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Mechanics

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Linearity and non-linearity

- Suppose there is a function or mapping that does this $f : x \longrightarrow y$. Function f takes x and gives y
- The function is linear if $f(x) = f(x_1) + f(x_2)$. where $x = x_1 + x_2$
- Suppose we know $f()$ and y but not x. Because the function is linear, we can construct solutions x that may be a combination of different solutions x_i. The idea is that f should not depend on x. It can have other parameters, but not the values of the solution itself. In that case, we would have to find the solution x and also the mapping f

Example

Take function or map ()

Structural non-linearity introduction

- What does non-linearity mean ?
- What makes a structure non-linear ?
- How do you model non-linearity and solve it?

- There are different types of non-linearity in a structure
  - Geometrical - $\mathbf{K}(\mathbf{x})\mathbf{x} = \mathbf{F}$ where here $\mathbf{K}$ depends on $\mathbf{x}$ because $\mathbf{K}$ changes as you chagne x due to the change in geometry and the large deformation.
  - Material - $\mathbf{K}(\mathbf{x})\mathbf{x} = \mathbf{F}$ because $\mathbf{K}$ changes due to the changes in the material property. For eg Youngs modulus may be a function of x.
- $\mathbf{K}$ is now a function of (x), it means that it can't be solved by using a linear method (Inverting a matrix).
- But the system of equations now has to be linearised about a point. We keep the system of equations as an equality equation $\mathbf{K}(\mathbf{x})\mathbf{x} - \mathbf{F} = 0$.
- We linearise and try to solve the root. At every point we construct the tangent and find a solution, but when we construct the solution at that point, we find the residual and we then use to iterate until convergence.

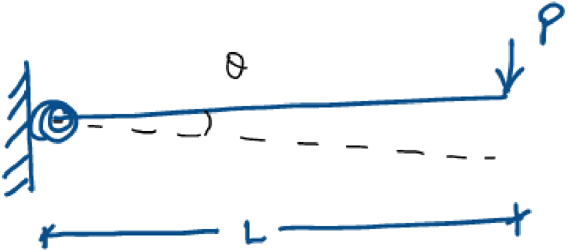


Figure:

Taking a weightless rigid bar with a torsional spring that resists any moment and a load P at the end. Since the bar is rigid so vertical equilibrium is satisfied at the support without any deformation. However, it can rigidly rotate through the spring.

The moment generated at the end support is

$$PL cos\theta = M \tag{1}$$

Taking the equilibrium equation with the spring as $M = K\theta$ we get

$$PL cos\theta = K\theta$$
$$\frac{PL}{K} = \frac{\theta}{cos\theta}$$

- If we take $\theta \rightarrow 0$ then $cos\theta \rightarrow 1$
- So $P = \frac{K}{L}\theta$ which is linear wrt θ

- Linearity : $\frac{PL}{K} = \theta$
- Geometric nonlinearity only : $\frac{PL}{K} = \frac{\theta}{cos\theta}$
- Material nonlinearity only : $\frac{PL}{K(\theta)} = \theta$ [1]
- Geometric + Material nonlinearity : $\frac{PL}{K(\theta)} = \frac{\theta}{cos\theta}$

[1] We can say the spring stiffness as a material parameter K that also depends on θ introducing the material nonlinearity. For eg, we can model $K(\theta) = K_o(1 - c\theta)$

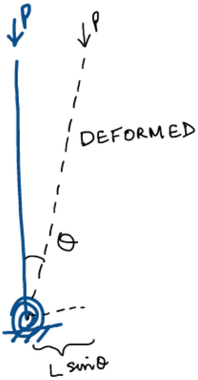


Figure:

Same problem but now the rigid bar is vertical. This is a very common problem for nonlinearity (I think the previos one is better tho!). Nicely represents buckling of columns

Same equilibrium equation but now the lever arm is different (Because the load along the bar)

$$PLsin\theta = M \tag{2}$$

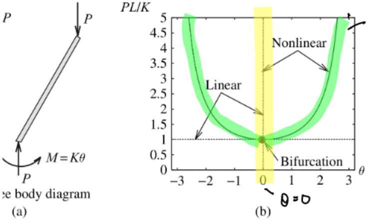
Taking the equilibrium equation with the spring as $M = K\theta$ we get

$$PLsin\theta = K\theta$$

$$\frac{PL}{K} = \frac{\theta}{sin\theta}$$

- If we take $\theta \rightarrow 0$ then $sin\theta \rightarrow 0$ so $M = 0$ (This is one possible equilibrium)
- $\frac{PL}{K} = \frac{\theta}{sin\theta}$: This is the other
- The load $\frac{PL}{K}$ where these two equilibrium equations are possible for the same structure is called the bifurcation point.
- You can imagine that when the load is smaller, it will not buckle and only deform axially so only one equilibrium position ($\theta = 0$) is possible. When $\frac{PL}{K} > 1$ then the other equilibrium comes to play.

Therefore two solutions are there :



imple column.

$\frac{PL}{K} = \frac{\theta}{\sin\theta}$

- Below $\frac{PL}{K} < 1$, Only Equilibrium is $\theta = 0$

  Because $\frac{PL}{K} < 1$, $\frac{\theta}{\sin\theta} \not< 1$ so $\theta = 0$

- Above $\frac{PL}{K} > 1$, we get two possible solutions
  - $\theta = 0$: $M = 0$
  - $\theta \neq 0$: $\frac{PL}{K} = \frac{\theta}{\sin\theta}$.

- If we linearise our equilibrium equation for small $\theta \to sin\theta$
- We get $(Pl - K)\theta = 0$ and we get our linear eigen value problem with a trivial solution of $\theta = 0$ and a nontrivial solution of $PL = K$. Again we get two equilibrium solutions. $PL = K$ being the buckling load.
- So when we reach that load, it means we have reached the bifurcation point, where multiple equilibrium solutions exist and the rod may buckle dependant on imperfection, lateral load etc.

Non-Linear strain measures introduction

- Now suppose of being rigid, the body is deformable that is the relative deformation is introducing strain and stresses
- LARGE DISPLACEMENTS + LARGE STRAINS
- The first deals with displacements that are large, and therefore while finding the strains, we have to use higher orders of the displacement derivatives
- The second deals with ????????????????????????????

OneD measures

- Emphasize its only for One D! But same theory for other dimensions
- A strain measure need not be fixed. Sometimes the strain measure we usually use may not be able to model the correct behaviour. When we choose any strain measure, the proper corresponding stress and the constitutive relationship ($\sigma = \mathbf{C}\varepsilon$) has to be taken.
- The stress and strain have to be "work compatible". That is they are together used in the strain energy density function.

Engineering strain

- Engineering strain $\varepsilon_E = \dfrac{l - L}{L} = \dfrac{\Delta}{L}$. l is deformed length, L is initial undeformed
- We could have also divided Δ by l (Change by deformed length). If $l \approx L$ then it would not matter.
- ε_E is the small infinitesimal strain, where the deformed and undeformed lengths are very similar.

Logarithimic strain

- The instantaneous strain increment can be thought as $\varepsilon_L = \frac{\Delta_1}{L} + \frac{\Delta_2}{l_1} \ldots$
- Or $d\varepsilon_L = \frac{dl}{l}$
- $\varepsilon_L = \int_L^l \dfrac{dl}{l} = ln \frac{l}{L}$
- The integration is done between two configurations $L \longrightarrow l$

These strains are more easily extrapolated to continuum (3d cases)

Green strain

- $\varepsilon_G = \dfrac{l^2 - L^2}{2L^2}$

Almansi strain

- $\varepsilon_A = \dfrac{l^2 - L^2}{2l^2}$

- Suppose $l \approx L$ and therefore Δ is small
- And $l = (L + \Delta)$
- $\varepsilon_G = \dfrac{(L+\Delta)^2 - L^2}{2L^2} = \dfrac{(L^2 + \Delta^2 + 2L\Delta - L^2)}{2L^2} \approx \dfrac{\Delta}{L}$ (As Δ is very small and so Δ^2 vanishes)

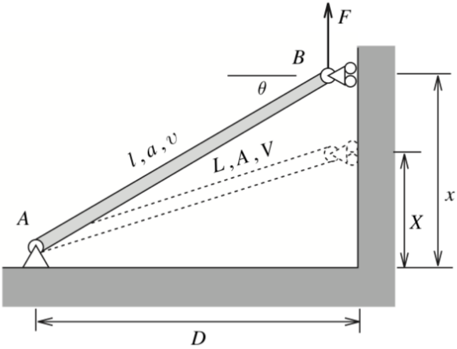


Figure:

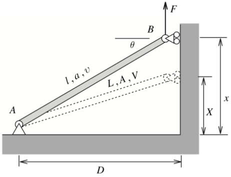
- Initial length L, area A, volume V
- Final length l, area a, volume v

- In defining the equilibrium (Froces = 0 , No moments). We will be defining the internal stress by different strain measures.
- Remember that a proper constitutive law has to be taken for a particularly strain measure
- Here we have chosen the Cauchy stress and E randomly and not dependant on work compatibility. The cauchy stress is the actual/true stress in the deformed state. (Or it is the stress in the deformed state which is in equilibrium)

Using two strain measures

Green and logarithmic.

- Cauchy stress (True stress) $\sigma = E\varepsilon$ can be :
- $\sigma = E\dfrac{l^2 - L^2}{L^2}$
- $\sigma = Eln\dfrac{l}{L}$



- The bar will keep moving up until the vertical equilibrium is reached.
- Vertical equilibrium at B is $F - T(x) sin\theta(x) = 0$, where $T(x)$ is the internal force and depends on x. θ is also dependant on x
- Now we can construct a residual function $R(x) = F - T(x) sin\theta(x)$ where the residual becomes zero for a particular solution of x. [2] So

$$R(x) = \sigma a sin\theta - F = \sigma(x) a \frac{x}{l} - F \tag{3}$$

Stress dependant on different strain measures

- $T(x) = E \frac{l^2 - L^2}{L^2} a \frac{x}{l}$ $\quad (\sigma = E\varepsilon_G)$
- $T(x) = E ln \frac{l}{L} a \frac{x}{l}$ $\quad (\sigma = E\varepsilon_L)$

[2] Note that $\frac{dR}{dx}$ is the tangent stifness K_{Bx} or force in direction B due to displacement x.

Stress dependant on different strain measures

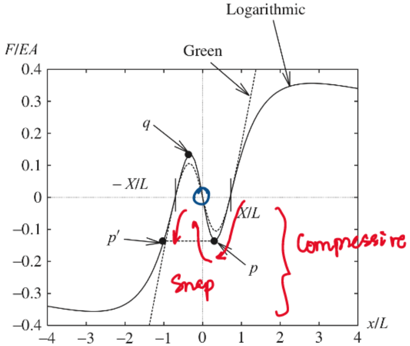
- $T(x) = E\frac{l^2 - L^2}{L^2}a\frac{x}{l}$ $\qquad (\sigma = E\varepsilon_G)$
- $T(x) = Eln\frac{l}{L}a\frac{x}{l}$ $\qquad (\sigma = E\varepsilon_L)$

- l is a function of x, $l^2 = D^2 + x^2$
- R(x) is therefore very nonlinear with respect to x. In R(x), F is not dependant on x. But sometimes it can be the case that the load is also nonlinear.

Solving

We need to solve the nonlinear equation R(x) = 0

- So we use NR, or first order taylor series to linearise R and solve it iteratively
- $R(x_{i+1}) = R(x_i) + \frac{dR}{dx}|_{x_i}(x_{i+1} - xi)$
- We want R = 0 , so the value $R(x_{i+1}) = 0$
- $0 = R(x_i) + \frac{dR}{dx}|_{x_i}(x_{i+1} - xi)$



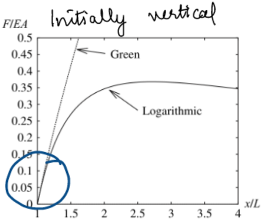


FIGURE 1.6 Large strain rod: load deflection behavior.

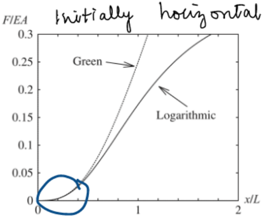


FIGURE 1.7 Horizontal truss: tension stiffening.

- ◯ ← When the strains are small the different measures give similar equilib profile
- As you increase compressive (-ve) you get snap through behaviour from $p \to q \to p'$ (imagine a wrinkled coco cola bottle you push in).

- Regions where x is small, different mesasures gives okay results
- We see different behaviour behaviours between the strain measures at higher strain
- Snap through behaviour if we increase compressive load too much. Imagine you are pushing the truss down (-x) and suddenly it will roll to the other side.
- If truss is initially vertical (Like a column in tension, Therefore no rotation) : Same E should have not been used for both different strain measures. (It seems that the green strain looks good as we expect it to be linear in axial)
- Initially horizontal : Stiffening due to tension

- A comment was made that E should not have been used.
- The vertical stiffness K_{Bx} is the chagne in equilibrium at B in direction x. $K_{Bx} = \frac{dR}{dx}$. If F is constant then $\frac{dR}{dx} = \frac{dT}{dx}$
- Without the inclusion of the strain measures, internal force is $T(x) = \sigma a \frac{x}{l}$. (All three are a function of x)
- Since both strain measures are function of l we can write $\sigma = f(l)$
- Using the incompressibility codition [3], we can replace a with $a = \frac{V}{l}$, and using chain rule:

$$\frac{dT}{dx} = \frac{d}{dx}\left(\frac{\sigma Vx}{l^2}\right) = \frac{Vx}{l^2}\frac{\partial\sigma}{\partial l}\frac{\partial l}{\partial x} - \frac{2\sigma Vx}{l^3}\frac{\partial l}{\partial x} + \frac{\sigma V}{l^2}$$

$$\frac{dT}{dx} = \frac{ax}{l}\frac{\partial\sigma}{\partial l}\frac{\partial l}{\partial x} - \frac{2\sigma ax}{l^2}\frac{\partial l}{\partial x} + \frac{\sigma a}{l}$$

[3] The condition states that volume cant change under deformation and so $al = AL$

$$\frac{dT}{dx} = \frac{ax}{l}\frac{\partial\sigma}{\partial l}\frac{\partial l}{\partial x} - \frac{2\sigma ax}{l^2}\frac{\partial l}{\partial x} + \frac{\sigma a}{l}$$

Stress gradient

- So we need to find $\frac{\partial\sigma}{\partial l}$
- Green : $\left(\frac{\partial\sigma}{\partial l}\right)_G = E\frac{\partial\varepsilon_G}{\partial l} = E2l/2L^2 = \frac{El}{L^2}$
- Logarithmic : $\left(\frac{\partial\sigma}{\partial l}\right)_L = E\frac{\partial\varepsilon_L}{\partial l} = E\frac{d}{dl}\left(ln(l) - ln(L)\right) = \frac{E}{l}$

l gradient

- $l^2 = D^2 + x^2$
- $2l\frac{dl}{dx} = 2x$
- $\frac{dl}{dx} = \frac{x}{l}$

- $K_{Bx} = \frac{dR}{dx} = \frac{dT}{dx}$
- Green : $K_G = \frac{A}{L}\left(E - 2\sigma\frac{L^2}{l^2}\right)\frac{x^2}{l^2} + \frac{\sigma a}{l}$ [4]
- Logirthmic : $K_L = \frac{a}{l}\left(E - 2\sigma\right)\frac{x^2}{l^2} + \frac{\sigma a}{l}$

- They look similar but the causal consitutive relation chosen has led to the different results
- We will write K_G as with the idea of getting an insight:

$$K_G = \frac{A}{L}\left(E - 2S\right)\frac{x^2}{l^2} + \frac{SA}{l}$$

$$\text{where } S = \sigma\frac{L^2}{l^2}$$

- Where S is the second-Piola Kirchoff stress which gives the force per unit underformed area transformed by the deformation gradient inverse $(l/L)^{-1}$

[4] V = AL

- S is actually associated with ε_G
- $(x/l)^2$ is the transformation from local to global forces.
- Therefore K_G shows that we can express the stiffness in initial underformed configuration
- If x is close to X and l is close to L then both the stiffness would be the same. The second term contains the change $\frac{\partial l}{\partial x}$, so this term disappears.
- The third term is the initial stress or geometric stiffness. This is unconcerned with the change in cross sectional area and associated only with the change in rigid body rotation. A very negative value can cause instability and singular K. The third term actually came from the derivative of the direction cosines (x/L).

Continuum measures - 2D

- Strain ε has components ε_x, y, xy
- This strain is a measurement at a point!!!
- Infinitesimal strains

$$\varepsilon_x = \frac{\partial u}{\partial x}$$
$$\varepsilon_y = \frac{\partial v}{\partial y}$$
$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

- Displacements are small, so only linear orders of displacement gradients are available

- Notation for different configurations undeformed : x, and deformed : X

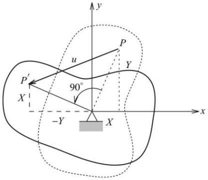


FIGURE 1.8 90° rotation of a two-dimensional body.

- Suppose there is a rotation in any solid by 90 ^{o} , No deformation!. So:

$$u = -X - Y$$
$$v = X - Y$$

- So our infinitesimal strains are :

$$\varepsilon_x = \varepsilon_y = -1$$
$$\varepsilon_{xy} = 0$$

- So we still get strain, when we should have not

- Using the same Green strain, we extend it in some way for 2D
- Taking the differential length dS (Undeformed) and ds (Deformed)
- Take a small element dX initially parallel to *x* axis

$$ds^2 = \left(dX + \frac{\partial u}{\partial X} dX\right)^2 + \left(\frac{\partial v}{\partial X} dX\right)^2$$

$$E_{xx} = \frac{ds^2 - dX^2}{2dX^2} = \frac{1}{2}\left(\left(1 + \frac{\partial u}{\partial X}\right)^2 + \left(\frac{\partial v}{\partial X}\right)^2\right) - 1$$

Similarly we get the Green strains equations :

- Thse strain components = 0 for the rigid rotation case
- Nonlinear strains are better, but they coincide with the infinitesimal strains when x and X are close to each other. [5]

[5] Here x and X are vectors that define the total position of a body in the deformed and undeformed

Stress and equilibrium

- We are dealing with different configurations. One configuration is maybe unstressed and the deformed one is. So at the deformed x we should get an equilibrium of stresses and the external loads
- Now, the actual stresses at the deformed or current configuration is the Cauchy stress : defined as the force in different directions by the area in different planes
- Stresses can also be defined with respect to the initial configuration X

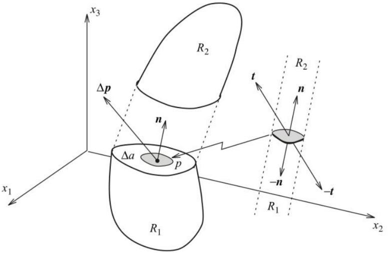


FIGURE 5.1 Traction vector.

At the deformed configuration :

- See two bodies R_1 and R_2 free body with force acting on them
- Imagine the traction vector on a small area element : $t(n) = \frac{\Delta p}{\Delta a}$ as $\lim \Delta a \to 0$ where Δp is the resultant force
- Obviously t and n will depend on the surface it acts on. Here on the right we can see that based on the surface we get opposite forces. (In the negative normal , we will get negative force which is positive in that direction!)

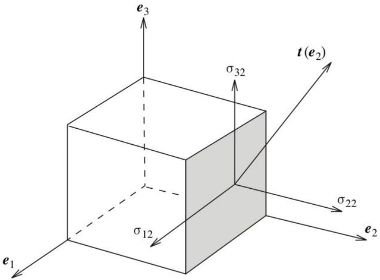
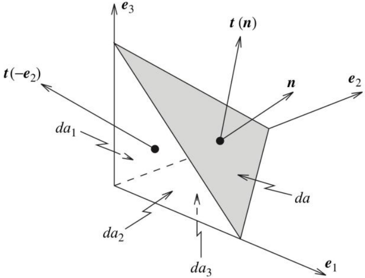


FIGURE 5.2 Stress components.

- Let us denote the traction acting on the surface having normals denoted by e_1, e_2, e_3
- Remember in the other slice we will have an opposite reaction

$$\begin{aligned} \mathbf{t}(\mathbf{e_1}) &= \sigma_{\mathbf{j1}}\mathbf{e_j} \\ \mathbf{t}(\mathbf{e_2}) &= \sigma_{\mathbf{j2}}\mathbf{e_j} \\ \mathbf{t}(\mathbf{e_3}) &= \sigma_{\mathbf{j3}}\mathbf{e_j} \end{aligned} \tag{4}$$

- Or $\mathbf{t_i} = \sigma_{\mathbf{ji}}\mathbf{e_j}$ or $\mathbf{t} = \sigma^{\mathbf{T}}\mathbf{e}$



Now let us look if we take a plane cut of that sphere. Again by context of opposite reactions. All the forces should be equal. So we will use here the concept of equilibrium between the traction vector we have defined in the last slide with respect to some basis and the traction vector defined on the angled plane.

Equilibrium

$$\mathbf{t}(\mathbf{n})da + t(-\mathbf{e_i})da_i + \mathbf{f}dv = 0 \tag{5}$$

This states that the force vector on the inclined cut should be in equilibrium with the opposite forces defined on the negative sufraces and the body force

- Now the areas (Because they are with defined respect to the basis vectors) can be written as the projection of the inclined area

$$da_i = da(\mathbf{n}.\mathbf{e_i}) \tag{6}$$

- Diving by da we get

$$\mathbf{t}(\mathbf{n}) + t(-\mathbf{e_i})\frac{da(\mathbf{n}.\mathbf{e_i})}{da} + \mathbf{f}\frac{dv}{da} = 0 \tag{7}$$

- $\frac{dv}{da} \rightarrow 0$ (I don't know why????????????????)
- We get :

$$\begin{aligned} \mathbf{t}(\mathbf{n}) = -t(-\mathbf{e_i})(\mathbf{n}.\mathbf{e_i}) = t(\mathbf{e_i})(\mathbf{n}.\mathbf{e_i}) \\ \mathbf{t}(\mathbf{n}) = (\sigma_{\mathbf{ji}}\mathbf{e_j})(\mathbf{n}.\mathbf{e_i}) \\ \mathbf{t}(\mathbf{n}) = (\sigma_{\mathbf{ij}}\mathbf{e_i})(\mathbf{n}.\mathbf{e_j})\text{Replacing indexs} \end{aligned} \tag{8}$$

- Very interesting, we started off with a statement that the resultant force on the plane is equal to the summation of the opposite forces
- Then we got the traction vector is equal to the traction vectors multiplied by some scalar product (Think of ratio)
- $t(\mathbf{e_i})(\mathbf{n}.\mathbf{e_i})$ states the traction in e_i direction multiplied by the projection of planar area for i =1,2,3
- Now we can replace the traction vector by the components of stress vectors in the basis direction $\mathbf{t}(\mathbf{n}) = (\sigma_{\mathbf{ij}}\mathbf{e_i})(\mathbf{n}.\mathbf{e_j})$
- Here we have to point out that σ_{ij} has not been described as a tensor yet

- If we look at $\mathbf{t}(\mathbf{n}) = (\sigma_{ij}\mathbf{e_i})(\mathbf{n}.\mathbf{e_j})$, we can see that σ_{ij} is just a component and $\mathbf{n}.\mathbf{e_j}$ is a scalar (Or a projection).
- That scalar value then becomes the component of e_i. Lets see what that means

$$\sigma_{12}\mathbf{e_1}(\mathbf{n}.\mathbf{e_2}) = \sigma_{12}\begin{bmatrix}1\\0\\0\end{bmatrix}\left(\begin{bmatrix}n_1 & n_2 & n_3\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \sigma_{12}n_2\begin{bmatrix}1\\0\\0\end{bmatrix} \tag{9}$$

- So we took the second component of n and added to the result of linear map in e_1.This is what we do when we multiply the first row of a matrix and a vector. We add all the components of the vector to keep in the first component of the output
- So we can write it therefore as

$$\mathbf{e_i}(\mathbf{n}.\mathbf{e_j}) = (\mathbf{e_i} \otimes \mathbf{e_j}).\mathbf{n} \tag{10}$$

  which states that the tensor takes the projection of e_j in n and maps as components of e_i
- This then allows us to understand that $\sigma_{ij}(\mathbf{e_i} \otimes \mathbf{e_j})$ is a tensor σ

- In simplicity $\mathbf{t}(\mathbf{n}) = (\sigma_{ij}\mathbf{e_i})(\mathbf{n}.\mathbf{e_j})$ says that for every cut, $\mathbf{n}$

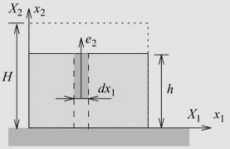
$$\begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \begin{bmatrix} \sigma_{11}\mathbf{e_1}(\mathbf{n}.\mathbf{e_1}) + \sigma_{12}\mathbf{e_1}(\mathbf{n}.\mathbf{e_2}) + \sigma_{13}\mathbf{e_1}(\mathbf{n}.\mathbf{e_3}) \\ \sigma_{21}\mathbf{e_2}(\mathbf{n}.\mathbf{e_1}) + \sigma_{22}\mathbf{e_2}(\mathbf{n}.\mathbf{e_2}) + \sigma_{23}\mathbf{e_2}(\mathbf{n}.\mathbf{e_3}) \\ \sigma_{31}\mathbf{e_3}(\mathbf{n}.\mathbf{e_1}) + \sigma_{32}\mathbf{e_3}(\mathbf{n}.\mathbf{e_2}) + \sigma_{33}\mathbf{e_3}(\mathbf{n}.\mathbf{e_3}) \end{bmatrix} \tag{11}$$

  FIXXX. These are not components yet!

- And now you can see that $\mathbf{e_i} \otimes \mathbf{e_j}$ just makes the tensor take the right projection and keeps in the component in the output

- $\sigma_{ij}n_j$ is therefore something like taking the projection of j for every component i

Problem #1



A simple example of a two-dimensional stress tensor results from the self-weight of a block of uniform initial density ρ_0 resting on a frictionless surface as shown in the figure above. For simplicity we will assume that there is no lateral deformation (in linear elasticity this would imply that the Poisson ratio $\nu = 0$).

- $t(\mathbf{e_2}) = \dfrac{\left(-\int_y^h \rho g dx_2\right) \mathbf{e_2} dx_1}{dx_1}$
- Mass conservation $\rho dx_1 dx_2 = \rho_o dX_1 dX_2$ + poisson = 0 give :

$$t(\mathbf{e_2}) = \rho_o g(H - X_2)\mathbf{e_2} \tag{12}$$

$$t(\mathbf{e_2}) = \sigma_{12}\mathbf{e_1} + \sigma_{22}\mathbf{e_2} \tag{13}$$

so $\sigma_{12} = 0$, so you can construct σ. Check Bonet Pge 138

- Obviously the Cauchy stress components can be described with respect to its principal directions ϕ_1, ϕ_2, ϕ_3 with principal stresses $\sigma_{\lambda_1}, \sigma_{\lambda_2}, \sigma_{\lambda_3}$
- So we can write in tensor notation : $\sigma = \sigma_{\lambda_\mathbf{i}} (\lambda_\mathbf{i} \otimes \lambda_\mathbf{i})$ (Only diagonals, the tensor is $i \otimes i$ which only is for the diagonal components)
- The cauchy stress is a spatial tensor (In the deformed configuration), and is symmetric because of the rotational equilibrium

- The idea is that the same stress should be the same as measured by different observers
- The same problem is equivalent to if we applied a rigid body motion
- So the stress tensor should not change it's property when there is a rigid body motion etc.
- ????????????????

Equilibrium

- The spatial configuration of the body has to be in equilibrium having volume v and boundary Γ
- At equilibrium, the body is under forces f and traction forces t
- Looking at the translational equilibrium of the structure, we get :

$$\int_{\delta\Gamma} \mathbf{t} da + \int_{v} \mathbf{d} dv = 0 \tag{14}$$

- In terms of The Cauchy stress we get

$$\int_{\delta\Gamma} \sigma \mathbf{n} da + \int_{v} \mathbf{d} dv = 0 \tag{15}$$

- If we use the Gauss theorem to convert the area to volume integral we get

$$\int_{\delta v} (DIV\sigma + \mathbf{d})\, dv = 0 \tag{16}$$

- As the above region can be applied to any closed region, the integrand must vanish to get $DIV\sigma + \mathbf{f} = \mathbf{0}$

- This is the equilibrium equation at a very smalllll level :

$$\frac{\partial \sigma_{ij}}{\partial xj} + f_i = 0 \tag{17}$$

- This equation is the *local* or spatial (deformed) equilibrium.
- While solving this equation may not be satisfied, and we have a pointwise out of balance or residual given as

$$r = DIV\sigma + f \tag{18}$$

- Not gonna explain. The rotational equilibrium gives you the fact that the Cauchy stress is symmetric. $\sigma^T = \sigma$
- See Bonet Page 142

Principle of virtual work

- FEM is usually based in terms of a weak form of the differential equations
- Let δv denote an arbitrary virtual velocity
- Virtual work, δw per unit volume and time by a residual force r during the virtual work as $r.\delta v$

$$\delta w = r.\delta v = 0 \tag{19}$$

- The equation above is equivalent to the equation $r = 0$
- Since δv is arbitary, we can get the seperate components of **r** if we take $\delta v = \begin{bmatrix}1 & 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0 & 1\end{bmatrix}$

The weak statement of static equilibrium of a body over it's volume is given then as

$$\delta W = \int_v (DIV\sigma + f)\,.\delta v dv = 0 \tag{20}$$

- $DIV\,(\sigma\delta v) = (DIV\sigma).\delta v + \sigma : \nabla\delta v$

$$DIV(\sigma\delta v) = DIV\left(\delta v_1 \begin{bmatrix}\sigma_{11}\\ \sigma_{21}\\ \sigma_{31}\end{bmatrix} + \delta v_2 \begin{bmatrix}\sigma_{12}\\ \sigma_{22}\\ \sigma_{32}\end{bmatrix} + \delta v_3 \begin{bmatrix}\sigma_{13}\\ \sigma_{23}\\ \sigma_{33}\end{bmatrix}\right) \tag{21}$$

- We get the equilibrium therefore as :

$$\int_\Gamma \mathbf{n}.\sigma\delta\mathbf{vda} - \int_\mathbf{v} \sigma : \nabla\delta\mathbf{vdv} + \int_\mathbf{v} \mathbf{f}.\delta\mathbf{vdv} = \mathbf{0} \tag{22}$$

- Check Bonet page 143 for symmetric velocity defined equilibrium $\delta W = \int_\mathbf{v} \sigma : \delta\mathbf{ddv} - \int_\mathbf{v} \mathbf{f}.\delta\mathbf{vdv} - \int_\Gamma \mathbf{t}.\delta\mathbf{vda} = \mathbf{0}$

Alternate stress definitions

Internal virtual work given as : $\delta W = \int_{\mathbf{v}} \sigma : \delta \mathbf{d} dv$

- Now σ and d are said to be work conjugate with respect to the current deformed volume. Product gives the work per unit current volume
- If we defined the stress with respect to the undeformed(material) coordinates, alternative work conjugete pairs are needed
- The virtual work with respect to the intial volume and area by transforming the integrals is given as

$$\int_V J\sigma : \delta d \; dV = \int_V f_o.\delta v dV + \int_{\Gamma_o} t_o.\delta v dA \tag{23}$$

where $f_0 = J f$ and $t_o = t\frac{da}{dA}$ and $\frac{da}{dA} = J\sqrt{N.C^{-1}N}$

The internal virtual work then expressed as the Kirchoff tensor is

$$\delta W_{int} = \int_V \tau : \delta d dV \tag{24}$$

where $\tau = J\sigma$, we can see that τ is work conjugate to the rate of the defomration tensor with respsect to the initial volume (Rember same work density per unit mass should be invariant or $\frac{1}{\rho}\sigma : d = \frac{1}{\rho_o}\tau : d$ [J takes care of the density])

The previous internal work definition still relied on the spatial (current configuration) quantities τ and d

- Remember in the internal virtual work, the stress is conjugate with the gradient of the virtual displacements : $\int \sigma : \nabla \delta v dv = \int \sigma : \delta l dv$
- Writing as

$$
\begin{aligned}
\int_V J\sigma : \delta l dv \\
\int_V J\sigma : (\delta \dot{F} F^{-1}) dV \\
\int_V J \; tr(\sigma(F^{-T} \delta \dot{F})) dV \\
\int_V J\sigma F^{-T} : \delta \dot{F} dV \\
\int_V P : \delta \dot{F} dV
\end{aligned}
\tag{25}
$$

where P is the first Piola-Kirchhoff stress conjugate with the deformation rate

- Unsymmetric two-point tensor with components

$$P = P_{iJ}\mathbf{e_i} \otimes \mathbf{E_j}$$
$$P_{iJ} = J\sigma_{ik}(F^{-1})_{Jk} \tag{26}$$

Virtual work is then

$$\int_V P : \delta\dot{F}dV \int_V f_o.\delta v dV + \int_\Gamma t_o.\delta v dA \tag{27}$$

If we had reversed the virtual displacement to get the equilrium differential equations we get

$$r_o = DIVP + f_o = Jr \tag{28}$$

where DIV P is withre repect to the intial configuration given as DIV P = $\nabla_o P : I$ (Diagonal elemnts basically!)

- In the current configuration a force vector dp acting on an element area $\mathbf{da} = \mathbf{n}.da$

$$d\mathbf{p} = \mathbf{t}da = \sigma da \tag{29}$$

- The Cauchy stress gives the current force per unit deformed area
- Now d$\mathbf{p}$ can be written in erms of the undedformed area dA giving

$$d\mathbf{p} = \sigma \mathbf{F}^{-\mathbf{T}} \mathbf{dA} J = P d\mathbf{A} \tag{30}$$

so P relates an area vector in the intial configuration to a force vector in the current one

- Unsymmetric two-point tensor that is not at all related to the material (intial) configuration
- We need to pull back the force from the spatial to material configuration $d\mathbf{p} \rightarrow d\mathbf{P}$

$$d\mathbf{P} = \phi^{-1}[d\mathbf{p}] = F^{-1} d\mathbf{p} \tag{31}$$

- Now we define the second piola kirchoff as

$$\begin{aligned} d\mathbf{P} &= \mathbf{SdA} \\ \mathbf{F^{-1}}\sigma\mathbf{da} &= S\frac{1}{J}\mathbf{F^T da} \\ S &= J\mathbf{F^{-1}}\sigma\mathbf{F^{-T}} \end{aligned} \tag{32}$$

- The spatial virtual rate of deformation is related to the material as

$$\delta d = F^{-T}\delta\dot{E}F^{-1} \tag{33}$$

- Keeping it in the internal virtual work

$$\begin{aligned} \delta W_{int} &= \int_v \sigma : ddv \\ &\int_V J\sigma : ddV \\ &\int_V J\sigma : F^{-T}\delta\dot{E}F^{-1}dV \\ &\int_V J\mathrm{tr}(\sigma(F^{-T}\delta\dot{E}F^{-1})^T)dV \\ \int_V J\mathrm{tr}(\sigma F^{-T}\delta\dot{E}F^{-1})dV &= \int_V J\mathrm{tr}(\sigma F^{-1}\delta\dot{E}F^{-T})dV \quad [\mathrm{tr(ABCD) = tr(DABC)}] \\ &= \int_V S : \delta\dot{E}dV \end{aligned} \tag{34}$$

- Threfore S is work conjugate to $\dot{E}$ and we have to total material description

$$\int_V S : \delta \dot{E} dV = \int_V f_o . \delta v dV + \int_{\Gamma_o} t_o . \delta v dA \tag{35}$$

We get then the relation ship between the two piola stresses and cauchy stress

$$\begin{aligned} \sigma &= J^{-1} P F^T \\ \sigma &= J^{-1} F S F^T \end{aligned} \tag{36}$$

And we also get push forward and pull back operations

- $S = J \phi_*^{-1} [\sigma]$
- $\sigma = J^{-1} \phi_* [S]$

- In case of rigid body motion, the polar decomposition of the defomration gradient gives F = R and J = 1
- S = $\mathbf{R^T Q R}$
- Second piola kirchoff components coincide with the Cauchy components given in a different basis rotated by R!!
- S is also objective and independant from reimposed rotations Q
- Check bonet Page 151 for biot stress

- It is practical to decompose the stress tensor to its deviatoric and pressure components
- This is useful as both these tensors play a different role in failure theory
- $\sigma = \sigma_{\mathbf{D}} + \sigma_{\mathbf{H}}$ whre p = 1/3 tr(σ) and tr($\sigma_D = 0$)
- Also we can do

$$P = P_D + pJF^{-T} \quad P_D = J\sigma_D F^{-T}$$
$$S = S_D + pJC^{-T} \quad S_D = JF^{-1}\sigma_D F^{-T} \tag{37}$$

  where trace of S_D and P_D need not be zero
- Check Bonet page 151 for other relations

Check Bonet Page 152

Linearised equilibrium

- The virtual work gives the linear variation in work. Meaning it is linear with respect to the variations
- The equilibrium equation however will still be nonlinear with respect to the geometry and material
- Now this equilibrium equation will then need to be linearised

- The principle of virtual work is as follows

$$\delta W(\phi, \delta v) = \int_v \sigma : \delta d dv - \int_v f.\delta v dv - \int_\Gamma t.\delta v da = 0 \tag{38}$$

where ϕ is the trial solution

- Linaerising f agains meaning f + D f

$$\delta W(\phi, \delta v) + D\delta W(\phi, \delta v)[u] = 0 \tag{39}$$

- So we are finding the directional derivative of the virtual work equation, i.e at ϕ at a direction u
- Remember to derive the equilibrium, we set up a virtual work equation about a position x. Here we are then trying to find that position x, making the non-linear equilibrium equations linear

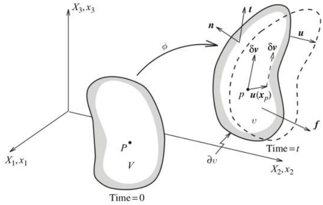


FIGURE 8.1 Linearized equilibrium.

- As shown in the figure, the virtual displacement/velocity is still the same but now the actual quilibrium configuration is changing
- At a trial solution ϕ_k, the virtual work $\delta W(\phi, \delta v) \neq 0$
- Therefore $D\delta W(\phi_k, \delta v)[u]$ is the change in δW due to ϕ_k change to $\phi_k + u$.
- Since δv is not changing, therefore it is r or the internal forces that are changing as the solution changes due to u
- NR therfore makes the internal and external forces in equilibrium by changing the configuration to the equilibrium configuration

$$D\delta W(\phi, \delta v)[u] = D\delta W_{int}(\phi, \delta v)[u] - D\delta W_{ext}(\phi, \delta v)[u] \tag{40}$$

$$D\delta W(\phi, \delta v)[u] = D\left(\int_v \sigma : \delta d \; dv\right)[u] - D\left(\int_\Gamma t.\delta v \; da\right)[u] \tag{41}$$

- Linearisation of the equilibrium equations with respect to the material description
- Simpler as we have the material description , and we know the dV integral to integrate over
- We can push forward the equations to the spatial description

- Internal virtual work is given as

$$\delta W_{int}(\phi, \delta v) = \int_V S : \delta\dot{E} dV \tag{42}$$

- Using product rule for directional derivatives, we get

$$\begin{aligned} D\delta W_{int}(\phi, \delta v)[u] = D\left(\int_V S : \delta\dot{E}\right)[u]dV \\ \left(\int_V DS : \delta\dot{E}\right)[u]dV \\ \left(\int_V \delta\dot{E} : DS[u]\right)dV + \left(\int_V S : D\delta\dot{E}[u]\right)dV \\ \left(\int_V \delta\dot{E} : C : DS[u]\right)dV + \left(\int_V S : D\delta\dot{E}[u]\right)dV \end{aligned} \tag{43}$$

(Have to check why can we take the derivative inside the integral?????????????/)

- Where we can then find $D\dot{E}[u]$ Bonet

- $D\dot{\delta E}[u]$ is a function of δv and also of configuration ϕ

$$\dot{\delta E} = \frac{1}{2}\left(\delta\dot{F}^T F + F^T \delta\dot{F}\right) \tag{44}$$

and $\delta\dot{F} = \frac{\partial \delta v}{\partial X} = \nabla_o \delta v$ and $DF[u] = \nabla_o u$

- So

$$D\dot{\delta E}[u] = \frac{1}{2}\left(\nabla(\delta v)^T \nabla_o u + \nabla(\delta u)^T \nabla_o v\right) \tag{45}$$

Here $\nabla_o v$ remains constant, as they are not functions of the configuration

$$D\delta W_{int}(\phi, \delta v)[u] = \int_V \dot{\delta E} : C : DE[u] dV + \int_V S : [(\nabla_o u)^T \nabla_o \delta v] dV \tag{46}$$

and $\delta\dot{E}$ can be written as $DE[\delta v]$ which gives a symmetric form:

$$D\delta W_{int}(\phi, \delta v)[u] = \int_V DE[\delta v] : C : DE[u] dV + \int_V S : [(\nabla_o u)^T \nabla_o \delta v] dV \tag{47}$$

- The formulations are pushed forward, check Bonet page 219

- Body forces
- Surface forces

- The advantage of finding a stationary problem with respect to displacements, is the advantage that such a treatment gives a uniform framework to find
  - Incompressibility, contact boundary conditions and finite element methods
  - Done by use of lagrangian multipliers or penalty methods where the variational principle incorporates eg internal pressure

- The potential energy whose directional derivative gives the virtual work is

$$\Pi(\phi) = \int_V \Psi(C) dV - \int_V f_o.\phi dV - \int_{\Gamma_o} t.\phi dA \tag{48}$$

- Assuming that the body force and traction not a function of ϕ (Not actuall for traction). The directional derivative is

$$\Pi(\phi)[\delta v] = \int_V S : DC[\delta v] dV - \int_V f_o.\delta v dV - \int_{\Gamma_o} t.\delta v dA \tag{49}$$

which is similar to the theory of virtual work that is $D\Pi(\phi)[\delta v] = \delta W(\phi, \delta v)$

- The stationary conition of $\Pi(\phi)$ gives the equilibirum and known as the varitational statement of equilibrium. The lienarised equation (NR) can be taken as the second derivative of the energy functional

$$D\delta W(\phi, \delta v)[u] = D^2\Pi(\phi)[\delta v, u] \tag{50}$$

- We need to keep a incompressibility constraint J =1
- Lagrange multiplier term

$$\Pi_L(\phi, p) = \hat{\Pi}(\phi) + \int_V p(J-1)\,dV \tag{51}$$

where p is the lagrange multiplier with knowing that it will be the internal pressure. $\hat{\Pi}$ is the strain energy given as a function of the distrotion component of the right Cauchy tensor

- We will obviously have to find $\Pi_L(\phi, p)[\delta v]$ and $\Pi_L(\phi, p)\delta p$
- And linearise the equilibirum equations again with respect to p and u
- Check bonet page 226 for details

- Different alternative to Lagrangian methods
  - Eliminates pressure as an independant variable keeping a large value of the bulk modulus
  - Perturb the lagrangian functional with a penalty allowing the pressure to be associated with the deformation
    - The perturbed lagrangian is

$$\Pi_P(\phi, p) = \Pi_L(\phi, p) - \int_V \frac{1}{2k} p^2 dV \tag{52}$$

as $k \longrightarrow \infty : \Pi_P \longrightarrow \Pi_L$
with the stationary condition as :

$$D\Pi_P(\phi, p)[\delta p] = \int_V \delta p \left((J-1) - \frac{p}{k} \right) dV = 0 \tag{53}$$

The (J-1) comes from the Π_L term. Now this euilibrium equation gives us a relationship between p and J as p = k(J-1).

    - This represents a nearly incompresible material with he penalty number as the bulk modulus (So same thing like increasing bulk modulus)

Check bonet page 229

content

Discretisation

- Using either configuration, the resulting linearised quantity will be the same
- Spatial is easier though

- Isoparametric elemnents, discretise the initial geometry X, defomrned geometry x, virtual velocity $v, \delta v$ and linearised displacment u

$$\begin{aligned} X &= N_a X_a \\ x &= N_a x_a \\ v &= N_a v_a \\ \delta v &= N_a \delta v_a \\ u &= N_a u_a \end{aligned} \tag{54}$$

where a = 1...n Number of nodes in element, and $N(\xi_1, \xi_2, \xi_3)$ are the standard shape parametric shape function

- The deformation gradiend F is interpolated over an element

$$\mathbf{F} = \sum_{\mathbf{i}}^{\mathbf{n}} \mathbf{x_a} \otimes \nabla_{\mathbf{o}} \mathbf{N_a} \tag{55}$$

$$F = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix} \qquad \mathbf{x} = \mathbf{N_a x_a} \text{ , so } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{N_a} \begin{bmatrix} x_{a_1} \\ x_{a_2} \\ x_{a_3} \end{bmatrix}$$

and $\frac{\partial x_1}{\partial X_1} = \frac{\partial N_a x_{a_1}}{\partial X_1}$

- $F_{iJ} = \sum_a^n x_{a,i} \frac{\partial N_a}{\partial X_J}$ *i* is the component of the deflection given as descritised, and j is the component of the deformed geometry. AGAIN descritised!

- where $\nabla_o N_a = \frac{\partial N_a}{\partial \mathbf{X}}$

- We can relate it to $\nabla_\xi N_a = \frac{\partial N_a}{\partial \xi}$ using chain rule as

$$\frac{\partial N_a}{\partial \mathbf{X}} = \left(\frac{\partial X}{\partial \xi}\right)^{-T} \frac{\partial N_a}{\partial \xi} \qquad \left(\frac{\partial N}{\partial \xi} = \frac{\partial N}{\partial \mathbf{X}} \frac{\partial \mathbf{X}}{\partial \xi}\right) \qquad \left(\frac{\partial N_1}{\partial \xi_1} = \frac{\partial N_1}{\partial X_1}\frac{\partial X_1}{\partial \xi_1} + \frac{\partial N_1}{\partial X_3}\frac{\partial X_3}{\partial \xi_1} + \frac{\partial N_1}{\partial X_2}\frac{\partial X_2}{\partial \xi_1}\right)$$

$$\frac{\partial \mathbf{X}}{\partial \xi} = \sum_i^n X_a \otimes \nabla_\xi N_a \qquad \frac{\partial \mathbf{X}}{\partial \xi} = \begin{bmatrix} \frac{\partial X_1}{\partial \xi_1} & \frac{\partial X_1}{\partial \xi_2} & \frac{\partial X_1}{\partial \xi_3} \\ \frac{\partial X_2}{\partial \xi_1} & \frac{\partial X_2}{\partial \xi_2} & \frac{\partial X_2}{\partial \xi_3} \\ \frac{\partial X_3}{\partial \xi_1} & \frac{\partial X_3}{\partial \xi_2} & \frac{\partial X_3}{\partial \xi_3} \end{bmatrix}$$

- We can work also with the right and left Cauchy tensors **C** and **b**

$$
\begin{aligned}
\mathbf{C} &= \mathbf{F}^{\mathbf{T}}\mathbf{F} = \sum_{\mathbf{a},\mathbf{b}} (\mathbf{x_a}.\mathbf{x_b})\nabla_{\mathbf{o}}\mathbf{N_a} \otimes \nabla_{\mathbf{o}}\mathbf{N_b} \\
\mathbf{b} &= \mathbf{F}\mathbf{F}^{\mathbf{T}} = \sum_{\mathbf{a},\mathbf{b}} (\nabla_{\mathbf{o}}\mathbf{N_a}.\nabla_{\mathbf{o}}\mathbf{N_b})\mathbf{x_a} \otimes \mathbf{x_b}
\end{aligned}
\tag{56}
$$

- We also know that the velocity gradient $\mathbf{d} = \frac{1}{2}\left(\mathbf{l} + \mathbf{l}^{\mathbf{T}}\right)$ can be given as

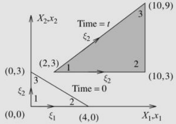
$$
\begin{aligned}
\mathbf{d} &= \frac{\mathbf{1}}{\mathbf{2}}\left(\mathbf{v_a} \otimes \nabla\mathbf{N_a} + \nabla\mathbf{N_a} \otimes \mathbf{v_a}\right) \\
\delta\mathbf{d} &= \frac{\mathbf{1}}{\mathbf{2}}\left(\delta\mathbf{v_a} \otimes \nabla\mathbf{N_a} + \nabla\mathbf{N_a} \otimes \delta\mathbf{v_a}\right)^6 \\
\varepsilon &= \frac{1}{2}\left(u_a \otimes \nabla N_a + \nabla N_a \otimes u_a\right)
\end{aligned}
\tag{57}
$$

So basically whenever we are finding the derivative of the continuous funciton, we replace it with the discretise one! Representing the derivative of N in parametric form, as:

$$
\frac{\partial N_a}{\partial \mathbf{x}} = \left(\frac{\partial \mathbf{x}}{\partial \xi}\right)^{-T}; \frac{\partial N_a}{\partial \xi} = \sum_a^n x_a \otimes \nabla_\xi N_a; \frac{\partial x_i}{\partial \xi_\alpha} = \sum_a^n x_{a,i}\frac{\partial N_a}{\partial \xi_\alpha}
\tag{58}
$$

Problem #1

This simple example illustrates the discretization and subsequent calculation of key shape function derivatives. Because the initial and current geometries comprise right-angled triangles, these are easily checked.



The initial $\boldsymbol{X}$ and current $\boldsymbol{x}$ nodal coordinates are

$$
\begin{aligned}
&X_{1,1} = 0; \quad X_{2,1} = 4; \quad X_{3,1} = 0;\\
&X_{1,2} = 0; \quad X_{2,2} = 0; \quad X_{3,2} = 3;\\
&x_{1,1} = 2; \quad x_{2,1} = 10; \quad x_{3,1} = 10;\\
&x_{1,2} = 3; \quad x_{2,2} = 3; \quad x_{3,2} = 9.
\end{aligned}
$$

(continued)

- The shape function are

$$
N_1 = 1 - \xi_1 + \xi_2; \qquad N_2 = \xi_2; \qquad N_3 = \xi_2
$$
$$
\frac{\partial N_1}{\partial \xi} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \qquad \frac{\partial N_2}{\partial \xi} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \frac{\partial N_3}{\partial \xi} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{59}
$$

Also

$$
X = N_a(\xi) X_a \qquad X_1 = 4\xi_1 \qquad X_2 = 3\xi_2 \tag{60}
$$

Think of $N(\xi)$ like some scaling that takes X to the actual value X_a. Like a mapper (maybe lienar or not) from 0 to 1 that makes you actually move in the X space.

- $\frac{\partial \mathbf{X}}{\partial \xi} = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix} \qquad \left(\frac{\partial \mathbf{X}}{\partial \xi}\right)^{-T} = \frac{1}{12}\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$
- $\frac{\partial N_a}{\partial \mathbf{X}} = \left(\frac{\partial X}{\partial \xi}\right)^{-T} \frac{\partial N_a}{\partial \xi}$

$$\frac{\partial N_1}{\partial \mathbf{X}} = \frac{1}{12}\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} -1 \\ -1 \end{bmatrix} = \frac{-1}{12}\begin{bmatrix} 3 \\ 4 \end{bmatrix} \qquad \frac{\partial N_2}{\partial \mathbf{X}} = \frac{1}{12}\begin{bmatrix} 3 \\ 0 \end{bmatrix} \qquad \frac{\partial N_3}{\partial \mathbf{X}} = \frac{1}{12}\begin{bmatrix} 0 \\ 4 \end{bmatrix} \tag{61}$$

- We can do the same with respect to the spatial coordinates giving us

$$\frac{\partial N_1}{\partial \mathbf{x}} = \frac{1}{24}\begin{bmatrix} 3 & 0 \\ -4 & 4 \end{bmatrix}\begin{bmatrix} -1 \\ -1 \end{bmatrix} = \frac{-1}{24}\begin{bmatrix} 3 \\ 0 \end{bmatrix} \qquad \frac{\partial N_2}{\partial \mathbf{x}} = \frac{1}{24}\begin{bmatrix} 3 \\ -4 \end{bmatrix} \qquad \frac{\partial N_3}{\partial \mathbf{x}} = \frac{1}{24}\begin{bmatrix} 0 \\ 4 \end{bmatrix} \tag{62}$$

- $F_{iJ} = \sum_a^n x_{a,i} \frac{\partial N_a}{\partial X_J}$
- $F_{11} = x_{1,1}\frac{\partial N_1}{\partial X1} + x_{2,1}\frac{\partial N_2}{\partial X1} + x_{3,1}\frac{\partial N_3}{\partial X1} = \frac{1}{12}\left(-2.3 + 10.3 + 10.0\right) = \frac{6}{3}$
- $\mathbf{F} = \frac{1}{3}\begin{bmatrix} 6 & 8 \\ 0 & 6 \end{bmatrix}$
- $\mathbf{C} = \mathbf{F^T F} \qquad \mathbf{b} = \mathbf{F F^T} \qquad \mathrm{J} = \det \mathbf{F}$

Spatial description

$$\delta W(\phi, \delta v) = \int_v \sigma : \delta d dv - \int_v f.\delta v dv - \int_\Gamma t.\delta v da \tag{63}$$

which is the virtual work done by the residual **r**.
A pretty neat thing is that we can consider a single virtual nodal velocity δv_a occuring at node a in element e. We will assemble the elements later but the velocity at the node will be consistent.

$$\delta W^e(\phi, N_a \delta v_a) = \int_{v^e} \sigma : (\delta v_a \otimes \nabla N_a) dv - \int_{v^e} f.(N_a \delta v_a) dv - \int_{\Gamma^e} t.(N_a \delta v_a) da \tag{64}$$

* Very interesting, this gives us the equilibrium at a node a. The ∇N_a almost acts like a component term
* The first term because σ is symmetric so the 1/2 in velocity gradient disappears

Insight :

$$d = \frac{1}{2}\left(l + l^T\right) = \frac{1}{2}\left(\nabla v + (\nabla v)^T\right) \tag{65}$$
$$\nabla v = n_a \otimes \nabla N_a \text{CHECK????}$$

We know that $\sigma : (\mathbf{u} \otimes \mathbf{v}) = \mathbf{u}.\sigma\mathbf{v}$ for any vectors $\mathbf{v}, \mathbf{u}$. Almost like when you take a scalar product and you only want the diagonals added

So we get

$$\delta W^e(\phi, N_a \delta v_a) = \delta v_a. \left(\int_{v^e} \sigma \nabla N_a dv - \int_{v^e} N_a f dv - \int_{\Gamma^e} N_a t da \right) \tag{66}$$

$$\delta v_a = \begin{bmatrix} \delta v_{a,1} \\ \delta v_{a,2} \\ \delta v_{a,3} \end{bmatrix}$$

- So per element the virtual work is expressed in terms of the internal and external nodal forces $\mathbf{T_a^e}$ and $\mathbf{F_a^e}$
- $\delta W^e(\phi, N_a \delta v_a) = \delta \mathbf{v}.(\mathbf{T_a^e} - \mathbf{F_a^e})$
  $\mathbf{T_a^e}$ is the internal force with different components , $T^e_{a,i} = \sum_j^3 \int_{v^e} \sigma_{ij} \frac{\partial N_a}{\partial x_j} dv$ ($\sigma \nabla N_a$ is a linear map where ∇N_a is a vector, but with respect to the global directions!. But it's not a unit vector)
- The cauchy stress is found from the constitutive relationship and the left cauchy tensor
- The virtual work allows you to say that the components of the inernal forces will be zero

- Same example as last where $b = \frac{1}{9}\begin{bmatrix} 100 & 48 & 0 \\ 48 & 36 & 0 \\ 0 & 0 & 9 \end{bmatrix}$ and J = 4
- $\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix} = \frac{\mu}{J}(b - I) + \frac{\lambda}{J}(lnJ)I = \begin{bmatrix} 8 & 4 & 0 \\ 4 & 3 & 0 \\ 0 & 0 & 0.8 \end{bmatrix}$
- $T_{a,i} = \int_{v^e} \left(\sigma_{i1} \frac{\partial N_a}{\partial x_1} + \sigma_{i2} \frac{\partial N_a}{\partial x_2} \right) \mathrm{dv}$
- $T_{1,1} = -24t \qquad T_{1,1} = 8t \qquad T_{1,1} = 16t$
  $T_{1,2} = -12t \qquad T_{2,2} = 0 \qquad T_{3,2} = 12t$

Equilibrium at a global level :

- From all elements e (1 to m_1) containing node a
  $\delta W(\phi, N_a \delta v) = \sum_{e=1, e \ni a}^{m_a} \delta W^e(\phi, N_a \delta \mathbf{v_a}) = \delta \mathbf{v_a}.(\mathbf{T_a} - \mathbf{F_a})$
- where assembled equivalent nodal forces are:
  $\mathbf{T_a} = \sum_{\mathbf{e=1,e \ni a}}^{\mathbf{m}} \mathbf{T_a^e} \qquad \mathbf{F_a} = \sum_{\mathbf{e=1,e \ni a}}^{\mathbf{m}} \mathbf{F_a^e}$
- And then for all nodes we get $\delta W(\phi, \delta v) = \sum_a^n \delta \mathbf{v_a}.\, (\mathbf{T_a} - \mathbf{F_a})$
- Since the virtual work should be satisifed for any virtual nodal velocity, we get the Residual force with respect to the whole system $\mathbf{R_a} = \mathbf{T_a} - \mathbf{F_a}$

- Organise R in an array $\mathbf{T} = [\mathbf{T_1}\ \mathbf{T_2} ... \mathbf{T_N}]$, $\mathbf{F} = [...]$, $\mathbf{R} = [....]$ (I think each T_1 contains three components)
- Virtual work equation is : $\delta \mathbf{v}^{\mathbf{T}} \mathbf{R} = \delta \mathbf{v}^{\mathbf{T}} (\mathbf{T} - \mathbf{F}) = \mathbf{0}$
  where $\delta \mathbf{v}^{\mathbf{T}} = [\delta \mathbf{v}_{\mathbf{1}}^{\mathbf{T}} \delta \mathbf{v}_{\mathbf{2}}^{\mathbf{T}} ....]$
- Since the internal forces are nonlinear functions of the current nodal positions $\mathbf{x} = [\mathbf{x_1 x_2 x_3} ...]$
- In matrix notation, we keep the symmetric tensor as
  $\sigma' = [\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{13}, \sigma_{23}]^T$ and $\mathbf{d}$ as $\mathbf{d} = [d_{11}, d_{22}, d_{33}, 2d_{12}, 2d_{13}, 2d_{23}]^T$
  (Where off diagonals is twice to make sure $d^T \sigma$ gives the correct internal energy)
- $\int \sigma : \mathbf{d} = \int \mathbf{d}^{\mathbf{T}} \sigma dv$
- $\mathbf{d} = \sum_{\mathbf{a}}^{\mathbf{n}} \mathbf{B_a v_a}$ where $B_a = \begin{bmatrix} \frac{\partial N_a}{\partial x_1} & 0 & 0 \\ 0 & \frac{\partial N_a}{\partial x_2} & 0 \\ 0 & 0 & \frac{\partial N_a}{\partial x_3} \\ \frac{\partial N_a}{\partial x_2} & \frac{\partial N_a}{\partial x_1} & 0 \\ \frac{\partial N_a}{\partial x_3} & 0 & \frac{\partial N_a}{\partial x_1} \\ 0 & \frac{\partial N_a}{\partial x_3} & \frac{\partial N_a}{\partial x_2} \end{bmatrix}$

- So : $\delta W = \int (B_a \delta v_a)^T \sigma dv - \int f.(N_a \delta v_a) dv - \int t.(N_a \delta v_a) da$
- We can also write the internal force as : $T_a^e = \int_{v^e} B_a^T \sigma' dv$

- The equilibrium equations are still nonlinear with respect to the nodal positions. A NR is used to solve it
- The linear virtual work components is found using the directional derivative as

$$D\delta W(\phi, \delta v)[u] = D\delta W_{int}(\phi, \delta v)[u] - D\delta W_{ext}(\phi, \delta v)[u] \tag{67}$$

- The internal work linearisation can be decomposed into the constitutive and intial stress components

$$D\delta W_{int}(\phi, \delta v)[u] = D\delta W_C(\phi, \delta v)[u] + D\delta W_\sigma(\phi, \delta v)[u] \tag{68}$$

$$= \int_{\mathbf{v}} \delta \mathbf{d} : \mathbf{c} : \varepsilon dv + \int_{\mathbf{v}} \sigma : \left((\nabla \mathbf{u})^{\mathbf{T}} (\nabla \delta \mathbf{v}) \right) dv \tag{69}$$

which is the tangent stiffness matrix

- Remember that at each node, we get the residual due to the nodal equivalent forces at a due to the whole equilibrium of node a (R = T -F)
- F may be dependant on a and so linearisation in the direction of u_b or $N_b u_b$ with $N_a v_a$ constant, gives only the change of the residual force at node a due to the change u_b in the current position of node b

$$D\delta W^e(\phi, N_a \delta v_a)[N_b u_b] = D(\delta v_a.(T_a^e - F_a^e))[N_b u_b] \tag{70}$$

$$= \delta v_a.D(T_a^e - F_a^e)[N_b u_b] = \delta v_a.K_{ab}^e u_b \tag{71}$$

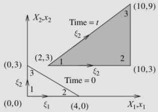
- Change in force at node a due to change in current position of node b
- This is not the full stiffness matrix, but each component. When we do the whole assembly, we get the full tangen stiffness matrix

$$\frac{\partial \mathbf{R}}{\partial \mathbf{U}} = \begin{bmatrix} \frac{\partial R_1}{\partial u_1} & \frac{\partial R_1}{\partial u_2} & \cdots\cdots & \frac{\partial R_1}{\partial u_n} \\ \frac{\partial R_2}{\partial u_1} & \frac{\partial R_2}{\partial u_2} & \cdots\cdots & \frac{\partial R_2}{\partial u_n} \\ \frac{\partial R_n}{\partial u_1} & \frac{\partial R_n}{\partial u_2} & \cdots\cdots & \frac{\partial R_n}{\partial u_n} \end{bmatrix} \tag{72}$$

So one component here is attained by finding the linearisation in one direction node with the virtual work equilibrium in another node seperately

- Check bonet page 248
-
$$D\delta W_c^e(\phi, N_a\delta v_a)[N_b u_b] = \delta va.K^e_{c,ab}u_b \tag{73}$$
- where
$$[K_{c,ab}]_{ij} = \int_{v^e} \sum_{k,l=1}^{3} \frac{\partial N_a}{\partial x_k} Cikjl \frac{\partial N_b}{\partial x_l} dv \qquad i,j = 1,2,3 \tag{74}$$

This simple example illustrates the discretization and subsequent calculation of key shape function derivatives. Because the initial and current geometries comprise right-angled triangles, these are easily checked.



The initial $\boldsymbol{X}$ and current $\boldsymbol{x}$ nodal coordinates are

$$X_{1,1} = 0;\quad X_{2,1} = 4;\quad X_{3,1} = 0;$$
$$X_{1,2} = 0;\quad X_{2,2} = 0;\quad X_{3,2} = 3;$$
$$x_{1,1} = 2;\quad x_{2,1} = 10;\quad x_{3,1} = 10;$$
$$x_{1,2} = 3;\quad x_{2,2} = 3;\quad x_{3,2} = 9.$$

(continued)

$$[\boldsymbol{K}_{c,23}]_{11} = \left(\frac{1}{8}\right)(\mathbb{c}_{1112})\left(\frac{1}{6}\right) - \left(\frac{1}{6}\right)(\mathbb{c}_{1212})\left(\frac{1}{6}\right)(24t);$$
$$[\boldsymbol{K}_{c,23}]_{12} = \left(\frac{1}{8}\right)(\mathbb{c}_{1122})\left(\frac{1}{6}\right) - \left(\frac{1}{6}\right)(\mathbb{c}_{1222})\left(\frac{1}{6}\right)(24t);$$
$$[\boldsymbol{K}_{c,23}]_{21} = \left(\frac{1}{8}\right)(\mathbb{c}_{2112})\left(\frac{1}{6}\right) - \left(\frac{1}{6}\right)(\mathbb{c}_{2212})\left(\frac{1}{6}\right)(24t);$$
$$[\boldsymbol{K}_{c,23}]_{22} = \left(\frac{1}{8}\right)(\mathbb{c}_{2122})\left(\frac{1}{6}\right) - \left(\frac{1}{6}\right)(\mathbb{c}_{2222})\left(\frac{1}{6}\right)(24t);$$

where t is the thickness of the "element. Substituting for $\mathbb{c}_{ijkl}$ from Equations (6.40) and (6.41) yields the stiffness" coefficients as

$$[\boldsymbol{K}_{c,23}]_{11} = -\frac{2}{3}\lambda' t;\quad [\boldsymbol{K}_{c,23}]_{12} = \frac{1}{2}\mu' t;\quad [\boldsymbol{K}_{c,23}]_{21} = \frac{1}{2}\mu' t;$$
$$[\boldsymbol{K}_{c,23}]_{22} = -\frac{2}{3}(\lambda' + 2\mu')t;$$

where $\lambda' = \lambda/J$ and $\mu' = (\mu - \lambda \ln J)/J$.

$$\frac{\partial N_1}{\partial \mathbf{x}} = \frac{1}{24}\begin{bmatrix} 3 & 0 \\ -4 & 4 \end{bmatrix}\begin{bmatrix} -1 \\ -1 \end{bmatrix} = \frac{-1}{24}\begin{bmatrix} 3 \\ 0 \end{bmatrix} \qquad \frac{\partial N_2}{\partial \mathbf{x}} = \frac{1}{24}\begin{bmatrix} 3 \\ -4 \end{bmatrix} \qquad \frac{\partial N_3}{\partial \mathbf{x}} = \frac{1}{24}\begin{bmatrix} 0 \\ 4 \end{bmatrix} \tag{75}$$

K_{11} means equilibrium in 2 due to change in 3

- Virtual work for element e expressed in matrix notation by a small starin vector as $\varepsilon' = [\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, 2\varepsilon_{12}, 2\varepsilon_{13}, 2\varepsilon_{23}]^T$
- $\varepsilon' = B_a u_a$
- Now the constitutive component of the linearised virtual work can be written as

$$D\delta W_C(\phi, \delta v)[u] = \int_V \delta d : c : \varepsilon dv = \int_v \delta \mathbf{d}^{\mathbf{T}} \mathbf{D} \varepsilon' dv \quad (76)$$

- Where the later part is the matrix componets received from the tensor contraction

$$D = \begin{bmatrix} c_{1111} & c_{1122} & c_{1133} & c_{1112} & c_{1113} & c_{1123} \\ c_{2211} & c_{2222} & c_{2233} & c_{2212} & c_{2213} & c_{2223} \\ . & . & . & . & . & . \\ c_{2311} & c_{2322} & c_{2333} & c_{2312} & c_{2313} & c_{2323} \end{bmatrix} \quad (77)$$

- See bonet page 250 for neo-Hookean model
- For node a and b

$$D\delta W_c^e(\phi, N_a \delta v_a)[N_b u_b] = \int_{v^e} (\mathbf{B_a} \delta \mathbf{v_a})^{\mathbf{T}} \mathbf{D} (\mathbf{B_b} \mathbf{u_b}) dv = \int_{v^e} \delta \mathbf{v_a} . \underbrace{(\mathbf{B_a^T} \mathbf{D} \mathbf{B_b})}_{Tangent\ K} . \mathbf{u_b} dv \quad (78)$$

- Remember that the gradients of u and δv can be found as

$$\begin{aligned}\nabla\delta\mathbf{v} &= \delta v_a \otimes \nabla N_a \\ \nabla\delta\mathbf{u} &= \delta u_b \otimes \nabla N_b\end{aligned} \tag{79}$$

- We've seen the intial stress component as (Check the linearising equilib equations)

$$\begin{aligned}D\delta W_\sigma(\phi, N_a\delta v_a)[N_b\delta u_b] &= \int_v \sigma : [(\nabla\mathbf{u_b})^{\mathbf{T}}\nabla\delta\mathbf{v_a}]\mathbf{dv} \\ &= \int_v \sigma : [(\delta\mathbf{v_a}.\mathbf{u_b})\nabla N_b \otimes \nabla N_a]dv \\ &= (\delta v_a.u_b)\int_{v^e} \nabla N_a.\sigma\nabla N_b dv\end{aligned} \tag{80}$$

- As we have $\delta v_a.u_b = \delta v_a.Iu_b$
- We get $\delta v_a.K_{\sigma,ab}u_b$

$$\begin{aligned}\mathbf{K}^{\mathbf{e}}_{\sigma,\mathbf{ab}} &= \int_{\mathbf{v^e}}(\nabla\mathbf{N_a}.\sigma\nabla\mathbf{N_b})\mathbf{I}\mathbf{dv} \\ [K^e_{\sigma,ab}]_{ij} &= \int_{v^e}\sum_{k,l=1}^{3}(\frac{\partial N_a}{\partial x_k}\sigma_{kl}\frac{\partial N_b}{\partial x_l}\delta_{ij})dv\end{aligned} \tag{81}$$

- Find intial stiffness matrix joining node 1 and 2

$$[K_{\sigma,12}] = \int_{v^e} \begin{bmatrix} \frac{\partial N_1}{\partial x_1} & \frac{\partial N_1}{\partial x_2} \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \begin{bmatrix} \frac{\partial N_2}{\partial x_1} \\ \frac{\partial N_2}{\partial x_2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} dv \tag{82}$$

Check bonet : 252

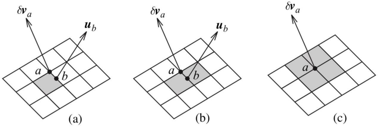


FIGURE 9.2 Assembly of linearized virtual work.

- For an element *e* linking nodes a and b we find $\mathbf{K}^{\mathbf{e}}_{\mathbf{ab}} = \mathbf{K}^{\mathbf{e}}_{\mathbf{c,ab}} + \mathbf{K}^{\mathbf{e}}_{\sigma\mathbf{,ab}} + \mathbf{K}^{\mathbf{e}}_{\mathbf{p,ab}}$ (Fig a)
- Then we find the assembly of the total linearized virtual work of contribution to a from b from all elements (Fig b)
- Find then for all nodes connecting to a
- Doing the above for all nodes

$$
\begin{aligned}
(i)\, D\delta W(\phi, N_a \delta v_a)\,[N_b u_b] &= \sum_{e=1, e \ni a,b}^{m_{a,b}} D\delta W^e(\phi, N_a \delta v_a)\,[N_b u_b] \\
(ii)\, D\delta W(\phi, N_a \delta v_a)\,[u] &= \sum_{b=1}^{n_a} D\delta W^e(\phi, N_a \delta v_a)\,[N_b u_b] \\
(iii)\, D\delta W(\phi, v)\,[u] &= \sum_{a=1}^{N} D\delta W^e(\phi, N_a \delta v_a)\,[u]
\end{aligned}
\tag{83}
$$

Check bonet 258 for

- NR
- Line search
- Arc-Length method