

DIFFERENTIAL GEOMETRY

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SURFACES

- Every surface S can be written as a function of two points ξ_1 and ξ_2 , where we get

$$x_1 = f_1(\xi_1, \xi_2) \quad y = f_2(\xi_1, \xi_2) \quad z = f_3(\xi_1, \xi_2) \quad (1)$$

where all the functions are single valued continuous functions of ξ_1, ξ_2 (Which are called curvilinear coordinates of the surface). By fixing one and changing the other, we get a family of curves called parametric curves on the surface.

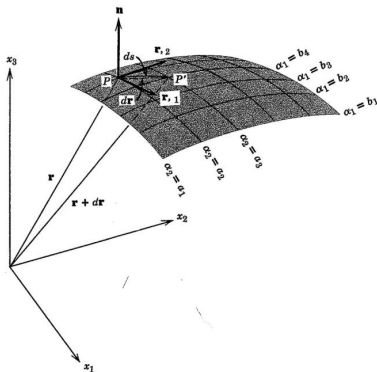


Figure: PATH NOT PROPER

- Now the position of a point on the curve is given as
 $\mathbf{r}(\xi_1, \xi_2) = \mathbf{f}_1(\xi_1, \xi_2)\mathbf{e}_1 + \mathbf{f}_2(\xi_1, \xi_2)\mathbf{e}_2 + \mathbf{f}_3(\xi_1, \xi_2)\mathbf{e}_3$
- Now a small differential change $d\mathbf{r}$ in the vector \mathbf{r} as we move from a point P to a very close point P' on the surface S can be found as

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial \xi_1} d\xi_1 + \frac{\partial \mathbf{r}}{\partial \xi_2} d\xi_2 \quad (2)$$

where we can also write $\mathbf{r}_{,i} = \frac{\partial \mathbf{r}}{\partial \xi_i}$ with $i = 1, 2$

for partial derivatives of vectors. The square of the magnitude of the differential change vector $d\mathbf{r}$ is found by taking the scalar product of $d\mathbf{r}$ with itself

$$\begin{aligned} (ds)^2 = d\mathbf{r} \cdot d\mathbf{r} &= \mathbf{r}_{,1} \cdot \mathbf{r}_{,1} (d\xi_1)^2 + \mathbf{r}_{,1} \cdot \mathbf{r}_{,2} (d\xi_1)(d\xi_2) + \mathbf{r}_{,2} \cdot \mathbf{r}_{,2} (d\xi_2)^2 \\ &= E(d\xi_1)^2 + F(d\xi_1)(d\xi_2) + G(d\xi_2)^2 \end{aligned} \quad (3)$$

which is the fundamental form of the surface by knowing the fundamental magnitudes E, F, G.

- We note that $\mathbf{r}_{,i}$ are the tangents to the curves of constant ξ_1, ξ_2 . F will be zero if the parametric curves form an orthogonal net. We can write it also as

$$(ds)^2 = A_1^2 (d\xi_1)^2 + A_2^2 (d\xi_2)^2 \quad (4)$$

where $A_1 = \sqrt{E}$, $A_2 = \sqrt{G}$

- At every point P, there exists a unit normal vector $\mathbf{n}(\xi_1, \xi_2)$ which is perpendicular to $\mathbf{r}_{,1}$, $\mathbf{r}_{,2}$ and hence to the tangent plane at P. We get \mathbf{n} by the familiar expression

$$\mathbf{n}(\xi_1, \xi_2) = \mathbf{r}_{,1} \times \mathbf{r}_{,2} / |\mathbf{r}_{,1} \times \mathbf{r}_{,2}| \quad (5)$$

where we have

$$\begin{aligned} |\mathbf{r}_{,1} \times \mathbf{r}_{,2}| &= |\mathbf{r}_{,1}| |\mathbf{r}_{,2}| \sin\theta \\ \mathbf{r}_{,1} \cdot \mathbf{r}_{,2} &= |\mathbf{r}_{,1}| |\mathbf{r}_{,2}| \cos\theta \end{aligned} \quad (6)$$

and we have expressions by manipulation :

$\cos\theta = F/\sqrt{EG}$, $\sin\theta = \sqrt{(EG - F^2)}/EG = H/\sqrt{G}$. We can also have $\mathbf{n}(\xi_1, \xi_2) = \mathbf{r}_{,1} \times \mathbf{r}_{,2}/H$ provided H does not vanish

- The principal normal N does not need to be normal to the surface ($\mathbf{N} \cdot \mathbf{n} \neq 1$)

We have described the curvature vector \mathbf{k} of a space curve. We shall consider a curve on a surface and use the properties of the curvature vector to derive an important features of surfaces of the second fundamental form

- Recall that the curvature vector \mathbf{k} is given by $\frac{d\mathbf{T}}{ds}$ where \mathbf{T} is the unit vector tangent to the curve.
- The curvature vector \mathbf{k} of the curve into its normal and tangential components to the surface. Thus

$$\mathbf{k} = \frac{d\mathbf{T}}{ds} = \mathbf{k}_n + \mathbf{k}_t \quad (7)$$

k_n and k_t are the normal and tangential curvature vector.

- Since \mathbf{k}_n is in the direction of the normal to the surface, it is proportional to \mathbf{n} and can be expressed in terms as

$$\mathbf{k}_n = -K_n \mathbf{n} \quad (8)$$

The minus sign takes into account the fact that the sense of the curvature vector \mathbf{k} is opposite to that of the normal vector \mathbf{n}

- Since \mathbf{n} is perpendicular to \mathbf{T} , differentiation with respect to s along the surface gives

$$\frac{d\mathbf{n}}{ds} \cdot \mathbf{T} = -\mathbf{n} \cdot \frac{d\mathbf{T}}{s} \quad (9)$$

- If we form the scalar of $\mathbf{k}_n = -K_n \mathbf{n}$, where K_n , we get

$$-(\mathbf{k}_n \cdot \mathbf{n}) = K_n \quad (10)$$

- And finally we will get

$$K_n = \frac{d\mathbf{r} \cdot d\mathbf{n}}{d\mathbf{r} \cdot d\mathbf{r}} \quad (11)$$

where we use

$(ds)^2 = d\mathbf{r} \cdot d\mathbf{r}$, now if we keep

$$d\mathbf{n} = \mathbf{n}_{,1} d\xi_1 + \mathbf{n}_{,2} d\xi_2$$

$$d\mathbf{r} = \mathbf{r}_{,1} d\xi_1 + \mathbf{r}_{,2} d\xi_2$$

$$K_n = \frac{II}{I} = \frac{L(d\xi_1)^2 + 2M(d\xi_1 d\xi_2) + N(d\xi_2)^2}{E(d\xi_1)^2 + 2F(d\xi_1 d\xi_2) + G(d\xi_2)^2} \quad (12)$$

where $L = \mathbf{r}_{,1} \cdot \mathbf{n}_{,1}$ $2M = \mathbf{r}_{,1} \cdot \mathbf{n}_{,2} + \mathbf{r}_{,2} \cdot \mathbf{n}_{,1}$ $N = \mathbf{r}_{,2} \cdot \mathbf{n}_{,2}$

- Now since we have the principal curvatures, we want to find the direction ($\alpha = d\xi_1/d\xi_2$) for which the normal curvature K_n has a maximum or minimum force.
- We then get the curvature as

$$K(\alpha) = \frac{L + 2M\lambda + N\lambda^2}{E + 2F\lambda + G\lambda^2} \quad (13)$$

- The curvature attains an extremum in some direction λ if $\frac{dK}{d\lambda} = 0$
- Solving the quadratic equation will result into two roots corresponding to the two directions $(d\xi_1/d\xi_2)_1$ and $(d\xi_1/d\xi_2)_2$ of

$$(MG - NF)\lambda^2 + (LG - NE)\lambda + (LF - ME) = 0 \quad (14)$$

One solution is the maximum while the other is the minimum curvature

- Now $K_1 = 1/R_1$ $K_2 = 1/R_2$ are called principal curvatures.
- These two are actually orthogonal and it can be proven (Check Harry Kraus)
- Suppose the lines of curvature are taken as the parametric lines. Therefore $d\xi_1/d\xi_2 = 0$ and $d\xi_2/d\xi_1 = 0$ should satisfy the eigen value problem
- It will come out that $F = M = 0$ and we can get

$$K_1 = 1/R_1 = L/E \quad K_2 = 1/R_2 = N/G \quad (15)$$

- Consider a triplet of orthogonal unit vectors ($\mathbf{t}_1, \mathbf{t}_2, \mathbf{n}$) so that they are tangent to ξ_1, ξ_2 and normal to the surface respectively.
- As we move on the surface, we end up changing the orientation of these unit vector, but they remain constant at unity and orthogonal
- Let us firstly define these vectors

$$\begin{aligned}\mathbf{t}_1 &= \frac{\mathbf{r}_{,1}}{|\mathbf{r}_{,1}|} \\ \mathbf{t}_2 &= \frac{\mathbf{r}_{,2}}{|\mathbf{r}_{,2}|} \\ \mathbf{n} &= \mathbf{t}_1 \times \mathbf{t}_2\end{aligned}\tag{16}$$

- Interesting thing is that since $\mathbf{n}_{,1}$ and $\mathbf{n}_{,2}$ are perpendicular to \mathbf{n} , they lie in the plane formed by \mathbf{t}_1 and \mathbf{t}_2 . Therefore we can decompose like $\mathbf{n}_{,1} = a\mathbf{t}_1 + b\mathbf{t}_2$
- To find a, b we dot product with $\mathbf{t}_1, \mathbf{t}_2$ to get two equations
- Since the system is orthogonal $M = \mathbf{t}_1 \cdot \mathbf{t}_2 = 0$

$$a = \frac{L}{A_1} \quad b = 0 \quad (17)$$

- And interestingly we get that it is in the direction of the tangents itself given as

$$\begin{aligned} \mathbf{n}_{,1} &= \frac{A_1}{R_1} \mathbf{t}_1 \\ \mathbf{n}_{,2} &= \frac{A_2}{R_2} \mathbf{t}_1 \end{aligned} \quad (18)$$

So we have found the derivatives of the normal along the parametric lines.

- To find the derivatives of $\mathbf{t}_1, \mathbf{t}_2$ along the parametric lines we proceed as we did for the case of the derivatives of \mathbf{n}
- For continuous with continuous second derivatives we get $\mathbf{r}_{12} = \mathbf{r}_{21}$ allowing us to write

$$\begin{aligned} (A_1 t_1)_{,2} &= (A_2 t_2)_{,1} \\ \mathbf{t}_{2,1} &= \frac{1}{A_2} [A_1 \mathbf{t}_{1,2} + A_{1,2} \mathbf{t}_1 - A_{2,1} \mathbf{t}_2] \end{aligned} \quad (19)$$

- As state we want to find $\mathbf{t}_{1,1}$ $\mathbf{t}_{1,2}$
- We observe that this derivative will be perpendicular to \mathbf{t}_1 and will lie in the plane formed by \mathbf{t}_2 and \mathbf{n} . We can therefore express it as

$$\mathbf{t}_{1,1} = c\mathbf{n} + d\mathbf{t}_2 \quad (20)$$

where we can find c and d by finding the dot product with \mathbf{n} and \mathbf{t}_2

- We then get

$$\mathbf{t}_{1,1} = -\frac{A_1}{R_1} \mathbf{n} - \frac{1}{A_2} A_{1,2} \mathbf{t}_2 \quad (21)$$

Other directions, you can find from Harry Kraus

- The first condition is

$$\frac{1}{R_1} A_{1,2} = \left(\frac{A_1}{R_1} \right)_{,2}, \quad \frac{1}{R_2} A_{2,2} = \left(\frac{A_2}{R_2} \right)_{,1} \quad (22)$$

- The other is

$$\left(\frac{1}{A_1} A_{2,1} \right)_{,1} + \left(\frac{1}{A_2} A_{1,2} \right)_{,2} = - \frac{A_1 A_2}{R_1 R_2} \quad (23)$$

- If E, G, L and N are given as functions of the real curvilinear coordinates ξ_1, ξ_2 and are sufficiently differentiable and satisfy Gauss codazzi equations while $E > 0$ and $G > 0$. Then there exist a real surface with 1 and 11 form as

$$I = E(d\xi_1)^2 + G(d\xi_2)^2 \quad II = L(d\xi_1)^2 + N(d\xi_2)^2 \quad (24)$$

- This is for the simplified cases where the lines of principle curvature are the parametric lines ($F=M=0$). Now more general Gauss Codazzi equations also exist.

- Suppose that there is a surface which is given as

$$\mathbf{r}(x_3, \theta) = R_o(x_3)\cos\theta\mathbf{e}_1 + R_o(x_3)\sin\theta\mathbf{e}_2 + x_3\mathbf{e}_3 \quad (25)$$

- Now in finding g we get, keeping x_3, θ as ξ_1 and θ as ξ_2 , we get:

$$\begin{aligned} \frac{d\mathbf{r}}{d\xi_1} &= \mathbf{r}_{,1} = R'_o(x_3)\cos\theta\mathbf{e}_1 + R'_o(x_3)\sin\theta\mathbf{e}_2 + \mathbf{e}_3 \\ \frac{d\mathbf{r}}{d\xi_2} &= \mathbf{r}_{,2} = -R_o(x_3)\sin\theta\mathbf{e}_1 + R_o(x_3)\cos\theta\mathbf{e}_2 + 0\mathbf{e}_3 \end{aligned} \quad (26)$$

- $E = \mathbf{r}_{,1} \cdot \mathbf{r}_{,1} = R_o'^2 \cos^2 \theta + R_o'^2 \sin^2 \theta + 1 = 1 + R_o'^2$
- $F = \mathbf{r}_{,1} \cdot \mathbf{r}_{,2} = -R_o^2 \cos \theta \sin \theta + R_o^2 \cos \theta \sin \theta = 0$
- $G = \mathbf{r}_{,2} \cdot \mathbf{r}_{,2} = R_o^2 \sin^2 \theta + R_o^2 \cos^2 \theta = R_o^2$
- $H = \sqrt{EG - F^2} = R_o \sqrt{1 + R_o'^2}$
- $A_1 = \sqrt{E} \quad A_2 = \sqrt{G}$
- First form : $(ds)^2 = \left(1 + R_o'^2\right) dx_3^2 + R_o^2 d\theta^2$
- Normal to surface $\mathbf{n} = \frac{\mathbf{r}_{,1} \times \mathbf{r}_{,2}}{H} = \frac{-R_o}{H} (\cos \theta \mathbf{e}_1 + \sin \theta \mathbf{e}_1 - R_o' \mathbf{e}_3)$
It acts as assumed from concave to convex side

$$\blacksquare L = -\mathbf{r}_{,11} \cdot \mathbf{n} = \frac{-R_o}{H} \begin{bmatrix} R_o'' \cos\theta & R_o'' \sin\theta & 0 \end{bmatrix} \cdot \begin{bmatrix} \cos\theta \\ \sin\theta \\ -R_o' \end{bmatrix} = -R_o R_o H'' / H$$

$$\blacksquare M = -\mathbf{r}_{,12} = 0$$

$$\blacksquare N = -\mathbf{r}_{,22} = R_o^2 / H$$

$$\blacksquare \text{Principal curvatures } R_1 = E/L \quad R_2 = G/N$$

■ We can check the gauss codazzi to test the surface

■ An alternate description is if we use ϕ, θ where the first is the angle between the axis revolution of the surface and a normal to the surface of the point. (See Kraus)