Differential Geometry

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Differential geometry : Curved computing

- A shell is comprised of its reference surface, thickness and edges. The thickness of the shell at a point is found as the distance between the bounding surface (Top and bottom surfaces) as measured along a normal to the reference surface that passes through the point.
- FIX ME???????????? New things also have to be added

- A 3d curve in a rectangular coordinate system (x_1, x_2, x_3) can be represented by the locus of the end point of the position vector

$$\mathbf{x} = x_1(t)\mathbf{e_1} + x_2(t)\mathbf{e_2} + x_3(t)\mathbf{e_3} \tag{1}$$

for all the values of the parameter t that are in the interval $t_1 < t < t_2$. Suppose that $\mathbf{x}$ takes only a single value of the parameter t, then we insure it's uniqueness.

- We take s as the variable of the arc length along the space curve. The derivative of the position vector x with respect to s is given as

$$\frac{d\mathbf{x}}{ds} = \frac{dx_1}{ds}\mathbf{e_1} + \frac{dx_2}{ds}\mathbf{e_2} + \frac{dx_3}{ds}\mathbf{e_3} \tag{2}$$

- If we form a scalar product with itself we get

$$\frac{d\mathbf{x}}{ds} \cdot \frac{d\mathbf{x}}{ds} = \left(\frac{dx_1}{ds}\right)^2 + \left(\frac{dx_2}{ds}\right)^2 + \left(\frac{dx_3}{ds}\right)^2 \tag{3}$$

- From differential calculus we know that

$$(ds)^2 = (dx_1)^2 + (dx_2)^2 + (dx_3)^2 \tag{4}$$

Hence

$$\frac{d\mathbf{x}}{ds} \cdot \frac{d\mathbf{x}}{ds} = 1 \tag{5}$$

This confuese me how the magnitude of the tangent becomes 1, because sometimes in calculus some mainpulations seem very devious

- We can think of a vector $\Delta\mathbf{x}$ joining two points Q and Q' on a curve. The vector $\Delta\mathbf{x}/\Delta s$ has the same directiona as $\Delta\mathbf{x}$ and as Δs approaches 0, $\Delta\mathbf{x}/\Delta s$ becomes the tangent vector to the curve at he poin Q.
- The vector $\mathbf{T}$ is as the limit of $\Delta s \to 0$ we get

$$\mathbf{T} = \frac{d\mathbf{x}}{ds} \tag{6}$$

- We also note that

$$x' = \frac{d\mathbf{x}}{dt} = \frac{d\mathbf{x}}{ds}\frac{ds}{dt} \tag{7}$$

  This comes from chain rule by the way the parametrisation is done orginally. This is als a tangent vector but it is not of unit length

- The tangent to the Curve at Q is found to be the limiting position of the line connecting the points Q and Q'. Now if we consider the limiting position of a plane passing through three consequitve places. We will get the osculating plane joining the normal and the tangent of a curve.
- The tripple scalar produc of three coplanar vectors is zero and the expression of the osculating plane is as

$$(\mathbf{X} - \mathbf{x}) \cdot (\mathbf{x}' \times \mathbf{x}'') \tag{8}$$

- By taking the dot product with itself $\mathbf{T} \cdot \mathbf{T} = \mathbf{1}$. If we differentiate this scalar product with respect to the arc length we get

$$\frac{d}{ds}(\mathbf{T} \cdot \mathbf{T}) = 2\mathbf{T} \cdot \mathbf{T}^{o} = 0 \tag{9}$$

where $\mathbf{T}^{o}$ is the derivative with respect to s. Now this tells us that the tangent is prependicular to the curvature. We can also then write

$$\mathbf{T} = \frac{d\mathbf{x}}{ds} = \frac{d\mathbf{x}}{dt}\frac{dt}{ds} = \mathbf{x}' t^{o} \tag{10}$$

where this comes from chain rule but the parameterisation is messed up, originally it is $\frac{d\mathbf{x}}{dt} = \frac{d\mathbf{x}}{ds}\frac{ds}{dt}$. I don't know when things can be used like fractions!! But it checks out here. Also note that s and t will be different functions for different components of $\mathbf{x}$

- Therefore we can say

$$\mathbf{T}^{\mathbf{o}} = \mathbf{x}' t^{oo} + \mathbf{x}''(t^{o})^{2} \tag{11}$$

Which shoes that the curvature also lies in the plane of $\mathbf{x}'$ and $\mathbf{x}''$, or the osculating plane.

- Since the curvature has been shown to be perpendicular to the tangent **T** we can say that the curvature is parallel to the principal normal given by

$$\mathbf{T^o} = k\mathbf{N} = \mathbf{k} \tag{12}$$

  where **N** is a unit normal vector in the direction of the principal normal to the curve at a point (1/R). We can see that, k is the curvature and $k\mathbf{N}$ is the curvature vector.

- Although the sense of T^o is determined solely by the curve, the sense of the principal normal **N** is arbitary. k therefore depends on the sense of **N**.