1 Non-Linear strain measures introduction : February 18, 2021

2 OneD measures

3 Continuum measures - 2D

Non-Linear strain measures introduction : February 18, 2021

- Now suppose of being rigid, the body is deformable that is the relative deformation is introducing strain and stresses
- LARGE DISPLACEMENTS + LARGE STRAINS
- The first deals with displacements that are large, and therefore while finding the strains, we have to use higher orders of the displacement derivatives
- The second deals with ????????????????????????????

OneD measures

- Emphasize its only for One D! But same theory for other dimensions
- A strain measure need not be fixed. Sometimes the strain measure we usually use may not be able to model the correct behaviour. When we choose any strain measure, the proper corresponding stress and the constitutive relationship ($\sigma = \mathbf{C}\varepsilon$) has to be taken.
- The stress and strain have to be "work compatible". That is they are together used in the strain energy density function.

Engineering strain

- Engineering strain $\varepsilon_E = \dfrac{l - L}{L} = \dfrac{\Delta}{L}$. l is deformed length, L is initial undeformed
- We could have also divided Δ by l (Change by deformed length). If $l \approx L$ then it would not matter.
- ε_E is the small infinitesimal strain, where the deformed and undeformed lengths are very similar.

Logarithimic strain

- The instantaneous strain increment can be thought as $\varepsilon_L = \frac{\Delta_1}{L} + \frac{\Delta_2}{l_1} \ldots$
- Or $d\varepsilon_L = \frac{dl}{l}$
- $\varepsilon_L = \int_L^l \dfrac{dl}{l} = ln\frac{l}{L}$
- The integration is done between two configurations $L \longrightarrow l$

These strains are more easily extrapolated to continuum (3d cases)

Green strain

- $\varepsilon_G = \dfrac{l^2 - L^2}{2L^2}$

Almansi strain

- $\varepsilon_A = \dfrac{l^2 - L^2}{2l^2}$

- Suppose $l \approx L$ and therefore Δ is small
- And $l = (L + \Delta)$
- $\varepsilon_G = \dfrac{(L+\Delta)^2 - L^2}{2L^2} = \dfrac{(L^2 + \Delta^2 + 2L\Delta - L^2)}{2L^2} \approx \dfrac{\Delta}{L}$ (As Δ is very small and so Δ^2 vanishes)

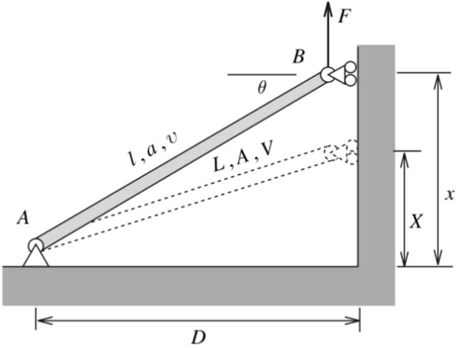


Figure:

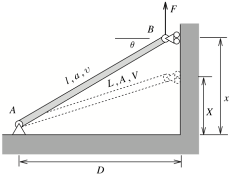
- Initial length L, area A, volume V
- Final length l, area a, volume v

- In defining the equilibrium (Froces = 0 , No moments). We will be defining the internal stress by different strain measures.
- Remember that a proper constitutive law has to be taken for a particularly strain measure
- Here we have chosen the Cauchy stress and E randomly and not dependant on work compatibility. The cauchy stress is the actual/true stress in the deformed state. (Or it is the stress in the deformed state which is in equilibrium)

Using two strain measures

Green and logarithmic.

- Cauchy stress (True stress) $\sigma = E\varepsilon$ can be :
- $\sigma = E\frac{l^2 - L^2}{L^2}$
- $\sigma = Eln\frac{l}{L}$



- The bar will keep moving up until the vertical equilibrium is reached.
- Vertical equilibrium at B is $F - T(x)sin\theta(x) = 0$, where $T(x)$ is the internal force and depends on x. θ is also dependant on x
- Now we can construct a residual function $R(x) = F - T(x)sin\theta(x)$ where the residual becomes zero for a particular solution of x. [1] So

$$R(x) = \sigma a sin\theta - F = \sigma(x) a \frac{x}{l} - F \tag{1}$$

Stress dependant on different strain measures

- $T(x) = E\frac{l^2 - L^2}{L^2} a \frac{x}{l}$ $\qquad (\sigma = E\varepsilon_G)$
- $T(x) = Eln\frac{l}{L} a \frac{x}{l}$ $\qquad (\sigma = E\varepsilon_L)$

[1] Note that $\frac{dR}{dx}$ is the tangent stifness K_{Bx} or force in direction B due to displacement x.

Stress dependant on different strain measures

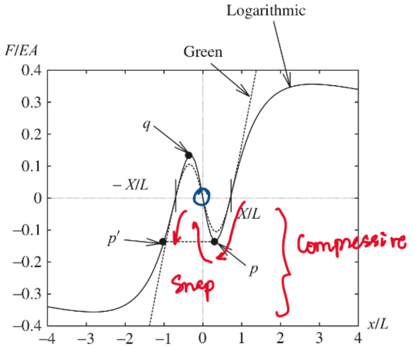
- $T(x) = E\frac{l^2 - L^2}{L^2} a \frac{x}{l}$ $\qquad (\sigma = E\varepsilon_G)$
- $T(x) = Eln\frac{l}{L} a \frac{x}{l}$ $\qquad (\sigma = E\varepsilon_L)$

- l is a function of x, $l^2 = D^2 + x^2$
- R(x) is therefore very nonlinear with respect to x. In R(x), F is not dependant on x. But sometimes it can be the case that the load is also nonlinear.

Solving

We need to solve the nonlinear equation R(x) = 0

- So we use NR, or first order taylor series to linearise R and solve it iteratively
- $R(x_{i+1}) = R(x_i) + \frac{dR}{dx}|_{x_i}(x_{i+1} - xi)$
- We want R = 0 , so the value $R(x_{i+1}) = 0$
- $0 = R(x_i) + \frac{dR}{dx}|_{x_i}(x_{i+1} - xi)$



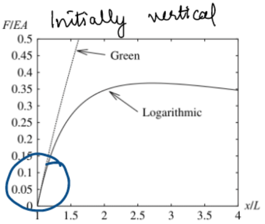


FIGURE 1.6 Large strain rod: load deflection behavior.

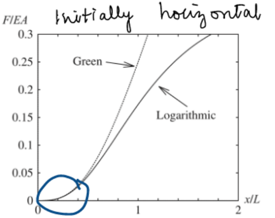


FIGURE 1.7 Horizontal truss: tension stiffening.

- ◯ ← When the strains are small the different measures give similar equilib profile

- As you increase compressive (-ve) you get snap through behaviour from $p \to q \to p'$ (imagine a wrinkled coca cola bottle you push in).

- Regions where x is small, different mesasures gives okay results
- We see different behaviour behaviours between the strain measures at higher strain
- Snap through behaviour if we increase compressive load too much. Imagine you are pushing the truss down (-x) and suddenly it will roll to the other side.
- If truss is initially vertical (Like a column in tension, Therefore no rotation) : Same E should have not been used for both different strain measures. (It seems that the green strain looks good as we expect it to be linear in axial)
- Initially horizontal : Stiffening due to tension

- A comment was made that E should not have been used.
- The vertical stiffness K_{Bx} is the chagne in equilibrium at B in direction x. $K_{Bx} = \frac{dR}{dx}$. If F is constant then $\frac{dR}{dx} = \frac{dT}{dx}$
- Without the inclusion of the strain measures, internal force is $T(x) = \sigma a \frac{x}{l}$. (All three are a function of x)
- Since both strain measures are function of l we can write $\sigma = f(l)$
- Using the incompressibility codition [2], we can replace a with $a = \frac{V}{l}$, and using chain rule:

$$\frac{dT}{dx} = \frac{d}{dx}\left(\frac{\sigma V x}{l^2}\right) = \frac{Vx}{l^2}\frac{\partial \sigma}{\partial l}\frac{\partial l}{\partial x} - \frac{2\sigma V x}{l^3}\frac{\partial l}{\partial x} + \frac{\sigma V}{l^2}$$

$$\frac{dT}{dx} = \frac{ax}{l}\frac{\partial \sigma}{\partial l}\frac{\partial l}{\partial x} - \frac{2\sigma a x}{l^2}\frac{\partial l}{\partial x} + \frac{\sigma a}{l}$$

[2] The condition states that volume cant change under deformation and so $al = AL$

$$\frac{dT}{dx} = \frac{ax}{l}\frac{\partial\sigma}{\partial l}\frac{\partial l}{\partial x} - \frac{2\sigma ax}{l^2}\frac{\partial l}{\partial x} + \frac{\sigma a}{l}$$

Stress gradient

- So we need to find $\frac{\partial\sigma}{\partial l}$
- Green : $\left(\frac{\partial\sigma}{\partial l}\right)_G = E\frac{\partial\varepsilon_G}{\partial l} = E2l/2L^2 = \frac{El}{L^2}$
- Logarithmic : $\left(\frac{\partial\sigma}{\partial l}\right)_L = E\frac{\partial\varepsilon_L}{\partial l} = E\frac{d}{dl}\left(ln(l) - ln(L)\right) = \frac{E}{l}$

l gradient

- $l^2 = D^2 + x^2$
- $2l\frac{dl}{dx} = 2x$
- $\frac{dl}{dx} = \frac{x}{l}$

- $K_{Bx} = \frac{dR}{dx} = \frac{dT}{dx}$
- Green : $K_G = \frac{A}{L}\left(E - 2\sigma\frac{L^2}{l^2}\right)\frac{x^2}{l^2} + \frac{\sigma a}{l}$ [3]
- Logirthmic : $K_L = \frac{a}{l}(E - 2\sigma)\frac{x^2}{l^2} + \frac{\sigma a}{l}$

- They look similar but the causal consitutive relation chosen has led to the different results
- We will write K_G as with the idea of getting an insight:

$$K_G = \frac{A}{L}(E - 2S)\frac{x^2}{l^2} + \frac{SA}{l}$$
$$\text{where } S = \sigma\frac{L^2}{l^2}$$

- Where S is the second-Piola Kirchoff stress which gives the force per unit underformed area transformed by the deformation gradient inverse $(l/L)^{-1}$

[3] V = AL

- S is actually associated with ε_G
- $(x/l)^2$ is the transformation from local to global forces.
- Therefore K_G shows that we can express the stiffness in initial underformed configuration
- If x is close to X and l is close to L then both the stiffness would be the same. The second term contains the change $\frac{\partial l}{\partial x}$, so this term disappears.
- The third term is the initial stress or geometric stiffness. This is unconcerned with the change in cross sectional area and associated only with the change in rigid body rotation. A very negative value can cause instability and singular K. The third term actually came from the derivative of the direction cosines (x/L).

Continuum measures - 2D

- Strain ε has components ε_x, y, xy
- This strain is a measurement at a point!!!
- Infinitesimal strains

$$\varepsilon_x = \frac{\partial u}{\partial x}$$

$$\varepsilon_y = \frac{\partial v}{\partial y}$$

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

- Displacements are small, so only linear orders of displacement gradients are available
- Notation for different configurations undeformed : x, and deformed : X

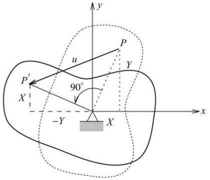


FIGURE 1.8 90° rotation of a two-dimensional body.

- Suppose there is a rotation in any solid by 90 ^{o} , No deformation!. So:

$$u = -X - Y$$
$$v = X - Y$$

- So our infinitesimal strains are :

$$\varepsilon_x = \varepsilon_y = -1$$
$$\varepsilon_{xy} = 0$$

- So we still get strain, when we should have not

- Using the same Green strain, we extend it in some way for 2D
- Taking the differential length dS (Undeformed) and ds (Deformed)
- Take a small element dX initially parallel to *x* axis

$$ds^2 = \left(dX + \frac{\partial u}{\partial X} dX\right)^2 + \left(\frac{\partial v}{\partial X} dX\right)^2$$

$$E_{xx} = \frac{ds^2 - dX^2}{2dX^2} = \frac{1}{2}\left(\left(1 + \frac{\partial u}{\partial X}\right)^2 + \left(\frac{\partial v}{\partial X}\right)^2\right) - 1$$

Similarly we get the Green strains equations :

- Thse strain components = 0 for the rigid rotation case
- Nonlinear strains are better, but they coincide with the infinitesimal strains when x and X are close to each other. [4]

[4] Here x and X are vectors that define the total position of a body in the deformed and undeformed