



**N-FEM**

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# INTRODUCTION

- If a response is a function of space and time, we need to move in space and time.
- Partial discretisation, where the differential equation at each node is now already discretised by space and only dependant on time. FDM will do this at a point where at each point we have a differential equation that is now numerically defined
- $\hat{f}(x, t) \approx f(x, t)$
- There are methods developed that are dependant on  
Minimisation Error =  $\hat{f}(x, t) - f(x, t)$
- We can also minimise the differential equation  
$$k \frac{\partial^2 \hat{f}}{\partial^2 x} - f(x) = R(x)$$
$$\int_{x_a^e} W(x) R(x) dx = 0$$

- Galerkin : Babnov, Petrov
- Before dynamics, if we go for statics (Steady state) without partial descritisation
- So we can  $\int_{x_a}^{x_b} W(x)R(x)dx = 0$   
It forces the function to be averagely zero. So some nodes will satisfy it exactly!!  
$$\int_{x_a}^{x_b} W(x) \cdot \left( k \frac{\partial^2 \hat{f}}{\partial x^2} - f(x) \right) dx = 0$$
- I will have to assume however the function  $f(\hat{x})$ . Easy to take polynomials, but the order should not vanish. Minimum requirement, but can we reduce this need of order??

- So we can keep the function like this :

$$\int_{x_a}^{x_b} W(x) \cdot k \frac{\partial^2 \hat{f}}{\partial x^2} dx - \int_{x_a}^{x_b} W(x) \cdot f(x) - dx = 0 \quad (1)$$

- And using integration by parts we get

$$\left( W(x) \cdot k \frac{\partial \hat{f}}{\partial x} \right) \Big|_{x_a}^{x_b} - \int_{x_a}^{x_b} \frac{dW(x)}{dx} \cdot k \frac{\partial \hat{f}}{\partial x} dx - \int_{x_a}^{x_b} W(x) \cdot f(x) - dx = 0$$

- Note we have reduce the order, and we also get the boundary term. (The flux that we describe at the end). And we have a weak weighted residual statement.
- It is an integral statement. Not point based in FDM.

- Choosing weight :
  - ▶ Any function : Petrov-Galerkin
  - ▶ Shape function : Bubnov - Galerkin
- Suppose we have actual  $f(x)$  and approx  $\hat{f}(x) = a_0 + a_1x$ . Knowing the boundary conditions at  $x = x_a$  and  $x = x_b$ , where  $\hat{f} = f_a$  and  $\hat{f} = f_b$
- We get  $\hat{f}(x) = \frac{x_b - x}{x_b - x_a}f_a + \frac{x - x_a}{x_b - x_a}f_b$ , which are shape linear functions. Where we get the boundary values if we keep x at boundary.

$$\hat{f}(x) = N_1(x)f_a + N_2(x)f_b \quad (2)$$

given by Ritz, Ritz approximation which gives us a way how to choose  $\hat{f}(x)$

- Petrov :  $W_i = N_i$  and suppose  $\hat{f}(x) = \sum_1^3 N_i(x)f_i$

Weak form

$$\left( W(x).k \frac{\partial \hat{f}}{\partial x} \right) \Big|_{x_a}^{x_b} - \int_{x_a}^{x_b} \frac{dW(x)}{dx} \cdot k \frac{\partial \hat{f}}{\partial x} dx - \int_{x_a}^{x_b} W(x).f(x) - dx = 0$$

$$B.T - \int_{x_a}^{x_b} \frac{dN_i}{dx} \cdot k \frac{\partial}{\partial x} (N_1f_1 + N_2f_2 + N_3f_3) dx - \int_{x_a}^{x_b} N_i.f(x) - dx = 0$$

Weight gives equation at each node. Only is differentiation in partial.

- If we keep the unkown boundary terms as a vector we get

$$\mathbf{0} = \mathbf{B}.T - \mathbf{K}\mathbf{f} - \mathbf{P} \quad (3)$$

where  $K_{ij} = \int_{x_a}^{x_b} \frac{dN_i}{dx} \cdot k \frac{\partial N_j}{\partial x} dx$

- FDM writes at every node. Here we minimise the governing differential in the domain (Integral). The weak form is valid over the entire domain. And now we write the equation at some points dependant on the approx funciton.

- For a three parameter approximation

$$\begin{bmatrix} 0 - BT \\ 0 - BT \\ 0 - BT \end{bmatrix} = - \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} - \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \quad (4)$$

where  $K_{ij} = \int_{x_a}^{x_b} \frac{dN_i}{dx} \cdot k \frac{\partial N_j}{\partial x} dx$  and  $p_i = \int_{x_a}^{x_b} N_i f(x) dx$

- $\mathbf{Kd} = (\mathbf{BT} - \mathbf{p}) = \mathbf{F}$  which is the descretised form.
- Differential system  $\rightarrow$  Alegebraic system
- We can take N as picewise also (T.Kant)
- B.T will disappear so for a three noded we get :

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} BT - 0 \\ 0 \\ BT - 0 \end{bmatrix} + \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \quad (5)$$

$B.T = (W \cdot k \frac{du}{dx})|_{x_1}^{x_2}$ , once u is known at boundary and prescribed, the weight will be zero. Weight is given only at points where we don't know

- Any general function can be approximated by linear terms in a smaller domain, while in a larger domain the function is more complicated.
- This is an integral method so we can always discretise it

$$0 = \int_{x_a}^{x_b} W(x) R(x) dx = \sum_{e=1}^n \int_{x_a^e}^{x_b^e} W^e(x) R^e(x) dx \quad (6)$$

- Now so this is the concept of finite element method form of weighted residual method
  - ▶ Continuity of the field variables must be maintained
  - ▶ So we can do the computation only on one element
- Boundary volume method : Only at the boundary (Dimension less)

- Again when we keep the full term we get

$$\left| \left( W(x) \cdot k \frac{\partial \hat{f}}{\partial x} \right) \right|_{x_a}^{x_b} - \int_{x_a}^{x_b} \frac{dW(x)}{dx} \cdot k \frac{\partial \hat{f}}{\partial x} dx - \int_{x_a}^{x_b} W(x) \cdot f(x) - dx = 0 \quad (7)$$

- In the interior nodes of a 2 element discretisation of 2 noded element, the boundary term from element 1 becomes  $BT_{x_2} - BT_{x_1}$  and from element 2 becomes  $BT_{x_3} - BT_{x_2}$
- As we Join node 2, we will get at node  $x_2$  that the boundary terms will cancel each other.
- The way I like to look at this is that the B.T. is the internal force from each element and corresponding face. Each element will give a boundary term pointing corresponding to the face.

- So in descritised form for a 2 noded element we get :

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} BT_1 \\ BT_2 \end{bmatrix} + \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \quad (8)$$

where  $K_{ij} = \int_{x_a^e}^{x_b^e} \frac{\partial N_i^e}{\partial x} k \frac{\partial N_j^e}{\partial x} dx$

$p_i^e = \int_{x_a^e}^{x_b^e} N_i^e f(x) dx$

- And then we can write the equations for each element, and rearrange the dof as global dof in the same vector and we get our global matrix!
- The advantage is always that all the integrals are done in the sub domain

- With the local weighted residual method, we can discretise it into simpler domains etc.
- Second order - Axial. Bending is forth order, (If we do second order, then two equations with two unknowns)
- Axial rod subjected to axial deformation

$$EA \frac{\partial^2 u}{\partial x^2} = p(x) \quad (9)$$

$$EI \frac{\partial^4 u}{\partial x^4} = p(x) \text{ (Bending)}$$

with boundary conditions.

- Unknown per node is 1, but varies over x and y

- $\frac{\partial}{\partial x} \left( k_x \frac{\partial \phi}{\partial x} \right) + \frac{\partial}{\partial y} \left( k_y \frac{\partial \phi}{\partial y} \right) + p(x, y) = 0$

- Mixed boundary conditions

- ▶ Dirichlet :  $\phi = \hat{\phi}$  in some boundary
- ▶ General Newman :  $k_x \frac{\partial \phi}{\partial x} L_x + k_y \frac{\partial \phi}{\partial y} Y + q + \alpha(\phi - \phi_a) = 0$  : In some portion of the boundary
- ▶ For isotropic material

$$k \frac{\partial \phi}{\partial n} + q + \alpha(\phi - \phi_a) \quad (10)$$

## 2D QUASI-HARMONIC EQUATION

- 1 dof/node
- Wide range of physical problems governed by Q-H equation.

<u>PHYSICAL PROB.</u>	<u>UNKNOWN, <math>\phi</math></u>	<u>RHS By</u>	<u><math>p(x,y)</math></u>
Heat cond.	Temperature	Conductivity	Int. heat gen.
Gas diffusion	Concentration	Diffusivity	
Seepage	Pressure head	Permeability	
Incompressible flow	Streamfunction	1	
Compressible flow	Velocity	Density	
Magnetostatics	Mag. potential	Reluctivity	Current density
Torsion	Stress function	(Shear mod.)	Twist
Torsion	Warping function	Shear mod.	
Lubrication	Pressure	(Film thickness) <sup>3</sup> /Viscosity	Lubricant supply

- So  $R(x, y) = k \frac{\partial^2 \phi}{\partial x^2} + k \frac{\partial^2 \phi}{\partial y^2} + p$

- Global error =  $\sum_{e=1}^N \int_{\Omega_e} local$

- WRM

$$\int \int_{\Omega} W \left( k \frac{\partial^2 \phi}{\partial x^2} + k \frac{\partial^2 \phi}{\partial y^2} + p \right) dx dy = 0 \quad (Strong statement) \quad (11)$$

- Greens theorem :

$$\int \int_A \left( \frac{\partial C}{\partial x} \frac{\partial D}{\partial x} - C \frac{\partial^2 D}{\partial x^2} \right) dA = \int_S C \frac{\partial D}{\partial x} L_x ds \quad (12)$$

So we use this we get :

$$\int \int_A C \frac{\partial^2 D}{\partial x^2} dA = \int \int_A \frac{\partial C}{\partial x} \frac{\partial D}{\partial x} - \int_S C \frac{\partial D}{\partial x} L_x ds \quad (13)$$

The first term is like  $W(k \frac{\partial^2 \phi}{\partial x^2})$

- $\int \int_A \frac{\partial W}{\partial x} k \frac{\partial \hat{\phi}}{\partial x} dA - \int_S W \cdot k \frac{\partial \hat{\phi}}{\partial x} L_x ds + \int \int_A \frac{\partial W}{\partial y} k \frac{\partial \hat{\phi}}{\partial y} dA - \int_S W \cdot k \frac{\partial \hat{\phi}}{\partial y} L_y ds + \int \int W p dA = 0$   
(Weak statement)
- Where  $L_x$  is direction cosine of s with respect to x
- $\hat{\phi} = \sum_{i=1}^n N_i(x, y) \phi_i$  (Ritz, bubonov)
- For  $j^{th}$  node
 
$$\int \int_A \frac{\partial N_j}{\partial x} k \frac{\partial \sum_{i=1}^n N_i \phi_i}{\partial x} dA - \int_S N_j k \frac{\partial \sum_{i=1}^n N_i \phi_i}{\partial x} L_x ds + \int \int_A \frac{\partial N_j}{\partial y} k \frac{\partial \sum_{i=1}^n N_i \phi_i}{\partial y} dA -$$

$$\int_S N_j k \frac{\partial \sum_{i=1}^n N_i \phi_i}{\partial y} L_y ds + \int \int_A N_j p dA = 0$$
- So  $K_{ij} = \int \int_A k \left( \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial y} \right) dA$
- **Kd = P + X<sub>i</sub>(Direct) + Newman = F**

- Anticlockwise noded. Each node has (x,y)
- $\hat{\phi} = N_1(x, y)\phi_1 + N_2(x, y)\phi_2 + N_3(x, y)\phi_3$
- Let  $\hat{\phi} = a + bx + cy$  (Three unknowns, you need three for a plane)
  - $$\begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$
  - Then we can find the shape functions
  - Then we can find the element! Stress and strain is constant

- In plane deformation : Plane stress, plane strain, axisymmetric
- Integration over the triangle domain (area) poses problems
- Then they developed area coordinates, which is the area sections that they develop for the shape functions  $L_i = N_i$ . Some area integration based on these area coordinates.
- The stiffness is constant, and area can be formed while the force can be found

The steps of a finite element method are

- Divide the whole domain into finite parts
- For each element develop relations between pairs of dual variables (primary and secondary, eg forces and displacements)
- Assemble elements together to get the relationship of the variables for the whole system

We are going to look at second differential equations and how to solve them using fem

# EXAMPLE OF PRIMARY AND SECONDARY VARIABLES

**Table 3.2.1:** List of fields in which the model equation in Eq. (3.2.1) arises, with meaning of various parameters and variables; see the bottom of the table for the meaning of the parameters\*.

Field of study	Primary variable $u$	Coefficient $a$	Coefficient $c$	Source term $f$	Secondary variable $Q$
Heat transfer	Temperature $T - T_{\infty}$	Thermal conductance $kA$	Surface convection $p\beta$	Heat generation $f$	Heat $Q$
Flow through porous medium	Fluid head $\phi$	Permeability $\mu$	0	Infiltration $f$	Point source $Q$
Flow through pipes	Pressure $P$	Pipe resistance $1/R$	--	--	Point source $Q$
Flow of viscous fluids	Velocity $v_x$	Viscosity $\mu$	0	Pressure gradient $-dP/dx$	Shear stress $\sigma_{xx}$
Elastic cables	Displacement $u$	Tension $T$	--	Transverse force $f$	Point force $P$
Elastic bars	Displacement $u$	Axial stiffness $EA$	0	f	P
Torsion of bars	Angle of twist $\theta$	Shear stiffness $GJ$	--	--	Torque $T$
Electrostatics	Electrical potential $\phi$	Dielectric constant $\epsilon$	--	Charge density $\rho$	Electric flux $E$

\*  $k$  = thermal conductance;  $\beta$  = convective film conductance;  $p$  = perimeter;  $P$  = pressure or force;  $T_{\infty}$  = ambient temperature of the surrounding fluid medium;  $R = 128\mu h/(\pi d^4)$  with  $\mu$  being the viscosity;  $h$ , the length and  $d$  the diameter of the pipe;  $E$  = Young's modulus;  $A$  = area of cross-section;  $J$  = polar moment of inertia.

- Consider the equation

$$-\frac{d}{dx} \left( a \frac{u}{x} \right) + cu = f \quad \text{for } 0 < x < L \quad (14)$$

- where  $a = a(x)$ ,  $c = c(x)$ ,  $f = f(x)$  are the known quantities and  $u(x)$  has to be found

- The domain  $\Omega = (0, L)$  is discretised into a set of intervals with  $\Omega^e = (x_a^e, x_b^e)$  which denotes the end of the element
- The length of an element  $h_e = x_b^e - x_a^e$

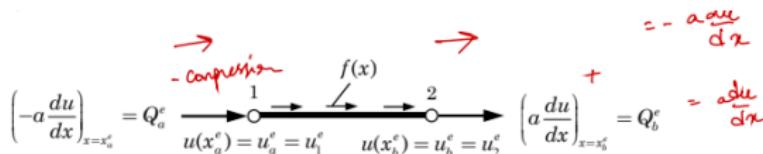


Fig. 3.2.1: A typical finite element in one dimension.

- We find an approx solution over each element  $\Omega^e$  and then we assemble it all together

$$u(x) \approx u_h^e(x) = \sum_j^n c_j^e N_j^e \quad (15)$$

where we choose the shape functions and then have to find the coefficients such that our approx solution is like the real one

- Since there are n unkown parameters (For each dof), we need n linearly independant equations

- Keeping the discretised DE equation we get

$$-\frac{d}{dx} \left( a \frac{u_h^e}{x} \right) + c u_h^e - f(x) = R^e(x, c_1^e, c_2^e, c_3^e, \dots, c_n^e) \neq 0 \quad (16)$$

What we want to do is find the coefficients such that the residual is zero. This is again equilibrium at a point!

- One way to make it zero is to set the weighted integral of the residual zero

$$\int_{x_a^e}^{x_b^e} w_i^e(x) R^e(x, c_1^e, c_2^e, c_3^e, \dots, c_n^e) dx = 0 \quad i = 1, 2, \dots, n \quad (17)$$

- where  $w_i^e$  are different weight functions giving us  $n$  equations for the coefficient parameters  $(c_1^e, c_2^e, c_3^e, \dots, c_n^e)$
- Now the weights however should all be independent and invertible. If we took  $w = 1$ , we would have only one equation
- These weights feel like the variation, but in the variation we choose also that the variations are the same function like the shape functions hmmm.

- If we choose  $w_i^e$  to be the shape functions. We get the Galerkin method. Which is exactly the same as the virtual work where  $\delta v = \sum_i \delta v_i N_i$  where you can say then that  $\delta v_i$  is 1, since the virtual disp magnitude comes out anyways
- Since the residual  $R_e$  has the same order derivatives of the dependant unkown  $u(x)$ , we need at least quadratic representation of  $u_h^e(x)$
- To reduce or weaken the differentiability of the shape functions (node disp are constant) , we distribute the order between the weights and  $u_h^e$

This is the weak form : Reducing the order of the dependant variable to the weight to make the order of the variable lesser

- The weights kind of give different component equilib equations

- Note that in usual structural mechanics we derive the virtual work equation from a potential functional. Most fem methods are based on an element wise application of the Ritz method
- The virtual displacement is an integral statement which is the same as the integral weak form found from the governing differential equations.
- But differential equations are easy to form, and most fem methods are based on de

- After getting the weighted residual statement, the next job is to weaken the differentiability of  $u_h^e$ . To make both the orders of  $u_h^e$  and  $w_i^e$  the same
- The steps are

- ▶ Write weighted residual statement

$$0 = \int_{x_a^e}^{x_b^e} \left[ w_i^e \left( -\frac{d}{dx} \left( a \frac{u_h^e}{x} \right) + cu_h^e - f(x) \right) \right] dx \quad (18)$$

We are taking the summation of the weighted residual over the whole element and saying that it is zero. The weight sort of gets the components

- ▶ Weakening the form using integration by parts

$$\begin{aligned} (uv)' &= uv' + u'v \\ \int_a^b uv' &= uv|_a^b - \int_a^b u'v \\ 0 &= \int_{x_a^e}^{x_b^e} \left[ w_i^e \left( -\frac{d}{dx} \left( a \frac{u_h^e}{x} \right) + cu_h^e - f(x) \right) \right] dx \\ 0 &= \int_{x_a^e}^{x_b^e} \left( a \frac{dw_i^e}{dx} \frac{du_h^e}{dx} + cw_i^e u_h^e - w_i^e f(x) \right) dx - \left[ w_i^e a \frac{du_h^e}{dx} \right]_{x_a^e}^{x_b^e} \end{aligned} \quad (19)$$

- Very interesting, we actually get the boundary terms too!

$$0 = \int_{x_a^e}^{x_b^e} \left( a \frac{dw_i^e}{dx} \frac{du_h^e}{dx} + cw_i^e u_h^e - w_i^e f(x) \right) dx - \left[ w_i^e \cdot a \frac{du_h^e}{dx} \right]_{x_a^e}^{x_b^e} \quad (20)$$

Direct boundary on the dependant: Dritchlet/essential  $u = 0$

Boundary on the derivatives of dependant : Newman/natural  $\frac{du}{dx} = p$

- The coefficient of the weight function which is  $a \frac{du}{dx}$  is the second variable
- We state the differenet variables

$$\text{Primary variable : } u \quad \text{Secondary variable: } n_x \left( a \frac{du}{dx} \right) = Q(x) \quad (21)$$

See that  $n_x = -1, 1$  on left and right end. ???WHye

- In the final weak form, we keep the secondary variables at the element ends as

$$Q_a^e = Q(x_a^e) = - \left( a \frac{du}{dx} \right)_{x_a^e} \quad Q_b^e = Q(x_b^e) = \left( a \frac{du}{dx} \right)_{x_b^e} \quad (22)$$

In the figure above we can think this of a FBD but in arbitary configruaion. The first one is a compressive, while later is a tensile force. In heat the first would be the heat input and later output

- Althout Q replaced  $a(\frac{du}{dx})$ , it is not consdiered as a function of u, but a variable dual to u?????

- The final expression for the weak form is

$$0 = \int_{x_a^e}^{x_b^e} \left( a \frac{dw_i^e}{dx} \frac{du_h^e}{dx} + cw_i^e u_h^e - w_i^e f(x) \right) dx - w_i^e(x_a^e) Q_a^e - w_i^e(x_b^e) Q_b^e \quad (23)$$

But even in the virtual work when you reduce the order of the strains, you get the boundary condition

- The remarks are:

- Integration by parts (i) reduces the degree of the fem approximation (ii) introduces the secondary variables that are physically meaningful as they can be specified at a point where the primary variable is not specified. If the secondary variable is not a physical quantity, then the integration by parts should not be carried out even to reduce the order of  $u_h^e$
- The terms containing both  $w_i^e$  and  $u_h^e$  are called bilinear functional

$$B(w_i^e, u_h^e) = \int_{x_a^e}^{x_b^e} \left( a \frac{dw_i^e}{dx} \frac{du_h^e}{dx} + cw_i^e u_h^e \right) dx \quad (24)$$

but has to be linear with respect to  $w_i^e$  and  $u_h^e$ . So it has to be bilinear map. Like a scalar product with metric tensor where  $u$  and  $v$  are the input and the metric tensor is the bilinear map. If  $a$  or/and  $c$  is a function of  $u$ . Then  $B$  is always linear in  $w$  but not  $u$ .

- Terms having only  $w_i^e$  are only linear functionals because they are only linear with respect to  $w_i^e$ .  $l(w_i^e)$

- Therefore the weak form can be expressed as

$$B(w_i^e, u_h^e) = l(w)i^2 \quad (25)$$

which is a variational problem where we find  $u^e \in U$  such that the equation is satisfied for all  $w_i^e \in U$  (See Reddy page 103 for hilbert spaces)

- The weak form is nothing but the statement of minimum total potential energy, or the variational minimum

$$\begin{aligned} & \Pi(u_h^e) \\ & \delta\Pi = B(\delta u_h^e, u_h^e) - l(\delta u_h^e) = 0 \\ & \Pi(u_h^e) = \frac{1}{2}B(\delta u_h^e, u_h^e) - l(\delta u_h^e) \quad (26) \\ & = \int_{x_a^e}^{x_b^e} \left[ \frac{a}{2} \left( \frac{du_i^e}{dx} \right)^2 + \frac{c}{2} (u_h^e)^2 - u_h^e f \right] dx - u_h^e(x_a^e)Q_a^e - u_h^e(x_b^e)Q_b^e \end{aligned}$$

This is when you have reduced the order of the derivative in the virtual work, (We get the Euler lagrange equilibrium).

- This equation  $\frac{1}{2}B(w_i^e, u_h^e) = l(w)i^2$ , B should be symmetric and the first term is the elastic energy while the later is the work done by the load and point loads.

- We have to satisfy the weak form of the differential equation along with the continuity and boundary conditions. We need to choose a function that satisfies the differentiability requirement and the end conditions  $u(x_i) = u_i^e$ . Any function with a non zero differentiation of the order of the weak form would be a candidate. We can therefore use interpolation.
- The interpolation is

$$u_h^e(x) = c_1^e + c_2^e x \quad (27)$$

is okay, since the differentiation  $\neq 0$ , but we only now need to make sure that  $c_1, c_2$  are such that the end displacements match

$$u_h^e(x_a^e) = c_1^e + c_2^e x_a^e = u_a^e \quad u_h^e(x_b^e) = c_1^e + c_2^e x_b^e = u_b^e \quad (28)$$

or

$$\begin{bmatrix} 1 & x_a^e \\ 1 & x_b^e \end{bmatrix} \begin{bmatrix} c_1^e \\ c_2^e \end{bmatrix} = \begin{bmatrix} u_a^e \\ u_b^e \end{bmatrix} \text{ and we get the interpolating functions } u_h^e(x) = \sum_{j=1}^2 N_j^e u_j^e$$

$$\blacksquare N_1^e(x) = \frac{x_b^e - x}{x_b^e - x_a^e} \quad N_2^e(x) = \frac{x - x_a^e}{x_b^e - x_a^e}$$

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<sup>1</sup>The book makes  $\phi$  for  $N$ . But I usually use that for eigen directions?

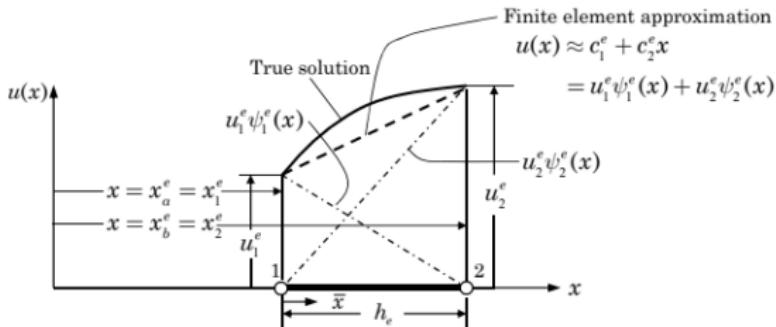


Fig. 3.2.2: Linear approximation over a finite element.

- These  $N$  are linear lagrange interpolation functions and  $u_1^e = u_a^e$        $u_2^e = u_b^e$  are the nodal values of the approx function at the ends.  $U_h$  belongs to a hilbert subspace spanned by  $N_1^e, N_2^e$
- Remember that  $N_i^e(x_j^e) = 1$    if  $i = j$ . They also satisfy the partition of unity where  $\sum_{j=1} N_j^e(x) = 1$

## QUADRATIC APPROXIMATIONS

For a quadratic approximation we choose

$$u_h^e(x) = c_1^e + c_2^e x + c_3^e x^2 \quad (29)$$

- Since there are three parameters we need to have three nodal points where we can relate the constants to.

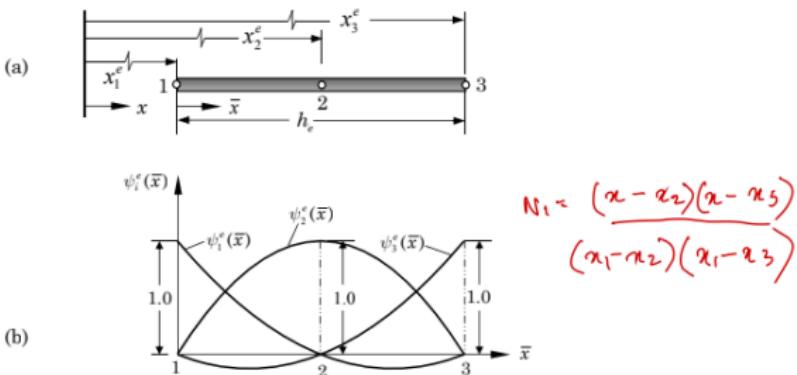
$$x_1^e = x_a^e \quad x_2^e = x_a^e + \frac{h_e}{2} \quad x_3^e = x_a^e + h_e = x_b^e \quad (30)$$

And we similarly get :

$$u_h^e(x) = \sum_{j=1} N_j^e u_j^e \quad (31)$$

where  $N$  are the quadratic lagrange interpolation functions. If they are expressed in local coordinate we get

$$N_1^e(x') = \left(1 - \frac{x'}{h_e}\right) \left(1 - \frac{2x'}{h_e}\right) \quad N_2^e(x') = 4 \frac{x'}{h_e} \left(1 - \frac{x'}{h_e}\right) \quad N_3^e(x') = -\frac{x'}{h_e} \left(1 - \frac{2x'}{h_e}\right) \quad (32)$$



- This is a quadratic element
- Any higher order lagrange interpolations can be developed. A  $(n-1)$ degree can be written as  $u_h^e \sum_{j=1}^n N_j^e u_j^e$
- Where the interpolating function can be given as

$$N_j^e(x) = \prod_{i=1, i \neq j} \left( \frac{x - x_i^e}{x_j^e - x_i^e} \right) \quad (33)$$

For example  $N_1(x) = \left( \frac{x - x_2}{x_1 - x_2} \right) \left( \frac{x - x_3}{x_1 - x_3} \right)$

- Approx solution should be continuous and differentiable as needed by the weak form. This ensures that every term in the differential equation does not have a zero coefficient.
- It should be a complete polynomial (Pascal's law). To capture all the actual deformation. Lower to higher!
- It should interpolate the primary variables at the nodes of the fem and at the end points. To ensure continuity of the primary variable across elements!

- Keeping the approximate solutions in the weak form gives us the algebraic equations
- The degree of the approx solution has to be decided a priori. If there are more than 2 nodes, then the number of non-zero secondary variables increases at the interior nodes

$$0 = \int_{x_a}^{x_b} \left( a \frac{dw_i^e}{dx} \frac{du_h^e}{dx} + cw_i^e u_h^e \right) dx - \int_{x_a}^{x_b} w_i^e f dx - \sum_{j=1}^n w_i^e(x_j^e) Q_j^e \quad (34)$$

- If 1 and n denote the end points then  $Q_1^e, Q_n^e$  denote the unknown point sources, while the other  $Q_j^e \quad (j = 2, 3 \dots n-1)$  are the externally applied and known point sources. So at the ends these are internal or external loads?????
- If we keep  $w_i^e = N_i^e$  into the weak form, we get n algebraic equations. This is the Galerkin method (Original was weighted of residual and not of the weak form, that would be exactly the same to Ritz method). The  $i$ th algebraic equation is the one obtained by keeping  $w_i^e$  as  $N_i^e$ . This is the same as the virtual work method, where each equation of a discretised system comes from the virtual displacement of each node

■ So we get

$$0 = \int_{x_a}^{x_b} \left( a \frac{dN_i^e}{dx} \sum_{j=1}^n u_j^e \frac{dN_j^e}{dx} + c N_i^e \sum_{j=1}^n u_j^e N_j^e \right) dx - \int_{x_a}^{x_b} N_i^e f dx - \sum_{j=1}^n N_i^e(x_j^e) Q_j^e \quad (35)$$

so for each equation i there will be a summation on the derivatives of the approx solution due to chain rule! and we get

$$0 = \sum_{j=1}^n \left[ \int_{x_a}^{x_b} \left( a \frac{dN_i^e}{dx} \frac{dN_j^e}{dx} + c N_i^e N_j^e \right) dx \right] u_j^e - \int_{x_a}^{x_b} N_i^e f dx - Q_i^e \quad (36)$$

where we have taken the summation of u magnitude coefficient for, shape functions outside. Also we see that for each shape function at each node we actually only get the boundary load at that variation. Because  $N_i^e(x_j^e) = 0$  when  $i \neq j$ . We get for each node!

$$0 = \sum_{j=1}^n K_{ij}^e u_j^e - f_i^e - Q_i^e \quad (37)$$

■ Where

$$K_{ij}^e = \int_{x_a}^{x_b} \left( a \frac{dN_i^e}{dx} \frac{dN_j^e}{dx} + c N_i^e N_j^e \right) dx = B(N_i^e, N_j^e) \quad (38)$$
$$f_i^e = \int_{x_a}^{x_b} f N_i^e dx = l(N_i^e)$$

So this is interesting the coefficient of the stiffness says basically states change in the shape functions!

Matrix form

$$\mathbf{K}^e \mathbf{u}^e = \mathbf{f}^e + \mathbf{Q}^e \approx \mathbf{F}^e$$

where

- $K^e$  is the symmetric coefficient, stiffness matrix
- $f^e$  is the source or force vector
- This method is the weak-form Galerkin or Ritz finite element method

## Matrix form

$$\mathbf{K}^e \mathbf{u}^e = \mathbf{f}^e + \mathbf{Q}^e \approx \mathbf{F}^e$$

- But for every element, we have  $n$  equations and  $n + 2$  unknowns. The 2 unknowns are the secondary nodal values that we don't know  $Q_a^e, Q_b^e$ .  $(u_1^e, u_2^e \dots u_n^e)$  are the element primary nodal degrees. Remember these values, we know if they are along the inside of the element as external forces.
- Assembling the elements by imposing the continuity of the elements. U2 of 1 is U1 of 2. We get the same number of equations and unknowns (Primary + Secondary).
- The stiffness and force matrix can be found for a certain value. And if the coefficients (a,c,f) are also functions of x, then we need to do numerical integration.

# GENERAL FEM ONE-D MATRIX FORM

Linear element (i.e. element with linear approximation)

$$\psi_1^e(\bar{x}) = 1 - \frac{\bar{x}}{h_e}, \quad \psi_2^e(\bar{x}) = \frac{\bar{x}}{h_e} \quad (3.2.37)$$

$$\left( \frac{a_e}{h_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{c_e h_e}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \right) \begin{Bmatrix} u_1^e \\ u_2^e \end{Bmatrix} = \frac{f_e h_e}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} + \begin{Bmatrix} Q_1^e \\ Q_2^e \end{Bmatrix} \quad (3.2.38)$$

Quadratic element (i.e. element with quadratic approximation)

$$\psi_1^e(\bar{x}) = \left(1 - \frac{2\bar{x}}{h_e}\right) \left(1 - \frac{\bar{x}}{h_e}\right), \quad \psi_2^e(\bar{x}) = \frac{4\bar{x}}{h_e} \left(1 - \frac{\bar{x}}{h_e}\right), \quad \psi_3^e(\bar{x}) = -\frac{\bar{x}}{h_e} \left(1 - \frac{2\bar{x}}{h_e}\right) \quad (3.2.39)$$

$$\left( \frac{a_e}{3h_e} \begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix} + \frac{c_e h_e}{30} \begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix} \right) \begin{Bmatrix} u_1^e \\ u_2^e \\ u_3^e \end{Bmatrix} = \frac{f_e h_e}{6} \begin{Bmatrix} 1 \\ 4 \\ 1 \end{Bmatrix} + \begin{Bmatrix} Q_1^e \\ Q_2^e \\ Q_3^e \end{Bmatrix} \quad (3.2.40)$$

- Where in linear element lets write the expansion for the first linear element equation

$$K_{11} u_1^e + K_{12} u_2^e = f_1 + Q_1^e$$

$$\left[ \int_0^{h_e} a \frac{-1}{h_e} \frac{-1}{h_e} + c \left(1 - \frac{x}{h_e}\right) \left(1 - \frac{x}{h_e}\right) \right] u_1^e + \left[ \int_0^{h_e} a \frac{-1}{h_e} \frac{1}{h_e} + c \left(1 - \frac{x}{h_e}\right) \left(\frac{x}{h_e}\right) \right] u_2^e = \int_0^{h_e} f \left(1 - \frac{x}{h_e}\right) + Q_1^e \quad (39)$$

2

---

We note that (i) In quad, the force vector is not just  $fh/3$  but it depends on the work done! Not same with 2 elements combined. (ii) There are also more unknowns than the no of equations. When one element is used however we have only n unknowns cause bcs will be applied on Q.

## PROBLEM #1

- Consider a homogeneous, isotropic bar of length  $L(m)$ , cross sectional area ( $A$ ) and conductivity  $k (W/(m^o C))$
- Ambient temperature is  $T_o (^oC)$
- No heat loss throughout the bar and the right end is exposed to ambient temperature of  $T_{inf}$
- Uniform heat of  $g_0$ , heat transfer with fin and air is  $\beta$
- Check Reddy Page 3.2.1

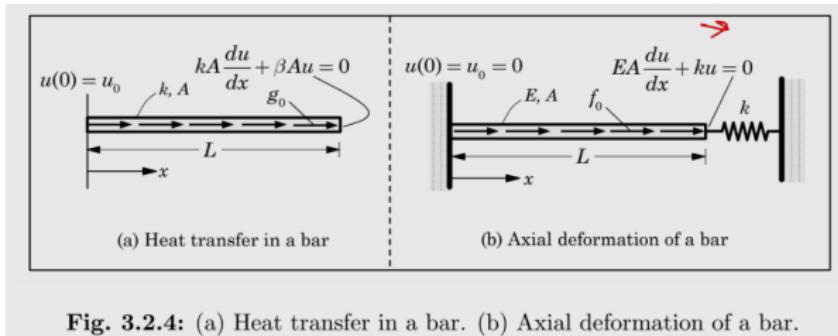
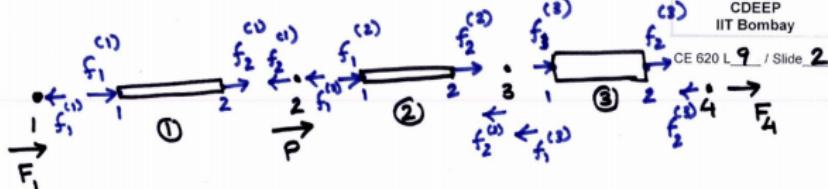


Fig. 3.2.4: (a) Heat transfer in a bar. (b) Axial deformation of a bar.

Step - 5

Assembly by nodal equilibrium



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- Remember when we draw the internal force signs as compressive for a tensile element. This we are drawing for the nodal equilibrium side cut of the fbd. Remember force, stresses are all defined with respect to the cut and face.
- In the Reddy example you will find that at the B.C., the internal force reaction at the node,  $Q_2^4$  and  $kU_5$  is a force in ←

- Check reddy 114 for origin at left side of element, natural  $\bar{x}$
- Origin at center denoted as  $\xi$ 
  - ▶  $\xi$  is -1 and +1 at LHS and RHS. Since your current basis is euclidian. The transformation is linear.
  - ▶  $x$  can be given as a function of  $\xi$  and found out
  - ▶ This is interesting. Suppose  $x = x_a + \frac{h}{2}(1 + \xi)$

$$\frac{d}{dx} = \frac{2}{h} \frac{d}{d\xi} \quad \frac{dN_i}{dx} = \frac{2}{h} \frac{dN_i}{d\xi}$$

$$dx = \frac{h}{2} d\xi$$

$$\int_{x_a^e}^{x_b^e} N_i(x) dx = \int_0^h N_i(\bar{x}) d\bar{x} = \frac{h}{2} \int_{-1}^1 N_i(\xi) d\xi \quad (40)$$

$$\int_{x_a^e}^{x_b^e} \frac{dN_i(x)}{x} dx = \int_0^h \frac{N_i(\bar{x})}{\bar{x}} d\bar{x} = \int_{-1}^1 \frac{N_i(\xi)}{d\xi} d\xi$$

$$\int_{x_a^e}^{x_b^e} \frac{dN_i(x)}{x} \frac{dN_j(x)}{x} dx = \int_0^h \frac{N_i(\bar{x})}{\bar{x}} \frac{N_j(\bar{x})}{\bar{x}} d\bar{x} = \frac{2}{h} \int_{-1}^1 \frac{N_i(\xi)}{d\xi} \frac{N_j(\xi)}{d\xi} d\xi$$

- Governing differential equation. Suppose a single field  $u(x, y)$  with the following partial differential equation varies over x and y
- $\frac{\partial}{\partial x} \left( k_x \frac{\partial \phi}{\partial x} \right) + \frac{\partial}{\partial y} \left( k_y \frac{\partial \phi}{\partial y} \right) + f(x, y) = 0$  where for a heat problem, k is the conucity in a orthotropic medium and u is the temperature and f is the internal heat generation.
- Mixed boundary conditions
  - ▶ Dritchlet :  $\phi = \hat{\phi}$  in some boundary
  - ▶ General Newman :  $k_x \frac{\partial u}{\partial x} n_x + k_y \frac{\partial u}{\partial y} n_y + q_c = \hat{q}_n$  : In some portion of the boundary.
    - $q_c$  represents the convective component of flux (heat problems  $q_c = h_c(u - u_c)$ )
    - $n_x = \cos(x, \mathbf{n})$  and  $n_y = \cos(y, \mathbf{n})$  which is the angle of the normal of the boundary and the axis
    - $u_c$  is the ambient temperature and  $h_c$  is the convective heat coefficient

- In FEM the domain is discretised into subdomains. Any shape qualifies as long as the approximating functions  $N_i^e$  can be derived uniquely for the shape. The discretisation may or may not represent the actual boundary though at really curved regions.
- Suppose the dependent unknown  $u$  is given by  $\hat{u}^e = \sum_{j=1}^n u_j^e N_j^e(x, y)$
- The interpolation functions depend not only on the number of nodes but also on the shape of the element
- A triangle will need two points given by  $\hat{u}^e = c_1 + c_2x + c_3y$
- A triangle with three nodes in each side is given by  
$$\hat{u}^e = c_1 + c_2x + c_3y + c_4xy + c_5x^2 + c_6y^2$$

- The n nodal values  $u_j^e$  must be found such that the approximating solution  $u_h^e(x)$  satisfies the governing differential equation in a weak sense. Steps are :

- Take non zeros of the G.D.E as  $R(x,y)$  and multiply by the weight function  $w_i^e$  from a set of linearly independant functions. We get then

$$\int \int_{\Omega} w_i^e \left( k \frac{\partial^2 u_h^e}{\partial x^2} + k \frac{\partial^2 u_h^e}{\partial y^2} - f(x, y) \right) dx dy = 0 \quad (\text{Strong statement}) \quad (41)$$

For n independant choices of  $w_i^e$ , we get n independant equations.

- Distribute so that for both  $u_h^e, w_i^e$  are required to be differentiated once. Using component form of the divergence theorem or Greens theorem :

$$\begin{aligned} \int \int_A \frac{\partial}{\partial x} (w_i^e F_1) dA &= \int_S (w_i^e F_i) n_x ds \\ \int \int_A \frac{\partial}{\partial y} (w_i^e F_2) dA &= \int_S (w_i^e F_i) n_y ds \end{aligned} \quad (42)$$

$$F_1 = k_x \frac{\partial u_h^e}{\partial x} \quad F_2 = k_y \frac{\partial u_h^e}{\partial y}$$

And from product rule we get

$$-w_i^e \frac{\partial F_1}{\partial x} = -\frac{\partial}{\partial x} (w_i^e F_1) + F_1 \frac{\partial w_i^e}{\partial x} \quad -w_i^e \frac{\partial F_2}{\partial x} = -\frac{\partial}{\partial x} (w_i^e F_2) + F_2 \frac{\partial w_i^e}{\partial x} \quad (43)$$

- And we get the weak form as

$$0 = \int \int_A \left( k_x \frac{\partial w_i^e}{\partial x} \frac{\partial u_h^e}{\partial x} + k_y \frac{\partial w_i^e}{\partial x} \frac{\partial u_h^e}{\partial y} - w_i^e f(x, y) \right) dA - \int_S w_i^e \left( k_x \frac{\partial u_h^e}{\partial x} n_x + k_y \frac{\partial u_h^e}{\partial y} n_y \right) dS$$

- Now the order of the differentiation has been reduced.
- Looking at the boundary terms, we see that  $u_h^e$  is the primary variable and essential boundary.  $q_n = \left( k_x \frac{\partial u_h^e}{\partial x} n_x + k_y \frac{\partial u_h^e}{\partial y} n_y \right)$  is the secondary variable and the natural boundary condition. It is positive as one travels counterclockwise in the boundary.
- This is why nodes are counted in counterclockwise and boundary integrals are carried in counter-clockwise sense.  $\mathbf{q}$  is the outward flux normal and the flux  $\mathbf{q} = q_x e_1 + q_y e_2$

$$q_x = k_x \frac{\partial u_h^e}{\partial x} \quad q_y = k_y \frac{\partial u_h^e}{\partial y}$$

- The normal flux is given by

$$q_n = \hat{n} \cdot \mathbf{q} = k_x \frac{\partial u_h^e}{\partial x} n_x + k_y \frac{\partial u_h^e}{\partial y} n_y \quad (44)$$

1. So the third step is to use the general Newmann boundary condition and write as

$$0 = \int \int_A \left( k_x \frac{\partial w_i^e}{\partial x} \frac{\partial u_h^e}{\partial x} + k_y \frac{\partial w_i^e}{\partial x} \frac{\partial u_h^e}{\partial y} - w_i^e f(x, y) \right) dA - \int_S w_i^e \left( \hat{q}_n - h_c (u_h^e - u_c) \right) dS \quad (45)$$

2. Rearranging we get

$$0 = \int \int_A \left( k_x \frac{\partial w_i^e}{\partial x} \frac{\partial u_h^e}{\partial x} + k_y \frac{\partial w_i^e}{\partial x} \frac{\partial u_h^e}{\partial y} - w_i^e f(x, y) \right) dA - \int_S h_c w_i^e u_h^e dS - \int_S w_i^e (\hat{q}_n + h_c u_c) dS \quad (46)$$

3. So we get the form  $B(w_i^e, u_h^e) = l(w_i^e)$

- Note that the variational problem is to find a  $u_h^e$  such that  $B(w_i^e, u_h^e) = l(w_i^e)$  for all  $w_i^e \in U_h$  a subspace span by polynomial basis functions.
- $B(w_i^e, u_h^e)$  is bilinear and symmetric and  $l(w_i^e)$  is linear in w. So we can construct a functional  $I = \frac{1}{2}B(u_h^e, u_h^e) - l(u_h^e)$  and the minimum is equivalent to solving the variation problem.
- It is not always possible to make a functional whose weak form whose first variation is equivalent to the weak form.

- The weak form in the above equation requires that the approx function to be at least linear in both x and y. Suppose that  $u_h^e = N_i u_i$
- The weak form is given as

$$\left( \int_A \int_A \frac{\partial N_j}{\partial x} k \frac{\partial \sum_{i=1}^n N_i}{\partial x} dA + \int_A \int_A \frac{\partial N_j}{\partial y} k \frac{\partial \sum_{i=1}^n N_i}{\partial y} dA + \int_S h_c w_i^e N_j^e \right) u_j^e = 0 \quad (47)$$

$$- \int_A N_i^e f dA - \int_S N_i^e (\hat{q} + h_c u_c) dS$$

- This is when w is taken as the virtual displacement of the dependant unknown or  $w_i^e = \delta u_h^e$ . Each w we get a separate equation and we get
- $K^e u^e = f^e + q^e$

- $u$  should be continuous as required in the weak form that is all the terms are represented as non zero values
- The polynomials must be complete and contain the same order of  $x$  and  $y$
- All the terms in the polynomial should be linearly independant. The no of linearly independent terms in representation  $u$  dictate the shape and no of nodes. It turns out only triangular and quad elements satisfy this.

- The lowest order polynomial that we can come up with is

$$u_h^e(x, y) = c_1^e + c_2^e x + c_3^e y \quad (48)$$

- The set  $1, x, y$  is also linearly independent forming the basis for the subspace of the H1 hilbert space
- Now we need to find out a geometry that we can also use the continuity conditions. And we get a triangle
- Check reddy 123 for shape functions
- The shape functions are lagrange interpolation functions, with the sum = 1
- If you mess up the aspect ratio, you will mess up the underlying physics. An intuition is as sometimes the stiffness can be decomposed to different matrixes that are dependant on its aspect ratio and the material properties.
- The boundary conditions  $q$  are not found for elements connected on all sides, but with nodes on the boundary,  $q_n^e$  is known and found by  $q_i^e = \int_S \hat{q}_n N_i^e(s) dS$ . The internal boundary forces are canceled, like the balance of internal forces.

- The next polynomial that meets the requirements on the approx solution is

$$u_h^e(x, y) = c_1^e + c_2^e x + c_3^e y + c_4^e xy \quad (49)$$

- The geometry is a quad element with a linear variation along two points in the element. We usually use isoparametric elements to represent the element.
- Also node is named counterclockwise.
- Reddy 126 for shape functions. Again we see that the aspect ratio mess up the stiffness matrix.

- For triangular elements, we can construct natural coordinates  $L_i$

- $N_i = L_i = \frac{A_i}{A}$

- $\mathbf{N} = \begin{bmatrix} L_1(2L_1 - 1) \\ L_2(2L_2 - 1) \\ L_3(2L_3 - 1) \\ 4L_1L_2 \\ 4L_2L_3 \\ 4L_3L_1 \end{bmatrix}$

- Formula to find the area also given using formula (Reddy 128)

- By multiplying from single linear we get four node
- Tensor product two quadratic one d we get quadratic quad element.
- Serendipity have no interior node. Not complete. They don't have the dual<sup>2</sup> term.

- Continuity of primary variable : Same nodes from different elements
- Equilibrium of secondary variables : At the interface between two elements, the flux or internal force fro the two elements is equal and opposite in sign.

- The same second order equation in the polar coordinates are given as

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r k_{rr} \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left( k_{\theta\theta} \frac{\partial u}{\partial \theta} \right) + \frac{\partial}{\partial z} \left( k_{zz} \frac{\partial u}{\partial z} \right) + f(r, \theta, z) = 0 \quad (50)$$

- We can remove the terms of  $z$  and  $\theta$  as if the problem is independant of these parameters.
- Reddy Page 137, One D and two D equations

- We use numerical integration. We use some parametric form to also do the integration on a square

- We do it on a square region dimension (2x2) with respect to  $(\xi, \eta)$  in domain  $-1 < \xi, \eta < 1$ .
- The element of the fem mesh is transformed, only for the purpose of numerical integration.
- We use a coordinate transformation of the form

$$x = \sum_{j=1}^m x_j^e N_j^e(\xi, \eta) \quad y = \sum_{j=1}^m y_j^e N_j^e(\xi, \eta) \quad (51)$$

- The shape functions are of the master element in the -1 to 1 coordinate.
- The master element is transformed in the linear transfmration to the quadrilateral element.
- The dependant variable is also approximated by the same

$$u_h^e = \sum_{j=1}^n u_j^e \phi_j^e(\xi, \eta) \quad (52)$$

- Superparametric : If  $m > n$ . Geometry approx is higher
- Isoparametric : If  $m = n$
- Subparametric : If  $m < n$ . If dependant variable is higher.

Transformation of the quad to the master is for numerical evaluation. The final equations are always in terms of the nodal values of the physical domain.

1. Different elements can be generated from the same master element. A master element can have different order.
2. A quad master can generate quad curvilinear elements.
3. But elements should not overlap each other.

- Consider the stiffness matrix

$$K_{ji}^e = \int \int_A \frac{\partial N_j}{\partial x} k_x \frac{\partial N_i}{\partial x} dA + \int \int_A \frac{\partial N_j}{\partial y} k_y \frac{\partial N_i}{\partial y} dA \quad (53)$$

- Now this is in the global coordinates and we want to write it in terms of  $\xi$  and  $\eta$ .
- We can write using chain rule

$$\begin{bmatrix} \frac{\partial N_i^e}{\partial \xi} \\ \frac{\partial N_i^e}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{bmatrix} \frac{\partial N_i^e}{\partial x} \\ \frac{\partial N_i^e}{\partial y} \end{bmatrix} \quad (54)$$

With the inner matrix the Jacobian  $\mathbf{J}$ , whose determinant should be  $>0$

- We can then find the inverse transformation for the derivatives as

$$\mathbf{N}_{(x,y)} = \mathbf{J}^{-1} \mathbf{N}_{(\xi,\eta)}$$

- The geometry can also be easily differentiated

$$\frac{\partial x}{\partial \xi} = \sum_{j=1}^m x_j^e \frac{\partial N_j}{\partial \xi} \quad \text{and so on} \quad (55)$$

- Obviously we also get  $dA = \det(J)\xi\eta$
- When all or some of the variables are approximated using hermite interpolating functions, linear approx of geometry and so subparametric or isoparametric forms are adopted.
- Full stiffness matrix, we keep the terms in intergration in the master coordinate. See Page 147
- Integration over a master element can also be looked at page 148 and 149. It is basically  $\int_{-1}^1 \int_{-1}^1 F(\xi, \eta) d\xi d\eta = \sum_{I=1}^M \sum_{J=1}^N F(\xi_I, \eta_J) W_i W_j$

# **ONE D PROBLEM : SINGLE VARIABLE**

- A(u(x)) = f(x) in interval  $0 < x < L$       B(u) = g

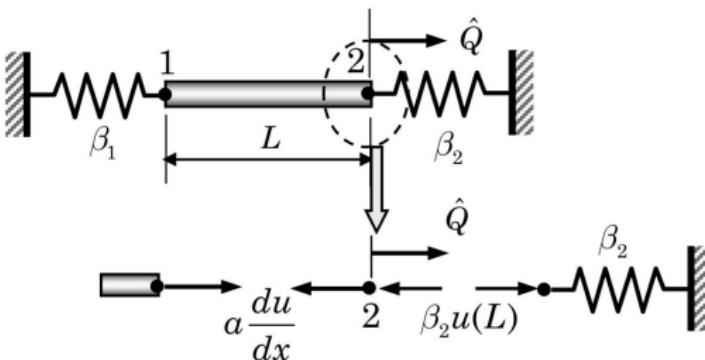
- Consider the differential equation

$$-\frac{d}{dx} \left( k(x, u) \frac{du}{dx} \right) + b(x, u) \frac{du}{dx} + c(x, u)u = f(x) \quad 0 < x < L$$

Boundary conditions (56)

$$n_x k \frac{du}{dx} + \beta(x, u)(u - u_\infty) = \hat{Q} \quad \text{or} \quad u = \hat{u}$$

- Note that  $n_x = -1, \beta = \beta_1$  at  $x = x_a$  and  $n_x = 1, \beta = \beta_2$  at  $x = x_b$
- For a bar with a spring,  $u_\infty = 0$  and we get the equation that the bar should be equal to the spring force  $\beta u$ . Or  $-k \frac{du}{dx} - \beta_2 u = Q_2$  where  $Q_2$  is the extrenal force.



- Therefore we can keep it generally and nicely as:

$$\begin{aligned} A(u) &= f \quad \text{in } 0 < x < L & B(u) &= g \quad \text{at } x = 0 \text{ or } L \\ A &= -\frac{d}{dx} \left( a \frac{du}{dx} \right) + \frac{d}{dx} + c. & B &= n_x a \frac{du}{dx} + \beta, \quad g = \beta u_\infty + \hat{Q} \end{aligned} \tag{57}$$

- If  $a, b, c$  are functions of  $u$  then  $A$  and  $B$  become nonlinear.
- In heat  $a = kA$ ,  $b = 0$  and  $c = \text{perimeter } \beta$

## 4.2 Weak Formulation

Suppose that the domain  $\Omega = (0, L)$  is divided into  $N$  line elements. A typical element from the collection of  $N$  elements is denoted as  $\Omega^e = (x_a, x_b)$ , where  $x_a$  and  $x_b$  denote the global coordinates of the end nodes of the line element. The weak form of Eq. (4.1.1) over the element can be developed as follows (see Section 3.2 for details):

$$\begin{aligned} 0 &= \int_{x_a}^{x_b} \left( a \frac{dw_i^e}{dx} \frac{du_h^e}{dx} + bw_i^e \frac{du_h^e}{dx} + cw_i^e u_h^e - w_i^e f \right) dx - \left[ w_i^e \left( a \frac{du_h^e}{dx} \right) \right]_{x_a}^{x_b} \\ &= \int_{x_a}^{x_b} \left[ a(x, u) \frac{dw_i^e}{dx} \frac{du_h^e}{dx} + b(x, u) w_i^e \frac{du_h^e}{dx} + c(x, u) w_i^e u_h^e - w_i^e f(x) \right] dx \\ &\quad - \left\{ Q_a^e - \beta_a [u_h^e(x_a) - u_\infty^a] \right\} w_i^e(x_a) - \left\{ Q_b^e - \beta_b [u_h^e(x_b) - u_\infty^b] \right\} w_i^e(x_b) \end{aligned} \quad (4.2.1)$$

where  $w_i^e(x)$  is the  $i$ th weight function. The number of weight functions is equal to the number of unknowns in the approximation of  $u_h$ . The first line of Eq. (4.2.1) suggests that  $u$  is the primary variable and  $Q = a(du/dx)$  is the secondary variable of the formulation. Using the mixed boundary condition in Eq. (4.1.2), we can express  $a(du/dx)$  in terms of  $(Q_a^e, Q_b^e)$  and  $(u_\infty^a, u_\infty^b)$  as

$$\begin{aligned} - \left[ a \frac{du_h^e}{dx} \right]_{x=x_a} &= Q_a^e - \beta_a [u_h^e(x_a) - u_\infty^a] \\ \left[ a \frac{du_h^e}{dx} \right]_{x=x_b} &= Q_b^e - \beta_b [u_h^e(x_b) - u_\infty^b] \end{aligned} \quad (4.2.2)$$

where  $(Q_a^e, Q_b^e)$  are the nodal values,  $(u_\infty^a, u_\infty^b)$  denote the values of the variable  $u_\infty$ , and  $(\beta_a, \beta_b)$  denote certain physical parameters (e.g. film conductances) at the left and right ends of the element, respectively. When a node is in the

- If we use the discretisation then we will get

$$K(U)U = F \quad (58)$$

where

$$\begin{aligned} K_{ij}^e &= \int_{x_a}^{x_b} \left[ a(x, u_h^e) \frac{d\psi_i^e}{dx} \frac{d\psi_j^e}{dx} + b(x, u_h^e) \psi_i^e \frac{d\psi_j^e}{dx} + c(x, u_h^e) \psi_i^e \psi_j^e \right] dx \\ &\quad + \beta_a \psi_i^e(x_a) \psi_j^e(x_a) + \beta_b \psi_i^e(x_b) \psi_j^e(x_b) \\ F_i^e &= \int_{x_a}^{x_b} f(x) \psi_i^e dx + \beta_a u_\infty^a \psi_i^e(x_a) + \beta_b u_\infty^b \psi_i^e(x_b) + Q_a \psi_i^e(x_a) + Q_b \psi_i^e(x_b) \end{aligned} \quad (4.3.3)$$

Note that the coefficient matrix  $\mathbf{K}^e$  is a function of the unknown nodal values  $u_j^e$ , and it is an unsymmetric matrix when  $b \neq 0$ ; when  $b = 0$ ,  $\mathbf{K}^e$  is a symmetric matrix. The term involving  $c$  is symmetric, independent of whether it depends on  $u$  and/or  $du/dx$ . Therefore, it is advisable to include nonlinear terms of the type  $u(du/dx)$  in a differential equation as the  $c$ -term in the equation by writing it as  $u(du/dx) = cu$ , with  $c = du/dx$ ; otherwise, it will be unsymmetric and convergence of the solution may become a problem. The coefficients involving  $\beta$  in  $\mathbf{K}^e$  and  $\mathbf{F}^e$  should be included only in elements that have end nodes with the convection type boundary condition. Example 4.3.1 provides more insight into the make-up of the coefficient matrix  $\mathbf{K}^e$ .

- The Boundary terms still confuses me. Especially the sign part. So here it is!  $Q_e$  is the external force.  $F$  is the nodal force with equivalent parts. We actually get the external force from the internal Boundary terms!

$\int_{\text{Force}} \bar{a} \quad \bar{Q}_A^E \quad \bar{A} \quad \bar{\frac{du}{dx}} \quad \bar{K}_n \quad \bar{B.T.}$

$\left. \frac{adu}{dx} \right|_a = - \frac{adu}{dx} \quad \left. \frac{adu}{dx} \right|_b$

Suppose  $\frac{du}{dx} = +ve$  (Tension) so

At  $\bar{a}$  Internal force  $= \frac{adu}{dx} = - \frac{adu}{dx}$

At  $\bar{a}$  Internal force  $= K_n$

At  $A$ :  $\bar{Q}_A^E - K_n = - \frac{adu}{dx}$

At  $B$ :  $\bar{Q}_B^E - K_n = \frac{adu}{dx}$ .

$$\begin{aligned} & \left[ w_i^e \left( a \frac{du^e}{dx} \right) \right]_{x_a}^{x_b} \\ &= w_i^e(x_b) \left[ a \frac{du^e}{dx}(x_b) \right] \\ &+ w_i^e(x_a) \left[ - \frac{adu^e(x_a)}{dx} \right] \\ &= (\bar{Q}_A^E - K_n a) + (\bar{Q}_B^E - K_n b) \end{aligned}$$

- Check problem from Reddy for nonlinear constraints etc.

- Direct iteration procedure
- Newton rhapsone method

## DIRECT ITERATION PROCEDURE

- We solve this system of equations using direct iteration, Picard iteration or method of successive substitutions

$$\mathbf{K}(\mathbf{U}^{(r-1)})\mathbf{U}^r = \mathbf{F}(\mathbf{U}^{(r-1)}) \quad (59)$$

**Box 4.4.1:** Steps involved in the direct iteration scheme.

1. **Initial solution vector.** Assume an initial solution vector  $\mathbf{U}^{(0)}$  such that it (a) satisfies the specified boundary conditions on  $\mathbf{U}$  and (b) does not make  $\mathbf{K}^e$  singular.
2. **Computation of K and F.** Use the latest known vector  $\mathbf{U}^{(r-1)}$  ( $\mathbf{U}^{(0)}$  during the first iteration) to evaluate  $\mathbf{K}^e$  and  $\mathbf{F}^e$ , assemble them to obtain global  $\mathbf{K}$  and  $\mathbf{F}$ , and apply the specified boundary conditions on the assembled system.
3. **Computation of  $\mathbf{U}^{(r)}$ .** Compute the solution at the  $r$ th iteration

$$\mathbf{U}^{(r)} = [\mathbf{K}(\mathbf{U}^{(r-1)})]^{-1}\mathbf{F}^{(r-1)}$$

4. **Convergence check.** Compute the residual

$$\mathbf{R}^{(r)} = \mathbf{K}(\mathbf{U}^{(r)})\mathbf{U}^{(r)} - \mathbf{F}^{(r)}$$

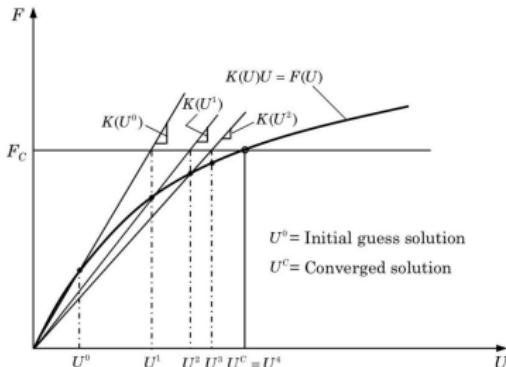
with the latest known solution and check if

$$\|\mathbf{R}^{(r)}\| \leq \epsilon \|\mathbf{F}^{(r)}\| \quad \text{or} \quad \|\mathbf{U}^{(r)} - \mathbf{U}^{(r-1)}\| \leq \epsilon \|\mathbf{U}^{(r)}\|$$

where  $\|\cdot\|$  denotes the euclidean norm and  $\epsilon$  is the convergence tolerance (read as an input). If the solution has converged, print the solution and move to the next “load” level or quit if it is the only or final load; otherwise, continue.

5. **Maximum iteration check.** Check if  $r < itmax$ , where  $itmax$  is the maximum number of iterations allowed (read as an input). If yes, set  $r \rightarrow r + 1$  and go to Step 2; if no, print a message that the iteration scheme did not converge and quit.

## DIRECT ITERATION PROCEDURE



- The method can be accelerated by using a weighted average of solutions from the last two iterations

$$\begin{aligned} \mathbf{U}^r &= \mathbf{K}(\mathbf{U})^{-1} \mathbf{F}(\mathbf{U}) \\ \mathbf{U} &= \beta \mathbf{U}^{(r-2)} + (1 - \beta) \mathbf{U}^{(r-1)} \end{aligned} \tag{60}$$

- Check Reddy 184 for some pages

## NEWTONS ITERATION PROCEDURE

- In NM we expand the residual vector  $\mathbf{R}^{(r)}$  in Taylor series about a known solution  $\mathbf{U}^{(r-1)}$  to get

$$\mathbf{R}^r = \mathbf{R}^{(r-1)} + \left( \frac{\partial \mathbf{R}}{\partial \mathbf{U}} \right)^{(r-1)} \cdot \Delta \mathbf{U} + \mathbf{O}(\mathbf{h}^2) \quad (61)$$

where  $\Delta \mathbf{U} = \mathbf{U}^r - \mathbf{U}^{(r-1)}$

- And saying that  $\mathbf{R}$  in the next iteration should be zero we get

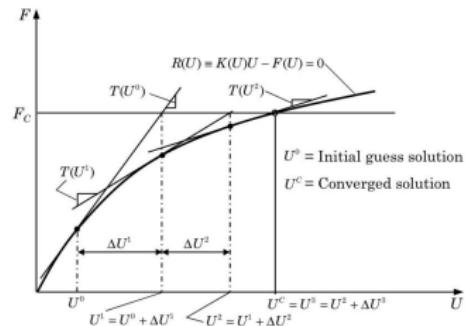
$$\begin{aligned} \left( \frac{\partial \mathbf{R}}{\partial \mathbf{U}} \right)^{(r-1)} \Delta \mathbf{U} &= -\mathbf{R}^{(r-1)} \\ T^{(r-1)} \mathbf{U}^r &= -R^{(r-1)} + T^{(r-1)} \mathbf{U}^{r-1} \end{aligned} \quad (62)$$

where  $T$  is the tangent matrix can be found at element level given as

$$T_{IJ} = \frac{\partial R_I}{\partial U_J} = K_{IJ} + \sum_{m=1}^N \left( \frac{\partial K_{Im}}{\partial U_J} U_m \right) - \frac{\partial F_I}{\partial U_J} \quad (63)$$

The force derivative is zero if it is not a function of the load

- 1. Initial solution vector.** Assume an initial solution vector  $\mathbf{U}^{(0)}$  such that: (a) it satisfies the specified boundary conditions on  $\mathbf{U}$  and (b) it does not make  $\mathbf{T}^e$  singular.
- 2. Computation of  $\mathbf{T}$  and  $\mathbf{R}$ .** Use the latest known vector  $\mathbf{U}^{(r-1)}$  ( $\mathbf{U}^{(0)}$  during the first iteration) to: (a) evaluate  $\mathbf{K}^e$ ,  $\mathbf{F}^e$ ,  $\mathbf{T}^e$ , and  $-\mathbf{R}^e = \mathbf{F}^e - \mathbf{K}^e \mathbf{U}^e$ , (b) assemble  $\mathbf{T}^e$  and  $\mathbf{R}^e$  to obtain global  $\mathbf{T}$  and  $\mathbf{R}$ , and (c) apply the specified *homogeneous* boundary conditions (since  $\mathbf{U}^{(0)}$  already satisfies the actual boundary conditions) on the assembled system,  $\mathbf{T}\mathbf{U} = -\mathbf{R}$ .
- 3. Computation of  $\mathbf{U}^{(r)}$ .** Compute the solution increment at the  $r$ th iteration
 
$$\Delta\mathbf{U} = -[\mathbf{T}(\mathbf{U}^{(r-1)})]^{-1} \mathbf{R}^{(r-1)}$$
 and update the total solution
 
$$\mathbf{U}^{(r)} = \mathbf{U}^{(r-1)} + \Delta\mathbf{U}$$
- 4. Convergence check.** Compute the residual
 
$$\mathbf{R}^{(r)} = \mathbf{K}(\mathbf{U}^{(r)})\mathbf{U}^{(r)} - \mathbf{F}^{(r)}$$
 and check if
 
$$\|\mathbf{R}^{(r)}\| \leq \epsilon \|\mathbf{F}^{(r)}\| \quad \text{or} \quad \|\Delta\mathbf{U}\| \leq \epsilon \|\mathbf{U}^{(r)}\|$$
 where  $\epsilon$  is the convergence tolerance (read as an input). If the solution has converged, print the solution and move to the next “load” level or quit. Otherwise, continue.
- 5. Maximum iteration check.** Check if  $r < itmax$ , where  $itmax$  is the maximum number of iterations allowed (read as an input). If yes, set  $r \rightarrow r + 1$  and go to Step 2; if no, print a message that the iteration scheme did not converge and quit.



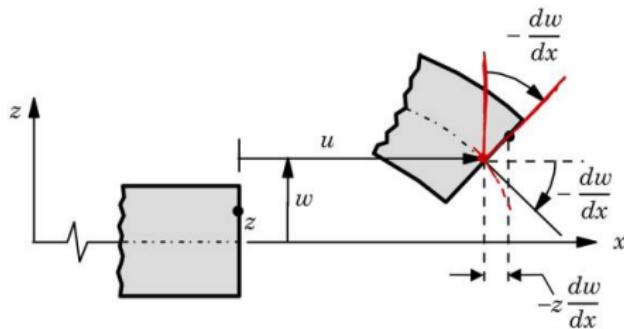
- In the direct iteration method, the actual bc are applied at each iteration. In NR we find the increment to the known solution. If previous displ satisfies, then the increment should be zero and satisfy the boundary condition.
- The symmetry of K and T depends on the weak form. even if K is symmetric, T may not be symmetric.
- T can be approximate, and convergence is only when the residual is small. If it is only updated once then it is called the modified Newtons method.
- See the problem in Reddy -188 . Finding T

# **NONLINEAR BENDING OF STRAIGHT BEAMS**

- Assuming the geometry does not change significantly, allows the principle of virtual work to be written over the undeformed body. So stress is force per unit undeformed area, strain measure of change in length w.r.t original length and shear as change in length from  $\pi/2$ . No distinction between Piola Kirchhoff and Cauchy stress
- Nonlinearity comes solely from inplane forces proportional to the square of the rotation of a transverse normal line in the beam
- There are two theories
  - ▶ Euler-Bernoulli beam (EBT)
  - ▶ Timoshenko beam (TBT)

### Assumptions:

- Plane sections perpendicular to the axis of the beam remain (a) plane (b) rigid (not deform) (c) rotate such that they remain plane to the deformed axis after deformation
- Assumptions amount to neglection poissons effect and transverse normal and shear strains



■ The displacement is given as :  $u_1 = u(x) - z \frac{dw}{dx}$        $u_2 = 0$        $u_3 = w(x)$   
 $u_1, u_2, u_3$  is the disp of any point while  $u, w$  are the disp of the centroidal axis

- So the total disp can be found from the neutral axis values.

- The greens theorem is

$$E_{ij} = \varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \frac{1}{2} \frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial x_j}$$

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1} + \frac{1}{2} \left[ \left( \frac{\partial u_1}{\partial x_1} \right)^2 + \left( \frac{\partial u_3}{\partial x_1} \right)^2 \right] \quad (64)$$

- Now the axial strains of higher orders we can ignore, but the rotation of the line perpendicular to the beam is pretty large so we have to retain it.
- Non-linear strains where only squares of the rotations are included are known as von Karman nonlinearity
- The zero strains are  $\varepsilon_{33} = \varepsilon_{13} = 0$  so only non zero strain is

$$\varepsilon_{11} = \frac{du}{dx} - z \frac{d^2 w}{dx^2} + \frac{1}{2} \left( \frac{dw}{dx} \right)^2 \quad (65)$$

- So we have two longitudinal von karman strains:

$$\varepsilon_{11}^0 = \frac{du}{dx} + \frac{1}{2} \left( \frac{dw}{dx} \right)^2 , \quad \varepsilon_{11}^1 = -z \frac{d^2 w}{dx^2} \quad (66)$$

- The weak form can be found without knowing the governing d.e
- If a body is in equilibrium, the total virtual work done in moving through the respective displacements is zero

$$\delta W^e = \delta W_I^e + \delta W_E^e \quad (67)$$

$\delta W_I^e$  denotes the virtual strain stored due to the actual Cauchy or second Piola stresses (Same geom)  $\sigma_{ij}$

$\delta W_E^e$  is the work done by external applied loads

■ So we have

$$\delta W_I^e = \int_V^e \delta \varepsilon_{xx} \sigma_{xx} dV = \int_{x_a}^{x_b} \int_A^e \left( \frac{d\delta u}{dx} + \frac{dw}{dx} \frac{d\delta w}{dx} - z \frac{d^2 \delta w}{dx^2} \right) dA dx$$

$$\delta W_E^e = - \left[ \int_{x_a}^{x_b} q \delta w dx + \int_{x_a}^{x_b} f \delta u dx + \sum_{i=1}^6 Q_i^e \delta \Delta_i^e \right] \quad (68)$$

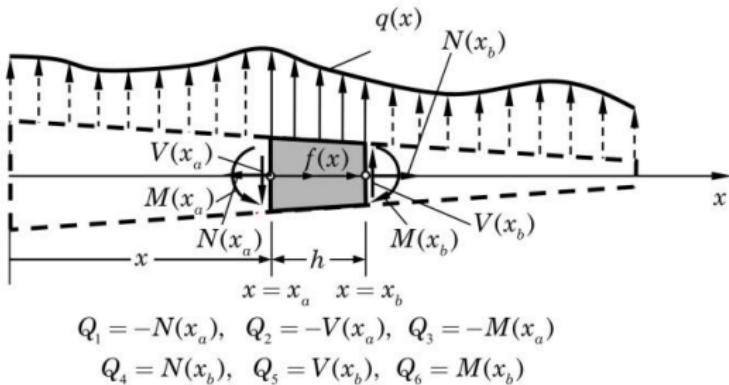
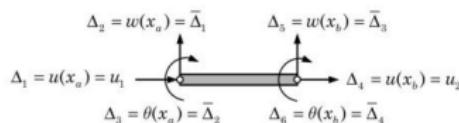


Fig. 5.2.2: A typical beam element,  $\Omega^e = (x_a, x_b)$ .

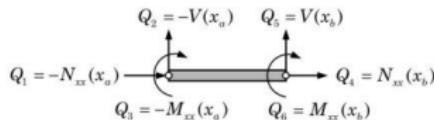
# SIGN CONVENTIONS

$$\begin{aligned}\Delta_1^e &= u(x_a), \quad \Delta_2^e = w(x_a), \quad \Delta_3^e = \left[ -\frac{dw}{dx} \right]_{x_a} \equiv \theta(x_a) \\ \Delta_4^e &= u(x_b), \quad \Delta_5^e = w(x_b), \quad \Delta_6^e = \left[ -\frac{dw}{dx} \right]_{x_b} \equiv \theta(x_b)\end{aligned}\quad (5.2.6)$$

$$\begin{aligned}Q_1^e &= -N_{xx}(x_a), \quad Q_4^e = N_{xx}(x_b), \quad Q_2^e = -\left[ \frac{dw}{dx} N_{xx} + \frac{dM_{xx}}{dx} \right]_{x_a} \equiv -V_x(x_a) \\ Q_5^e &= \left[ \frac{dw}{dx} N_{xx} + \frac{dM_{xx}}{dx} \right]_{x_b} \equiv V_x(x_b), \quad Q_3^e = -M_{xx}(x_a), \quad Q_6^e = M_{xx}(x_b)\end{aligned}\quad (5.2.7)$$



(a) Generalized displacement degrees of freedom



(b) Generalized force degrees of freedom

- The main thing is that for  $V_x$ , the component of  $N_x$  is  $\sin(\frac{dw}{dx}) \approx \frac{dw}{dx}$ , and also in moment equilibrium we get  $N_{xx} \frac{dw}{dx} \Delta x = N_{xx} \Delta w$

- The free body diagram is what it is. But we can take our generalised displacement and force dof signs based on what we want.
- Generalised is used to show that rotations and moments are treated as displacements and forces
- By integrating in the area we get the virtual work as

$$\delta W_I^e = \int_{x_a}^{x_b} \left[ \left( \frac{d\delta u}{dx} + \frac{dw}{dx} \frac{d\delta w}{dx} \right) N_{xx} - \frac{d^2 \delta w}{dx^2} M_{xx} \right] - \int_{x_a}^{x_b} \delta w q dx - \int_{x_a}^{x_b} \delta u f dx - \sum_{i=1}^6 \delta \Delta_i^e Q_i^e \quad (69)$$

where  $N_{xx} = \int_A \sigma_{xx} dA$        $M_{xx} = \int_A \sigma_{xx} z dA$

By taking the individual virtual terms, as the coefficients have to separately equal to zero we get

$$\int_{x_a}^{x_b} \left( \frac{d\delta u}{dx} N_{xx} - \delta u f \right) dx - \delta \Delta_1^e Q_1^e - \delta \Delta_4^e Q_4^e = 0$$

$$\int_{x_a}^{x_b} \left( \frac{d\delta w}{dx} \left( \frac{dw}{dx} N_{xx} \right) - \frac{d^2 \delta w}{dx^2} M_{xx} - \delta w q \right) dx - \delta \Delta_2^e Q_2^e - \delta \Delta_3^e Q_3^e - \delta \Delta_5^e Q_5^e - \delta \Delta_6^e Q_6^e = 0$$

■ By integration by parts we get the euler lagrange equilibrium equations : given as

$$\begin{aligned} & \int_{x_a}^{x_b} \delta u \left( -\frac{d\delta N_{xx}}{dx} - f \right) dx - \delta \Delta_1^e Q_1^e + [\delta u N_{xx}]_{x_b}^{x_a} - \delta \Delta_4^e Q_4^e = 0 \\ & \int_{x_a}^{x_b} -\delta w \left( \frac{d}{dx} \left( \frac{dw}{dx} N_{xx} \right) + \frac{d^2 M_{xx}}{dx^2} + q \right) dx - \left[ \frac{d\delta w}{dx} M_{xx} \right]_{x_b}^{x_a} \\ & + \left[ \delta w \left( \frac{dw}{dx} N_{xx} + \frac{dM_{xx}}{dx} \right) \right]_{x_b}^{x_a} - \delta \Delta_2^e Q_2^e - \delta \Delta_3^e Q_3^e - \delta \Delta_5^e Q_5^e - \delta \Delta_6^e Q_6^e = 0 \end{aligned} \quad (70)$$

- We get our euler equations from coefficients of the two variations

$$\begin{aligned} -\frac{dN_{xx}}{dx} &= f(x) \\ -\frac{d}{dx} \left( \frac{dw}{dx} N_{xx} \right) - \frac{d^2 M_{xx}}{dx^2} &= q(x) \end{aligned} \quad (71)$$

- We get our boundary conditions as follows :

$$\begin{aligned} Q_1^e &= -N_{xx}(x_a) & Q_4^e &= N_{xx}(x_b) \\ Q_2^e &= - \left[ \frac{dw}{dx} N_{xx} + \frac{dM_{xx}}{dx} \right]_{x_a} & Q_5^e &= \left[ \frac{dw}{dx} N_{xx} + \frac{dM_{xx}}{dx} \right]_{x_b} \\ Q_3^e &= M_{xx}(x_a) & Q_6^e &= M_{xx}(x_b) \end{aligned} \quad (72)$$

Again the - sign makes all the Qs positive in the equation

- We can also find the differential equations by writing the equilibrium equations ( $F_x, F_y, M_z = 0$ ) for a small element and then take  $\Delta x \rightarrow 0$ . But we don't get the boundary conditions by using this.
- After getting the Euler-Lagrange differential equations, we can again use the weighted residual method to find the stiffness forms

$$0 = \int_{x_a}^{x_b} v_1 \left( -\frac{dN_{xx}}{dx} - f \right) dx = \int_{x_a}^{x_b} \left( \frac{dv_1}{dx} N_{xx} - v_1 f \right) dx - [v_1 N_{xx}]_{x_a}^{x_b} \\ = \int_{x_a}^{x_b} \left( \frac{dv_1}{dx} N_{xx} - v_1 f \right) dx - v_1(x_a) [-N_{xx}(x_a)] - v_1(x_b) N_{xx}(x_b) \quad (5.2.16)$$

$$0 = \int_{x_a}^{x_b} v_2 \left[ -\frac{d}{dx} \left( \frac{dw}{dx} N_{xx} \right) - \frac{d^2 M_{xx}}{dx^2} - q \right] dx \\ = \int_{x_a}^{x_b} \left[ \frac{dv_2}{dx} \left( \frac{dw}{dx} N_{xx} \right) - \frac{d^2 v_2}{dx^2} M_{xx} - v_2 q \right] dx \\ - \left[ v_2 \left( \frac{dw}{dx} N_{xx} + \frac{dM_{xx}}{dx} \right) \right]_{x_a}^{x_b} - \left[ \left( -\frac{dv_2}{dx} \right) M_{xx} \right]_{x_a}^{x_b} \\ = \int_{x_a}^{x_b} \left[ \frac{dv_2}{dx} \left( \frac{dw}{dx} N_{xx} \right) - \frac{d^2 v_2}{dx^2} M_{xx} - v_2 q \right] dx \\ - v_2(x_a) \left[ -\left( \frac{dw}{dx} N_{xx} + \frac{dM_{xx}}{dx} \right) \right]_{x_a} - v_2(x_b) \left[ \frac{dw}{dx} N_{xx} + \frac{dM_{xx}}{dx} \right]_{x_b} \\ - \left[ -\frac{dv_2}{dx} \right]_{x_a} [-M_{xx}(x_a)] - \left[ -\frac{dv_2}{dx} \right]_{x_b} M_{xx}(x_b) \quad (5.2.17)$$

- We clearly see that the weights are nothing but the variations and the external load work has been replaced by the internal force equivalents!!!!

- Now the funny thing is that we kept as N and M and now we shall replace them by

$$\begin{aligned}
 N_{xx} &= \int_A \sigma_{xx} dA = \int_{A^e} E^e \left[ \frac{du}{dx} + \frac{1}{2} \left( \frac{dw}{dx} \right)^2 - z \frac{d^2 w}{dx^2} \right] dA \\
 &= A^e \left[ \frac{du}{dx} + \frac{1}{2} \left( \frac{dw}{dx} \right)^2 \right] - B^e z \frac{d^2 w}{dx^2} \\
 M_{xx} &= \int_{A^e} \sigma_{xx} z dA = B^e \left[ \frac{du}{dx} + \frac{1}{2} \left( \frac{dw}{dx} \right)^2 \right] - D^e \frac{d^2 w}{dx^2}
 \end{aligned} \tag{73}$$

$$\text{where } A = EA, \quad B = 0 \left( \int z dA = 0 \right), \quad D = EI \left( \int z^2 dA \right)$$

- Therefore the virtual work equations can be written as

$$\begin{aligned}
 0 &= \int_{x_a^e}^{x_b^e} \left( A \frac{d\delta u}{x} \left[ \frac{du}{dx} + \frac{1}{2} \left( \frac{dw}{dx} \right)^2 \right] - f \delta u \right) dx - \delta u(x_a) Q_1 - \delta u(x_b) Q_4 \\
 0 &= \int_{x_a^e}^{x_b^e} \left( \frac{d\delta w}{dx} \frac{dw}{dx} A \left( \frac{du}{dx} + \frac{1}{2} \left( \frac{dw}{dx} \right)^2 \right) + D \frac{d^2 \delta w}{dx^2} \frac{d^2 w}{dx^2} - q \delta w \right) dx \\
 &\quad - \delta w(x_a) Q_2 - \delta \theta(x_a) Q_3 - \delta w(x_b) Q_5 - \delta \theta(x_b) Q_6
 \end{aligned} \tag{74}$$

- We take the rotation as  $\theta = -\frac{dw}{dx}$
- We approximate the axial disp as linear lagrange and the transverse with hermite cubic interpolation functions

$$0 = \sum_{j=1}^2 K_{ij}^{11} u_j + \sum_{J=1}^4 K_{IJ}^{12} \bar{\Delta}_J - F_i^1 \quad (i = 1, 2) \quad - \text{Axial} \quad \mathbf{u} = [u_1 \ u_2]^T$$

$$0 = \sum_{j=1}^2 K_{Ij}^{21} u_j + \sum_{J=1}^4 K_{IJ}^{22} \bar{\Delta}_J - F_I^2 \quad (I = 1, 2, 3, 4) \quad - \text{Bending} \quad \Delta = \begin{bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{bmatrix}$$
(5.2.30)

where, for the case in which  $B_{xx} = 0$ , we have

$$K_{ij}^{11} = \int_{x_a}^{x_b} A_{xx} \frac{d\psi_i}{dx} \frac{d\psi_j}{dx} dx, \quad K_{ij}^{12} = \frac{1}{2} \int_{x_a}^{x_b} \left( A_{xx} \frac{dw}{dx} \right) \frac{d\psi_i}{dx} \frac{d\varphi_j}{dx} dx$$

$$K_{Ij}^{21} = \int_{x_a}^{x_b} A_{xx} \frac{dw}{dx} \frac{d\varphi_I}{dx} \frac{d\psi_j}{dx} dx, \quad K_{Ij}^{21} = 2K_{jI}^{12}$$

$$K_{IJ}^{22} = \int_{x_a}^{x_b} D_{xx} \frac{d^2\varphi_I}{dx^2} \frac{d^2\varphi_J}{dx^2} dx + \frac{1}{2} \int_{x_a}^{x_b} \left[ A_{xx} \left( \frac{dw}{dx} \right)^2 \right] \frac{d\varphi_I}{dx} \frac{d\varphi_J}{dx} dx$$

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$$F_i^1 = \int_{x_a}^{x_b} f \psi_i dx + \hat{Q}_i, \quad F_I^2 = \int_{x_a}^{x_b} q \varphi_I dx + \bar{Q}_I \quad (5.2.31)$$

for  $(i, j = 1, 2)$  and  $(I, J = 1, 2, 3, 4)$ , where  $\hat{Q}_1 = Q_1$ ,  $\hat{Q}_2 = Q_4$ ,  $\bar{Q}_1 = Q_2$ ,  $\bar{Q}_2 = Q_3$ ,  $\bar{Q}_3 = Q_5$ , and  $\bar{Q}_4 = Q_6$ , and the definition of  $Q_i$  is given Eq. (5.2.15). The pair of equations in Eq. (5.2.30) can be expressed in matrix form as

$$\begin{bmatrix} \mathbf{K}^{11} & \mathbf{K}^{12} \\ \mathbf{K}^{21} & \mathbf{K}^{22} \end{bmatrix} \begin{Bmatrix} \boldsymbol{\Delta}^1 \\ \boldsymbol{\Delta}^2 \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}^1 \\ \mathbf{F}^2 \end{Bmatrix} \quad \text{or} \quad \mathbf{K}^e \boldsymbol{\Delta}^e = \mathbf{F}^e \quad (5.2.32)$$

where

$$\Delta_i^1 = u_i, \quad i = 1, 2; \quad \Delta_I^2 = \bar{\Delta}_I, \quad I = 1, 2, 3, 4 \quad (5.2.33)$$

We also note that  $(\mathbf{K}^{12})^T \neq \mathbf{K}^{21}$ .

- Note that the Stiffness is not symmetric. When nonlinearity is not there then 12 and 21 (which depend on  $w$ ) become 0 and the system is uncoupled. That is axial is only dependant on  $u$  and bending only dependant on  $\Delta$ .
- See that we have tried to keep the shape functions symmetric
- When we find these coefficients from the previous iteration and then we say that the equations are linearised
- The coefficients can be different if we had accounted for the terms differently, and uncouple them to a system of equations . Read reddy page 223 for this (Did not understand as of now)
- We can also decompose the matrix because the 12 and 21 terms are very similar (Reddy 224)
- We obviously solve using direct iterative or NR method. Check reddy for the derivations of T

- For the von Karman nonlinearity, for two beams : roller roller and pinned pinned under transverse loading. Will not have  $u = 0$  because of the coupling and the solution will not be the same.
- Suppose the roller roller beam has a constraint on  $u$  in the middle to remove rigid body movement, and transverse load does not make axial strain because the beam can slide without making stresses. This will have higher transverse deflection because it can stretch.
- The pinned pinned however has constraints on  $x = 0$  and  $x = L$  so it will develop axial strains.
- To make sure that the roller roller does not have any axial strain

$$\varepsilon_{xx}^0 = \frac{du}{dx} + \frac{1}{2} \left( \frac{dw}{dx} \right)^2 = 0$$

And basically both should be of the same order  $-\frac{du}{dx} = \left( \frac{dw}{dx} \right)^2$

- So when  $w$  is cubic  $\frac{dw}{dx}$  is square and the power makes it quad. So  $u$  should be atleast order fifth. Taking  $u$  with any polynomial less , will make the constraint not satisfied and stiff, giving zero displacement field. (Membrane locking)
- We can also treat the axial strain as constant. Since  $\frac{du}{dx}$  is const, we can make  $\left( \frac{dw}{dx} \right)^2$  also constant. We can do this by reduced integration of all nonlinear stiffness coefficients (A,B,D).

- Strains and stresses are close when found at the Gauss points
- $u$  is approx with linear lagrange and  $w$  is approx with hermite cubic polynomials.
- Membrane strain  $\varepsilon_{xx}^0$  is assumed constant is evaluated using one point G.Q, and the bending strain  $\varepsilon_{xx}^1$  is also linear and done by one point G.Q.
- 

$$\begin{aligned}\varepsilon_{xx}^0 &= \frac{du}{dx} + \frac{1}{2} \left( \frac{dw}{dx} \right)^w \\ \varepsilon_{xx}^1 &= -\frac{d^2 w}{dx^2}\end{aligned}\tag{75}$$

And the stresses are given as follows :

- $\sigma_{xx} = \sigma_{xx}^0 + z\sigma_{xx}^1 = E\varepsilon_{xx}^0 + zE\varepsilon_{xx}^1$

- The Euler-Bernoulli beam is based on the fact that a straight line transverse to the axis before deformation remains (1) straight (2) inextensible (3) normal to mid-plane after deformation. In TBT, we say the last assumption, the rotation is independent of the slope
- So we get the displacement field with an independent slope

$$u_1 = u(x) + z\phi_x(x) \quad u_2 = 0 \quad u_3 = w(x) \quad (76)$$

- The non-zero strains are :

$$\begin{aligned} \varepsilon_{xx} &= \frac{du_1}{dx} + \frac{1}{2} \left( \frac{du_3}{dx} \right)^2 = \frac{du}{dx} + \frac{1}{2} \left( \frac{dw}{dx} \right)^2 + z \frac{d\phi}{dx} = \varepsilon_{xx}^0 + z\varepsilon_{xx}^1 \\ \gamma_{xz} &= \frac{du_1}{dz} \frac{du_3}{dx} = \phi + - \left( - \frac{dw}{dx} \right) = \gamma_{xz}^0 \end{aligned} \quad (77)$$

The last one we get because we remove the rotation due to  $w$

■ The virtual strains are :

$$\begin{aligned}\delta \varepsilon_{xx}^0 &= \frac{d\delta u}{dx} + \frac{dw}{dx} \frac{d\delta w}{dx} \\ \delta \varepsilon_{xx}^1 &= z \frac{d\phi}{dx} \\ \delta \gamma_{xz} &= \delta\phi + \frac{d\delta w}{dx}\end{aligned}\tag{78}$$

$$\gamma = \theta_A + \theta_B = \phi$$

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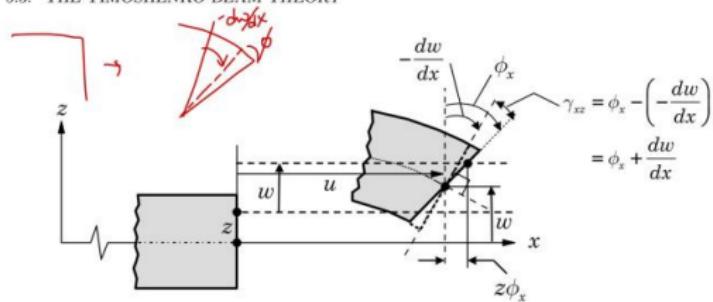


Fig. 5.3.1: Kinematics of a beam in the Timoshenko beam theory.

## WEAK FORM

- The weak form is similarly developed (Reddy 243)

$$\delta W = \int_{x_a^e}^{x_b^e} \int_{A^e} \left( N_{xx} \delta \varepsilon_{xx}^0 + M_{xx} \delta \varepsilon_{xx}^1 + Q_x \delta \varepsilon_{xz}^0 \right) - \delta W_E^e \quad (79)$$

Note that :

$$N = \int \sigma dA, \quad M = \int \sigma z dA \quad Q = \int \sigma_{xz} dA$$

$$\sigma = E\varepsilon \quad \sigma_{xz} = KG\gamma_{xz}$$

The K thing is cause we assume a constant shear over the cross section when we just say stress is G x strain. So it is a correcting factor! We compare the two energies and then we find K.

- Trying to keep all the variations in the same derivative order, we get the euler equilibrium equations, taking each coefficient as zero

$$\begin{aligned} \delta u : \quad & -\frac{dN}{dx} = f(x) \\ \delta w : \quad & -\frac{dQ}{dx} - \frac{d}{dx} \left( N \frac{dw}{dx} \right) = q(x) \\ \delta \phi : \quad & -\frac{dM}{dx} + Q_x = 0 \end{aligned} \quad (80)$$

$u, w, \phi$  are the primary and  $N, Q, M$  are the secondary variables

- We keep the secondary variables in terms of the independent primary variables

$$\begin{aligned}
 N &= A \left[ \frac{du}{dx} + \frac{1}{2} \left( \frac{dw}{dx} \right)^2 \right] + B \frac{d\phi}{dx} \\
 M &= B \left[ \frac{du}{dx} + \frac{1}{2} \left( \frac{dw}{dx} \right)^2 \right] + D \frac{d\phi}{dx} \\
 Q &= S \left( \frac{dw}{dx} + \phi \right)
 \end{aligned} \tag{81}$$

A,B,D are the previous terms moments of area giving the integral only in the x direction in the virtual work. S is the shear stiffness given as

$$S = K \int G dA = KGA = \frac{KEA}{2(1+\nu)}$$

- Obviously we can derive the governing differential equilibrium equations (Page 245). For homogeneous beams , we have B = 0. which are the equilibrium equations given by the variational principle

- The virtual work statement is then given as

$$0 = \int_{x_a}^{x_b} \left\{ A_{xx} \frac{d\delta u}{dx} \left[ \frac{du}{dx} + \frac{1}{2} \left( \frac{dw}{dx} \right)^2 \right] - f \delta u \right\} dx \\ - Q_1^e \delta u(x_a) - Q_4^e \delta u(x_b) \quad (5.3.18)$$

$$0 = \int_{x_a}^{x_b} \frac{d\delta w}{dx} \left\{ S_{xx} \left( \frac{dw}{dx} + \phi_x \right) + A_{xx} \frac{dw}{dx} \left[ \frac{du}{dx} + \frac{1}{2} \left( \frac{dw}{dx} \right)^2 \right] \right\} dx \\ - \int_{x_a}^{x_b} \delta w q dx - Q_2^e \delta w(x_a) - Q_5^e \delta w(x_b) \quad (5.3.19)$$

$$0 = \int_{x_a}^{x_b} \left[ D_{xx} \frac{d\delta \phi_x}{dx} \frac{d\phi_x}{dx} + S_{xx} \delta \phi_x \left( \frac{dw}{dx} + \phi_x \right) \right] dx \\ - Q_3^e \delta \phi_x(x_a) - Q_6^e \delta \phi_x(x_b) \quad (5.3.20)$$

- The Q's have the same meaning as the euler bernoulli element

$\phi_x$  may be used. Substitution of Eq. (5.3.22) for  $u$ ,  $w$ , and  $\phi_x$ , and  $\delta u = \psi_i^{(1)}$ ,  $\delta w = \psi_i^{(2)}$ , and  $\delta \phi_x = \psi_i^{(3)}$  into Eqs. (5.3.18)–(5.3.20) yields the finite element model

$$0 = \sum_{j=1}^m K_{ij}^{11} u_j^e + \sum_{j=1}^n K_{ij}^{12} w_j^e + \sum_{j=1}^p K_{ij}^{13} \phi_j^e - F_i^1 \quad (5.3.23)$$

$$0 = \sum_{j=1}^m K_{ij}^{21} u_j^e + \sum_{j=1}^n K_{ij}^{22} w_j^e + \sum_{j=1}^p K_{ij}^{23} \phi_j^e - F_i^2 \quad (5.3.24)$$

$$0 = \sum_{j=1}^m K_{ij}^{31} u_j^e + \sum_{j=1}^n K_{ij}^{32} w_j^e + \sum_{j=1}^p K_{ij}^{33} \phi_j^e - F_i^3 \quad (5.3.25)$$

The stiffness and force coefficients are

$$\begin{aligned} K_{ij}^{11} &= \int_{x_a}^{x_b} A_{xx} \frac{d\psi_i^{(1)}}{dx} \frac{d\psi_j^{(1)}}{dx} dx, \quad K_{ij}^{12} = \frac{1}{2} \int_{x_a}^{x_b} A_{xz} \frac{dw}{dx} \frac{d\psi_i^{(1)}}{dx} \frac{d\psi_j^{(2)}}{dx} dx \\ K_{ij}^{21} &= \int_{x_a}^{x_b} A_{xz} \frac{dw}{dx} \frac{d\psi_i^{(2)}}{dx} \frac{d\psi_j^{(1)}}{dx} dx, \quad K_{ij}^{13} = 0, \quad K_{ij}^{31} = 0 \end{aligned}$$

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$$\begin{aligned} K_{ij}^{22} &= \int_{x_a}^{x_b} S_{xz} \frac{d\psi_i^{(2)}}{dx} \frac{d\psi_j^{(2)}}{dx} dx + \frac{1}{2} \int_{x_a}^{x_b} A_{xz} \left( \frac{dw}{dx} \right)^2 \frac{d\psi_i^{(2)}}{dx} \frac{d\psi_j^{(2)}}{dx} dx \\ K_{ij}^{23} &= \int_{x_a}^{x_b} S_{xz} \frac{d\psi_i^{(2)}}{dx} \psi_j^{(3)} dx = K_{ji}^{32} \\ K_{ij}^{33} &= \int_{x_a}^{x_b} \left( D_{xz} \frac{d\psi_i^{(3)}}{dx} \frac{d\psi_j^{(3)}}{dx} + S_{xz} \psi_i^{(3)} \psi_j^{(3)} \right) dx \\ F_i^1 &= \int_{x_a}^{x_b} \psi_i^{(1)} f dx + Q_1^e \psi_i^{(1)}(x_a) + Q_4^e \psi_i^{(1)}(x_b) \\ F_i^2 &= \int_{x_a}^{x_b} \psi_i^{(2)} q dx + Q_2^e \psi_i^{(2)}(x_a) + Q_5^e \psi_i^{(2)}(x_b) \\ F_i^3 &= Q_3^e \psi_i^{(3)}(x_a) + Q_6^e \psi_i^{(3)}(x_b) \end{aligned} \quad (5.3.26)$$

- We then get  $\mathbf{Ku} = \mathbf{F}$  where  $\mathbf{u} = [u \ w \ \phi]^T$ . Check Reddy 249 for T stiffness derivation

- Timoshenko beam without von Karman nonlinearity differ from each other in the choice of the approx function for  $w$  and  $\phi$ . Some are equal and others different
- Linear interpolation of both  $w$  and  $\phi$  is the easiest. This makes the slope  $\frac{dw}{dx}$  constant. In a think beam, as the length to thickness ratio becomes large (100), the slope would be equal to  $-\phi$  which is linear instead of constant.
- On the other hand a constant  $\phi$  leads to zero bending energy while the transverse shear is nonzero.
- Check Reddy 248 ( Have not understood fully Locking issue)
- The primary variables may not be approximated by the same shape functions :

$$u(x) = N_i^1 u_i \quad w(x) = N_i^2 w_i \quad \phi(x) = N_i^3 \phi_i \quad (82)$$

Check reddy

## **TWO DIMENSIONAL PROBLEMS HAVING A SINGLE VARIABLE**

- Suppose we are trying to find the solution  $u(x, y)$  of the following partial differential equation

$$-\frac{d}{dx} \left( a_{xx} \frac{\partial u}{\partial x} + a_{xy} \frac{\partial u}{\partial y} \right) - \frac{d}{dy} \left( a_{yx} \frac{\partial u}{\partial x} + a_{yy} \frac{\partial u}{\partial y} \right) + a_{00} u = f(x, y) \quad \text{in } \Omega \quad (83)$$

The coefficients are also a function of  $u$ : eg  $a_{xx} = f(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y})$

- When we discretize it with  $\bar{\Omega}$  with a boundary  $\bar{\Gamma}$ , we get a residual given as :

$$R(u_h) = -\frac{d}{dx} \left( a_{xx} \frac{\partial u_h}{\partial x} + a_{xy} \frac{\partial u_h}{\partial y} \right) - \frac{d}{dy} \left( a_{yx} \frac{\partial u_h}{\partial x} + a_{yy} \frac{\partial u_h}{\partial y} \right) + a_{00} u_h = f(x, y) \quad (84)$$

The step is to multiply the residual with the  $i$ th weight function  $w_i(x, y)$  which should be differentiable too. We then set  $w_i R$  over the element domain  $\Omega^e = 0$

- $0 = \int_{\Omega_e} w_i \left[ -\frac{d}{dx} \left( a_{xx} \frac{\partial u_h}{\partial x} + a_{xy} \frac{\partial u_h}{\partial y} \right) - \frac{d}{dy} \left( a_{yx} \frac{\partial u_h}{\partial x} + a_{yy} \frac{\partial u_h}{\partial y} \right) + a_{00} u - f(x, y) \right] dx dy$

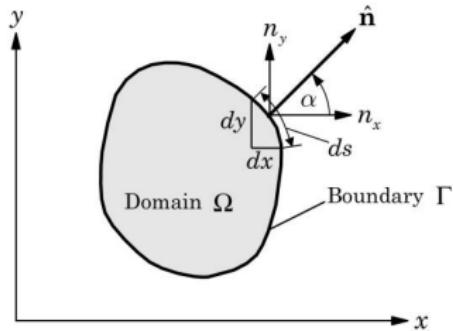
- Now we know :  $\frac{d}{dx} \left( w \frac{du}{dx} \right) = w \frac{d^2 u}{dx^2} + \frac{dw}{dx} \frac{du}{dx}$

and  $\int_A \frac{d}{dx} \left( w \frac{du}{dx} \right) dA = \int_S \left( w \frac{du}{dx} \right) dS$

- We get

$$0 = \int_A \left[ \frac{\partial w_i}{\partial x} \left( a_{xx} \frac{\partial u_h}{\partial x} + a_{xy} \frac{\partial u_h}{\partial y} \right) + \int_A \frac{\partial w_i}{\partial y} \left( a_{yx} \frac{\partial u_h}{\partial x} + a_{yy} \frac{\partial u_h}{\partial y} \right) + a_{00} w_i u_h - w_i f \right] dx dy - \int_S w_i \left[ \left( a_{xx} \frac{\partial u_h}{\partial x} + a_{xy} \frac{\partial u_h}{\partial y} \right) n_x + \left( a_{yx} \frac{\partial u_h}{\partial x} + a_{yy} \frac{\partial u_h}{\partial y} \right) n_y \right] dS \quad (85)$$

where  $\mathbf{n} = \mathbf{n}_x \mathbf{e}_1 + \mathbf{n}_y \mathbf{e}_2$  which gives the direction consies of the boundary  $\Gamma^e$ . The second term can also be written as  $- \int_S q_n dS$  which is the external flux normal as we move counter clockwise.



- In the case of heat transfer through an anisotropic medium,  $a_{ij}$  denotes the conductivity and  $q_n$  is the normal heat flux

- The weak form states that  $u$  should be atleast linear in both  $x$  and  $y$
- $u_h^e = \sum u_i N_i(x, y)$  with  $N_i(x_j, y_j) = \delta_{ij}$  and  $\sum_j N_j(x, y) = 1$

Substituting the finite element approximation in Eq. (6.3.1) for  $u_h$  into the weak form, Eq. (6.2.5), we obtain

$$0 = \sum_{j=1}^n u_j^e \int_{\Omega^e} \left[ \frac{\partial w_i}{\partial x} \left( a_{xx} \frac{\partial \psi_j^e}{\partial x} + a_{xy} \frac{\partial \psi_j^e}{\partial y} \right) + \frac{\partial w_i}{\partial y} \left( a_{yx} \frac{\partial \psi_j^e}{\partial x} + a_{yy} \frac{\partial \psi_j^e}{\partial y} \right) \right. \\ \left. + a_{00} w_i \psi_j^e \right] dx dy - \int_{\Omega^e} w_i f dx dy - \oint_{\Gamma^e} w_i q_n ds \quad (6.3.3)$$

For the weak-form Galerkin model, we replace the weight function with  $\psi_i^e$  and obtain

$$\sum_{j=1}^n K_{ij}^e u_j^e - f_i^e - Q_i^e = 0 \quad \text{or} \quad \mathbf{K}^e \mathbf{u}^e = \mathbf{f}^e + \mathbf{Q}^e \quad (6.3.4)$$

where

$$K_{ij}^e = \int_{\Omega^e} \left[ \frac{\partial \psi_i^e}{\partial x} \left( a_{xx} \frac{\partial \psi_j^e}{\partial x} + a_{xy} \frac{\partial \psi_j^e}{\partial y} \right) + \frac{\partial \psi_i^e}{\partial y} \left( a_{yx} \frac{\partial \psi_j^e}{\partial x} + a_{yy} \frac{\partial \psi_j^e}{\partial y} \right) \right. \\ \left. + a_{00} \psi_i^e \psi_j^e \right] dx dy \quad (6.3.5)$$

$$f_i^e = \int_{\Omega^e} \psi_i^e f dx dy, \quad Q^e = \oint_{\Gamma^e} \psi_i^e q_n ds$$

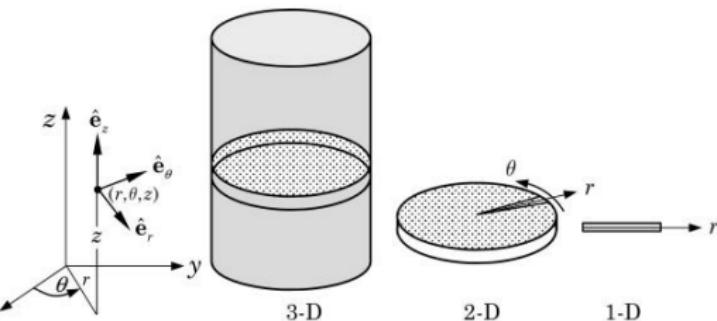
Note that  $K_{ij}^e \neq K_{ji}^e$  (i.e.  $\mathbf{K}^e$  is not symmetric) unless  $a_{xy} = a_{yx}$ . Equation (6.3.4) represents a set of  $n$  nonlinear algebraic equations.

- The equations have to be solved by nonlinear methods
- The tangent T is given in page 271

- The differential equation in cylindrical coordinate system  $(r, \theta, z)$

$$-\frac{1}{r} \frac{\partial}{\partial r} \left( r a_{rr} \frac{\partial u}{\partial r} \right) - \frac{1}{r^2} \frac{\partial}{\partial \theta} \left( a_{\theta\theta} \frac{\partial u}{\partial \theta} \right) - \frac{\partial}{\partial z} \left( a_{zz} \frac{\partial u}{\partial z} \right) = f \quad (86)$$

- where  $u, f$  and the coefficients are a function of  $(r, \theta, z)$
- Dependant on the coefficients, boundary conditions and load  $f$  the problem can be made to 2D or even 1D
- If the cylinder is very long and stuff dont vary and depend on  $z$ , then we can assume a disk. No also if there is independance from  $\theta$  then we can even just do a radial line <sup>3</sup>



<sup>3</sup>We just remove the derivatives in the equations where the change will be zero

- Suppose all the variables are not dependant on  $\theta$ , therefore we would get something like a plane and the Governing differential equation will be

$$-\frac{1}{r} \frac{\partial}{\partial r} \left( r a_{rr} \frac{\partial u}{\partial r} \right) - \frac{\partial}{\partial z} \left( a_{zz} \frac{\partial u}{\partial z} \right) = f(r, z) \quad (87)$$

- The weighted statement and weak form (Using greens theorem) will be given as :

$$\begin{aligned} 0 &= \int_A w_i \left[ -\frac{1}{r} \frac{\partial}{\partial r} \left( r a_{rr} \frac{\partial u}{\partial r} \right) - \frac{\partial}{\partial z} \left( a_{zz} \frac{\partial u}{\partial z} \right) - f(r, z) \right] r dr dz \\ &= \int_A \left[ a_{rr}(r, z, u_h) \frac{\partial w_i}{\partial r} \frac{\partial u_h}{\partial r} + a_{zz}(r, z, u_h) \frac{\partial w_i}{\partial z} \frac{\partial u_h}{\partial z} \right] r dr dz - \int_A w_i f(r, z) r dr dz - \int_S w_i q_n ds \end{aligned} \quad (88)$$

$$\text{where } q_n = r \left[ a_{rr} \frac{\partial u_h}{\partial r} n_r + a_{zz} \frac{\partial u_h}{\partial z} n_z \right]$$

- And we get  $\mathbf{Ku} = \mathbf{f} + \mathbf{Q} = \mathbf{F}$
- Remember the shape functions are functions of  $N(r, z)$

The transformation between  $\Omega^e$  and  $\hat{\Omega}$  is accomplished by a coordinate transformation of the form [see Eqs. (3.6.1) and (3.6.2)]

$$x = \sum_{j=1}^m x_j^e \hat{\psi}_j^e(\xi, \eta), \quad y = \sum_{j=1}^m y_j^e \hat{\psi}_j^e(\xi, \eta) \quad (6.6.1)$$

while a typical dependent variable  $u(x, y)$  is approximated by

$$u(x, y) = \sum_{j=1}^n u_j^e \psi_j^e(x, y) = \sum_{j=1}^n u_j^e \psi_j^e(x(\xi, \eta), y(\xi, \eta)) \quad (6.6.2)$$

where  $\hat{\psi}_j^e$  denote the interpolation functions of the master element  $\hat{\Omega}$  and  $\psi_j^e$  are the interpolation functions of a typical element  $\Omega^e$  over which  $u$  is approximated. The transformation in Eq. (6.6.1) maps a point  $(x, y)$  in a typical element  $\Omega^e$  of the mesh to a point  $(\xi, \eta)$  in the master element  $\hat{\Omega}$  and vice versa, if the Jacobian  $J_e$  of the transformation is positive-definite [see Eqs. (3.6.5)–(3.6.8)].

- Remember that we have to transform the integral domain to the master element so that Gauss quadrature can be used. The derivatives in the original geometry are also expressed with respect to  $\xi, \eta$  given by

$$\begin{bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{bmatrix} = J^{-1} \begin{bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{bmatrix} \quad (89)$$

where  $J = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \sum x_i \frac{\partial N_i}{\partial \xi} & \sum y_i \frac{\partial N_i}{\partial \xi} \\ \sum x_i \frac{\partial N_i}{\partial \eta} & \sum y_i \frac{\partial N_i}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \dots & \frac{\partial N_m}{\partial \xi} \\ \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \dots & \frac{\partial N_m}{\partial \eta} \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_m & y_m \end{bmatrix}$

- If the same shape functions are used for the geometry and field variable, we say its isoparametric

Returning to the coefficients  $K_{ij}^e$  in Eq. (6.6.3), we can write it now in terms of the natural coordinates  $(\xi, \eta)$  as

$$\begin{aligned} K_{ij}^e &= \int_{\hat{\Omega}} \left\{ \hat{a}_{xx}(\xi, \eta) \left( J_{11}^* \frac{\partial \psi_i^e}{\partial \xi} + J_{12}^* \frac{\partial \psi_i^e}{\partial \eta} \right) \left( J_{11}^* \frac{\partial \psi_j^e}{\partial \xi} + J_{12}^* \frac{\partial \psi_j^e}{\partial \eta} \right) \right. \\ &\quad + \hat{a}_{yy}(\xi, \eta) \left( J_{21}^* \frac{\partial \psi_i^e}{\partial \xi} + J_{22}^* \frac{\partial \psi_i^e}{\partial \eta} \right) \left( J_{21}^* \frac{\partial \psi_j^e}{\partial \xi} + J_{22}^* \frac{\partial \psi_j^e}{\partial \eta} \right) \\ &\quad \left. + \hat{a}_{00}(\xi, \eta) \psi_i^e \psi_j^e \right\} J_e d\xi d\eta \\ &\equiv \int_{\hat{\Omega}} F_{ij}^e(\xi, \eta) d\xi d\eta \end{aligned} \quad (6.6.9)$$

where the element area  $dA = dx dy$  in element  $\Omega^e$  is transformed to  $J_e d\xi d\eta$  in the master element  $\hat{\Omega}$ , and  $\hat{a}_{xx} = a_{xx}(x(\xi, \eta), y(\xi, \eta), u(\xi, \eta))$ , and so on.

Using the  $M \times N$  Gauss quadrature to evaluate integrals defined over a rectangular master element  $\hat{\Omega}$ , we obtain

$$\begin{aligned} \int_{\hat{\Omega}} F_{ij}^e(\xi, \eta) d\xi d\eta &= \int_{-1}^1 \left[ \int_{-1}^1 F_{ij}^e(\xi, \eta) d\eta \right] d\xi \approx \int_{-1}^1 \left[ \sum_{J=1}^N F_{ij}^e(\xi, \eta_J) W_J \right] d\xi \\ &\approx \sum_{I=1}^M \sum_{J=1}^N F_{ij}^e(\xi_I, \eta_J) W_I W_J \end{aligned} \quad (6.6.10)$$

where  $M$  and  $N$  denote the number of Gauss quadrature points in the  $\xi$  and  $\eta$  directions, respectively,  $(\xi_I, \eta_J)$  denote the Gauss points, and  $W_I$  and  $W_J$  denote the corresponding Gauss weights, as listed in Table 3.6.1.

As already discussed, if the integrand is a polynomial of degree  $p$  in a coordinate direction, it is integrated exactly by employing  $NGP \equiv N = \text{int}[\frac{1}{2}(p+1)]$  (the nearest equal or larger integer number) Gauss points in that direction. In most cases, the interpolation functions are of the same degree in both  $\xi$  and  $\eta$ , and we take  $N = M$ . For example, consider the expression involving  $a_{00}$  in  $K_{ij}^e$  of Eq. (6.6.3). When  $a_{00}$  is a linear function of  $\xi$  and  $\eta$ , it requires a  $2 \times 2$  Gauss rule when  $\psi_i^e$  are linear and a  $3 \times 3$  Gauss rule when  $\psi_i^e$  are quadratic to be evaluated exactly. When  $a_{00}$  is quadratic or cubic, it requires  $3 \times 3$  and  $4 \times 4$  Gauss rules for linear and quadratic *rectangular* elements, respectively.

where  $J^*$  are the components of the inverse jacobian

# **TIME DEPENDANT PROBLEMS**

Will have to read later!!

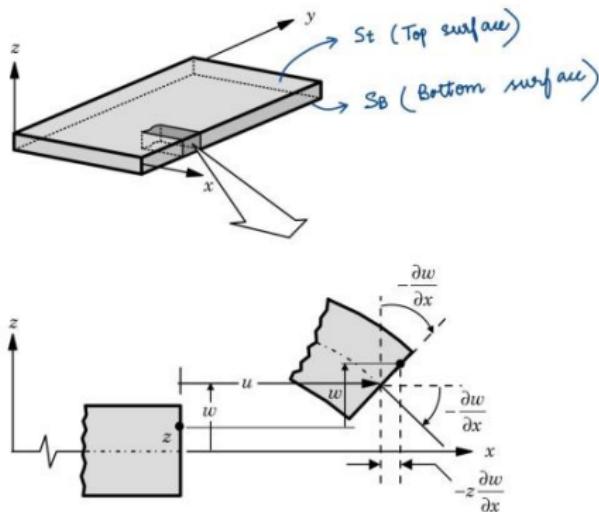
# PLATES

- Plate has large dimension compared to thickness. No need to use 3D, with study the deformations and stresses in plates, small rotations and large displacement ( $w/h > 1$ )
- Extension of Euler-Bernoulli is called Kirchhoff plate. Extension of Timoshenko beam is the first order or Mindlin shear deformation plate theory.
- **X** is used for the material coordinates and **x** is used for the spatial coordinates. No distinction is made between the material and spatial coordinates.

The disp satisfy the kirchoff rules which are an extension of the euler bernoulli hypothesis which are

- Straight lines perpendicular to the mid surface remain straight
- Transverse normals do not have elongation
- Cross sections remain perpendicular under rotation

- We have the domain of the plate as  $\Omega_o \times (-h/2, h/2)$ . The boundary of the top surface  $z = h/2$  and bottom surface  $z = -h/2$  with boundary  $\Gamma$  which is a curved surface with outward normal  $\mathbf{n} = n_x e_1 + n_y e_2$



**Fig. 7.2.1:** Undeformed and deformed geometries of an edge of a plate under the Kirchhoff assumptions.

- The kirchoff hypothesis implies the following displacement field

$$\begin{aligned} u_1(x, y, z, t) &= u(x, y, t) - z \frac{\partial w}{\partial x} \\ u_2(x, y, z, t) &= v(x, y, t) - z \frac{\partial w}{\partial y} \\ u_3(x, y, z, t) &= w(x, y, t) \end{aligned} \quad (90)$$

- Where  $u, v, w$  denote the material point in the undeformed of the nerutral axis  
wherees  $u_1, u_2, u_3$  denote any arbitrary point location
- The componenets of the Green-Lagrange strain tensor  $\mathbf{E}$  in terms of components of the total displacement vector  $u = x(x, t) - X$  (x and X here are the same) is

$$\begin{aligned} E_{11} &= \frac{\partial u_1}{\partial X_1} + \frac{1}{2} \left[ \left( \frac{\partial u_1}{\partial X_1} \right)^2 + \left( \frac{\partial u_2}{\partial X_1} \right)^2 + \left( \frac{\partial u_3}{\partial X_1} \right)^2 \right] \\ E_{22} &= \frac{\partial u_2}{\partial X_2} + \frac{1}{2} \left[ \left( \frac{\partial u_1}{\partial X_2} \right)^2 + \left( \frac{\partial u_2}{\partial X_2} \right)^2 + \left( \frac{\partial u_3}{\partial X_2} \right)^2 \right] \\ E_{33} &= \frac{\partial u_3}{\partial X_3} + \frac{1}{2} \left[ \left( \frac{\partial u_1}{\partial X_3} \right)^2 + \left( \frac{\partial u_2}{\partial X_3} \right)^2 + \left( \frac{\partial u_3}{\partial X_3} \right)^2 \right] \\ E_{12} &= \frac{1}{2} \left[ \frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1} + \frac{\partial u_1}{\partial X_1} \frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1} \frac{\partial u_2}{\partial X_2} + \frac{\partial u_3}{\partial X_1} \frac{\partial u_3}{\partial X_2} + \right] \\ E_{13} &= \frac{1}{2} \left[ \frac{\partial u_1}{\partial X_3} + \frac{\partial u_3}{\partial X_1} + \frac{\partial u_1}{\partial X_1} \frac{\partial u_1}{\partial X_3} + \frac{\partial u_2}{\partial X_1} \frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_1} \frac{\partial u_3}{\partial X_3} + \right] \\ E_{23} &= \frac{1}{2} \left[ \frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_2} + \frac{\partial u_1}{\partial X_2} \frac{\partial u_1}{\partial X_3} + \frac{\partial u_2}{\partial X_2} \frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_2} \frac{\partial u_3}{\partial X_3} + \right] \end{aligned} \quad (91)$$

- If the components of the displacement gradient are very small =  $O(\epsilon)$ , then the terms having  $O(\epsilon^2)$  can be omitted in the strains. However if the rotations of the transverse normals are moderate (10 -15 degrees), then the following strains are small but not negligible

$$\left( \frac{\partial u_3}{\partial X_1} \right)^2 \quad \left( \frac{\partial u_3}{\partial X_2} \right)^2 \quad \frac{\partial u_3}{\partial X_1} \frac{\partial u_3}{\partial X_2} \quad (92)$$

- Thus the strains take the following (Remember that  $E = \epsilon$  where we say its small strain but moderate rotations)

$$\begin{aligned} E_{11} &= \epsilon_{11} = \frac{\partial u_1}{\partial x} + \frac{1}{2} \left[ \left( \frac{\partial u_3}{\partial x} \right)^2 \right] & E_{22} &= \frac{\partial u_2}{\partial y} + \frac{1}{2} \left[ \left( \frac{\partial u_3}{\partial y} \right)^2 \right] \\ E_{33} &= \frac{\partial u_3}{\partial z} & E_{12} &= \frac{1}{2} \left[ \frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} + \frac{\partial u_3}{\partial x} \frac{\partial u_3}{\partial y} \right] \\ E_{13} &= \frac{1}{2} \left[ \frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x} \right] & E_{23} &= \frac{1}{2} \left[ \frac{\partial u_2}{\partial z} + \frac{\partial u_3}{\partial y} \right] \end{aligned} \quad (93)$$

- For this displacement field, we have  $\varepsilon_z = \frac{\partial u_3}{\partial z} = \frac{\partial w}{\partial z} = 0$  and taking the displacement fields. The strains then reduce to

$$\begin{aligned}\varepsilon_{xx} &= \frac{\partial u}{\partial x} + \frac{1}{2} \left[ \left( \frac{\partial w}{\partial x} \right)^2 - z \frac{\partial^2 w}{\partial x^2} \right] & \varepsilon_{yy} &= \frac{\partial v}{\partial y} + \frac{1}{2} \left[ \left( \frac{\partial w}{\partial y} \right)^2 - z \frac{\partial^2 w}{\partial y^2} \right] \\ 2\varepsilon_{xy} &= \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} - 2z \frac{\partial^2 w}{\partial x \partial y} \\ 2\varepsilon_{xz} &= -\frac{\partial w}{\partial x} + \frac{\partial w}{\partial x} = 0 & 2\varepsilon_{yz} &= -\frac{\partial w}{\partial y} + \frac{\partial w}{\partial y} = 0\end{aligned}\quad (94)$$

- These are called von Karman strains and called classical plate theory with von Karman strains. Note that the transverse strains are zero in the classical plate theory. The total strains can be written as membrane + bending strain

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx}^0 \\ \varepsilon_{yy}^0 \\ \gamma_{xy}^0 \end{bmatrix} + z \begin{bmatrix} \varepsilon_{xx}^1 \\ \varepsilon_{yy}^1 \\ \gamma_{xy}^1 \end{bmatrix} \quad (95)$$

■ The strains are expanded as :

$$\begin{bmatrix} \varepsilon_{xx}^0 \\ \varepsilon_{yy}^0 \\ \gamma_{xy}^0 \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \\ \frac{\partial v}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \end{bmatrix} \quad \begin{bmatrix} \varepsilon_{xx}^1 \\ \varepsilon_{yy}^1 \\ \gamma_{xy}^1 \end{bmatrix} = - \begin{bmatrix} \frac{\partial^2 w}{\partial x^2} \\ \frac{\partial^2 w}{\partial y^2} \\ 2 \frac{\partial^2 w}{\partial x \partial y} \end{bmatrix} \quad (96)$$

## WEAK FORM OF CLASSICAL PLATE THEORY

- Virtual work statement is used again to derive (Just like 4 beams). We account for thermal effects, where the material does not change with temperature which is known as a function of the position hence ( $\delta T = 0$ ), so temperature enters through the constitutive relations.
- Suppose the domain is represented by fem  $\Omega_e$  with distributed transverse loads  $q(x, y)$  at the top.  $(\sigma_{nn}, \sigma_{ns}, \sigma_{nz})$  are the stress components on the boundary of the plate

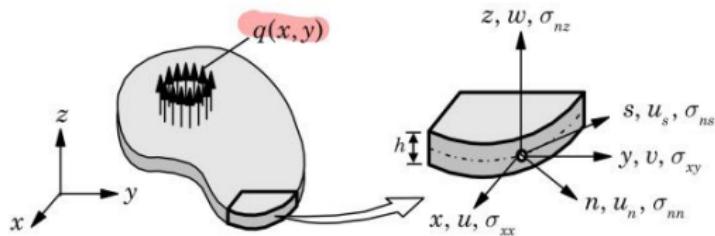


Fig. 7.3.1: Geometry of a plate element with curved boundary.

- The principle of virtual work states that  $0 = \delta W^e = \delta W_I^e + \delta W_E^e$
- As noted earlier, the transverse shears  $\gamma_{xz}, \gamma_{yz}, \epsilon_{zz}$  are zero. Therefore the transverse stresses ( $\sigma_{xz}, \sigma_{yz}, \sigma_{zz}$ ) do not enter the formulation because the strains due to these are zero. Even though they are not accounted, in reality they exist to maintain equilibrium, these components can also be specified at the boundary. So they have to be accounted in the equilibrium equations.
- The internal virtual strain is then given as

$$\begin{aligned}\delta W_I^e &= \int_A \int_{-\frac{h}{2}}^{\frac{h}{2}} (\sigma_{xx} \delta \varepsilon_{xx} + \sigma_{yy} \delta \varepsilon_{yy} + 2\sigma_{xy} \delta \varepsilon_{xy}) dz dx dy \\ &= \int_A (N_{xx} \delta \varepsilon_{xx}^0 + M_{xx} \delta \varepsilon_{xx}^1 + N_{yy} \delta \varepsilon_{yy}^0 + M_{yy} \delta \varepsilon_{yy}^1 + N_{xy} \delta \gamma_{xy}^0 + M_{xy} \delta \gamma_{xy}^1) dx dy\end{aligned}\quad (97)$$

where  $N$  and  $M$  are the axial and the moment internal forces per unit length.

## PLATE: INTERNAL FORCES

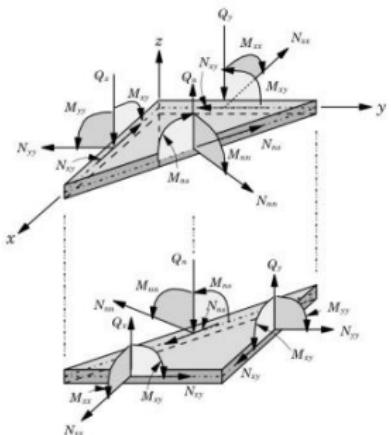


Fig. 7.3.2: Forces and moments per unit length on a plate element.

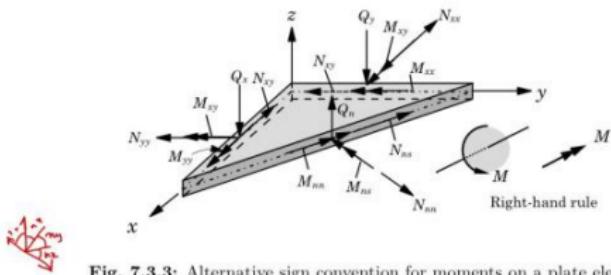


Fig. 7.3.3: Alternative sign convention for moments on a plate element.

- The virtual work by the distributed transverse load  $q(x,y)$ , reaction force of an elastic foundation, in plane normal stress  $\sigma_{nn}$ , in plane tangential stress  $\sigma_{ns}$ , transverse shear stress  $\sigma_{nz}$  is

$$\begin{aligned}
 \delta W_E^e &= -\left(\int_A q(x, y) \delta w(x, y, \frac{h}{2}) dx dy + \int_A F_s(x, y) \delta w(x, y, -\frac{h}{2}) dx dy \right. \\
 &\quad \left. + \int_S \int_{-\frac{h}{2}}^{\frac{h}{2}} \left[ \sigma_{nn} \left( \delta u_n - z \frac{\delta w}{n} \right) + \sigma_{ns} \left( \delta u_s - z \frac{\delta w}{s} \right) + \sigma_{nz} \delta w \right] dz ds \right) \\
 &= - \left[ \int_S \left( N_{nn} \delta u_n - M_{nn} \frac{\partial \delta w}{\partial n} + N_{ns} \delta u_s - M_{ns} \frac{\partial \delta w}{\partial s} + Q_n \delta w \right) ds + \int_A (q - kw) \delta w dx dy \right] \tag{98}
 \end{aligned}$$

- where  $F_s = -kw$  (Foundation force), the negative sign it is the force applied upwards, but you can think of it like the potential increases as the  $w$  increases.
- The last term in the work term is the virtual work of the transverse and normal forces on a boundary that is inclined. Where  $N = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma$ ,  $M = \int_{-\frac{h}{2}}^{\frac{h}{2}} z\sigma$ ,  $Q_n = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{nz}$

- We now relate the stresses in the direction in the boundary to the internal stresses given in cartesian using stress tensor coordinate transformation

$$\begin{bmatrix} \sigma_{nn} \\ \sigma_{ns} \end{bmatrix} = \begin{bmatrix} n_x^2 & n_y^2 & 2n_x n_y \\ -n_x n_y & n_x n_y & n_x^2 - n_y^2 \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} \quad (99)$$

- Keeping the weak forms in the full virtual work statement we get

$$\begin{aligned}
 0 &= \int_A \left( N_{xx} \delta \varepsilon_{xx}^0 + M_{xx} \delta \varepsilon_{xx}^1 + N_{yy} \delta \varepsilon_{yy}^0 + M_{yy} \delta \varepsilon_{yy}^1 + N_{xy} \delta \gamma_{xy}^0 + M_{xy} \delta \gamma_{xy}^1 \right) dx dy \\
 &- \left[ \int_S \left( N_{nn} \delta u_n - M_{nn} \frac{\partial \delta w}{\partial n} + N_{ns} \delta u_s - M_{ns} \frac{\partial \delta w}{\partial s} + Q_n \delta w \right) ds + \int_A (q - kw) \delta w dx dy \right]_4 \\
 &= \int_A \left( N_{xx} \delta \varepsilon_{xx}^0 + M_{xx} \delta \varepsilon_{xx}^1 + N_{yy} \delta \varepsilon_{yy}^0 + M_{yy} \delta \varepsilon_{yy}^1 + N_{xy} \delta \gamma_{xy}^0 + M_{xy} \delta \gamma_{xy}^1 \right) dx dy \\
 &- \left[ \int_S \left( N_{nn} \delta u_n - M_{nn} \frac{\partial \delta w}{\partial n} + N_{ns} \delta u_s - M_{ns} \frac{\partial \delta w}{\partial s} + Q_n \delta w \right) ds + \int_A (q - kw) \delta w dx dy \right] \tag{100}
 \end{aligned}$$

$$\begin{aligned}
 &= \int_{\Omega^e} \left[ \left( \frac{\partial \delta u}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial \delta w}{\partial x} \right) N_{xx} + \left( \frac{\partial \delta v}{\partial y} + \frac{\partial w}{\partial y} \frac{\partial \delta w}{\partial y} \right) N_{yy} \right. \\
 &\quad + \left( \frac{\partial \delta u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial \delta w}{\partial y} + \frac{\partial w}{\partial y} \frac{\partial \delta w}{\partial x} \right) N_{xy} \quad \text{uv}^t = \int u' V - [uv]_{xa}^{xb} \\
 &\quad \left. - \frac{\partial^2 \delta w}{\partial x^2} M_{xx} - \frac{\partial^2 \delta w}{\partial y^2} M_{yy} - 2 \frac{\partial^2 \delta w}{\partial x \partial y} M_{xy} + k \delta w w - \delta w q \right] dx dy \\
 &\quad - \oint_{\Gamma^e} \left( N_{nn} \delta u_n + N_{ns} \delta u_s - M_{nn} \frac{\partial \delta w}{\partial n} - M_{ns} \frac{\partial \delta w}{\partial s} + Q_n \delta w \right) ds \tag{7.3.9}
 \end{aligned}$$

The statement in Eq. (7.3.9) is equivalent to the following three statements:

$$0 = \int_{\Omega^e} \left( \frac{\partial \delta u}{\partial x} N_{xx} + \frac{\partial \delta u}{\partial y} N_{xy} \right) dx dy - \oint_{\Gamma^e} N_{nn} \delta u_n ds \tag{7.3.10}$$

$$0 = \int_{\Omega^e} \left( \frac{\partial \delta v}{\partial x} N_{zy} + \frac{\partial \delta v}{\partial y} N_{yy} \right) dx dy - \oint_{\Gamma^e} N_{ns} \delta u_s ds \tag{7.3.11}$$

$$\begin{aligned}
 0 &= \int_{\Omega^e} \left[ \frac{\partial \delta w}{\partial x} \left( \frac{\partial w}{\partial x} N_{xx} + \frac{\partial w}{\partial y} N_{xy} \right) + \frac{\partial \delta w}{\partial y} \left( \frac{\partial w}{\partial x} N_{xy} + \frac{\partial w}{\partial y} N_{yy} \right) \right. \\
 &\quad \left. - \frac{\partial^2 \delta w}{\partial x^2} M_{xx} - \frac{\partial^2 \delta w}{\partial y^2} M_{yy} - 2 \frac{\partial^2 \delta w}{\partial x \partial y} M_{xy} + k \delta w w - \delta w q \right] dx dy \\
 &\quad - \oint_{\Gamma^e} \left( -M_{nn} \frac{\partial \delta w}{\partial n} - M_{ns} \frac{\partial \delta w}{\partial s} + Q_n \delta w \right) ds \tag{7.3.12}
 \end{aligned}$$

<sup>4</sup> In the last three statements, It seems the equation has been kept according to the variation but how do you account for  $\delta u_n$   $\delta u_s$  as they will have components in both. Maybe the euler equations will make sense

# EQUILIBRIUM EQUATIONS

- Keeping the virtual parameters in the same order to get the euler equations we get

$$0 = \int_A \left[ -\left( N_{xx,x} + N_{xy,y} \right) \delta u - \left( N_{xy,x} + N_{yy,y} \right) \delta v - \left( M_{xx,xx} + 2M_{xy,xy} + M_{yy,yy} + N - kw + q \right) \delta w \right] dx dy \\ + \int_S \left( \left( N_{xx} n_x + N_{xy} n_y \right) \delta u + \left( N_{xy} n_x + N_{yy} n_y \right) \delta v + \left( M_{xx,x} n_x + M_{xy,y} n_x + M_{yy,y} n_y + M_{xy,x} n_y + P \right) \delta w \right) ds \quad (101)$$
$$- \left( M_{xx} n_x + M_{xy} n_x \right) \frac{\partial \delta w}{\partial x} - \left( M_{xy} n_x + M_{yy} n_y \right) \frac{\partial \delta w}{\partial y} ) ds - \int_S \left( N_{nn} \delta u_n + N_{ns} \delta u_s - M_{nn} \frac{\partial \delta w}{\partial n} - M_{ns} \frac{\partial \delta w}{\partial s} + Q_n \delta w \right) ds$$

- Where

$$N = \frac{\partial}{\partial x} \left( N_{xx} \frac{\partial w}{\partial x} + N_{xy} \frac{\partial w}{\partial y} \right) + \frac{\partial}{\partial y} \left( N_{xy} \frac{\partial w}{\partial x} + N_{yy} \frac{\partial w}{\partial y} \right) \quad (102)$$
$$P = \left( N_{xx} \frac{\partial w}{\partial x} + N_{xy} \frac{\partial w}{\partial y} \right) n_x + \frac{\partial}{\partial y} \left( N_{xy} \frac{\partial w}{\partial x} + N_{yy} \frac{\partial w}{\partial y} \right) n_y$$

INTEGRATION BY PARTS

Divergence theorem:

Suppose we have a vector field  $\mathbf{v}(x, y)$

$$\int_A \left( \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} \right) dA = \int_R \mathbf{v} \cdot \mathbf{n} ds = \int_R v_x n_x ds + \int_R v_y n_y ds.$$

So we can say  $\int_A \frac{\partial}{\partial x} (v_x) dA = \int_R v_x n_x ds$  etc.

Now:  $\int_A \frac{\partial}{\partial x} (AB) dA = \int_R (AB) \cdot \mathbf{n} ds$   
 and  $(AB) = AB' + B'A$  or  $AB' = (AB)' - B'A$ .

$$\therefore \int_A N_{xx} \frac{\partial v_x}{\partial x} = \int_A \left( N_{xx} \delta x \right) - \int_A \frac{\partial N_{xx}}{\partial x} \delta x \quad \left. \begin{array}{l} \text{Inside } \mathcal{N} \\ \int (N_{xx} \delta x) \cdot n_x ds - \int \frac{\partial N_{xx}}{\partial x} \delta x \end{array} \right\}$$

$$\int_A N_{xx} \frac{\partial w}{\partial x} \frac{\partial \delta w}{\partial x} = \int_A \frac{\partial}{\partial x} \left( N_{xx} \frac{\partial w}{\partial x} \delta w \right) - \int_A \frac{\partial}{\partial x} \left( N_{xx} \frac{\partial w}{\partial x} \right) \cdot \delta w \quad \left. \begin{array}{l} \text{Inside } \mathcal{P} \\ \int \left( N_{xx} \frac{\partial w}{\partial x} \delta w \right) \cdot n_x ds - \int \frac{\partial}{\partial x} \left( N_{xx} \frac{\partial w}{\partial x} \right) \delta w \end{array} \right\}$$

$$- \int_A M_{xy} \frac{\partial^2 \delta w}{\partial x \partial y} = -2 \left( \int_A \frac{\partial}{\partial x} \left( M_{xy} \frac{\partial \delta w}{\partial y} \right) - \int_A \frac{\partial M_{xy}}{\partial x} \frac{\partial \delta w}{\partial y} \right)$$

$$= -2 \left( \int_R \left( M_{xy} \frac{\partial \delta w}{\partial y} \right) \cdot n_x ds - \int_A \frac{\partial M_{xy}}{\partial x} \frac{\partial \delta w}{\partial y} \right)$$

$$(uv)' = u v' + v' u$$

$$uv = \int u v' ds + \int v' u ds$$

$$= -2 \left( \int_R \left( M_{xy} n_x \right) \frac{d \delta w}{dx} - \left( \int_R \frac{\partial (M_{xy} \delta w)}{\partial x} - \int_A \frac{\partial M_{xy}}{\partial x} \frac{\partial \delta w}{\partial y} \right) \right)$$

$$= -2 \left( \int_R \left( M_{xy} n_x \right) \frac{d \delta w}{dx} - \left( \int_R \frac{\partial M_{xy} \delta w}{\partial x} \right) n_y ds + \int_A \frac{\partial M_{xy}}{\partial x} \frac{\partial \delta w}{\partial y} \right)$$

- Keeping the coefficients of the variations (setting  $\delta u, \delta v, \delta w = 0$ ) we get the equilibrium equations given as

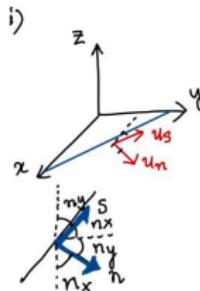
$$\begin{aligned}\delta u : \frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} &= 0 \\ \delta v : \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} &= 0\end{aligned}\tag{103}$$

$$\delta w : \frac{\partial^2 M_{xx}}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_{yy}}{\partial y^2} + N - kw + q = 0$$

## BOUNDRY CONDITIONS

- To cast the B.C. on an edge whose normal is  $\mathbf{n}$ , we express the generalised displacements  $(u, v, w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y})$  in  $x, y, z$  system in the corresponding displacements in normal, tangential and transverse directions. We get

$$\begin{aligned} u &= u_n n_x - u_s n_y & v &= u_n n_y + u_s n_x \\ \frac{\partial w}{\partial x} &= \frac{\partial w}{\partial n} n_x - \frac{\partial w}{\partial s} n_y & \frac{\partial w}{\partial y} &= \frac{\partial w}{\partial n} n_y + \frac{\partial w}{\partial s} n_x \end{aligned} \quad (104)$$



$$\left\{ \begin{array}{l} \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial y} \end{array} \right\} = N \left\{ \begin{array}{l} \frac{\partial w}{\partial n} \\ \frac{\partial w}{\partial s} \end{array} \right\}$$

Now the derivative of  $w$  is a vector which can undergo basis transformation

## BOUNDARY CONDITIONS

- The boundary conditions can be written in the normal, tangential coords

$$\int_S [ \left( N_{XX} n_X + N_{XY} n_Y \right) (\delta u_n n_X - \delta u_s n_Y) + \left( N_{XY} n_X + N_{YY} n_Y \right) (\delta u_n n_Y + \delta u_s n_X) \\ + \left( M_{XX,x} n_X + M_{XY,y} n_X + M_{YY,y} n_Y + M_{XY,x} n_Y + P \right) \delta w \\ - \left( M_{XX} n_X + M_{XY} n_Y \right) \left( \frac{\partial w}{\partial n} n_X - \frac{\partial w}{\partial s} n_Y \right) - \left( M_{XY} n_X + M_{YY} n_Y \right) \left( \frac{\partial w}{\partial n} n_Y + \frac{\partial w}{\partial s} n_X \right) ] ds \\ - \int_S \left( N_{nn} \delta u_n + N_{ns} \delta u_s - M_{nn} \frac{\partial \delta w}{\partial n} - M_{ns} \frac{\partial \delta w}{\partial s} + Q_n \delta w \right) ds \quad (105)$$

- $\delta u$  in natural does not change the direction cosines because here they are constant dependant on the geometry we are at and not the variation.

## BOUNDARY CONDITIONS

$$\begin{aligned}
 &= \oint_{\Gamma^e} \left\{ \left( N_{xx} n_x^2 + 2N_{xy} n_x n_y + N_{yy} n_y^2 - N_{nn} \right) \delta u_n \right. \\
 &\quad + \left[ (N_{yy} - N_{xx}) n_x n_y + N_{xy} (n_x^2 - n_y^2) - N_{ns} \right] \delta u_s \\
 &\quad + (M_{xx,x} n_x + M_{xy,y} n_x + M_{yy,y} n_y + M_{xy,x} n_y + \mathcal{P} - Q_n) \delta w \\
 &\quad - \left( M_{xx} n_x^2 + 2M_{xy} n_x n_y + M_{yy} n_y^2 - M_{nn} \right) \frac{\partial \delta w}{\partial n} \\
 &\quad \left. - \left[ (M_{yy} - M_{xx}) n_x n_y + M_{xy} (n_x^2 - n_y^2) - M_{ns} \right] \frac{\partial \delta w}{\partial s} \right\} ds \quad (7.3.19)
 \end{aligned}$$

The natural boundary conditions are obtained by setting the coefficients of  $\delta u_n$ ,  $\delta u_s$ ,  $\delta w$ ,  $\frac{\partial \delta w}{\partial n}$  and  $\frac{\partial \delta w}{\partial s}$  on  $\Gamma^e$  to zero: (interesting)

$$\delta u_n : N_{nn} = N_{xx} n_x^2 + 2N_{xy} n_x n_y + N_{yy} n_y^2 \quad (7.3.20a)$$

$$\delta u_s : N_{ns} = (N_{yy} - N_{xx}) n_x n_y + N_{xy} (n_x^2 - n_y^2) \quad (7.3.20a)$$

$$\delta w : Q_n = M_{xx,x} n_x + M_{xy,y} n_x + M_{yy,y} n_y + M_{xy,x} n_y + \mathcal{P}$$

$$\frac{\partial \delta w}{\partial n} : M_{nn} = M_{xx} n_x^2 + 2M_{xy} n_x n_y + M_{yy} n_y^2 \quad (7.3.20b)$$

$$\frac{\partial \delta w}{\partial s} : M_{ns} = (M_{yy} - M_{xx}) n_x n_y + M_{xy} (n_x^2 - n_y^2)$$

From Eq. (7.3.20), it is clear that the primary variables (i.e. generalized displacements) and secondary variables (i.e. generalized forces) of the theory are:

$$\text{Primary variables: } u_n, u_s, w, \frac{\partial w}{\partial n}, \frac{\partial w}{\partial s} \quad (7.3.21)$$

$$\text{Secondary variables: } N_{nn}, N_{ns}, Q_n, M_{nn}, M_{ns}$$