaged.
      - Deleting and adding is $O(1)$ (add to front)
    - Solution 2: Using an array
      - Have a pointer to first free man and another to last free man
      - If first man gets engaged, move pointer to the right
      - If someone becomes free, then add to end and change pointer
      - Since we never have more than n people free, can use mod n

5 Lecture 5 *<2017-09-21 Thu>*

5.1 Graphs

5.1.1 Undirected Graphs

- Notation $G = (V, E)$
  - V = nodes (or vertices)
  - E = edges (or arcs) between pairs of nodes.
  - Captures pairwise relationship between object
  - Graph size parameters: $n = |V|, m = |e|$

5.1.2 Example applications

Graph	Node	Edge
Communication	telephone,computer	fiber optic cable
Circuit	gate, register, processor	wire
mechanical	joint	rod, beam, spring
financial	stock, currency	transactions
transportation	street intersection, airport	highway, airway route
internet	class C network	connection
game	board position	legal move
social relationship	person, actor	friendship, movie cast
neural network	neuron	synapse
protein network	protein	protein-protein interaction
molecule	atom	bond

5.1.3 Ways of implementing in a program

1. Adjacency matrix n-by-n matrix with $A_{uv} = 1$ if (u, v) is an edge.

    - Two representations of each edge.
    - Space proportional to n^2
    - Checking if (u, v) is an edge takes $\Theta(1)$ time
    - Identifying all edges takes $\Theta(n^2)$ time
    - It's exactly symmetric

2. Adjacency list Node-indexed array of lists

    - Two representations of each edge
    - Space is $\Theta(m + n)$
    - Checking if (u, v) is an edge takes $O(degree(u))$ time
    - Identifying all edges takes $\Theta(m + n)$ time

5.1.4 Paths and connectivity

- Def. A **path** in an undirected graph $G = (V, E)$ is a sequence of nodes $v_1, v_2, \ldots, v_k$ with the property that each consecutive pair v_{i-1}, v_i is joined by an edge in E.
- Def. A path is **simple** if all nodes are distinct.

- Def. An undirected graph is **connected** if for every pair of nodes u and v, there is a path between u and v

5.1.5 Cycles

- Def. A **cycle** is a path $v_1, v_2, \ldots, v_k$ in which $v_1 = v_k$, $k > 2$, and the first $k - 1$ nodes are all distinct.

5.1.6 Trees

- Def. An undirected graph is a **tree** if it is connected and does not contain a cycle

1. Theorem Let G be an undirected graph on n nodes. Any two of the following statements imply the third:
    - G is connected
    - G does not contain a cycle
    - G has $n - 1$ edges
2. Rooted trees
    - Given a tree T, choose a root node r and orient each edge away from r.
    - Importance. Models hierarchical structure

5.1.7 Connectivity

- s-t connectivity problem. Given two nodes s and t, is there a path between s and t?
- s-t shortest path problem. Given two nodes s and t, what is the length of a shortest path between s and t?
- Applications.
    - Friendster
    - Maze traversal
    - Kevin Bacon number
    - Fewest hops in a communication network

5.1.8 Breadth-first search

1. BFS intuition Explore outward from s in all possible directions, adding nodes one "layer" at a time. At most n layers.

2. BFS algorithm

   - $L_0 = \{s\}$
   - $L_1 =$ all neighbors of L_0
   - $L_2 =$ all nodes that do not belong to L_0 or L_1, and that have an edge to a node in L_1
   - $L_{i+1} =$ all nodes that do not belong to an earlier layer, and that have an edge to a node in L_i

3. Theorem For each i, L_i consists of all nodes at distance exactly i from s. There is a path from s to t iff t appears in some layer.

4. Property Let T be a BFS tree of $G = (V, E)$, and let (x, y) be an edge of G. Then, the levels of x and y differ by at most 1.

5. Analysis

   (a) Theorem The above implementation of BFS runs in $O(m + n)$ time if the graph is given by its adjacency representation.

   (b) Proof

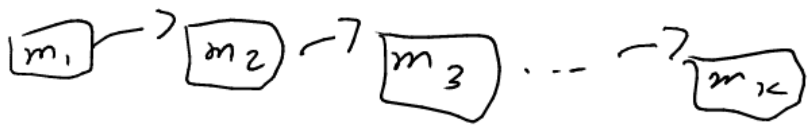
   - Easy to prove $O(n^2)$ running time:
     - At most n lists $L[i]$
     - Each node occurs on at most one list; for loop runs $\leq n$ times
     - When we consider node u, there are $\leq n$ incident edges (u, v), and we spend $O(1)$ processing each edge
   - Actually runs in $O(m + n)$ time:
     - When we consider node u, there are $degree(u)$ incident edges (u, v)
     - total time processing edges is $\sum_{u \in V} degree(u) = 2m$

6 Lecture 6 *<2017-09-26 Tue>*

6.1 Stable Marriage

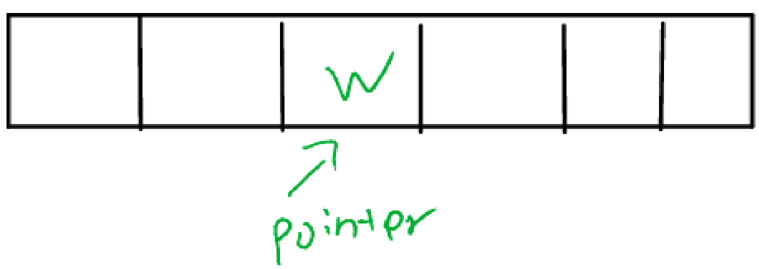
Continuation of Lecture 4: Stable Marriage algorithm analysis.

- Good data structure to tell if someone is free or not?
  - Can have a linked list of all the free men, remove them when they're no longer free.

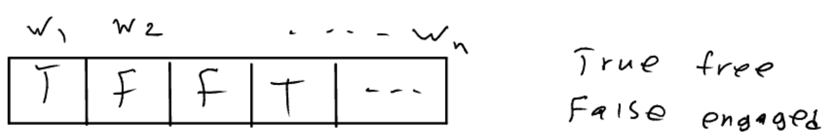


Initially all men are here. Finding a free man: $O(1)$

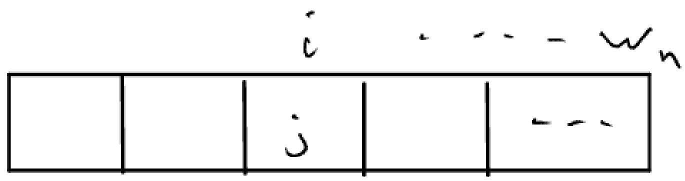
- Keep an ordered list of women sorted according to m's preference. Keep a pointer to the first person he has not proposed to yet. (can store as a linked list, array or stack) Move pointer along to the next after proposing.



Now we need to know if the woman is free. Make a boolean array of women with true or false.



- Array telling whom w is engaged to (j^{th} entry contains who w_j is engaged to)



- We keep a matrix $w[i, j]$ = the rank of m_j in the eye of w_i
- Example: $w_2 : m_4 > m_3 > m_5 > m_2 \ldots$, $w[2, 5] = 3$ (don't need to do linear time)
- If w_i prefers m_j to $mk \iff w[i, j] < w[i, k]$
  - Do some "preprocessing" in the beginning to make it easier during the algorithm

With all these data structures, our algorithm can run in $O(n^2)$

6.2 Priority Queue

Say we're running a clinic and new patients come. A nurse assesses them and gives them a priority so that we know who we should see next.

Dynamic Scenario

- Get elements with different priorities in an "online" matter (sometimes you get new data, not all given to you in the beginning)
- Once in awhile we can serve the element with the highest priority (and remove from the set)

We have a set S.

- Initially $S = \emptyset$
- At every step either
  - A new number is added to S.
  - or the smallest number is removed from S.

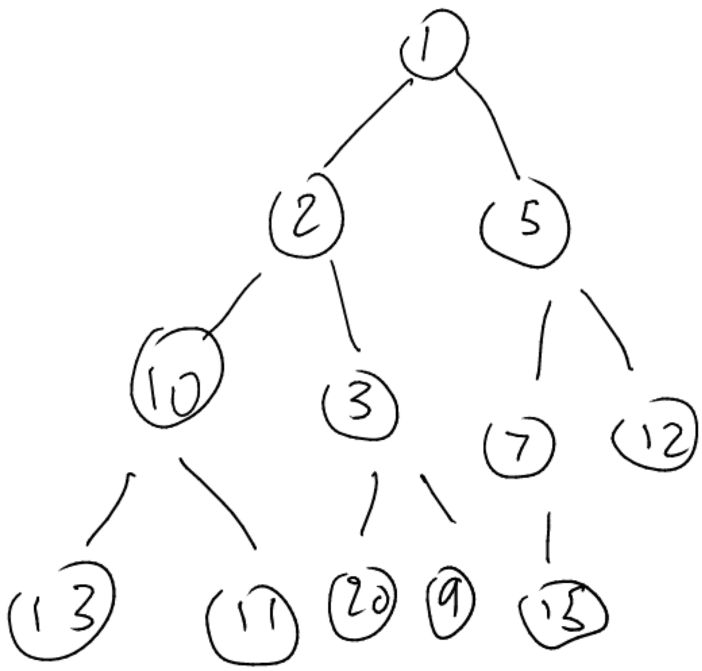
Some ideas:

- An unsorted list:
  - Inserting a new element $O(1)$
  - Removing the minimum: $O(n)$ (n elements in the list, have to find smallest)
  - Too costly, not good.
- Sorted list:
  - Inserting a new element $O(n)$
    - With an array, need to shift all elements.
    - Linked list (no binary search)
  - Removing the smallest $O(1)$.

6.2.1 Heap Data Structure

A balanced binary tree

- All levels are full except the last level which is filled **from left to right**
- Every node is $\geq$ its parent
- Ex:



Can be implemented with an array. Fill left to right.

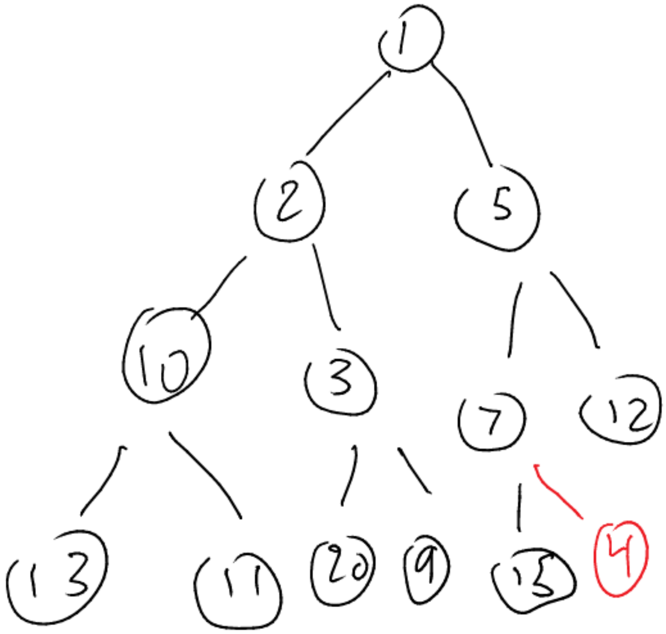
1	2	3	4	5	6	7	8	9	10	11	12
1	2	5	10	3	7	12	13	11	20	9	15

children

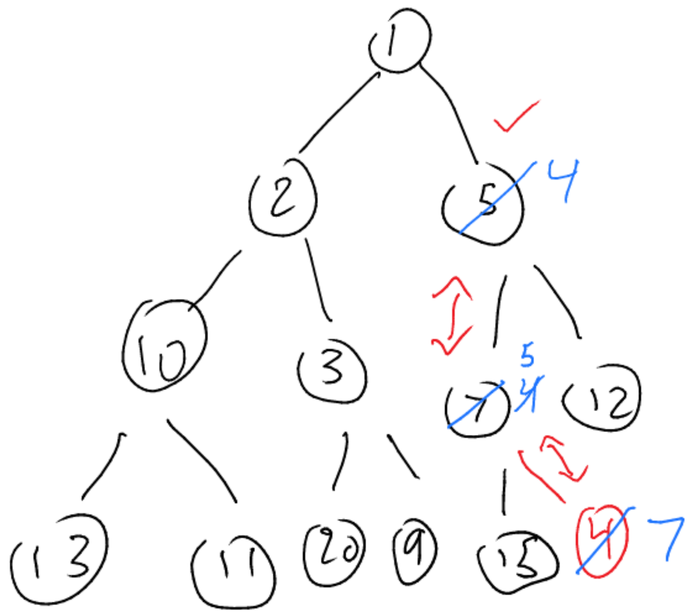
Where are the children of entry i? $2i, 2i+1$ (convenient)

What do we do when a new number arrives? Say insert(4)

- Naturally we want to put it in the next available place "n^{th}" if n is the updated # of nodes



But 4 is smaller than its parent. How to fix? Swap with parent.



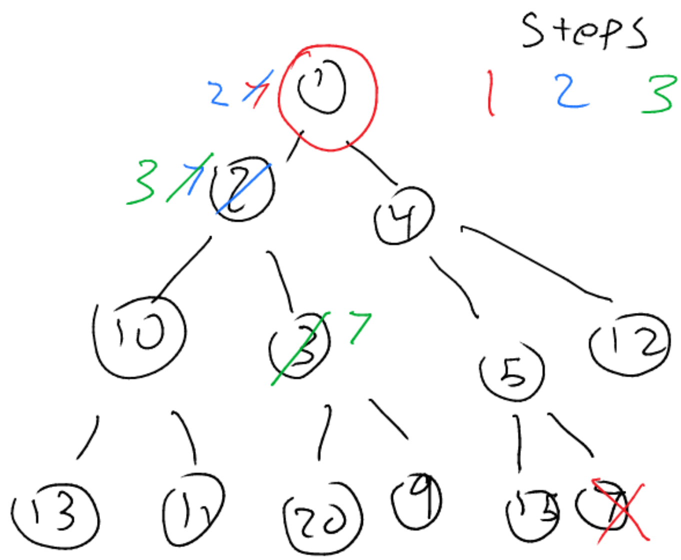
We will call this operation Heapify-Up.

Heapify-Up(H, i) // i is index
if $i > 1$ **then**
  let $j = parent(i) = \lfloor \frac{i}{2} \rfloor$
  if $H[i] < H[j]$ **then**
    swap$(H[i], H[j])$
    Heapify-Up(H, j)
  end if
end if

Running time of Heapify-Up: $O(\log n) = O(\text{Height of the tree})$

How do we remove the minimum?

- Insert last element at head and then swap with smallest child until the tree is balanced



Heapify-down(H, i)
$n = \text{length}(H)$
if $2i > n$ **then** // Elements $> n/2$ have no children
    Terminate
else if $2i + 1 \leq n$ **then**
    $left = 2i, right = 2i + 1$
    if $H[left] < H[right]$ **then**
        $j = left$
    else
        $j = right$
    end if
else//$(n = 2i)$
    $j = left = 2i$
end if
if $H[j] < H[i]$ **then**
    swap$(H[j], H[i])$
    Heapify-down(H, j)
end if

Q: How can we use this data structure to sort a list of n numbers?

Answer: Insert the elements one by one and then extract the minimums one by one.

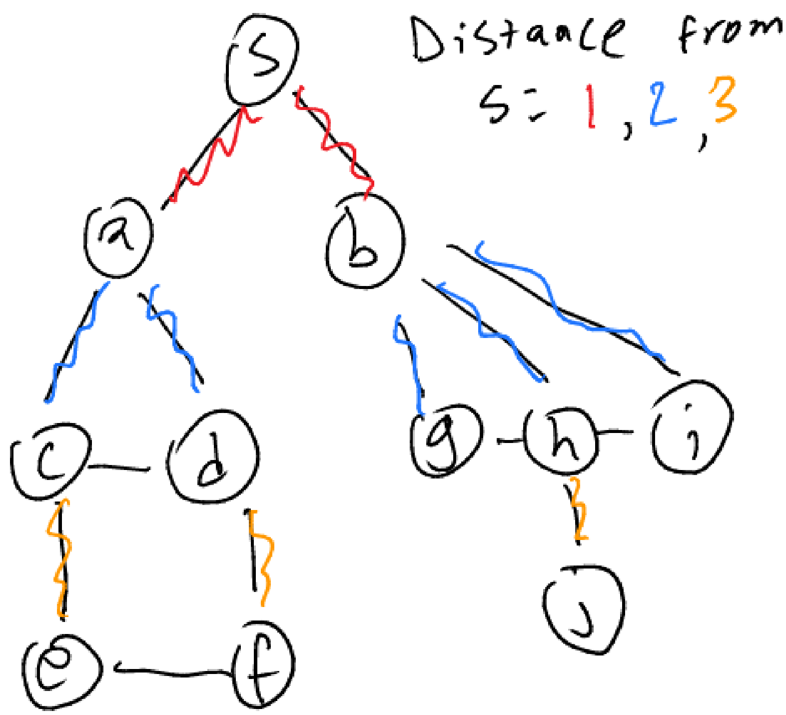
- Running time? $2nO(\log n)$

- $O(n \log n)$

7 Lecture 7 *<2017-09-28 Thu>*

7.1 Graph Exploration Algorithms

7.1.1 Breadth-First-Search (BFS)



Also tells you length of shortest path from s to any vertex.

- We explore according to the distance from s.
- How to implement this?

BFS(G)
for every vertex v in G **do**
  if v is unexplored **then**
    Mark v as explorerd
    BFS.vertex(v)

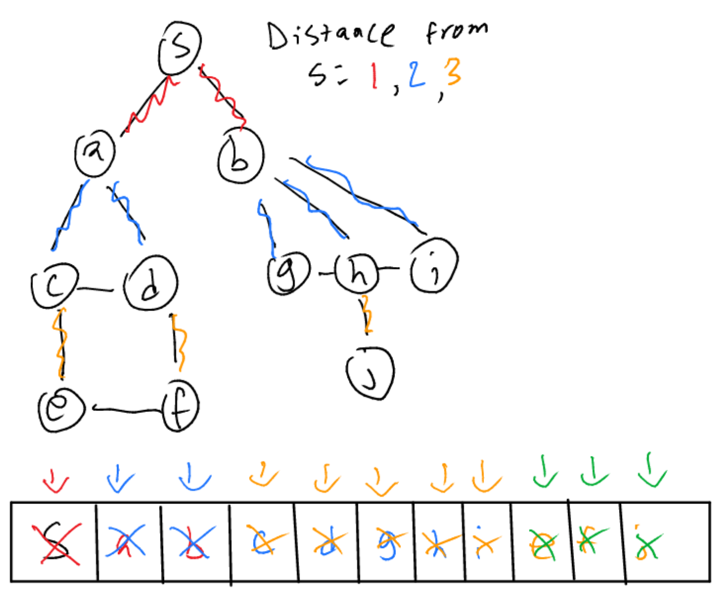
connected-comp++
end if
end for

BFS-vertex(v)
Make a list of all the unexplored neighbors of v.
Mark every vertex in this list as explored
for every u in this list **do**
BFS-Vertex(u)
end for

Recursive way above does not work?

A good way to implement this is to keep the newly discovered vertices in a queue (FIFO, first in first out).

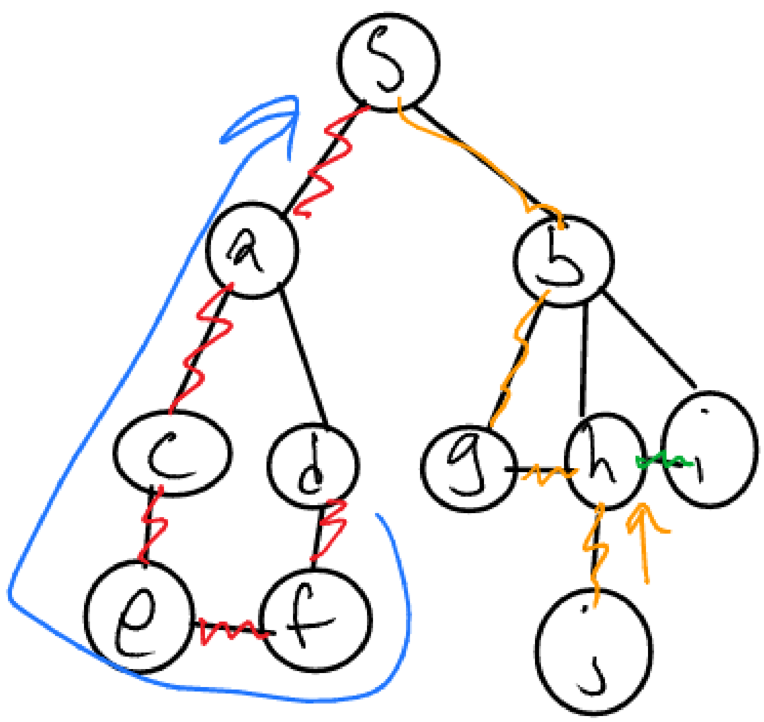
BFS-Vertex(v)
Add v to the queue
while queue is not empty **do**
Pick the first vertex u in the queue.
Mark all unexplored neighbors of u as explored and add them to the queue
end while



7.1.2 Depth-First-Search (DFS)

We go in a path discovering new vertices until we reach a dead-end, and then we step back ...

DFS(u)
for every edge (u,v) **do**
    if v is unexplored **then**
        mark v as explored
        DFS(v)
    end if
end for



Non-recursive DFS: Every time we discover a new vertex we put it at the top of a stack (FILO, first in last out).

7.2 Data Structure for Graphs

What data structure to use for graphs?

- Adjacency Matrix

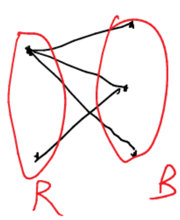
$$A[u,v] = \begin{cases} 1 & \text{if } (u,v) \in E \\ 0 & \text{if } (u,v) \notin E \end{cases}$$

  - Pros: very easy to see if u is connected to v
  - Cons: IF the graph has few edges it is wasteful. $O(n^2)$ bits of memory.

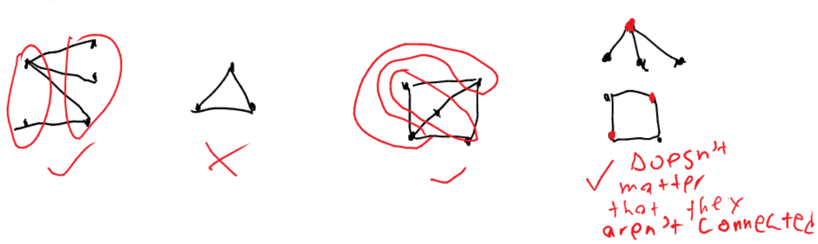
- For every vertex v we keep a list of all edges (u,v) incident to v
- Pros: easy to find the neighbors
  - Doesn't take much memory if the graph is sparse
- Cons: Takes $O(n)$ to see if u is adjacent to v.

7.3 Bipartites

An undirected graph is called bipartite if we can partition the vetices into two parts R and B such that all the edges are between R and B



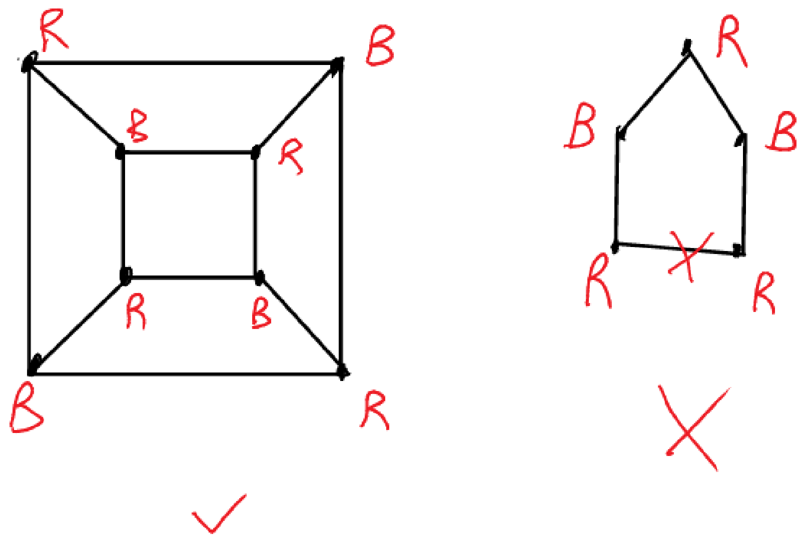
Ex:



7.3.1 Testing for bipartites

How can we test to see if G is bipartite? Label one vertex in R then:

- Look at neighbors to see if they're supposed to be R or B



DFS_Bipartitite(G)
for every vertex u in G **do**
  if u is not explored **then**
    color[u] = "R"
    mark u as explored
    DFS(u)
  end if
end for
if not declared "non-bipartite" yet **then**
  declare "bipartite"
end if

---

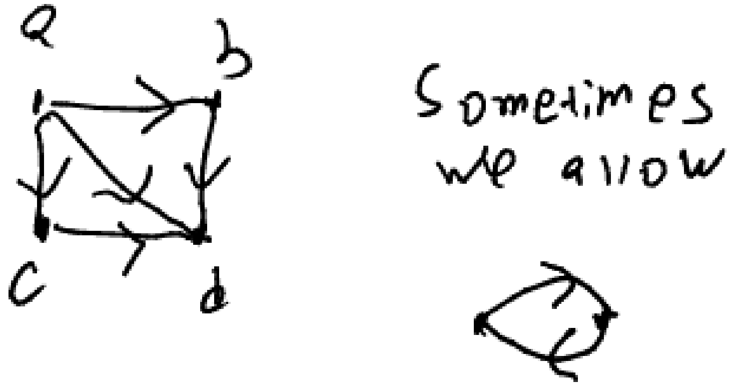
for each edge (u,v) **do**
  if v is not explored **then**
    Mark v as explored
    color v differently from color[u]
    DFS(v)

else if color[u]=color[v] **then**
declare "non-bipartite"
end if
end for

This is called proper two coloring of a graph.

7.4 Directed Graphs

Every edge has an orientation.



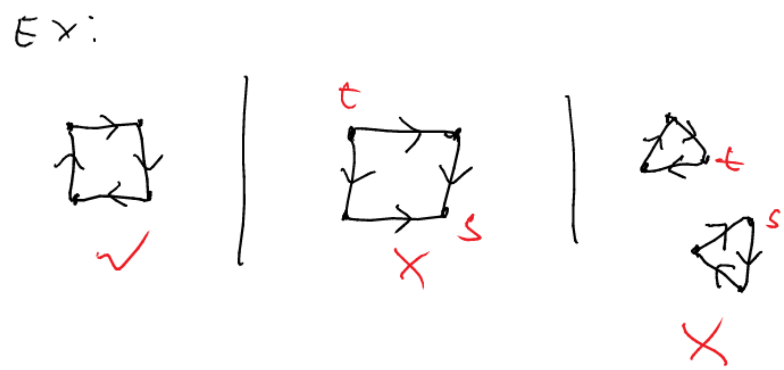
7.4.1 Data Structure:

For every vertex keep two lists: the edge going out, the edges coming into that vertex Given two vertices, s and t, is there a path from s to t?

- say s=a t=d
- yes in the graph
- But there is no path from d to a.

We can use the "directed" version of DFS to solve this problem: We run DFS(s) if t is explored then such a path exists otherwise it doesn't.

Def: A directed graph is called strongly connected if for every u and v there is a path from u to v. (can go from anywhere to anywhere)



Q: Given G, how can we tell if it is strongly connected?

- Pick a vertex s
- Run DFS(s) in G
- If there is any unexplored vertex then "not strongly connected"
- Run DFS(s) in G^{rev} (same as G, but with directions reversed)
- If $\exists$ any unexplored vertex then "not strongly connected"
- Otherwise declare "G is strongly connected"

8 Lecture 8 *<2017-10-03 Tue>*

8.1 Directed Graphs

- Each edge has a direction (seen last class)
- Not symmetric, edge from u to v means no edge from v to u.

8.1.1 Graph search

- Directed reachability
  - Find all nodes reachable from a given node
- Directed s-t shortest path problem

  - Given two nodes, what is length of shortest path between them

- BFS extends naturally to directed graphs
- Web crawler
  - Start from web page s. Find all web pages linked from s

8.1.2 Strong Connectivity

- Node u and v are **mutually reachable** if there is a path from u to v and also a path from v to u.
- A graph is **strongly connected** if every pair of nodes is mutually reachable.

1. Lemma Let s be any node. G is strongly connected iff every node is reachable from s, and s is reachable from every node.
   - Proof: $\implies$ Follows from definition
   - $\impliedby$ Path from u to v: concatenate u-s path with s-v path
     - Path from v to u: concatenate v-s path with s-u path
2. Algorithm
   (a) Theorem Can determine if G is strongly connected in $O(m+n)$ time.
   Proof:
   - Pick any node s
   - Run BFS from s in G
   - Run BFS from s in G^{rev}
   - Return true iff all nodes reached in both BFS executions
   - Correctness follows immediately from previous lemma
   - Has running time of BFS $O(m+n)$

8.2 Directed Acyclic Graphs

- A **DAG** is a directed graph that contains no directed cycles
  - Good for modeling dependencies, like a course's prerequisites
- Ex. Precedence constraints: edge (v_i, v_j) means v_i must precede v_j.

    - Precedence constraints imply no cycle

- A **topological order** of a directed graph $G = (V, E)$ is an ordering of its nodes as $v_1, v_2, \dots, v_n$ so that for every edge (v_i, v_j) we have $i < j$

8.2.1 Lemma

If G has a topological order, the G is a DAG.

Proof (by contradiction)

- Suppose G has a topological order $v_1, \dots, v_n$ and that G also has a directed cycle C.
- Let v_i be the lowest-indexed node in C and let v_j be the node just before v_i: thus (v_j, v_i) is an edge.
- By our choice of i, we have $i < j$
- On the other hand, since (v_j, v_i) is an edge and $v_1, v_2, \dots, v_n$ is a topological order, we must have a contradiction. ↯

8.2.2 Lemma

If G is a DAG, then G has a node with no incoming edges.

Proof (by contradiction)

- Suppose G is a DAG and every node has at least one incoming edge.
- Pick any node v, begin following edges backward from v. Since v has at least one incoming edge (u, v) we can walk backward to u.
- Since u has at least one incoming edge (x, u) we can walk backward to x
- Repeat until we visit a node, say w, twice.
- Let C denote the sequence of nodes encountered between successive visits to w. C is a cycle. ↯

8.2.3 Lemma

If G is a DAG, then G has a topological ordering.

Proof (by induction on n)

- Base case: true if $n = 1$

- Given DAG on $n > 1$ nodes, find a node v with no incoming edges
- $G \setminus \{v\}$ is a DAG, since deleting v cannot create cycles
- By inductive hypothesis, $G \setminus \{v\}$ has a topological ordering.
- Place v first in topological ordering: then append nodes of $G \setminus \{v\}$ in topological order. This is valid since v has no incoming edges.

1. Algorithm To compute a topological ordering of G
   - Find a node v with no incoming edges and order it first
   - Delete v from G
   - Recursively compute a topological ordering of $G \setminus \{v\}$ and append this order after v
   - Running time: $O(n)$ for each call, calling exactly n times. So algorithm runs in $O(n^2)$. Lots of running time if the graph is sparse, not many edges. If we reimplement this more carefully, we can get $O(m+n)$, with m being the number of edges. Note that making an algorithm run faster usually requires more space.'
2. Theorem Algorithm finds a topological order in $O(m+n)$ time

   Proof:
   - Maintain the following information:
     - $count[w]$ = remaining number of incoming edges
     - S = set of remaining nodes with no incoming edges
   - Initialization: $O(m+n)$ via single scan through graph.
   - Update: to delete v
     - Remove v from S
     - Decrement $count[w]$ for all edges from v to w and add w to S if $count[w]$ hits 0
     - This is $O(1)$ per edge

9 Lecture 9 *<2017-10-05 Thu>*

9.1 Greedy Algorithm

- In every step, it tries to be myopic and optimize its current goal/step
- Doesn't care about the future

9.1.1 Interval scheduling

- Have a class room and a microscope
- Every request has a starting time and finishing time $\{1, \ldots, n\}$, (s_i, f_i)
- Def. i and j are compatible $(i + j)$ when $f_i \leq s_j$ or $f_j \leq s_i$
- Subset of requests is compatible if every point of requests are compatible.
- Maximum sized compatible subset is the optimal subset

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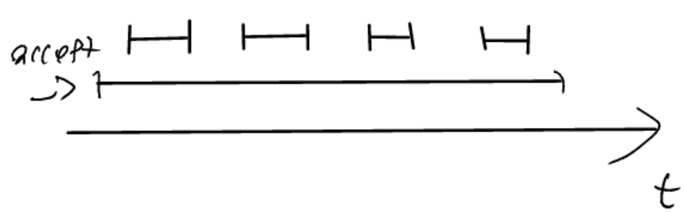
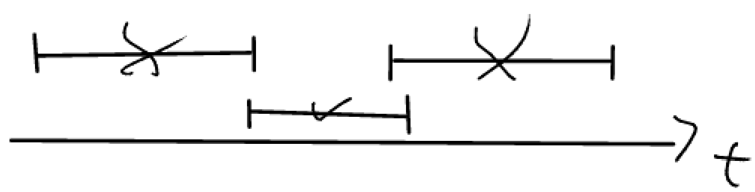
1. Pick $s(i)$ with earliest request
    - Might not give an optimal solution if the request that begins the earliest goes until the end, not allowing any of the other requests to be fulfilled.

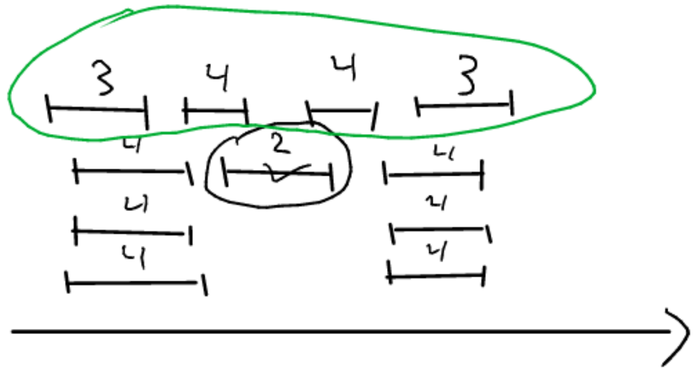


2. $f(i) - s(i)$ is the smallest
    - Can be problematic if the smallest is in between 2



3. For each request compute the # of requests it overlaps with. Pick the one with the smallest number.

- Still problematic



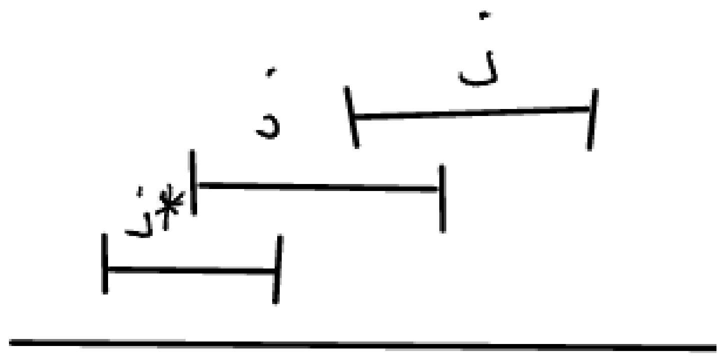
4. Accept (greedy rule) requests i for which $f(i)$ is the smallest.
    - Sort requests so that $f(i_1) \leq f(i_2) \leq \ldots \leq f(i_n)$
    - This one works

$A = \emptyset$
for $j = 1$ to n **do**
    if j is compatible with A **then**
        $A \leftarrow A \cup \{j\}$
    end if
end for
return A

9.1.2 TODO clean up this section

Running time of method 4:

- Sort : $O(n \log n)$
- $f(j) \geq f(j^*) \forall i \in A, f(i) \leq f(j^*) \leftarrow O(n)$



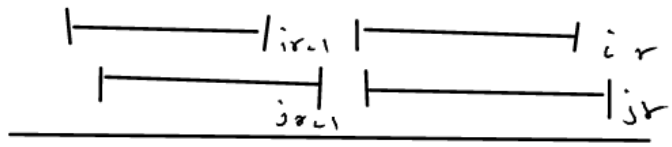
9.1.3 Theorem

This greedy algorithm returns the optimal subset.

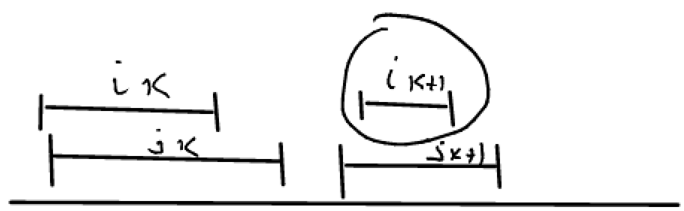
$$\underbrace{|A|}_{\text{optimal}} = |O| - \text{optimal subset}$$

- "stays ahead"
- $|A| = k, |O| = n$, assume $k < m$
- O is ordered by their starting and finishing time for every $j \in O$, $f(i_1 \in A) \leq f(j)$

1. Lemma. For all $r \leq k$, $f(i_r) \leq f(j_r)$
   (a) Proof
      - $r = 1 f(\underbrace{i_1}_{A}) \leq f(\underbrace{j_1}_{O})$ Works
      - This greedy algorithm returns the optimal subset $r - 1$ i.e. $f(i_{r-1}) \leq f(j_{r-1})$.
      - But this contradicts $f(i_r) \geq f(j_r) \implies f(i_r) \leq (j_r)$



- $A : i_1 \dots i_k$
- $O : j_1 \dots j_k j_{k+1} \dots$
- Apply the lemma with $r = k$ so $f(i_k) \leq f(j_k)$



This contradicts $k < m$! Thus $m = k$

- Sort O by starting time, it's also sorted by finishing time

9.1.4 Satisfying requests

Given requests, how many resources do we need to satisfy all of them?

- Def. depth is the maximum number of requests that have a common point in the time line.

1. Claim The # of resources is at least d. $\$\backslash \mathrm{I}_1, \ldots, \mathrm{I}_d\backslash\$$- requests with depth d.

10 Lecture 10 *<2017-10-10 Tue>*

10.1 Recall

Interval scheduling

- Input: Lectures s_j, f_n (start and finish) $j = 1, \dots n$

- Goal: Find the largest non-overlapping set.
- Alg: Always pick the job with earliest finish time.

10.2 Partition scheduling

Now we really want to accommodate all these jobs. How many rooms/resources do we need?

- Input: Same as above
- Goal: Smallest number of rooms that can accommodate all the lectures.

10.2.1 Greedy Template

Consider lectures in some **natural order**. Assign each lecture to an available room (how?). If none is available open a new room.

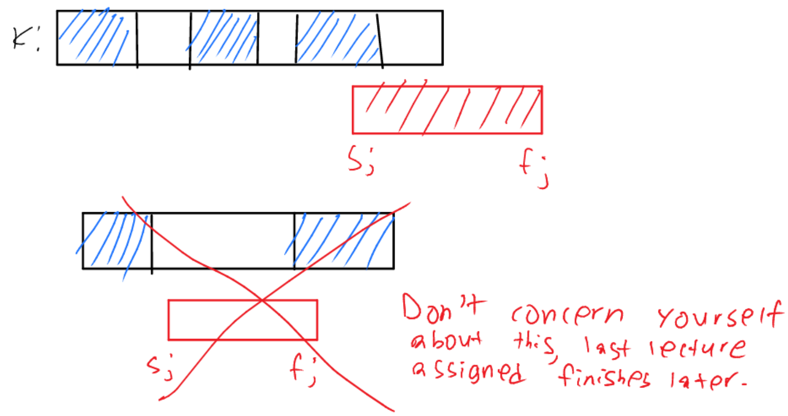
Earliest-Start-Time-first

- (n, s_1, ..., s_n, f_1, ..., f_n)

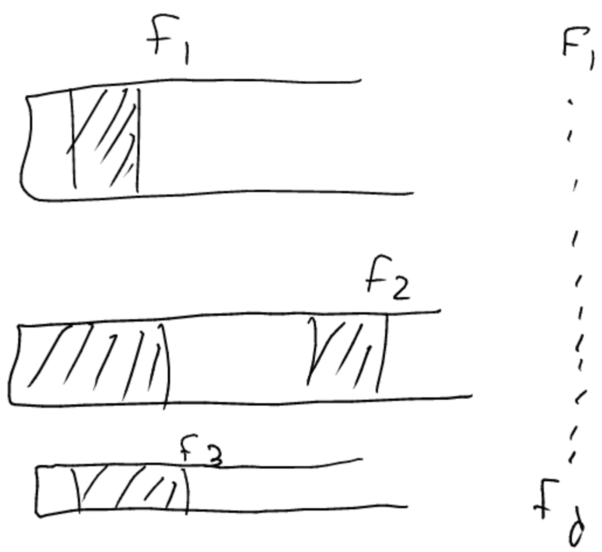
Sort the lectures so that $s_1 \leq s_2 \leq \ldots \leq s_n$
$d = 0$ (number of rooms)
for j = 1,. . . , n **do**
  if lecture j is compatible with room k **then**
    Assign j to room k
  elseAssign j to room $d + 1$
    set $d = d + 1$
  end if
end for

Now that we have the algorithm, we need to analyze its correctness and its running time.

1. Running Time
   - Sorting: $O(n \log n)$
   - For loop runs n times. Each time we check if a lecture is compatible with a room, so we must do this fast.
   - To see if lecture j is compatible with a room k we only need to compare s_j with the finishing time of the last lecture assigned to that room. (Since we know that none of the lectures in the room start after time s_j)



So for each room we keep a variable which tells us when the room becomes available.



We need to see if $s_j > \min(F_1, \ldots, F_d)$. (How to do this quickly? Priority queue.)

if Yes **then** The room with minimum F_k is available
else Open a new room
end if

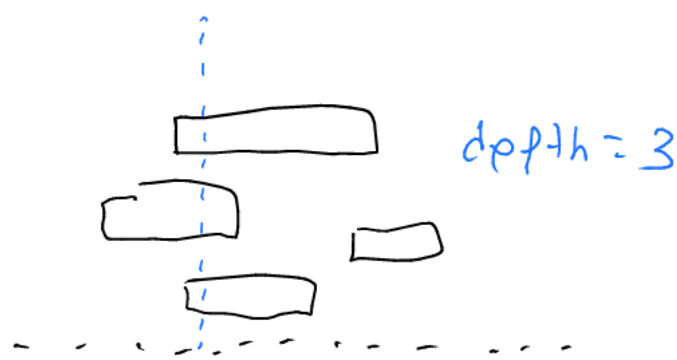
This is the priority queue problem: Always want to know the minimum (we can add or delete numbers from the list). Using a heap this can be implemented so that all insertions and deletions can be done in $O(\log n)$

Running Time:

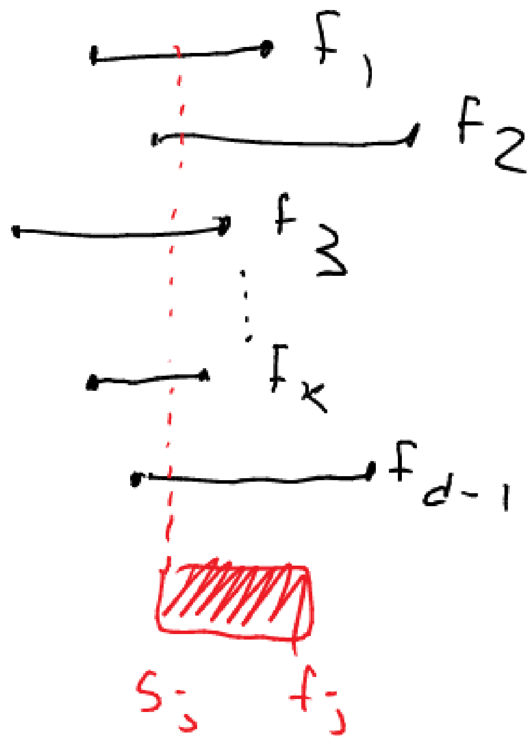
- $\underbrace{O(n\log n)}_{\text{Sort}} + \overbrace{n}^{\text{For loop}} \times \underbrace{O(\log n)}_{\text{Priority queue}} = O(n\log n)$

2. Correctness Why does this alg output the best solution?

   - Depth: Max number of intervals that contain any point on the timeline



   - Obviously: Optimal $\geq$ depth
   - Claim: When the algorithm opens a new room d then depth $\geq$ d
   - Proof: Room d is opened since lecture j was incompatible with $d-1$ other rooms.



In this case every room $1, \ldots, d-1$ has a lecture that ends after s_j (and starts before s_j, due to the way the algorithm works). These together with j show depth $\geq d$

What do we know? depth $\geq$ Output of algorithm d (just showed) $\geq$ optimal $\geq$ depth (earlier) $\implies$ depth = optimal = output of alg

10.3 Minimizing Lateness

- Input: n tasks.
  - Processing times: $t_1, \ldots, t_n$
  - Deadline: $d_1, \ldots, d_n$
- Goal: We have a single processor. Ideally we want to schedule all tasks so that they all finish before their deadlines.

Each task will be scheduled for some time $s_j = f_j - t_k$ to f_j (to finish at f_j we need to start at $f_j - t_k$).

- Lateness = $\$\mathrm{max}_\mathrm{j}\{\mathrm{f}_\mathrm{j}\text{ - }\mathrm{d}_\mathrm{j}\}$ (Time we finish - deadline for job)
- Goal: Minimize the lateness

10.3.1 Greedy Template

Sort the jobs according to some order and assign them to the processor according to this order.

Shortest job first?

- This doesn't work.

Process Time	Deadline
1	100
10	10

Optimal is $f = 10, f = 11$. But this alg gives us $f = 1, f = 11$.

Smallest slack $(d_j - t_j)$ first. But this might give us huge lateness.

t	d
1	2
10	10

Optimal: $f = 1, f = 11$, lateness = 1

Alg: $f = 10, f = 11$, lateness = 9

10.3.2 Optimal Alg

Sort by the deadline: $d_1 \leq d_2 \leq \ldots d_n$

  Set f $\leftarrow$ 0
  for $i = 1 \ldots n$ **do**
      Assign job j to $[f, f + t_j]$
      $f = f + t_j$
  end for

Running time: $O(n \log n)$ (sort)

Why is this optimal? Suppose the optimal is not sorted according to deadlines. Then we will have i and j:

What will switching these two jobs do? It can only improve the lateness.

10.4 Midterm

Next class is the midterm, will be split into 2 rooms.

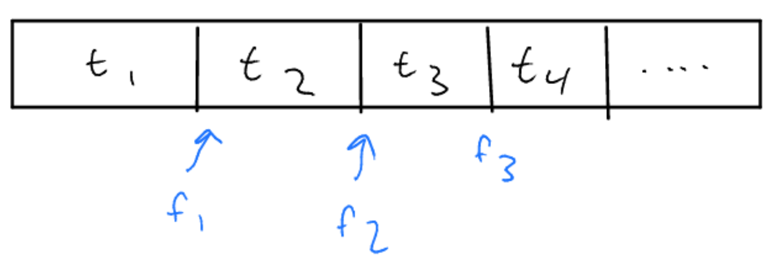
- Topics: Everything until today
- Format: Similar to assignments, 3-4 questions like on the assignments
- No crib sheets

11 Lecture 11 *<2017-10-17 Tue>*

- Recall: Minimize Lateness
  - Input: Jobs $(t_i, d_i), i = 1, \ldots, n$, where t_i is the process time and d_i is the deadline.
  - In which order should we proceed them in order to minimize
  - Lateness $= max_i f_i - d_i$
    - Where f_i is the finishing time of job i

Greedy alg: Process these jobs in increasing order of their deadlines

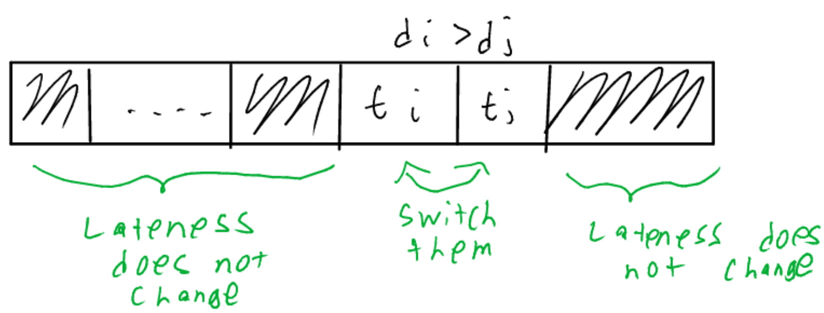
- (Earliest deadline first)
- Sort $d_1 \leq d_2 \leq \ldots \leq d_n$



How do we show that this is optimal?

- Most greedy algorithm proofs are similar, start with the optimal solution and then show that the algorithm keeps with it

Consider an optimal solution. If different from the output of the algorithm (not sorted), then we can find 2 jobs that are not sorted in order of deadline



- How does the lateness of the jobs we switch change?
- $f_i = T + t_i \to f_i' = T + t_i + t_j$
- $f_j = T + t_i + t_j \to f_j' T + t_i$
- f_j' has better lateness than before (smaller lateness), but f_i' lateness might increase
  - new lateness $= T + t_i + t_j - d_i$
- Why won't this increase lateness? This is smaller than the original lateness of the j^{th} job $= T + t_i + t_j - d_j$ (since d_i is larger than d_j)

A different way of writing this proof: Among all optimal solutions pick the one that agrees with the greedy algorithm for the longest period.

11.1 Optimal Caching

(Very complicated)

- Cache with some capacity to store items
- If someone requests an item that we have in the cache, we can show it to them
- If they request something we don't have in the cache, then we have to remove it from the cache
- Sequence of m requests: $d_1, d_2, \dots, d_m$

- Cache hit: The item is in the cache.
- Cache miss: Item not in cache when requested. (Must bring the item to the cache and evict some existing item) This is a costly operation.
- We want to make the cache optimal given the schedule beforehand
- We assume that we start with a full cache.

Example: $k = 2$, initial cache |a|b|

- Requests:

	cache	
1 ✓ a	ab	
2 ✓ b	ab	
3 miss × c	cb	$a \leftarrow c$
4 ✓ b	cb	
5 ✓ c	cb	
6 miss × a	ab	$c \leftarrow a$
7 ✓ a	ab	
8 ✓ b	ab	

We managed to do this one with 2 cache misses. How do we optimize this?

Greedy Alg: Evict the item that is needed farthest in the future. In the above example, in step 3, we see that a is needed in step 6 but b is needed in step 4, so we evict a.

Example: cache abc

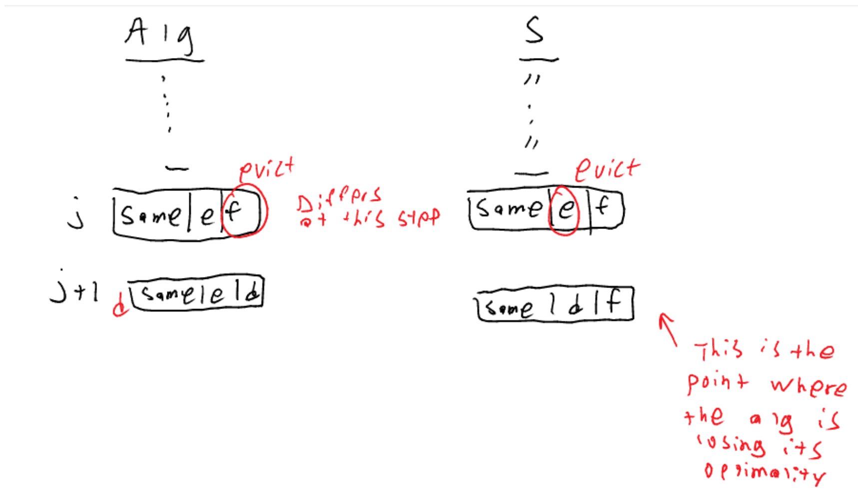
1 ✓a	abc	
2 ✓b	abc	
3 ✓c	ab**c**	
4 × d	abd	$c \leftarrow d$
5 ✓a	abd	
6 ✓d	a **b** d	
7 × e	aed	$b \leftarrow e$
8 ✓a	aed	
9 ✓d	aed	
10 × b	bed	$a \leftarrow b$
11 × c	ced	$b \leftarrow c$

Can do anything for steps 10 & 11, but are steps 4 and 7 unique? No, we can do $b \leftarrow d$ at step 4 instead. So the greedy algorithm is one solution, but it isn't the only solution, making it harder to prove.

Reminder: We will assume that we only evict items if there is a request that is not in the cache. (won't preemptively remove something)

- Read the book: There is no disadvantage in doing this

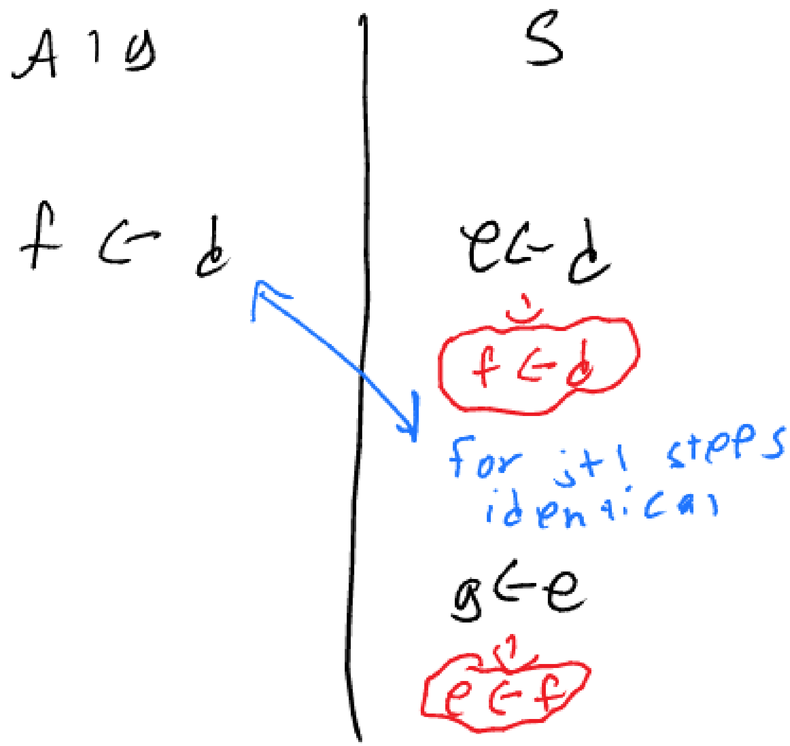
Proof: Among all the optimal solutions, pick the one that agrees with our algorithm for the longest period (assuming they all diverge eventually), call it solution S.



- From the algorithm, we know that e is requested earlier than f, say at step n.

- As for the optimal solution, at step n it must have e. Let t be the first time after j that S has $g \leftarrow e$ for some g.

  - t cannot be later than n so $t \leq n$
  - How do we satisfy t without increasing the number of cache misses in S and making the solution closer to our algorithm? Evict f at j instead of e



So for the proof, either assume that there's an optimal solution that remains stays the same for j steps and reach a contradiction showing that it is the same for $j+1$ steps or show that it keeps going on

12 Lecture 12 *<2017-10-19 Thu>*

12.1 Shortest Path in Graphs

- Input: Directed graph $G = (V, E)$, source s, destination t
- $\forall e, \ell e =$ length of edge e
- Goal: Find the length of the shortest path from $\underline{s}$ to $\underline{t}$.

12.2 Dijkstra's Algorithm

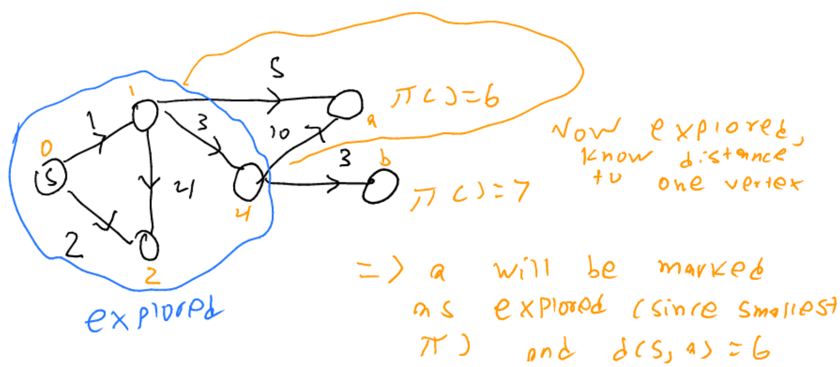
It will find shortest paths from s to all the other nodes in one go.

- Idea: We keep a list of all vertices (initially includes source)

- We already know the lengths of the shortest paths from s to all the explored vertices
- At the next step we choose the vertex with smallest

$$\pi(v) = \min_{\ell=(u,v)_{\text{u is explored}}} d(s,u) + \ell e$$

and mark that as explored and set $d(s,v) = \pi(v)$

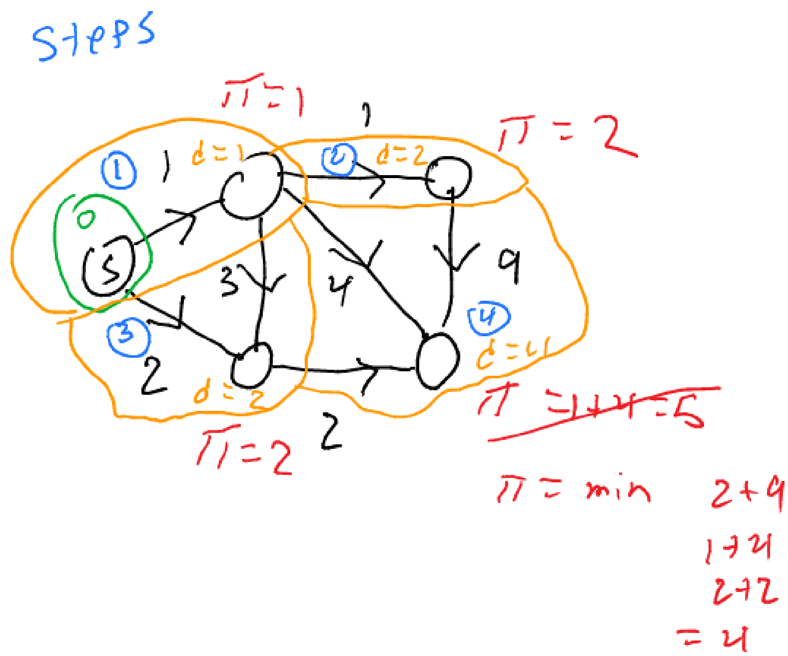


Alg: S = set of explored vertices

- $d(u)$ = distance from s to u for explored u

set $S = \{s\}, d(s) = 0$
while $S \neq V$ **do** choose $w \in V - S$ with minimum $\pi(w) = \min_{\ell=uw, u \in S} d(u) + \ell e$
  $S \leftarrow S \cup \{w\}$
  $d(w) = \pi(w)$
end while

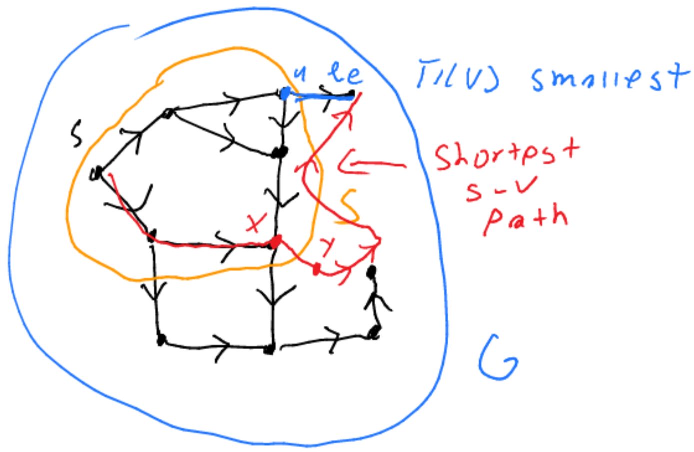
- Example:



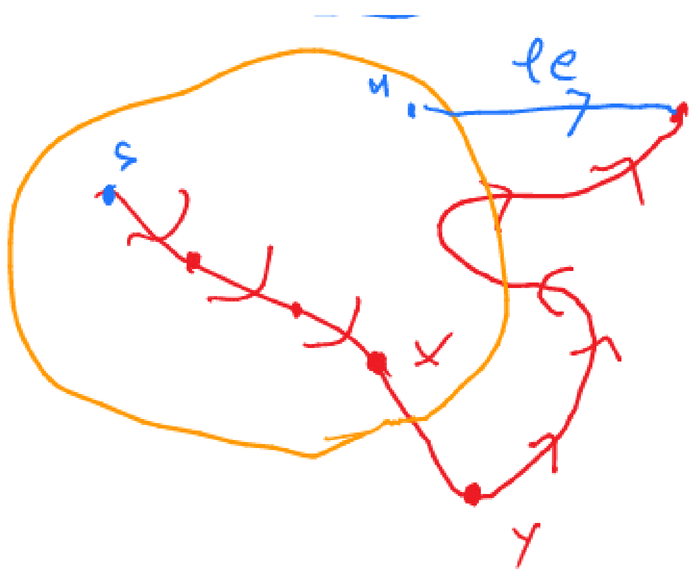
12.2.1 Correctness

Claim: During the execution of the algorithm for every $u \in S$, $d(u)$ is the length of the shortest path from s to u

- Proof: We use induction on size of S.
  - Base: Trivial, $S = \{s\}, d(s) = 0$
  - Induction Hypothesis: The claim remains true after adding next v.



- If $\pi(v)$ is not the length of the shortest $s - v$ path
  * Consider the shortest $s - v$ path on the red path
  * Consider first vertex y outside S on the path. Let x be the previous vertex.



- $\pi(y) \leq d(x) + \ell xy \leq$ length of the red path $< \pi(v)$ (because we assumed $\pi(v)$ is not the shortest path from s to v)
- Contradiction as we assumed $\pi(v)$ was the smallest (we want to

pick smallest π outside of explored area and we showed that $\pi(y)$ is clearly smaller)

Runtime of implementation

set $S = \{s\}, d(s) = 0$

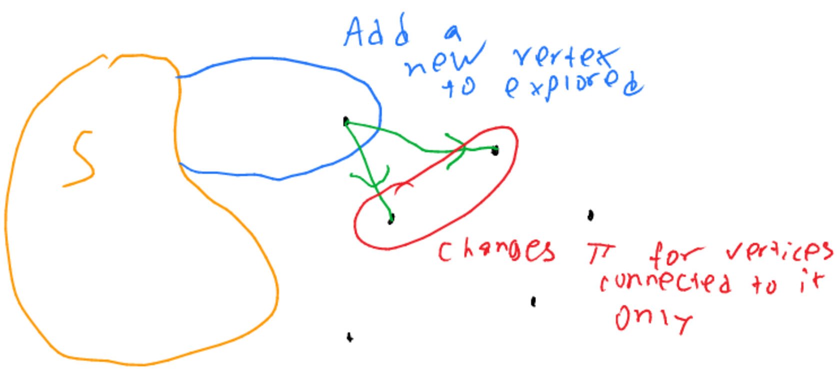
while $S \neq V$ **do** choose $w \in V - S$ with minimum $\pi(w) = \min_{\ell=uw, u \in S} d(u) + \ell e$

$S \leftarrow S \cup \{w\}$

$d(w) = \pi(w)$

end while

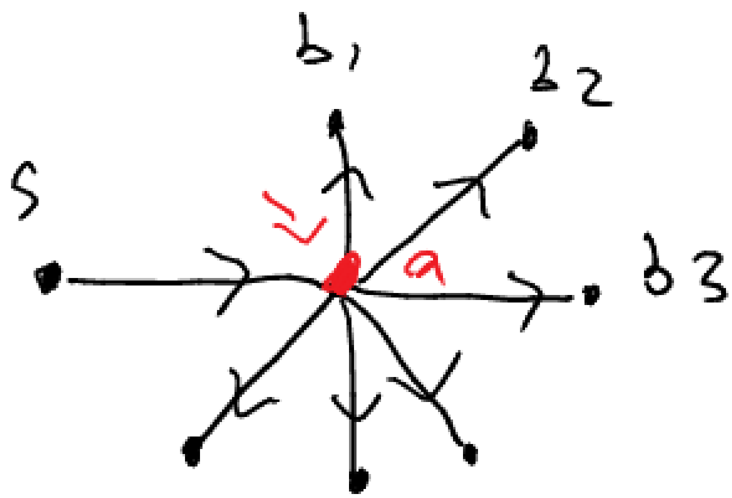
- While: $|V| = n$ iterations
  - Computing $\pi(w) \forall w \in V - S \rightarrow O(m) \rightarrow O(mn)$ after multiplying loop iterations
    - Might be costly to calculate one π, as we can have many incoming edges to a vertex, up to m incoming edges
  - Taking their min



When we add v to S we onlny need to update the pi value for all $w \in S - v$ with $vw \in E$

- If we use a binary heap to implement a priority queue for π values then
- Finding $\min \pi : O(\log n)$
  - (Extracting min from a binary heap)

- Updating the key ($\pi - value$) for all $w∈ V-S with $vw \in E$: Updating each one at these w's costs $O(\log n)$ (either heapify-up or heapify-down, depending on if we're increasing or lowering key)
  - $n \log n$ since a vertex might have linear amount of outward eges to unvisited vertices
- Note that each edge $vw \in E$ is causing at most one of those updates. It will never be visited again. Therefore total # of these key updates is at most $m = |E|$



So all these updates cost $O(m \log n)$

- Binary heap implementation $O(m \log n + n \log n)$
- Fibonacci Heap: $O(m+n \log n)$ (Won't be looking at this in this course as it's much more complicated)

13 Lecture 13 *<2017-10-24 Tue>*

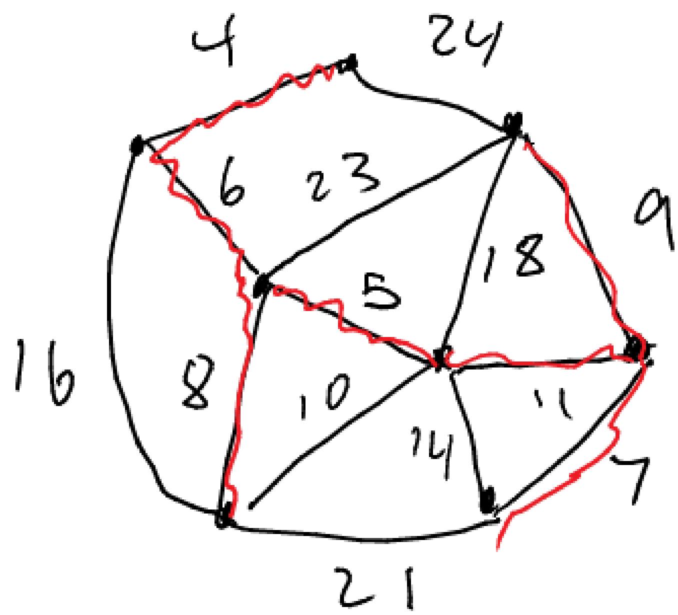
13.1 The Minimum Spanning Tree Problems (MST)

- Input: Undirected Connected Graph $G = (V, E)$

- To every edge <u>e</u> a positive cost $c_e > 0$ is assigned

- Goal: Find a spanning tree in G (i.e. a tree that includes all the vertices of G) with minimum cost.

$$cost = \sum_{e \text{ is an edge of the tree}} c_e$$



$$cost = 4 + 6 + 5 + 8 + 11 + 9 + 7$$

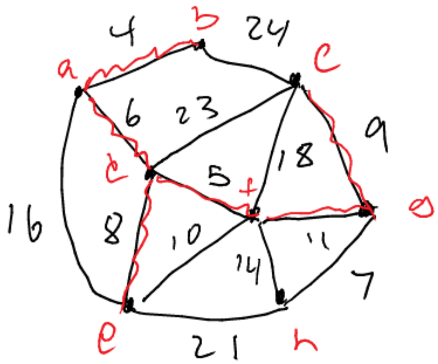
- Why not check all the spanning trees? Very costly.

- Cayley's Thm: Complete graph on n vertices have n^{n-2} spanning trees

- So checking all the spanning trees requires exponential time $\Omega(n^{n-2})$

13.1.1 Three Greedy Algorithms:

- <u>Kruskal</u>: Start with $T =$. At each step add the edge with minimum cost that does not create a cycle until we find a spanning tree (i.e. $n-1$ edges are added)



$T = \emptyset$

$T = \{ab\}$

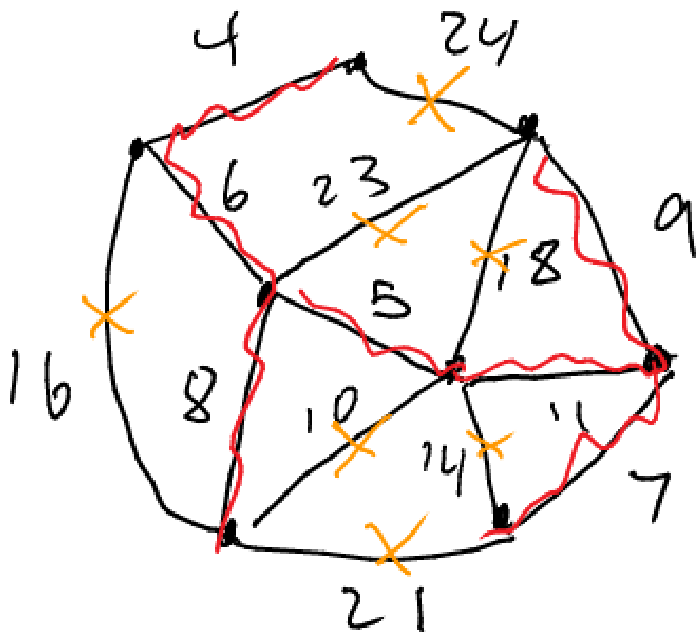
$T = \{ab, df\}$

$T = \{ab, df, ad\}$

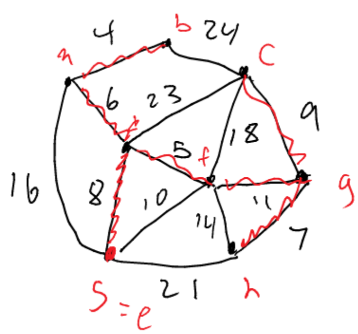
$\vdots$

Don't add ef
since it makes
a cycle

- Reverse Deletion: Now start with $T = E$ (all the edges). At every step we remove the most expensive edge from T that does not disconnect it until we arrive at a spanning tree



- Prims: Start with a node s (root) and greedily grow a tree from s outward by adding the cheapest edge that leaves T.



$T = \{ed\}$

$T = \{ed, df\}$

$T = \{ed, df, ad\}$

$T = \{ed, df, ad, ab\}$

$T = \{ed, df, ad, ab, fg\}$

$T = \{ed, df, ad, ab, fg, gh\}$

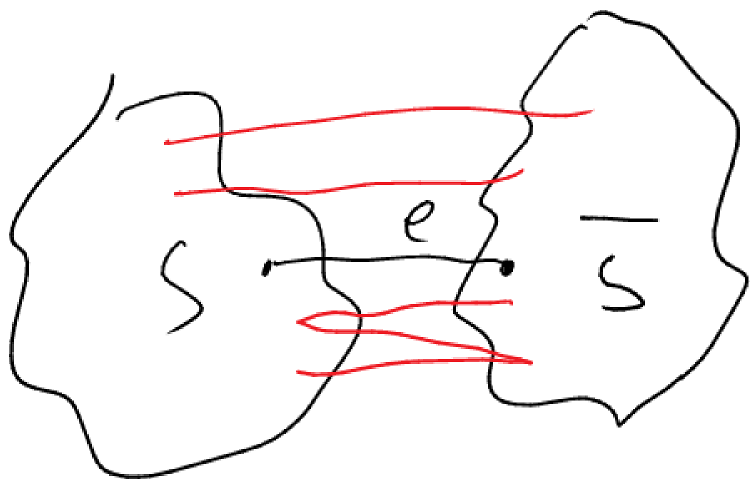
$T = \{ed, df, ad, ab, fg, gh, gc\}$

13.1.2 Correctness

Why do they all find the MST?

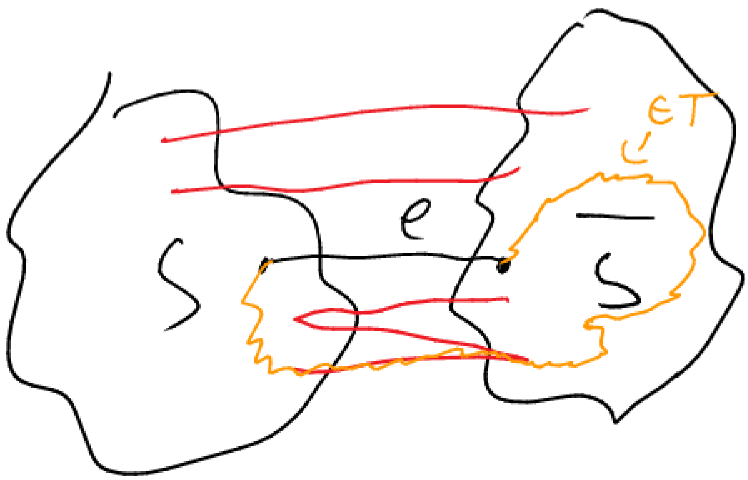
- Simplifying assumption: we assume that all c_e are different (just to simplify the presentation of the proof)

Cut Property: Let S be a subset of nodes and $\underline{e}$ be the minimum cost edge from S to $\overline{S}$. Then $\underline{e}$ is in every minimum spanning tree.



Proof: Suppose not. Let T be an MST that does not include $\underline{e}$

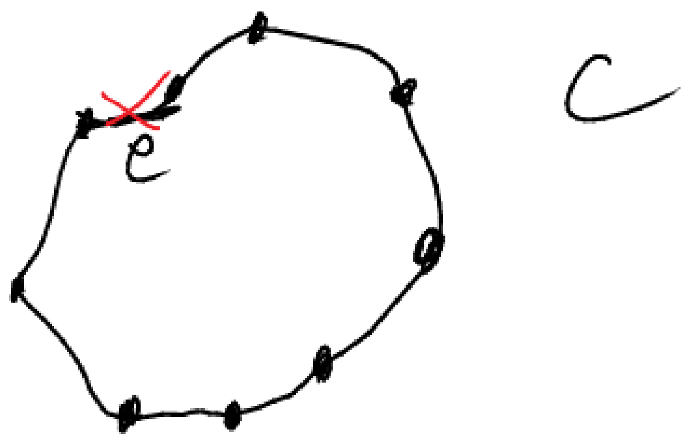
- Consider the path that connects u to v in T.



Pick an edge on this path that goes from S to $\overline{S}$ and replace it with e. Thus way we find another spanning tree with smaller cost. This contradicts the assumption that T is a MST.

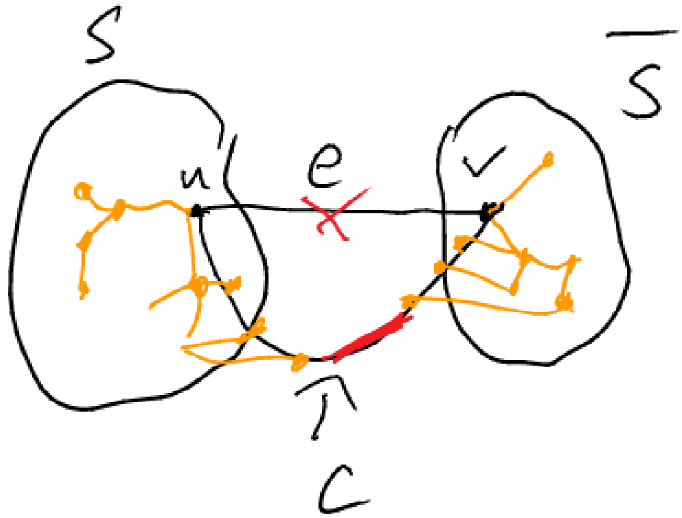
- Why am I allowed to do this? Why doesn't it create a cycle?
  - Can this create a cycle? If adding e made a cycle, then we had a cycle in the original T.
  - If there were two paths from u to v (such that adding e makes a cycle), then T already had a cycle.

Cycle Property: Let C be a cycle in G and let e be the most costly edge on this cycle. Then e does not belong to any spanning tree.



Proof: Suppose not. There is a MST "T" that contains e.

- Remove e from T. This will break T into two components S and $\overline{S}$.



Since C is a cycle it crosses the cycle at some other edge e'. Adding e' instead of e creates a better spanning tree. A contradiction!

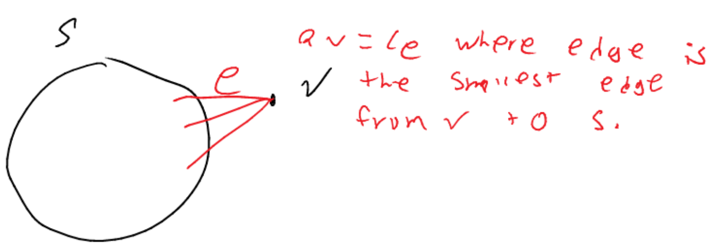
Prims: Each time add the smallest edge from T to the rest of the graph (starting from a root s).

- Theorem: If all costs are different then there's a unique MST and Prims Alg finds it.
- Pf: Consider a step of the alg. Let S be the component of the current T.

Prim's alg picks the smallest edge e between S and $\overline{S}$ and adds it to T. By cut property e is in every MST. So our alg indeed only picks edges that are in every MST. This finishes‘ the proof of the theorem.

Implementation: We maintain S (initially $S =$)

- For each $v \notin S$ maintain an attachment cost.
- $a_v =$ The cost of the cheapest edge from S to v



- At each step we add the vertex with smallest attachment cost to S and update attachment costs.

- Using a priority queue seems like a good idea.

Running time of updating attachment costs once v is added to S. Only vertices in $\overline{S}$ with edges to v need updates: There are $\leq deg(v)$ of these vertices w. If we use a binary heap to keep attachment costs then the updates have running time $deg(v)$.

$$dev(v) \times O(\log_n)$$

- $\log_n$ for updating key in binary heap
- So total running time:

$$\sum_{v \in V} deg(v))(\log n) = O(\log(n)) \times \sum_{v \in V} deg(v) = O(m \log n)$$