# MATH 222: Calculus III Review

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Adapted from Alexander Garver's Winter 2017 MATH222 lecture notes.

## Contents

# 1 Series

$n^{th}$ partial sum of a sequence. $a_n$, the terms of the series, must tend to 0, or else the series **diverges**.

## 1.1 Special Series

**Harmonic** $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

**Geometric** $\sum_{n=1}^{\infty} r^n$ converges to $\frac{1}{1-r} \iff -1 < r < 1$.

## 1.2 Tests

**Alternating Series Test/Leibniz Test** Sequence $a_1, a_2, \ldots$ is **decreasing** and has limit 0. Then $\sum_{n=1}^{\infty}(-1)^n a_n$ converges. In other words, absolute value of the alternating series forms a convergence sequence.

**Absolute Convergence Test** $\sum_{n=1}^{\infty} |a_n|$ converges $\implies \sum_{n=1}^{\infty} a_n$ converges.

**Ratio Test** Suppose $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = r$.
$r < 1 \implies \sum_{n=1}^{\infty} |a_n|$ converges and
$r > 1 \implies \sum_{n=1}^{\infty} a_n$ diverges.

**Root Test** Suppose $\lim_{n\to\infty} \sqrt[n]{|a_n|} = r$.
$r < 1 \implies \sum_{n=1}^{\infty} |a_n|$ converges and
$r > 1 \implies \sum_{n=1}^{\infty} a_n$ diverges.

**Comparison Test** Suppose fixed number $K$ s.t. $0 < a_n < Kb_n$, $\forall$ sufficiently large $n$.
$\sum_{n=1}^{\infty} b_n$ converges $\implies \sum_{n=1}^{\infty} a_n$ converges.
$\sum_{n=1}^{\infty} a_n$ diverges $\implies \sum_{n=1}^{\infty} b_n$ diverges.

**Limit Comparison Test** Suppose $a_n > 0, b_n > 0$ and $\lim_{n\to\infty} \frac{a_n}{b_n} = R \neq 0$. Then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ **both** converge or **both** diverge.

**Integral Test** Suppose $f(x)$ is **positive** and **decreasing**, $\forall$ large enough $x$. Then the following are equivalent:

1) $\int_1^{\infty} f(x)dx$ is finite, i.e. converges.

2) $\sum_{n=1}^{\infty} f(n)$ is finite, i.e. converges.

The p-test follows from this.

***p*-test** $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges $\iff p > 1$.

**Alternating Series Estimation Theorem** If $s = \sum_{n=1}^{\infty}(-1)^{n-1}b_n$ is the sum of an alternating series that satisfies:
$b_{n+1} \leq b_n$ and $\lim_{n\to\infty} b_n = 0$
then $|R_n| = |s - s_n| \leq b_{n+1}$.

## 1.3 Power Series

Series of the form $\sum_{n=0}^{\infty} c_n x^n$ or $\sum_{n=0}^{\infty} c_n(x-a)^n$

### 1.3.1 Important Power Series to Know

- $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, R = \infty$
- $\sin x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}, R = \infty$
- $\cos x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}, R = \infty$
- $\tan^{-1} x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}, R = 1$
- $\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n-1} \frac{x^n}{n}, R = 1$

### 1.3.2 Convergence

**Theorem 1.** $\sum_{n=0}^{\infty} c_n(x-a)^n$ *does exactly one of the following:*

*(i) Converges only when $x = a$.*

*(ii) Converges for all $x$.*

*(iii) $\exists\ R > 0$ s.t. $|x-a| < R$, the series converges and diverges if $|x-a| > R$.*

R is the **radius of convergence**. The values of $x$ where the series converges is called the **interval of convergence**. Radius of convergence **does not** tell you if endpoints are included, have to check both. Ratio test is usually a good tool to find the radius of convergence.

### 1.3.3 Representing Functions As Power Series

If $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ for some $c_0, c_1, \dots$ then $f'(x)$ and $\int f(x)dx$ can also be represented by a power series.
If $f(x) = \sum_{n=0}^{\infty} c_n(x-a)$ then $c_n = \frac{f^n(a)}{n!}$
Work with a familiar power series: $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $|x| < 1$.

**Theorem 2.** *Suppose $\sum_{n=0}^{\infty} c_n(x-a)^n$ with $R > 0$. Then $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ is differentiable on $(a-R, a+R)$ and*

*1) $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \dots$*

*2) $\int f(x)dx = c + c_0(x-a) + \frac{c_1}{2}(x-a)^2 + \frac{c_2}{3}(x-a)^3 + \dots$*

*The radius of convergence of $f'(x)$ and $\int f(x)dx$ is $R$.*

Otherwise said, you can easily differentiate & integrate series.

**Theorem 3.** *If $f(x)$ has a power series representation $\sum_{n=0}^{\infty} c_n(x-a)^n$ then $c_n = \frac{f^n(a)}{n!}$. Called the $n!$ **Taylor series** of $f$ at $a$.*

How to show a function is represented by a power series?

**Theorem 4.** *Suppose $\sum_{n=0}^{\infty} c_n(x-a)^n$ is the Taylor series of $f(x)$ with $R > 0$.
If $\lim_{n\to\infty}(f(x) - \sum_{i=0}^{\infty} c_i(x-a)^i) = 0$ for $|x-a| < R$, then $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ for $|x-a| < R$.*

# 2 3-Dimensional Coordinate System

$XYZ$ plane.

**Distance Between 2 Points, $P$ and $Q$**

$$|PQ| = \sqrt{(a_2-a_1)^2 + (b_2-b_1)^2 + (c_2-c_1)^2}$$

## 2.1 Vectors

**Definition 1.** A **vector** $\underline{v}$ is a quantity with a **magnitude** and **direction**. Vectors are **equal** if they have the same magnitude and direction. There is a **zero vector**, denoted $\underline{0}$. It has no magnitude or direction.

**Vector Addition**

**Definition 2.** Sum of $\underline{u}, \underline{v}$ denoted $\underline{u} + \underline{v}$ is the vector whose initial point is that of $\underline{u}$ and whose terminal point is that of $\underline{v}$. $\underline{u} + \underline{v} = \underline{v} + \underline{u}$.

**Scalar Multiplication**

**Definition 3.** If $c$ is a **scalar**, i.e. $c \in \mathbb{R}$, then $c\underline{v}$ is the vector whose length is $|c|$ times the length of $\underline{v}$ and whose direction is the same as $\underline{v}$ if $c > 0$ and opposite if $c < 0$. $c = 0 \implies c\underline{v} = \underline{0}$.

**Vectors in Coordinates**
$\underline{v} = \langle a_1, a_2, a_3 \rangle = a_1\underline{i} + a_2\underline{j} + a_3\underline{k}$, where
$\underline{i} = \langle 1, 0, 0 \rangle$
$\underline{j} = \langle 0, 1, 0 \rangle$
$\underline{k} = \langle 0, 0, 1 \rangle$

**Magnitude** The **magnitude** of $\underline{v}$ is $|\underline{v}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$.

### 2.1.1 Dot Product

"Multiplying" vectors.

**Definition 4.** Given $\underline{v} = \langle a_1, a_2, a_3 \rangle, \underline{u} = \langle b_1, b_2, b_3 \rangle$, their **dot product** is defined as:

$$\underline{v} \cdot \underline{u} = a_1b_1 + a_2b_2 + a_3b_3$$

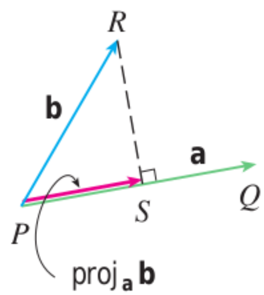
**Theorem 5.** *i)* $\underline{v} \cdot \underline{v} = |\underline{v}|^2$

*ii)* $\underline{v} \cdot \underline{u} = |\underline{v}||\underline{u}| \cos\theta$, *where* $\theta$ *is the angle between* $\underline{v}, \underline{u}$ *with* $0 \leq \theta \leq \pi$

*iii)* $\underline{v}$ *and* $\underline{u}$ *are* ***orthogonal*** *(or* ***perpendicular****)* $\iff \underline{v} \cdot \underline{u} = 0$, *and* $\underline{v} \cdot \underline{u} \iff \theta = \frac{\pi}{2}$

Note that $\underline{v} \cdot \underline{u} > 0 \implies \theta < \frac{\pi}{2}$ (acute) and $\underline{v} \cdot \underline{u} < 0 \implies \theta > \frac{\pi}{2}$ (obtuse).

### 2.1.2 Projections




**Scalar Projection**

**Definition 5. Scalar Projection** of $\underline{v}$ onto $\underline{u}$ is given by: $comp_{\underline{u}}(\underline{v}) = \frac{\underline{u}\cdot\underline{v}}{|\underline{u}|}$

**Vector Projection**

**Definition 6. Vector Projection** of $\underline{v}$ onto $\underline{u}$ is given by: $proj_{\underline{u}}(\underline{v}) = \left(\frac{\underline{u}\cdot\underline{v}}{|\underline{u}|^2}\right)\underline{u}$

### 2.1.3 Cross Product

**Definition 7.** Let $\underline{v}_1 = \langle a_1, a_2, a_3\rangle, \underline{v}_2 = \langle b_1, b_2, b_3\rangle$. The **cross product** of $\underline{v}_1, \underline{v}_2$ is given by $\underline{v}_1 \times \underline{v}_2 = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1\rangle$

Can be obtained from the determinant of:

$$\begin{bmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$$

**Theorem 6.** *i)* $|\underline{v}_1 \times \underline{v}_2| = |\underline{v}_1||\underline{v}_2|\sin\theta, 0 \le \theta \le \pi$

*In fact,* $|\underline{v}_1||\underline{v}_2|\sin\theta$ *is the area of the parallelogram determined by* $\underline{v}_1, \underline{v}_2$

*ii) Two nonzero vectors* $\underline{v}_1, \underline{v}_2$ *are parallel if and only if* $\underline{v}_1 \times \underline{v}_2 = 0$.

## 2.2 Lines

**Equation of a Line**

**Definition 8.** The equation of a line is given by: $\underline{r} = \underline{r}_0 - t\underline{v}$.
Now let $\underline{r} = \langle x, y, z\rangle, \underline{r}_0 = \langle x_0, y_0, z_0\rangle, \underline{v} = \langle a, b, c\rangle$.
The **parametric equations** of the line L passing through $(x_0, y_0, z_0)$ and parallel to $\underline{v} =$

$\langle a, b, c\rangle$ is given by: $x = x_0 + at, y = y_0 + bt, z = z_0 + ct$
Solving for $t$ produces the **symmetric equations** of the line L: $\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$

**Definition 9.** 2 lines are **skew lines** if they are not parallel and do not intersect.

## 2.3 Planes

What determines a plane in 3-D?

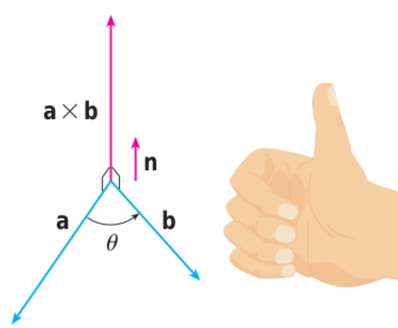
- 3 noncolinear points in the plane.
- 2 nonparallel vectors and a point $p_0$ in the plane.
- a point $p_0$ in the plane and a vector $\underline{n}$ (**normal vector**) that is perpendicular to the plane.

**Definition 10.** Let $p_0 = (x_0, y_0, z_0)$ and $p = (x, y, z)$.
$\underline{n} \cdot (\underline{r} - \underline{r}_0) = 0$ is the **vector equation** of the plane.
$\underline{r} = \langle x, y, z\rangle, \underline{r}_0 = \langle x_0, y_0, z_0\rangle, \underline{n} = \langle a, b, c\rangle$
$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$ is the **scalar equation** of the plane that contains $p_0 = (x_0, y_0, z_0)$ and is perpendicular to $\underline{n}$.
$ax + by + cz + d = 0$ is the **linear equation for the plane**.

**Theorem 7.** *$\underline{v}_1 \times \underline{v}_2$ is orthogonal to $\underline{v}_1$ and $\underline{v}_2$.*

## 2.4 Right-Hand Rule

If the finger of your right hand curl in the direction of rotation from $\underline{a}$ to $\underline{b}$ through $\theta$ ($0^\circ \leq \theta \leq 180^\circ$), then your thumb points in the direction of $\underline{a} \times \underline{b}$.




## 2.5 Vector Functions and Space Curves

### Vector Functions

**Definition 11.** We say $\underline{r}(t) = \langle f(t), g(t), h(t)\rangle = f(t)\underline{i} + g(t)\underline{j} + h(t)\underline{k}$ is a **vector function**.

**Definition 12.** $f(t), g(t), h(t)$ are the **component functions** of $\underline{r}(t)$. The **domain** is the set $t \in \mathbb{R}$ s.t $f, g, h$ are defined at $t$.

**Definition 13.** The **limit** of $\underline{r}$ is defined by taking the limits of its component functions, that is:

$$\lim_{t\to a} \underline{r}(t) = \langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t)\rangle$$

**Definition 14.** A vector function $\underline{r}$ is **continuous** at a if

$$\lim_{t\to a} \underline{r}(t) = \underline{r}(a)$$

$\underline{r}$ is **continuous** at a if and only if $f, g, h$ also are.

**Definition 15.** Let $f, g, h$ be continuous on an interval $I$. Let $C$ be the set of points $(x, y, z)$ satisfying

$$x = f(t), y = g(t), z = h(t) \tag{1}$$

for any $t$ in $I$. We say $C$ is a space curve and the equations given by equation (1) are its **parametric equations**.
We say $t$ is a **parameter**.

## 2.6 Arc Length, Curvature and the TNB Frame

**Definition 16.** The **derivative** of a vector function $\underline{r}(t)$ is given by:

$$\lim_{h\to 0} \frac{\underline{r}(t+h) - \underline{r}(t)}{h} = \underline{r}'(t) = \frac{d\underline{r}}{dt}$$

if it exists.

**Theorem 8.** *If* $\underline{r}(t) = \langle f(t), g(t), h(t)\rangle$ *and* $f, g, h$ *are differentiable, then* $\underline{r}'(t) = \langle f'(t), g'(t), h'(t)\rangle$

**Definition 17.** We say $\underline{r}'(t)$ is the **tangent vector** of $\underline{r}(t)$ at $t$.

### Arc Length

**Definition 18.** Suppose we have a curve given by $\underline{r}(t) = \langle f(t), g(t), h(t)\rangle$ with $a \le t \le b$ and $f', g', h'$ are continuous. The **arc length** is defined as

$$\int_a^b |\underline{r}'(t)| dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt$$

The **arc length function** is given by:

$$s(t) = \int_a^t |\underline{r}'(u)| du$$

**Definition 19.** A **parametrization** of a curve $C$ is a representation of $C$ by a vector function using arc length.

### Curvature

**Definition 20.** A parametrization $\underline{r}(t)$ of $C$ is **smooth** on an interval $I$ if $\underline{r}'(t)$ is continuous and $\underline{r}'(t) \neq 0$ on $I$. A curve $C$ is **smooth** if it has a smooth parametrization.

### TNB Vectors

**Definition 21.** The **unit tangent vector** of $\underline{r}(t)$ is given by

$$\underline{T}(t) = \frac{\underline{r}'(t)}{|\underline{r}'(t)|}$$

The **unit normal vector** of $\underline{r}(t)$ is given by

$$\underline{N}(t) = \frac{\underline{T}'(t)}{|\underline{T}'(t)|}$$

The **binormal vector** of $\underline{r}(t)$ is given by

$$\underline{B}(t) = \underline{T}(t) \times \underline{N}(t)$$

They are all pairwise orthogonal and are of unit length.

**Definition 22.** The **curvature** $\kappa$ of $C$ is the length of the derivative of $\underline{T}(s)$, given by:

$$\kappa = \left| \frac{dI}{dS} \right|$$

$$\kappa(t) = \left| \frac{\underline{T}'(t)}{\underline{r}'(t)} \right| = \frac{|\underline{r}'(t) \times \underline{r}''(t)|}{|\underline{r}'(t)|^3}$$

## 2.7 Velocity & Acceleration

**Definition 23.** Given a curve $C$ denoted by $\underline{r}(t)$, the **velocity** of $\underline{r}(t)$ is given by:

$$\underline{r}'(t) = \lim_{h\to 0} \frac{\underline{r}(t+h) - \underline{r}(t)}{h} = \underline{v}(t)$$

Note that speed is given by $|\underline{r}'(t)| = |\underline{v}(t)|$

**Definition 24.** The **acceleration** of $\underline{r}(t)$ is

$$\underline{a}(t) = \underline{r}''(t) = \underline{v}'(t)$$

**Components of Acceleration**
$\underline{a}(t)$ can be expressed purely in terms of $\underline{T}$ and $\underline{N}$ like so:

$$\underline{a} = \underbrace{v'}_{a_T}\underline{T} + \underbrace{\kappa v^2}_{a_N}\underline{N}$$

One can also show:

$$a_T = \frac{\underline{r}'(t) \cdot \underline{r}''(t)}{|\underline{r}'(t)|}$$

$$a_N = \frac{|\underline{r}'(t) \times r''(t)|}{|\underline{r}'(t)|}$$

# 3 Multi-variable Functions

**Definition 25.** A **function of two variables** is a rule that assigns to each ordered pair of real numbers $(x, y)$ a real number $f(x, y)$ when $(x, y)$ is in the **domain** $D$ of $f$.
**Domain** of $f$ is $D = \{(x, y) : f(x, y) \text{ is defined}\} \subseteq \mathbb{R}^2$
**Range** of $f$ is $\{f(x, y) : (x, y) \in D\} \subseteq \mathbb{R}$
**Graph** of $f$ is the set $\{(x, y, z) \in D \text{ and } z = f(x, y)\} \subseteq \mathbb{R}^3$

## 3.1 Contour Maps

**Definition 26.** We can represent functions $f(x, y)$ by taking horizontal slices of their graphs. These slices indicate height. The slices or **level curves** of $f(x, y)$ are the curves with equations $f(x, y) = k$ where $k$ is a constant in the range of $f$. If we draw the level curves we obtain a **contour map** of $f$.

## 3.2 Level Surfaces

To understand graphs of functions of 3 variables, we draw **level surfaces**.

## 3.3 Limits and Continuity

**Definition 27.** Let $f$ be a function of two variables whose domain $D$ includes points that are arbitrarily close to $(a, b)$. We say the **limit** of $f(x, y)$ as $(x, y)$ approaches $(a, b)$ is $L$:

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

if $\forall\varepsilon > 0\ \exists\delta > 0$ s.t. if $(x, y)$ is in $D$ and $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \implies |f(x, y) - L| < \varepsilon$

**Limit Laws**

**Theorem 9.** *If the limits* $\lim_{(x,y)\to(a,b)} f(x, y)$ *and* $lim_{(x,y)\to(a,b)} g(x, y)$ *exist, then*

*i)* $\lim_{(x,y)\to(a,b)} c(f(x, y)) = c \lim_{(x,y)\to(a,b)} f(x, y)$

*ii)* $\lim_{(x,y)\to(a,b)}(f(x, y) + g(x, y)) = \lim_{(x,y)\to(a,b)} f(x, y) + \lim_{(x,y)\to(a,b)} g(x, y)$

*iii)* $\lim_{(x,y)\to(a,b)} f(x, y)g(x, y) = (\lim_{(x,y)\to(a,b)} f(x, y))(\lim_{(x,y)\to(a,b)} g(x, y))$

*iv)* $\lim_{(x,y)\to(a,b)} \frac{f(x,y)}{g(x,y)} = \frac{\lim_{(x,y)\to(a,b)} f(x,y)}{\lim_{(x,y)\to(a,b)} g(x,y)}$, *where denominator is nonzero.*

**Continuity**

**Definition 28.** A function $f$ is **continuous** at $(a, b)$ if $\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$.
A function $f$ is **continuous** on a set $D$ if it is continuous at each $(a, b)$ in $D$.

**Theorem 10.** $\frac{f}{g}$ *is continuous if* $f, g$ *are continuous.*
*We can also show that polynomials and rational functions are continuous on their domains.*

## 3.4 Partial Derivatives

**Definition 29.** The **partial derivative** of $f(x, y)$ with respect to $x$ is

$$f(x, y) = \lim_{h\to 0} \frac{f(x + h, y) - f(x, y)}{h}$$

In order to evaluate these limits, fix $y$ and differentiate wrt $x$ to obtain $f_x(x, y)$ or fix $x$ and wrt $y$ to get $f_y(x, y)$.

Notation:

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial z}{\partial x} = D_x f$$
$$f_y = \frac{\partial f}{\partial y} = \frac{\partial z}{\partial y} = D_y f$$

**Higher Order Derivatives** We can differentiate $f_x$ and $f_y$ to obtain

$$(f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$
$$(f_x)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{(\partial x)^2}$$
$$(f_y)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$
$$(f_y)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial f^2}{(\partial y)^2}$$

Note that, with $\partial$ notation, we derive from right to left, but with $f_x$ notation we derive from left to right.

**Clairaut's Theorem:**

**Theorem 11.** *Suppose $f$ is defined on a disk $D$ that contains point $(a, b)$. If the functions $f_{xy}, f_{yx}$ are continuous on $D$, then $f_{xy}(a, b) = f_{yx}(a, b)$*

## 3.5 Tangent Planes

Let $f(x, y)$ be a function and let $S$ be the surface $z = f(x, y)$.
$T_1$ : tangent line in $x$-direction at $(x_0, y_0, f(x_0, y_0))$.
$T_2$ : tangent line in $y$-direction at $(x_0, y_0, f(x_0, y_0))$.

**Definition 30.** Define the **tangent plane** to $S$ at $(x_0, y_0, f(x_0, y_0))$ to be the plane that contains both $T_1, T_2$, given by:

$$z = z_0 + a(x - x_0) + b(y - y_0)$$

Its intersection with the plane $y = y_0$ (or $x = x_0$) is $T_1$ (or $T_2$)

$$\implies T_1 = z - z_0 = a(x - x_0), T_2 = z - z_0 = b(y - y_0)$$

**Theorem 12.** *If $f$ has continuous partial derivatives, an equation of the tangent plane to $z = f(x, y)$ at $(x_0, y_0, f(x_0, y_0))$ is*

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

**Approximation Using Tangent Planes**

**Definition 31.** The **linearization** of $f(x, y)$ at $(a, b)$ is defined as

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

The approximation

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called the **linear approximation** or **tangent plane approximation** of $f$ at $(a, b)$.

**Theorem 13.** *If $f_x$,$f_y$ exist near $(a, b)$ and are continuous at $(a, b)$, then $f$ is differentiable at $(a, b)$.*

# 4 How to Solve Problems

## 4.1 Series

**Representing a Function as a Power Series** Look for a familiar function that has a power series representation, plug it in and simplify. Integrate and differentiate as required.

**Finding the Radius of Convergence** Usually involves the ratio test and checking when $r < 1$.

**Finding the Interval of Convergence** Use the radius of convergence and check if end-points converge.

**Finding the Sum of a Series** Look for a familiar series that can be represented as a function.

**Using the Definition of Taylor Series** To find a power series representation or to find first few terms, just derive and use $c_n = \frac{f^n(a)}{n!}$.

**Evaluating an Indefinite Integral with Series** Replace known function by a familiar series, try to cancel out other terms, integrate the series.

**Evaluating a Limit with Series** Same as above for integrating.

## 4.2 Vectors

**Compute Something** Compute what it asks for given the corresponding formula, whether it's the dot product, cross product, projection, etc.

**Angle Between 2 Vectors** Use either the dot product or cross product.

**Values for x Such that 2 Vectors are Orthogonal** Use the dot product and solve for it being 0.

**Finding a Parametric Equation of a Line** Use a point and a direction vector.

**Finding the Equation of a Plane** Use a point and a normal vector.

**Find Where a Line Intersects a Plane** Plug in parameters $(x, y, z)$ from line into the equation of a plane and solve for $t$ and get the corresponding point from the line with that value of $t$.

**Distance from a Line to the Origin** Take $DV = \underline{a}$ and $\underline{b}$ some point on the line (usually $t = 0$). Then $d = \frac{|\underline{a} \times \underline{b}|}{|\underline{a}|}$.

**Are 2 Lines Skew, Parallel or Intersecting?** If $DV$ are multiples of each other, parallel. If you equate each component $x = x, y = y, z = z$ from both lines and you can solve the system, then they intersect. Otherwise, skew.

**Angles Between Planes/Parallel or Perpendicular** To show parallel, compare $NV$. To show perpendicular, use the dot product. If neither, the angle can be computed with the dot product.

**Line of Intersection of Two Planes** Find an intersecting point and use the cross product with both $NV$ to get a $DV$.

**Distance Between 2 Parallel Planes** Given plane equations of the form $ax+by+cz=d$, then distance $D = \frac{|d_1 - d_2|}{\sqrt{a^2+b^2+c^2}}$

**Distance Between a Point and a Plane** $p = (x_1, y_1, z_1)$, then $D = \frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$

**Diagonals of a Parallelogram** Given $\underline{u}$ and $\underline{v}$ that form the sides of a parallelogram, lengths of two diagonals are $|\underline{u}+\underline{v}|$ and $|\underline{u}-\underline{v}|$.

## 4.3 Vector Functions

**Find the Domain of a Vector Function** Check where it isn't defined.

**Limit of a Vector Function** Take the limit of each component.

**Integral of a Vector Function** Take the integral of each component.

**Curve of Intersection Between Cylinder and Plane** If you have a projection of a cylinder onto a circle like $x^2+y^2=16, z=0$, then you can write $x = 4\cos t, y = 4\sin t, 0 \leq t \leq 2\pi$. Take the plane, isolate for $z$ and plug in $x, y$ from circle. Then your vector function is given by $x\&y$ from circle and $z$ from plane with plugged in $x, y$.

**Where does a Curve Intersect a Plane?** $xz$-plane $\implies y=0$, $xy$-plane $\implies z=0$, etc.

**Parametric Equation of a Line at a Certain Point** Get $\underline{r}'(t)$ and plug in $t$ to get $DV$. Can use this $DV$ as a $NV$ for a normal plane to the curve.

**Length of the Curve** Use arc length formula.

**Angle of Intersection of 2 Curves** Get the point where they intersect, then find tangents at those points and use dot product.

**Reparametrizing a Curve** Given a point, get the corresponding $t$ value. Then measure arc length from 0 to $t$ and solve for $t$ wrt $s$ and plug it into arc length formula wrt $t$, getting $r(t(s))$.

**Computing $\underline{T}$ $\underline{N}$ $\underline{B}$, $\kappa$** Use the formulas.

**Particle Velocity, Speed and Acceleration** Compute with formulas, note that speed is $|v(t)|$. Might have to work backwards by integrating if given acceleration and/or velocity to get position, don't forget constant.

**Acceleration and Normal Components of Acceleration Vector** Formulas.

### 4.4 Multi-variable Functions

**Showing Limits Don't Exist** Approach from different lines, show that they approach different values.

**Where is a Function Continuous** Check if polynomial, rational function, composition of continuous functions and check domain.

# 5 Problems

## 5.1 Important Problems

### 5.1.1 Assignment 1

14, 15, 16, 17, 18, 19

### 5.1.2 Assignment 2

2, 3, 5, 7, 8, 9, 10, 11, 12, 14, 16, 17

## 5.2 Review Problems

- p.811-812: 5-16, 35-44, 53-56, 61-65, 73-80
- p.882-883: 4-7, 9, 15-25, 27
- p.922: 2, 3, 5, 6, 8, 9, 10, 11, 12, 13, 17, 19, 22
- Section 14.2: 9, 11, 15, 21, 29-38

# 6 Misc

$$\lim_{n\to\infty} arctan(n) = \frac{\pi}{2} \tag{2}$$

$$\frac{d}{dx}(a^x) = a^x log(a) \tag{3}$$