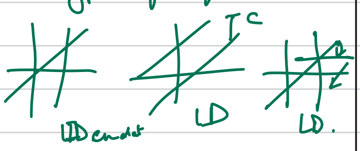
LINEAR ALGEBRA

- Prof. Gilbert Strang

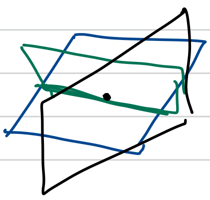
Types of System of equation 2va



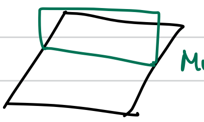
When we Linear regression

Points > Line.

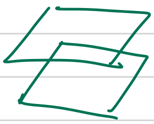
Types in 3 var.
x y z.



Unique.



Multiple.



L. Dependent

Lecture 1.

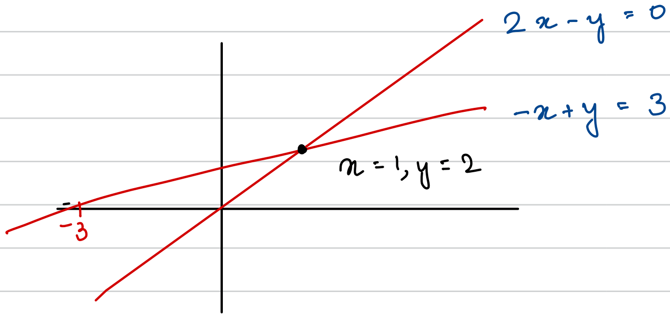
- Row picture - Column picture

- $2x - y = 0$
  $-x + 2y = 3$

Matrix $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 3 \end{Bmatrix}$

So $\underset{\approx}{A}\,\underset{\sim}{x} = \underset{\sim}{b}$

Row picture - Take one row at a time

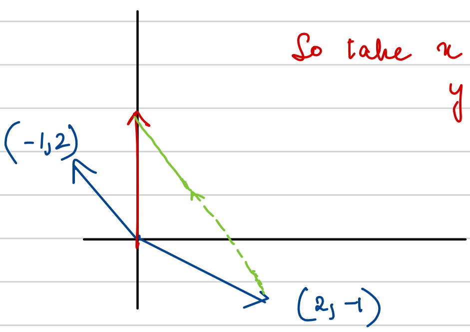


Column picture

$$x \begin{bmatrix} 2 \\ -1 \end{bmatrix} + y \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

Linear combination

So take $x = 1$
$y = 2$



What are all the combinations?

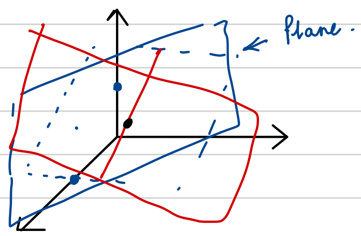
- It will fill the whole plane.

Eg: 3×3 example

$$\begin{aligned} 2x - y \quad &= 0 \\ -x - 2y - z &= -1 \\ -3y + 4z &= 4 \end{aligned}$$

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & -2 & -1 \\ 0 & -3 & +4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

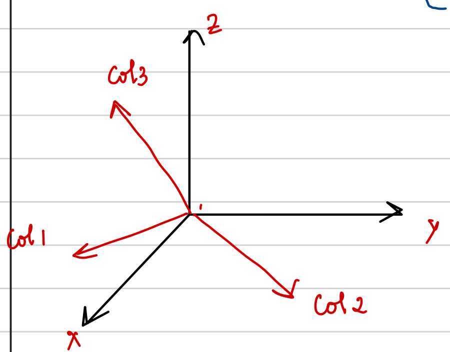
Row picture:



- Each row in a 3×3 gives a plane.
- Two rows meet in a line
- Three in a point

Column picture:

$$x\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + y\begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} + z\begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$



- Combine these three vectors with the right combination.

- $x, y = 0$ and $z = 1$.

The big picture, let us change RHS.

$$x\begin{bmatrix}2\\-1\\0\end{bmatrix} + y\begin{bmatrix}-1\\2\\-3\end{bmatrix} + z\begin{bmatrix}0\\-1\\4\end{bmatrix} = \begin{bmatrix}1\\1\\-3\end{bmatrix}$$

$x=1, y=1, z=0$

- Think about all RHS.
- Can I solve $Ax=b$ for every b?
- Or do the L.C of the column fill 3D space?

What could go wrong?
- When could I not get b
- If the columns are in same plane, then combinations are in the plane.
  - So matrix is singular

Imagine 9 dimensions.

- vector $= \begin{bmatrix}9\\ \vdots & \text{col} \\ \vdots\end{bmatrix}$

- So depends on the matrix.

- If the 9^{th} column is dependant on 8^{th}, then it would be a kind of plane

$$\boxed{Ax = b}$$

- How do you multiply a matrix by vector.

  - Column way (Ax is a combn of columns)

$$\begin{bmatrix}2 & 5\\1 & 3\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix} = 1\begin{bmatrix}2\\1\end{bmatrix} + 2\begin{bmatrix}5\\3\end{bmatrix} = \begin{bmatrix}12\\7\end{bmatrix}$$

- Dot product

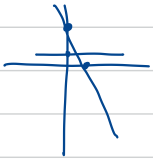
$$= \begin{bmatrix}2\cdot1 + 5\cdot2\\1\cdot1 + 3\cdot2\end{bmatrix}$$

Recitation 1

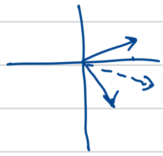
$$\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

Solve $\begin{cases} 2x + y = 3 \\ x - 2y = -1 \end{cases}$

Row picture



Col picture



$A = [v_1 \ v_2] = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$

$A^{-1}A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad \begin{bmatrix} x \\ y \end{bmatrix} = A^{-1}\begin{bmatrix} 3 \\ -1 \end{bmatrix}$

$5x = 5$

$\therefore x = 1$

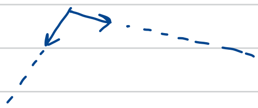
$y = 1$

Review of L.A.

- Vectors / Matrices / Subspaces.

Basic operations : Scale, Add, Subtract. $\quad x_1 u + x_2 v + x_3 w = b$

$$u = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad v = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$



If I took all their combinations?

→ We get a plane. Not fill 3D.

$$u = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad v = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \quad w = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} x_2 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} x_3$$

$$= \underbrace{\begin{bmatrix} x_1 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix}}_{b}$$

(Some sort of difference matrix)

Eg:

$$A\begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

(Difference of the squares are odd nos).

- Most of the time, I have input - output
- So if I have B what is x.

$$\begin{bmatrix} x_1 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Solution: Lower Triangular so easy.

$$x_1 = b_1$$
$$x_2 = b_2 + b_1$$
$$x_3 = b_3 + b_1 + b_2.$$

- Think of the solution, you have matrix X b.

$$x = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix}$$

- The matrix is the inverse matrix.
- $Ax = b$
  $x = A^{-1} b.$

- The inverse takes from b to x.

- Difference transform, it takes sum matrix.

$$b = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \quad x = \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix}$$

- $Ax = b$ has one solution
- A^{-1} perfect map.

- If all the $b = 0$, then $x = 0$

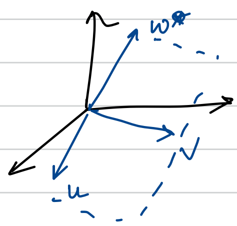
2nd example.

$$C = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_3 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix}$$

- there there is a solution x, which gives $b = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$x = c\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} \to b = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

- Differences of constant $= 0$
- If I add all LHS, $0 = b_1 + b_2 + b_3$. (So this is the condition for the solution)
- If RHS $= 0$, then we are not.
- $Cx = 0$
  - C^{-1} cant bring x.
- Trouble different ways.



- They are in the same plane.
- Dependant.

Basis → • Matrix invertible
• Independant.
• Span the whole space.

All comb of u, v, w* gives a plane

= all vectors Cx.

$= \begin{bmatrix} u & v & w^* \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

- What b, do we get.

- We get b on the plane that $b_1 + b_2 + b_3 = 0$

$u + v + w = 0$.

- A subspace is a vector space in the whole space

- Subspace in $\mathbb{R}^3$
  - Zero $\underset{\sim}{0}$
  - Line
  - Plane
  - Whole space

Rectangular A
- Not invertible.
- A^T comes in.

$A^TA \Rightarrow$ Square
$\Rightarrow$ Symmetric.

- They team up together $A^T A$.

Rec.

A is a matrix $Ax = \begin{bmatrix} 1 \\ 4 \\ 1 \\ 1 \end{bmatrix}$ $\quad x = \underbrace{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}_{xp} + c \underbrace{\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}}_{xs}$

What say about columns of A?

- b : 4 x 1 So A - 4 x 3
- x = 3 x 1

$$A \begin{bmatrix} c_1 & c_2 & c_3 \\ & & \\ & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 1 \\ 1 \end{bmatrix}$$

$$c_1 x_1 + c_2 x_2 + c_3 x_3 = \begin{bmatrix} 1 \\ 4 \\ 1 \\ 1 \end{bmatrix}$$

$x = \begin{pmatrix} 1 \\ \cdot \\ \cdot \end{pmatrix} + c \begin{pmatrix} 1 \\ \xi \\ \cdot \end{pmatrix}$

- $A(x_p + c x_s) = b$
- $c = 0$ $\quad \boxed{Ax_p = b}$ → Particular
- $c = 1$ $\quad \underbrace{Ax_p}_{b} + Ax_s = b.$

$\therefore \boxed{Ax_s = 0}$ → Homogeneous.

⇒ $c_1 0 + c_2 1 + c_3 = b$
$c_2 + c_3 = b$

⇒ $c_1 0 + 2c_2 + c_3 = 0.$
$2c_2 + c_3 = 0$

$c_3 = -2c_2$
$c_2 - 2c_2 = b$

$$\boxed{\begin{aligned} \therefore\ c_2 &= -b \\ c_3 &= 2b \end{aligned}}$$

b is col(A)
$c_2 = -b$
$c_3 = 2b.$

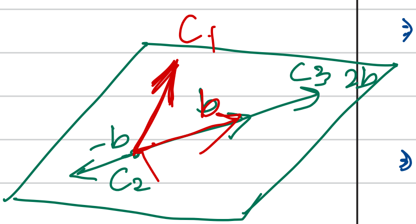
$x = x_p + c \underbrace{x_s}_{Ax=0}$ Nullspace A

dim (N(A)) = 1

rank (A) = 3 - 1 = 2

$\therefore c_1$ not a multiple of b.

(c_2 and c_3 are dependent)



Anything

$(c_1 \, c_2 \, c_3)(x) = (b)$

$b = x_1 c_1 + x_2 c_2 + c_2 c_3$
$b = 0 \, c_1 + \cdot \, c_2 + \cdot \, c_3$

$$\begin{pmatrix} c_1 & -2 & 4 \\ & -3 & 6 \\ & -4 & 8 \end{pmatrix} \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$$

Lecture 2 :

- Elimination < Succes / Failure
- Back substitution
- Elimination matrices.
- Multiplication

Eg : $Ax = b$

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix}$$
A

- Multiply 1st equation by the right no and subtract from the second.
- Purpose is to eliminate x in ation 2.

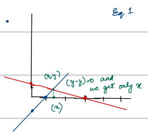
$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} \xrightarrow{(2,1)} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix} \xrightarrow{(3,2)} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix}$$

1st Pivot

A U (Upper Triangular)

- Pivots can't be 0
- Determinant is $\prod$ pivots
- Failure
  - 1st Pivot zero
  - Exchange zero.
  - If my last pivot would be failure.
- Matrix would not be invertible, nothing below to row exchang

What happens!



equ

$x - y = 1 \times 2$ — 1
$x + 2y = 2$ — 2

$2x - 2y = 2$ — 3
$x + 2y = 2$
$3x = 4$
$x = \frac{4}{3} = 1.33$

When we subtract 2 from 3

Use large row for pivot

$$\begin{pmatrix} 0.00001 & 32 \\ 32 & 6.1 \end{pmatrix} = \begin{pmatrix} b \end{pmatrix}$$

$$\begin{pmatrix} 32 & 6.1 \\ 0.00001 & 32 \end{pmatrix} = \begin{pmatrix} b' \end{pmatrix}$$

Augmented

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 3 & 8 & 1 & 12 \\ 0 & 4 & 1 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 4 & 1 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 0 & 5 & -10 \end{array}\right]$$

b $\qquad$ U $\quad$ c

And now $Ux = c$ (Back substitution)

$$\therefore z = -2, \; y = -1, \; x = 2$$

Now, I want to keep the elimination steps as matrices:

Matrices: $\begin{bmatrix} \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = xC_1 + yC_2 + zC_3$ (Matrix $\times$ col) = col

Now with row operation

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix} = xR_1 + yR_2 + zR_3$$ (Row $\times$ Matrix) = row

- Using Elimination matrices:

$$\begin{matrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{matrix} \xrightarrow{\text{Subtract } -3R_1 + R_2} \begin{matrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{matrix} \xrightarrow{\text{Subtract } -2R_2 + R_3} \begin{matrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{matrix}$$

$$E_{21} = \begin{matrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \qquad E_{32} = \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{matrix}$$

- If we have a Identity matrix:

$$IA = I$$

$$U = E_{32}(E_{21}A)$$

$$= (E_{32}E_{21})A$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = EA = U$$

ASSOCIATIVE LAW.

$$E_1(E_2A)$$
$$= (E_1E_2)A$$

PERMUTATION MATRIX:

Suppose we want to exchange row 1 and row 2

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

Row matrix

Permutation matrix

Exchage colums

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & a \\ d & c \end{bmatrix}$$

Exchange colums of identity.

Now :

$$(E_{32} E_{21}) A = U$$ (Commutative is FALSE)

Think of how I get from U to A?

Inverse : 1^{st} // Inv of Subtract $R_2 - 3R_1$ (So Add $3R_1 + R_2$)

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E^{-1}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{I}$$

LECTURE 3 :

1 //

$$\underbrace{\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}}_{B} = \begin{bmatrix} \quad & \overset{c_{ij}}{\bullet} & \quad \\ \quad & \quad & \quad \end{bmatrix}$$

$C = AB$

- c_{ij} comes from Row i of A and Row j of B.
- c_{34} = (row 3 of A) . (Col 4 of B)

$$= a_{31} b_{41} + a_{32} b_{42} + \dots\dots$$

$$= \sum_{k_1}^{n} a_{3k} b_{k4}$$

When do we allow to multiply matrices :

- $A^{m \times n}$ $B^{n \times p}$.
- Shape of $C^{m \times p}$

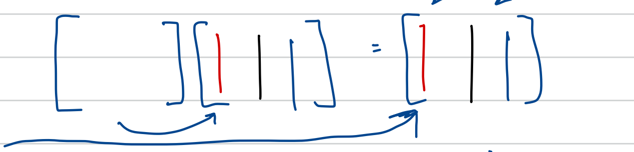
$$A\begin{pmatrix}x\\y\\z\end{pmatrix} = xC_1 + yC_2 + zC_3$$

2/ Other ways:

$$[A][B] = [C]$$

$m\times n \quad n\times p \qquad m\times p.$

A(col 1) A(col 2)



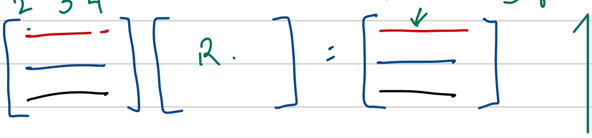
- The columns of C are combinations of columns of A.

eg
$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$
$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$
$$C = \left[\, b_{11}\begin{pmatrix}a_{11}\\a_{21}\end{pmatrix} + b_{21}\begin{pmatrix}a_{12}\\a_{22}\end{pmatrix},\ b_{12}\begin{pmatrix}a_{11}\\a_{21}\end{pmatrix} + b_{22}\begin{pmatrix}a_{12}\\a_{22}\end{pmatrix} \right.$$

3/

2 3 4

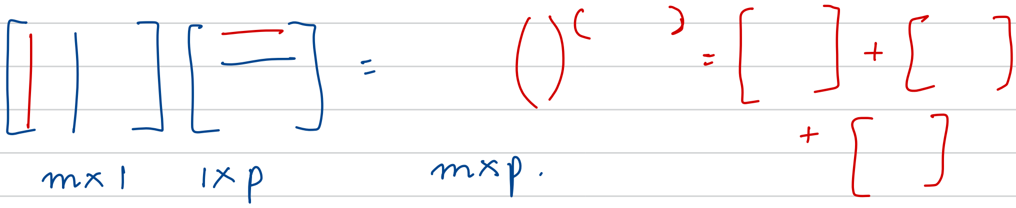
$2R_1 + 3R_2 + 4R_3$ of B.



$$C = \Big(a_{11}(b_{11}, b_{12}) + a_{12}(b_{21}, b_{22})$$

- Rows of C are combinations of rows of B.

4/



$m\times 1 \quad 1\times p \qquad m\times p.$

$$\begin{Bmatrix}2\\3\\4\end{Bmatrix}\{1 \;\; 6\} = \begin{bmatrix} 2 & 12 \\ 3 & 18 \\ 4 & 24 \end{bmatrix}$$

AB is a sum of (col A) × (rows b)

$$\begin{bmatrix} 2 & 7 \\ 3 & 8 \\ 4 & 9 \end{bmatrix}\begin{bmatrix} 1 & 6 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix}2\\3\\4\end{bmatrix}\begin{bmatrix}1 & 6\end{bmatrix} + \begin{bmatrix}7\\8\\9\end{bmatrix}\begin{bmatrix}0 & 0\end{bmatrix}$$

$3\times 2 \qquad 2\times 2$

$$\begin{bmatrix} 2 & 12 \\ 3 & 18 \\ 4 & 24 \end{bmatrix}$$

- Rows same direction $(1, 6)$ → Row space line
- Column " " $(2, 3, 4)$. → Col " line

5 Blocks

$A_1 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ ..

$$\underset{A}{\left[\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right]} \left[\begin{array}{c|c} B_1 & B_2 \\ \hline B_3 & B_4 \end{array}\right] = \left[\begin{array}{c|c} A_1B_1 + A_2B_3 & A_1B_2 + A_2B_4 \\ \hline A_3B_1 + A_4B_3 & \end{array}\right]$$

Inverse (Square matrices)
(Invertible / Non singular)

- $A^{-1}A = I = AA^{-1}$ (For square matrix)
  ↑ if A^{-1} exists.

Singular (No inverse)

- $A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$. Columns are dependant.

1 reason

I suppose we multiply $A^{-1}A = I$

- such that A^{-1} col 1 (A) + A^{-1} (col 2 (A)) = I
- But columns of A are same direction, so can't get $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

2 reason

- Square matrix no inverse, I can find vector x
  $Ax = 0$

$$1x + 3y = 0$$
$$2x + 6y = 0.$$
$$y = c$$
$$x = -3c$$

$$\bar{x} = \begin{bmatrix} -3c \\ c \end{bmatrix}$$

$Ax = 0$
only $x = \underset{\sim}{0}$

- If $Ax = 0$
  $\Rightarrow A^{-1}Ax = 0$
  $\Rightarrow x = 0$
  But x is not zero!

Singular matrix takes $x = 0$ and cant recover it

Column point in diff direction

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$A \qquad A^{-1}$

$A \times$ colum j of A^{-1} = colum j of I.

$$\begin{bmatrix} A \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

GAUSS JORDAN. (Solve 2 eqns at once) $\qquad AI = A.$

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

$A \quad I \qquad\qquad I \quad A^{-1}$

Why do we get A^{-1}

- $E(A \mid I) = [I \ ?]$

(Es) - Elimination matrix

$EA = I$ tells us $\therefore E = A^{-1}$

$\therefore EI = A^{-1}$

$E(A) \qquad E(I)$

I

What is

$E = A^{-1}$

Rec :

1. Solve using elimination

$$\begin{aligned} x - y - z + u &= 0 \\ 2x + 2z &= 8 \\ -y - 2z &= -8 \\ 3x - 3y - 2z + 4u &= 7 \end{aligned} \rightarrow \left[\begin{array}{cccc|c} 1 & -1 & -1 & 1 & 0 \\ 2 & 0 & 2 & 0 & 8 \\ 0 & -1 & -2 & 0 & -8 \\ 3 & -3 & -2 & 4 & 7 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & 4 & -2 & 8 \\ 0 & -1 & -2 & 0 & -8 \\ 0 & 0 & 1 & 1 & 7 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & 4 & 0 & 8 \\ 0 & 0 & 0 & -1 & -4 \\ 0 & 0 & 1 & 1 & 7 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & 4 & 0 & 8 \\ 0 & 0 & 1 & 1 & 7 \\ 0 & 0 & 0 & -1 & -4 \end{array}\right]$$

$u = 4$, $z = 3$, $y = 2$, $x = 1$

2. What are conditions a and b that would make A invertible $A = \begin{bmatrix} a & b & b \\ a & a & b \\ a & a & a \end{bmatrix}$

- $a = 0$
- $a = b$
-

$$\left[\begin{array}{ccc|ccc} a & b & b & 1 & 0 & 0 \\ a & a & b & 0 & 1 & 0 \\ a & a & a & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & b/a & b/a & 1/a & 0 & 0 \\ 0 & 1 & 0 & \frac{-1}{a-b} & \frac{1}{a-b} & 0 \\ 0 & 0 & 1 & 0 & \frac{-1}{a-b} & \frac{1}{a-b} \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{a-b} & 0 & \frac{-b}{a(a-b)} \\ 0 & 1 & 0 & \frac{-1}{a-b} & \frac{1}{a-b} & 0 \\ 0 & 0 & 1 & 0 & -\frac{1}{a-b} & \frac{1}{a-b} \end{array}\right]$$

$= \frac{1}{a-b}$

$a - b \neq 0$
$a \neq 0$

Lecture 4: Elimination

- $A = LU$ (Decomposition)

$AA^{-1} = I = A^{-1}A$

$AA^{-1} = I$
$AA^{-1} = I$

- $(A\underbrace{BB^{-1}}_{I}A^{-1}) = I$

- $B^{-1}A^{-1}AB = I$

Supp : from $AA^{-1} = I$
$(A^{-1})^T A^T = I$

$\Rightarrow A^T (A^T)^{-1} = I$
$\Rightarrow AA^{-1} = I \quad (A^{-1})^T A^T =$
$\therefore (A^T)^I = (A^I)^T$.

If I transpose of a matrix, what is its inverse

$(A^T)^{-1} = (A^{-1})^T$

- $A = LU$ (Simplest factorization)

$A = U$

$$\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix} \xrightarrow{\underset{E_{21}}{\begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}}} \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$$

- $L = E^{-1}$

$E^{-1} = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$

- L is lower triangle, ones in terminal

$$A = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix}$$

LD

if A is sym Then $A = LDL^T$

Suppose 3×3 → We need transformations

Suppose.

$\underbrace{E_{32}E_{31}E_{21}}\ A = U$ (No row exchange)

- $EA = U$
- $A = E^{-1}U$

$= \underbrace{E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}}_{\text{Simpler.}}U$

$= LU$

Eg

$$E = \overset{E_{31}}{\begin{bmatrix}1&0&0\\0&1&0\\0&-5&1\end{bmatrix}} \overset{E_{21}}{\begin{bmatrix}1&0&0\\-2&1&0\\0&0&1\end{bmatrix}}$$

$E^{-1} = E_{21}^{-1}E_{31}^{-1}$

Inverse :

$E^{-1} = E_{21}^{-1}\ E_{32}^{-1}$

$$= \begin{bmatrix}1&0&0\\2&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&5&1\end{bmatrix} = \begin{bmatrix}1&0&0\\2&1&0\\0&5&1\end{bmatrix} = L$$

+ve +ve

$A = LU$

The multiplier just stay in L.

Don't mix up in E.

- If there are no row exchanges, multiplier go directly to L.
- So you can forget A
- How expensive, operation on $N\times N$ matrix :

Suppose $n = 100$ (No zeros) $n, n^2, n^3, n!$

(multiply + subtract)

- $n^2 + (n-1)^2 + \ldots\ldots 1^2$
- n^3.

About $\frac{1}{3}n^3$ like $\int n^2$ | Cost of b: RHS ? : n^2

TRANSPOSES + PERMUTATIONS

3x3 List. $6P's$.

Exchange rows

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \quad P_{12} = \begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix} \quad P_{13} = \begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix} \quad P_{23} = \begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}$$

All Rows = $\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}$ $\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}$

If I multiply we get row exchange
Inverse is again the same thing.

$P^{-1} = P^T$ Orthogonality.

4X4 : Guess : 24.

Eg $A = \begin{pmatrix}1&0&1\\a&a&a\\b&b&a\end{pmatrix}$ $\quad \begin{matrix}1&0&1\\0&a&0\\0&b&a-b\end{matrix} \rightarrow$

$E_{21} = \begin{bmatrix}1&0&0\\-a&1&0\\0&0&1\end{bmatrix}$

$$U = \begin{pmatrix}1&0&1\\0&a&0\\0&0&a-b\end{pmatrix}$$

$$L = E_{21}^{-1} E_{31}^{-1} E_{32}^{-1}$$

$$= \begin{pmatrix}1&&\\a&1&\\0&0&1\end{pmatrix}\begin{pmatrix}1&&\\0&1&\\b&0&1\end{pmatrix}\begin{pmatrix}1&&\\0&1&\\0&b/a&1\end{pmatrix}$$

$$= \begin{pmatrix}1&&\\a&1&\\b&b/a&1\end{pmatrix}$$

Exists if $a \neq 0$
$a-b$ can be zero

Lecture 5: Transpose, Permutations & Subspaces

Permutation P: Execute row exchange for pivot $\neq 0$

- So what happens to LU

$$A = LU = \begin{bmatrix} 1 & & 0 \\ - & 1 & \\ - & & 1 \end{bmatrix} \begin{bmatrix} \ddots & - & - \\ & \ddots & - \\ 0 & & \ddots \end{bmatrix}$$

- Matlab will also check if pivot $> \epsilon$.

$A = LU$ becomes

$\boxed{: PA = LU}$ // For any invertible A.

Permutations P

P = Identity matrix with reordered rows.

- $n!$ possibilities.
- Counts reorderings
  $n \times n$ permutations

$\boxed{P^{-1} = P^T}$ $\quad P^T P = I$

Eg:

$$\begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 3 & 1 \end{bmatrix}$$

Transpose

$$(A^T)_{ij} = A_{ji}$$

$A.B = a_{kj} b_{jl}$.
$C_{kl} = a_{kj} b_{jl}$.
$A^T B = a_{jk} b_{jl}$.
$A^T A = a_{jk} a_{jl}$
$C_{12} = a_{j1} a_{j2}$
$C_{21} = a_{j2} a_{j1}$

Symmetric matrices.

$A^T = A$. $\quad$ Eg $\begin{bmatrix} 3 & 1 & 7 \\ 1 & 2 & 9 \\ 7 & 9 & 4 \end{bmatrix}$

$$\underbrace{\begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}}_{A^T} \overset{\text{same}}{\vphantom{x}}^{T} \underbrace{\begin{bmatrix} 1 & 2 & 4 \\ 3 & 3 & 1 \end{bmatrix}}_{A} = \text{Symmetric matrices.}$$

$R^T R$ is always symmetric

$$= \begin{bmatrix} 10 & 11 & 7 \\ 11 & - & - \\ 7 & - & - \end{bmatrix}$$

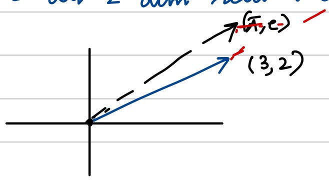
$((R^T)R)^T = R^T R$ (Got it the same).

Vector spaces & Subspace.

- What do we do with vectors. We add, scale. and some other rules

Eg: Space means a bunch of vectors. Make linear combinations

$\mathbb{R}^2$ = all 2 dim real vectors $\begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \pi \\ e \end{bmatrix} \ldots\ldots$



- Suppose I remove $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$. I need to multiply $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and get $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

or $\begin{bmatrix} 3 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \end{bmatrix} \rightarrow \underset{\sim}{0}$.

- Add, x $\Rightarrow$ still in $\mathbb{R}^2$

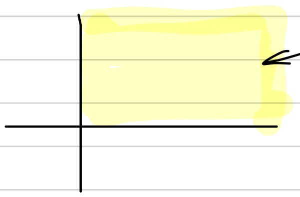
eg: $\mathbb{R}^3$ - vectors with 3 real components.

$\begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$ • Still in $\mathbb{R}^2$

$\mathbb{R}^n$ - all vectors with n components. Real vectors.

- We need to satisfy the rules. 8 rules

Eg: Not a vector space

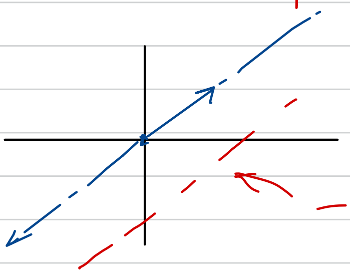


1/4 type. space

$\left(\frac{3}{2}\right) \times -5 \Rightarrow$ Not a vector space.

- It should be closed under addition + multip

Smaller vector space in $\mathbb{R}^2$, (Not that 1/4 space)?



- So I have a whole line of vectors.
- Line in $\mathbb{R}^2$ (Must go through in zero vector)
- Not subspace, doesn't have $\underset{\sim}{0}$

- Subspaces of $\mathbb{R}^2$

1) all of $\mathbb{R}^2$
2) line through $\underset{\sim}{0}$ (Not same as $\mathbb{R}^1$ ← this has only component)
3) $\underset{\sim}{0}$ vector alone only.

- Subspaces of $\mathbb{R}^3$
1) all of $\mathbb{R}^3$
2) Plane through origin
3) $\underset{\sim}{0}$ vector alone

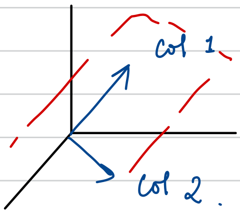
How do subspaces come from matrices.

$A = \begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}$ Columns in $\mathbb{R}^3$.

- I have to take all Linear combinations of column formed subspace

$x c_1 + y c_2$ = column space. (Zero should also be there)

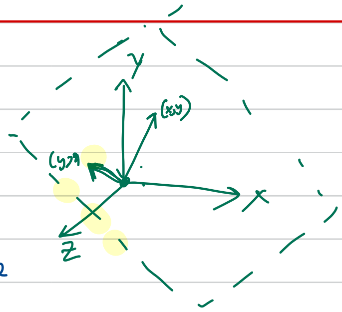
- We want a space of their combinations.



$\Rightarrow$ Combination of Col 1 + Col 2 $\Rightarrow$ Plane

$\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$ $\begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix}$

Rec : Eg : $x_1 = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$ $x_2 = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}$



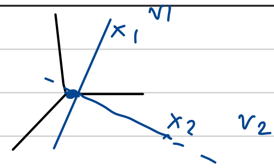
- Find v_1 = subspace generated by x_1
  v_2 = " " " x_2
  Describe $v_1 \cap v_2$

- Find v_3 = subspace of $\{x_1, x_2\}$ = $\alpha x_1 + \beta x_2$.
  $v_3 = v_1 \cup v_2$?
  Find subspace S of v_3 s.t $x_1 \in S$ $x_2 \in S$

- What is $v_3 \cap \{xy \text{ plane}\}$?

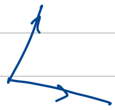
Condition : $\Rightarrow$ Sum is closed.
$\Rightarrow$ Mul is closed

1. $V_1 = \alpha x_1$ $\quad V_2 = \beta x_2$



$V_1 \cap V_2 = \underset{\sim}{0}$

2. $V_3 = \alpha x_1 + \beta x_2$

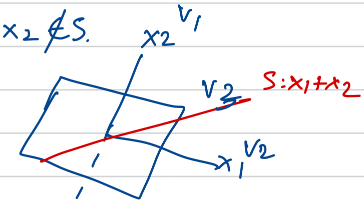


. Union on this line or not.

$V_3 = V_1 \cup V_2$ (No)

S of V_3 such $x_1 \notin S$, $x_2 \notin S$.

S $\underset{\sim\sim}{x_1 + x_2}$.



3. $V_3 \cap \{xy \text{ plane}\}$

$= V_3$

Lecture 6 :

- Vector space
- Column space
- Null space

Vector space requirement :

(W, V)

- Add vector and stay in space
- Combinations $cV + dW$ are in space.

Eg : vector space : $\mathbb{R}^3$

subspace.

Subspace : Eg of $\mathbb{R}^3$

- Vector space inside vector space.
- $\{\underset{\sim}{0}\}$ has to be there
- Plane with $\{0,0,0\}$
- Line with $\{0,0,0\}$

Suppose 2 subspaces : P and L

$P \cup L$ = all vectors in P or L or both.

- Is it a subspace? Not. Because if I add, I will be outside the union

Suppose $P \cap L$, is it is a subspace? It is a subspace.

b and w in P and L.
b+w also in P and L (subspace).

Column space:

$$\begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{pmatrix}$$

- column space of A subspace of $\mathbb{R}^4$
- $C(A)$

- Check for subspace.
- $C(A)$ all Lineal combination of columns.
- Does it fill the full space? No. Only 3 C
- So how many dim.
- Does $Ax = b$ have sol for every b? NO.
- Which RHS b is A okay.

4 equation, 3 unknowns.

$$Ax = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

- So b has to be in some lower dim.

- Which RHS allow me to solve it.
(b)

- One RHS, $b = \{\underset{\sim}{0}\}$

$b = \{1\ 2\ 3\ 4\}^T \qquad x = \{1\ 0\ 0\}$

$b = \{1\ 1\ 1\ 1\}^T \qquad x = \{0\ 1\ 0\}$

$b = \{2\ 3\ 4\ 5\}^T \qquad x = \{0\ 0\ 1\}$

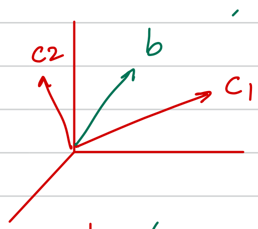
- Think of any $x := x_1, x_2, x_3$
- I can solve $Ax = b$ exactly when b is in the column space.
- If I take the comb of col do I get the columnspace. $C(3) = C(1) + (2)$

So 1, 2 is a pivot column (Depends which one is pivot)

So your column space is a 2D of $\mathbb{R}^4$.

NULL SPACE: $N(A)$

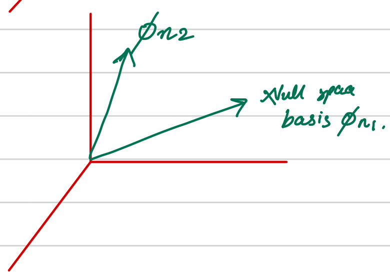
All solutions x, who $A(x) = \underset{\sim}{0}$



$$A \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ Null space is subspace of $\mathbb{R}^3$

Colun space is " " $\mathbb{R}^4$



$$A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

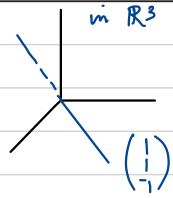
Has $x = \{\underset{\sim}{0}\}$.

$x = (1\ 1\ -1)$

$x = (c\ c\ -c)^T$

Any multiple $c \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

• So my Null space is a line in $\mathbb{R}^3$



• Why is a Null space vector space?

Check $Ax = 0$ always give a subspace.

If $Av = 0$, $Aw = 0$ $\quad A(v+w) = 0$

$Av + Aw = 0$ Distributive Law

$A(\alpha v) = 0$.

$$Ax = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \quad \text{So } x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{ or } x = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

Do the solⁿ form a vector space. The zero vector does not solve.

Rec: Which are subspaces of $\mathbb{R}^3 = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

1) $b_1 + b_2 - b_3 = 0$ ✓ Null space

2) $b_1 b_2 - b_3 = 0$ ✗ $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ ✓ $\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$ ✗

3) $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ ✓

4) $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ ✗

Lecture 7 :

- Computing null space
- Pivot variables
- Special soln.

So finding Algorithm to find null space

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}$$

$x + y = 3$
$2x + 6y = 2.$
$x - y$ ⇐ Disturbs.
x column space

When I do elimination, I am not changing the soln or the null space

Column space is

$$\begin{pmatrix} 1 & 2 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + z\begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$$

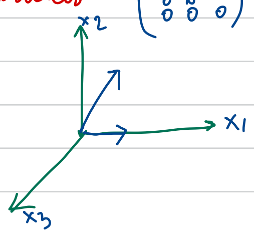
No of pivots

$$A = \begin{pmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} = U \quad \text{(Echelon form, Staircase form)}$$

No of free variables.

$$\begin{pmatrix} 1 & 2 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

- Rank of a matrix = No of pivots = 2.
- Seperate pivot columns, others free.
- Free columns, I can decide anything I like



$$x = c\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \begin{matrix} x_1 + 2x_2 + 2x_3 + 2x_4 = 0 \\ 2x_3 + 4x_4 = 0. \end{matrix}$$

Free

Is it full null space? No.

$$x = d\begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} \quad \begin{matrix} 1x_1 + 2x_3 + 2x_4 = 0 \\ 2x_3 + 4x_4 = 0 \end{matrix}$$

$$X = c\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + d\begin{pmatrix} 2 \\ 0 \\ -2 \\ 1 \end{pmatrix}$$

$$Ux = 0$$

If we take $x_2 = 1$, $x_4 = 1$ we get.

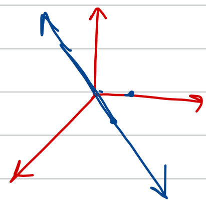
$$\begin{pmatrix} 0 \\ 1 \\ -2 \\ 1 \end{pmatrix}$$

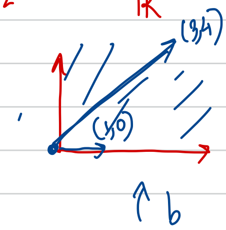
- Null space contains all combinations of the special soln
- How many special soln = No of free variables
  $= n - r = 4 - 2 = 2$

①

$$\begin{pmatrix} 1 & 3 & 2 \\ 2 & 2 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 2 \\ 0 & 4 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$\mathbb{R}^3 \to \mathbb{R}^2 \qquad \mathbb{R}^3 \to \mathbb{R}^2.$





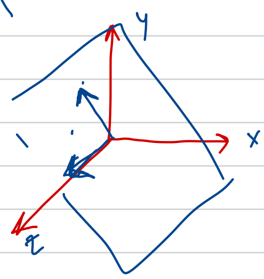
Null space

$$x = c\begin{Bmatrix} 2 \\ 0 \\ -1 \end{Bmatrix}$$

②

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix} \qquad N = c\begin{pmatrix} -1/2 \\ 1 \\ 0 \end{pmatrix} + d\begin{pmatrix} -1/3 \\ 0 \\ 1 \end{pmatrix}$$

$\mathbb{R}^2$

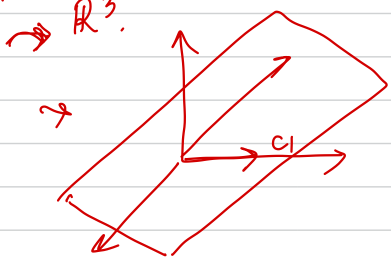


③

$$\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 2 & 6 \end{pmatrix} \to \begin{pmatrix} 1 & 4 \\ 0 & 3 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

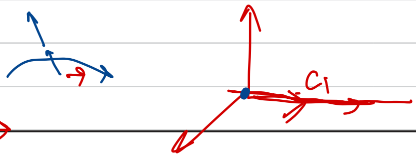
span full col space.

$\mathbb{R}^2 \qquad \mathbb{R}^3.$



④

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 2 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad N = c\begin{pmatrix} -1/2 \\ 1 \end{pmatrix}$$



$r = 2$ (Only 2 independant solutions).
Rank. - No of pivots.

Clean up the matrix more = Reduced row echelon

- I got u, $\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ ← This row was a L.C of top 2

So find zeros above and below the pivot

$$= \begin{pmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad R_1 \to R_1 - R_2$$

$$R. = \begin{pmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad R_2 \to R_2/2$$

$$\begin{pmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \underline{x_1} \\ x_2 \\ \underline{x_3} \\ x_4 \end{pmatrix} \begin{matrix} \\ \to \text{Free} \\ \\ \to \text{Free} \end{matrix}$$

$$= \begin{pmatrix} 1 & 0 & 2 & -2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_3 \\ x_2 \\ x_4 \end{pmatrix}$$

Reduced echelon (rref).

Info : Pivot rows, columns.
Free colums
Notice

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ I in pivot rows & colums.

- So the special soln I can read off.

$Ax = 0 \Rightarrow Rx = 0 \quad Ux = 0.$

So

$$\underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{\text{Pivot columns}} \qquad \underbrace{\begin{matrix} 2 & -2 \\ 0 & 2 \end{matrix}}_{\text{Free coluns.}}$$

$$\begin{array}{cccc} I & & F & \\ \end{array}$$
$$\begin{pmatrix} 1 & 0 & 2 & -2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_3 \\ x_2 \\ x_4 \end{pmatrix}$$
I

$N \rightarrow$
$$\begin{pmatrix} x_1 \\ x_3 \\ x_2 \\ x_4 \end{pmatrix} \rightarrow \begin{pmatrix} -2 \\ -1 \\ 1 \\ 0 \end{pmatrix}$$

$x_1 + 2x_2^{\to 1} - 2x_4^{\to 0}$ $\quad x_1 = -2$
$x_3 + x_2^{\to 1} + 2x_4^{\to 0}$ $\quad x_2 = -1$

$$\begin{pmatrix} x \\ \\ \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 0 \\ 1 \end{pmatrix}$$

$x_1 + 0 - 2x_4^{\to 1} = 2.$
$x_3 + 0 + 2x_4 = -2$
$x_4 \to 1$

$$\text{rref}(A) = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} \text{ - } r \text{ pivot rows.}$$
r pivot colums, $\quad n - r$ free columns.

• Null space matrix (Column = special solution)

$$N = \begin{bmatrix} -F \\ I \end{bmatrix} \leftarrow \begin{bmatrix} -F \\ I \end{bmatrix} \quad (\text{Matlab})$$

• $Rx = 0$

$$\begin{bmatrix} I & F \end{bmatrix}\begin{bmatrix} x_{pv} \\ x_{free} \end{bmatrix} = 0$$

$x_{pv} = -F x_{free}.$

Eg: $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 6 & 8 \\ 2 & 8 & 10 \end{bmatrix}$, Expect 2 columns, or rank 2 or 2 pivots

$$= \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 4 & 4 \end{pmatrix} - \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 4 & 4 \end{pmatrix} - \underset{U}{\begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}} - \underset{R}{\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}$$

• 1 special solution. (free column)
• $3 - 2 = 1$ free column.

$$x_1 + 2x_2 + 3x_3 = 0$$
$$2x_2 + 2x_3 = 0$$

Free/
$x_3 = 1, \quad x_2 = -1, \quad x_1 = -1.$

$-1c_1 - 1c_2 + 1c_3 = \underset{\sim}{0}$

$$\boxed{x = c\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}}$$

Suppose rref - $\begin{array}{c} \quad I \quad\quad F \\ \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ 0 \quad 0 & 0 \\ 0 \quad 0 & 0 \end{pmatrix} \end{array} = R.$

Plane

$x = \begin{bmatrix} x_p \\ x_f \end{bmatrix} = c \begin{bmatrix} -F \\ I \end{bmatrix}$

$x - y + z = 0.$

1 $z=0, x=y$
2 $z=1, y=1+x.$
3 $z=2, y=2+x$

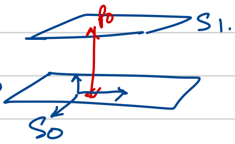
Rec: The set S of points $P(x,y,z)$ st $x - 5y + 2z = 9$ is a plane in $\mathbb{R}^3$. It is parallel. to the plane

- So of $P(x,y,z)$ st $x - 5y + 2z = 0$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underbrace{\begin{pmatrix} ? \\ 0 \\ 0 \end{pmatrix}}_{p_0} + \underbrace{c_1 \begin{pmatrix} ? \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} ? \\ 0 \\ 1 \end{pmatrix}}_{S_0}$

- Suppose $c_1, c_2 = 0$.

$\therefore \quad y = 0, z = 0 \quad x = 9.$



$= \underbrace{\begin{bmatrix} 9 \\ 0 \\ 0 \end{bmatrix}}_{p_0} + \underbrace{c_1 \begin{pmatrix} \quad \end{pmatrix} + c_2 \begin{pmatrix} \quad \end{pmatrix}}_{S_0}$

$S_0 = \begin{bmatrix} 1 & -5 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$

$\uparrow$ Pivot $\quad \uparrow$ Free.

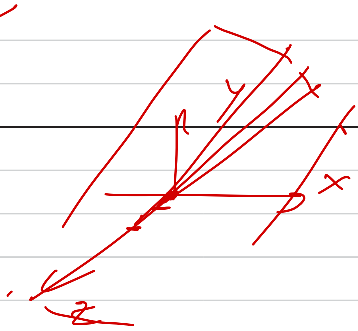
* $y = 1, z = 0.$
$x = 5.$

* $y = 0, z = 1$
$x = -2.$

So $\begin{bmatrix} 9 \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

$\uparrow$ Not V subspace no $\underset{\sim}{0}$.

i) $\underbrace{x - 5y + 2z = 0}$



$z = 0.$

$x = 5y$

$z = 1$

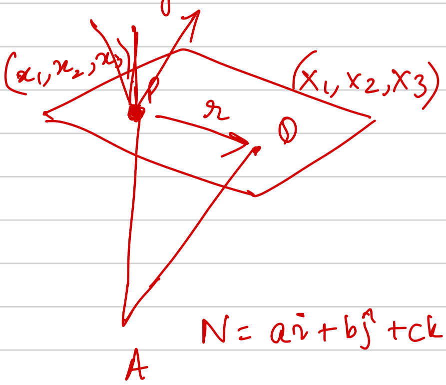
$x = 5y - 2$

$z = z.$

$x = 5y - 2z \cdot N$

$X_p + X_h$

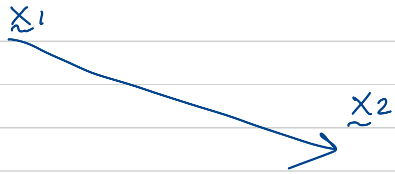
Row space.



$EQP = QA - PA.$

$(X_1, X_2, X_3) - (x_1, x_2, x_3)$

$N = a\hat{i} + b\hat{j} + c\hat{k}.$



$\underset{\sim}{v} = \underset{\sim}{x}_2 - \underset{\sim}{x}_1$

Line $= x_1 + (\underset{\sim}{x}_2 - \underset{\sim}{x}_1)t$.

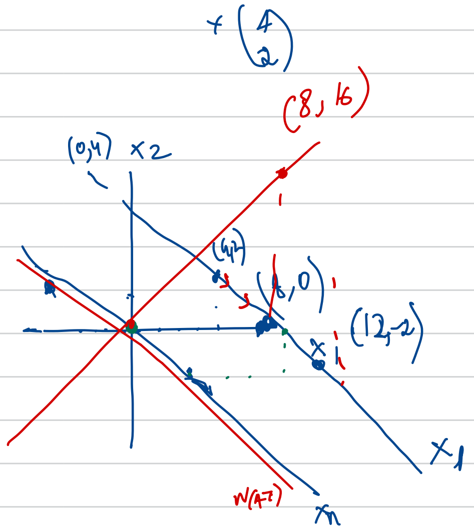
$$x = x_1 + (x_2 - x_1)t$$
$$y = y_1 + (y_2 - y_1)t$$
$$z = z_1 + (z_2 - z_1)t.$$

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$ Equation.

$c\begin{pmatrix}-2 & 1\end{pmatrix}\begin{pmatrix}1 & 2\\ 2 & 4\end{pmatrix}$

1) $\begin{pmatrix}1 & 2\\ 2 & 4\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} = \begin{pmatrix}8\\ 16\end{pmatrix}$

$\rightarrow \begin{pmatrix}1 & 2\\ 0 & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} = \begin{pmatrix}8\\ 0\end{pmatrix}$

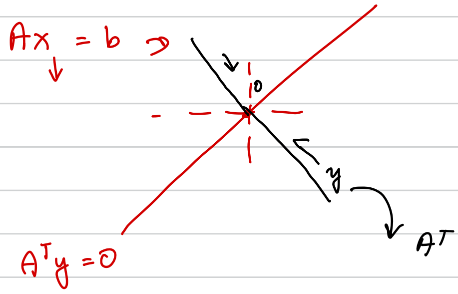


$x = \begin{pmatrix}8\\ 0\end{pmatrix} + c_1\begin{pmatrix}-2\\ 1\end{pmatrix}$

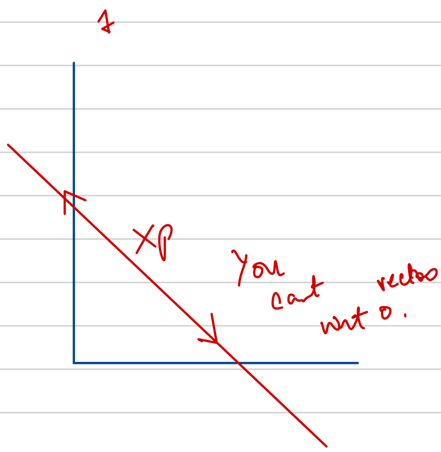
$x_p = \begin{pmatrix}4\\ 2\end{pmatrix},\ \begin{pmatrix}12\\ -2\end{pmatrix}.$

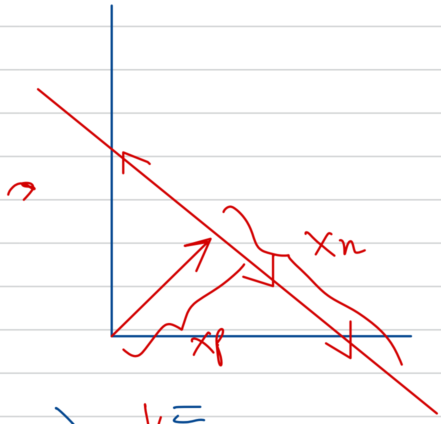
$\therefore\ x_1 + 2x_2 = 0.$

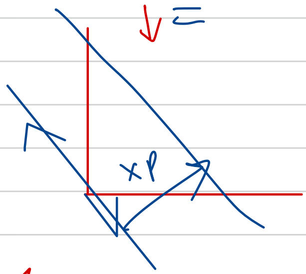
$\underset{\sim}{\begin{pmatrix}4\\ 2\end{pmatrix}} + c_1\underset{\sim}{\begin{pmatrix}-2\\ 1\end{pmatrix}}$



2)







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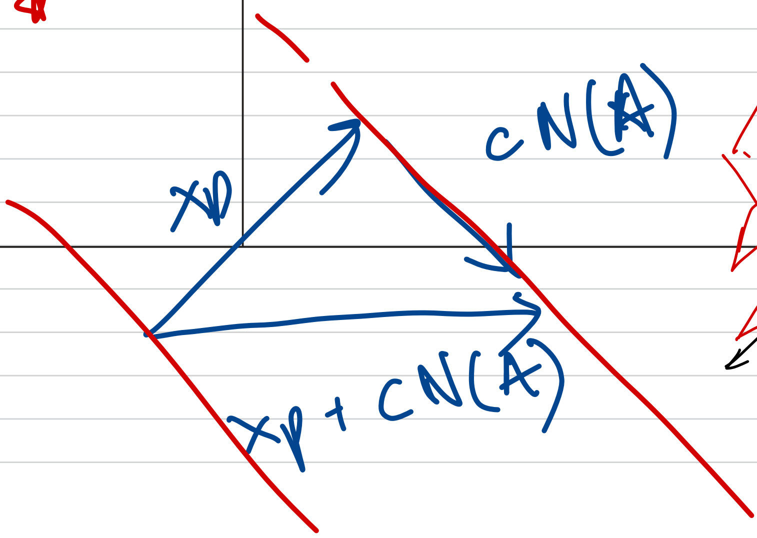

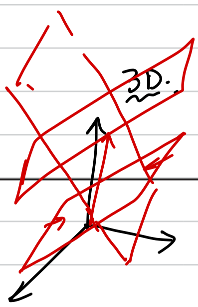

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Lecture 8 : 9:15

Solving $Ax = b$, if there is a solution.

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

- We can see $R_1 + R_2 = 3$.

- What condition on b_1, b_2, b_3 that gives solution

$$\left(\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2 - 2b_1 \\ 0 & 0 & 2 & 4 & b_3 - 3b_1 \end{array}\right)$$ Augmented matrix

$$= \left(\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 - 3b_1 - (b_2) \end{array}\right)$$

Pivot.

$$= b_3 - 3b_1 - b_2 + 2b_1$$

$$0 = b_3 - b_2 - b_1$$

$$\boxed{0 = b_3 - b_2 - b_1 \quad (\text{Constraint})}$$

Eg : $b = \begin{pmatrix} 1 \\ 5 \\ 6 \end{pmatrix}$

$\therefore$ After elimination $= \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} = 0.$

Solvability : Condition on b.

$Ax = b$ solvable. when b is the column space of A.

OR if comb of rows of A gives zero row, then the same comb of entry of $B = 0$

- How are they equivalent.

To find the complete solution to $Ax = b$

last equation $= 0$, 2 eqns, 4 unknowns.

① X_p : • Particular solution : Set all free variables to zero
• solve $Ax = b$ for free variables.

- Free : x_2, x_4
Pivot : x_1, x_3

• $\begin{pmatrix} 1 & 2 \\ & 2 \end{pmatrix} \begin{matrix} x_1 = \\ x_3 \end{matrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix}$: $x_3 = 3/2$, $x_1 = -2$

$x_p = \begin{pmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{pmatrix}$

$12 - 9 - 3$
$11 - 8 - 2$
$10 - 7 - 1$

$A x_p$

$\begin{pmatrix} 1 & \cdot & 2 & \cdot \\ 2 & \cdot & 6 & \cdot \\ 3 & \cdot & 8 & \cdot \end{pmatrix} \begin{pmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ -4 \\ -6 \end{pmatrix} + \begin{pmatrix} 3 \\ 9 \\ 12 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 6 \end{pmatrix}$ ✓

② • X null space.

① + ②

$X = x_p + x_n$

$Ax_p = b$
$Ax_n = 0$

$A(x_p + x_n) = b$

• If I had one solution, I can add anything in the null space because $Ax_n = 0$

$$X \text{ complete} = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \\ -2 \\ 0 \end{bmatrix}$$

$A = \begin{pmatrix} 1 & 2 \\ 2 & 6 \\ 3 & 8 \end{pmatrix}$

WHY THE CONNECTION!

$\begin{pmatrix} 1 & 2 \\ & 2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 - 2b_1 \end{pmatrix}$

$x_2 = (b_2 - 2b_1)/2$
$x_1 = b_1 - b_2 + 2b_1$
$= 3b_1 - b_2$.

$\begin{pmatrix} 1 & 2 \\ 2 & 6 \\ 3 & 8 \end{pmatrix}$

$(3b_1 - b_2)\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + (b_2 - 2b_1)\begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$

$= 3b_1 - b_2 + b_2 - 2b_1 =$
$= 6b_1 - 2b_2 + 3b_2 - 6b_1$
$= 9b_1 - 3b_2 + 4b_2 - 8b_1$

$= \begin{pmatrix} b_1 \\ b_2 \\ b_1 + b_2 \end{pmatrix}$

$b_3 - b_2 - b_1 = 0$

• X_p solves $Ax_p = b$. So no scalar.

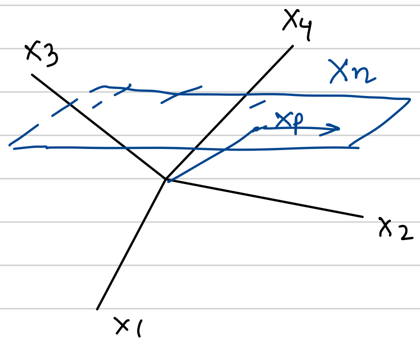
Plot all solutions.

$x = x_p + x_n$ a subspace?

Is $Ax = b$ a subspace

- No way.

x in $\mathbb{R}^4$



$\underbrace{c_1(x) + c_2(x)}_{x_p}$

• It's like a shifted subspace (Does not have zero $\underset{\sim}{0}$)

---

m by n matrix A of rank r.

$(r \le m)$
$(r \le n)$ – If Full Rank.

$A = \left(\quad \right)_{m \times n}$

Rows – m

Columns – n ←

$m > n$	$n > m$	$m = n$
$r < n < m$	$r < m$	$r < m$ and n.
$r = n$	$r = m$	

FULL COLUMN RANK $(r = n)$

- No free variables
- All pivot columns.

- $N(A) = \underset{\sim}{0}$

◦ Solution $Ax = b$

- x is only x_p. - Unique soln if it exists (Either it exists or not)

Eg: Full column rank matrix

$$\begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 6 & 1 \\ 5 & 1 \end{bmatrix}$$ Pivot.

$\rightarrow \begin{pmatrix} 1 & 3 \\ 0 & 5 \\ 0 & 17 \\ 0 & 14 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ (Full rank but there not be any solution)

(No comb of column give $\underset{\sim}{0}$)

- Is there a soln to $Ax = b$. ?

→ No, 4 soln, 2 variables.

→ RHS that has soln.

One type that has soln. := $\begin{pmatrix} 4 \\ 3 \\ 7 \\ 6 \end{pmatrix}$ $\therefore$ $x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\text{RHS} = x_1 C_1 + x_2 C_2$$
$$= C_1 + C_2$$

Full row rank ($r = m$)

- r = Pivots $= m$
- Every row has a pivot
- I can solve $Ax = b$ for whic RHS ?
- Can solve $Ax = b$ for every b.

Left with $\underbrace{n-r}_{= n-m}$ free variables

My note

$$\begin{pmatrix} 1 & 0 & a & c \\ 0 & 1 & b & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_p + c(N)$$

Null space.

Already span

if $F = 0 \Rightarrow 1$ soln

$= \alpha$ soln

$r = m < n$

$$A = \begin{pmatrix} 1 & 2 & 6 & 5 \\ 3 & 1 & 1 & 1 \end{pmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_p \end{bmatrix} =$$

Ensures whole 2 dimension is spaned by $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ so yes.

Rank = 2.

$$\text{Rref} = \left(\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \begin{pmatrix} - & - \\ - & - \end{pmatrix} \right) F$$

$r = m = n$. [Full Rank]

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$$

It's invertible !!.

rref $R = I$.

$N(A) = \underset{\sim}{0}$

What are the conditions on b.

- No cond, unique

$L: m > n$

$m \longrightarrow n$

$\therefore \longrightarrow 0$,

$(I\ 0)\begin{pmatrix} x_p \\ x_f \end{pmatrix}$ $x_f = c.$ $x_p I + c\,0 = b$

$r = m = n$	Column	Row
	$r = n < m$	$r = m < n$
$R = I$	$R = \begin{bmatrix} I \\ 0 \end{bmatrix}$	$R = [\, I \; F \,]$
1 solution $Ax = b$	0 or 1 solution (Either in col^n space or not)	(∞ solution) (Depends on F).

$r < m < n$ (Normal case with null sol^n)

$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$

(0 sol^n or ∞ solution)

- The rank tells you everything about the sol^n

Ex: Find all sol^n depending on b_1, b_2, b_3

$$\begin{pmatrix} 1 & -2 & -2 \\ 2 & -5 & -4 \\ 4 & -9 & -8 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 & -2 \\ 0 & -1 & 0 \\ 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 - 2b_1 \\ b_3 - 4b_1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 & -2 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 - 2b_1 \\ b_3 - 4b_1 - b_2 + 2b_1 \end{pmatrix}$$

$-2b_1 - b_2 + b_3$

- If $-2b_1 - b_2 + b_3 \neq 0$
  $\therefore$ No sol^n.

- If $-2b_1 - b_2 + b_3 = 0$
  sol^n

rref $R = \begin{pmatrix} 1 & -2 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} b_1 \\ 2b_1 - b_2 \\ -2b_1 - b_2 + b_3 \end{pmatrix}$

$$- \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} b_1 + 2(2b_1 - b_2) \\ 2b_1 - b_2 \\ -2b_1 + b_2 + b_3 \end{pmatrix}$$

Pivot Free

z

Particular sol^n.

$= Ax = b$ $\quad x_p = \begin{bmatrix} 5b_1 - 2b_2 \\ 2b_1 - b_2 \\ 0 \end{bmatrix}$

$z = 0$

Special $\quad x = x_p + c_1 \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$

$Ax = 0$

$= c_1 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

Lecture 9: 12:06

- Linear independance
- Subspace
- Dimension.

We talk about vectors:

Suppose A is m by n with $m < n$.
(Then there are more unknown than equation)

Conclusion $Ax = 0$

There is $N(A)$.

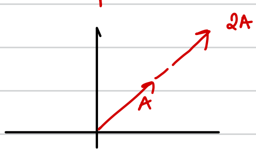
- Reason, there will be free variables!!!
  Atleast $n-m$ free variables.

Independance:

Vectors $x_1, x_2 \dots x_n$ are independant if.
No combination gives zero vector (except the zero comb)
all $c_i = 0$.

$$c_1 x_1 + c_2 x_2 \dots + c_n x_n \neq 0$$

Eg: 2D space.



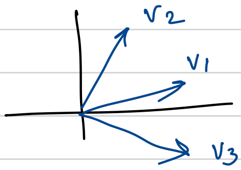
1) $v = A$
$w = 2A$

$2v - w = 0$.

2) $v_1 = A$ (So dependandce)
$v_2 = 0$.

$$0 v_1 + 1 v_2 = 0$$

$c_i \neq 0$.

3) 

$c_1 v_1 + c_2 v_2 \neq 0.$

$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$ for some c.

- I know because.

$$A = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$$

$$\text{eg} = \begin{pmatrix} 2 & 1 & 2.5 \\ 1 & 2 & -1 \end{pmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 0$$

- When $\underline{v_1, \dots v_n}$ are columns of A.
  - They are independant if nullspace A is $\underset{\sim}{0}$ (Rank $= n$) no free vari $N(A) = \underset{\sim}{0}$
  - They are dependant if $Ac = 0$ for some nonzero c. (Rank $< n$) Yes free variables

---

Span a space. Vectors $v_1 \dots v_l$ span a vector space.

- Spanning a space means the space consists of all combinations of those vectors.
- Columns of matrix spans column space.

- Are the colums independant? Depends, yes or no.
- I need to find the right amt. Basis.

Basis for a space is a sequence of vectors.

$v_1, v_2, \dots, v_d$ with prop:

1. They are independant.
2. They span the whole space

Eg:

Space is $\mathbb{R}^3$

- One basis is $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$

  - Are they independant? $\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix} = b.$

    b is 0 only if $c = 0$.

- Another basis: $\begin{pmatrix}1\\1\\2\end{pmatrix}, \begin{pmatrix}2\\2\\5\end{pmatrix}, \begin{pmatrix}3\\4\\8\end{pmatrix}$

  Not sum of $\alpha_1 c_1 + \alpha_2 c_2$

  - Check $\begin{pmatrix}1&2&3\\1&2&3\\2&5&8\end{pmatrix}$ See if we get full Rank or not.

- $\mathbb{R}^n$ n vectors give a basis nxn with their col is invertible.

- Is $\begin{pmatrix}1\\1\\2\end{pmatrix}, \begin{pmatrix}2\\2\\5\end{pmatrix}$ a basis? Yes of a plane.

  $\begin{pmatrix}3\\3\\7\end{pmatrix}$ would be on the plane

---

If we have 3 vec to span $\mathbb{R}^3$. All the diff basis will need 3 vectors.

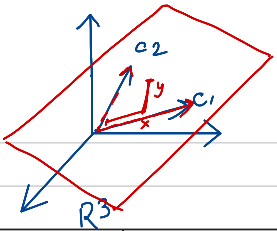
- Basis are not unique.
  - There is something in common:
    - They all have same no of vectors.

Every basis, given a space $(\mathbb{R}^3, \mathbb{R}^n)$, has the same no of vectors.

The no of vectors say how big is the space.

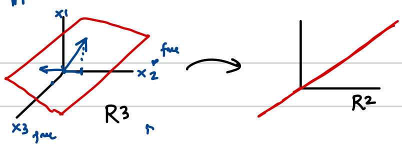
That no is the dimension of the space.

DIMENSION



$\begin{pmatrix} c_1 & \big| & N \end{pmatrix}_{3\times 2} \begin{pmatrix} \ \end{pmatrix}_{2\times 1} = \begin{pmatrix} \ \end{pmatrix}_{2\times 1}$

Suppose $r = 1$



$N(A) = c_1(v_1) + c_2(v_2)$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Summary: Independence, spanning, dimension

Eg: Space is $C(A)$

$$\begin{pmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{pmatrix} \qquad N(A) = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

1 Col3 = 1 col 1 + 1 col 2 +

So x with = 0.

- What is the basis - $C(1) + C(2) \rightarrow$ Pivot columns. (Original column)
- Rank matrix = 2.
- rank (A) = Pivot columns = dimension of $C(A) = 2$
- Whats a basis? $C(1)$ and $C(2) \rightarrow$ col space basis

  $N(A)$ special soln. $\rightarrow$ Nullspace basis.

  another basis :- $\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 6 \\ 4 \\ 6 \end{pmatrix}$: $\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}\begin{pmatrix} 7 \\ 5 \\ 7 \end{pmatrix}$ for col space.

- dim $C(A) = 2$ = rank.

dim $N(A) = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ = Yes = 2.

Two special soln. (basis)

dim $N(A)$ = free variables = $n - r$

Rec: Fin dimension of vector space spanned by the vectors.

- $v_1 = (1,1,-2,0,-1)$ $v_2 = (1,2,0,-4,1)$, $v_3 = (0,1,3,-3,2)$ $v_4 = (2,3,0,-2,0)$

1

$$\begin{pmatrix} 1 & 1 & 0 & 2 \\ 1 & 2 & 1 & 3 \\ -2 & 0 & 3 & 0 \\ 0 & -4 & -3 & -2 \\ -1 & 1 & 2 & 0 \end{pmatrix}^T \to \begin{pmatrix} 1 & 1 & -2 & 0 & -1 \\ 0 & 1 & 2 & -4 & 2 \\ 0 & 1 & 3 & -3 & 2 \\ 0 & 1 & 4 & -2 & 2 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 & -2 & 0 & -1 \\ 0 & 1 & 2 & -4 & 2 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 2 & 0 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 1 & -2 & 0 & -1 \\ 0 & 1 & 2 & -4 & 2 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{array}{l} \leftarrow \text{Rows.} \\ \leftarrow \\ \leftarrow \\ \end{array}$$

Basis.

If you keep in col matrix. → You cant use elimination of initial colum space. Use original colums then

But in row matrix → when you do elimination, you dont chage the row space. or you take the orginal rows also

2

If I do in Col space?

$$\begin{pmatrix} & & \\ & & \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

1 2 3

→ There are colums

→ We get 3 independ pivots.
→ But I can't use the elements, as basis cause I chaged the col ~~space~~. Cant use
— Have 1, 2, 3 c pivots

Lecture 10 : Four dimensional subspaces.

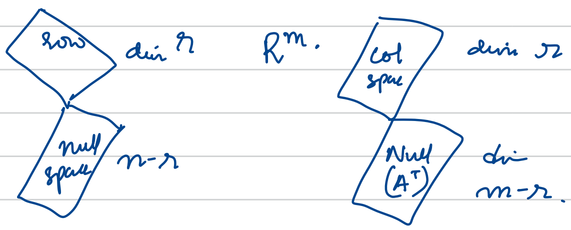
Connection of column space and row space.

4 Subspaces:

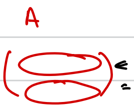
- Column space. $C(A)$
- Null space $N(A)$
- Row space = $R(A)$ all combinations of rows of A.
  = all combo of columns of A^T
  = $C(A^T)$
- Null space $N(A^T)$ [Left null space of A]

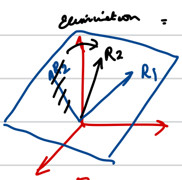
- Suppose A is $m \times n$
  $C(A)$ in R^m
  $N(A)$ in R^n
  $C(A^T)$ in R^n
  $N(A^T)$ in R^m

$A = \left(\quad \right)_{m \times n} \left(\, \right)_{n \times 1}$

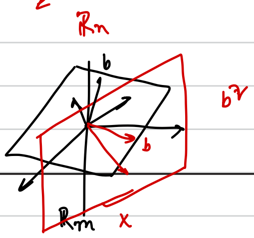
- 

I would like to know • basis of spaces
• dimensions.





- - dim $C(A)$ = rank r = no of indep. col.
  - Basis of col space = Pivot columns. of original A, rows of (A^T) pivot. echelon
- dim $C(A^T)$ = rank r = .
  Basis of row space = $A \to_{\text{echelon form}} A'$ (Basis pivot rows)
- dim $N(A)$ = $n-r$.
  basis of $N(A)$ = special solutions.
- dim $N(A^T)$ = $m-r$
  basis of $N(A^T)$ = .



✱ ✱ ↓↓

Basis of.

∴ Row space - We can take (A^T), find the $C(A^T)$ and get basis of row space. (Original columns.)
But we can use R. Because A and R are the same row space

$$A = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = R$$

so we can use the same.

- The column $C(R) \neq C(A)$ [diff column spaces]

• Same row space!

• Basis for row space is first r rows of R.

⇒ Why in row space ⇒ Because all the combinations still make us stay in Row space.

Null space (A^T) : why left null space.

$$A^T y = 0.$$

• y Null space of A^T

Or $y^T A = 0$

$$\begin{bmatrix} y^T \end{bmatrix} \begin{bmatrix} A \end{bmatrix} = 0.$$

↑ Null space left.

• But stay with $A^T y = 0$

• Looking from $A \to R$. So we need $y^T A = 0$

- Use Gauss Jordan. R is reduced echelon form.

$$\text{rref}.\ [A_{m \times n} \mid I_{m \times m}] \to [R_{m \times n} \mid E_{m \times m}]\ m$$

$$EA = R$$

In chapter 2, $R = I$ (if invertible, the $E = A^{-1}$)

• Elimination changes C space not row space.

• Suppose we have $\begin{pmatrix} 1 & 5 \\ 2 & 10 \end{pmatrix}$ which = rank 1
dim $(C) = 1$
$\phi = c\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

GE ⇒ $\begin{pmatrix} 1 & 5 \\ 0 & 0 \end{pmatrix}$ which is diff
$\phi = c\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

It works because b also undergoes G.E.

• However for rows we have $E_{21} = [\,\vdots\,]$ where rows are combinations of others. So in the same row space.

• Null (A), basis works because again we solve by the rows. (I think)

$\Rightarrow A$

$\Rightarrow EA = R$ which is Row echelon form.

if A is invertible

then E is A^{-1} $\Rightarrow$ $E \ A \ | EI$

$I \ | A^{-1}$

Same $E = A^{-1}$

if $\det(A)$ is $\neq 0$

But. $EA = R$

$$\Rightarrow \begin{pmatrix} -1 & 2 & 0 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{pmatrix} = \text{rref}(A) = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = R.$$

$\leftarrow -1R_1 + 0R_2 + 1R_3 \Rightarrow 0.$

Now we see that bottom $m-r$ rows show linear dependency of rows of A, because $m-r$ rows $= R$ (last row) $= 0$.

So $y^T A = \begin{pmatrix} -1 & 0 & 1 \end{pmatrix} (A) = 0$

So $m-r$ rows satisfy $EA_{m-r} = 0$ and form basis of $y^T A$ or left nullspace of A.

basis is $= c\begin{pmatrix} -1 & 0 & 1 \end{pmatrix}$

A^T

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 3 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

free.

$$y = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

New vector space.

All 3×3 matrices!!

- I can add, scalar and still in matrices space.

- Only $A+B$, cA (not AB).

Subspaces - 1. All upper triangular.
2. Symmetric.
3. $A\cap B \Rightarrow$ subspace. (Diagonal matrices)

Diagonal matrix - dim of subspace is 3.

$$\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}1&0&0\\0&3&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}$$

$\mathbb{R}^n \rightarrow \mathbb{R}^{n\times n}$

Rec.:

$$B = \begin{array}{cc} L & U \\ \begin{pmatrix}1&&\\2&1&\\-1&0&1\end{pmatrix} & \begin{pmatrix}5&0&3\\0&1&1\\0&0&0\end{pmatrix}\end{array}$$

Find the basis for and find the dimension of each 4 fundam. subspaces

$B = LU$

$EB = U$

$$E = \begin{pmatrix}1&&\\-2&1&\\1&0&1\end{pmatrix}$$

$B = U$ (Row echelon form).

$$= LU = \begin{pmatrix}5&0&3\\10&1&7\\-5&0&-3\end{pmatrix}$$

So $\dim(B) = 2$, basis $= (5,10,-5,\ 0,1,0)$ or $(1,-2,1)\ (0,1,0)$

$\dim N(B) = 1$, basis $= (-3/5, -1, 1)$

$\dim(B^T)$, basis 1st rows)

$\dim N(B^T) = 1$, $(1,0,1)$

Lecture 11 :

- Bases of new vector spaces.
- Rank one matrices.
- Small world gap.

Matrix space : Symm subspace, Diag subspace] + and α.

- Dim ? Basis ?

---

1. Basis → All $c = 0$, we will get $\underset{\sim}{0}$
→ • Span the whole space

- Basis for M = 3×3's dim M = 9

$$\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix} \cdots\cdots \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}$$

→ M . 3×3 ,

1. Vector space :
→ $\alpha v_1 + \beta v_2 = v \in \mathcal{V}$
→ 7 rules, $\underset{\sim}{0}$, $-1 \cdot \underset{\sim}{v} = -\underset{\sim}{v}$

- Basis of Symm matrix dim S = 6

$$\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}, \begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix}$$

Upper triangular, dim = 6

- We need to think again basis for subspace.

$S \cap U$ = diagonal matrix
dim$(S \cap U)$ = 3.

$S \cup U$ – Not subspace
so
$S + U$ – Makes sense.

$S + U$ – Any element in S + U.

• $S + U$ = all 3×3.

$\dim(S+U) = 9$ All space.
• $\dim S + \dim U = \underline{\dim S \cup U} + \dim S \cap U$.
• $6 + 6 = 9 + 3$
• $\cup = \cap +$ sum.

$\dfrac{d^2y}{dx^2} \cdot y$.
$\dfrac{d^2y}{dx}\,\dfrac{dy}{dx}$

D.E $\quad \dfrac{d^2y}{dx^2} + y = 0$

$y = a\cos x + b\sin x$
$\dfrac{d^2}{dx^2}(a\cos x + b\sin x) + a\cos x + b\sin x = 0$
$\rightarrow \left(\dfrac{d^2 a\cos x}{dx^2} + a\cos x\right) + \left(\dfrac{d^2 b\sin x}{dx} + b\sin x\right) = 0$

Som $y = \cos x$, $y = \sin x$ $\quad y = e^{ix}$

$a y_1 + b y_2$

Complete soln. $\quad y = \underbrace{a\cos x + b\sin x}_{\text{basis vector}}$ (That's a vector space).

$y = a\cos + b\sin x$

$= \int_{-\infty}^{\infty} \cos \cdot \sin x = 0$

cos x
sin x

• A basis means combinations → $\cos x$, $\sin x$
(Special solution)

dim (Solution space) = 2. (Also order of the differential equation)

• These things dont look like vectors.
- But we can add and scale.

KEY NO - RANK
- rank $< m < n$

Rank 1
• $\underset{2\times 3}{A} = \begin{pmatrix} 1 & 4 & 5 \\ & & \end{pmatrix}$ So : $A = \begin{pmatrix} 1 & 4 & 5 \\ 2 & 8 & 10 \end{pmatrix}$

$\dim C(A)$ = rank = $\dim C(A^T)$ (Row space)
= 1 =

Rank 1 Matrix

$$A = \begin{pmatrix} 1 & 4 & 5 \\ 2 & 8 & 10 \end{pmatrix} \Rightarrow \begin{array}{c} c_1 \\ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \end{array} \begin{pmatrix} 1 & 4 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 4 & 5 \\ 2 & 8 & 10 \end{pmatrix}$$

$C_2 = 4C_1 \quad C_3 = 5C_1 \quad \therefore$ Only c_i is dependent.

$$\underset{2\times 1}{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} \underset{1\times 3}{\begin{bmatrix} 1 & 4 & 5 \end{bmatrix}}$$

Rank 1 matrix has form:

Dependance.

$$A = \underset{\sim}{u}\, \underset{\sim}{v}^T$$

Independent of

$A' = (\text{Rank } 4)$
$B' = (\text{Rank } 4)$
$+ = C'$ might not be Rank 4.

eg. Rank 4 Matrices

M = all 5×17 matrices. $\quad$ 17. $\quad 5[\quad] \quad 4[$

- Subset of rank 4 matrices. even if we include $\underset{\sim}{0}$.

  Subspace?

  Sum, Add 2 rank 4 $\rightarrow$ Not always rank 4.

- Subset of rank 1 matrices. Not a subspace.

  Rank 1 + Rank 1 $\rightarrow$ may Not Rank 1

  A $\quad$ B $\quad$ C

How do we get a subspace?

1. We have a vector space V.
2. $S \in V$
3. Then prove S is a vector space.

Suppose in $\mathbb{R}^4$

set of all $\underset{\sim}{v} = \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix}$ such that $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0$

if $A = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \qquad \underset{\approx}{A}\,\underset{\sim}{v} = 0$

- v is $N(A)$

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}$$

Pivot.

So rank 1

$\dim(N(A)) = 4 - 1 = 3 \quad (n - r)$

$$\text{Now } N(A) = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \leftarrow \text{Free variables}$$

$\dim(N(A^T)) = \underset{\sim}{0} = 0. \qquad$ (which is 1 row)

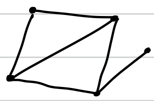
$\dim(A^T) = 1.$

which is 1 row only.

$\dim(A) = 1$

What's a graph?

Graph = {nodes, edges}



How many steps you take one graph to another.

Rec: Show that set of 2x3 matrices whose nullspace contains $\begin{pmatrix}2\\1\\1\end{pmatrix}$ is a vector subspace, and find a basis for it. $\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}$ does not have

What about set that contains $\begin{pmatrix}2\\1\end{pmatrix}$? $\begin{bmatrix} & 2\\ & 1\end{bmatrix}$

One matrix

Answer:

$$\underset{2\times 3}{[A]}\begin{pmatrix}2\\1\\1\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix},\quad [B]\begin{pmatrix}2\\1\\1\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

$$(A+B)\begin{pmatrix}2\\1\\1\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\quad \left(A\begin{pmatrix}2\\1\\1\end{pmatrix}+B\begin{pmatrix}2\\1\\1\end{pmatrix}=\begin{matrix}0\\0\end{matrix}\right) \checkmark$$

$$(cA)\begin{pmatrix}2\\1\\1\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\quad \ldots \text{So it's a subspace.} \checkmark$$

$Ay=0$

$\begin{pmatrix}\text{———}\\ \text{———}\\ a\ b\ c\end{pmatrix}\begin{pmatrix}|\\y\\|\end{pmatrix}=0$

$ay_1+by_2+cy_3=0$

$(a\ \ b\ \ -2a-b)\begin{pmatrix}2\\1\\1\end{pmatrix}=0$

$=a(1\,0\,-2)+b(0\,1\,-1)$

- All a, b, c that satisfy
- All a b that satisfy

• Each row must satisfy

$\underset{\uparrow \text{ basis of ?}}{A}\begin{pmatrix}2\\1\\-1\end{pmatrix}$

$$(a\ \ b\ \ c)\begin{pmatrix}2\\1\\1\end{pmatrix}=0$$

$c=-2a-b$

$= 2a+b+c=0 \Leftarrow$

$= (a\ \ b\ \ -2a-b) = (a\ \ 0\ \ -2a)+(0\ \ b\ \ -b)$

$= (a\ 0\ \ -2a)+(0\ b\ -b)$

$= a(1\ 0\ -2)+b(0\ 1\ -1)$

Must be a l.c of $(1\,0\,-2)$, $(0\,1\,-1)$

Basis.

$$? \begin{pmatrix}1&0&-2\\0&0&0\end{pmatrix},\begin{pmatrix}0&0&0\\1&0&-2\end{pmatrix},\begin{pmatrix}0&1&-1\\0&0&0\end{pmatrix},\begin{pmatrix}0&0&0\\0&1&-1\end{pmatrix}$$

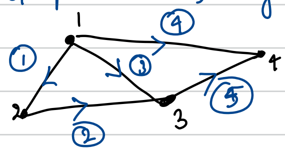
dim (subspace) is 4.

• $\left(c\begin{pmatrix}2\\1\end{pmatrix}\right)$ $\begin{pmatrix}0&0&0\\0&0&0\end{pmatrix}$ is in $\begin{pmatrix}2\\1\end{pmatrix}$ in Column space NO.

Lecture 12 - Graphs, Networks, Incidence matrixes

- Graph + networks.
  Incidence matrices
  Kirchoff's law

- Graph: Nodes, Edges.



- no of nodes $n = 4$
  edges $e = 5$.

rref $A = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Incidence matrix:

$$A = \begin{array}{c|cccc} \text{node} & 1 & 2 & 3 & 4 \\ \hline 1 & -1 & 1 & 0 & 0 \\ 2 & 0 & -1 & 1 & 0 \\ 3 & -1 & 0 & 1 & 0 \\ 4 & -1 & 0 & 0 & 1 \\ 5 & 0 & 0 & -1 & 1 \end{array} \quad \text{edge.}$$

elements $\qquad m \times n$

Make Loop → Are the rows dependent. Yes!

$C_1 + C_2 + C_3 + C_4 = 0$

$R_1 + R_2 - R_3 = 0$

$R_4 + R_5 - R_3 = 0$

$\Rightarrow R_1 + R_2 + R_4 + R_5 = 0$.

Relative displacement.

Suppose $b: \begin{pmatrix} 3 \\ -2 \\ 0 \\ 1 \\ 2 \end{pmatrix}$ Extends / Compress

x: displacement

$\underbrace{Ax}_{\text{Relative displacement}} = b$

What Q:

- Lots of zeros: No of zeros $= 2 \times m$
- Null space: (Tells us how to combine Columns to get $\underset{\sim}{0}$)

Suppose:
$Ax = 0$

$$\begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \rightarrow \begin{pmatrix} x_2 - x_1 \\ x_3 - x_2 \\ x_3 - x_1 \\ x_4 - x_1 \\ x_4 - x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$X = x_1, x_2, x_3, x_4$ (Potential at nodes) (Displacements)

$Ax \rightarrow$ (Potential differences across edges)

$x = c \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, If potential is const diff is const.

$\dim N(A) = 1$.

- $\dim N(A) = 1$.
- What does it mean. Potential constant, problem.
- We need to set one as grounded =

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} \\ \\ \\ \end{pmatrix} \quad x_4 = 0 / \text{some val.}$$

VOLTAGE DIFF → CURRENT → Equilibrium of current (Kirchoff's law)

DEFORMATION → FORCES. → Equilib of forces.

$C = EA$

Ax → CAx → FORCES →

- Rank = 3.

- $e = Ax$ ← Constitutive
- $y = Ce$ ✓

$Ax = x_2 - x_1$

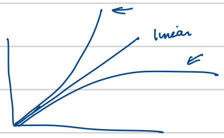
$e = \frac{x_2 - x_1}{L}$

Strains

→ Linear = $\frac{\Delta u}{L}$ «

→ Green Lagrange = $\frac{1}{2}\left(\frac{\partial u}{\partial x_i} + \frac{\partial u}{\partial x_j}\right) + \frac{1}{2}\left(\left(\frac{\partial u}{\partial x}\right)^2 \cdots\right)$

→ Euler Almasi strain



→ $u_x - u_1$ -

- $C(A) = aC_1 + bC_2 + cC_3$
- $y = CAx$ (if y = stress, $C = EAL^{-1}$ (young modulus × area), $L^{-1} = \begin{bmatrix} 1/L_1 & & \\ & 1/L_2 & \\ & & 1/L_3 \ldots \end{bmatrix}$, $A = [a_1\ a_2\ a_3]$ ← area)
- $A^T y = 0$ ← Where y is the current. Nullspace of A^T gives the collection that satisfy Kirchoff's law.

$\dim N(A^T) = m - r$
$= 5 - 3 = 2$

$$A^T = \begin{pmatrix} -1 & 0 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$N(A^T)$

Potentials $x = x_1, x_2, x_3, \cancel{x_4}$

$\downarrow A$

$x_2 - x_1, x_3$ $\xrightarrow{\text{OHM'S LAW}}$ $e = Ax$
P. Difference

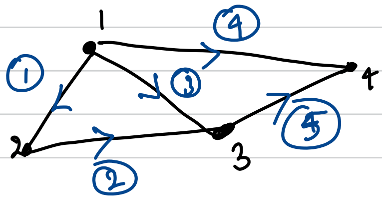
CURRENT. on edge.
Some no $\left(\frac{1}{R}\right)$ × Potential diff. $\quad y = Ce$

↓

$A^T y = 0$ (Kirchoffs current law).

$N(A^T)$ dim = 2.

- Vectors in nullspace of A^T correspond to collections of currents that satisfy Kirchoff's law



Vector in it.
Kirchoff's law :-

- Net flow is zero
Balance law.

$$-y_1 - y_3 - y_4 = 0$$
$$y_1 - y_2 = 0$$
$$y_2 + y_3 - y_5 = 0$$
$$y_4 + y_5 = 0$$

(A^T) Pivots
(1, 2, 4)

3 and 5 are free.

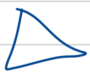
(Can I figure the null space ?)

- Basis for $N(A^T)$, Suppose $y_1 = 1$, then $y_2 = 1$
then do equilibrium. in 1st loop

1) $\begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$ Satisfies 1st loop, I current flowing in 1st loop.
we can multiply

the 2nd loop. $y_5 = 1$, then $y = 4$.

2) $\begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{pmatrix}$

3rd loop ?  (Big one) ?

→ Its not independant.
It is dependant on inner loops.

** If you had voltages between nodes. So then

$A^T y = f$. ← Charges.

$e = Ax$, $y = Ce$, $A^T y = f$.

→ $\boxed{A^T C A x = f}$ WOW!!

If we ground,

$$\begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_2 - x_1 \\ x_3 - x_2 \\ x_3 - x_1 \\ x_4 - x_1 \\ x_4 - x_3 \end{pmatrix}$$

Incidence — Field Variable

$A = (\text{no of member} \times \text{no of nodes})$ $\quad x = (\text{No of nodes} \times 1)$

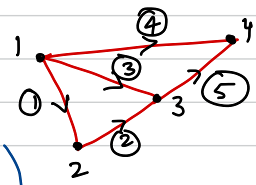
- If we equate $Ax = 0$, the $N(A) = c(1\ 1\ 1\ 1)^T$
  - Because rank $(A) = 3$.
  - So if we give a value to one x_i in $\underset{\sim}{x}$, we remove that dof from the system of linear equations.

eg: $x_1 + x_2 = c$ $\quad$ we have to keep x_2, x_1 = const
$2x_1 + 2x_2 = 2c$ $\quad$ to get an exact mapping.

- Now well look at A^T

$\Rightarrow A : x \longrightarrow$ member property y (Relative difference of member nodes)
Field variable

$\Rightarrow A^T$: Member prop y $\longrightarrow$ Conditions at each node.



- A^T: $\begin{pmatrix} -1 & 0 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{pmatrix} = \begin{pmatrix} -y_1 - y_3 - y_4 \\ y_1 - y_2 \\ y_2 + y_3 - y_4 \\ y_4 + y_5 . \end{pmatrix}$

$(n \times m)$ $\quad (m \times 1)$

if $(A^T y) = 0$ (This is the condition that at each node the member property equilibrium should be there if no external property (force))

left nullspace

$\Rightarrow \underset{\sim}{N(A^T)}$: $\underset{\sim}{y}^T \underset{\sim}{A} = 0$

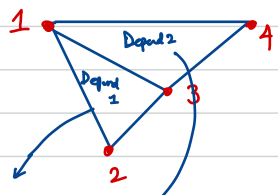
EA: Row $E \Rightarrow [1 \cdots]\,[A] = [0\ 0\ 0\ 0]$ $\quad A^T$

- $N(A)$
  $Ax = 0$
  $x = ?$

$N(A^T)$
$= A^T y = 0$
$y = ?$ $\quad$ Same.
or $y^T A$
$y = ?$

What is

$$(y_1\, y_2\, y_3\, y_4\, y_5) \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$



Look at Row dependance

$R_1 - R_3 + R_2 = 0$ $\quad (1\ \ 1\ -1\ \ 0\ \ 0) = y_1^T$
$-R_4 + R_5 + R_3 = 0$ $\quad (0\ \ 0\ \ 1\ -1\ \ 1) = y_2^T$

$$\boxed{\begin{aligned} A^T y &= f \\ A^T C e &= f \\ \underline{A^T C A}\, x &= f \end{aligned}}$$

$A x = e$

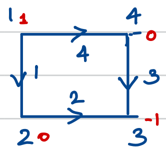
$y = C e$ ← Constitutive matrix.

$A^T y = 0$ ← if no external f.

$A^T y = f$ ← if there is external

Electrical	Structures
x - voltages	x - disp.
e - voltage diff.	e - relative disp
$y = Ce$ - ohms law	$y = Ce$ - hookes law
$A^T y$ - Kirchoffs law (Incoming - +ve, Outg - -ive). Each node $\Sigma I + 0 = 0$) $\to f$	$A^T y \Rightarrow$ Force equilibrium (

eg:



$$A = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

$f = (1, 0, -1, 0)$

Our $C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

(This just means
y_1 = relative disp of memb 1
$y_2 = 2x$ " " " " 2.

$\Rightarrow A^T C A x = f.$

$$= \begin{pmatrix} -1 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & 3 & -3 \\ -1 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 3 & -2 & 0 \\ 0 & -2 & 4 & -2 \\ -1 & 0 & -2 & 3 \end{pmatrix}$$

If we write in Augmented:

$$\left(\begin{array}{cccc|c} 2 & -1 & 0 & -1 & 1 \\ -1 & 3 & -2 & 0 & 0 \\ 0 & -2 & 4 & -2 & -1 \\ -1 & 0 & -2 & 3 & 0 \end{array}\right) \xrightarrow[\text{rowred}]{\text{GE}} \left[\begin{array}{cccc|c} -1 & 0 & 0 & -1 & 1 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & +1 & 1 & 0 \end{array}\right]$$

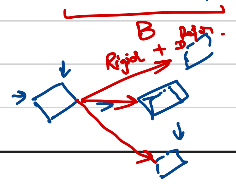
$c_1 + c_2 - c_3 = c_4$

$\underbrace{\qquad}_{B} \quad \underbrace{\qquad}_{\underset{\sim}{f}.}$

if $Bx = f.$

$x = \underbrace{x_p}_{\downarrow} + \underbrace{x_n}_{\downarrow}.$

$\therefore \Rightarrow$ Extension. Rigid body.



if we say $x_3 = 0$, so its grounded.

$$\begin{aligned} -x_1 - x_4 &= 1 \\ x_1 - x_2 &= 0 \\ x_2 &= -1 \\ +x_4 &= 0 \end{aligned}$$

if we rearrng

$$\begin{bmatrix} A_{inv} & A' \\ A' & A_{fixed} \end{bmatrix} \begin{Bmatrix} x_{free} \\ \hline x_{fixed} \end{Bmatrix} = \begin{Bmatrix} f \end{Bmatrix}$$

$A_{ii}\, x_{free} = f'$

$C \Rightarrow$ Incidence matrix.

$\Rightarrow Cx = u$ (differences)

$\Rightarrow U = \begin{bmatrix} u_1 & & & \\ & u_2 & & \\ & & \ddots & \\ & & & u_m \end{bmatrix}$ $e = \frac{EA}{l_0}(u)$

$F \Rightarrow EAL^{-1}(u - l) = \begin{bmatrix} \frac{EA_1(u_1 - l_1)}{l_1} & & \\ & \frac{EA_2(u_2 - l_2)}{l_2} & \\ & & \ddots \end{bmatrix} = \begin{bmatrix} f_1 & & & \\ & f_2 & & \\ & & f_3 & \\ & & & \ddots \end{bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \\ \vdots \end{Bmatrix}$

It's in the direction of the member.

$\Rightarrow UW^{-1}f \overset{?}{=} t_x \quad \left(F \times \frac{(x_2 - x_1)}{L} \right)$ $L \Rightarrow$ deformed length

$t_x = \begin{bmatrix} \frac{u_1}{W_2} f_1 & & \\ & \frac{u_2 f_2}{W_2} & \\ & & \ddots \end{bmatrix}$ x.

X Direction.

$\Rightarrow C^T t_x = r_x$ (external forces) comp.

$\boxed{C^T U W^{-1} f = r_x}$ ← Determinate. A^T

$\Rightarrow C^T U W^{-1} EA L^{-1} C x = r_x.$

Node i: Member i

$$EA \frac{(x_i - x_i^\circ)}{l_i^\circ} \times \left(\frac{u_i}{l_i}\right) + EA \frac{(x_{i+1} - x_{i+1}^\circ)}{l_{i+1}^\circ} \times \frac{(u_{i+1})}{l_{i+0}} = P_x$$

$\sum$ Component of Internal forces $= P_x$ (External load)

↓

Deformed shape.

$\left(e_{21} = \sqrt{?} , \; \vdots \right)$

If we say small displacements.

Then $(UW^{-1}) = (U_0 L_0^{-1})$

↓ Difference of initial geom

↘ Length of initial geometry.

Row space. = Independent c_1, c_2, c_4.
1^{st} three not independent.

If you see null space $\begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$ Combi of C = 0.

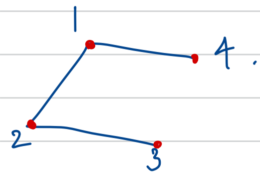
Pivot.

$$A^T = \begin{array}{c} \\ \\ \\ \\ \\ \end{array}\begin{array}{ccccc} \downarrow & \downarrow & & \downarrow & \\ \begin{matrix} -1 \\ 1 \\ 0 \\ 0 \end{matrix} & \begin{matrix} 0 \\ -1 \\ 1 \\ 0 \end{matrix} & \begin{matrix} -1 \\ 0 \\ 1 \\ 0 \end{matrix} & \begin{matrix} -1 \\ 0 \\ 0 \\ 1 \end{matrix} & \begin{matrix} 0 \\ 0 \\ -1 \\ 1 \end{matrix} \\ \uparrow & & & & \\ 1 & 2 & & 3 & \end{array}$$

Edge

- Independants dont have loop.
- Dependants have a loop

- Tree has no loop.



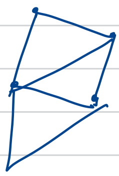
• $\dim N(A^T) = m - r$.

$$\boxed{\#\text{ loops} = \#\text{ edges} - (\text{no of nodes} - 1)}$$

$\#$ nodes $-$ $\#$ edges $+$ $\#$ loops $= 1$.

Eulers formula.

$5 - 7 + 3 = 1$



$$\boxed{\pi = Ax \qquad y = C\pi \qquad A^T y = 0}$$

Equation of current.

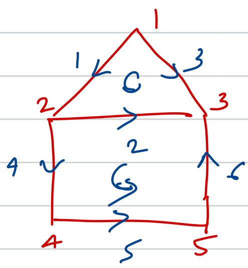
- Battery edges they go π.
- Current sources go to $A^T y = f$.

$$\boxed{A^T C A x = f}$$

$A^T A$ - Symmetric !!

---

rec:



$$A \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 \end{bmatrix}$$

$$N(A) = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

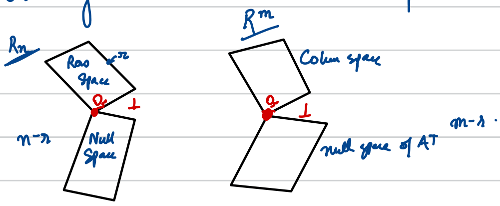
$N^T y = 0$

$A^T y$ Take y_1 in loop,

Trace $(A^T A) = |Col|^2 = 2 + 3$ (No of edges connect we have in dim m. there are n cols.

Lecture 14

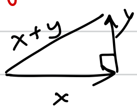
Orthogonal vect and subspaces



- Angle between these subspaces is 90°
- What does it mean for subspaces orthogonal - 90°

- Most subspaces orthogonal

Orthogonal vectors



• Pythagoras.

Dot product

- $x^T y = 0$
- Why? RHT if $x+y$ $\|x\|^2 + \|y\|^2 = \|x+y\|^2$

Length squared of $x = (1,2,3)$ $1^2 + 2^2 + 3$

$= x^T x \rightarrow$ Length squared. ← $x^T x$.

$x^T x = 1+4+9 = 14.$

$y = (2\ -1\ 0)$ $x+y = (3\ 1\ 3)^T$

$\|y\|^2 = 5$

$\|x+y\|^2 = 19$ ✓ Check.

$\|\ \|$ ←

$\|x\|_1 = |x|$

1 dim vector space.

$\|x\|_2 - \sqrt{(\ )^2 + (\ )^2}$

$\|x\|_p = (\sum |x|^p)^{1/p}$.

limit $p \to \infty$

$\|x\|_\infty = \max |x_i|$

= max(abs value)

supremum, infin

Pythagorean theorem

Distance $\|\ \|_2$

Vectors

$x+y$, y, x

$(x^T x) + (y^T y) = (x+y)^T (x+y)$

$$x^T \cancel{x} + \cancel{y}^T y = (x+y)^T (x+y)$$
$$= x^T \cancel{x} + \cancel{y}^T \cancel{y} + x^T y + y^T x$$

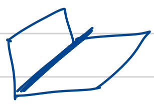
So $x^T y + y^T x = 0$.
$2x^T y = 0$.

Dot product of orthogonal $= 0$

If $x = \underset{\sim}{0}$, $x^T y$ orthogonal? Sure.

Subspace S is orthogonal to subspace T means :
- Every vector in S $\perp$ to vector in T.

- They should not intersect



Eg of orthogonal spaces.

- When does we have orthogonal subspace in plane
  - Line + plane X
  - line + zero ✓
  - line + line Sometimes (90°)

- They don't intersect.

$A \cap B = \underset{\sim}{0}$

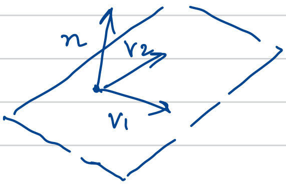
$$\begin{bmatrix} R_1 & \text{---} \\ R_2 & \text{---} \end{bmatrix}_{m\times n}$$

Row space space span by the rows.

$R(A) = a\{R_1\} + b\{R_2\}$] n.

$N(A)$

$$[C_1\ C_2 \dots C_n]\begin{Bmatrix} x \\ \vdots \\ x_n \end{Bmatrix}$$



$n \cdot V_1 = 0$

$n \cdot V_2 = 0.$

$n^T \cdot V_1 = 0$

$n^T \cdot V_2 = 0$

$V = \alpha(V_1) + \beta(V_2).$

$n^T \cdot V$
$= n^T(\alpha V_1 + \beta V_2)$
$= \alpha n^T V_1 + \beta n^T V_2$
$= 0.$

Row space is orthogonal to nullspace.

Why? Every row in row space is $\perp$ to every vector in null space.
$N(A) = Ax = 0$

- $$\begin{bmatrix} \text{row } 1 \\ \text{row } 2 \\ \text{row } n \end{bmatrix}\begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

so row 1 $\perp$ x
row 2 $\perp$ x

- So rows of A $\perp$ to X. ←

Col space $\perp$ A^T because $C(A) = R(A^T) \perp N(A^T)$ Same.
$\therefore C(A) \perp N(A^T)$.

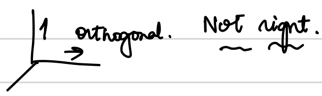
- Have to check combination
  $C(A) \perp N(A^T)$.
  $\Rightarrow R(A^T) \perp N(A^T)$
  $\Rightarrow A^T \cdot y = 0$

Column space $\perp$ $N(A^T)$ - Same.

R^n R^m

$n-r$ $m-r$.

- Just like 3D



- Tell orthogonal spaces that don't span 3D.
  → 2 lines. subspace. → No

If $A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 4 & 10 \end{bmatrix}$

Rank $r = 1$. $m = 2$ $n = 3$. (Basis of row space)

dim $N(A) = 3 - 1 = 2$ ϕ of $R(A) = (1,2,5)$

Null space is
- Plane $\perp$ to $(1,2,5)$

- So the row space is $\perp$ to $N(A)$

Eg: (125)

- Null space and row space are othorgonal and dimension fill the whole space in R^n
  Orthogonal complements.

- Null space contains all vectors $\perp$ to row space.

$Ax = b$ (I would like to solve when there is no solution)

- $m > n$.

- More information (NOISE)

- One way is to say some info (eq) are useful.

- A is usually rectangular.
- This is the matrix:

→ $A^T A$
  - Square
  - Symmetric
    $(A^T A)^T = A^T A$

Good equation:
$$A^T A x = A^T b.$$

eg: $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 5 \end{pmatrix}$

- $m = 3$ $n = 2$.
  $r = 2$

Can I solve:
$$A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = b$$

- I can't.

$A^T A$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 5 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 5 \end{pmatrix} = \begin{pmatrix} 3 & 8 \\ 8 & 30 \end{pmatrix}$$

- $A^T A$ is not allways invertible.

$$\begin{pmatrix} 1 & 1 & 1 \\ 3 & 3 & 3 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 1 & 3 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 9 \\ 9 & 27 \end{pmatrix}$$

- Rank of $A^T A$ = A
- Null space $A^T A$ = Null space A

$A^T A$ is invertible if $N(A) = \underline{0}$. Independent columns. in A.

Rec: S spanned by $(1\ 2\ 2\ 3)$ and $(1\ 3\ 3\ 2)$

- Find basis for $S^\perp$ ($S^\perp$ is $N(A^T)$)
- Can every v in $\mathbb{R}^4$ be writen uniquely writen in term of S and $S^\perp$

$$\begin{pmatrix} 1 & 1 \\ 2 & 3 \\ 2 & 3 \\ 3 & 2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 2 \\ 2 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 3 \\ 3 \\ 2 \end{pmatrix}$$

Yes

$$v = c \begin{pmatrix} 1 \\ 2 \\ 2 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 3 \\ 3 \\ 2 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} + c_4 \begin{pmatrix} -5 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$\Rightarrow$ LI.

- $(1\ 2\ 2\ 3) . x = 0$
  $(1\ 3\ 3\ 2) . x = 0$

$C(A) \perp N(A^T)$ or

- $\begin{pmatrix} 1 & 2 & 2 & 3 \\ 1 & 3 & 3 & 2 \end{pmatrix} x = 0$

Find x in $N(A^T)$

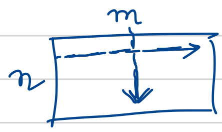
$$\text{rref}(A) = \begin{pmatrix} 1 & 2 & 2 & 3 \\ 0 & 1 & 1 & -1 \end{pmatrix} \Rightarrow \quad x_3 = 1,\ x_4 = 0$$

$$x_3 = 0,\ x_4 = 1.$$

$$c_1 \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix},\ c_2 \begin{pmatrix} -5 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

Morning Lecture:

$\mathbb{R}^n$ | $\mathbb{R}^m$



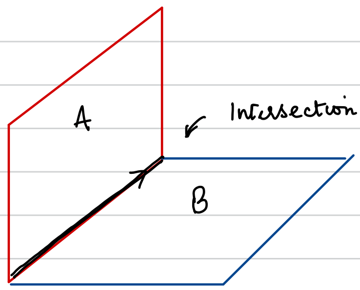
Row space $R(A)$ or $C(A^T) = r$. | Column $C(A) = r$

$\perp$ | $\perp$

Null space $N(A) = n-r$ | $N(A^T)$ or Left null space $= m-r$.

- For a vector space V to be $\perp$ to another space S, all the $v \perp s$ for all $v \in V$ and $s \in S$.

Now



Intersection $\Rightarrow$ In.

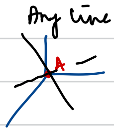
Are the spaces $\perp$?

$a \in I_n$ $b \in I_n$

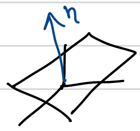
$a^T b \neq 0$.

- So example of $\perp$ vector spaces.

- $A = \underset{\sim}{0}$ $B =$ any line (through origin). $A \perp B$



- $A = \{x + y = 0\}$ the origin $B = \hat{n}$ line through origin

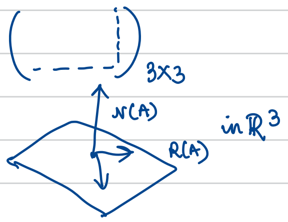


- A is line, B is the whole space $\cancel{\perp}$

- Two $\perp$ lines, ✓. $\perp$

(Row space $\perp$ Null space) of A

- $Ax = 0$
- All rows of $A \perp x$

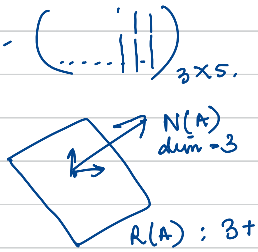


- The row space and null space subdivide $\mathbb{R}^n$ into two perpendicular subspaces.

$$A = \begin{pmatrix} 1 & 2 & 5 \\ 2 & 4 & 10 \end{pmatrix} \xrightarrow{\text{Gauss.}} A = \begin{pmatrix} 1 & 2 & 5 \\ 0 & 0 & 0 \end{pmatrix}$$

Pivot. Free

x_2 and x_3 free



$R(A)$: $3 + r = 5$.

in $\mathbb{R}^5$.

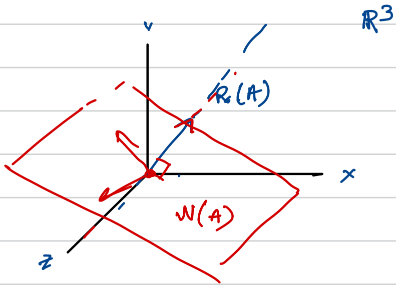
$R(A) = 1\ 2\ 5.$

$$N(A) = a\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, b\begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix} = a\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix}$$

$x_2 = 1$, $x_3 = 0$, $x_1 = ?$

$x_2 = 0$, $x_3 = 1$, $x_1 = ?$

- $(1\ 2\ 5)^T \cdot N(A)$.

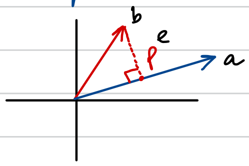


$\mathbb{R}^n$ is a linear comb of $R(A)$ and $N(A)$.

- We say the null space and row space are orthogonal compliments in $\mathbb{R}^n$.

Lec 15.

- Projections !
- Least square



$e = b - p$

Find point on line a closest to b?

Where does orthogonality come?

- e is ⊥ to a

- $p = xa$ (p is some multiple of a)

- $a^T(b - xa) = 0$

$\Rightarrow a^T b = a^T x a$

$\Rightarrow a^T b = x\, a^T a$ is No.

$$\boxed{\begin{aligned} &\Rightarrow x = \frac{a^T b}{a^T a} \\ &p = ax \end{aligned}}$$

$$p = a\frac{a^T b}{a^T a}$$

- Suppose b is 2b.

  Then p is projected twice

- Suppose a is 2a.

  Nothing changes.

Projection = P b.

$$P = \begin{pmatrix} a_1^2 & a_1a_2 & a_1a_3 \\ a_2a_1 & a_2^2 & a_2a_3 \\ a_3a_1 & a_3a_2 & a_3^2 \end{pmatrix}$$

$\downarrow$

$a_1 \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \leftarrow$ In $\underset{\sim}{a}$.

Rank

$\underset{a_3}{\text{Col}\,3} = \underset{a_1}{\text{Col}\,1} + \underset{a_2}{\text{Col}\,2}$.

Projection matrix.

$P = \dfrac{aa^T}{\underbrace{a^Ta}_{\text{No.}}}$; $p = Pb$.

- Properties: Pb.
  - So column space of P.

    $C(P)$ = line through a ✓

    rank $(P) = 1$. ✓

  - P is symmetric. (Anytime we A^TA)
  - What happens if I do projection twice. → I stay put.

    $P^2 = P$ or $P^2 - P = 0$ $P(P-I) = 0$.

x, p and P.

Why do I want this projection?

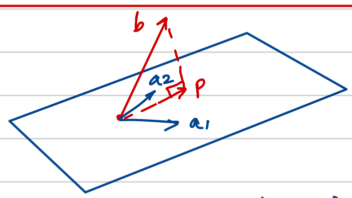
- $Ax = b$ may have no solution.

  Ax always has to be in col (A)

  b isn't

- I change b, Solve $A\hat{x} = p$. → Projection of p in $C(A)$

  Projection of b in A. or $C(A)$



- Gotta find what the plane is.
  - Find basis, a_1 & a_2

    a_1, a_2 col space of what matrix. ?

    $A = \begin{pmatrix} | & | \\ a_1 & a_2 \\ | & | \end{pmatrix}$

$e = b - p$

- $e = \perp$ to plane
- Part of b $\perp$ plane

$p = x_1 a_1 + x_2 a_2$
$\quad = A\hat{x}$

So the problem is:
- Find $\hat{x}$ for $A\hat{x} = p$.
- Such that $e = b - p \perp$ plane.

$p = A\hat{x}$, find $\hat{x}$
Key: $b - A\hat{x}$ perp. to plane.

$a_1^T(b - A\hat{x}) = 0$ & $a_2^T(b - A\hat{x}) = 0$

$$\begin{pmatrix} a_1^T \\ a_2^T \end{pmatrix}(b - A\hat{x}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- $A^T(b - A\hat{x}) = 0$ (Same thing for 1D).
  $A^T e = 0$

So e is in the $N(A^T)$
- $e \perp C(A)$ YES!!

$$\boxed{A^TA\hat{x} = A^Tb.}$$

$\hat{x} = (A^TA)^{-1}A^Tb$

Projection, $p = A\hat{x} = \underbrace{A(A^TA)^{-1}A^T}_{P}b$.

Tempting to simplify $\rightarrow$ $AA^{-1}A^{-T}A^T b$.

$P = A(A^TA)^{-1}A^T$

May be inv

Shall I do it: Video tape. $A\,A^{-1}A^{T^{-1}}A^T = I$???

expand?

Not possible, A not square.

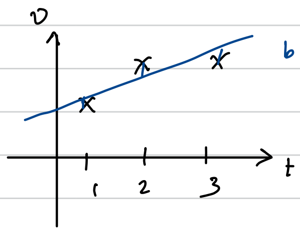
Why, projecting b in whole A (if square invertible) so its I.

Properties P

- Symmetric ($P^T = P$)
- $P^2 = P$.

Eg: Least square Fitting

Fit. (1,1), (2,2), (3,2)



$C + D = 1$

$C + 2D = 2$.

$C + 3D = 2$.

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

$A \quad x \quad = b$.

$\hat{x} = (A^TA)^{-1}A^Tb$.

Rec: Find the orthogonal projection matrix on plane. $x+y-z=0$

$v = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$x+y-z=0.$ $\quad \begin{pmatrix} 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \end{pmatrix}$

$A\hat{x} = p.$

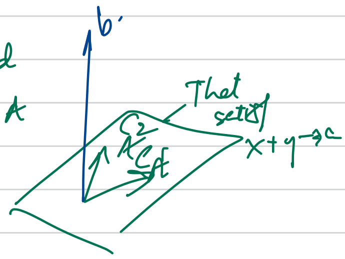
$(b - A\hat{x}) \perp A\hat{x}$

Need to find Col of A

$a_1^T\,(b - A\hat{x}) = 0$

$a_2^T\,(b - A\hat{x}) = 0$

$A = \begin{pmatrix} 1 & 1 \\ -1 & 0 \\ 0 & 1 \end{pmatrix}$



$A^T(b - A\hat{x}) = 0$ $\quad x+y-z=0 \quad x+y-z=0$

$\hat{x} = (A^TA)^{-1}A^Tb$

$p = A\hat{x} = A(A^TA)^{-1}A^T.$

Another way $P_{Normal} = \underset{\sim}{Q}$

$Ib = P_N b + P_p b.$ $\quad$ Now $v = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

$I = P_N + P_p.$

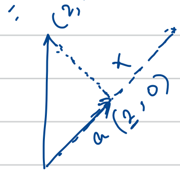
$P_p = I - P_N.$

$P_N = N(N^TN)^{-1}N^T = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \frac{1}{3} \begin{pmatrix} 1 & 1 & -1 \end{pmatrix}$

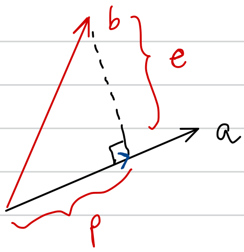
$P_p = I - P_N.$

Lockdown lectures.

$$\begin{bmatrix}2 & 2\end{bmatrix}\begin{bmatrix}2\\0\end{bmatrix} = 4 + 0 = 4.$$



$= 2x$



$b - p = e$

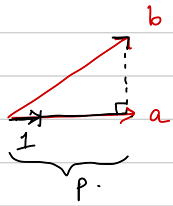
$p = xa$

OK $e \perp p$. and $\underline{e \perp a}$.

$\Rightarrow \underset{\sim}{a}^T(\underset{\sim}{b} - \underset{\sim}{p}) = 0.$

$\Rightarrow \underset{\sim}{a}^T(\underset{\sim}{b} - x\underset{\sim}{a}) = 0$

$\Rightarrow \underset{\sim}{a}^T\underset{\sim}{b} - x\underset{\sim}{a}^T\underset{\sim}{a} = 0$



$\Rightarrow \underset{\sim}{x} = \left(\dfrac{\underset{\sim}{a}^T\underset{\sim}{b}}{\underset{\sim}{a}^T\underset{\sim}{a}}\right) \leftarrow b$

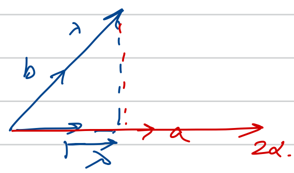
Unit vector in direction a

Projection

$p = xa, \; ax$

$= \left(\dfrac{\underset{\sim}{a}\underset{\sim}{a}^T}{\underset{\sim}{a}^T\underset{\sim}{a}}\right) b$

$\underset{\approx}{P}$



$p = \underset{\approx}{P}\, b$

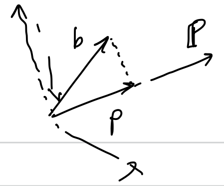
if we double b, p will get doubled $\alpha p = (\quad)\alpha b$

" " " a, does not affect p $= \dfrac{a a^T}{a^T a}$

$= \dfrac{\cancel{\alpha^2}}{\cancel{\alpha^2}}\left(\dfrac{a a^T}{a^T a}\right)$

$P = \underset{\sim}{x a}$
$= \underset{\sim}{a x}$
$=$

$\underset{\sim}{p} = \underset{\sim}{a} x$



$\mathbb{P}$ a.

$\mathbb{P}(b) \rightarrow$ Projection.

$x = \dfrac{a^T b}{a^T a}$

$p = a x \quad = x a = \dfrac{a^T b \, a}{a^T a}$

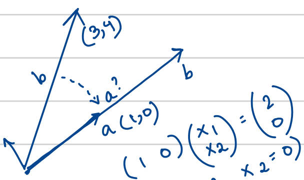
$p = \dfrac{a a^T b}{a^T a}$

$\dfrac{\{a_1 b_1 + a_2 b_2 + a_3 b_3\}}{a_1^2 + a_2^2 + a_3^2} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$

$= \dfrac{1}{a^T a} = \begin{pmatrix} a_1^2 b_1 + a_2 a_1 b_2 + a_1 a_3 b_3 \\ a_2 a_1 b_1 + a_2^2 b_2 + a_2 a_3 b_3 \\ a_3 a_1 b_1 + a_2 a_3 b_2 + a_3^2 b_3 \end{pmatrix}$

$= \dfrac{1}{a^T a} \begin{Bmatrix} a_1^2 & a_1 a_2 & a_1 a_3 \\ a_2 a_1 & a_2^2 & a_2 a_3 \\ a_1 a_3 & a_2 a_3 & a_3^3 \end{Bmatrix} \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix}$

↑ Rank 1.



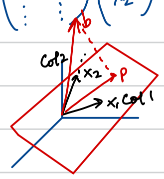
$(1 \;\; 0)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

$(x_1 = 2, \; x_2 = 0)$

$\underset{a}{(1 \;\; 0)}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

x can't solve.

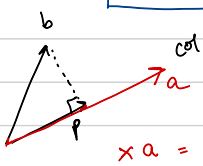
$\begin{pmatrix} \text{Col}_1 & \text{Col}_2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{Bmatrix} b \end{Bmatrix}$



$\underset{\sim}{p} = \mathbb{P}\underset{\sim}{b}$

$(\mathbb{P}\underset{\sim}{b} = \underset{\sim}{p})$

$\boxed{A \hat{\underset{\sim}{x}} = \underset{\sim}{p}}$



$\underset{\sim}{x} a = p.$

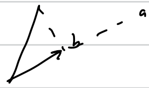
↑

$\underset{\sim}{\mathbb{P}} = \dfrac{\underset{\sim}{a} \underset{\sim}{a}^T}{\underset{\sim}{a}^T \underset{\sim}{a}} \qquad \underset{\sim}{a} = (a_1 \;\; a_2 \;\; a_3)$

$(\;)(\quad)^T$

$\mathbb{P} = \dfrac{1}{(a^T a)} \begin{pmatrix} a_1^2 & a_1 a_2 & a_1 a_3 \\ a_2 a_1 & a_2^2 & a_2 a_3 \\ a_3 a_1 & a_3 a_2 & a_3^2 \end{pmatrix}$

→ Rank $(P) = 1$. $\left(\dfrac{c_1}{a_1} a_2 = c_2 \right)$



→ Col $(\mathbb{P})$ will be in a.

→ $\mathbb{P}$ is symmetric. $\quad \mathbb{P}^2 b = \mathbb{P} b$

$\mathbb{P}^2 = \mathbb{P}$

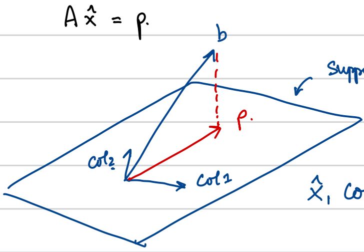
- $\mathbb{P}^T = \mathbb{P}$
- $\mathbb{P}^2 = \mathbb{P}$

Why projection?

$Ax = b.$ ← b is not lying in col of A. (most times)

What we want, we project b on col of A. → p.

$A\hat{x} = p.$



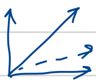
$\hat{x}_1 \,\text{Col}\,1 + \hat{x}\,\text{Col}\,2 \doteq p.$

Sub Reduced modelling.

$Kx = f$

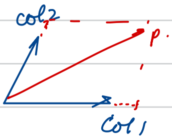
$K\hat{x} = p.$

↑ Subhajit



Q What x, y to get b. ← Exact
↙
x to get bx.

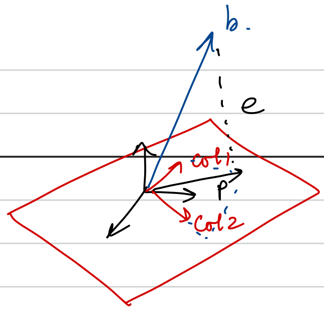
$\hat{x}\;\hat{y}$ to get. b projection.



→ Model Reduction

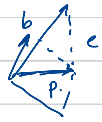
→ Important DOF (Projection over imp DOF)

$A = \begin{pmatrix} \vdots & \vdots \\ a_1 & a_2 \\ \vdots & \vdots \end{pmatrix} \qquad b = \begin{pmatrix} \vdots \\ \vdots \end{pmatrix}$

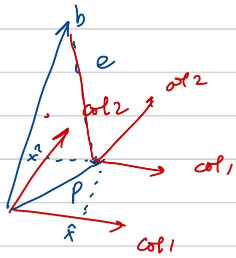


$p = A\hat{x}. \quad = \hat{x}_1 c_1 + \hat{x}_2 c_2$

$e = (b - \underbrace{A\hat{x}}_{p})$



$e = (b - A\hat{x})$



- Columns space ⊥ e.
- Columns ⊥ e.

$A = \begin{pmatrix} \vdots & \vdots \\ c_1 & c_2 \\ \vdots & \vdots \end{pmatrix}$

$\Rightarrow c_1^T e = 0$

$\Rightarrow c_2^T e = 0$

$\Rightarrow \begin{cases} c_1^T \cdot e = 0 \\ c_2^T \cdot e = 0 \end{cases}$

$A^T \cdot e = 0$

$\frac{x \cdot a}{a^T a}$

$A^T (b - A\hat{x}) = 0$

$A^T b - A^T A \hat{x} = 0$

$\underbrace{\hat{x} = (A^T A)^{-1} A^T b.}$ ← Mag of projection.

$\underset{n \times m}{A^T} \; \underset{m \times n}{A}$ ← $\underbrace{A_{n \times n}}$

↳ symme

⇒ +ve definite

⇒ might not be invertible.

$p = A\hat{x}$

$p = \underbrace{A (A^T A)^{-1} A^T}_{P} b.$

A is inventible.

⟺ unique sol'n of b and a.

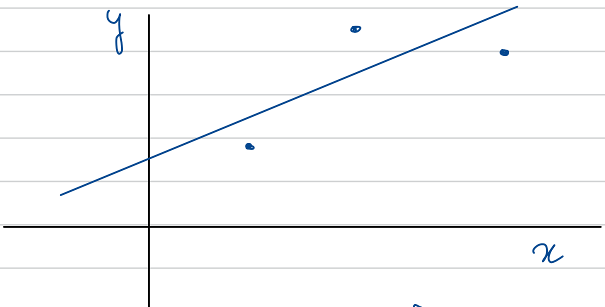
A^{-1} exist!

$P = A A^{-1} A^T A^T b$

$= I b$

A is not invertible

$(A^T A)^{-1}$



$(x,y) = \{(1,1), (2,2), (3,2)\}$.

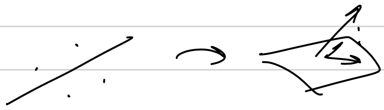
We want to fit: $y = a + bx$

$$a + b = 1$$
$$a + 2b = 2$$
$$a + 3b = 2$$

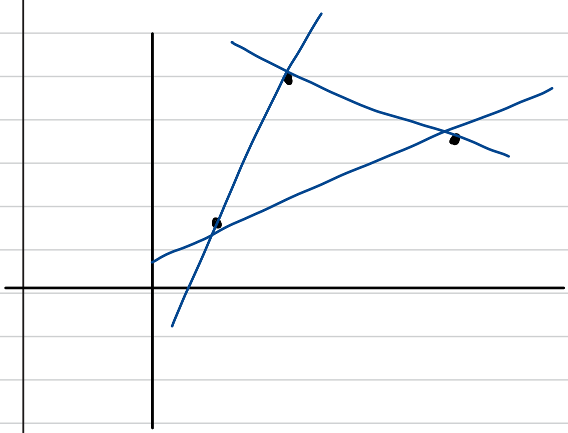
$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

$$\hat{x} = (A^T A)^{-1} A^T b.$$

- Think



- $(A^T A)^{-1}$.



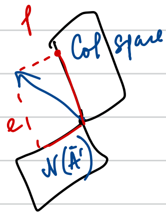
Lecture 16: Projection + Least square.

$$P = A(A^TA)^{-1}A^T$$

- Pb = project b to nearest point on column space (of A)

if $b \perp$ to column space $Pb = 0$ (Null space of A^T) $(P.Ax) = b$.
if b is in column space $Pb = b$ $(A(A^TA)^{-1}A(A^Tb) = 0$

b is in colum space of A
or Ax., $PAx = b$.



$p + e = b$ $\quad Ib - Pb = e$.

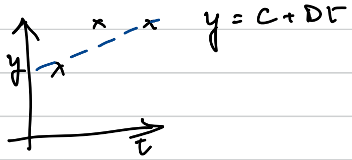
$Pb + (I-P)b = b$

Proj to $\perp$ space

so $e = (I-P)b$.
Eb

P sym - $I-P$ sym
$P^2 = P$ - $(I-P)^2 = I-P$.

Find the best straight line



- Make line & minimize error.

$C + D = 1$
$C + 2D = 2$
$C + 3D = 2$

- No unique solution but there is best solution

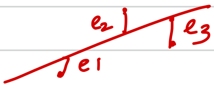
$Ax = b$.

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

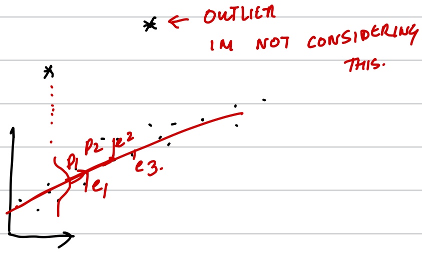
Minimize = Sum of square)
$= (Ax-b) = e$

Small $= \|Ax-b\|^2 = \|e\|^2$ (length).

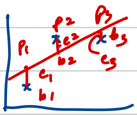
Two pictures
i) 3 point in line



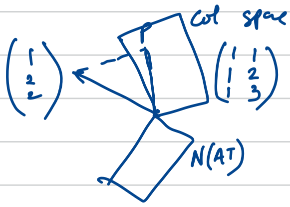
Overall error $e_1^2 + e_2^2 + e_3^2$



→ If my data was way off (OUTLIER).
→ What are pts on line



If I keep (p_1, p_2, p_3) in $Ax = p$.
- p will be in columnspace of A.



Find $\hat{x} = \begin{pmatrix}\hat{C}\\ \hat{D}\end{pmatrix}$, P

- $A^TA\hat{x} = A^Tb$.

- $\begin{bmatrix}3 & 6\\6 & 14\end{bmatrix}\hat{x} = A^Tb$

$\begin{bmatrix}3 & 6\\6 & 14\end{bmatrix}\hat{x} = \begin{bmatrix}5\\11\end{bmatrix}$

$3C + 6D = 5$
$6C + 14D = 11.$

$C + Dx = y$.

$(C + Dx - y)^2 = (\text{error})^2$

$$\begin{pmatrix} x = 1 & y = 1 \\ x = 2 & y = 2 \\ x = 3 & y = 2 \end{pmatrix}$$

Min calculus.

$\mathcal{E} = e_1^2 + e_2^2 + e_3^2$

$= (C + D - 1)^2 + (C + 2D - 2)^2 + (C + 3D - 2)^2$

$\frac{\partial \mathcal{E}}{\partial C} =$ 1st eq $\quad \frac{\partial \mathcal{E}}{\partial D} =$ 2nd eq.

$C = 2/3 \quad D = 1/2$

$\therefore D = 1/2, \quad C = 2/3$.

$P = \frac{2}{3} + \frac{1}{2}x$

Best line $= \frac{2}{3} + \frac{1}{2}x$

$\frac{2}{3} + \frac{1}{2} 3$

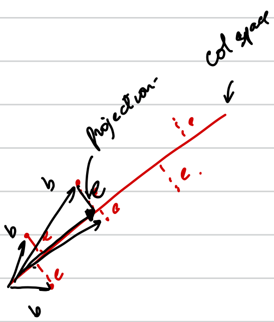
And $e_1 = -1/6 \quad e_2 = +2/6, \quad e_3 - 1/6$

$\frac{2}{3} + \frac{3}{2} = \frac{4 + 9}{6} = \frac{13}{6}$

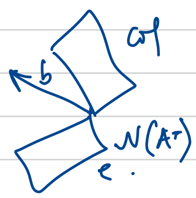
And $e + p = b$. !!!

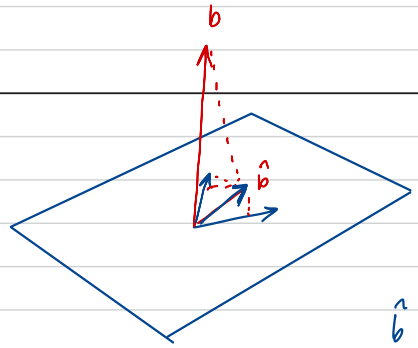
$$\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 7/6 \\ 5/3 \\ 13/6 \end{pmatrix} + \begin{pmatrix} -1/6 \\ 2/6 \\ -1/6 \end{pmatrix} = \frac{1}{36}\begin{pmatrix} -7 + 20 - 13 \end{pmatrix}$$

- e & P are $\perp$. $(e \cdot p = 0)$



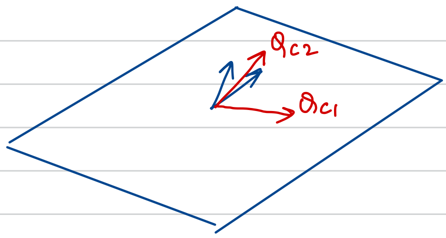
$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}.$$





$$Ax = b.$$
$$A^TA\hat{x} = A^Tb.$$
$$\hat{x} = (A^TA)^{-1}A^Tb.$$

$$\hat{b} = \overbrace{\left(\frac{A^Tb}{A^TA}\right)}^{\hat{x}} A.$$



$A \to Q$

$$Qx = b$$
$$Q^TQ\hat{x} = Q^Tb.$$
$$\hat{x} = Q^Tb.$$

$$Q^T = Q^{-1}$$

A^TA. rank (A^TA) – rank(A)

- Invertible.

↳ If A has independant columns.

To proove ↑

· Suppose $A^TAx = 0$

To show $x = \underset{\sim}{0}$.

IDEA:

$$x^TA^TAx = 0$$

$$(Ax)^T(Ax) = 0$$

$$(y^Ty = 0 \text{ (Length)})$$

$$\therefore Ax = 0$$

· A independant & $Ax = 0$

$\therefore x = \underset{\sim}{0}$.

One case when colums are certain to be independant.

- if they are ⊥ unit vectors.

$$\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$$

· A^Tx Then A^TA → symmetric, invertible

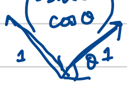
$Ix = 0$

Orthonormal basis

⊥ 1 unit

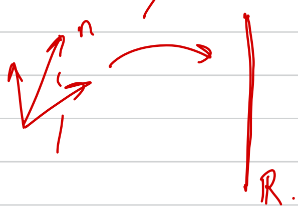
Matrices. (unit)

$\begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}$ $\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}$



$$\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$$

R^n Bilinear



R.

$R^n \to R$.

Metric Tensors

Bilinear form.

$a^T \cdot b = a \, [\;\;] \, b$.

- $a^T [\;\;\;] a = \|a\|$ in some basis.
- $a^T [\;\;\;] b = a \cdot b$. in some basis.

Metric tensor,

→ basis you are in.

$$g = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = \begin{bmatrix} e_1 e_1 & e_1 e_2 \\ e_2 e_1 & e_{22} \end{bmatrix}$$

Basis.

Dot Product $a^T a$.

$(a_1 e_1 + a_2 e_2)(a_1 e_1 + a_2 e_2)$

$= (a_1)^2 e_1 e_1 + 2 a_1 a_2 e_1 \cdot e_2 + (a_2)^2 (e_2)^2$

$a^T (\;\;) a > 0$

length of a in the basis defined by me > 0.

$\frac{1}{2} [u] \, [K] \, [u] \to$ energy term.

$\frac{1}{2} K u^2 \to$ positive. $= 0$.

rec: Find the quadratic eq $(1,1)$ $(2,5)$ $(-1,-2)$ through origin.

$Y = a + bx + cx^2$

Now $0 = a$

$1 = b + c$

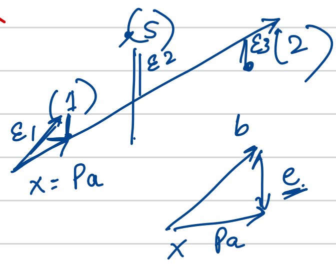
$5 = 2 + 4c$

$-2 = -1 + c$

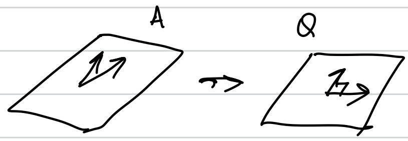
$$\begin{pmatrix} 1 & 1 \\ 2 & 4 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} b \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix}$$

$x = (A^T A)^{-1} A^T b$

$Y = 1\tfrac{1}{2}t - 5/2 t^2$



Lecture 17: Orthogonal Matrices & Gram - Schmidt.



Orthonormal vectors

$$q_i^T q_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

- There dot product $= 0$
- Their length $= 1$.
- They will be independent

How does having othonormal vector nice? Q - Orthonormal matrix

- Suppose $A = (v_1 \; v_2 \; v_3) \xrightarrow{GS} A = (q_1 \; q_2 \; q_3)$

Gram - Schmidt $\cdot \; A \to Q$.

$Q = (q_1 \; q_2 \cdots q_n)$ (Q does not have to be square)

$$Q^T Q = \begin{pmatrix} q_1^T \\ \vdots \\ q_n^T \end{pmatrix} \begin{pmatrix} q_1 & q_2 & \cdots & q_n \end{pmatrix} = I$$

$A^T A >$ (Asks for their dot products)

Orthogonal matrix : Only when its square.

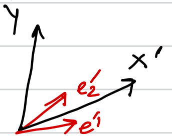
then $Q^T Q = I$, tells us $Q^T = Q^{-1}$

$$(a_1 \;\; a_2) \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = a_1 V_1 B_{11} + a_1 V_2 B_{12} + a_2 V_1 B_{21} + a_2 V_2 B_{22}$$

Sym

$L'(x') = y'$

We want.



$x' \to y'$

But we know $x \xrightarrow{[L]} y$.

Le

$[\;][\;][\;](w)_{e'}$

Le

$[\;][L_e]_{we} \; we$.

Ve

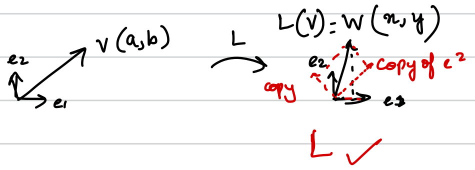
$[\;]_{ve} \to ve'$ in e' basis

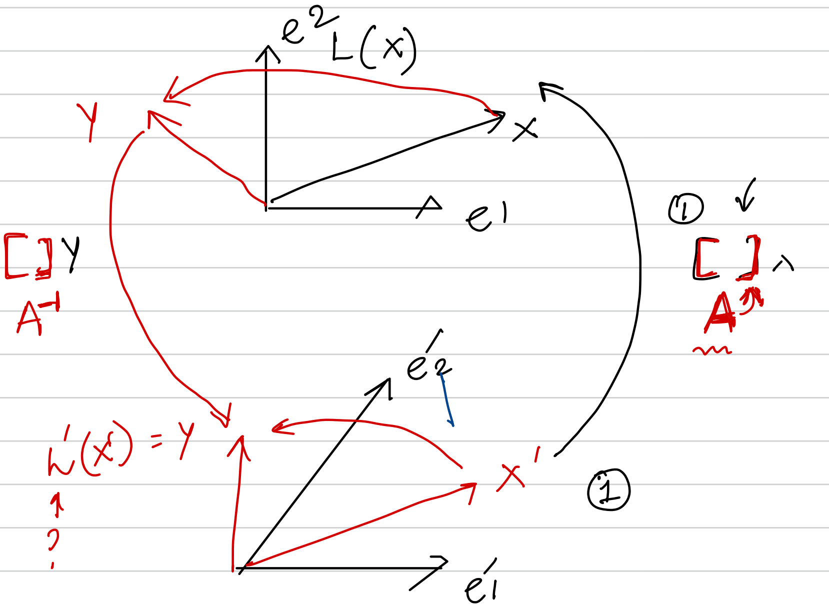
(L_e)

$[\;][\;][\;]$

Le

Linear maps change space, NOT THE BASIS





L linear map.
ansform.

$v = \bigcirc e_1 + \bigcirc e_2$

$v = \bigcirc e_1' + \bigcirc e_2'$

[ ] X' gives components of vector X in $e_1\, e_2$.

Inverse

$[\ ]^{-1} [L] [\ ]$

= What happens if [ ] is orthogonal?
Transform

Orthonormal matrix:

Example: perm $Q = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

1)

$$Q^T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & \cancel{0} & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$QQ^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

2) $Q = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

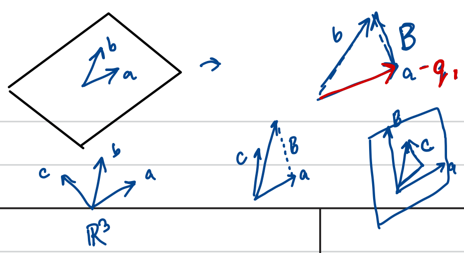
3) $Q = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ $\quad 1.1 + 1 \cdot -1 = 0$

$(1)^2 + (1)^2 = \sqrt{2} / \sqrt{2} = 1$

4) $Q = \frac{1}{\sqrt{4}} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$

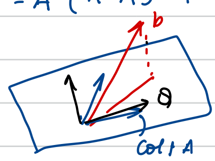
5) $Q = \frac{1}{\sqrt{9}} \begin{pmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{pmatrix}$

Gram schmidt (square roots)



. Q has orthonormal colums.

· Project onto its column space



$$P = Q(Q^TQ)^{-1}Q^T$$
$$= QQ^T$$

Suppose matrix is square, then Q is the whole space:

then $P \to I$. (if Q is square)

$P \to$ symm, $P^2 = P$

• $\underbrace{(QQ^T)^T}_{P}\underbrace{(QQ^T)}_{P} = QQ^T$.

Eq become simple

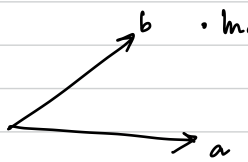
$A^TA\hat{x} = A^Tb$.

• $Q^TQ\,\hat{x} = Q^Tb$.

• $\hat{x} = Q^Tb$. (Very close the form of Subhrajit's paper).

$\hat{x}_i = q_i^T b$.

Gram-schmidth (Goal - Make matrix orthonormal)



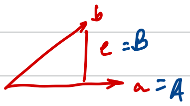
• Independant vectors a, b.

• I take the first a.

• b should be orthogonal to a.

Orthogonal vec (A, B)

$q_1 = \frac{A}{\|A\|}$ $q_2 = \frac{B}{\|B\|}$



e vector is $\perp$ to a

$$e = b - p b$$

- b is independant so we will have e.

$$e = b - \frac{A^T b}{A^T A} A$$

$$B = b - \frac{A^T b}{A^T A} A.$$

Check for $\perp$

$$A^T B = A^T\left(b - \frac{A^T b A}{A^T A}\right)$$

$$\cancel{A^T} b - \frac{\cancel{A^T} b A}{A^T A} = 0$$

Check 3D,

if I have independent vectors a, b, c

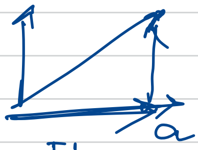
→ Find orthogonal A, B, C.

$$q_1 = \frac{A}{\|A\|} \quad q_2 = \frac{B}{\|B\|}, \quad q_3 = \frac{C}{\|C\|}$$

- I know A and B.

$$C = c - \frac{A^T c A}{A^T A} - \frac{B^T c B}{B^T B}$$ (· Subtract components in 2 directions)

Eg: $a = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $b = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$



$\frac{a^Tb}{a^Ta}$

$A = a$

$$B = b - \frac{a^T b a}{a^T a}$$

$$= \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} - \frac{3}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \sim B$$

$A + B = 0$

$$Q = \begin{pmatrix} 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & -1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{2} \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 2 \end{pmatrix} \rightarrow Q.$$

How is col (Q) related to col (A)

- They are the same!

- In matrix we get $A = LU$. $\qquad (E)\overset{E^{-1}=L}{A} \to U.$

- So we need to find $A = QR$

LL^T

Gram schmidth.

$$A = \begin{pmatrix} a_1, a_2 \end{pmatrix} = \begin{pmatrix} q_1 & q_2 \end{pmatrix} \begin{pmatrix} a_1^T q_1 & a_2^T q_2 \\ a_1^T q_1 & a_2^T q_2 \end{pmatrix}$$

R ⇒ upper triagle

0 because $a_1 = q_1$

$A = QR$ (R is triangular matrix)

sec: Find q_1, q_2, q_3 (orthonormal) from a, b, c (colums of A). Then write A as QR. (Q orthagonal, R upper triangle)

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 3 & 6 \end{pmatrix}$$

a b c are the orthonormal vectors.

$$a = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad b = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} - \frac{(1\,0\,0)\begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}}{(1\,0\,0)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$$

$$c = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} - \frac{(1\,0\,0)\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}}{1} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \frac{(0\,0\,3)\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}}{9} \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$$

$A = QR$.

$$\begin{array}{c} \begin{pmatrix} 1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 3 & 6 \end{pmatrix} \\ a \quad b \quad c \end{array} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}}_{\text{Permutation}} \begin{pmatrix} 1 & 2 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 5 \end{pmatrix}$$

$a = q_1$

$b = 2q_1 + 3q_2$

$c = 4q_1 + 6q_2 + 5q_3$

3616 :

[ ]

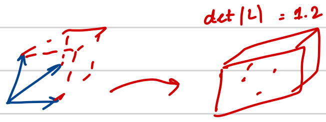
Linear map.

$\mathbb{R}^n \xrightarrow{L:} \mathbb{R}^n$

$L \neq I$ det = 1

det = -1.

det |L| = 1.2



$R^m \to R^n$.

no of cols   no of rows.

det = 0

3D → 2D

2D → 3D

[ ]

[ ] → Change of

[ ] → Metric

$\alpha L \cdot \cdot = L$

(α) $L : \alpha = 2$

Lec 18 : Properties of determinants

- Every square matrix has a determinant.
- det A = |A|.
- Matrix is singular if det A = 0

Properties:

- det I = 1 $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$
- det Exchange rows : Reverse sign of det. (det of permutation $= 1$ even, -1 odd)
  $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$
- $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ $\begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix}$, → t

These prop will give rule for $n \times n$ matrix

$\begin{bmatrix} ta & tb \\ c & d \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} = x \begin{Bmatrix} ta \\ c \end{Bmatrix} + y \begin{Bmatrix} t \\ d \end{Bmatrix}$

- $\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

- $\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$

Det is a linear func of 1^{st} row if others same.

- Not saying $\det(A+B) \neq \det A + \det B$.

⇒ LINEAR FOR EACH ROW.

- 2 equal rows ⇒ det = 0

⇒ Use prop 2, exchange rows ⇒ same matrix, same det.
but 2 is wrong so det = 0.

- Subtract $l \times$ row i from row k (Gauss elimination).

⇒ Determinant does not change.

$\Rightarrow \begin{vmatrix} a & b \\ c & d \end{vmatrix} \xrightarrow[-lR_1]{R_2 - R_2} \begin{vmatrix} a & b \\ c-la & d-lb \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a & b \\ -la & -lb \end{vmatrix}$ (3b)

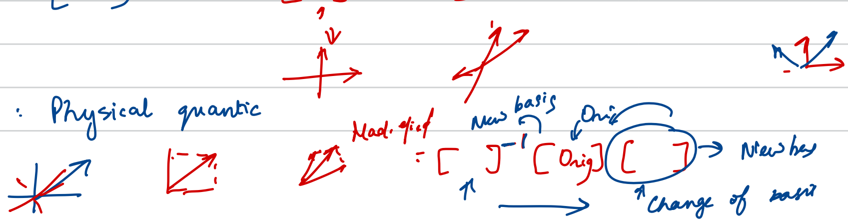
$= \begin{vmatrix} a & b \\ c & d \end{vmatrix} - l \begin{vmatrix} a & b \\ a & b \end{vmatrix}$

$= \begin{vmatrix} a & b \\ c & d \end{vmatrix} - 0.$

[ ] → [ ]

Defined certain basis

[ ] → [ ]



: Physical quantity

Mod. quantity New basis Orig

$= [\;]^{-1} [\text{Orig}] ([\;]) \to$ New basis

Change of basis

- Row of zero $\Rightarrow \det(A) = 0$

$$\begin{vmatrix} a & b \\ 0 & 0 \end{vmatrix} \Rightarrow \begin{vmatrix} a & b \\ a & b \end{vmatrix} \Rightarrow 0$$

or $\begin{vmatrix} ta+tb \\ 0 \quad 0 \end{vmatrix} \Rightarrow t \begin{vmatrix} a & b \\ 0 & 0 \end{vmatrix}$

- $U = \begin{pmatrix} d_1 & * & \cdot & \cdot \\ & d_2 & \cdot & \cdot \\ & & d_3 & \cdot \\ 0 & & & d_4 \end{pmatrix} = d_1 \times d_2 \times d_3 \ldots \times d_n$

Elimination, then product of pivots. if there is no row exchange.

- $= d_1 d_2 d_3 d_4 \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix} = d_1 d_2 d_3 d_4 . 1.$

What if one $d = 0$, the $|\det| = 0$.

- $\det A = 0$. when A is singular.
- $\det A \neq 0$ " " " invertible

$A \Rightarrow U \Rightarrow d_1 d_2 d \ldots d_n$.

Eg $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \Rightarrow \begin{vmatrix} a & b \\ 0 & d - c/a\, b \end{vmatrix}$ det - $(ad - bc)$

a $\Rightarrow$ should not be zero, exchange.
$\Rightarrow$ if it doesn't work - singular.

- $\det AB = \det A . \det B$.

$\det A^{-1}$

$A^{-1} A = I$

- $\det(A^{-1} A) = 1$.
- $(\det A^{-1})(\det A) = 1$

$$\det A^{-1} = \frac{1}{\det A}.$$

(if $A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ $A^{-1} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/3 \end{pmatrix}$, proof for diag)

$\det A^2 = (\det A)^2$

$\det(2A) = 2^n \det A$. (2 is factored from every row)

Eg: If I have a box, increase each side by 2.
Then volume $= 2^n$ (n dim)
$= 8$ (3D)

∄ A^{-1} then $\det A = 0$
so $\det A^{-1} = 1/\det A$ is void.

- $\det A^T = \det A$.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix}$$

So if a column is zero. $= 0$.
And if I exchange column $=$ change sign.

$|A^T| = |A|$

$|U^T L| = |LU|$

$|U^T||L| = |L||U|$ $\quad L = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$

$\det U^T = \det U$

- If you do row exchange - change sign.
  - $\Rightarrow$ 7 row exchange $-$ $-$ve
  - $\Rightarrow$ 10 (even) $-$ $+$ve

Properties of determinants

Determinants

Now halfway through the course, we leave behind rectangular matrices and focus on square ones. Our next big topics are determinants and eigenvalues.

The *determinant* is a number associated with any square matrix; we'll write it as $\det A$ or $|A|$. The determinant encodes a lot of information about the matrix; the matrix is invertible exactly when the determinant is non-zero.

A is singular $|A| = 0$

A : change in area

Properties

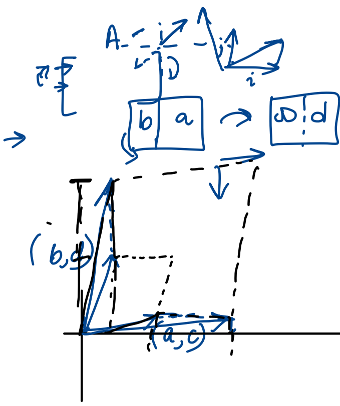
Rather than start with a big formula, we'll list the properties of the determinant. We already know that $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$; these properties will give us a formula for the determinant of square matrices of all sizes.

det = 1

det = >1

1. $\det I = 1$
2. If you exchange two rows of a matrix, you reverse the sign of its determinant from positive to negative or from negative to positive.
3. (a) If we multiply one row of a matrix by t, the determinant is multiplied by t: $\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t \begin{vmatrix} a & b \\ c & d \end{vmatrix}$. $\qquad \begin{vmatrix} ta & tb \\ tc & td \end{vmatrix} = t^2 |A|$

   (b) The determinant behaves like a linear function on the rows of the matrix:
$$\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}.$$



Property 1 tells us that $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$. Property 2 tells us that $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$. The determinant of a permutation matrix P is 1 or -1 depending on whether P exchanges an even or odd number of rows.

From these three properties we can deduce many others:

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ det = (-1)

4. If two rows of a matrix are equal, its determinant is zero.

   $\begin{pmatrix} a & b \\ a & b \end{pmatrix} = \times$

   This is because of property 2, the exchange rule. On the one hand, exchanging the two identical rows does not change the determinant. On the other hand, exchanging the two rows changes the sign of the determinant. Therefore the determinant must be 0.

5. If $i \neq j$, subtracting t times row i from row j doesn't change the determinant.

$\alpha(a+b) = (\alpha a + \alpha b)$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$(u, \alpha + u)$ $(\text{Row}_1, \text{Row}_2, \text{Row}_3)$ $\alpha + y$

In two dimensions, this argument looks like:

$$\begin{aligned}
\begin{vmatrix} a & b \\ c-ta & d-tb \end{vmatrix} &= \begin{vmatrix} a & b \\ c & d \end{vmatrix} - \begin{vmatrix} a & b \\ ta & tb \end{vmatrix} \quad \text{property 3(b)} \\
&= \begin{vmatrix} a & b \\ c & d \end{vmatrix} - t \begin{vmatrix} a & b \\ a & b \end{vmatrix} \quad \text{property 3(a)} \\
&= \begin{vmatrix} a & b \\ c & d \end{vmatrix} \quad \text{property 4.}
\end{aligned}$$

$= \alpha(\text{Row}_1, \alpha\text{Row}, \text{Row}_3)$
$+ \beta(\text{Row}_1, \beta\text{Row}, \text{Row}_3)$

The proof for higher dimensional matrices is similar.

6. If A has a row that is all zeros, then $\det A = 0$.

   We get this from property 3 (a) by letting $t = 0$.

   $\begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} = t^{=0} \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

7. The determinant of a triangular matrix is the product of the diagonal entries (pivots) $d_1, d_2, ..., d_n$.

   Property 5 tells us that the determinant of the triangular matrix won't change if we use elimination to convert it to a diagonal matrix with the entries d_i on its diagonal. Then property 3 (a) tells us that the determinant of this diagonal matrix is the product $d_1 d_2 \cdots d_n$ times the determinant of the identity matrix. Property 1 completes the argument.

   Note that we cannot use elimination to get a diagonal matrix if one of the d_i is zero. In that case elimination will give us a row of zeros and property 6 gives us the conclusion we want.

$\begin{pmatrix} -1 & 0 & 0 \\ 4 & 2 & 3 \\ -5 & 6 & 7 \end{pmatrix}$

$= \begin{pmatrix} p & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & d \end{pmatrix} = p \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & d \end{pmatrix} = pcd \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

8. $\det A = 0$ exactly when A is singular.

   If A is singular, then we can use elimination to get a row of zeros, and property 6 tells us that the determinant is zero.

   If A is not singular, then elimination produces a full set of pivots $d_1, d_2, ..., d_n$ and the determinant is $d_1 d_2 \cdots d_n \neq 0$ (with minus signs from row exchanges).

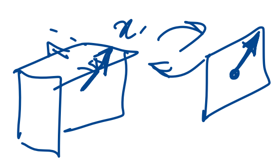
$A = \begin{pmatrix} \ddots & \\ 0\,0\,0\,0 \end{pmatrix}$

$r = n = m \quad R = I$

$r = n \quad R = \begin{bmatrix} I \\ 0 \end{bmatrix}$

We now have a very practical formula for the determinant of a non-singular matrix. In fact, the way computers find the determinants of large matrices is to first perform elimination (keeping track of whether the number of row exchanges is odd or even) and then multiply the pivots:

$Ax = b$



$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longrightarrow \begin{bmatrix} a & b \\ 0 & d - \frac{c}{a}b \end{bmatrix}, \text{ if } a \neq 0, \text{ so}$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = a\left(d - \frac{c}{a}b\right) = ad - bc.$$

9. $\det AB = (\det A)(\det B)$

   This is very useful. Although the determinant of a sum does not equal the sum of the determinants, it is true that the determinant of a product equals the product of the determinants.

$\det(A+B) \neq \det(A) + \det(B)$

$A^{-1}A = I$

$\det(A^{-1}A) = \det I$

$\det(A^{-1})(\det A) = 1.$

$\left(A^{-1} = \frac{1}{\det A}.\right)$

$\det 2A.$

$= \begin{vmatrix} 2a & 2b \\ 2c & 2d \end{vmatrix}$

$= 2^2 \det A$

$= 2^{n\ \text{rows}} \det A.$

For example:

$$\det A^{-1} = \frac{1}{\det A},$$

because $A^{-1}A = 1$. (Note that if A is singular then A^{-1} does not exist and $\det A^{-1}$ is undefined.) Also, $\det A^2 = (\det A)^2$ and $\det 2A = 2^n \det A$ (applying property 3 to each row of the matrix). This reminds us of volume – if we double the length, width and height of a three dimensional box, we increase its volume by a multiple of $2^3 = 8$.

10. $\det A^T = \det A$

$\det(A^T) = A^T$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - bc.$$

This lets us translate properties (2, 3, 4, 5, 6) involving rows into statements about columns. For instance, if a column of a matrix is all zeros then the determinant of that matrix is zero.

To see why $|A^T| = |A|$, use elimination to write $A = LU$. The statement becomes $|U^TL^T| = |LU|$. Rule 9 then tells us $|U^T||L^T| = |L||U|$.

Matrix L is a lower triangular matrix with 1's on the diagonal, so rule 5 tells us that $|L| = |L^T| = 1$. Because U is upper triangular, rule 5 tells us that $|U| = |U^T|$. Therefore $|U^T||L^T| = |L||U|$ and $|A^T| = |A|$.

det.

$[u]$

$\begin{bmatrix} 1 & 2 & 3 \\ & 4 & 5 \\ & & 6 \end{bmatrix}$

Row exchange

$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 5 & 6 \end{bmatrix}$

We have one loose end to worry about. Rule 2 told us that a row exchange changes the sign of the determinant. If it's possible to do seven row exchanges and get the same matrix you would by doing ten row exchanges, then we could prove that the determinant equals its negative. To complete the proof that the determinant is well defined by properties 1, 2 and 3 we'd need to show that the result of an odd number of row exchanges (odd permutation) can never be the same as the result of an even number of row exchanges (even permutation).

1. $\det I = 1.$
2. Flip, change in sign when change rows.
3. a. linearity in rows. $t|\ | = t$

   b.

- $\det A^{-1} = \frac{1}{\det A}$
- $\det(AB) = \det(A)(B)$

  $\det(A^{n}) = n^{\text{rows}} \det A$
- $\det(A^T) \overset{\det}{=} A.$

ex: $A = \begin{pmatrix} 101 & 201 & 301 \\ 102 & 202 & 302 \\ 103 & 203 & 303 \end{pmatrix} = \begin{pmatrix} 101 & 201 & 301 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = 0$

$B = \begin{pmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{pmatrix} =$

$D = \begin{pmatrix} 0 & 1 & 3 \\ -1 & 0 & 4 \\ -3 & -4 & 0 \end{pmatrix}$

$C = \begin{vmatrix} 1 \\ 2 \\ 3 \end{vmatrix} (1 \; -4 \; 5)$ $\quad$ det (0)

pull for each row

$|D^T| = |-D| = (-1)^3 |D|$

$|D| = 0$

even : ok $\quad$ odd = -1.

Lect 19: Determinant formula and cofactors.

- Cofactor, Triangonal matrics

① $\det I = 1$

② $\det(\text{exch})$ $\begin{matrix} \text{even} = 1 \\ \text{odd} = -1 \end{matrix}$

③ det linear with each row

- $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix} = \underbrace{\begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix}}_{=0} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix}$

$+ \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} + \underbrace{\begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix}}_{=0}$

$= \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \underbrace{\begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix}}_{\text{Flip} \cdot -ve} = (ad - bc)$

- $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & 0 & 0 \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} 0 & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$

$= \begin{vmatrix} a & 0 & 0 \\ 0 & e & 0 \\ g & h & i \end{vmatrix} + \begin{vmatrix} a & 0 & 0 \\ d & 0 & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} \; \end{vmatrix} \; \cdots$

Survivor : $\begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \end{vmatrix} \cdots = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{33}$

... det formula

Whats used:

$$\begin{vmatrix} a_{11} & & \\ & a_{22} & \\ & & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \\ a_{31} & 0 & 0 \end{vmatrix}$$

$\bar{4}$

a-ve: One Row ex $-$ $+$

to get to I

To get I
even - +ve
odd - -ve

$$+ \begin{vmatrix} 0 & 0 & a_{12} \\ 0 & a_{23} & 0 \\ a_{31} & 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ a_{24} & 0 & 0 \\ 0 & a_{31} & 0 \end{vmatrix}$$

$-$ $+$

BIG FORMULA:

$\det A = \sum \pm a_{1\alpha} a_{2\beta} a_{3\gamma} \ldots a_{n\omega}$
$n!$ terms $(\alpha \beta, \ldots \omega)$ = Perm of $(1, 2 \ldots)$

$n!$ terms - n ways + $(n-1)$ ways + $(n-2)$
$\rightarrow$

Eg:
$$\begin{vmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{vmatrix}$$
· 24 terms $n!$ $4 \times 3 \times 2 = 24$

det $(4,2,3,1)$ $= +1$ det $(3,2,1,4)$ -1 $= 0$

$22 = 0.$

Cofactors: $n \times n \to$ smaller.

Eg 3×3

$$\det = a_{11}(a_{22}a_{33} - a_{23}a_{32}) + a_{12}(\quad) + a_{13}(\quad).$$

$\downarrow$ determinant of smaller eg

$$\begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & a_{23} \\ a_{31} & 0 & a_{33} \end{vmatrix}$$

$$- a_{12}(a_{21}a_{33} - a_{23}a_{31})$$

Cofactor of $a_{ij} = \pm \det$ ($n-1$ matrix with row i, col j erased)

$= \pm C_{ij}$

$(+)$ $i+j$ = even
$(-)$ $i+j$ = odd.

Cofactor formula.

$$\boxed{\det = a_{11}C_{11} + a_{12}C_{12} + a_{1n}C_{1n}.}$$

Eg $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

Eg: A_4: $\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$ $|A_1| = 1$ $|A_2| = 0$ $|A_3| = -1$

$\det A = 1\,|A_3| - 1\,|0| = -1$

$|A_n| = |A_{n-1}| - |A_{n-2}|$

$|A_5| = |A_4| - |A_3|$
$= 0$

$|A_6| = 1$ $\quad |A_7| = 1$

1
0
-1
1
0
1
1

ex: Find determinants of

$$A = \begin{pmatrix} x & y & 0 & 0 & 0 \\ 0 & x & y & 0 & 0 \\ 0 & 0 & x & y & 0 \\ 0 & 0 & 0 & x & y \\ y & 0 & 0 & 0 & x \end{pmatrix} \to \begin{pmatrix} x & y & 0 & 0 & 0 \\ 0 & x & y & 0 & 0 \\ 0 & 0 & x & y & 0 \\ 0 & 0 & 0 & x & y \\ 0 & -\frac{y^2}{x} & 0 & 0 & x \end{pmatrix} \to \begin{pmatrix} x & y & 0 & 0 & 0 \\ 0 & x & y & 0 & 0 \\ 0 & 0 & x & y & 0 \\ 0 & 0 & 0 & x & y \\ 0 & 0 & \frac{y^3}{x} & 0 & x \end{pmatrix}$$

$y - \frac{y}{x}$

$-\frac{y^2}{x} + \frac{y^2}{x}$

$$\downarrow$$

$$\begin{pmatrix} & & & x & y \\ 0 & 0 & 0 & -\frac{y^4}{x} & x \end{pmatrix}$$

$-\frac{y^4}{x} + \frac{y^4}{x}$ $\quad x + \frac{y^5}{x}$

Inside

Also $= x(x^4) + y(y^4)$

$$B = \begin{pmatrix} x & y & y & y & y \\ y & x & y & y & y \\ y & y & x & y & y \\ y & y & y & x & y \\ y & y & y & y & x \end{pmatrix} - \left(\quad \right)$$

↓ Do elimination.

Lecture 20:

- Formula for A^{-1}
- Cramers Rule for $x = A^{-1}b$
- $|\text{Det } A|$ = Volume of box.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Think cofactors.

$$A^{-1} = \frac{1}{\det} C^T$$

- Why is it?

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = C^T \leftarrow \text{product of } n-1 \text{ entries.}$$

det $\leftarrow$ product of n entries

- Check $AA^{-1} = (\det A) I$?

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{1n} & \cdots & a_{nn} \end{bmatrix}\begin{bmatrix} C_{11} & & C_{n1} \\ C_{12} & & \vdots \\ \vdots & & \\ C_{1N} & & C_{nn} \end{bmatrix} = \text{cofactor formula for the det.}$$

$$= \begin{bmatrix} \det A & & 0 \\ & \det A & \\ 0 & & \det A \end{bmatrix}$$

why is it

if $i \neq j$ we get 0

- $$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$A_s = \begin{pmatrix} a & b \\ a & b \end{pmatrix}$$

$$\det A_s = ab + b(-a) = 0$$

Wrong column

$Ax = b$

- $x = A^{-1} b$

$$= \frac{1}{\det A} C^T b$$

- Play with it further. Cramer's rule:

$x : \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix}$ $\quad x_1 = \frac{\det B_1}{\det A}$ $\quad$ (Multiply cofactor x something – det)

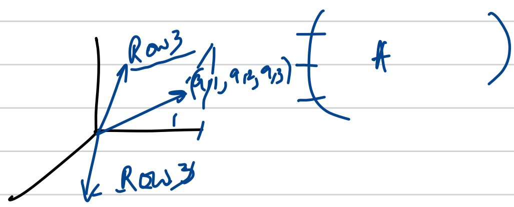
$x_2 = \frac{\det B_2}{\det A}$ .... $\underline{\underline{C^T b}}$.

$B_1 = \begin{bmatrix} b & n-1 \text{ col of } A \end{bmatrix}$ $\quad$ A with col 1 by b.

B_j = A with C_j replaced by b.

(3x3)

det A gives the volume of box



Volume = $|\det A|$

If det = -ve. (LHanded Box)

(R " ")

$A = I$

$\det A = 1$

- We can prove if vol has 3 rules. then its the determinant

Orthogonal matrix

$A = Q$

$\det|A| = 1$ (Rotated cube in space)

Prop: $Q^T Q = I$

$\det(Q^T)\det(Q) = 1.$

$\det|Q|^2 = 1$

$= 1$ or -1

• Double edge

Volume doubles, Det doubles (Rule 3a)

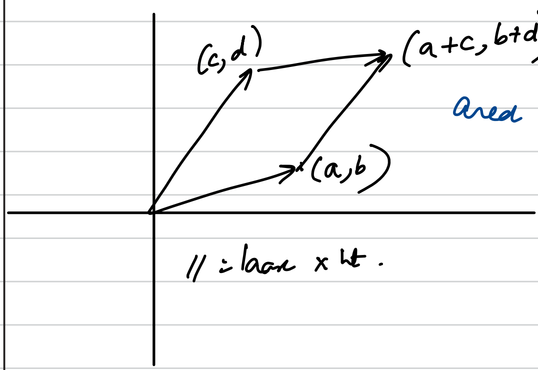
One row can factor.

• Rule

$(\det A) = $ vol of box $-(1, 2, 3a, 3b)$

Don't care

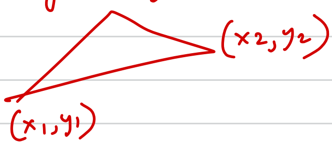
3b $\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$



area $= \det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

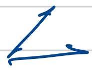
// = base × ht.

Triangle



$$\text{area} = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$\begin{pmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{pmatrix}$$



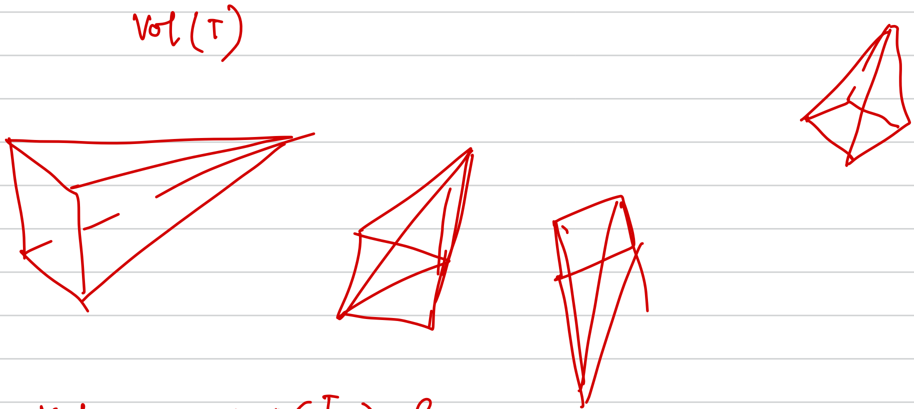
$$(x_2 - x_1)(y_3 - y_1) - (y_2 - y_1)(x_3 - x_1)$$

$$= x_2y_3 - x_2y_1 - x_1y_3 + x_1y_1 - y_2x_3 + y_2x_1 + y_1x_3 - y_1x_1$$

$$= (x_2y_3 - y_2x_3) - (x_1y_3 - y_1x_3) + (x_1y_2 - x_2y_1)$$

Eg Tetrahedron $O(0,0,0)$ $A(2,2,-1)$ $A_2(1,3,0)$ $A_3(-1,1,4)$

Vol(T)



$$\text{Vol } T = \frac{1}{3} A(\text{base})\, h.$$

$$\text{Vol } P = 2A(\text{Tria base})\, h$$

$$\text{Vol } T = \frac{1}{6}\text{Vol } P = \frac{1}{6}\begin{pmatrix} 2 & 2 & -1 \\ 1 & 3 & 0 \\ -1 & 1 & 4 \end{pmatrix} = 2$$

Determinant formulas and cofactors

Now that we know the properties of the determinant, it's time to learn some (rather messy) formulas for computing it.

Formula for the determinant

We know that the determinant has the following three properties:

1. $\det I = 1$
2. Exchanging rows reverses the sign of the determinant.
3. The determinant is linear in each row separately.

Last class we listed seven consequences of these properties. We can use these ten properties to find a formula for the determinant of a 2 by 2 matrix:

$$\begin{aligned}
\begin{vmatrix} a+0 & b+0 \\ c & d \end{vmatrix} &= \begin{vmatrix} a & 0 \\ c+0 & 0\text{ or } d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c+0 & 0\text{ or } d \end{vmatrix} \\
\begin{vmatrix} a+0 & 0+b \\ c & d \end{vmatrix} &= \begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} + \begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix} \\
&= 0 + ad + (-cb) + 0 \\
&= ad - bc.
\end{aligned}$$

By applying property 3 to separate the individual entries of each row we could get a formula for any other square matrix. However, for a 3 by 3 matrix we'll have to add the determinants of twenty seven different matrices! Many of those determinants are zero. The non-zero pieces are:

$$\begin{aligned}
\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} &= \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & a_{33} \end{vmatrix} \\
&+ \begin{vmatrix} 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \\ a_{31} & 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & 0 & 0 \\ 0 & a_{32} & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & 0 \end{vmatrix} \\
&= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{33} - a_{12}a_{21}a_{33} \\
&\quad + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}.
\end{aligned}$$

$\rightarrow a_{11}(a_{22}a_{33} - a_{23}a_{32}) +$

$(3,1,2) =$ even.

$\#(1,3,2)$

Each of the non-zero pieces has one entry from each row in each column, as in a permutation matrix. Since the determinant of a permutation matrix is either 1 or -1, we can again use property 3 to find the determinants of each of these summands and obtain our formula.

One way to remember this formula is that the positive terms are products of entries going down and to the right in our original matrix, and the negative terms are products going down and to the left. This rule of thumb doesn't work for matrices larger than 3 by 3.

3 ways.

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} a & 0 & 0 \\ d & e & f \\ g & h & i \end{pmatrix} + \begin{pmatrix} 0 & b & 0 \\ d & e & f \\ g & h & i \end{pmatrix} + \begin{vmatrix} 0 & 0 & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

$[\ ]_{2\times 2} \rightarrow$ 2 Non zero

The number of parts with non-zero determinants was 2 in the 2 by 2 case, 6 in the 3 by 3 case, and will be $24 = 4!$ in the 4 by 4 case. This is because there are n ways to choose an element from the first row (i.e. a value for α), after which there are only $n-1$ ways to choose an element from the second row that avoids a zero determinant. Then there are $n-2$ choices from the third row, $n-3$ from the fourth, and so on.

The big formula for computing the determinant of any square matrix is:

$$\det A = \sum_{n! \text{ terms}} \pm a_{1\alpha} a_{2\beta} a_{3\gamma} \ldots a_{n\omega}$$

where $(\alpha, \beta, \gamma, \ldots \omega)$ is some permutation of $(1, 2, 3, \ldots, n)$. If we test this on the identity matrix, we find that all the terms are zero except the one corresponding to the trivial permutation $\alpha = 1$, $\beta = 2$, ..., $\omega = n$. This agrees with the first property: $\det I = 1$. It's possible to check all the other properties as well, but we won't do that here.

Applying the method of elimination and multiplying the diagonal entries of the result (the pivots) is another good way to find the determinant of a matrix.

$\Sigma\, a_{13}\, a_{22}\, a_{31}\, a_{44}$

$+\, a_{14}\, a$

Example

In a matrix with many zero entries, many terms in the formula are zero. We can compute the determinant of:

$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

by choosing a non-zero entry from each row and column, multiplying those entries, giving the product the appropriate sign, then adding the results.

The permutation corresponding to the diagonal running from a_{14} to a_{41} is $(4, 3, 2, 1)$. This contributes 1 to the determinant of the matrix; the contribution is positive because it takes two row exchanges to convert the permutation $(4, 3, 2, 1)$ to the identity $(1, 2, 3, 4)$.

Another non-zero term of $\sum \pm a_{1\alpha} a_{2\beta} a_{3\gamma} a_{4\omega}$ comes from the permutation $(3, 2, 1, 4)$. This contributes -1 to the sum, because one exchange (of the first and third rows) leads to the identity.

These are the only two non-zero terms in the sum, so the determinant is 0. We can confirm this by noting that row 1 minus row 2 plus row 3 minus row 4 equals zero.

Cofactor formula

The cofactor formula rewrites the big formula for the determinant of an n by n matrix in terms of the determinants of smaller matrices.

In the 3×3 case, the formula looks like:

$$\det A = a_{11}(a_{22}a_{33} - a_{23}a_{32}) + a_{12}(-a_{21}a_{33} + a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

1 row separate term

$$= \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & a_{23} \\ a_{31} & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 0 \end{vmatrix}$$

This comes from grouping all the multiples of a_{ij} in the big formula. Each element is multiplied by the *cofactors* in the parentheses following it. Note that each cofactor is (plus or minus) the determinant of a two by two matrix. That determinant is made up of products of elements in the rows and columns NOT containing a_{1j}.

$$\begin{vmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix}$$

In general, the cofactor C_{ij} of a_{ij} can be found by looking at all the terms in the big formula that contain a_{ij}. C_{ij} equals $(-1)^{i+j}$ times the determinant of the $n-1$ by $n-1$ square matrix obtained by removing row i and column j. (C_{ij} is positive if $i+j$ is even and negative if $i+j$ is odd.)

$(-1)^{i+j}$

$i+j$: even $= 1$.
odd $= -1$

For $n \times n$ matrices, the cofactor formula is:

$$\boxed{\det A = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}.}$$

Applying this to a 2×2 matrix gives us:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad + b(-c).$$

Tridiagonal matrix

A *tridiagonal matrix* is one for which the only non-zero entries lie on or adjacent to the diagonal. For example, the 4×4 tridiagonal matrix of 1's is:

$C_{11} + C_{12}$

$$A_4 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

What is the determinant of an $n \times n$ tridiagonal matrix of 1's?

$$|A_1| = 1, |A_2| = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0, |A_3| = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = -1$$

$(i=1, j=2)$

$$|A_4| = 1\begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = |A_3| - 1|A_2| = -1$$

In fact, $|A_n| = |A_{n-1}| - |A_{n-2}|$. We get a sequence which repeats every six terms:

$$|A_1| = 1, |A_2| = 0, |A_3| = -1, |A_4| = -1, |A_5| = 0, |A_6| = 1, |A_7| = 1.$$

18.06SC Linear Algebra
Fall 2011

$$\left|\quad\right| = \begin{vmatrix} a & 0 & 0 \\ & & \\ & & \end{vmatrix} + \begin{vmatrix} 0 & b & 0 \\ & & \\ & & \end{vmatrix} + \begin{vmatrix} 0 & 0 & c \\ & & \\ & & \end{vmatrix}$$

$$(a_{1\alpha}\, a_{2\beta}\, a_{3\gamma})$$

C^T

$a_{11}(a_{22}a_{33} - a_{23}a_{32}) + a_{22}$

1 Row +

Cramer's rule, inverse matrix, and volume

We know a formula for and some properties of the determinant. Now we see how the determinant can be used. $C_{ij} = (-1)^{i+j} M_{ij}$

Formula for A^{-1}

$C^T \to$ matrix of cofactors

We know:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

$\frac{1}{\det A} \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}^T$

Can we get a formula for the inverse of a 3 by 3 or n by n matrix? We expect that $\frac{1}{\det A}$ will be involved, as it is in the 2 by 2 example, and by looking at the cofactor matrix $\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$ we might guess that cofactors will be involved.

In fact:

$$A^{-1} = \frac{1}{\det A} C^T$$

where C is the matrix of cofactors – please notice the transpose! Cofactors of row one of A go into column 1 of A^{-1}, and then we divide by the determinant.

The determinant of A involves products with n terms and the cofactor matrix involves products of $n-1$ terms. A and $\frac{1}{\det A} C^T$ might cancel each other. This is much easier to see from our formula for the determinant than when using Gauss-Jordan elimination.

$AC^T = \det A \, I$

To more formally verify the formula, we'll check that $AC^T = (\det A)I$.

$$AC^T = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} C_{11} & \cdots & C_{n1} \\ \vdots & \ddots & \vdots \\ C_{1n} & \cdots & C_{nn} \end{bmatrix}. = \begin{bmatrix} \det A & & \\ & \det A & \\ & & \ddots \ \det A \end{bmatrix}$$

The entry in the first row and first column of the product matrix is:

$$\sum_{j=1}^{n} a_{1j} C_{j1} = \det A.$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ & & \end{bmatrix}\begin{bmatrix} C_{11} & C_{21} \\ C_{12} & C_{22} \\ C_{13} & C_{23} \end{bmatrix}$$

(This is just the cofactor formula for the determinant.) This happens for every entry on the diagonal of AC^T.

To finish proving that $AC^T = (\det A)I$, we just need to check that the off-diagonal entries of AC^T are zero. In the two by two case, multiplying the entries in row 1 of A by the entries in column 2 of C^T gives $a(-b) + b(a) = 0$. This is the determinant of $A_s = \begin{bmatrix} a & b \\ a & b \end{bmatrix}$. In higher dimensions, the product of the first row of A and the last column of C^T equals the determinant of a matrix whose first and last rows are identical. This happens with all the off diagonal matrices, which confirms that $A^{-1} = \frac{1}{\det A} C^T$.

$a_{11}(a_{22}a_{33} - a_{23}a_{32})$

$a_{11}(a_{12}a_{33} - a_{13}a_{32}) +$

What is this det of

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ & a_{12} & a_{13} \\ & a_{32} & a_{33} \end{bmatrix}$$

$a_{12}(a_{11}a_{33} - a_{13}a_{31})$

and so.

Det is to find

$\to a_{11}(a_{22}a_{33} - a_{23}a_{32}) + a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) = \det A.$

$\to a_{11}(a_{12}a_{33} - a_{13}a_{32}) + a_{12}(a_{11}a_{33} - a_{13}a_{31}) + a_{13}(a_{11}a_{32} - a_{12}a_{31})$

if we multiply 1st row × 2nd col we get

But it means that $a_{12} = a_{22}$, $a_{11} = a_{21}$, $a_{13} = a_{23}$.

det of A having 2 rows the same

$\underline{\det \sum C \times a}$ $\quad$ $\sum C^T a = \det$. $\quad$ $\det = a\, C^T$

$\det A =$
$\det A = \sum a_{i1} C^T$

$\det A = a_{11}C_{11} + \dots$
$= a_{11}C_{21} +$ $\quad$ $x = \frac{1}{\det A} C^T = A^{-1}$

This formula helps us answer questions about how the inverse changes when the matrix changes.

$A^{-1} = \frac{C^T}{\det B} = C^T b.$

$Ax = b$

Cramer's Rule for $\mathbf{x} = A^{-1}\mathbf{b}$

We know that if $A\mathbf{x} = \mathbf{b}$ and A is nonsingular, then $\mathbf{x} = A^{-1}\mathbf{b}$. Applying the formula $A^{-1} = C^T/\det A$ gives us:

C^Tb - give some det

$$\mathbf{x} = \frac{1}{\det A} C^T \mathbf{b}.$$

→ some determinant.

Cramer's rule gives us another way of looking at this equation. To derive this rule we break $\mathbf{x}$ down into its components. Because the i'th component of $C^T\mathbf{b}$ is a sum of cofactors times some number, it is the determinant of some matrix B_j.

= det B_1
= det B_2

like det of something

$i = 1^{st}$ eq → $C_{11}b_1 + C_{21} b_2 + C_{31} b_3$
$i = 2$ → $C_{12}b_1 + C_{22}b_2 + C_{32}b_3$

$$x_j = \frac{\det B_j}{\det A},$$

where B_j is the matrix created by starting with A and then replacing column j with $\mathbf{b}$, so:

$$B_1 = \begin{bmatrix} \mathbf{b} & \begin{matrix}\text{last n-1}\\ \text{columns}\\ \text{of } A\end{matrix} \end{bmatrix} \quad \text{and}$$

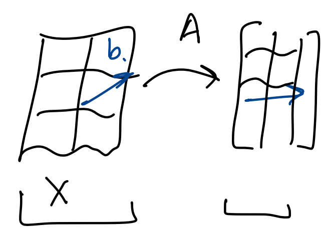
$$B_n = \begin{bmatrix} \begin{matrix}\text{first n-1}\\ \text{columns}\\ \text{of } A\end{matrix} & \mathbf{b} \end{bmatrix}$$

det of

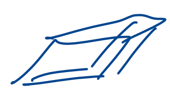
Need 3 values of x

Have to do |det| × 3.

This agrees with our formula $x_1 = \frac{\det B_1}{\det A}$. When taking the determinant of B_1 we get a sum whose first term is b_1 times the cofactor C_{11} of A.

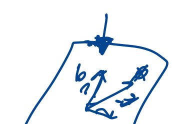


Computing inverses using Cramer's rule is usually less efficient than using elimination.



$|\det A|$ = volume of box

Claim: $|\det A|$ is the volume of the box (*parallelepiped*) whose edges are the column vectors of A. (We could equally well use the row vectors, forming a different box with the same volume.)

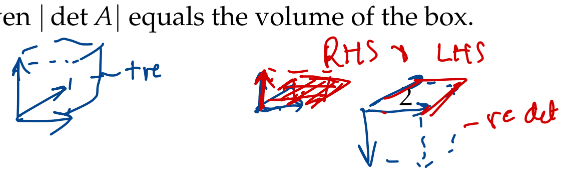


If $A = I$, then the box is a unit cube and its volume is 1. Because this agrees with our claim, we can conclude that the volume obeys determinant property 1.

If $A = Q$ is an orthogonal matrix then the box is a unit cube in a different orientation with volume $1 = |\det Q|$. (Because Q is an orthogonal matrix, $Q^TQ = I$ and so $\det Q = \pm 1$.)

Swapping two columns of A does not change the volume of the box or (remembering that $\det A = \det A^T$) the absolute value of the determinant (property 2). If we show that the volume of the box also obeys property 3 we'll have proven $|\det A|$ equals the volume of the box.

$(u \times w) \cdot w.$



+ve $\quad$ RHS → LHS $\quad$ -ve det.

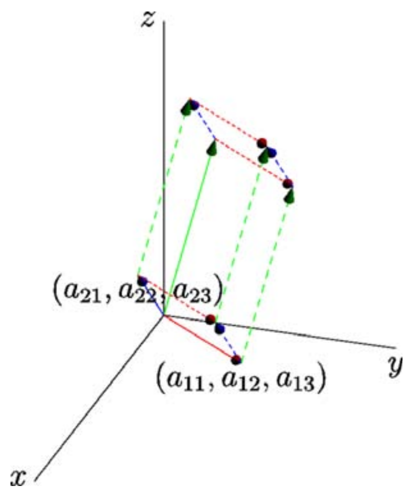


Figure 1: The box whose edges are the column vectors of A.

If we double the length of one column of A, we double the volume of the box formed by its columns. Volume satisfies property 3(a).

Property 3(b) says that the determinant is linear in the rows of the matrix:

$$\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}.$$

$\begin{matrix} x_1 & y_1 \\ x_2 & y_2 \end{matrix}]$

Figure 2 illustrates why this should be true.

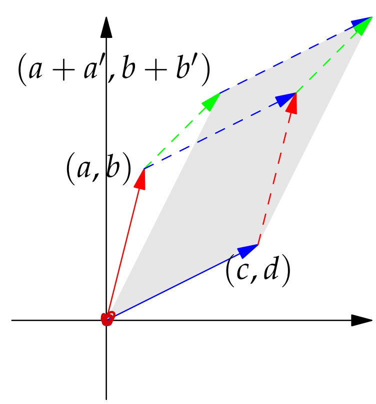


Figure 2: Volume obeys property 3(b).

Although it's not needed for our proof, we can also see that determinants obey property 4. If two edges of a box are equal, the box flattens out and has no volume.

Important note: If you know the coordinates for the corners of a box, then computing the volume of the box is as easy as calculating a determinant. In particular, the area of a parallelogram with edges $\begin{bmatrix} a \\ b \end{bmatrix}$ and $\begin{bmatrix} c \\ d \end{bmatrix}$ is $ad - bc$. The area of a triangle with edges $\begin{bmatrix} a \\ b \end{bmatrix}$ and $\begin{bmatrix} c \\ d \end{bmatrix}$ is half the area of that parallelogram, or $\frac{1}{2}(ad - bc)$. The area of a triangle with vertices at (x_1, y_1), (x_2, y_2) and (x_3, y_3) is:

$$\underbrace{\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}}.$$

$$A = \frac{1}{2}\begin{vmatrix} (x_1 - x_3) & (y_1 - y_3) \\ (x_2 - x_3) & (y_2 - y_3) \end{vmatrix}$$

18.06SC Linear Algebra
Fall 2011

$\underbrace{\{\ \}}_{x}$ $\begin{bmatrix}1&4\\5&6\end{bmatrix}\{x\} = \lambda\{x\}$.

Lecture 21 : Eigen values + vectors.

$Ax = \lambda I x$

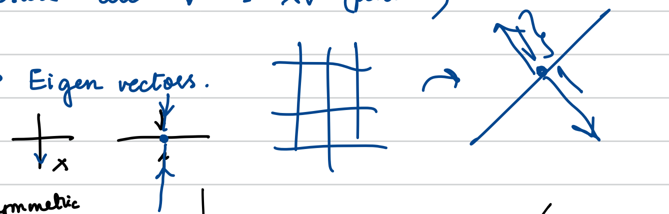
$\underbrace{(A - \lambda I)}\ \underset{\sim}{x}$

$\underbrace{(A - \lambda I)}\ \underline{x} = 0$

- Eigen values - Eigen vectors
- $(A - \lambda)v = 0$.
- Trace $= \lambda_1 + \lambda_2 + \dots \lambda_n$
- det $A = \lambda_1 \lambda_2 \lambda_3 \dots \lambda_n$.
- Square matrices

- A - What does a matrix do. It acts on a vector.
  linear map.
- It's like a function - $x \xrightarrow{A} Ax$.
- The vectors that we interested are $v \xrightarrow{A} \lambda v$ (parallel)
- Those Ax parallel to x → Eigen vectors.

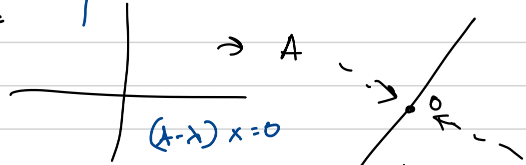
$Ax = 0x$



$\boxed{Ax = \lambda x}$

+ve definite, indefinite, skew symmetric singular.

- λ - +ve, -ve, complex, 0



$(A-\lambda)x = 0$

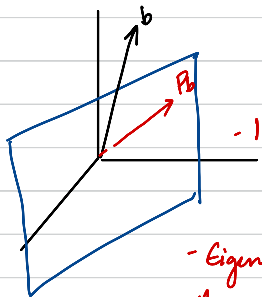
$\lambda = 0$

- $Ax = \lambda x$
  - → What $\lambda = 0$, what are x? $(A - 0I)\underline{x} =$ $N(A)$
  - → Eigen vectors are in $N(A)$

and ϕ: $N(A)$

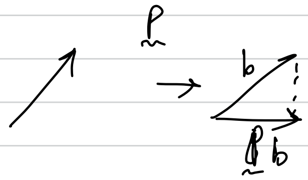
If A is singular, $\lambda = 0$ is an eigen values.

- Can't use elimination.

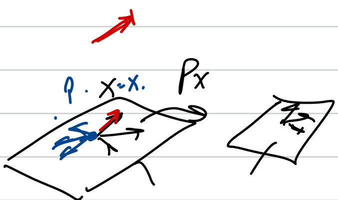


- What are x λ for.
  Projection P

P



- Is b eigen vector? for $\lambda \geq 0$
  Pb is in the other direction.
- Eigen vector if it was on that plane.
  Any x in plane: $Px = x$.



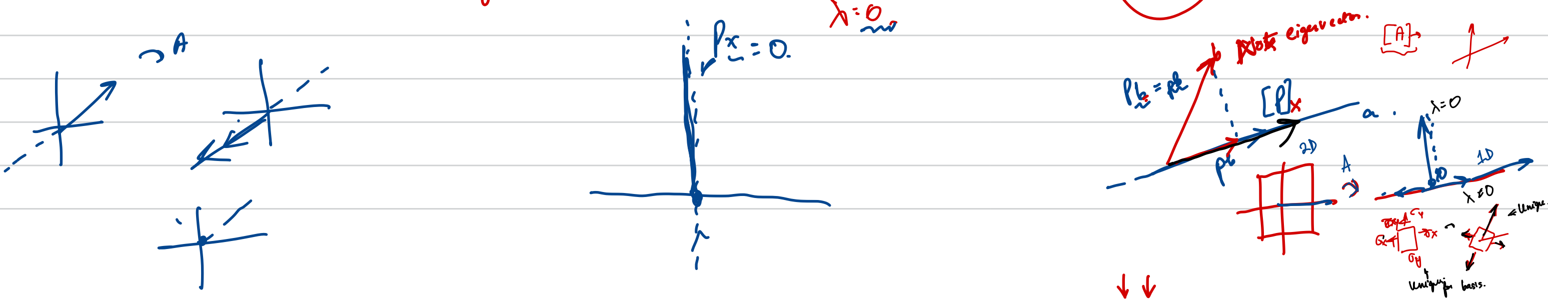
P. $x = x$. Px

$=1$ All eigen vectors $\lambda = 1$

and $\lambda = 1$.

- Another in the plane.
- Any $x \perp$ to Plane $Px = \underset{\sim}{0}$, null space. ∴ $\lambda = 1$ $\lambda = 0$.

$\lambda = \underset{\sim}{0}$



$P\underset{\sim}{x} = 0$.

$Pb = pb$ ~~Not~~ Not eigenvector. [A] [P]x. $\lambda = 0$ 2D 1D A $\lambda \neq 0$ unique basis.

$A x = 0$ A is A^{-1}: $x \neq 0$, $A, N(A)$

$x = 0$

$(A - \lambda)x = 0$. $\det = 0$

$\det A = -1$

$\det A = -1 = \lambda_1 \cdot \lambda_2$.

trace $A = 0+0 = \lambda_1 + \lambda_2$

$\lambda_1 = 1, \lambda_2 = -1$.

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \\ 1 & 2 & 5 \end{pmatrix} = \begin{pmatrix} 1-\lambda & 2 & 3 \\ 1 & 2-\lambda & 4 \\ 1 & 2 & 5-\lambda \end{pmatrix}$$

$(1-\lambda)[(2-\lambda)(5-\lambda) - 4\cdot 2]$
$-2[(5-\lambda) - 4] + 3[2 - 2 - \lambda]$

$\lambda = 0$,

$A : Q \Sigma Q^T$ - Square.

When do we

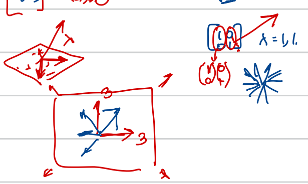
Independent eigen vectors

Dont have independent

$[3\times3] \to (3)(2)(1)$

λ 2, 2.

$\lambda = 1, 1$



New basis

$$Q^T \{\} \begin{bmatrix} a_{xx} & a_{xy} & c_{xy} \\ & & \end{bmatrix} \{Q\}$$

A, xy, W

Eg 2. $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $Ax = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\lambda = 1$

$x = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ $Ax = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $\lambda = -1$

- $n \times n$ matrices have n eigen values.
- $\sum_{i}^{n} \lambda_i = \text{trace } A = a_{ii}$ $\quad (\lambda^2 + \text{tra } \lambda + \det A) \lessgtr$ sign?

$\prod \lambda_i = \det A$.

How to solve $Ax = \lambda x$

Rewrite : $(A - \lambda I)x = 0$

: Some x, not $\underset{\sim}{0}$.

: A shifted by λI should be singular.

- $\det |A - \lambda I| = 0$

→ Characteristic equation. Find λ.
→ Repeated λ is confusing.

After finding λ that makes $|A - \lambda I| = 0$
→ Then we need to find $N(A - \lambda I)$

$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$ (Special prop of matrix → Special prop of eigen values)

$\det |A - \lambda I|$

$I_1^2 - I_{II}\ \ I_{III}$ $\quad I_I = \det \sigma$

$I_{II} = \text{tr}\, \sigma$

$= \frac{1}{2}(\text{tr}(\sigma^2)) + (\text{tr}\,\sigma)^2$

$$\begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 - 1 = 0$$

$\Rightarrow 9 + \lambda^2 - 6\lambda - 1 = 0$

$\Rightarrow \lambda^2 - \underbrace{6\lambda}_{\text{Trace}} + \underbrace{8}_{\text{Determinant}}$

$\Rightarrow (\lambda - 4)(\lambda - 2)$

$= \lambda^2 - \text{Trace}\,\lambda + \text{Det}$. $\quad \therefore \lambda = 4$

$\lambda = 2$

$(A - \lambda)x = 0$

Eigen vectors.

$(A - \lambda I)\vec{x}$

$A = \begin{pmatrix} 3-4 & 1 \\ 1 & 3-4 \end{pmatrix} x = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$ /Singular.

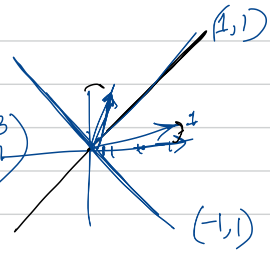
$N(A - \lambda I)$

G.E $= \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}$

And $N_{v_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$\begin{pmatrix} & \\ & \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} =$



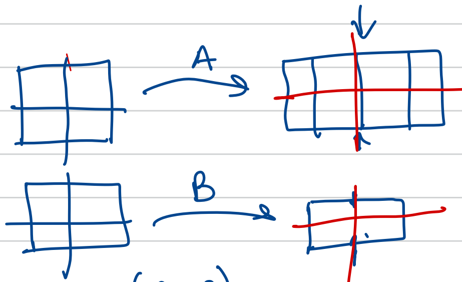
$A = \begin{pmatrix} 3-2 & 1 \\ 1 & 3-2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ $N = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$ $Q\,(v_1 | v_2 | v_3)$

$Q \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} Q^T$

So Eigen value. $\lambda_1 = 4$, $\vec{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_2 = 2$, $\vec{x} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$



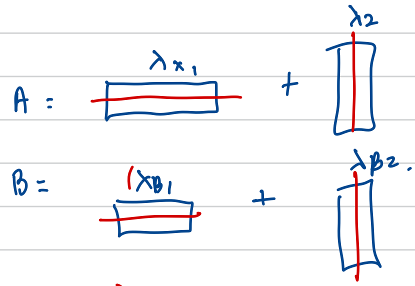
$(A+B)$, (λ added up)

$B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\lambda_1 = 1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\lambda_2 = -1 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} = 3I + B$, $3 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $-2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

= Eigen vectors same.

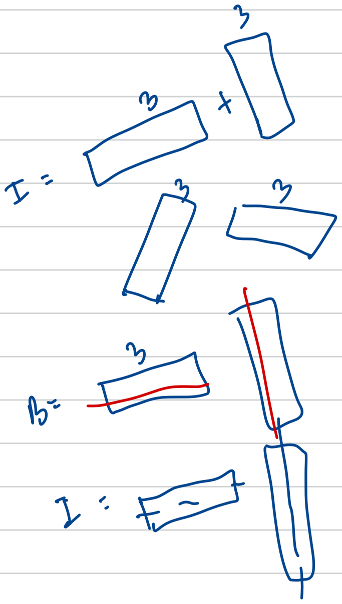
= Eigen values $+3$



$Ax = \lambda x$, $Bx = \alpha x$.

$(A + 3I)x = \lambda x + 3x$ $\qquad (A+B)x = (\lambda + \alpha)x$

$(A + 3I)x = (\lambda + 3)x$



A 2 eigen vectors,
B 3 eigen vectors.

NOT so great

If $Ax = \lambda x$

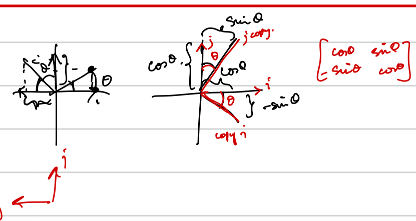
B has eigen values $\alpha_1, \alpha_2 \ldots$

$(A+B)x = (\lambda+\alpha)x \rightarrow$ Wrong. (Eigen values don't get added up).

Bx diff.

Example $Q = \begin{bmatrix} \cos 90 & -\sin 90 \\ \sin 90 & \cos 90 \end{bmatrix}$

90° rotat



$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Sum $\lambda = 0 + 0 = 0$ (+ve −ve)

det $= 1 = \lambda_1 \lambda_2$ (+1 ?) ?

What $v = \lambda v$ for rotation ?

$\det(Q - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1$

$= \lambda_1 = i$ Whatever that is

$\lambda_2 = -i$

So real matrix can lead to complex nos.

→ i and $-i$ are complex conjugate.

$A = A^T$

$A = -A^T$

$\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$

if matrix → Symmetric → eign → real.

Anti " → $i, -i$ pure imaginary

In between − partially real, + i

$A = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$ = Matrix is triangular, Then $\lambda \to$ eigen values.

$\lambda_1 = 3 \quad \lambda_2 = 3$

$\begin{pmatrix} 3-\lambda & 1 \\ 0 & 3-\lambda \end{pmatrix} = (3-\lambda)(3-\lambda)$

$\lambda = 3 \quad \lambda = 3.$

Eigen vectors.

$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$x_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ There isn't one.

$x_2 =$ No 2^{nd} ind

- Eigen values repeat, degenerate eigen vectors. NOT complete story.

---

see: Give the $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 0 & 1 & 4 \end{pmatrix}$

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 6 \end{pmatrix}$ $\lambda = 1, 1, 6.$ ✗

Changing elimination changes λ and v. because

$\in A$

Elimination

changes the basis

$A^2 = Q\Sigma \cancel{Q^T Q} \Sigma Q^T$
$= Q \Sigma^2 Q^T.$

- $A^2 v = A(Av) = A(\lambda v) = \lambda(Av) = \lambda^2 v$
- $A^{-1} v = A^{-1}\left(\frac{Av}{\lambda}\right) = A^{-1}A \frac{v}{\lambda} = \frac{1}{\lambda} v.$
  $\lambda \neq 0$

$A^{3/2}$

$A = Q \Sigma^{3/4} Q^T$

$A^2 v = A(Av)$
$= A(\lambda v) = \lambda(Av) = \lambda^2 v$

$A^2 v = \lambda^2 v.$

$A^2 \to \lambda^2$

$[A]_n$ = has n independant eigenvectors.

$$\begin{bmatrix} x_1 & & \\ & x_2 & \\ & & x_3 \end{bmatrix}$$

$$\begin{bmatrix} a_x & b_{x} & c_x \end{bmatrix}\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} \quad \begin{Bmatrix} ax \\ by \\ cz \end{Bmatrix} \qquad \begin{bmatrix} c_1 & & \\ & c_2 & \\ & & c_3 \end{bmatrix}\begin{Bmatrix} x \\ y \\ z \end{Bmatrix}$$

Lecture 22: Diagonalisation and Powers of A.

- $S^{-1}AS = \Lambda$

  S- eigen vectors

- Powers of A.

- Suppose n independant eigen vectors of A.

Put them in columns of S

$$Ax = \lambda x$$

$$A\begin{pmatrix} x_1 & x_2 & x_3 & x_n \end{pmatrix} = \begin{pmatrix} Ax_1 & Ax_2 & Ax_3 & Ax_n \end{pmatrix}$$

$$\begin{pmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \lambda_3 x_3 & \lambda_n x_n \end{pmatrix}$$

$$AS = A\overbrace{\begin{pmatrix} x_1 & x_2 & \dots & x_n \end{pmatrix}}^{S} = \begin{pmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \dots & \lambda_n x_n \end{pmatrix}$$

eigen vectors

$$= \begin{bmatrix} x_1 \dots x_2 & \dots & x_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & & 0 \\ 0 & 0 & & \vdots \\ 0 & 0 & & \lambda_n \end{bmatrix} = S\Lambda$$

Diagonal eigen value matrix

$$AS = \begin{pmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \lambda_3 x_3 \end{pmatrix}$$

$$= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}\begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix}$$

$$AS = S\Lambda.$$

$$S^{-1}AS = \Lambda.$$

$$A = (S\Lambda S^{-1}) \text{ spe.}$$

$$A = \underbrace{Q\Lambda Q^T}_{}$$

- I want to invert S, S should be invertible

$$S^{-1}AS = \Lambda$$

- There are small no of matrices that don't have n eigen vectors

$$\left(\begin{array}{c} S^{-1}AS = \Lambda \\ \text{or } A = S\Lambda S^{-1} \end{array} \right)$$

If $Ax = \lambda x$

$$A(Ax) = A(\lambda x) = \lambda Ax = \lambda^2 x.$$

$$\text{Eig}(A^2) = \lambda^2.$$

$$\text{Eig}_v\ x = \text{same } x$$

$$A = S\Lambda S^{-1} S\Lambda S.$$

$$A^2 = S\Lambda^2 S^{-1}$$

$$A^2 + A = S^{-1}S\Lambda^2 S^{-1}S + S^{-1}S\Lambda S^{-1}S$$

$$S^{-1}(\quad)S \qquad \Lambda^2 + \Lambda$$

$A^2 = S\Lambda S^{-1} S \Lambda S^{-1} = \underbrace{S\Lambda^2 S^{-1}}$

So $A^2 = \lambda^2$.

$A^k = S\Lambda^k S^{-1}$

In other words, eigen values gives us a sense of the power of the matrix.

A. A^k $k \to \infty$

$A \to 0$. $|\lambda| < 1$.

> 1.

re cursive

Eg: Theorem $A^k \to 0$ as $k \to \infty$

if all $|\lambda_i| < 1$.

- Assumption full set of n independent eigen vectors.

- A is sure to have n independent eigen vectors (and be diagonisable) if all the λ's are different.

(no repeated λ's) eig(rand (10,10))

Repeated λ possibility:

- May or may not have n independent eigenvectors.

eg $A = I$, eigen values = 1, 1, 1.

$S^{-1} A S = I$

Suppose $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ $\lambda_{1,2} = 2, 2.$

$$\begin{vmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 \Rightarrow \lambda = 2.$$

Eigen vector = $\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}$

$N(A-\lambda I)$ = One dimension. $\lambda : \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Equation

$u_{k+1} = A u_k$
$u_1 = A u_0.$
$u_2 = A u_1 = A . A u_0$
$= A^2 u_0.$

$u_k = A^k u_0.$

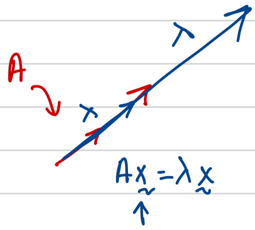
- Start with given vector u_0

$$\underbrace{u_{k+1}} = A \underbrace{u_k}.$$

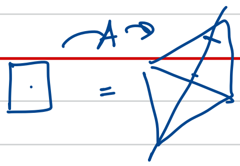
$\Rightarrow u_1 = A u_0$

$\Rightarrow u_2 = A^2 u_1 \ldots$

$$\boxed{u_k = A^k u_0}$$



$A\underset{\sim}{x} = \lambda \underset{\sim}{x}$



To really solve :

$u_0 : c_1 x_1 + c_2 x_2 + c_n x_n$

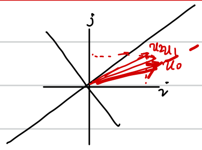
$(A\cancel{x} = \cancel{\lambda x})$

$A u_0 : c_1 \lambda_1 x_1 + c_2 \lambda_2 x_2 + c_n \lambda_n x_n = \Lambda S c.$

$$\begin{array}{c} x_1 \; x_2 \\ \begin{bmatrix} x_1 & x_2 & x_3 \\ \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \end{array}$$

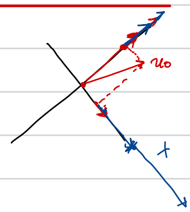
Suppose $u^{100} = A^{100} u_0 = c_1 \lambda_1^{100} x_1 + c_2 \lambda_2^{100} x_2 + c_n \lambda_n^{100} x_n.$

$= \Lambda^{100} S c$



$u = 3i + 4j$

$Au = 3Ai + 4Aj$



$u = 3x_1 + 4x_2$

$Au = 3Ax_1 + 4x_2$

$= 3\lambda x_1 + 4.\lambda_2 x_2$

$\frac{\partial^2 u}{\partial t^2} = a$

$\downarrow$

$\frac{dv}{dt} = w$

$\frac{du}{dt} = w$

$F_{k+2} = F_{k+1} + F_k.$

$\begin{Bmatrix} F_{k+2} \\ F_{k+1} \end{Bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$

$\{u_{k+1}\} = A \{u_k\}$

Example: Fibonacci : 0, 1, 1, 2, 3, 5, 8, 13, 21 .... F_{100} = ?

$F_{k+2} = F_{k+1} + F_k$ (2nd order).

- So we want to find 0, 1, ...
- $F_{k+2} = F_{k+1} + F_k$

$u_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$ (2x2 system 1st order)

But we keep:

$F_{k+1} = F_{k+1}$

$u_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$ and so a doF has $\begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$

$u_{k+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$

u_{k+1} or $\begin{bmatrix} F_{k+2} \\ F_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$

$\therefore u_{k+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$

$\begin{bmatrix} F_{k+3} \\ F_{k+2} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{k+2} \\ F_{k+1} \end{bmatrix}$

$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ - sym, λ real, λ orthogonal.

$(A - \lambda I) \begin{pmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = \det \; (1-\lambda)\lambda - 1$

$= \lambda - \lambda^2 - 1$

$= \lambda^2 - \lambda - 1$

F_{k+1}, $\lambda_1 = 1.618$

$\lambda_2 = -.618.$

$= \lambda_1 = \frac{1}{2}(1 + \sqrt{5}) \approx 1.618 ..$ $x_1 =$

$\lambda_2 = \frac{1}{2}(1 - \sqrt{5}) \approx -.618...$ $x_2 =$

How fast is Fibonacci increasing

$F_{100} \approx c_1 \lambda_1^{100} \underset{\sim}{x_1} + \underbrace{c_2 \lambda_2^{100}}_{\text{small no.}} x_2.$

$\approx c_1 \left(\frac{1+\sqrt{5}}{2}\right)^{100}.$

$u_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} = c_1 \lambda_1^k x_1 + c_2 \lambda_2^k x_2.$

$\Rightarrow u_k$ in $c_1 x_1$ $c_2 x_2$

$u_{k+1} = A^k u_k = c_1 \lambda_1^k x_1 + c_2 \lambda_2^k x_2$

$F_{K+4} = F_{K+3} + F_{K+2} + F_{K+1}$

$$\begin{Bmatrix} F_{K+4} \\ F_{K+3} \\ F_{K+2} \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} F_{K+3} \\ F_{K+2} \\ F_{K+1} \end{Bmatrix}$$

$F_{K+2} = F_{K+1} + F_K$

$\{\ \} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

$u_0 = \begin{Bmatrix} F_{K+1} \\ F_K \end{Bmatrix}.$

$u_K = A^K u_0 \Rightarrow (-)$

$= \lambda^K u_0 \Rightarrow$ Eigenbasis

- We are doing dynamic. Ax

Find eigen vectors:

$$(A - \lambda I) = \begin{pmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

For 2×2 easy has to be $\begin{pmatrix} \lambda \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$u_K = \begin{pmatrix} F_{K+1} \\ F_K \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\underset{\sim}{x_1} = \begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix} \quad \underset{\sim}{x_2} = \begin{pmatrix} \lambda_2 \\ 1 \end{pmatrix}$

$u_0 = c_1 \underset{\sim}{x_1} + c_2 \underset{\sim}{x_2}$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \underset{\frac{1}{\sqrt{5}}}{c_1} \begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix} + \underset{-\frac{1}{\sqrt{5}}}{c_2} \begin{pmatrix} \lambda_2 \\ 1 \end{pmatrix}$$

$u_0 = \begin{pmatrix} F_1 \\ F_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\begin{bmatrix} F_{K+1} \\ F_K \end{bmatrix} = c_1 \lambda_1^K \begin{pmatrix} x_1 \\ 1 \end{pmatrix} + c_2 \lambda_2^K \begin{pmatrix} x_2 \\ 1 \end{pmatrix}$$

$$F_K = \underset{c_1}{\frac{1}{\sqrt{5}}} \underset{\lambda^K}{\left(\frac{1+\sqrt{5}}{2}\right)^K} - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^K$$

So: $c_1 x_1 + c_2 x_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} x_1 & \lambda_2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

- Key idea:
  - When things are evolving in time, by a first order system.
  - Find the eigen values of A, what its doing.
  - Take u_0 keep it as $c_1 \underset{\sim}{x_1} + \ldots$
  - And see A^K.

rec: Find a formula for C^k when $C = \begin{pmatrix} 2b-a & a-b \\ 2b-2a & 2a-b \end{pmatrix}$ $a, b = -1$

$\det(C - \lambda I) = (\lambda - a)(\lambda - b)$ $\lambda = a, b.$

1ˢᵗ ve $= \begin{pmatrix} 2b-2a & a-b \\ 2b-2a & a-b \end{pmatrix}$ $v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

2ⁿᵈ ve $= \begin{pmatrix} b-a & a-b \\ 2b-2a & 2a-2b \end{pmatrix} = v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$C = S \Lambda S^{-1} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -2 & 1 \end{pmatrix} \frac{1}{(-1)}$

$C^k = \begin{pmatrix} a^k & 0 \\ 0 & b^k \end{pmatrix}$

$C^{100} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$ $(a, b = -1)$

$$\begin{pmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{pmatrix}$$

$(1-\lambda)x_1 + x_2 = 0$

$x_1 - \lambda x_2 = 0$

$x_1 = \lambda x_2$

$x_1 = \lambda.$ $x_2 = 1$

$(1-\lambda)\lambda \cancel{x_1} + \cancel{x_2}\,1 = 0.$

$\lambda - \lambda^2 + 1 = 0$

$\frac{1}{2}(1+\sqrt{5}) - \left(\frac{1}{2}(1+\sqrt{5})\right)^2 + 1.$

$= \frac{1}{2}(1+\sqrt{5}) - \frac{1}{4}(1+2\sqrt{5}+5) + 1.$

$= \frac{1}{2} + \cancel{\frac{\sqrt{5}}{2}} - \frac{1}{4} - \cancel{\frac{\sqrt{5}}{2}} \frac{5}{4} + 1 = \frac{-6+4+2}{4}$

In a linear D.E, superposition is allowed.

$y_1 =$ sol $\quad y_2 =$ soln.

$y_1 + y_2$

$\frac{d^2y}{dx^2} +$ $\quad \frac{dy}{dx}$ $\quad e^y \quad$ $\ln y$ $\quad \sin y$

D.

$$\frac{d}{dx}(\beta a + \alpha b) = \beta \frac{da}{dx} + \alpha \frac{db}{dx}$$

$D u = A u$

eigenfunctions

Lecture 23. Differential equation.

- $\frac{du}{dt} = Au$ (Const coeff differential eqn)
- Exponential e^{At} of a matrix.

$\frac{d}{dt} e^{At} = A e^{At}$

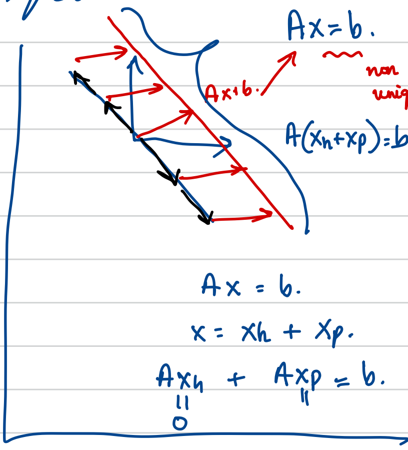
Ex: $\frac{du_1}{dt} = -u_1 + 2u_2$

$\frac{du_2}{dt} = u_1 - 2u_2$

2 Differential equations

Here I already have a system of D.E.

So $\frac{d}{dt}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$



$Ax = b.$

$x = x_h + x_p.$

$\underset{0}{Ax_h} + \underset{\|}{Ax_p} = b.$

Initial conditions

$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}_0 = u(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\frac{du}{dt} = A u.$

$A = \begin{pmatrix} -1 & 2 \\ 1 & -2 \end{pmatrix}$

Eigen values of matrix A:

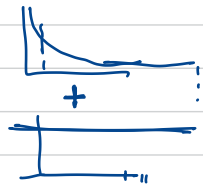
Some properties of λ and the matrix to find λ.

- A is Singular, so one $\lambda_1 = 0$ $\quad |A| = \lambda(3+\lambda)$
- trace(λ) $= -3$ $\quad$ so $\lambda_2 = -3$ $\quad \lambda = 0$ $\quad \lambda = -3$

$\det(A) = -3 \times 0 = 0$

$-e^{\alpha t}$

G.S. $\Rightarrow = \underbrace{\left(e^{0t} + e^{-3t}\right)}$



- $\lambda = 0, -3.$

(Steady state end)

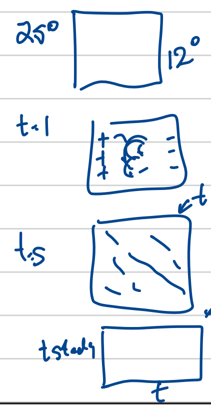
General soln to D.E.

I expect the soln to have e^{0t} and e^{-3t}

↑ Const. $\quad$ ↑ Disappear

- I have a steady state when I have a 0λ.

$\frac{du}{dt} = Au$



Find eigen value
+ eigen vector.

$\lambda = 0$ Null space of $\begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} x_1$ $\quad Ax_1 = 0x_1$

$\lambda = -3$ Null space of $\begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix}$ $\quad N(A) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \to x_2$ $\quad Ax_2 = -3x_2$

Sol^n $u(t) = \underbrace{c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2}$ $\leftarrow$ General solution

$Ax_1 = \lambda x_1$

Check: $\frac{du}{dt} = Au$ Plug in $e^{\lambda_1 t} x_1$ $\quad \frac{du}{dt} = Au$

$$\lambda_1 \cancel{e^{\lambda_1 t}} x_1 = A \cancel{e^{\lambda_1 t}} x_1$$

$$Ax_1 = \lambda_1 x_1$$

Pure sol^n.

Last time. $\approx c_1 \lambda_1^k x_1 + c_2 \lambda_2^k x_2$. $(u_{k+1} = Au_k)$

what are c_1 and c_2

$$c_1 e^{0t} x_1 + e^{-3t} x_2$$
$$= c_1 x_1 + c_2 e^{-3t} x_2$$

$\therefore u(t) = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{-3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$u_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$c_1, c_2 = 1/3$$

As time when on.

$t \to \infty$ steady state.

$$u(t)_{t \to \infty} = \frac{1}{3} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

When do we get

(1) Stability $u(t) \to 0$.

need $e^{\lambda t} \to 0$.

as $u(t) \to 0$, what if λ are complex. $\Big|\, e^{(-3+6i)t}$

$= e^{-3it}$.

$|e^{6it}| = 1$

$\mathrm{Re}\,\lambda < 0$. So we take Real part. < 0

(2) Steady state

$\lambda_1 = 0$, and other λ Real part < 0.

(3) Blow up.

if any $\lambda > 0$.

$$\boxed{\mathrm{eig}(A) = -\mathrm{eig}(-A)}$$

For 2×2, we can pin down more:

2×2, stability. $\mathrm{Re}\,\lambda_1 < 0$ $\mathrm{Re}\,\lambda_2 < 0$

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$; • trace $a + d = \lambda_1 + \lambda_2 < 0$

eg: • trace < 0, but still blow up $\begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}$

• Another condition:

$\Rightarrow$ det should > 0. $(\lambda_1 \lambda_2 > 0)$

$c_1\begin{pmatrix}2\\1\end{pmatrix} + c_2\begin{pmatrix}1\\-1\end{pmatrix} = 0$

$Sc = 0$

$\Rightarrow \begin{pmatrix}2 & 1\\1 & -1\end{pmatrix} c = 0.$

So I originally had $\frac{du}{dt} = Au$. (Coupled)

So its coupled the original DE $\frac{du_1}{dt} = u_2 \ldots$ $\frac{du_2}{dt} = u_2.$

When I find the eigen values. then the eq gets decoupled

And the eigen vectors uncoupled.

eg: I can also write $u(t)$: solution

A couples.

$\frac{du}{dt} = Au$ [A - couples]

$u = Sv$.

set $u = Sv$ (Uncouple) $S \leftarrow$ eigen vectors.

$\rightarrow$ Diagonalising

$S\frac{dv}{dt} = ASv$ $\leftarrow$ eign.

$\frac{dv}{dt} = \underbrace{S^{-1}AS}\,v = \Lambda v.$ (So if I use eigen vector as basis, I have uncoupled).

If I use eigen vectors is uncoupled.

Along eigen vector direction.

$\frac{dv_1}{dt} = \lambda_1 v_1$

$\vdots$ $\vdots$

System of eq, easy to solve, not connected

$v(t) = e^{\lambda t}v(0)$ $e^{At} = Se^{\lambda t}S^{-1}$

$u(t) = \underbrace{Se^{\lambda t}S^{-1}}_{e^{At}}u(0)$ $\leftarrow$ what happened.

What do we mean e^{At}.

Actual soln without diagonalising

$\frac{du}{dt} = Au$ = Actual soln. $e^{At}u(0)$

Matrix exponential.

e^{At}.

Taylor series

$e^x = \sum x^n/n!$

Power series-

$\frac{1}{1-x} = \sum_0^{\infty} x^n$ (Geom series)

$e^{At} = I + At + \frac{At^2}{2} + \frac{At^3}{6} + \dots \frac{A^n}{n!}$ ← Doesn't fail

$(I-At)^{-1} = I + (At) + (At)^2 + (At)^3 \dots\dots$ ← Can blow.

$\lambda(At) < 1$ — Then converges.

$e^{At} = Se^{\Lambda t}S^{-1}$??

$= I + S\Lambda S^{-1}t + \frac{S\Lambda^2 S^{-1}}{2}t^2 + \dots$

$= SS^{-1} + S\Lambda S^{-1} + \frac{1}{2}S\Lambda^2 S^{-1}$

$= S\left(I + \Lambda + \frac{1}{2}\Lambda^2 \dots\right)S^{-1}$

$= S e^{\Lambda t} S^{-1}$

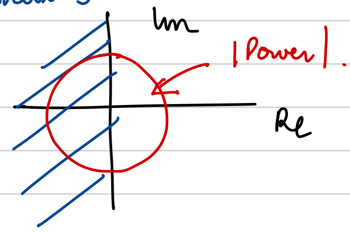
Some matrices are non independent & can't get S^{-1}

$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & e^{\lambda_2 t} & \\ 0 & 0 \ddots & e^{\lambda_n t} \end{bmatrix}$ ← Uncoupled.

Exponential. $S e^{\Lambda t} S^{-1} =$ $\qquad e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix}$

All $\lambda_n < 0$ to converge.

Stability



$y'' + by' + ky = 0$ $\qquad$ 1 2nd order $\rightarrow$ 2 first order.

Let u be $\begin{bmatrix} y' \\ y \end{bmatrix}$

$$u' = \begin{bmatrix} y'' \\ y' \end{bmatrix}$$

So $\begin{pmatrix} y'' \\ y' \end{pmatrix} = \begin{bmatrix} -b & -k \\ 1 & 0 \end{bmatrix} \begin{pmatrix} y' \\ y \end{pmatrix}$

If I had 5th order eq., 5×5 matrix

$$\begin{pmatrix} y''''' \\ \\ \\ \\ \end{pmatrix} \begin{bmatrix} - & - & - & - \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \vdots \end{bmatrix}$$

Solve d.E. $y''' + 2y'' - y' - 2y = 0$. For the general soln. What is A. Find the first col of $\exp(At)$

3 order $\to$ 3 1^{st} orders.

$$u = \begin{pmatrix} y'' \\ y' \\ y \end{pmatrix}$$

$$\frac{du}{dt} = \begin{bmatrix} y''' \\ y'' \\ y' \end{bmatrix} = \underbrace{\begin{pmatrix} -2 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}}_{A.} \begin{bmatrix} y'' \\ y' \\ y \end{bmatrix}$$

General solution.

$$x_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad x_2 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \quad x_3 \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix}$$

$$u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2 + c_3 e^{\lambda_3 t} x_3$$

$$\lambda = 1 \quad \lambda = -1 \quad \lambda = -2.$$

$y = u(3) = 3^{rd}$ component of $u(t)$.

$$e^{At} = S e^{\Lambda t} S^{-1}$$

$$= \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 4 \\ 1 & -1 & -2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} e^{t} & & \\ & e^{-t} & \\ & & e^{-2t} \end{pmatrix}$$

Only 1^{st} col. $= \begin{pmatrix} e^{t} x_1 & e^{-t} x_2 & e^{-2t} x_3 \end{pmatrix} \begin{pmatrix} S^{-1} \end{pmatrix}$

Lecture 24: Markov Matrices : Fourier Series

$$A = \begin{pmatrix} .1 & .01 & .3 \\ .2 & .99 & .3 \\ .7 & & .4 \end{pmatrix}$$

① All entries ≥ 0 (Connected to probability ideas)
② All columns add to 1
③ Power of markov matrix = markov matrix

Steady state (Power series) $\lambda = 1$ [In DE $\lambda = 0$].

④ Havi column = 1, gives $\lambda = 1$.

① $\lambda = 1$ is an eigenvalue

② All other $|\lambda_i| < 1$

$u_k = A^k u_0 = c_1 \lambda_1^k x_1 + c_2 \lambda_2^k x_2 \ldots$
A should be invertible

if $\lambda_1 = 1$, $\lambda_2 < 1$.

- → And we get steady state.

  x_1 part.

- The eigen vector x_1 and all comp ≥ 0.

How can I say if $\Sigma c = 1$, it leads to $\lambda = 1$ is an eigenvalue

$$A - 1I = \begin{bmatrix} -0.9 & .01 & 0.3 \\ 0.2 & -0.1 & 0.3 \\ .7 & 0 & -0.6 \end{bmatrix} \leftarrow \text{I believe it is singular.}$$

- why is it singular?
- Now All columns add to zero. $\longrightarrow$ AI is singular.

$\Rightarrow c_1 C_1 + c_2 C_2 + c_3 C_3 = 0$ for $c_1, c_2, c_3 = 1$.

$\Rightarrow$ Rows also dependant. $R_3 + R_2 + R_1 = 0$
because $(1,1,1)$ is in left nullspace $N(A^T)$
then $x_1 (\quad)$ is in $N(A)$.

What can you tell eigenvalues of A and A^T? SAME

$\det(A - \lambda I) = 0$ $\quad (\det A = \det A^T)$
$\det(A^T - \lambda I) = 0$

But eigen vector of A and A^T diff.

What is application of Markov matrices?

$u_{k+1} = A u_k$ (A is Markov)

Suppose people from California going to MIT. , Only fractions the. people.
So Col = 1, because all pop accounted for.

$$\begin{pmatrix} u_{cal} \\ u_{MIT} \end{pmatrix}_{t=k+1} = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix} \begin{pmatrix} u_{cal} \\ u_{MIT} \end{pmatrix}_{t=k}$$

What is the steady state?

→ If everybody started in MIT.

and at T_{100}.

$$\begin{pmatrix} u_{cal} \\ u_{MIT} \end{pmatrix}_0 = \begin{bmatrix} 0 \\ 1000 \end{bmatrix}$$ ← No people in cal.

∴ $u_{cal} + u_{mass} = 1000$

$$\begin{bmatrix} u_{cal} \\ u_{mass} \end{bmatrix}_1 = \begin{bmatrix} 200 \\ 800 \end{bmatrix}$$

$$\begin{bmatrix} u_{cal} \\ u_{mass} \end{bmatrix}_{200} =$$ So I need eigen values etc. cause people leaving + coming so I need to figure it out in that way.

$\pi = -\int \sigma d\varepsilon = 0$ ← cauch

$= \frac{P}{A} \int d\varepsilon$

$\varepsilon = \frac{l^2 - l}{l}$

$\pi = \int \sigma : \varepsilon.$

$\int \sigma : \delta\varepsilon$

Now $\sigma \int \delta\varepsilon.$

$= \sigma \int B \,\delta\varepsilon \, dv.$

$= \sigma \int A$

$$\begin{pmatrix} .9 & .2 \\ .1 & .8 \end{pmatrix} \quad \lambda_1 = 1, \quad \lambda_2 = 0.9 + 0.8 - 1 = 0.7.$$

$\cancel{.2} x_1 = -\cancel{.2}.$

$$x_1 = \begin{pmatrix} -.1 & .2 \\ .1 & -.2 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad x_2 = \begin{pmatrix} .2 & .2 \\ .1 & .1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

x_1 the.

Steady state $= \begin{pmatrix} 2/3 \times 1000 \\ 1/3 \times 1000 \end{pmatrix}$

$$u_k = c_1 1^k \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2\, 0.7^k \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$ → Disappear. 0.7^k.

$$u_0 = \begin{pmatrix} 0 \\ 1000 \end{pmatrix} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Find $c_1 = 1000/3$, $c_2 = 2000/3$.

- Sometimes they want to use row vectors. Rows add to 1.

Projections with orthonormal basis $q_1, q_2 .. q_n$

$$V = c_1 q_1 + c_2 q_2 + c_3 q_3 + \cdots\cdots$$

- $q_1^T v = c_1$
- $q_2^T v = c_2$.

$Q x = V$

$x = Q^{-1} v = Q^T v$.

If I have an orthonormal basis

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_n \end{bmatrix} = \begin{bmatrix} - q_1 - \\ - q_2 - \\ - q_3 \end{bmatrix} V. = \begin{bmatrix} & \end{bmatrix} \begin{bmatrix} q_1 & q & \cdots \end{bmatrix}$$

$x = Q^T V$

$x = Q^{-1} V.$

Fourier series. $f(x)$ - periodic function. $x - 2\pi$.

$$f(x) = a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + a_3 \cos 3x + b_3 \sin 2x + \dots$$

Infinite dimension.

Basis $(1, \cos x, \sin x, \cos 2x, \sin 2x, \dots)$

$f^T \quad g$

We say basis orthogonal.

- Vectors : $v^T w = v_1 w_1 + \dots v_n w_n$.
- Functions $f^T g = \int_0^{2\pi} f(x)\, g(x)\, dx$

$$= \int_0^{2\pi} \sin x \cos x \, dx = \frac{1}{2} \sin x^2 \Big|_0^{2\pi} = 0$$

Find coefficient of a_1, $\sin x$, $\cos x$. [a_0 - average value of f]

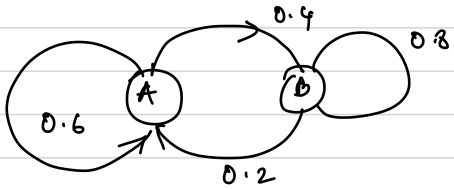
$$\int_0^{2\pi} f(x) \cos x \, dx = a_1 \int_0^{2\pi} \cos^2 x \, dx$$

$$= \pi a_1$$

$$a_1 = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos x \, dx.$$

---

sec: A particle jumps between A & B with foll prob:



$$\begin{pmatrix} A \\ B \end{pmatrix}_{k+1} = \begin{pmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}_k$$

$$\begin{pmatrix} A \\ B \end{pmatrix}_{k0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad A = S D S^{-1}$$

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix} \qquad \text{ii) } \infty \text{ steps}$$

$$u_\infty = c_1 \lambda^\infty x_1$$

$$u_\infty = c_1 1 x_1$$

$$\begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix} = c_1 \lambda_1 x_1 + c_2 \lambda_2 x_2$$

Lecture 25: Symmetric matrices.

- Eigen values / vectors.
- Positive Define Matrices.

- Eigen values are REAL
- Eigen values are PERPENDICULAR
  - ↳ can be "shown" -

There is also a full set of them :

Usual case : $A = S \Lambda S^{-1}$ ↙ orthonormal eigen vectors.

Symmetric case : $A = Q \Lambda Q^{-1}$

$= Q \Lambda Q^T$ - Spectral theorem.

Principal axis theorem: Dimensions dont couple.

Why real eigenvalues? $\overline{a+ib} = a - ib$

$Ax = \lambda x \overset{\text{always}}{\Rightarrow} \bar{A}\bar{x} = \bar{\lambda}\bar{x} = A\bar{x} = \bar{\lambda}x$

↑ Real

λ can be complex

x can be complex

- If x and λ are complex.

$Ax = \lambda x \qquad A\bar{x} = \bar{\lambda}\bar{x}$

$\bar{x}^T A x = \bar{x}^T \lambda x \qquad \bar{x}^T A^T = \bar{\lambda}\bar{x}^T$

$\bar{x}^T A x = \bar{\lambda}\bar{x}^T x$

SAME

- $\lambda = \bar{\lambda}$

If $\bar{x}^T x \neq 0$.

If A is real & symmetric, λ is real.

So if $\bar{x}^T x = (\bar{x}_1\ \bar{x}_2\ x \ldots \bar{x}_n)\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

- Length of $\bar{x}^T x$.

$= \bar{x}_1 x_1 + \bar{x}_2 x_2 + \bar{x}_3 x_3 \ldots$

$= (a+ib)(a-ib) \ldots$

$= (a^2+b^2) \ldots \quad$ (Real).

So $x^T x \Rightarrow$ Positive

Right thing.

Suppose A is complex. $\bar{A}^T = \bar{A}$??

Good matrices.
- Real λs $\longrightarrow$ $A = A^T$ if A is real.
- Perpendicular vs $\quad A = \bar{A}^T$ A is complex. !!

$A = Q \Lambda Q^T \quad (A = A^T)$

$= \begin{bmatrix} q_1\ q_2 \ldots q_n \end{bmatrix}\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \\ \vdots \end{bmatrix}^T$

Imp - have to check!

$= (q_1\lambda_1\ q_2\lambda_2 \ldots q_n\lambda_n)\begin{pmatrix} q_1^T \\ q_2^T \\ q_n^T \end{pmatrix} = \lambda_1 q_1 q_1^T + \lambda_2 q_2 q_2^T + \lambda_n q_n q_n^T.$

$= \begin{bmatrix} q_1\lambda_1 & q_2\lambda_2 & | & | \\ | & | & | & | \end{bmatrix}\begin{bmatrix} - & - & - \\ - & - & \\ - & - & \end{bmatrix}$

$(A)(|\ |\ |)$

$= (A\text{col}_1\ A\text{col}_2\ A\text{col}_3)$

$(A)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 1\ \text{col}_1(AQ) + 0\ \text{col}_2(AQ) + 0 c.. = 1\ A\text{col}_1$

or:

$\approx \begin{bmatrix} q_1 & \ldots & q_n \\ | & | & | \\ | & | & | \end{bmatrix}\begin{bmatrix} q_1^T\lambda_1 \\ q_2^T\lambda_2 \\ \vdots \\ q_n^T\lambda_n \end{bmatrix} = q_1 q_1^T \lambda_1 + q_2 q_2^T \lambda_2.$

(Col $\times$ Rows)

Its a matrix.

$$= \lambda \underbrace{q_1 q_1^T}_{\text{Projection}} + \lambda q_2 q_2^T + \dots$$

Every symmetric matrix is a comb of per. projection matrix.

Sym matrix - Real. - Are they +ve or -ve.

- det $|A - \lambda I|$ is bad way. Unstable.
- Signs of pivots are signs of eigen values (For symmetric matrics)
  \# +ve pivots = +ve eigen values.

$\prod$ Pivots $= \prod$ eign $=$ det.

Positive definite matrix

- It is symmetric !
- All eigen values are +ve.
- All pivots are $\therefore$ +ve.

→ All the det = +ve. and sub dets

$$\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 0 & +1/5 \end{bmatrix}$$

Product of pivots $=$ det
$= 10 - 9 = 1.$

$\lambda^2 - 7\lambda + 1$

$\lambda^2 -$ trace λ + det λ.

rec:

1) Every +ve def matrix is invertible

Well pivots are +ve.
det = pivots. So det $\neq 0$.

2) Only +ve def projection is $P = I$

Well, P is proj, eigenvalues of P = 0 or 1.
- P is +ve definite, eigen = 1.
$P = U I U^{-1}$, $UU^T = I$

3) D is diag with +ve entries is +ve def: $x^T \begin{pmatrix} \cdot & & \\ & \cdot & \\ & & \cdot \end{pmatrix} x \Rightarrow$

$= d_1 x_1^2 + d_2 x_2^2 + \dots d_n x_n^2 > 0$

Symm. for = positive diagonal = λ.

4) S symm with det $S > 0$ might not be +ve definite

$$\begin{pmatrix} -2 & 0 \\ 0 & -3 \end{pmatrix}$$

$$x^T (S) x = \quad x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}^T$$

$-3 \ 1^2 +$

Lecture 26: Complex matrices, Fast fourier transform (FFT)

Usually multiplication $\Rightarrow n^2$ mults.
FFT $\Rightarrow n \log_2 n$.

Length.

$$z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} \text{ in } \mathbb{C}^n$$

$z^T z \rightarrow$ No good.

- Doesn't give me anything.
- Cause length should be +ve.

$$\begin{pmatrix} 1 & i \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} = 0. \quad \neq \text{ Why.}$$

What I want.

$\bar{z}^T z = |z_1|^2$ is good.

When we ^{T} we $\bar{z}$

$$\begin{pmatrix} 1 & -i \end{pmatrix}\begin{pmatrix} 1 \\ i \end{pmatrix} = 1+1 = 2$$

- $z^H z$ - z Hermitian z. $-|z_1|^2 + \dots |z_n|^2$
- Innerproduct $y^T x$.

So its $y^H x \rightarrow$ length squared.

Symmetric means $A = A^T$ if A is not complex

$\bar{A}^T = A$ or $A = A^H$ even if A is complex.

$$\begin{pmatrix} 2 & 3+i \\ 3-i & 5 \end{pmatrix}$$ - Hermetian matrices

- Real igen value
- $\perp$ eigen vectors.

$(q_1, q_2 \dots q_n)$ $\quad q_i^H q_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

othonormal basis

$$Q = \begin{pmatrix} q_1 & q_2 & \dots & q_n \\ \vdots & \vdots & & \end{pmatrix}$$

$\cancel{Q^T Q = I}$

now $Q^H Q = I$ if complex.

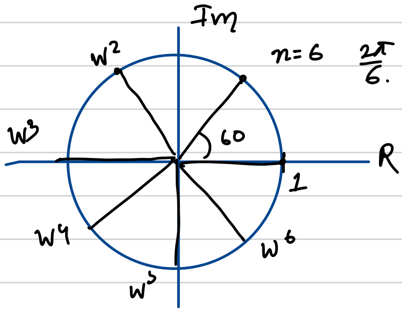
$\mathbb{R}^n \rightarrow \mathbb{C}^n$

Symmetric $\rightarrow$ Hermetian.
Orthogonal $\rightarrow$ Unitary like orthogonal matrix, unit vector. complete

Fourier Matrix :-

$$F_n = \begin{array}{c} \text{col} \\ \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & w & w^2 & & w^{n-1} \\ 1 & w^2 & w^4 & & w^{2(n-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & w^{n-1} & w^{2(n-1)} & & w^{(n-1)^2} \end{pmatrix} \end{array}$$

$(F_n)_{ij} = w^{ij}$



$w^n = 1$

$W = e^{i2\pi/n} = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$

$n = 4$
$w^4 = e^{2\pi i/4}$
$= i$
$i^2 = -1 \quad i^3 = -i$
$i^4 = 1.$

$$F = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}$$
$\qquad\qquad 2 \qquad 4$

- Columns are orthogonal.

$\bar{q}^2 q^4 = q_2^H q_4 = 0.$

- Columns are othonormal.

$|1^2| + |1^2| + |1|^2 + |1^2| = \sqrt{4} = 1/2$

$(1+i)(1-i) = 1 - i + i - i^2$
$= 2 + 2 + 2 + 2$
$= 8$??

Othonal colms mean

$$F_4^H F_4 = I$$

Connection

$$W_{64} = e^{i2\pi/64}$$

$$(W_{64})^2 = W_{32}$$

$$[F_{64}] = \begin{bmatrix} I & D \\ I & -D \end{bmatrix} \begin{bmatrix} F_{32} & 0 \\ 0 & F_{32} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Put then together

P = Takes even and then odd

Cost $2 \times (32)^2 + 32.$

$$D = \begin{bmatrix} 1 & & & \\ & w & & \\ & & w^2 & \\ & & & \ddots \\ & & & & w^{31} \end{bmatrix}$$

$$F_{32} = \begin{bmatrix} & \\ & \end{bmatrix} \begin{bmatrix} F_{16} & \\ & F_{16} \end{bmatrix} \begin{bmatrix} P \end{bmatrix}$$

$$\text{So } F_{64} = \begin{bmatrix} I & D \\ D & I \end{bmatrix} \begin{bmatrix} I\,D & 0 \\ I\,D & \\ 0 & I\,D \\ & I\,D \end{bmatrix} \begin{bmatrix} F_{16} & F_{16} & & \\ & & F_{16} & F_{16} \end{bmatrix} \begin{bmatrix} P & \\ & P \end{bmatrix} \begin{bmatrix} P \end{bmatrix}$$

So cost 2 $= 2\left[2[16]^2 + 16\right] + 32$ –

3 $2 [2 [$ –

Final cost $\rightarrow 6 \times 32$

$$= \log_2 64 \cdot \frac{64}{2}$$

$$\left(\frac{1}{2} n \log_2 n\right).$$

$n = 1024 = 2^{10}$

$n^2 = 1{,}000{,}000 = 1024 \times 1024$

$$\frac{1}{2} n \log_2 n = (1024)\frac{10}{2} = 5 \times 1024$$

rec: Diagonalizing A by constructing Λ and eigenvector matrix S.

$$A = \begin{pmatrix} 2 & 1-i \\ 1+i & 3 \end{pmatrix} \rightarrow \text{Its } H$$

so has real Eigen value.

$$(A - \lambda I) = 0$$

$$\begin{vmatrix} 2-\lambda & 1-i \\ 1+i & 3-\lambda \end{vmatrix} = 0 \quad (2-\lambda)(3-\lambda) - (1+i)(1-i)$$

$$\lambda = \underline{1, 4}.$$

Find eig vectors.

$$\lambda = 1 \quad \begin{vmatrix} 1 & 1-i \\ 1+i & 2 \end{vmatrix} \begin{vmatrix} \\ \\ \end{vmatrix} = 0 \quad R_2 = (1+i)R_1 - R_2$$

$$= (1-i)(1+i) - 2$$

$$= 2 - 2 = 0$$

$$\begin{vmatrix} 1 & 1-i \\ 0 & 0 \end{vmatrix} = \begin{pmatrix} 0 \end{pmatrix} \qquad x =$$

$$= \begin{pmatrix} i-1 \\ 1 \end{pmatrix}$$

$$\lambda = 4 \quad \begin{vmatrix} -2 & 1-i \\ 1+i & -1 \end{vmatrix} \quad R_{12} = 2(1+i) + (i+1)\bar{2} = 0$$

$R_{22} =$

$$= 2(-1) + (1-i)(i+1)$$

$$= -2 + (i + 1 - i^2 - i)$$

$$= -2 + (2) = 0$$

$$-2x_1 + 1 - i = 0$$

$$0$$

$$= \begin{pmatrix} (1-i)/2 \\ 1 \end{pmatrix} \times (i+1) = \begin{pmatrix} 2/2 \\ (1+i) \end{pmatrix}$$

Length $\; |1|^2 + |1+i|^2 = \sqrt{3}$

$$u = \begin{pmatrix} 1/\sqrt{3} \\ 1+i/\sqrt{3} \end{pmatrix} \qquad S = \frac{1}{\sqrt{3}} \begin{pmatrix} 1-i & 1 \\ -1 & 1+i \end{pmatrix}$$

$$S^{-1} = \bar{S}^T \quad \text{(Unitary)}$$

like Orthogonal.

Lecture 27: Positive Definite Matrices and Minima

- Positive definite matrix (Tests)
- Test for minimum. ($x^TAx > 0$)
- Ellipsoids in R^n

Matrix is Symmetric; Positive definite?

$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

1.) $\lambda_1 > 0 \quad \lambda_2 > 0$.
2) all dets $> 0 \quad a > 0 \quad ac - b^2 > 0$
3) Pivots $a > 0 \quad \frac{ac - b}{a} > 0$.

4) $x^TAx > 0$

Eg. $\begin{pmatrix} 2 & 6 \\ 6 & 19 \end{pmatrix}$

if I have $19 \rightarrow 18$. well in a borderline.

pos semidefinite.

1) λ values $= 0$, other $= 20$. ≥ 0. $\rightarrow$ semidefinite

2) Pivots 2, (One pivot cause singular)

3) x^TAx

$$\underset{x^t}{(x_1 \; x_2)} \underset{A}{\begin{pmatrix} 2 & 6 \\ 6 & 18 \end{pmatrix}} \underset{x}{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}$$

$$= 2x_1x_1 + 6x_1x_2 + 18x_2^2 + 6x_1x_2$$

$$= 2x_1^2 + 12x_1x_2 + 18x_2^2 > 0.$$

Quadratic form. > 0

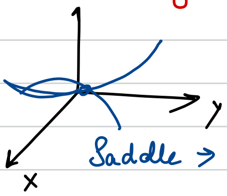
$\begin{pmatrix}2 & 6\\ 6 & 7\end{pmatrix} \to$ pivots $\begin{pmatrix}2 & \\ & -11\end{pmatrix}$

$2x_1^2 + 12x_1x_2 + 7x_2^2 > 0.$

$x = (1\ \ -1)$

Negative.

Graph $f(x,y) = \vec{x}^T A \vec{x}$
$= ax^2 + 2bxy + cy^2$

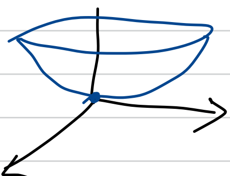


Saddle →

$\begin{pmatrix}2 & 6\\ 6 & 20\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$

det = 4   trace = 22.

$2x_1^2 + 12x_1x_2 + 20x_2^2.$



$f(x,y) = 2x^2 + 12xy + 20y^2$

1st der = 0
2nd der = +ve.

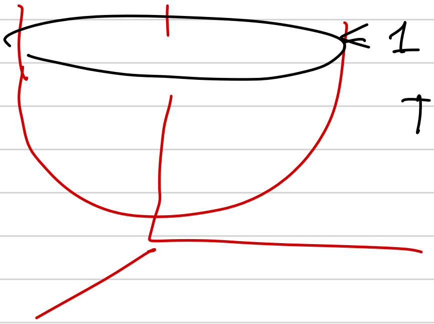
$= 2(x+3y)^2 + 2y^2$
+ve
All squares.

Suppose 7.
$2(x+3y)^2$
−T

If its positive for all x. Then positive.

Calculus: MIN ~ $\frac{d^2u}{dx^2} > 0$

18.06: MIN ~ $f(x_1, x_2 \ldots x_n)$ MATRIX of 2nd DE. is positive definite.



1

Then ellipse $x^TAx = 1$

Cut through saddle I get hyperbola

$\begin{pmatrix}2 & 6\\ 6 & 20\end{pmatrix} \to \begin{pmatrix}2 & 6\\ 0 & 2\end{pmatrix}$ U

$L = \begin{pmatrix}1 & 0\\ 3 & 1\end{pmatrix}$

$2(x+3y)^2 + 2y^2$

PIVOTS   PIVOTS

So +ve pivots give +ve definite.

Suppose 18
$(\quad)^2$ − Exact.

MATRIX OF 2^{nd} derivative

$$\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} \rightarrow \text{+ve definite}$$

Function of 2 variable

$$f_{xx} f_{yy} - f_{xy}^2 > 0.$$

3×3 example.

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

det 1 = 2
det 2×2 = 3
det 3×3 = 4.

Pivots, $2, 3/2, 4/3$

eigen values - All +ve,
$2-\sqrt{2}, 2, 2+\sqrt{2}$.

$x^T A x$

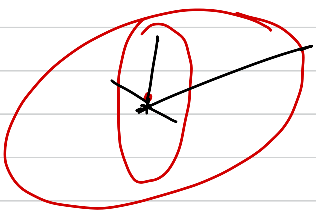
$x = (x_1 \; x_2 \; x_3)$

$$x^T A x = 2x_1^2 - 2x_1 x_2 + 2x_2^2 - 2x_2 x_3 + 2x_3^2 > 0$$

$$2(x_1 + x_2 + x_3)^2 + 3/2\, x_2^2 + 4/3\, x_3^2$$

If I cut at height 1 (____) = 1

→ Gives me an ellipsoid.



1 eg Repeated
1 none

$$A = Q \Lambda Q^T$$

Exc: Find which values of c is

$$B = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2+c \end{pmatrix}$$

- positive definite
- " semidefinite

det B, det bigger =

$= 3c$

- $c > 0$
- $c \geq 0$.

Pivot test

Check $x^T A x$

$$= 2x_1^2 + 2x_2^2 + (2+c)x_3^2 - 2x_1x_2 - 2x_1x_3 - 2x_2x_3.$$

$$(a+b+c)^2 = a^2+b^2+c^2 + 2ab + 2ac + 2bc$$

$$= \underset{\text{Pivot}}{2}\underbrace{\left(x - \frac{1}{2}y - \frac{1}{2}z\right)^2} + \underset{\text{Pivot}}{\frac{3}{2}}\underbrace{(y-z)^2} + \underset{\text{Pivot}}{c}\,\underbrace{z^2} \geq 0.$$

$c > 0$. Then

$c = 0$.

Check, Didn't Understand!

$x^T A x > 0$ (except for $x = 0$)

• Where do +ve definite matrix come from.

→ Suppose A is +ve definite, is the A^{-1} also +ve definite

$$\lambda(A) = \frac{1}{\lambda(A^{-1})}$$

So A (+ve def.) → A^{-1} (+ve.)

• If $A + B$ are pos def.

$x^T (A+B) x$

$x^T A x > 0$ $\quad x^T B x > 0$.

So $A+B$ ✓.

Now A is m by n.

→ Not +ve definite, not even symmetric

$A^T A$ – square
– symmetric
– pos definite ??

$x^T (A^T A) x$

→ Can never be negative. $= (x^T A^T)(Ax)$
$= (Ax)^T (Ax)$ length $\|Ax\|^2 \geq 0$.

$\|Ax\|^2 = 0$ only if length $= 0$ and $A = 0$.

$\therefore \|Ax\|^2 > 0$.

m by n, (rank n) $m > n$.

$\therefore$ pos definite.

rank n, then A^TA is invertible.

+ve definite matrix - never have to do row exchanges.

A and B are similar.

nxn matrices

- means: for some M

$$B = M^{-1}AM.$$

Eg:

$$S^{-1}AS = \Lambda$$

if I have M, similar to each other.

Outstanding grey is 1

eg: $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ $\Lambda = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$ $\lambda_A = 3,1$

$$\begin{pmatrix} 1 & -4 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\overset{M}{\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}} = \begin{pmatrix} 1 & -4 \\ 0 & 1 \end{pmatrix}\begin{bmatrix} 2 & 9 \\ 1 & 6 \end{bmatrix} = \begin{bmatrix} -2 & -15 \\ 1 & 6 \end{bmatrix}$$

B

$\lambda_B = 3,1.$

• They have something in common.

$$B = M^{-1}AM.$$

Same eigen value

$\begin{bmatrix} 3 & 7 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 7 \\ 0 & 3 \end{bmatrix}$

$M_2^{-1}AM_2$ $M_3^{-1}AM_3$

Similar matrices have the same eigen value. λ's !!

- $Ax = \lambda x$

- $B = M^{-1} A M$.

$\rightarrow \quad Ax = \lambda x$

$$A M M^{-1} x = \lambda x$$

$$M^{-1} A M M^{-1} x = \lambda M^{-1} x$$

$$\Rightarrow \quad B \underline{M^{-1} x} = \lambda \underline{M^{-1} x}$$

$\therefore$ Eigen value same
vectors different.

Eigen vector $B = M^{-1}$ eigen vector A.

Just like
$S^{-1} A S = \Lambda$
Eigen vector of
$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

- If eigen values same, we may not have full set of eigen vectors and be able to diagonalize.

BAD CASE : $\lambda_1 = \lambda_2$

- The matrix might not be diagonizable

Suppose $\lambda_1 = \lambda_2 = 4$.

One family : $\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} \quad \begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}$

Not same family.

Big family includes

The only matrix $4I$ is I

$$M^{-1} \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} M = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

So cI is one small family.

Other family contains all matrices that are similar.

eg $\begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}$

I can't make it diagonizable.

- $\begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}$ → Eigen values are 4, 4 but we can't diagonalize it because this has just one eigen vector.

$\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

The best guy in the family is this $\begin{pmatrix} a & 1 \\ 0 & b \end{pmatrix}$

- Not completely diagonal.

- Other members in the family. Jordan found the best looking matrix including the non-diagonizable (less eigen-vectors)/

- Not easy to find jordan form for a general matrix

→ Depend on eigen values.
→ Know exactly rank.

Eg: More members of family.

$\begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}, \begin{pmatrix} 5 & 1 \\ -1 & 3 \end{pmatrix}, \begin{pmatrix} 4 & 0 \\ 17 & 4 \end{pmatrix}$

↳ Can't get to diagonal.

$$\begin{pmatrix} a & 4 \\ & 8-a \end{pmatrix}$$ trace = 8, det = 16

- Same no of eigenvectors [Not enough).
- Same eigen values .!

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$ $\lambda = 0, 0, 0, 0.$

dim (null space) . = 2.

No (independent rows) = 2
col = 2.

$\therefore$ 4-2 = 2. (N)

2 eigen vectors.

$$\begin{pmatrix} 0 & 1 & 7 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$ Same - jordan block

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

4 eigen values $= 0$.

Rank is 2.

- 2 eigen vectors
- But its not similar to previous one.
- Same count of eigen vectors !! but still not similar
- So just same no of eigen vectors does not ensure they are all similar for non-diagonizable matrices.

Jordan block:

$$J_i = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \lambda_i & 1 & \\ & & \lambda_i & \ddots \\ 0 & & & \lambda_i \end{bmatrix} \text{ - 1 eigen vector.}$$

Each block has one eigen vector.

$$\left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right) \qquad \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

λ: 0,0,0,0 — λ: 0,0,0,0.

No $v_\lambda = 2$ — No $v_\lambda = 2$

$N(A) = 2$ — $N(A) = 2$

But with Jordan block, not similar

Every square A is similar to a Jordan matrix J.

$$J = \begin{bmatrix} J_1 & & \\ & J_2 & \\ & & J_n \end{bmatrix}$$

No of blocks = No of eigen vectors.

Steps

Start with A

: If it's λ distinct, then similar to diagonal matrix $J = \text{diag}(\lambda)$

If we start with A which has λ similar, then we can use jordan form

Rec: Which are true?

a) A & B are similar then $2A^3 + A - 3I$, $2B^3 + B - 3I$ are similar

b) A, B are 3x3 matrice, with $\lambda = 1, 0, -1$, A & B are similar.

c) $J_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$ $J_2 = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ are similar.

a) $C = \_$ $D = \_$

$M^{-1}(2A^3 + A - 3I)M^{-1}$

$= 2MAM^{-1}MAM^{-1}MAM^{-1} + MAM^{-1} - 3MIM^{-1}$

$= 2BBB + B - 3B$ ✓.

A B similar, Any polynomial similar!

b) True $A = S\Lambda S^{-1}$, $\Lambda = \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}$

$B = T\Lambda T^{-1}$

$\Lambda = T^{-1}BT$

$A = ST^{-1}BTS^{-1}$

?? c) False. $J_1 + I = \begin{pmatrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{pmatrix} - N(J_1 + I) = 1$

Not sure about this one $J_2 + I = \begin{pmatrix} 0 & 1 & \\ & 0 & \\ & & 0 \end{pmatrix} - N(J_1 + I) = 2$

Lecture 29. Singular Value Decomposition (SVD)

$$A = \underset{\text{orthogonal}}{U}\overset{\text{Diag}}{\Sigma} V^T \text{ orthogonal}$$

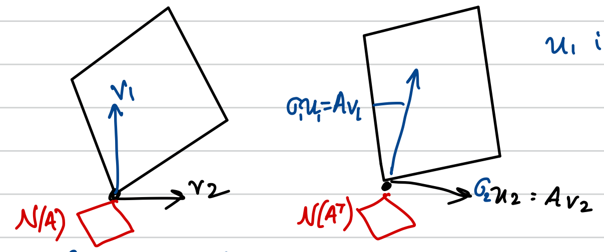
- Any matrix what soever.

$A = Q \Lambda Q^T$ (Eigen vector, eigen value)

$A = S \Lambda S^{-1}$ (Sometimes eigen vectors aren't orthogonal)

R^n row space — Column space R^m.

u_1 is unit.



In SVD Im looking for orthogonal basis in row space go to ortho basis in column space.

$$A \begin{bmatrix} v_1 & v_2 & \dots & v_r \end{bmatrix}$$

- Make them also orthonormal $A(v_1) \to \sigma_1 u_1$

$$A \left(v_1 \, v_2 \, \cdots \, v_r \right) = \begin{bmatrix} u_1 & u_2 & \cdots & u_r \end{bmatrix} \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_r \end{bmatrix}$$

$$\therefore AV = U\Sigma$$

Orthonormal basis in column space

Find orthonormal basis in Row space

Eg: $A = \begin{pmatrix} 4 & 4 \\ -3 & 3 \end{pmatrix}$

Find v_1, v_2 in Row space $\mathbb{R}^2$

u_1, u_2 in Col space $\mathbb{R}^2$

$\sigma_1 > 0 \quad \sigma_2 > 0$

• If there's a null space :-

Suppose

$$\begin{bmatrix} v_1 \cdots v_r\, v_n \end{bmatrix} = \begin{bmatrix} u_1 \cdots u_r & \overbrace{u_{r-n} \quad u_n}^{\text{Null space}} \end{bmatrix} \begin{bmatrix} \sigma_1 & & & & \\ & \ddots & & & \\ & & \sigma_r & & \\ & & & 0 & \\ & & & & 0 \end{bmatrix}$$

My Goal : $Av_1 = \sigma_1 u_1$

$Av_2 = \sigma_2 u_2$

$$\begin{aligned} AV &= U\Sigma \\ &= U\Sigma V^{-1} \\ &= U\Sigma V^T \quad (V = \text{orthonormal}) \end{aligned}$$

I want to make U disappear.

• $$\begin{aligned} A^TA &= V\Sigma^T U^T U \Sigma V^T \\ &= V\Sigma^T\Sigma V^T \quad (U^TU = I) \\ &= V\Sigma^2 V^T \end{aligned}$$

$A^TA = V \begin{bmatrix} \sigma_1^2 & & \\ & \sigma_2^2 & \\ & & \sigma_3^2 \ldots \end{bmatrix} V^T$ - Perfect $A^TA - Q\Lambda Q^T$

$$\begin{aligned} AA^T &= U\Sigma V^T V \Sigma^T U^T \\ &= U\Sigma^2 U^T \end{aligned}$$

So V is eigen vectors of A^TA

U " " " of AA^T

σ are $\sqrt{\sigma}$ (Positive)

Example:

$$A^TA = \begin{pmatrix} 4 & -3 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 4 & 4 \\ -3 & 3 \end{pmatrix} = \begin{pmatrix} 25 & 7 \\ 7 & 25 \end{pmatrix}$$

$$V_{\lambda_1} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \qquad \lambda_1 = 32$$

$$V_{\lambda_2} = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} \qquad \lambda_2 = 18.$$

$$\begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} = \underset{U}{} \ \underset{\Sigma}{\begin{bmatrix} \sqrt{32} & 0 \\ 0 & \sqrt{18} \end{bmatrix}} \ \underset{V^T}{\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}}$$

$$AA^T = \begin{pmatrix} 4 & 4 \\ -3 & 3 \end{pmatrix}\begin{pmatrix} 4 & -3 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} 32 & 0 \\ 0 & 18 \end{pmatrix}$$

$$\lambda_1 = 32 \qquad V_{\lambda_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\lambda_2 = 18 \qquad V_{\lambda_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Same Eigen values.

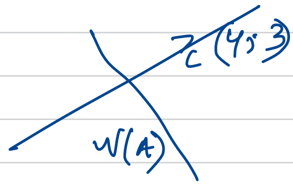
$(AA^T = A^TA)$ — Same
AB same as BA.

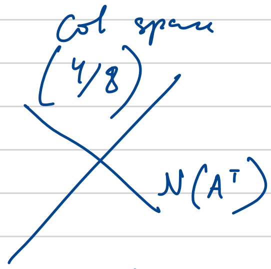
$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{bmatrix} \sqrt{32} & 0 \\ 0 & \sqrt{18} \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

whof = -

Row space.

Ex 2 $A = \begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix}$





$v_1 = \begin{pmatrix} .8 \\ .6 \end{pmatrix}$

$u_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \frac{1}{\sqrt{5}}.$

$A^TA = \begin{pmatrix} 4 & 8 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} 4 & 3 \\ 8 & 6 \end{pmatrix} = \begin{pmatrix} 80 & 60 \\ 60 & 45 \end{pmatrix}$

Rank 1, so $\lambda = 0$

other $\lambda = 125$. (Trace)

$$\begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} & \\ 2/\sqrt{5} & \end{bmatrix} \underbrace{\begin{bmatrix} \sigma_1^2 & 0 \\ 0 & 0 \end{bmatrix}}_{\Sigma} \Big[$$

$$= \begin{pmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & -1/\sqrt{5} \end{pmatrix} \begin{bmatrix} \sqrt{125} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} .8 & .6 \\ .6 & -.8 \end{bmatrix}$$

$$= \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & +2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} \sqrt{125} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} .8 & .6 \\ .6 & -.8 \end{pmatrix}$$

0.

$U \quad \Sigma \quad V^T$

- $v_1 \ldots v_r$ are orthonormal basis for row space.
- $u_1, \ldots u_r$ " " " col "
- v_{r+1}, v_n " " " $N(A)$
- $u_{r+1} \ldots u_m$ — — — $N(A^T)$

Ne: SVD

Find singular decomp of $C = \begin{pmatrix} 5 & 5 \\ -1 & 7 \end{pmatrix}$

$$C^TC = \begin{pmatrix} 5 & -1 \\ 5 & 7 \end{pmatrix}\begin{pmatrix} 5 & 5 \\ -1 & 7 \end{pmatrix} = \begin{pmatrix} 26 & 18 \\ 18 & 74 \end{pmatrix}$$

$$\lambda^2 - 100\lambda + 1600$$
$$(\lambda - 80)(\lambda - 20)$$

$\lambda = 80$ $\quad \begin{pmatrix} 26-80 & 18 \\ 18 & 74-80 \end{pmatrix} = \begin{pmatrix} -54 & 18 \\ 18 & -6 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{10} \\ 3/\sqrt{10} \end{pmatrix}$

$\lambda = 20$ $\quad \begin{pmatrix} 26-20 & 18 \\ 18 & 74-20 \end{pmatrix} = \begin{pmatrix} 6 & 18 \\ 18 & 54 \end{pmatrix}$ $\quad v_1 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$ $\quad \begin{pmatrix} -3/\sqrt{10} \\ 1/\sqrt{30} \end{pmatrix}$

$$V = \begin{pmatrix} -3/\sqrt{10} & 1/\sqrt{10} \\ 1/\sqrt{10} & 3/\sqrt{10} \end{pmatrix}\begin{pmatrix} \sqrt{20} & 0 \\ 0 & \sqrt{80} \end{pmatrix}$$

$Av = \sigma$

$$\begin{pmatrix} 5 & 5 \\ -1 & 7 \end{pmatrix}\begin{pmatrix} -3/\sqrt{10} & 1/\sqrt{10} \\ 1/\sqrt{10} & 3/\sqrt{10} \end{pmatrix} = \begin{pmatrix} -\sqrt{10} & 2\sqrt{10} \\ \sqrt{10} & 2\sqrt{10} \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}}_{U}\underbrace{\begin{pmatrix} \sqrt{20} & 0 \\ 0 & \sqrt{80} \end{pmatrix}}_{\Sigma}$$

Lecture 30: L.T and their Matrices

- Without coords: no matrix
  with coords: Matrix.

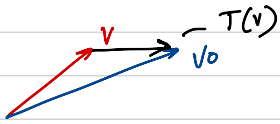
Example 1: Projection:

$T: \mathbb{R}^2 \to \mathbb{R}^2$

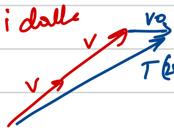
L.T: $T(v+w) = T(v) + T(w)$
$T(cv) = cT(v)$

$$T(\alpha v + \beta w) = \alpha T(v) + \beta T(w)$$

Not.
Example 2: Shift whole plane by v_0



Is it linear? No.



- $T(0) = 0$.

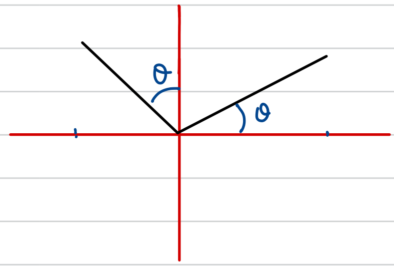
Not Example
- $T(v) = \|v\|$
  $T: \mathbb{R}^3 - \mathbb{R}^1$

$\rightarrow T(0) = 0$
$T(2v) = 2\|v\|$
$T(-2v) = 2\|v\|$

Not satisfied.

• Yes example: Rot by 45°

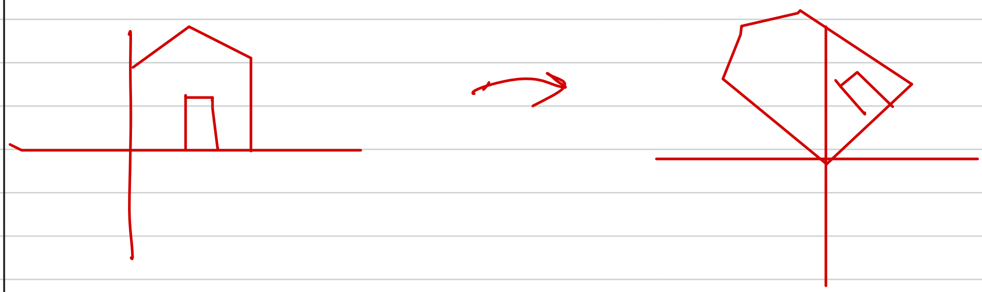
$T: \mathbb{R}^2 \to \mathbb{R}^2$



$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

$T(2v) = 2v.$

$T(a+b) = T(c) + T(b)$



Example 3.11 Matrix A.

$T(v) = Av.$

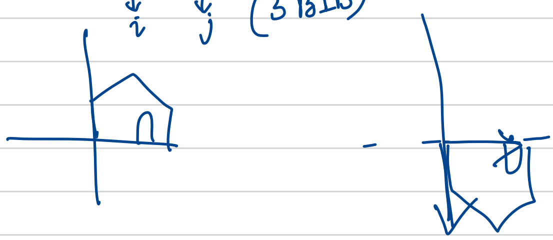
$A(v+w) = A(v) + A(w)$

$A(cw) = cAw.$

• Apply A on the whole plane.

$$A = \begin{array}{c} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ \quad\ \downarrow \quad \downarrow \\ \quad\ i \quad\ j \end{array}$$

(3B1B)



- LT is abstract description.
- Find the matrix behind.

Start: $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

Example: $\underbrace{\begin{pmatrix} & & \\ & & \end{pmatrix}}_{2\times 3}\begin{pmatrix} v \end{pmatrix}$

Information needed to know $T(v)$ for all inputs.

$T(v_1)$, $T(v_2)$. Do I need to know more?

- With basis v_1, v_2 I know whole thing if v_1 & v_2 are independant.
- So $T(v_1) \dots T(v_n)$ for any input basis $v_1 \dots v_n$.

Every $v = c_1 v_1 + \dots + c_n v_n$.

If I know what $T(v_1) \dots T(v_n)$

Know $T(v) = c_1 T(v_1) \dots c_n T(v_n)$ - Because of linearity.

L.T $\rightarrow$ Matrix.

Once you decide on a basis.

- Coordinates come from a basis.

" of $v = c_1 v_1 + \dots + c_n v_n$.

$$v = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 4\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

I might have some other basis

- I can use $\lambda, V\lambda$ as basis.

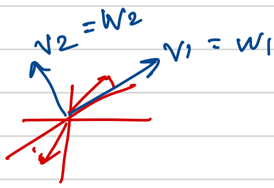
Construct matrix A that represents L.T. $T(x)$.

$$T: \mathbb{R}^n \to \mathbb{R}^m$$

I need an input basis
output.

Choose basis $v_1, \dots v_n$ for input $\mathbb{R}^n$
" $w_1, \dots w_m$ for output $\mathbb{R}^m$

Step: Matrix A



Eg: Projection : $\mathbb{R}^2 \to \mathbb{R}^1$

- I will choose basis on input and output.

$$V = \underset{\text{basis}}{c_1 V_1} + \underset{\text{basis}}{c_2 V_2}$$

Projection leaves $v_2 = 0$.

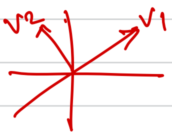
$$T(V) = c_1 V_1$$

Input (c_1, c_2)
Output $(c_1, 0)$

$$\underset{A.}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}} \underset{\text{input.}}{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}} = \underset{\text{output}}{\begin{bmatrix} c_1 \\ 0 \end{bmatrix}}$$

$\downarrow \lambda$

Here: input + output basis same



Eigen vector basis leads to diagonal matrix Λ.

$$\begin{array}{c} \lambda_1 \\ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \lambda_2 \end{array}$$

P in standard basis 45° wiy.

$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = w_1 \qquad v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} w_2.$

$$P = \frac{aa^T}{a^Ta} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

$$P\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} \qquad P\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$$

In the best bases.

Rule to find A. Given $v_1 \ldots v_n$, $w_1 \ldots w_m$.

1st col - Write $T(v_1) = a_{11}w_1 + a_{21}w_2 + a_{m1}w_m$

2nd col - $T(v_2) = a_{12}w_1 + a_{22}w_2 + \ldots a_{m2}w_m$

A (input) = (out coords)

If we put $v = (1\ 0\ 0 \ldots)$

Then $T(v_1)$ = coords of v_1

$T = \frac{d}{dx}$ (Linear!)

Input : $c_1 + c_2x + c_3x^2$

basis : $1, x, x^2$

Output = $c_2 + 2c_3x$

basis = $1, x$.

$\mathbb{R}^3 - \mathbb{R}^2$,

$$A\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} c_2 \\ 2c_3 \end{bmatrix} \qquad A = \underset{2\times 3}{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}}$$

A^{-1} gives the inverse transformation sec : LT

Let $T(A) = A^T$, A is 2x2

1) why is T linear? What is T^{-1}

2) Write down T in

$v_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $v_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $v_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $v_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$w_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $w_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ $w_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $w_4 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

3) Eigenvalues/ vectors of T?

1) $T(A+B) = (A+B)^T = B^T + A^T = T(B) + T(A)$

$T(2A) = 2A^T = 2T(A)$ $T^2 = I = T^{-1} = T.$

2) $Tv_1 = v_1$
$Tv_2 = v_3$
$Tv_3 = v_2$
$Tv_4 = v_4$

$$= \begin{array}{cccc} Tv_1 & Tv_2 & Tv_3 & Tv_4 \end{array}$$
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

3) $T\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$

$\downarrow$ $\qquad \downarrow T$

$\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \\ 2 \\ 4 \end{pmatrix}$

$Tw_1 = w_1$
$Tw_2 = w_2$
$Tw_3 = w_3$
$Tw_4 = -w_4$

We have four eigen values.

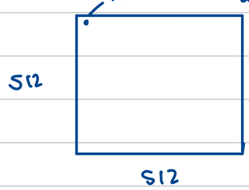
4) Eig / Eigvector of T.

Lecture 31

- Change of basis
- Compression images
- Transformation $\Leftrightarrow$ Matrix.

Compression - - Compressed images
- Lossless compression.

Pixel. - 0 - 255.



2^8 possi

$0 \leq x_i < 255$

$x \in \mathbb{R}^n$

$n = (512)^2$

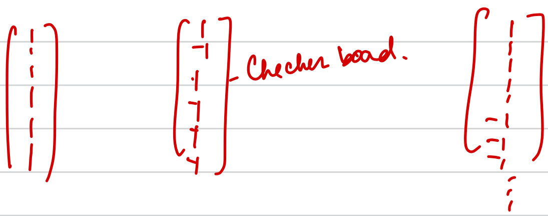
Standard basis - $x = \begin{bmatrix} \\ \\ 121 \\ 124 \end{bmatrix}^{512}$

- Colors are close. and correlated.
- Standard basis lousy.
- Neighbouring pixels.

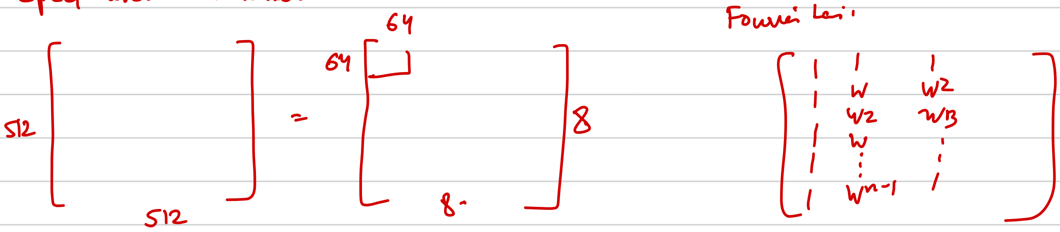
Standard basis.

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \end{bmatrix}$$

Better basis



JPEG uses - Fourier basis. 8x8



Change basis in 64 dimensional step.

$$\text{Signal} \xrightarrow[\text{lossless}]{} \overset{64}{\text{coeff } C} \xrightarrow[\text{loss.}]{\text{compression}} \text{(Throw away compression)}$$

$\downarrow$

$\hat{C}$ (reduced c) many zeros

$\begin{pmatrix} \vdots \end{pmatrix} \leftarrow$ more $\qquad \begin{pmatrix} \vdots \end{pmatrix} \leftarrow$ c less.

64-4.

$$\hat{x} = \sum_{1}^{4} \hat{c}_i v_i \quad \text{(signal)}$$

Video → Sequence of images

One image ⟷ Next image.

Correlated (Using correlation)

Wavelets

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}$$

$$\begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ \\ \\ \end{bmatrix} \rightarrow c_1 \begin{pmatrix} \\ \\ \\ \end{pmatrix} + c_2 \begin{pmatrix} \\ \\ \\ \end{pmatrix} + c_3 \begin{pmatrix} \\ \\ \\ \end{pmatrix} + \dots$$

Pixel

Standard.

$$x = c_1 w_1 + c_2 w_2 + c_3 w_3 + c_4 w_4 + \dots c_8 w_8.$$

$$x = \begin{bmatrix} w_1 & w_2 & w_2 & w_4 & w_5 & w_6 & w_7 & w_8 \\ \\ \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \end{bmatrix}$$

$$X = WC$$

$$C = W^{-1}X$$

Good basis: $W \Rightarrow$ fast inverse.

2) Fourier transform FFT

FWaveletT.

3) Wavelet $Q^T = Q^{-1}$

Throw away $c_5\ c_6\ c_7\ c_8 \Leftarrow$ Good compression

- Jpeg 2000 - use Wavelets.

Change of basis:

Col of W = new basis vector.

$[x]_{\text{old basis}} \longrightarrow [c]$ new basis.

$x = Wc$.

LT : T : with respect $v_1, \dots, v_8$ it has matrix $\underline{A}$.

T with " $w_1, \dots w_8$ it " " $\underline{B}$.

- A and B are similar.

- $B = M^{-1} A M$

- If I change basis, every vector has a new coord.

What is A? Using basis $v_1 \dots v_8$.

Know T completely from $T(v_1) \dots T(v_8)$

$x = c_1 v_1 + \dots$

$T(x) = c_1 T(v_1) \dots$

Write $T(v_1) = a_{11} v_1 + a_{21} v_2 + a \dots$

$T(v_2) = a_{12} v_1 + a_{22} v_2 + \dots$

$$[A] = \begin{bmatrix} a_{11} & a_{21} & & \\ \vdots & \vdots & & \\ a_{18} & a_{28} & & \end{bmatrix}$$

eg: Eigen vector basis

$T(v_i) = \lambda_i v_i$

What is A?

$$A = \begin{bmatrix} \lambda_1 & 0 & \cdots & & \\ 0 & \lambda_2 & \cdots & & \\ 0 & 0 & & \ddots & \\ 0 & 0 & & & \\ 0 & 0 & \cdots & \cdots & \lambda_n \end{bmatrix}$$

1st input is v_1
$\to \lambda_1 v_1$
Input u v_2
Out v_2

Rec: $y(x) = 1, x, x^2$.

Let w_1, w_2, w_3 be diff basis chose values at .

$x = -1, 0, 1$ are

x	w_1	w_2	w_3	y	1	x	x^2
−1	1	0	0	6	1	−1	1
0	0	1	0	5	1	0	0
1	0	0	1	4	1	−1	1

These are the values.

$5 - 1 \Rightarrow 6\ 5\ 4$

a) Express $y(x) = -x+5$ in this basis!
b) Change of basis $(1, x, x^2) \sim (w_1, w_2, w_3)$
c) Find the matrix of "taking derivative" in both basis!

$$y(x) = \alpha w_1(x) + \beta w_2(x) + \gamma w_3(x) \Big/ \begin{matrix} x=-1 \\ x=0 \\ x=1 \end{matrix}$$

$$y(-1) = \alpha w_1(-1) + \beta \cdot w_2(-1) + \gamma w_3(-1)$$
$$(0) = \alpha w_1(0)$$
$$\vdots$$
$$(1) =$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \\ 4 \end{pmatrix}$$

$$A' = \begin{array}{c} \begin{matrix} w_1 & w_2 & w_3 \end{matrix} \\ \begin{pmatrix} 0 & 1 & 0 \\ -1/2 & 0 & 1/2 \\ 1/2 & -1 & 1/2 \end{pmatrix} \end{array} \begin{matrix} 1 \\ x \\ x^2 \end{matrix}$$

b) $1 = w_1 + w_2 + w_3$
$x = -w_1 - w_3$
$x^2 = w_1 + w_3$

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{pmatrix} 5 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \\ 4 \end{pmatrix}$$

$=$

c)

$$D_x = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{matrix} 1 \\ x \\ x^2 \end{matrix}$$

$$D_w = A . D . A^{-1} \Leftarrow .$$

back to w

D

w to x

Lecture 33 : Left & Right Inverse, Psuedo inverse.

Left Inverse.
Right Inverse.
Psuedo Inverse.

18.03 LA.3: Complete Solutions, Nullspace, Space, Dimension, Basis

[1] Particular solutions

Matrix Example

Consider the matrix equation

$$\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 8 \end{bmatrix}$$

The complete solution to this equation is the line $x_1 + x_2 = 8$. The homogeneous solution, or the *nullspace* is the set of solutions $x_1 + x_2 = 0$. This is all of the points on the line through the origin. The homogeneous and complete solutions are picture in the figure below.

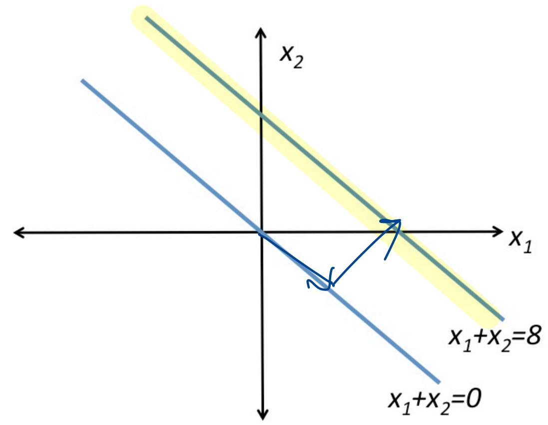


Figure 1: The homogeneous and complete solutions

To describe a complete solution it suffices to choose one particular solution, and add to it, any homogeneous solution. For our particular solution, we might choose

$$\begin{bmatrix} 8 \\ 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 \\ 8 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

If we add any homogeneous solution to this particular solution, you move along the line $x_1 + x_2 = 8$. All this equation does is take the equation for the homogeneous line, and move the origin of that line to the particular solution!

How do solve this equation in Matlab? We type

```
x = [1 1] \ [8]
```

In general we write

```
x = A \ b
```

Differential Equations Example

Let's consider the linear differential equation with initial condition given:

$$\frac{dy}{dt} + y = 1$$
$$y(0)$$

To solve this equation, we can find one particular solution and add to it any homogeneous solution. The homogeneous solution that satisfies the initial condition is $x_h = y(0)e^{-t}$. So then a particular solution must satisfy $y_p(0) = 0$ so that $x_p(0) + x_h(0) = y(0)$, and such a particular solution is $y_p = 1 - e^{-t}$. The complete solution is then:

complete solution		**particular solution**		**homogeneous solution**
y	$=$	$1 - e^{-t}$	$+$	$y(0)e^{-t}$

However, maybe you prefer to take the steady state solution. The steady state solution is when the derivative term vanishes, $\frac{dy}{dt} = 0$. So instead we can choose the particular solution $y_p = 1$. That's an excellent solution to choose. Then in order to add to this an homogeneous solution, we add some multiple of e^{-t} so that at $t = 0$ the complete solution is equal to $y(0)$ and we find

$$\begin{array}{ccccc} \textbf{complete solution} & & \textbf{particular solution} & & \textbf{homogeneous solution} \\ \\ y & = & 1 & + & (y(0)-1)e^{-t} \\ & & \uparrow & & \uparrow \\ & & \textit{steady state solution} & & \textit{transient solution} \end{array}$$

The solution 1 is an important solution, because all solutions, no matter what initial condition, will approach the steady state solution $y = 1$.

There is not only 1 particular solution. There are many, but we have to choose 1 and live with it. But any particular solution will do.

[2] Complete Solutions

Matrix Example

Let's solve the system:

$$\begin{array}{llll} x_1 & & +cx_3 & = b_1 \\ & x_2 & +dx_3 & = b_2 \end{array}$$

What is the matrix for this system of equations?

$$A = \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & d \end{bmatrix}$$

Notice that A is already in row echelon form! But we could start with any system

$$\begin{array}{llll} x_1 & +3x_2 & +5x_3 & = b_1 \\ 4x_1 & +7x_2 & +19x_3 & = b_2 \end{array}$$

and first do a sequence of row operations to obtain a row echelon matrix. (Don't forget to do the same operations to b_1 and b_2:

$$\begin{bmatrix} 2 & 3 & 5 & \vdots & b_1 \\ 4 & 7 & 17 & \vdots & b_2 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 3 & 5 & \vdots & b_1 \\ 0 & 1 & 9 & \vdots & b_2 - 2b_1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3/2 & 5/2 & \vdots & b_1/2 \\ 0 & 1 & 9 & \vdots & b_2 - 2b_1 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & -11 & \vdots & 5b_1/2 - 3b_2/2 \\ 0 & 1 & 9 & \vdots & b_2 - 2b_1 \end{bmatrix}$$

Let's find the complete solution to $Ax = b$ for the matrix

$$A = \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & d \end{bmatrix}.$$

Geometrically, what are we talking about?

The solution to each equation is a plane, and the planes intersect in a line. That line is the complete solution. It doesn't go through 0! Only solutions to the equation $A\mathbf{x} = \mathbf{0}$ will go through 0!

So let's find 1 particular solution, and all homogeneous solutions.

Recommended particular solution: Set the free variable $\mathbf{x}_3 = 0$. Then

$$\mathbf{x}_p = \begin{bmatrix} b_1 \\ b_2 \\ 0 \end{bmatrix}.$$

We could let the free variable be any value, but 0 is a nice choice because with a reduced echelon matrix, it is easy to read off the solution.

So what about the homogenous, or null solution. I will write $\mathbf{x}_n$ instead of $\mathbf{x}_h$ for the null solution of a linear system, but this is the same as the homogeneous solution. So now we are solving $A\mathbf{x} = \mathbf{0}$. The only bad choice is $x_3 = 0$, since that is the zero solution, which we already know. So instead we choose $\mathbf{x}_3 = 1$. We get

$$\mathbf{x}_n = C \begin{bmatrix} -c \\ -d \\ 1 \end{bmatrix}$$

The complete solution is

$$x_{complete} = \begin{bmatrix} b_1 \\ b_2 \\ 0 \end{bmatrix} + C \begin{bmatrix} -c \\ -d \\ 1 \end{bmatrix}.$$

This is the power of the row reduced echelon form. Once in this form, you can read everything off!

Differential Equations Example

Let's consider the differential equation $y'' + y = 1$. We can choose the steady state solution for the particular solution $y_p = 1$.

Let's focus on solving $y'' + y = 0$. What is the nullspace of this equation?

We can't say vectors here. We have to say functions. But that's OK. We can add functions and we can multiply them by constants. That's all we could do with vectors too. Linear combinations are the key.

So what are the homogeneous solutions to this equation? Give me just enough, but not too many.

One answer is $y_h = c_1 \cos(t) + c_2 \sin(t)$. Using linear algebra terminology, I would say there is a *2-dimensional* nullspace. There are two independent solutions $\cos(t)$ and $\sin(t)$, and linear combinations of these two solutions gives all solutions!

$\sin(t)$ and $\cos(t)$ are a *basis* for the nullspace.

A **basis** means each element of the basis is a solution to $A\mathbf{x} = \mathbf{0}$. Can multiply by a constant and we still get a solution. And we can add together and still get a solution. Together we get all solutions, but the $\sin(t)$ and $\cos(t)$ are different or *indepenedent* solutions.

What's another description of the nullspace?

$$C_1 e^{it} + C_2 e^{-it}$$

This description is just as good. Better in some ways (fulfills the pattern better), not as good in others (involves complex numers). The basis in this case is e^{it} and e^{-it}. They are independent solutions, but linear combinations give all null solutions.

If you wanted to mess with your TA, you could choose $y_h = Ce^{it} + D\cos(t)$. This is just as good.

We've introduced some important words. The *basis for the nullspace*. In this example, the beauty is that the nullspace will always have 2 functions in it. 2 is a very important number.

- The degree of the ODE is 2
- There are 2 constants
- 2 initial conditions are needed
- The dimension of the nullspace is 2.

[3] The nullspace

Suppose we have the equation $R\mathbf{x} = 0$. The collection of $\mathbf{x}$ that solve this equation form the *nullspace*. The nullspace always goes through the origin.

Example

Suppose we have a 5 by 5 matrix. Does it have an inverse or doesn't it? Look at the nullspace! If only solution in the nullspace is $\mathbf{0}$, then yes, it is invertible. However, if there is some nonzero solution, then the matrix is not invertible.

The other importan work we used is space.

Matrix Example

Let NR denote the nullspace of R:

$$R = \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & d \end{bmatrix}$$

What's a basis for the nullspace? A basis could be $\begin{bmatrix} -c \\ -d \\ 1 \end{bmatrix}$. Or we could take $\begin{bmatrix} -2c \\ -2d \\ 2 \end{bmatrix}$. The dimension is $3 - 2 = 1$. So there is only one element in the basis.

Why can't we take 2 vectors in the basis?

Because they won't be independent elements!

Differential Equations Example

For example, $Ce^{it} + D\cos(t) + E\sin(t)$ does not form a basis because they are not independent! Euler's formula tells us that $e^{it} = \cos(t) + i\sin(t)$, so e^{it} *depends* on $\cos(t)$ and $\sin(t)$.

[4] **Space, Basis, Dimension** There are a lot of important words that have been introduced.

- Space
- Basis for a Space
- Dimension of a Space

We have been looking at small sized examples, but these ideas are not small, they are very central to what we are studying.

First let's consider the word space. We have two main examples. The column space and the nullspace.

A	**Column Space**	**Nullspace**
Definition	All linear cominations of the columns of A	All solutions to $A\mathbf{x} = \mathbf{0}$
50×70 matrix	Column space lives in $\mathbb{R}^{50}$	Nullspace lives in $\mathbb{R}^{70}$
$m \times n$ matrix	Column space lives in $\mathbb{R}^m$	Nullspace lives in $\mathbb{R}^n$

Definition V is a space (or *vector* space) when: if x and y are in the space, then for any constant c, cx is in the space, and $x + y$ is also in the space. That's superposition!

Let's make sense of these terms for a larger matrix that is in row echelon form.

Larger Matrix Example

$$R = \begin{bmatrix} =1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The first and third columns are the *pivot* columns. The second and forth are *free* columns.

What is the column space, $C(R)$?

All linear combinations of the columns. Is $\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$ in the column space? No it's not. The column space is the xy-plane, all vectors $\begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$. The dimension is 2, and a basis for the column space can be taken to be the pivot columns.

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Note, if your original matrix wasn't in rref form, you must take the original form of the pivot columns as your basis, not the row reduced form of them!

What is a basis for the nullspace, $N(R)$?

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 4 \\ 1 \end{bmatrix} \right\}$$

The reduced echelon form makes explicit the linear relations between the columns.

The relationships between the columns of A are the same as the linear relationships between the columns of any row-equivalent matrix, such as the reduced echelon form R. So a pivot indicates that this column is independent of the previous columns; and, for example, the 2 in the second column in this reduced form is a record of the fact that the second column is 2 times the first. This is why the reduced row echelon form is so useful to us. It allows us to immediately read off a basis for both the independent columns, and the nullspace.

Note that this line of thought is how you see that the reduced echelon form is well-defined, independent of the sequence of row operations used to obtain it.

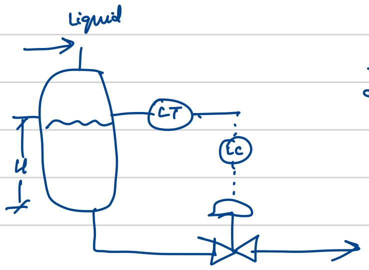
M.I.T. 18.03 Ordinary Differential Equations
18.03 Extra Notes and Exercises

M.I.T. 18.03 Ordinary Differential Equations
18.03 Extra Notes and Exercises

Eigen : control systems

Reservoir management:

- Water needs to control water.



$$\frac{dV}{dt} = A\frac{dh}{dt} = F_{in} - F_{out}.$$

$$= F_{in} - au\sqrt{2gh}.$$

Control problems:-

- Maintain level $h = h_{set}$. when $F(t)$ is fluctuate in an unknown manner.) Regulat
- Keep $h \to h_{set_1} \to h_{set_2}$. when $F(t)$ is fluctuating) Server

- Measure states, level, temp. (Disturbance)

(Compare) with desired

(Manipulation)

Feedback Control System -