1 Stress and equilibrium

2 Equilibrium

3 Principle of virtual work

Mechanics

Allan Marbaniang

Updated : Dec 6 2020

Stress and equilibrium

- We are dealing with different configurations. One configuration is maybe unstressed and the deformed one is. So at the deformed x we should get an equilibrium of stresses and the external loads
- Now, the actual stresses at the deformed or current configuration is the Cauchy stress : defined as the force in different directions by the area in different planes
- Stresses can also be defined with respect to the initial configuration X

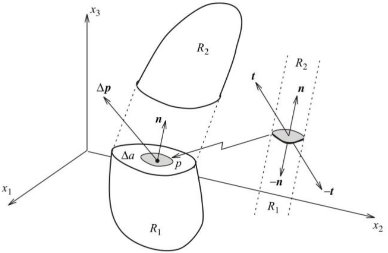


FIGURE 5.1 Traction vector.

At the deformed configuration :

- See two bodies R_1 and R_2 free body with force acting on them
- Imagine the traction vector on a small area element : $t(n) = \frac{\Delta p}{\Delta a}$ as $\lim \Delta a \to 0$ where Δp is the resultant force
- Obviously t and n will depend on the surface it acts on. Here on the right we can see that based on the surface we get opposite forces. (In the negative normal , we will get negative force which is positive in that direction!)

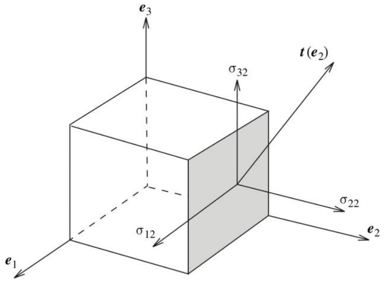
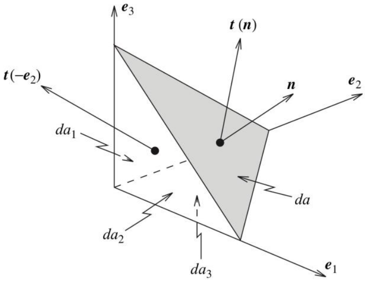


FIGURE 5.2 Stress components.

- Let us denote the traction acting on the surface having normals denoted by e_1, e_2, e_3
- Remember in the other slice we will have an opposite reaction

$$\begin{aligned} \mathbf{t}(\mathbf{e_1}) &= \sigma_{\mathbf{j1}}\mathbf{e_j} \\ \mathbf{t}(\mathbf{e_2}) &= \sigma_{\mathbf{j2}}\mathbf{e_j} \\ \mathbf{t}(\mathbf{e_3}) &= \sigma_{\mathbf{j3}}\mathbf{e_j} \end{aligned} \tag{1}$$

- Or $\mathbf{t_i} = \sigma_{\mathbf{ji}}\mathbf{e_j}$ or $\mathbf{t} = \sigma^{\mathbf{T}}\mathbf{e}$



Now let us look if we take a plane cut of that sphere. Again by context of opposite reactions. All the forces should be equal. So we will use here the concept of equilibrium between the traction vector we have defined in the last slide with respect to some basis and the traction vector defined on the angled plane.

Equilibrium

$$\mathbf{t}(\mathbf{n})da + t(-\mathbf{e_i})da_i + \mathbf{f}dv = 0 \tag{2}$$

This states that the force vector on the inclined cut should be in equilibrium with the opposite forces defined on the negative sufraces and the body force

- Now the areas (Because they are with defined respect to the basis vectors) can be written as the projection of the inclined area

$$da_i = da(\mathbf{n}.\mathbf{e_i}) \tag{3}$$

- Diving by da we get

$$\mathbf{t}(\mathbf{n}) + t(-\mathbf{e_i})\frac{da(\mathbf{n}.\mathbf{e_i})}{da} + \mathbf{f}\frac{dv}{da} = 0 \tag{4}$$

- $\frac{dv}{da} \to 0$ (I don't know why????????????????)
- We get :

$$\begin{aligned} \mathbf{t}(\mathbf{n}) = -t(-\mathbf{e_i})(\mathbf{n}.\mathbf{e_i}) = t(\mathbf{e_i})(\mathbf{n}.\mathbf{e_i}) \\ \mathbf{t}(\mathbf{n}) = (\sigma_{\mathbf{ji}}\mathbf{e_j})(\mathbf{n}.\mathbf{e_i}) \\ \mathbf{t}(\mathbf{n}) = (\sigma_{\mathbf{ij}}\mathbf{e_i})(\mathbf{n}.\mathbf{e_j})\text{Replacing indexs} \end{aligned} \tag{5}$$

- Very interesting, we started off with a statement that the resultant force on the plane is equal to the summation of the opposite forces
- Then we got the traction vector is equal to the traction vectors multiplied by some scalar product (Think of ratio)
- $t(\mathbf{e_i})(\mathbf{n}.\mathbf{e_i})$ states the traction in e_i direction multiplied by the projection of planar area for i =1,2,3
- Now we can replace the traction vector by the components of stress vectors in the basis direction $\mathbf{t}(\mathbf{n}) = (\sigma_{\mathbf{ij}}\mathbf{e_i})(\mathbf{n}.\mathbf{e_j})$
- Here we have to point out that σ_{ij} has not been described as a tensor yet

- If we look at $\mathbf{t}(\mathbf{n}) = (\sigma_{ij}\mathbf{e_i})(\mathbf{n}.\mathbf{e_j})$, we can see that σ_{ij} is just a component and $\mathbf{n}.\mathbf{e_j}$ is a scalar (Or a projection).
- That scalar value then becomes the component of e_i. Lets see what that means

$$\sigma_{12}\mathbf{e_1}(\mathbf{n}.\mathbf{e_2}) = \sigma_{12}\begin{bmatrix}1\\0\\0\end{bmatrix}\left(\begin{bmatrix}n_1 & n_2 & n_3\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \sigma_{12}n_2\begin{bmatrix}1\\0\\0\end{bmatrix} \tag{6}$$

- So we took the second component of n and added to the result of linear map in e_1.This is what we do when we multiply the first row of a matrix and a vector. We add all the components of the vector to keep in the first component of the output
- So we can write it therefore as

$$\mathbf{e_i}(\mathbf{n}.\mathbf{e_j}) = (\mathbf{e_i} \otimes \mathbf{e_j}).\mathbf{n} \tag{7}$$

  which states that the tensor takes the projection of e_j in n and maps as components of e_i
- This then allows us to understand that $\sigma_{ij}(\mathbf{e_i} \otimes \mathbf{e_j})$ is a tensor σ

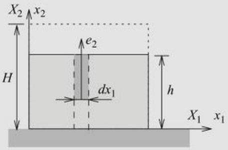
- In simplicity $\mathbf{t}(\mathbf{n}) = (\sigma_{ij}\mathbf{e_i})(\mathbf{n}.\mathbf{e_j})$ says that for every cut, $\mathbf{n}$

$$\begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \begin{bmatrix} \sigma_{11}\mathbf{e_1}(\mathbf{n}.\mathbf{e_1}) + \sigma_{12}\mathbf{e_1}(\mathbf{n}.\mathbf{e_2}) + \sigma_{13}\mathbf{e_1}(\mathbf{n}.\mathbf{e_3}) \\ \sigma_{21}\mathbf{e_2}(\mathbf{n}.\mathbf{e_1}) + \sigma_{22}\mathbf{e_2}(\mathbf{n}.\mathbf{e_2}) + \sigma_{23}\mathbf{e_2}(\mathbf{n}.\mathbf{e_3}) \\ \sigma_{31}\mathbf{e_3}(\mathbf{n}.\mathbf{e_1}) + \sigma_{32}\mathbf{e_3}(\mathbf{n}.\mathbf{e_2}) + \sigma_{33}\mathbf{e_3}(\mathbf{n}.\mathbf{e_3}) \end{bmatrix} \tag{8}$$

  FIXXX. These are not components yet!

- And now you can see that $\mathbf{e_i} \otimes \mathbf{e_j}$ just makes the tensor take the right projection and keeps in the component in the output

- $\sigma_{ij} n_j$ is therefore something like taking the projection of j for every component i



A simple example of a two-dimensional stress tensor results from the self-weight of a block of uniform initial density ρ_0 resting on a frictionless surface as shown in the figure above. For simplicity we will assume that there is no lateral deformation (in linear elasticity this would imply that the Poisson ratio $\nu = 0$).

- $t(\mathbf{e_2}) = \dfrac{\left(-\int_y^h \rho g dx_2\right)\mathbf{e_2} dx_1}{dx_1}$
- Mass conservation $\rho dx_1 dx_2 = \rho_o dX_1 dX_2$ + poisson = 0 give :

$$t(\mathbf{e_2}) = \rho_o g(H - X_2)\mathbf{e_2} \tag{9}$$

$$t(\mathbf{e_2}) = \sigma_{12}\mathbf{e_1} + \sigma_{22}\mathbf{e_2} \tag{10}$$

so $\sigma_{12} = 0$, so you can construct σ. Check Bonet Pge 138

- Obviously the Cauchy stress components can be described with respect to its principal directions ϕ_1, ϕ_2, ϕ_3 with principal stresses $\sigma_{\lambda_1}, \sigma_{\lambda_2}, \sigma_{\lambda_3}$
- So we can write in tensor notation : $\sigma = \sigma_{\lambda_\mathbf{i}} (\lambda_\mathbf{i} \otimes \lambda_\mathbf{i})$ (Only diagonals, the tensor is $i \otimes i$ which only is for the diagonal components)
- The cauchy stress is a spatial tensor (In the deformed configuration), and is symmetric because of the rotational equilibrium

- So the stress tensor should not change it's property when there is a rigid body motion etc.
- ????????????????

Equilibrium

- The spatial configuration of the body has to be in equilibrium having volume v and boundary Γ
- At equilibrium, the body is under forces f and traction forces t
- Looking at the translational equilibrium of the structure, we get :

$$\int_{\delta\Gamma} \mathbf{t} da + \int_{v} \mathbf{d} dv = 0 \tag{11}$$

- In terms of The Cauchy stress we get

$$\int_{\delta\Gamma} \sigma \mathbf{n} da + \int_{v} \mathbf{d} dv = 0 \tag{12}$$

- If we use the Gauss theorem to convert the area to volume integral we get

$$\int_{\delta v} \left(DIV\sigma + \mathbf{d} \right) dv = 0 \tag{13}$$

- As the above region can be applied to any closed region, the integrand must vanish to get $DIV\sigma + \mathbf{f} = \mathbf{0}$

- This is the equilibrium equation at a very smalllll level :

$$\frac{\partial \sigma_{ij}}{\partial xj} + f_i = 0 \tag{14}$$

- This equation is the *local* or spatial (deformed) equilibrium.
- While solving this equation may not be satisfied, and we have a pointwise out of balance or residual given as

$$r = DIV\sigma + f \tag{15}$$

- Not gonna explain. The rotational equilibrium gives you the fact that the Cauchy stress is symmetric. $\sigma^T = \sigma$
- See Bonet Page 142

Principle of virtual work

- FEM is usually based in terms of a weak form of the differential equations