MATH 222: Calculus III Review

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Adapted from Alexander Garver's Winter 2017 MATH222 lecture notes.

Contents

1 Series

n^{th} partial sum of a sequence. a_n, the terms of the series, must tend to 0, or else the series **diverges**.

1.1 Special Series

Harmonic $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Geometric $\sum_{n=1}^{\infty} r^n$ converges to $\frac{1}{1-r} \iff -1 < r < 1$.

1.2 Tests

Alternating Series Test/Leibniz Test Sequence $a_1, a_2, \ldots$ is **decreasing** and has limit 0. Then $\sum_{n=1}^{\infty}(-1)^n a_n$ converges. In other words, absolute value of the alternating series forms a convergence sequence.

Absolute Convergence Test $\sum_{n=1}^{\infty} |a_n|$ converges $\implies \sum_{n=1}^{\infty} a_n$ converges.

Ratio Test Suppose $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = r$.
$r < 1 \implies \sum_{n=1}^{\infty} |a_n|$ converges and
$r > 1 \implies \sum_{n=1}^{\infty} a_n$ diverges.

Root Test Suppose $\lim_{n\to\infty} \sqrt[n]{|a_n|} = r$.
$r < 1 \implies \sum_{n=1}^{\infty} |a_n|$ converges and
$r > 1 \implies \sum_{n=1}^{\infty} a_n$ diverges.

Comparison Test Suppose fixed number K s.t. $0 < a_n < Kb_n$, $\forall$ sufficiently large n.
$\sum_{n=1}^{\infty} b_n$ converges $\implies \sum_{n=1}^{\infty} a_n$ converges.
$\sum_{n=1}^{\infty} a_n$ diverges $\implies \sum_{n=1}^{\infty} b_n$ diverges.

Limit Comparison Test Suppose $a_n > 0, b_n > 0$ and $\lim_{n\to\infty} \frac{a_n}{b_n} = R \neq 0$. Then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ **both** converge or **both** diverge.

Integral Test Suppose $f(x)$ is **positive** and **decreasing**, $\forall$ large enough x. Then the following are equivalent:

1) $\int_1^{\infty} f(x)dx$ is finite, i.e. converges.

2) $\sum_{n=1}^{\infty} f(n)$ is finite, i.e. converges.

The p-test follows from this.

***p*-test** $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges $\iff$ $p > 1$.

Alternating Series Estimation Theorem If $s = \sum_{n=1}^{\infty}(-1)^{n-1}b_n$ is the sum of an alternating series that satisfies:
$b_{n+1} \leq b_n$ and $\lim_{n\to\infty} b_n = 0$
then $|R_n| = |s - s_n| \leq b_{n+1}$.

1.3 Power Series

Series of the form $\sum_{n=0}^{\infty} c_n x^n$ or $\sum_{n=0}^{\infty} c_n(x-a)^n$

1.3.1 Important Power Series to Know

- $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, R = \infty$
- $\sin x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}, R = \infty$
- $\cos x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}, R = \infty$
- $\tan^{-1} x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}, R = 1$
- $\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n-1} \frac{x^n}{n}, R = 1$

1.3.2 Convergence

Theorem 1. $\sum_{n=0}^{\infty} c_n(x-a)^n$ *does exactly one of the following:*

(i) Converges only when $x = a$.

(ii) Converges for all x.

(iii) $\exists\ R > 0$ s.t. $|x-a| < R$, the series converges and diverges if $|x-a| > R$.

R is the **radius of convergence**. The values of x where the series converges is called the **interval of convergence**. Radius of convergence **does not** tell you if endpoints are included, have to check both. Ratio test is usually a good tool to find the radius of convergence.

1.3.3 Representing Functions As Power Series

If $f(x)=\sum_{n=0}^{\infty}c_n(x-a)^n$ for some $c_0,c_1,\ldots$ then $f'(x)$ and $\int f(x)dx$ can also be represented by a power series.
If $f(x)=\sum_{n=0}^{\infty}c_n(x-a)$ then $c_n=\frac{f^n(a)}{n!}$
Work with a familiar power series: $\sum_{n=0}^{\infty}x^n=\frac{1}{1-x}$ for $|x|<1$.

Theorem 2. *Suppose* $\sum_{n=0}^{\infty}c_n(x-a)^n$ *with* $R>0$. *Then* $f(x)=\sum_{n=0}^{\infty}c_n(x-a)^n$ *is differentiable on* $(a-R,a+R)$ *and*

1) $f'(x)=c_1+2c_2(x-a)+3c_3(x-a)^2+\ldots$

2) $\int f(x)dx=c+c_0(x-a)+\frac{c_1}{2}(x-a)^2+\frac{c_2}{3}(x-a)^3+\ldots$

The radius of convergence of $f'(x)$ *and* $\int f(x)dx$ *is* R.

Otherwise said, you can easily differentiate & integrate series.

Theorem 3. *If* $f(x)$ *has a power series representation* $\sum_{n=0}^{\infty}c_n(x-a)^n$ *then* $c_n=\frac{f^n(a)}{n!}$. *Called the* $n!$ ***Taylor series*** *of* f *at* a.

How to show a function is represented by a power series?

Theorem 4. *Suppose* $\sum_{n=0}^{\infty}c_n(x-a)^n$ *is the Taylor series of* $f(x)$ *with* $R>0$.
If $\lim_{n\to\infty}(f(x)-\sum_{i=0}^{\infty}c_i(x-a)^i)=0$ *for* $|x-a|<R$, *then* $f(x)=\sum_{n=0}^{\infty}c_n(x-a)^n$ *for* $|x-a|<R$.

2 3-Dimensional Coordinate System

XYZ plane.

Distance Between 2 Points, P and Q

$$|PQ|=\sqrt{(a_2-a_1)^2+(b_2-b_1)^2+(c_2-c_1)^2}$$

2.1 Vectors

Definition 1. A **vector** $\underline{v}$ is a quantity with a **magnitude** and **direction**. Vectors are **equal** if they have the same magnitude and direction. There is a **zero vector**, denoted $\underline{0}$. It has no magnitude or direction.

Vector Addition

Definition 2. Sum of $\underline{u}, \underline{v}$ denoted $\underline{u} + \underline{v}$ is the vector whose initial point is that of $\underline{u}$ and whose terminal point is that of $\underline{v}$. $\underline{u} + \underline{v} = \underline{v} + \underline{u}$.

Scalar Multiplication

Definition 3. If c is a **scalar**, i.e. $c \in \mathbb{R}$, then $c\underline{v}$ is the vector whose length is $|c|$ times the length of $\underline{v}$ and whose direction is the same as $\underline{v}$ if $c > 0$ and opposite if $c < 0$. $c = 0 \implies c\underline{v} = \underline{0}$.

Vectors in Coordinates
$\underline{v} = \langle a_1, a_2, a_3 \rangle = a_1\underline{i} + a_2\underline{j} + a_3\underline{k}$, where
$\underline{i} = \langle 1, 0, 0 \rangle$
$\underline{j} = \langle 0, 1, 0 \rangle$
$\underline{k} = \langle 0, 0, 1 \rangle$

Magnitude The **magnitude** of $\underline{v}$ is $|\underline{v}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$.

2.1.1 Dot Product

"Multiplying" vectors.

Definition 4. Given $\underline{v} = \langle a_1, a_2, a_3 \rangle, \underline{u} = \langle b_1, b_2, b_3 \rangle$, their **dot product** is defined as:

$$\underline{v} \cdot \underline{u} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

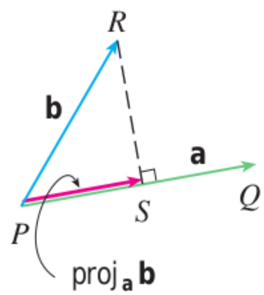
Theorem 5. *i)* $\underline{v} \cdot \underline{v} = |\underline{v}|^2$

ii) $\underline{v} \cdot \underline{u} = |\underline{v}||\underline{u}| \cos\theta$, *where* θ *is the angle between* $\underline{v}, \underline{u}$ *with* $0 \leq \theta \leq \pi$

iii) $\underline{v}$ *and* $\underline{u}$ *are* ***orthogonal*** *(or* ***perpendicular****)* $\iff \underline{v} \cdot \underline{u} = 0$, *and* $\underline{v} \cdot \underline{u} \iff \theta = \frac{\pi}{2}$

Note that $\underline{v} \cdot \underline{u} > 0 \implies \theta < \frac{\pi}{2}$ (acute) and $\underline{v} \cdot \underline{u} < 0 \implies \theta > \frac{\pi}{2}$ (obtuse).

2.1.2 Projections



Scalar Projection

Definition 5. **Scalar Projection** of $\underline{v}$ onto $\underline{u}$ is given by: $comp_{\underline{u}}(\underline{v}) = \frac{\underline{u}\cdot\underline{v}}{|\underline{u}|}$

Vector Projection

Definition 6. **Vector Projection** of $\underline{v}$ onto $\underline{u}$ is given by: $proj_{\underline{u}}(\underline{v}) = \left(\frac{\underline{u}\cdot\underline{v}}{|\underline{u}|^2}\right)\underline{u}$

2.1.3 Cross Product

Definition 7. Let $\underline{v}_1 = \langle a_1, a_2, a_3\rangle, \underline{v}_2 = \langle b_1, b_2, b_3\rangle$. The **cross product** of $\underline{v}_1, \underline{v}_2$ is given by $\underline{v}_1 \times \underline{v}_2 = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1\rangle$

Can be obtained from the determinant of:

$$\begin{bmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$$

Theorem 6. *i)* $|\underline{v}_1 \times \underline{v}_2| = |\underline{v}_1||\underline{v}_2|\sin\theta, 0 \leq \theta \leq \pi$
In fact, $|\underline{v}_1||\underline{v}_2|\sin\theta$ *is the area of the parallelogram determined by* $\underline{v}_1, \underline{v}_2$

ii) Two nonzero vectors $\underline{v}_1, \underline{v}_2$ *are parallel if and only if* $\underline{v}_1 \times \underline{v}_2 = 0$.

2.2 Lines

Equation of a Line

Definition 8. The equation of a line is given by: $\underline{r} = \underline{r}_0 - t\underline{v}$.
Now let $\underline{r} = \langle x, y, z\rangle, \underline{r}_0 = \langle x_0, y_0, z_0\rangle, \underline{v} = \langle a, b, c\rangle$.
The **parametric equations** of the line L passing through (x_0, y_0, z_0) and parallel to $\underline{v} =$

$\langle a,b,c\rangle$ is given by: $x = x_0 + at, y = y_0 + bt, z = z_0 + ct$
Solving for t produces the **symmetric equations** of the line L: $\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$

Definition 9. 2 lines are **skew lines** if they are not parallel and do not intersect.

2.3 Planes

What determines a plane in 3-D?

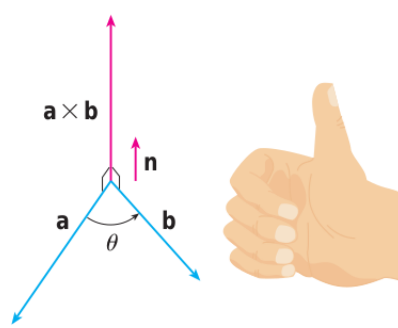
- 3 noncolinear points in the plane.
- 2 nonparallel vectors and a point p_0 in the plane.
- a point p_0 in the plane and a vector $\underline{n}$ (**normal vector**) that is perpendicular to the plane.

Definition 10. Let $p_0 = (x_0, y_0, z_0)$ and $p = (x, y, z)$.
$\underline{n} \cdot (\underline{r} - \underline{r}_0) = 0$ is the **vector equation** of the plane.
$\underline{r} = \langle x, y, z\rangle, \underline{r}_0 = \langle x_0, y_0, z_0\rangle, \underline{n} = \langle a, b, c\rangle$
$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$ is the **scalar equation** of the plane that contains $p_0 = (x_0, y_0, z_0)$ and is perpendicular to $\underline{n}$.
$ax + by + cz + d = 0$ is the **linear equation for the plane**.

Theorem 7. *$\underline{v}_1 \times \underline{v}_2$ is orthogonal to $\underline{v}_1$ and $\underline{v}_2$.*

2.4 Right-Hand Rule

If the finger of your right hand curl in the direction of rotation from $\underline{a}$ to $\underline{b}$ through θ ($0^\circ \leq \theta \leq 180^\circ$), then your thumb points in the direction of $\underline{a} \times \underline{b}$.



2.5 Vector Functions and Space Curves

Vector Functions

Definition 11. We say $\underline{r}(t) = \langle f(t), g(t), h(t)\rangle = f(t)\underline{i} + g(t)\underline{j} + h(t)\underline{k}$ is a **vector function**.

Definition 12. $f(t), g(t), h(t)$ are the **component functions** of $\underline{r}(t)$. The **domain** is the set $t \in \mathbb{R}$ s.t f, g, h are defined at t.

Definition 13. The **limit** of $\underline{r}$ is defined by taking the limits of its component functions, that is:

$$\lim_{t\to a} \underline{r}(t) = \langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t)\rangle$$

Definition 14. A vector function $\underline{r}$ is **continuous** at a if

$$\lim_{t\to a} \underline{r}(t) = \underline{r}(a)$$

$\underline{r}$ is **continuous** at a if and only if f, g, h also are.

Definition 15. Let f, g, h be continuous on an interval I. Let C be the set of points (x, y, z) satisfying

$$x = f(t), y = g(t), z = h(t) \tag{1}$$

for any t in I. We say C is a space curve and the equations given by equation (1) are its **parametric equations**.
We say t is a **parameter**.

2.6 Arc Length, Curvature and the TNB Frame

Definition 16. The **derivative** of a vector function $\underline{r}(t)$ is given by:

$$\lim_{h\to 0} \frac{\underline{r}(t+h) - \underline{r}(t)}{h} = \underline{r}'(t) = \frac{d\underline{r}}{dt}$$

if it exists.

Theorem 8. *If $\underline{r}(t) = \langle f(t), g(t), h(t)\rangle$ and f, g, h are differentiable, then $\underline{r}'(t) = \langle f'(t), g'(t), h'(t)\rangle$*

Definition 17. We say $\underline{r}'(t)$ is the **tangent vector** of $\underline{r}(t)$ at t.

Arc Length

Definition 18. Suppose we have a curve given by $\underline{r}(t) = \langle f(t), g(t), h(t)\rangle$ with $a \leq t \leq b$ and f', g', h' are continuous. The **arc length** is defined as

$$\int_a^b |\underline{r}'(t)| dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt$$

The **arc length function** is given by:

$$s(t) = \int_a^t |\underline{r}'(u)| du$$

Definition 19. A **parametrization** of a curve C is a representation of C by a vector function using arc length.

Curvature

Definition 20. A parametrization $\underline{r}(t)$ of C is **smooth** on an interval I if $\underline{r}'(t)$ is continuous and $\underline{r}'(t) \neq 0$ on I. A curve C is **smooth** if it has a smooth parametrization.

TNB Vectors

Definition 21. The **unit tangent vector** of $\underline{r}(t)$ is given by

$$\underline{T}(t) = \frac{\underline{r}'(t)}{|\underline{r}'(t)|}$$

The **unit normal vector** of $\underline{r}(t)$ is given by

$$\underline{N}(t) = \frac{\underline{T}'(t)}{|\underline{T}'(t)|}$$

The **binormal vector** of $\underline{r}(t)$ is given by

$$\underline{B}(t) = \underline{T}(t) \times \underline{N}(t)$$

They are all pairwise orthogonal and are of unit length.

Definition 22. The **curvature** κ of C is the length of the derivative of $\underline{T}(s)$, given by:

$$\kappa = \left| \frac{dI}{dS} \right|$$

$$\kappa(t) = \left| \frac{\underline{T}'(t)}{\underline{r}'(t)} \right| = \frac{|\underline{r}'(t) \times \underline{r}''(t)|}{|\underline{r}'(t)|^3}$$

2.7 Velocity & Acceleration

Definition 23. Given a curve C denoted by $\underline{r}(t)$, the **velocity** of $\underline{r}(t)$ is given by:

$$\underline{r}'(t) = \lim_{h\to 0} \frac{\underline{r}(t+h) - \underline{r}(t)}{h} = \underline{v}(t)$$

Note that speed is given by $|\underline{r}'(t)| = |\underline{v}(t)|$

Definition 24. The **acceleration** of $\underline{r}(t)$ is

$$\underline{a}(t) = \underline{r}''(t) = \underline{v}'(t)$$

Components of Acceleration

$\underline{a}(t)$ can be expressed purely in terms of $\underline{T}$ and $\underline{N}$ like so:

$$\underline{a} = \underbrace{v'}_{a_T}\underline{T} + \underbrace{\kappa v^2}_{a_N}\underline{N}$$

One can also show:

$$a_T = \frac{\underline{r}'(t) \cdot \underline{r}''(t)}{|\underline{r}'(t)|}$$

$$a_N = \frac{|\underline{r}'(t) \times r''(t)|}{|\underline{r}'(t)|}$$

3 Multi-variable Functions

Definition 25. A **function of two variables** is a rule that assigns to each ordered pair of real numbers (x, y) a real number $f(x, y)$ when (x, y) is in the **domain** D of f.
Domain of f is $D = \{(x, y) : f(x, y) \text{ is defined}\} \subseteq \mathbb{R}^2$
Range of f is $\{f(x, y) : (x, y) \in D\} \subseteq \mathbb{R}$
Graph of f is the set $\{(x, y, z) \in D \text{ and } z = f(x, y)\} \subseteq \mathbb{R}^3$

3.1 Contour Maps

Definition 26. We can represent functions $f(x, y)$ by taking horizontal slices of their graphs. These slices indicate height. The slices or **level curves** of $f(x, y)$ are the curves with equations $f(x, y) = k$ where k is a constant in the range of f. If we draw the level curves we obtain a **contour map** of f.

3.2 Level Surfaces

To understand graphs of functions of 3 variables, we draw **level surfaces**.

3.3 Limits and Continuity

Definition 27. Let f be a function of two variables whose domain D includes points that are arbitrarily close to (a, b). We say the **limit** of $f(x, y)$ as (x, y) approaches (a, b) is L:

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

if $\forall\varepsilon > 0\ \exists\delta > 0$ s.t. if (x, y) is in D and $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \implies |f(x, y)-L| < \varepsilon$

Limit Laws

Theorem 9. *If the limits* $\lim_{(x,y)\to(a,b)} f(x, y)$ *and* $lim_{(x,y)\to(a,b)}g(x, y)$ *exist, then*

i) $\lim_{(x,y)\to(a,b)} c(f(x, y)) = c\lim_{(x,y)\to(a,b)} f(x, y)$

ii) $\lim_{(x,y)\to(a,b)}(f(x, y) + g(x, y)) = \lim_{(x,y)\to(a,b)} f(x, y) + \lim_{(x,y)\to(a,b)} g(x, y)$

iii) $\lim_{(x,y)\to(a,b)} f(x, y)g(x, y) = (\lim_{(x,y)\to(a,b)} f(x, y))(\lim_{(x,y)\to(a,b)} g(x, y))$

iv) $\lim_{(x,y)\to(a,b)} \frac{f(x,y)}{g(x,y)} = \frac{\lim_{(x,y)\to(a,b)} f(x,y)}{\lim_{(x,y)\to(a,b)} g(x,y)}$, *where denominator is nonzero.*

Continuity

Definition 28. A function f is **continuous** at (a, b) if $\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$.
A function f is **continuous** on a set D if it is continuous at each (a, b) in D.

Theorem 10. $\frac{f}{g}$ *is continuous if* f, g *are continuous.*
We can also show that polynomials and rational functions are continuous on their domains.

3.4 Partial Derivatives

Definition 29. The **partial derivative** of $f(x, y)$ with respect to x is

$$f(x, y) = \lim_{h\to0} \frac{f(x + h, y) - f(x, y)}{h}$$

In order to evaluate these limits, fix y and differentiate wrt x to obtain $f_x(x, y)$ or fix x and wrt y to get $f_y(x, y)$.

Notation:

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial z}{\partial x} = D_x f$$
$$f_y = \frac{\partial f}{\partial y} = \frac{\partial z}{\partial y} = D_y f$$

Higher Order Derivatives We can differentiate f_x and f_y to obtain

$$(f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$
$$(f_x)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{(\partial x)^2}$$
$$(f_y)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$
$$(f_y)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial f^2}{(\partial y)^2}$$

Note that, with ∂ notation, we derive from right to left, but with f_x notation we derive from left to right.

Clairaut's Theorem:

Theorem 11. *Suppose f is defined on a disk D that contains point (a, b). If the functions f_{xy}, f_{yx} are continuous on D, then $f_{xy}(a, b) = f_{yx}(a, b)$*

3.5 Tangent Planes

Let $f(x, y)$ be a function and let S be the surface $z = f(x, y)$.
T_1 : tangent line in x-direction at $(x_0, y_0, f(x_0, y_0))$.
T_2 : tangent line in y-direction at $(x_0, y_0, f(x_0, y_0))$.

Definition 30. Define the **tangent plane** to S at $(x_0, y_0, f(x_0, y_0))$ to be the plane that contains both T_1, T_2, given by:

$$z = z_0 + a(x - x_0) + b(y - y_0)$$

Its intersection with the plane $y = y_0$ (or $x = x_0$) is T_1 (or T_2)

$$\implies T_1 = z - z_0 = a(x - x_0), T_2 = z - z_0 = b(y - y_0)$$

Theorem 12. *If f has continuous partial derivatives, an equation of the tangent plane to $z = f(x, y)$ at $(x_0, y_0, f(x_0, y_0))$ is*

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Approximation Using Tangent Planes

Definition 31. The **linearization** of $f(x, y)$ at (a, b) is defined as

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

The approximation

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called the **linear approximation** or **tangent plane approximation** of f at (a, b).

Theorem 13. *If f_x,f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).*

3.6 The Chain Rule

Theorem 14. *Suppose $z = f(x, y)$ is a differentiable function and $x = x(t)$, $y = y(t)$ are differentiable. Then z is a diffentiable function of t and*

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Theorem 15. *Suppose u is a differentiable function of $t_1, \ldots, t_m$. Then*

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \ldots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

for each $i = 1, \ldots, m$

3.7 Direction Derivatives

Definition 32. Let $z = f(x, y)$ be the surface s. Let $p = f(x_0, y_0, z_0)$ be a point on s, and let $\underline{u} = \langle a, b\rangle$ be any unit vector. The **directional derivative** of f in direction $\underline{u}$ at (x_0, y_0) is

$$D_{\underline{u}}f(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if it exists.

Theorem 16. *If f is a differentiable function of x and y, then f has a directional derivative in any direction and*

$$D_{\underline{u}}f(x,y) = f_x(x,y)a + f_y(x,y)b$$

3.8 Gradient Vectors

Theorem 17. *If $f(x,y)$ $(f(x,y,z))$ is a differentiable function of x and y, then f has a directional derivative in the direction of* ***any unit vector*** *$\underline{u} = \langle a,b\rangle$ and $D_{\underline{u}}f(x,y) = f_x(x,y)a + f_y(x,y)b = \nabla f \cdot \underline{u}$ ($\underline{u} = \langle a,b,c\rangle$ and $D_{\underline{u}}f(x,y,z) = f_x(x,y,z)a + f_y(x,y,z)b + f_z(x,y,z)c = \nabla f \cdot \underline{u}$)*

Theorem 18. *Suppose f is a differentiable function of two or three variables. Then the maximum value of $D_{\underline{u}}(\underline{x})$ is $|\nabla f(\underline{x})|$ and it occurs when $\underline{u}$ has the same direction as $\nabla f(\underline{x})$.*

3.9 Extreme Values

Definition 33. A function $f(x,y)$ has a **local maximum** at (a,b) if $f(x,y) \le f(a,b)$ when (x,y) is **near** (a,b) (i.e. $f(x,y) \le f(a,b)$ for any (x,y) inside some disk with center (a,b)). We call $f(a,b)$ a **local maximum value** of f.

Definition 34. If $f(x,y) \ge f(a,b)$ when (x,y) is near (a,b) then f has a **local minimum value**.

Definition 35. If $f(x,y) \le f(a,b)$ for all (x,y) in the domain of f, we say f has an **absolute maximum** at (a,b).

Theorem 19. *If f has a local maximum or a local minimum at (a,b), then $f_x(a,b) = 0$ and $f_y(a,b) = 0$. In other words, the tangent planes at those points $(a,b,f(a,b))$ are horizontal $z = f(a,b)$ and $\nabla f(a,b) = \underline{0}$*

Definition 36. A point (a,b) is a **critical point** of f if $f_x(a,b) = 0$ and $f_y(a,b) = 0$, or one of the partial derivatives does not exist.

Second Derivative Test

Theorem 20. *Suppose the second partial derivatives of f are continuous on a disk centered at (a,b) and suppose that $f_x(a,b) = 0,, f_y(a,b) = 0$. Let*

$$D = D(a,b) = \begin{vmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{vmatrix} = f_{xx}(a,b)f_{yy}(a,b) - \underbrace{(f_{xy}(a,b))^2}_{\text{Using Clairaut's thm}}$$

a) If $D > 0$ and $f_{xx}(a,b) > 0$, then $f(a,b)$ is a local minimum value.

b) If $D > 0$ and $f_{xx}(a,b) < 0$, then $f(a,b)$ is a local maximum value.

c) If $D < 0$, then $f(a,b)$ is neither (it's a saddle point).

d) If $D = 0$, the test is inconclusive.

Absolute Maxima and Minima Recall that if $f(x)$ is continuous on $[a,b]$, then $f(x)$ has an absolute maximum and an absolute minimum. The analog of $[a,b]$ for $f(x,y)$ are closed and bounded subsets of $\mathbb{R}^2$.

Definition 37. A subset of $\mathbb{R}^2$ is **closed** if it contains all of its **boundary points** (i.e. A boundary point of a subset $D \subseteq \mathbb{R}^2$ is a point (a,b) such that every disk with center (a,b) contains points in D **and** also from outside D).

Definition 38. A **bounded** subset of $\mathbb{R}^2$ is one that is contained in some disk of finite radius.

Extreme Value Theorem

Theorem 21. *If f is continuous on a closed bounded set $D \subseteq \mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ for some points $(x_1, y_1), (x_2, y_2) \in D$.*

Techniques for finding extreme values of continuous functions on closed, bounded sets

1. Find the values of f at its critical points that are **in** D.
2. Find the extreme values of f on the **boundary** of D.
3. The **largest** and **smallest** of these values of f are its extreme values on D.

3.10 Lagrange Multipliers

Method of Lagrange Multipliers To find the maximum and minimum values of $f(x,y,z)$ subject to $g(x,y,z) = k$ (assuming these extreme values exist and $\nabla g \neq 0$ on $g(x,y,z) = k$):

a) Find all values x, y, z, λ where

$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z) \text{ and } g(x,y,z) = k$$

b) Evaluate f at all these points (x, y, z). The largest and smallest are the maximum and minimum values of f.

For 2 variables:
For $f(x, y), g(x, y) = k$, find x, y, λ, satisfying $\nabla f(x, y) = \lambda \nabla g(x, y)$ and $g(x, y) = k$ and so on.

4 Multivariable Integration

4.1 Integration over Rectangles

We have n approximating rectangles for the sum, and each rectangle has width Δx and height $f(x_i^*)$. The X_i^* are sample points.

$$\int_a^b f(x)\ dx \approx \sum_{i=1}^{n} f(x_i^*)\Delta x$$

The following limit makes rectangles infinitely narrow:

$$\int_a^b f(x)\ dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

We would like to integrate $f(x, y)$ over

$$R = \underbrace{[a, b]}_{x} \times \underbrace{[c, d]}_{y} = \{(x, y) : \mathbb{R}^2 : a \leq x \leq b, c \leq y \leq d\}$$

To calculate volume, we'll further split up $a\&b$, getting:

$$\text{Volume of } S \approx \sum_{i=1}^{n} \sum_{j=1}^{n} f(x_i^*, y_j^*)\Delta A$$

We then have:

Definition 39.

$$\textbf{Volume of S} = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*)\Delta A$$

Definition 40. Given $f(x,y)$ we define the **double integral** of f over rectangle R to be:

$$\iint_R f(x,y)\ dA = \lim_{m,n\to\infty} \underbrace{\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i^*, y_j^*)\Delta A}_{\text{Double Riemann Sum}}$$

if it exists. If it exists, we say f is integrable.

Midpoint Rule is as follows:

$$\iint_R f(x,y)\ dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(\overline{x_i}, \overline{y_j})\Delta A$$

, where $\overline{x_i}$ is the midpoint of $[x_{i-1}, x_i]$ and $\overline{y_j}$ is the midpoint of $[y_{j-1}, y_j]$.

4.2 Iterated Integrals

Evaluate integrals without using the definition.

Definition 41. Suppose $f(x,y)$ is integrable on $R = [a,b]\times[c,d]$. The expression $\int_a^b f(x,y)\ dx$ means we integrate $f(x,y)$ with respect to x and fix y. The expression $\int_c^a f(x,y)\ dy$ means we integrate $f(x,y)$ with respect to y and fix x. We therefore have:

$$\int_a^b \int_c^d f(x,y)\ dy\ dx = \int_a^b \left(\int_c^d f(x,y)\ dy\right) dx$$

$$\int_c^d \int_a^b f(x,y)\ dx\ dy = \int_c^d \left(\int_a^b f(x,y)\ dx\right) dy$$

Theorem 22 (Fubini's Theorem)**.** *If f is continuous on $R = [a,b]\times[c,d]$, then*

$$\iint_R f(x,y) dA = \int_a^b \int_c^d f(x,y)\ dy\ dx = \int_c^d \int_a^b f(x,y)\ dx\ dy$$

4.3 Double Integrals over General Regions

Definition 42. Suppose we want to integrate $f(x,y)$ over D. Let R be a rectangle containing D.

$$F(x,y) = \begin{cases} f(x,y): & (x,y) \text{ in } D \\ 0: & (x,y) \text{ not in } D \text{ (but in } R) \end{cases}$$

If $f(x, y)$ is integrable over D we define

$$\iint_D f(x, y)\ dA = \iint_R F(x, y)\ dA$$

as the **double integral** of f **over** D.

Remark: If f is continuous, it is integrable. If f is **bounded** on D and f is continuous on D except possibly on finite number of smooth curves, then f is integrable.

Definition 43 (Type I Regions)**.** A region D is **type I** if it is bounded by the graphs of continuous functions $g_1(x), g_2(x)$.

$$D = \{(x, y) : a \le x \le b, g_1(x) \le y \le g_2(x)\}$$

Let $R = [a, b] \times [c, d]$ (rectangle containing D). Assume f is integrable. Then:

$$\iint_D f(x, y)\ dA = \iint_R F(x, y)\ dA \overset{\text{Fubini's Thm}}{=} \int_a^b \int_c^d F(x, y)\ dy\ dx = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\ dy\ dx$$

Definition 44 (Type II Regions)**.** A region D is **type II** if it is bounded by the graphs of continuous functions $h_1(y), h_2(y)$.

$$D = \{(x, y) : a \le x \le b, g_1(x) \le y \le g_2(x)\}$$

As above:

$$\iint_D f(x, y)\ dA = \iint_R F(x, y)\ dA \overset{\text{Fubini's Thm}}{=} \int_c^d \int_a^b F(x, y)\ dx\ dy = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\ dx\ dy$$

Theorem 23 (Properties of Integrals)**.** *Assume the following integrals exist:*

i)

$$\iint_D (f(x, y) + g(x, y))\ dA = \iint_D f(x, y)\ dA + \iint_D g(x, y)\ dA$$

ii)

$$\iint_D cF(x, y)\ dA = c \iint_D F(x, y)\ dA, \quad c \textit{ is any cosntant}$$

iii) If $g(x, y) \le f(x, y)$ for any (x, y) in D, then

$$\iint_D g(x, y)\ dA \le f(x, y)\ dA$$

iv) If $D = D_1 \cup D_2$ where D_1 and D_2 may only overlap on their boudnaries, then

$$\iint_D f(x,y)\ dA = \iint_{D_1} f(x,y)\ dA + \iint_{D_2} f(x,y)\ dA$$

4.4 Polar Coordinates & Double Integrals

A point P can be represented as (x,y) in **rectangular coordinates**.
P can also be represented by **polar coordinates** as (r,θ).
$|r|$ is the distance from P to the origin.
θ is the angle between the positive part of the x-Axis and the line between origin and f if $r > 0$, else, θ is the angle between the negative part of the x-Axis and the line between P and the origin if $r < 0$.
One can changed between polar and rectangular coordinates using the following equations:

$$\begin{aligned} x &= r\cos\theta & r &= \sqrt{x^2+y^2} \\ y &= r\sin\theta & \tan\theta &= \frac{y}{x} \end{aligned}$$

Definition 45. We say a region R in $\mathbb{R}^2$ is a **polar rectangle** if

$$R = \{(r,\theta) | 0 \le r \le b, \alpha \le \theta \le \beta\}$$

Theorem 24. *Let f be a continuous function.*

$$\begin{aligned} \iint_R f(x,y)\ dA &= \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^*\cos\theta_j^*, r_i^*\sin\theta_j^*)\Delta A \\ &= \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} \underbrace{f(r_i^*\cos\theta_j^*, r_i^*\sin\theta_j^*)r_i^*}_{g(r_i^*,\theta_j^*)}\Delta r\ \Delta\theta \\ &= \int_\alpha^\beta \int_a^b \underbrace{g(r,\theta)}_{=f(r\cos\theta, r\sin\theta)\cdot r}\ dr\ d\theta \\ &= \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta) \underbrace{r}_{!!}\ dr\ d\theta \end{aligned}$$

Theorem 25. *If f is continuous on $R = \{(r,\theta) | \alpha \le \theta \le \beta, h_1(\theta) \le r \le h_2(\theta)\}$, then*

$$\iint_R f(x,y)\ dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta) r\ dr\ d\theta$$

5 How to Solve Problems

5.1 Series

Representing a Function as a Power Series Look for a familiar function that has a power series representation, plug it in and simplify. Integrate and differentiate as required.

Finding the Radius of Convergence Usually involves the ratio test and checking when $r < 1$.

Finding the Interval of Convergence Use the radius of convergence and check if end-points converge.

Finding the Sum of a Series Look for a familiar series that can be represented as a function.

Using the Definition of Taylor Series To find a power series representation or to find first few terms, just derive and use $c_n = \frac{f^n(a)}{n!}$.

Evaluating an Indefinite Integral with Series Replace known function by a familiar series, try to cancel out other terms, integrate the series.

Evaluating a Limit with Series Same as above for integrating.

5.2 Vectors

Compute Something Compute what it asks for given the corresponding formula, whether it's the dot product, cross product, projection, etc.

Angle Between 2 Vectors Use either the dot product or cross product.

Values for x Such that 2 Vectors are Orthogonal Use the dot product and solve for it being 0.

Finding a Parametric Equation of a Line Use a point and a direction vector.

Finding the Equation of a Plane Use a point and a normal vector.

Find Where a Line Intersects a Plane Plug in parameters (x, y, z) from line into the equation of a plane and solve for t and get the corresponding point from the line with that value of t.

Distance from a Line to the Origin Take $DV = \underline{a}$ and $\underline{b}$ some point on the line (usually $t = 0$). Then $d = \frac{|\underline{a}\times\underline{b}|}{|\underline{a}|}$.

Are 2 Lines Skew, Parallel or Intersecting? If DV are multiples of each other, parallel. If you equate each component $x = x, y = y, z = z$ from both lines and you can solve the system, then they intersect. Otherwise, skew.

Angles Between Planes/Parallel or Perpendicular To show parallel, compare NV. To show perpendicular, use the dot product. If neither, the angle can be computed with the dot product.

Line of Intersection of Two Planes Find an intersecting point and use the cross product with both NV to get a DV.

Distance Between 2 Parallel Planes Given plane equations of the form $ax+by+cz = d$, then distance $D = \frac{|d_1-d_2|}{\sqrt{a^2+b^2+c^2}}$

Distance Between a Point and a Plane $p = (x_1, y_1, z_1)$, then $D = \frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$

Diagonals of a Parallelogram Given $\underline{u}$ and $\underline{v}$ that form the sides of a parallelogram, lengths of two diagonals are $|\underline{u} + \underline{v}|$ and $|\underline{u} - \underline{v}|$.

5.3 Vector Functions

Find the Domain of a Vector Function Check where it isn't defined.

Limit of a Vector Function Take the limit of each component.

Integral of a Vector Function Take the integral of each component.

Curve of Intersection Between Cylinder and Plane If you have a projection of a cylinder onto a circle like $x^2 + y^2 = 16, z = 0$, then you can write $x = 4\cos t, y = 4\sin t, 0 \leq t \leq 2\pi$. Take the plane, isolate for z and plug in x, y from circle. Then your vector function is given by $x\&y$ from circle and z from plane with plugged in x, y.

Where does a Curve Intersect a Plane? xz-plane $\implies y = 0$, xy-plane $\implies z = 0$, etc.

Parametric Equation of a Line at a Certain Point Get $\underline{r}'(t)$ and plug in t to get DV. Can use this DV as a NV for a normal plane to the curve.

Length of the Curve Use arc length formula.

Angle of Intersection of 2 Curves Get the point where they intersect, then find tangents at those points and use dot product.

Reparametrizing a Curve Given a point, get the corresponding t value. Then measure arc length from 0 to t and solve for t wrt s and plug it into arc length formula wrt t, getting $r(t(s))$.

Computing $\underline{T}$ $\underline{N}$ $\underline{B}$, κ Use the formulas.

Particle Velocity, Speed and Acceleration Compute with formulas, note that speed is $|v(t)|$. Might have to work backwards by integrating if given acceleration and/or velocity to get position, don't forget constant.

Acceleration and Normal Components of Acceleration Vector Formulas.

5.4 Multi-variable Functions

Showing Limits Don't Exist Approach from different lines, show that they approach different values.

Where is a Function Continuous Check if polynomial, rational function, composition of continuous functions and check domain.

6 Problems

6.1 Important Problems

6.1.1 Assignment 1

14, 15, 16, 17, 18, 19

6.1.2 Assignment 2

2, 3, 5, 7, 8, 9, 10, 11, 12, 14, 16, 17

6.2 Review Problems

- p.811-812: 5-16, 35-44, 53-56, 61-65, 73-80
- p.882-883: 4-7, 9, 15-25, 27
- p.922: 2, 3, 5, 6, 8, 9, 10, 11, 12, 13, 17, 19, 22
- Section 14.2: 9, 11, 15, 21, 29-38
- Section 15.7 (Cylindrical coordinates): 15-26, 29-30
- Section 15.8 (Spherical coordinates): 9-30, 41-43, 48

7 Misc

$$\lim_{n\to\infty} arctan(n) = \frac{\pi}{2} \tag{2}$$

$$\frac{d}{dx}\left(a^x\right) = a^x log(a) \tag{3}$$

Integration by Parts

$$\int u\ dv = uv - \int v\ du$$