MATH 324: Statistics

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Notes from Masoud Asgharian's Winter 2018 lectures.

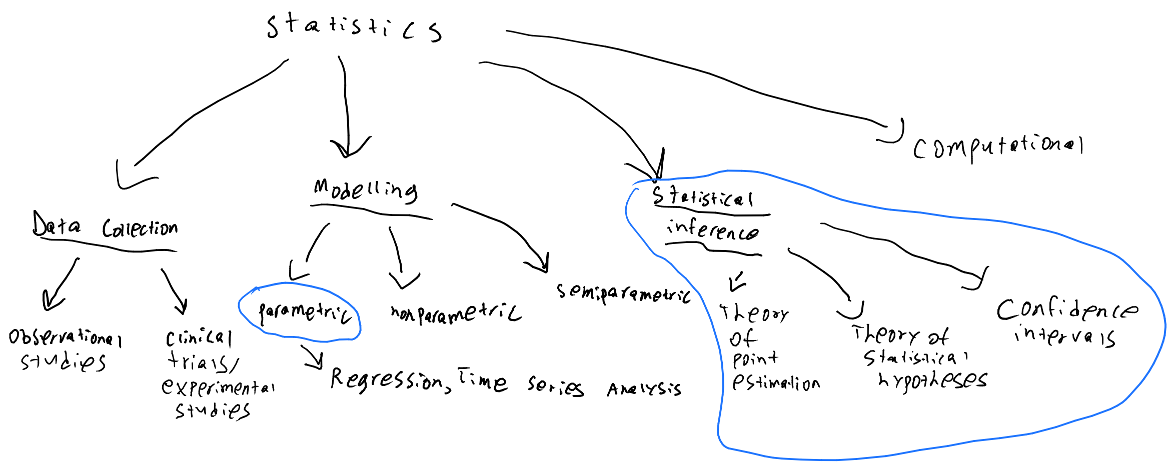
Contents

1 01/09/18

What we will cover this semester Will essentially cover chapter 8,9,10. For chapter 11, he will give us his own notes. The first 6 sections of chapter 13 and a few sections from chapter 14. Occasionally we will go back to chapter 7 to revisit things like the t distribution. In 323, we made probabilistic models. Statistics is the breach in which we connect these models to real life. Otherwise, those are just models. A core part of data analysis and data sciences is statistics and computer science.

1.1 Overview - What is Statistics?

Inductive logic, we have a sample from the population we want to make inference about. With this data, we want to extend the results to the whole population. From small to large, sample to the population.



- Observational studies: we go to the population and make observations.
- Experimental studies: give test subjects something, i.e. give them cigarettes when trying to test for if cigarettes lead to cancer. Need to account for causation, other factors that can affect outcome. In order to do so we have to keep their diet and other factors controlled. We must also have some sort of randomization, we can't send all males to one group and all females to another, as males may have a tendency to smoke or something of the like. These are also called clinical trials.
- When we have data, the next step is modeling. May occasionally speak of this, but this is not part of the course. There are different approaches to modeling, can be split into 3 parts.
  - Parametric: the salary is distributed like a distribution (ex. Gamma), but we don't know the parameters. Take for example, we always know that the normal distribution is a bell curve, but we don't know where it's centered. Very useful, but we might have a miss-specification. How do we know our models are correct? Most of the time we will be talking about **parametric** models.
  - Semiparametric
  - Nonparametric: since we don't know if parametric models are correct, we make no assumption about the distribution. We just assume that $X \sim F$, all we assume about F is that it's continuous, nothing more. This is an infinite dimensional vector. Why? How do we know a function? We have a vector for F, like $F(1), F(2, \ldots)$. How do we approximate this? $X_i \overset{iid}{\sim} F, i = 1, 2, \ldots, n$. n patients, with all the same distribution. So $F(t) = P(X \leq t)$. What does this

tell us? The proportion of time that x falls below t. So with n samples, how do we mimic this? We count the number of observations below t, i.e. $\frac{\# X_i \leq t}{n}$, which is an approximation of the above. This is an empirical observation. More mathematically:

$$\varepsilon(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

So we have $\hat{F}_n(t) = \frac{1}{n}\sum_{i=1}^{n} \varepsilon(t - x_i)$. This gives us a binomial distribution. But we are assuming they are all the same distribution.
Nonparametric approaches are good for functions of single variables, but not for multi variables, which is what semiparametric was made for.

- Bayesian inference: when you learn that $X \sim N(\mu, \sigma^2)$, X is normally distributed and μ is the average of the whole population. Bayes' approach says that these parameters are not constants, these are random variables themselves. Bayes did not look at probability as a frequentest approach, not the proportion of when something arrives (frequentest approach works when we have a huge sample). The other approach that Bayes had was an updating approach, that our parameters are unknown. This is good for when you have a stream of data (machine learning is a prime example). We have a lack of knowledge and then we update it using Bayesian's approach. $\rightarrow X|\mu, \sigma^2 \sim N(\mu, \sigma^2)$, i.e. the parameters are also normally distributed.

Most of the time we'll be at parametric modeling and statistical inference.

1.2 Point Estimation

What do we mean by point estimation? A scientific guess about the unknown parameter of the population. Consider the following situation:
$x_1, \ldots, x_n \sim N(\mu, 1)$ (usually interested in the normal distribution, binomial and poisson). Suppose this is the IQ of high school graduates in Canada (the x_i are numbers). Why do we call this distribution normal? Because for a healthy population, most of the weight should be in the middle, just like the bell curve. The Normal distribution is especially important for modeling error. For insurance companies, we see at the tails that there aren't many large claims.
We want to find μ. Recall that $E(X_i) = \mu, i = 1, 2, \ldots, n$ (if they all have the same observations, they have the same mean).
First, what is a point estimation? What properties should it have? If we know the value of μ, we have the whole thing, can calculate everything. How do we estimate this? The whole

population is huge, so we take a sample part of the population, mimicking the real μ, getting $\overline{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$. $\overline{X}_n$ is useful, but $\overline{X}_n - \mu$ isn't, as there's an unknown we have here.

Statistic A function of observations that does not depend on any unknown parameter.

Ex $\overline{X}_n$ is a statistic. $\overline{X}_n - \mu$ is not.

Estimator A statistic that aims at estimating an unknown parameter (we want to work with it). For example, if μ moves from $-\infty$ to ∞, we want to have an estimator that also has the same range, not one that is strictly positive. Example: $\overline{X}_n$ is an estimator. However, consider:

$$S^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \overline{X}_n)^2$$

This is a statistic, but not an estimator, it always returns a positive value. Also, take for example in physics, where each measure has a unit of measurement. This statistic wouldn't even be the same unit, so it once again is a bad estimator.
When we take a mean and try to estimate it, the next step is to figure out how we quantify possible bias.

$$\varepsilon = |\overline{X}_n - \mu|$$

We can use Tchbyshev's inequality to put a bound on the error.

$$P(|X - \underbrace{E(X)}_{\mu_x}| > k\sqrt{\underbrace{Var(X)}_{\sigma_x^2}})$$

$$P(|X - \mu_x > k\sigma_x) \leq \frac{1}{k^2}$$

Very useful, assume very little but get lots of information. One of the big hammers of probability and statistics. The only thing we assume here is the existence of the second moment.
Consider $k = 3$.

$$P(|X - \mu_x| > 3\sigma_x) \leq \frac{1}{9}$$

$$P(|X - \mu_x| \leq 3\sigma_x) \geq 1 - \frac{1}{9} \approx \%89$$

Without knowing anything else about the distribution, this tells us that about 89% of the population is within 3 times the variance of the mean.