1 Discretisation

Mechanics

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Discretisation

- Using either configuration, the resulting linearised quantity will be the same
- Spatial is easier though

- Isoparametric elemnents, discretise the initial geometry X, defomrned geometry x, virtual velocity $v, \delta v$ and linearised displacment u

$$
\begin{aligned}
X &= N_a X_a \\
x &= N_a x_a \\
v &= N_a v_a \\
\delta v &= N_a \delta v_a \\
u &= N_a u_a
\end{aligned}
\tag{1}
$$

where a = 1...n Number of nodes in element, and $N(\xi_1, \xi_2, \xi_3)$ are the standard shape parametric shape function

- The deformation gradiend F is interpolated over an element

$$\mathbf{F} = \sum_{\mathbf{i}}^{\mathbf{n}} \mathbf{x_a} \otimes \nabla_{\mathbf{o}} \mathbf{N_a} \tag{2}$$

$$F = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix} \qquad \mathbf{x} = \mathbf{N_a} \mathbf{x_a} \text{ , so } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{N_a} \begin{bmatrix} x_{a_1} \\ x_{a_2} \\ x_{a_3} \end{bmatrix}$$

and $\frac{\partial x_1}{\partial X_1} = \frac{\partial N_a x_{a_1}}{\partial X_1}$

- $F_{iJ} = \sum_a^n x_{a,i} \frac{\partial N_a}{\partial X_J}$ i is the component of the deflection given as descritised, and j is the component of the deformed geometry. AGAIN descritised!
- where $\nabla_o N_a = \frac{\partial N_a}{\partial \mathbf{X}}$
- We can relate it to $\nabla_\xi N_a = \frac{\partial N_a}{\partial \xi}$ using chain rule as

$$\frac{\partial N_a}{\partial \mathbf{X}} = \left(\frac{\partial X}{\partial \xi}\right)^{-T} \frac{\partial N_a}{\partial \xi} \qquad \left(\frac{\partial N}{\partial \xi} = \frac{\partial N}{\partial \mathbf{X}} \frac{\partial \mathbf{X}}{\partial \xi}\right) \qquad \left(\frac{\partial N_1}{\partial \xi_1} = \frac{\partial N_1}{\partial X_1}\frac{\partial X_1}{\partial \xi_1} + \frac{\partial N_1}{\partial X_3}\frac{\partial X_3}{\partial \xi_1} + \frac{\partial N_1}{\partial X_2}\frac{\partial X_2}{\partial \xi_1}\right)$$

$$\frac{\partial \mathbf{X}}{\partial \xi} = \sum_i^n X_a \otimes \nabla_\xi N_a \qquad \frac{\partial \mathbf{X}}{\partial \xi} = \begin{bmatrix} \frac{\partial X_1}{\partial \xi_1} & \frac{\partial X_1}{\partial \xi_2} & \frac{\partial X_1}{\partial \xi_3} \\ \frac{\partial X_2}{\partial \xi_1} & \frac{\partial X_2}{\partial \xi_2} & \frac{\partial X_2}{\partial \xi_3} \\ \frac{\partial X_3}{\partial \xi_1} & \frac{\partial X_3}{\partial \xi_2} & \frac{\partial X_3}{\partial \xi_3} \end{bmatrix}$$

- We can work also with the right and left Cauchy tensors $\mathbf{C}$ and $\mathbf{b}$

$$
\begin{aligned}
\mathbf{C} = \mathbf{F}^{\mathbf{T}}\mathbf{F} &= \sum_{\mathbf{a},\mathbf{b}} (\mathbf{x_a}.\mathbf{x_b}) \nabla_{\mathbf{o}} \mathbf{N_a} \otimes \nabla_{\mathbf{o}} \mathbf{N_b} \\
\mathbf{b} = \mathbf{F}\mathbf{F}^{\mathbf{T}} &= \sum_{\mathbf{a},\mathbf{b}} (\nabla_{\mathbf{o}} \mathbf{N_a}.\nabla_{\mathbf{o}} \mathbf{N_b}) \mathbf{x_a} \otimes \mathbf{x_b}
\end{aligned} \tag{3}
$$

- We also know that the velocity gradient $\mathbf{d} = \frac{1}{2}\left(\mathbf{l} + \mathbf{l}^{\mathbf{T}}\right)$ can be given as

$$
\begin{aligned}
\mathbf{d} &= \frac{1}{2}\left(\mathbf{v_a} \otimes \nabla \mathbf{N_a} + \nabla \mathbf{N_a} \otimes \mathbf{v_a}\right) \\
\delta\mathbf{d} &= \frac{1}{2}\left(\delta\mathbf{v_a} \otimes \nabla \mathbf{N_a} + \nabla \mathbf{N_a} \otimes \delta\mathbf{v_a}\right)^1 \\
\varepsilon &= \frac{1}{2}\left(u_a \otimes \nabla N_a + \nabla N_a \otimes u_a\right)
\end{aligned} \tag{4}
$$

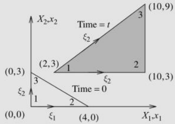
So basically whenever we are finding the derivative of the continuous funciton, we replace it with the discretise one! Representing the derivative of N in parametric form, as:

$$
\frac{\partial N_a}{\partial \mathbf{x}} = \left(\frac{\partial \mathbf{x}}{\partial \xi}\right)^{-T}; \frac{\partial N_a}{\partial \xi} = \sum_{a}^{n} x_a \otimes \nabla_{\xi} N_a; \frac{\partial x_i}{\partial \xi_\alpha} = \sum_{a}^{n} x_{a,i} \frac{\partial N_a}{\partial \xi_\alpha} \tag{5}
$$

[1] …DOF not in N

Problem #1

This simple example illustrates the discretization and subsequent calculation of key shape function derivatives. Because the initial and current geometries comprise right-angled triangles, these are easily checked.



The initial $\boldsymbol{X}$ and current $\boldsymbol{x}$ nodal coordinates are

$$
\begin{aligned}
&X_{1,1} = 0; \quad X_{2,1} = 4; \quad X_{3,1} = 0;\\
&X_{1,2} = 0; \quad X_{2,2} = 0; \quad X_{3,2} = 3;\\
&x_{1,1} = 2; \quad x_{2,1} = 10; \quad x_{3,1} = 10;\\
&x_{1,2} = 3; \quad x_{2,2} = 3; \quad x_{3,2} = 9.
\end{aligned}
$$

(continued)

- The shape function are

$$
N_1 = 1 - \xi_1 + \xi_2; \qquad N_2 = \xi_2; \qquad N_3 = \xi_2
$$
$$
\frac{\partial N_1}{\partial \xi} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \qquad \frac{\partial N_2}{\partial \xi} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \frac{\partial N_3}{\partial \xi} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{6}
$$

Also

$$
X = N_a(\xi) X_a \qquad X_1 = 4\xi_1 \qquad X_2 = 3\xi_2 \tag{7}
$$

Think of $N(\xi)$ like some scaling that takes X to the actual value X_a. Like a mapper (maybe lienar or not) from 0 to 1 that makes you actually move in the X space.

- $\frac{\partial \mathbf{X}}{\partial \xi} = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix} \qquad \left(\frac{\partial \mathbf{X}}{\partial \xi}\right)^{-T} = \frac{1}{12}\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$
- $\frac{\partial N_a}{\partial \mathbf{X}} = \left(\frac{\partial X}{\partial \xi}\right)^{-T} \frac{\partial N_a}{\partial \xi}$

$$\frac{\partial N_1}{\partial \mathbf{X}} = \frac{1}{12}\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} -1 \\ -1 \end{bmatrix} = \frac{-1}{12}\begin{bmatrix} 3 \\ 4 \end{bmatrix} \qquad \frac{\partial N_2}{\partial \mathbf{X}} = \frac{1}{12}\begin{bmatrix} 3 \\ 0 \end{bmatrix} \qquad \frac{\partial N_3}{\partial \mathbf{X}} = \frac{1}{12}\begin{bmatrix} 0 \\ 4 \end{bmatrix} \tag{8}$$

- We can do the same with respect to the spatial coordinates giving us

$$\frac{\partial N_1}{\partial \mathbf{x}} = \frac{1}{24}\begin{bmatrix} 3 & 0 \\ -4 & 4 \end{bmatrix}\begin{bmatrix} -1 \\ -1 \end{bmatrix} = \frac{-1}{24}\begin{bmatrix} 3 \\ 0 \end{bmatrix} \qquad \frac{\partial N_2}{\partial \mathbf{x}} = \frac{1}{24}\begin{bmatrix} 3 \\ -4 \end{bmatrix} \qquad \frac{\partial N_3}{\partial \mathbf{x}} = \frac{1}{24}\begin{bmatrix} 0 \\ 4 \end{bmatrix} \tag{9}$$

- $F_{iJ} = \sum_a^n x_{a,i} \frac{\partial N_a}{\partial X_J}$
- $F_{11} = x_{1,1}\frac{\partial N_1}{\partial X1} + x_{2,1}\frac{\partial N_2}{\partial X1} + x_{3,1}\frac{\partial N_3}{\partial X1} = \frac{1}{12}\left(-2.3 + 10.3 + 10.0\right) = \frac{6}{3}$
- $\mathbf{F} = \frac{1}{3}\begin{bmatrix} 6 & 8 \\ 0 & 6 \end{bmatrix}$
- $\mathbf{C} = \mathbf{F}^{\mathbf{T}}\mathbf{F} \qquad \mathbf{b} = \mathbf{F}\mathbf{F}^{\mathbf{T}} \qquad \mathrm{J} = \det\mathbf{F}$

Spatial description

$$\delta W(\phi, \delta v) = \int_v \sigma : \delta d dv - \int_v f.\delta v dv - \int_\Gamma t.\delta v da \tag{10}$$

which is the virtual work done by the residual **r**.
A pretty neat thing is that we can consider a single virtual nodal velocity δv_a occuring at node a in element e. We will assemble the elements later but the velocity at the node will be consistent.

$$\delta W^e(\phi, N_a \delta v_a) = \int_{v^e} \sigma : (\delta v_a \otimes \nabla N_a) dv - \int_{v^e} f.(N_a \delta v_a) dv - \int_{\Gamma^e} t.(N_a \delta v_a) da \tag{11}$$

* Very interesting, this gives us the equilibrium at a node a. The ∇N_a almost acts like a component term
* The first term because σ is symmetric so the 1/2 in velocity gradient disappears

Insight :

$$\begin{gathered} d = \frac{1}{2}\left(l + l^T\right) = \frac{1}{2}\left(\nabla v + (\nabla v)^T\right) \\ \nabla v = n_a \otimes \nabla N_a \text{CHECK????} \end{gathered} \tag{12}$$

We know that $\sigma : (\mathbf{u} \otimes \mathbf{v}) = \mathbf{u}.\sigma\mathbf{v}$ for any vectors $\mathbf{v}, \mathbf{u}$. Almost like when you take a scalar product and you only want the diagonals added

So we get

$$\delta W^e(\phi, N_a\delta v_a) = \delta v_a.\left(\int_{v^e} \sigma\nabla N_a dv - \int_{v^e} N_a f dv - \int_{\Gamma^e} N_a t da\right) \tag{13}$$

$$\delta v_a = \begin{bmatrix} \delta v_{a,1} \\ \delta v_{a,2} \\ \delta v_{a,3} \end{bmatrix}$$

- So per element the virtual work is expressed in terms of the internal and external nodal forces $\mathbf{T_a^e}$ and $\mathbf{F_a^e}$
- $\delta W^e(\phi, N_a\delta v_a) = \delta\mathbf{v}.(\mathbf{T_a^e} - \mathbf{F_a^e})$
  $\mathbf{T_a^e}$ is the internal force with different components , $T^e_{a,i} = \sum_j^3 \int_{v^e} \sigma_{ij}\frac{\partial N_a}{\partial x_j} dv$ ($\sigma\nabla N_a$ is a linear map where ∇N_a is a vector, but with respect to the global directions!. But it's not a unit vector)
- The cauchy stress is found from the constitutive relationship and the left cauchy tensor
- The virtual work allows you to say that the components of the inernal forces will be zero

- Same example as last where $b = \frac{1}{9}\begin{bmatrix} 100 & 48 & 0 \\ 48 & 36 & 0 \\ 0 & 0 & 9 \end{bmatrix}$ and J = 4
- $\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix} = \frac{\mu}{J}(b - I) + \frac{\lambda}{J}(lnJ)I = \begin{bmatrix} 8 & 4 & 0 \\ 4 & 3 & 0 \\ 0 & 0 & 0.8 \end{bmatrix}$
- $T_{a,i} = \int_{v^e} \left(\sigma_{i1} \frac{\partial N_a}{\partial x_1} + \sigma_{i2} \frac{\partial N_a}{\partial x_2} \right)$ dv
- $T_{1,1} = -24t \qquad T_{1,1} = 8t \qquad T_{1,1} = 16t$
  $T_{1,2} = -12t \qquad T_{2,2} = 0 \qquad T_{3,2} = 12t$

Equilibrium at a global level :

- From all elements e (1 to m_1) containing node a $\delta W(\phi, N_a\delta v) = \sum_{e=1, e\ni a}^{m_a} \delta W^e(\phi, N_a\delta \mathbf{v_a}) = \delta \mathbf{v_a}.(\mathbf{T_a} - \mathbf{F_a})$
- where assembled equivalent nodal forces are: $\mathbf{T_a} = \sum_{\mathbf{e=1, e\ni a}}^{\mathbf{m}} \mathbf{T_a^e}$ $\quad \mathbf{F_a} = \sum_{\mathbf{e=1, e\ni a}}^{\mathbf{m}} \mathbf{F_a^e}$
- And then for all nodes we get $\delta W(\phi, \delta v) = \sum_a^n \delta \mathbf{v_a}.\, (\mathbf{T_a} - \mathbf{F_a})$
- Since the virtual work should be satisifed for any virtual nodal velocity, we get the Residual force with respect to the whole system $\mathbf{R_a} = \mathbf{T_a} - \mathbf{F_a}$

- Organise R in an array $\mathbf{T} = [\mathbf{T_1}\ \mathbf{T_2}...\mathbf{T_N}]$, $\mathbf{F} = [...]$, $\mathbf{R} = [....]$ (I think each T_1 contains three components)
- Virtual work equation is : $\delta\mathbf{v}^\mathbf{T}\mathbf{R} = \delta\mathbf{v}^\mathbf{T}(\mathbf{T} - \mathbf{F}) = \mathbf{0}$
  where $\delta\mathbf{v}^\mathbf{T} = [\delta\mathbf{v}_\mathbf{1}^\mathbf{T}\delta\mathbf{v}_\mathbf{2}^\mathbf{T}....]$
- Since the internal forces are nonlinear functions of the current nodal positions $\mathbf{x} = [\mathbf{x_1 x_2 x_3}...]$
- In matrix notation, we keep the symmetric tensor as
  $\sigma' = [\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{13}, \sigma_{23}]^T$ and $\mathbf{d}$ as $\mathbf{d} = [d_{11}, d_{22}, d_{33}, 2d_{12}, 2d_{13}, 2d_{23}]^T$
  (Where off diagonals is twice to make sure $d^T\sigma$ gives the correct internal energy)
- $\int \sigma : \mathbf{d} = \int \mathbf{d^T}\sigma dv$
- $\mathbf{d} = \sum_\mathbf{a}^\mathbf{n} \mathbf{B_a v_a}$ where $B_a = \begin{bmatrix} \frac{\partial N_a}{\partial x_1} & 0 & 0 \\ 0 & \frac{\partial N_a}{\partial x_2} & 0 \\ 0 & 0 & \frac{\partial N_a}{\partial x_3} \\ \frac{\partial N_a}{\partial x_2} & \frac{\partial N_a}{\partial x_1} & 0 \\ \\ \frac{\partial N_a}{\partial x_3} & 0 & \frac{\partial N_a}{\partial x_1} \\ 0 & \frac{\partial N_a}{\partial x_3} & \frac{\partial N_a}{\partial x_2} \end{bmatrix}$

- So : $\delta W = \int (B_a \delta v_a)^T \sigma dv - \int f.(N_a \delta v_a) dv - \int t.(N_a \delta v_a) da$
- We can also write the internal force as : $T_a^e = \int_{v^e} B_a^T \sigma' dv$

- The equilibrium equations are still nonlinear with respect to the nodal positions. A NR is used to solve it
- The linear virtual work components is found using the directional derivative as

$$D\delta W(\phi,\delta v)[u] = D\delta W_{int}(\phi,\delta v)[u] - D\delta W_{ext}(\phi,\delta v)[u] \tag{14}$$

- The internal work linearisation can be decomposed into the constitutive and intial stress components

$$D\delta W_{int}(\phi,\delta v)[u] = D\delta W_{C}(\phi,\delta v)[u] + D\delta W_{\sigma}(\phi,\delta v)[u] \tag{15}$$

$$= \int_{\mathbf{v}} \delta\mathbf{d} : \mathbf{c} : \varepsilon dv + \int_{\mathbf{v}} \sigma : \left((\nabla\mathbf{u})^{\mathbf{T}} (\nabla\delta\mathbf{v}) \right) dv \tag{16}$$

which is the tangent stiffness matrix

- Remember that at each node, we get the residual due to the nodal equivalent forces at a due to the whole equilibrium of node a (R = T -F)
- F may be dependant on a and so linearisation in the direction of u_b or $N_b u_b$ with $N_a v_a$ constant, gives only the change of the residual force at node a due to the change u_b in the current position of node b

$$D\delta W^e(\phi, N_a \delta v_a)[N_b u_b] = D(\delta v_a.(T_a^e - F_a^e))[N_b u_b] \tag{17}$$

$$= \delta v_a.D(T_a^e - F_a^e)[N_b u_b] = \delta v_a.K_{ab}^e u_b \tag{18}$$

- Change in force at node a due to change in current position of node b
- This is not the full stiffness matrix, but each component. When we do the whole assembly, we get the full tangen stiffness matrix

$$\frac{\partial \mathbf{R}}{\partial \mathbf{U}} = \begin{bmatrix} \frac{\partial R_1}{\partial u_1} & \frac{\partial R_1}{\partial u_2} & \cdots\cdots & \frac{\partial R_1}{\partial u_n} \\ \frac{\partial R_2}{\partial u_1} & \frac{\partial R_2}{\partial u_2} & \cdots\cdots & \frac{\partial R_2}{\partial u_n} \\ \frac{\partial R_n}{\partial u_1} & \frac{\partial R_n}{\partial u_2} & \cdots\cdots & \frac{\partial R_n}{\partial u_n} \end{bmatrix} \tag{19}$$

So one component here is attained by finding the linearisation in one direction node with the virtual work equilibrium in another node seperately

- Check bonet page 248
-
$$D\delta W_c^e(\phi, N_a\delta v_a)[N_b u_b] = \delta va.K^e_{c,ab} u_b \tag{20}$$
- where
$$[K_{c,ab}]_{ij} = \int_{v^e} \sum_{k,l=1}^{3} \frac{\partial N_a}{\partial x_k} Cikjl \frac{\partial N_b}{\partial x_l} dv \qquad i, j = 1, 2, 3 \tag{21}$$

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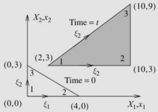

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$$x_{1,2}=3;\quad x_{2,2}=3;\quad x_{3,2}=9.$$

(continued)

$$[\boldsymbol{K}_{c,23}]_{11}=\left(\frac{1}{8}\right)(\mathfrak{c}_{1112})\left(\frac{1}{6}\right)-\left(\frac{1}{6}\right)(\mathfrak{c}_{1212})\left(\frac{1}{6}\right)(24t);$$
$$[\boldsymbol{K}_{c,23}]_{12}=\left(\frac{1}{8}\right)(\mathfrak{c}_{1122})\left(\frac{1}{6}\right)-\left(\frac{1}{6}\right)(\mathfrak{c}_{1222})\left(\frac{1}{6}\right)(24t);$$
$$[\boldsymbol{K}_{c,23}]_{21}=\left(\frac{1}{8}\right)(\mathfrak{c}_{2112})\left(\frac{1}{6}\right)-\left(\frac{1}{6}\right)(\mathfrak{c}_{2212})\left(\frac{1}{6}\right)(24t);$$
$$[\boldsymbol{K}_{c,23}]_{22}=\left(\frac{1}{8}\right)(\mathfrak{c}_{2122})\left(\frac{1}{6}\right)-\left(\frac{1}{6}\right)(\mathfrak{c}_{2222})\left(\frac{1}{6}\right)(24t);$$

where t is the thickness of the "element. Substituting for $\mathfrak{c}_{ijkl}$ from Equations (6.40) and (6.41) yields the stiffness" coefficients as

$$[\boldsymbol{K}_{c,23}]_{11}=-\frac{2}{3}\lambda' t;\quad [\boldsymbol{K}_{c,23}]_{12}=\frac{1}{2}\mu' t;\quad [\boldsymbol{K}_{c,23}]_{21}=\frac{1}{2}\mu' t;$$
$$[\boldsymbol{K}_{c,23}]_{22}=-\frac{2}{3}(\lambda'+2\mu')t;$$

where $\lambda'=\lambda/J$ and $\mu'=(\mu-\lambda\ln J)/J$.

$$\frac{\partial N_1}{\partial \mathbf{x}}=\frac{1}{24}\begin{bmatrix}3 & 0\\ -4 & 4\end{bmatrix}\begin{bmatrix}-1\\ -1\end{bmatrix}=\frac{-1}{24}\begin{bmatrix}3\\ 0\end{bmatrix}\qquad \frac{\partial N_2}{\partial \mathbf{x}}=\frac{1}{24}\begin{bmatrix}3\\ -4\end{bmatrix}\qquad \frac{\partial N_3}{\partial \mathbf{x}}=\frac{1}{24}\begin{bmatrix}0\\ 4\end{bmatrix} \tag{22}$$

K_{11} means equilibrium in 2 due to change in 3

- Virtual work for element e expressed in matrix notation by a small starin vector as $\varepsilon' = [\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, 2\varepsilon_{12}, 2\varepsilon_{13}, 2\varepsilon_{23}]^T$
- $\varepsilon' = B_a u_a$
- Now the constitutive component of the linearised virtual work can be written as

$$D\delta W_C(\phi, \delta v)[u] = \int_V \delta d : c : \varepsilon dv = \int_v \delta \mathbf{d}^{\mathbf{T}} \mathbf{D} \varepsilon' dv \tag{23}$$

- Where the later part is the matrix componets received from the tensor contraction

$$D = \begin{bmatrix} c_{1111} & c_{1122} & c_{1133} & c_{1112} & c_{1113} & c_{1123} \\ c_{2211} & c_{2222} & c_{2233} & c_{2212} & c_{2213} & c_{2223} \\ . & . & . & . & . & . \\ c_{2311} & c_{2322} & c_{2333} & c_{2312} & c_{2313} & c_{2323} \end{bmatrix} \tag{24}$$

- See bonet page 250 for neo-Hookean model
- For node a and b

$$D\delta W_c^e(\phi, N_a \delta v_a)[N_b u_b] = \int_{v^e} (\mathbf{B_a} \delta \mathbf{v_a})^{\mathbf{T}} \mathbf{D} (\mathbf{B_b} \mathbf{u_b}) dv = \int_{v^e} \delta \mathbf{v_a} . \underbrace{(\mathbf{B_a^T} \mathbf{D} \mathbf{B_b})}_{Tangent\ K} . \mathbf{u_b} dv \tag{25}$$

- Remember that the gradients of u and δv can be found as

$$content \tag{26}$$