1 Plates

Plates : February 18, 2021

- Plate has large dimension compared to thickness. No need to use 3D, with study the deformations and stresses in plates, smal rotations and large displacement (w/h>1)
- Extension of Euler berunoulli is called Kirchoff plate. Extension of timoshenko beam is the first order or mindlin shear deformation plate theory.
- **X** is used for the material coordinates and **x** is used for the spatial coordinates. No distinction is made between the material and spatial coordinates.

The disp satisfy the kirchoff rules which are an extension of the euler bernoulli hypothesis which are

- Straight lines perpendicular to the mid surface remain straight
- Transverse normals do not have elongation
- Cross sections remain perpendiuclar under rotation

- We have the domain of the plate as $\Omega_o \times (-h/2, h/2)$. The boundary of the top surface $z = h/2$ and bottom surface $z = -h/2$ with boundary Γ which is a curved surface with outward normal $\mathbf{n} = n_x e_1 + n_y e_2$

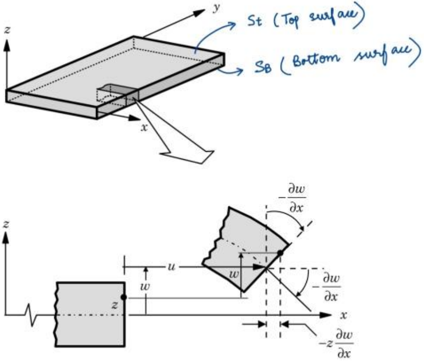


Fig. 7.2.1: Undeformed and deformed geometries of an edge of a plate under the Kirchhoff assumptions.

- The kirchoff hypotehesis implies the following displacement field

$$\begin{aligned} u_1(x,y,z,t) &= u(x,y,t) - z\frac{\partial w}{\partial x} \\ u_2(x,y,z,t) &= v(x,y,t) - z\frac{\partial w}{\partial y} \\ u_3(x,y,z,t) &= w(x,y,t) \end{aligned} \tag{1}$$

- Where u, v, w denote the material point in the undeformed of the nerutral axis wherees u_1, u_2, u_3 denote any aribitary point location
- The componenets of the Green-Lagrange strain tensor **E** in terms of components of the total displacement vector $u = x(x,t) - X$ (x and X here are the same) is

$$\begin{aligned} E_{11} &= \frac{\partial u_1}{\partial X_1} + \frac{1}{2}\left[\left(\frac{\partial u_1}{\partial X_1}\right)^2 + \left(\frac{\partial u_2}{\partial X_1}\right)^2 + \left(\frac{\partial u_3}{\partial X_1}\right)^2\right] \\ E_{22} &= \frac{\partial u_2}{\partial X_2} + \frac{1}{2}\left[\left(\frac{\partial u_1}{\partial X_2}\right)^2 + \left(\frac{\partial u_2}{\partial X_2}\right)^2 + \left(\frac{\partial u_3}{\partial X_2}\right)^2\right] \\ E_{33} &= \frac{\partial u_3}{\partial X_3} + \frac{1}{2}\left[\left(\frac{\partial u_1}{\partial X_3}\right)^2 + \left(\frac{\partial u_2}{\partial X_3}\right)^2 + \left(\frac{\partial u_3}{\partial X_3}\right)^2\right] \\ E_{12} &= \frac{1}{2}\left[\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1} + \frac{\partial u_1}{\partial X_1}\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1}\frac{\partial u_2}{\partial X_2} + \frac{\partial u_3}{\partial X_1}\frac{\partial u_3}{\partial X_2} + \right] \\ E_{13} &= \frac{1}{2}\left[\frac{\partial u_1}{\partial X_3} + \frac{\partial u_3}{\partial X_1} + \frac{\partial u_1}{\partial X_1}\frac{\partial u_1}{\partial X_3} + \frac{\partial u_2}{\partial X_1}\frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_1}\frac{\partial u_3}{\partial X_3} + \right] \\ E_{23} &= \frac{1}{2}\left[\frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_2} + \frac{\partial u_1}{\partial X_2}\frac{\partial u_1}{\partial X_3} + \frac{\partial u_2}{\partial X_2}\frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_2}\frac{\partial u_3}{\partial X_3} + \right] \end{aligned} \tag{2}$$

- If the components of the displacement gradient are very small = $O(\epsilon)$, then the terms having $O(\epsilon^2)$ can be omitted in the strains. However if the rotations of the transverse normals are moderate (10 -15 degrees), then the following strains are small but not negligible

$$\left(\frac{\partial u_3}{\partial X_1}\right)^2 \qquad \left(\frac{\partial u_3}{\partial X_2}\right)^2 \qquad \frac{\partial u_3}{\partial X_1}\frac{\partial u_3}{\partial X_2} \tag{3}$$

- Thus the strains take the following (Remember that $E = \varepsilon$ where we say its small strain but moderate rotations)

$$\begin{aligned} E_{11} = \varepsilon_{11} = \frac{\partial u_1}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u_3}{\partial x}\right)^2\right] \qquad & E_{22} = \frac{\partial u_2}{\partial y} + \frac{1}{2}\left[\left(\frac{\partial u_3}{\partial y}\right)^2\right] \\ E_{33} = \frac{\partial u_3}{\partial z} \qquad & E_{12} = \frac{1}{2}\left[\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} + \frac{\partial u_3}{\partial x}\frac{\partial u_3}{\partial y}\right] \\ E_{13} = \frac{1}{2}\left[\frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x}\right] \qquad & E_{23} = \frac{1}{2}\left[\frac{\partial u_2}{\partial z} + \frac{\partial u_3}{\partial y}\right] \end{aligned} \tag{4}$$

- For this displacement field, we have $\varepsilon_z = \frac{\partial u_3}{\partial z} = \frac{\partial w}{\partial z} = 0$ and taking the displacement fields. The strains then reduce to

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial w}{\partial x}\right)^2 - z\frac{\partial^2 w}{\partial x^2}\right] \qquad \varepsilon_{yy} = \frac{\partial v}{\partial y} + \frac{1}{2}\left[\left(\frac{\partial w}{\partial y}\right)^2 - z\frac{\partial^2 w}{\partial y^2}\right]$$

$$2\varepsilon_{xy} = \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} - 2z\frac{\partial^2 w}{\partial x \partial y} \tag{5}$$

$$2\varepsilon_{xz} = -\frac{\partial w}{\partial x} + \frac{\partial w}{\partial x} = 0 \qquad 2\varepsilon_{yz} = -\frac{\partial w}{\partial y} + \frac{\partial w}{\partial y} = 0$$

- These are called von Karman strains and called classical plate theory with von karman strains. Note that the transverse strains are zero in the classical plate theory. The total strains can be written as membrane + bending strain

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx}^0 \\ \varepsilon_{yy}^0 \\ \gamma_{xy}^0 \end{bmatrix} + z \begin{bmatrix} \varepsilon_{xx}^1 \\ \varepsilon_{yy}^1 \\ \gamma_{xy}^1 \end{bmatrix} \tag{6}$$

- The strains are expanded as :

$$\begin{bmatrix} \varepsilon_{xx}^0 \\ \varepsilon_{yy}^0 \\ \gamma_{xy}^0 \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \\ \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \end{bmatrix} \qquad \begin{bmatrix} \varepsilon_{xx}^1 \\ \varepsilon_{yy}^1 \\ \gamma_{xy}^1 \end{bmatrix} = - \begin{bmatrix} \frac{\partial^2 w}{\partial x^2} \\ \frac{\partial^2 w}{\partial y^2} \\ 2\frac{\partial^2 w}{\partial x \partial y} \end{bmatrix} \tag{7}$$

- Virtual work statement is used again to derive (Just like 4 beams). We account for thermal effects, where the material does not change with tempearture which is knowon as a fuction of the position hence ($\delta T = 0$), so temperature eneters throught the constitutive relations.
- Suppose the domain is represented by fem Ω_e with distributed transverse loads $q(x, y)$ at the top. ($\sigma_{nn}, \sigma_{ns}, \sigma_{nz}$) are the stress components on the boundary of the plate

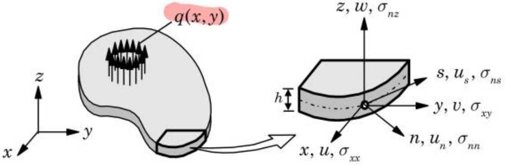


Fig. 7.3.1: Geometry of a plate element with curved boundary.

- The principle of virtual work states that $0 = \delta W^e = \delta W_I^e + \delta W_E^e$
- As noted earlier, the transverse shears $\gamma_{xz}, \gamma_{yz}, \varepsilon_{zz}$ are zero. Therefore the transverse stresses ($\sigma_{xz}, \sigma_{yz}, \sigma_{zz}$) do not eneter the formulatin because the strains due to these are zero. Even though they are not accounted, in reality they exist to maintain equilibirum, these components can also be specified at the boudary. So they have to be accounted in the equilibirum equations.
- The internal virtual strain is then given as

$$\begin{aligned} \delta W_I^e &= \int_A \int_{-\frac{h}{2}}^{\frac{h}{2}} \left(\sigma_{xx}\delta\varepsilon_{xx} + \sigma_{yy}\delta\varepsilon_{yy} + 2\sigma_{xy}\delta\varepsilon_{xy}\right) dz\; dx\; dy \\ &= \int_A \left(N_{xx}\delta\varepsilon_{xx}^0 + M_{xx}\delta\varepsilon_{xx}^1 + N_{yy}\delta\varepsilon_{yy}^0 + M_{yy}\delta\varepsilon_{yy}^1 + N_{xy}\delta\gamma_{xy}^0 + M_{xy}\delta\gamma_{xy}^1\right) dx\; dy \end{aligned} \tag{8}$$

where N and M are the axial and the moment internal forces per unit length.

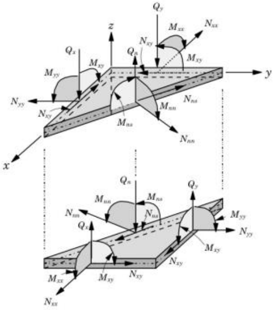


Fig. 7.3.2: Forces and moments per unit length on a plate element.

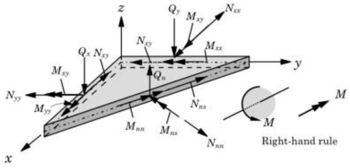


Fig. 7.3.3: Alternative sign convention for moments on a plate element.

- The virtual work by the distributed transverse load q(x,y), reaction force of an elastic foundation, in plane normal stress σ_{nn}, in plane tangential stress σ_{ns}, transverse shear stress σ_{nz} is

$$\begin{aligned}\delta W_E^e = -(\int_A q(x,y)\delta w(x,y,\frac{h}{2})dx\,dy + \int_A F_s(x,y)\delta w(x,y,-\frac{h}{2})dx\,dy \\ + \int_S \int_{-\frac{h}{2}}^{\frac{h}{2}} \left[\sigma_{nn}\left(\delta u_n - z\frac{\delta w}{n}\right) + \sigma_{ns}\left(\delta u_s - z\frac{\delta w}{s}\right) + \sigma_{nz}\delta w\right] dz\,ds) \\ = -\left[\int_S \left(N_{nn}\delta u_n - M_{nn}\frac{\partial \delta w}{\partial n} + N_{ns}\delta u_s - M_{ns}\frac{\partial \delta w}{\partial s} + Q_n \delta w\right) ds + \int_A (q-kw)\delta w dx\,dy\right]\end{aligned} \tag{9}$$

- where Fs = -kw (Foundation force), the negative sign it is the force applied upwards, but you can think of it like the potential increases as the w increases.
- The last term in the work term is the virtual work of the transverse and normal forces on a boundary that is inclined. Where N = $\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma$, M = $\int_{-\frac{h}{2}}^{\frac{h}{2}} z\sigma$, $Q_n = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{nz}$

- We now relate the stresses in the direction in the boundary to the internal stresses given in cartesion using stress tensor coordinate transformation

$$\begin{bmatrix}\sigma_{nn}\\ \sigma_{ns}\end{bmatrix} = \begin{bmatrix} n_x^2 & n_y^2 & 2n_x n_y \\ -n_x n_y & n_x n_y & n_x^2 - n_y^2 \end{bmatrix} \begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{bmatrix} \tag{10}$$

- Keeping the weak forms in the full virtual work statement we get

$$
\begin{aligned}
0 = &\int_A \left(N_{xx}\delta\varepsilon^0_{xx} + M_{xx}\delta\varepsilon^1_{xx} + N_{yy}\delta\varepsilon^0_{yy} + M_{yy}\delta\varepsilon^1_{yy} + N_{xy}\delta\gamma^0_{xy} + M_{xy}\delta\gamma^1_{xy} \right) dx\, dy \\
- &\left[\int_S \left(N_{nn}\delta u_n - M_{nn}\frac{\partial \delta w}{\partial n} + N_{ns}\delta u_s - M_{ns}\frac{\partial \delta w}{\partial s} + Q_n \delta w \right) ds + \int_A (q - kw)\delta w dx\, dy \right]_1 \\
= &\int_A \left(N_{xx}\delta\varepsilon^0_{xx} + M_{xx}\delta\varepsilon^1_{xx} + N_{yy}\delta\varepsilon^0_{yy} + M_{yy}\delta\varepsilon^1_{yy} + N_{xy}\delta\gamma^0_{xy} + M_{xy}\delta\gamma^1_{xy} \right) dx\, dy \\
- &\left[\int_S \left(N_{nn}\delta u_n - M_{nn}\frac{\partial \delta w}{\partial n} + N_{ns}\delta u_s - M_{ns}\frac{\partial \delta w}{\partial s} + Q_n \delta w \right) ds + \int_A (q - kw)\delta w dx\, dy \right]
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
= &\int_{\Omega^e} \left[\left(\frac{\partial \delta u}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial \delta w}{\partial x} \right) N_{xx} + \left(\frac{\partial \delta v}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial \delta w}{\partial y} \right) N_{yy} \right. \\
&+ \left(\frac{\partial \delta u}{\partial y} + \frac{\partial \delta v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial \delta w}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial \delta w}{\partial x} \right) N_{xy} \\
&\left. - \frac{\partial^2 \delta w}{\partial x^2} M_{xx} - \frac{\partial^2 \delta w}{\partial y^2} M_{yy} - 2\frac{\partial^2 \delta w}{\partial x \partial y} M_{xy} + k\delta w w - \delta w q \right] dx\, dy \\
&- \oint_{\Gamma^e} \left(N_{nn}\delta u_n + N_{ns}\delta u_s - M_{nn}\frac{\partial \delta w}{\partial n} - M_{ns}\frac{\partial \delta w}{\partial s} + Q_n \delta w \right) ds
\end{aligned}
\tag{7.3.9}
$$

$uv' = \int u'v - [uv]_{xa}^{xb}$

The statement in Eq. (7.3.9) is equivalent to the following three statements:

$$
0 = \int_{\Omega^e} \left(\frac{\partial \delta u}{\partial x} N_{xx} + \frac{\partial \delta u}{\partial y} N_{xy} \right) dx\, dy - \oint_{\Gamma^e} N_{nn}\delta u_n \, ds \tag{7.3.10}
$$

$$
0 = \int_{\Omega^e} \left(\frac{\partial \delta v}{\partial x} N_{xy} + \frac{\partial \delta v}{\partial y} N_{yy} \right) dx\, dy - \oint_{\Gamma^e} N_{ns}\delta u_s \, ds \tag{7.3.11}
$$

$$
\begin{aligned}
0 = &\int_{\Omega^e} \left[\frac{\partial \delta w}{\partial x} \left(\frac{\partial w}{\partial x} N_{xx} + \frac{\partial w}{\partial y} N_{xy} \right) + \frac{\partial \delta w}{\partial y} \left(\frac{\partial w}{\partial x} N_{xy} + \frac{\partial w}{\partial y} N_{yy} \right) \right. \\
&\left. - \frac{\partial^2 \delta w}{\partial x^2} M_{xx} - \frac{\partial^2 \delta w}{\partial y^2} M_{yy} - 2\frac{\partial^2 \delta w}{\partial x \partial y} M_{xy} + k\delta w w - \delta w q \right] dx\, dy \\
&- \oint_{\Gamma^e} \left(-M_{nn}\frac{\partial \delta w}{\partial n} - M_{ns}\frac{\partial \delta w}{\partial s} + Q_n \delta w \right) ds
\end{aligned}
\tag{7.3.12}
$$

[1] In the last three statements, It seems the equation has been kept according to the variation but how do you account for $\delta u_n \quad \delta u_s$ as they will have components in both. Maybe the euler equations will make sense

- Keeping the virtual parameters in the same order to get the euler equations we get

$$\begin{aligned}
0 = \int_A \Big[-\left(N_{xx,x} + N_{xy,y}\right)\delta u - \left(N_{xy,x} + N_{yy,y}\right)\delta v - \left(M_{xx,xx} + 2M_{xy,xy} + M_{yy,yy} + \mathcal{N} - kw + q\right)\delta w \Big]\, dx\, dy \\
+ \int_S (\left(N_{xx} n_x + N_{xy} n_y\right)\delta u + \left(N_{xy} n_x + N_{yy} n_y\right)\delta v + \left(M_{xx,x} n_x + M_{xy,y} n_x + M_{yy,y} n_y + M_{xy,x} n_y + \mathcal{P}\right)\delta w \\
- \left(M_{xx} n_x + M_{xy} n_x\right)\frac{\partial \delta w}{\partial x} - \left(M_{xy} n_x + M_{yy} n_y\right)\frac{\partial \delta w}{\partial y})\, ds - \int_S \left(N_{nn}\delta u_n + N_{ns}\delta u_s - M_{nn}\frac{\partial \delta w}{\partial n} - M_{ns}\frac{\partial \delta w}{\partial s} + Q_n \delta w\right) ds
\end{aligned} \tag{12}$$

- Where

$$\begin{aligned}
\mathcal{N} &= \frac{\partial}{\partial x}\left(N_{xx}\frac{\partial w}{\partial dx} + N_{xy}\frac{\partial w}{\partial dy}\right) + \frac{\partial}{\partial y}\left(N_{xy}\frac{\partial w}{\partial x} + N_{yy}\frac{\partial w}{\partial y}\right) \\
\mathcal{P} &= \left(N_{xx}\frac{\partial w}{\partial x} + N_{xy}\frac{\partial w}{\partial y}\right) n_x + \frac{\partial}{\partial y}\left(N_{xy}\frac{\partial w}{\partial x} + N_{yy}\frac{\partial w}{\partial y}\right) n_y
\end{aligned} \tag{13}$$

INTEGRATION BY PARTS

Divergance theorem:

· Suppose we have a vector field $v(x,y)$

$$\int_A \left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\right) dA = \int_\Gamma v \cdot n \, ds = \int_\Gamma v_x n_x \, ds + \int_\Gamma v_y n_y \, ds.$$

So we can say $\int_A \frac{\partial}{\partial x}(v)\, dA = \int_\Gamma v_x n_x \, ds$ etc.

Now: $\int_A \frac{\partial}{\partial x}(AB)\, dA = \int_\Gamma (AB) . n_x \, ds$

and $(AB)' = AB' + B'A$ or $AB' = (AB)' - B'A$.

$$\therefore \int_A N_{xx} \frac{\partial \delta u}{\partial x} = \int_A \frac{\partial}{\partial x}(N_{xx}\delta u) - \int_A \frac{\partial N_{xx}}{\partial x} \delta u$$

$$= \int_\Gamma (N_{xx}\delta u).n_x \, ds - \int \frac{\partial N_{xx}}{\partial x}\delta u$$

inside $\sqrt{V}$

$$\int_A N_{xx} \frac{\partial w}{\partial x} \frac{\partial \delta w}{\partial x} = \int_A \frac{\partial}{\partial x}\left(N_{xx}\frac{\partial w}{\partial x}\delta w\right) - \int_A \frac{\partial}{\partial x}\left(N_{xx}\frac{\partial w}{\partial x}\right).\delta w$$

$$= \int_\Gamma \left(N_{xx}\frac{\partial w}{\partial x}\delta w\right).n_x \, ds. - \int_{dA} \frac{\partial}{\partial x}\left(N_{xx}\frac{\partial w}{\partial x}\right)\delta w$$

– inside P

$$-\int_A 2M_{xy}\frac{\partial^2 \delta w}{\partial x \partial y} = -2\left(\int_A \frac{\partial}{\partial x}\left(M_{xy}\frac{d\delta w}{\partial y}\right) - \int_A \frac{\partial M_{xy}}{\partial x}\frac{\partial \delta w}{\partial y}\right)$$

$$= -2\left(\int_\Gamma \left(M_{xy}\frac{\partial \delta w}{\partial y}\right).n_x \, ds - \int_A \frac{\partial M_{xy}}{\partial x}\frac{\partial \delta w}{\partial y}\right)$$

$(uv)' = uv' + v'u$

$uv| = \int uv' + \int v' u$

$$= -2\left(\int_\Gamma (M_{xy} n_x)\frac{d\delta w}{dx} - \left(\int \frac{d}{dy}\left(\frac{M_{xy}}{dx}\partial \delta w\right) - \int \frac{\partial M_{xy}}{\partial x \partial y}\partial \delta w\right)\right)$$

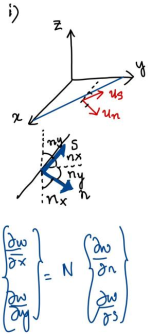
$$= -2\left(\int_\Gamma (M_{xy} n_x)\frac{d\delta w}{dx} - \int_\Gamma \left(\frac{\partial M_{xy}}{\partial x} d\delta w\right) n_y \, ds + \int_A \frac{\partial M_{xy}}{\partial x \partial y} d\delta w\right)$$

- Keeping the coefficients of the variations (seting $\delta u, \delta v, \delta w = 0$)we get the equilibirum equations given as

$$
\begin{aligned}
\delta u : \frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} &= 0 \\
\delta v : \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} &= 0 \\
\delta w : \frac{\partial^2 M_{xx}}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_{yy}}{\partial y^2} + \mathcal{N} - kw + q &= 0
\end{aligned}
\tag{14}
$$

- To cast the B.C. on an edge whose normal is **n**, we express the generalised displacements (u, v, w, $\frac{\partial w}{\partial x}$, $\frac{\partial w}{\partial y}$) in x,y,z system in the corresponding displacements in normal, tangential and transverse directions. We get

$$
\begin{gathered}
u = u_n n_x - u_s n_y \qquad v = u_n n_y + u_s n_x \\
\frac{\partial w}{\partial x} = \frac{\partial w}{\partial n} n_x - \frac{\partial w}{\partial s} n_y \qquad \frac{\partial w}{\partial y} = \frac{\partial w}{\partial n} n_y + \frac{\partial w}{\partial s} n_x
\end{gathered}
\tag{15}
$$



Now the derivative of w is a vector which can undergo basis transformation

- The boundary conditions can be written in the normal, tangenetial coords

$$
\begin{aligned}
&\int_S [\left(N_{xx} n_x + N_{xy} n_y\right)\left(\delta u_n n_x - \delta u_s n_y\right) + \left(N_{xy} n_x + N_{yy} n_y\right)\left(\delta u_n n_y + \delta u_s n_x\right) \\
&\qquad + \left(M_{xx,x} n_x + M_{xy,y} n_x + M_{yy,y} n_y + M_{xy,x} n_y + \mathcal{P}\right)\delta w \\
&- \left(M_{xx} n_x + M_{xy} n_y\right)\left(\frac{\partial w}{\partial n} n_x - \frac{\partial w}{\partial s} n_y\right) - \left(M_{xy} n_x + M_{yy} n_y\right)\left(\frac{\partial w}{\partial n} n_y + \frac{\partial w}{\partial s} n_x\right)] ds \\
&\qquad - \int_S \left(N_{nn}\delta u_n + N_{ns}\delta u_s - M_{nn}\frac{\partial \delta w}{\partial n} - M_{ns}\frac{\partial \delta w}{\partial s} + Q_n \delta w\right) ds
\end{aligned}
\tag{16}
$$

- δu in natural does not change the direction cosines because here they are constant dependant on the geometry we are at and not the variation.

$$
\begin{aligned}
= \oint_{\Gamma^e} \Big\{ & \left(N_{xx}n_x^2 + 2N_{xy}n_xn_y + N_{yy}n_y^2 - N_{nn}\right)\delta u_n \\
& + \left[(N_{yy} - N_{xx})n_xn_y + N_{xy}(n_x^2 - n_y^2) - N_{ns}\right]\delta u_s \\
& + \left(M_{xx,x}n_x + M_{xy,y}n_x + M_{yy,y}n_y + M_{xy,x}n_y + \mathcal{P} - Q_n\right)\delta w \\
& - \left(M_{xx}n_x^2 + 2M_{xy}n_xn_y + M_{yy}n_y^2 - M_{nn}\right)\frac{\partial \delta w}{\partial n} \\
& - \left[(M_{yy} - M_{xx})n_xn_y + M_{xy}(n_x^2 - n_y^2) - M_{ns}\right]\frac{\partial \delta w}{\partial s}\Big\} ds
\end{aligned}
\qquad (7.3.19)
$$

The natural boundary conditions are obtained by setting the coefficients of δu_n, δu_s, δw, $\frac{\partial \delta w}{\partial n}$ and $\frac{\partial \delta w}{\partial s}$ on Γ^e to zero: (Interesting)

$$
\begin{aligned}
\delta u_n: \; & N_{nn} = N_{xx}n_x^2 + 2N_{xy}n_xn_y + N_{yy}n_y^2 \\
\delta u_s: \; & N_{ns} = (N_{yy} - N_{xx})n_xn_y + N_{xy}(n_x^2 - n_y^2) \\
\delta w: \; & Q_n = M_{xx,x}n_x + M_{xy,y}n_x + M_{yy,y}n_y + M_{xy,x}n_y + \mathcal{P}
\end{aligned}
\qquad (7.3.20a)
$$

$$
\begin{aligned}
\frac{\partial \delta w}{\partial n}: \; & M_{nn} = M_{xx}n_x^2 + 2M_{xy}n_xn_y + M_{yy}n_y^2 \\
\frac{\partial \delta w}{\partial s}: \; & M_{ns} = (M_{yy} - M_{xx})n_xn_y + M_{xy}(n_x^2 - n_y^2)
\end{aligned}
\qquad (7.3.20b)
$$

From Eq. (7.3.20), it is clear that the primary variables (i.e. generalized displacements) and secondary variables (i.e. generalized forces) of the theory are:

$$
\begin{aligned}
&\text{Primary variables:} \quad && u_n, \quad u_s, \quad w, \quad \frac{\partial w}{\partial n}, \quad \frac{\partial w}{\partial s} \\
&\text{Secondary variables:} \quad && N_{nn}, \; N_{ns}, \; Q_n, \; M_{nn}, \; M_{ns}
\end{aligned}
\qquad (7.3.21)
$$

- I find this interesting to keep the varitaion with derivatives equal to zero.
- We see that from the equilibrium equations, if we keep them in displacements we woul get a DE having second order spatial derivatives of u, v and fourth order of w. Therefore we need eight boundary conditions (4 primary and 4 natural). But we have eight

- To remove this problem, one can integrate the tangential derivative term by parts

$$-\int_S M_{ns}\frac{\partial\delta w}{\partial s}ds = \int_S \frac{\partial M_{ns}}{\partial s}\delta w ds - [M_{ns}\delta w]_\Gamma \tag{17}$$

- The second term $[M_{ns}\delta w]$ is zero when the end points of two curves meet or when $M_{ns} = 0$. If M_{ns} is not specified at the corners, then concentrated forces $F = -2M_{ns}$ are produced at the corners. 2 appears from the two sides of the corner.
- The remaining boundary term is added to the shear force Q_n (Also having a coefficien of δw in integral S)to get

$$V_n = Q_n + \frac{\partial M_{ns}}{\partial s} \tag{18}$$

This specification of theshare force is known a sthe Kirchoff free edge condition. The final boundary conditions are

$$\begin{aligned}\text{Genearlised dispalcements} &: u_n, u_s, w, \frac{\partial w}{\partial n}\\ \text{Generalised forces} &: N_{nn}, N_{ns}, V_n, M_{nn}\end{aligned} \tag{19}$$

2

[2]So you known either one he displacement of the forces. On a side parallel to x asix (s = x and n =y). $u_n = v, u_s = u, w, \frac{\partial w}{\partial n} = \frac{\partial w}{\partial y}, N_{nn} = N_{yy}, N_{ns} = N_{yx}, V_n = V_y, M_{nn} = M_{yy}$

We discuss for some common boundary with edges parallel to x and y coordinates

- **Free edge with normal n** : We don't know the disp but we know the force/moment

$$
\begin{aligned}
u_n \neq 0, \quad u_s \neq 0, \quad w \neq 0, \quad \frac{\partial w}{\partial n} \neq 0 \\
N_{nn} = \hat{N_{nn}}, \quad N_{ns} = \hat{N_{ns}}, V_n = Q_n + \frac{\partial M_{ns}}{\partial s} = \hat{V_n}, M_{nn} = \hat{M_{nn}}
\end{aligned}
\tag{20}
$$

- **Fixed with normal n** : Fixed edge with primary values known. But we dont know the reaction forces and moments. Given as
  $u_n = 0, \quad u_s = 0, \quad w = 0, \quad \frac{\partial w}{\partial n} = 0.$
- **Simply suppored** : This is not unize especially when both inplane and bending are coupled. Here showing two types
  1. SS1: $u_s = 0, \quad w = 0 \quad, N_{nn} = \hat{N_{nn}}, \quad M_{nn} = \hat{M_{nn}}$
  2. SS2: $u_n = 0, \quad w = 0 \quad, N_{ns} = \hat{N_{ns}}, \quad M_{nn} = \hat{M_{nn}}$

- To express the foces and moments (N,M) per unit length in terms of the generalized displacements we need to bring the correct stress-strain relations. In the classical plate theory all the transverse strain components ($\varepsilon_{xx,xz,yz}$) are zero.
- Since $\varepsilon_{zz} = 0$, the transverse normal stress σ_{zz} even though not zero, does not appear in the virtual work statement and equation of motion. Therefore it is like we are neglecting the transverse normal stress. So we have a case of both plane strain and plane sterss.
- From practical consideration however a thin/moderatel thick plate is in plane stress because the thickness is smaller.

- Orthotropic material with principal axes (x_1, y_1, z_1) coincident with the plate coordinates (x,y,z), we get

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} - \alpha_1 \Delta T \\ \varepsilon_{yy} - \alpha_2 \Delta T \\ \gamma_{xy} \end{bmatrix} \tag{21}$$

where

$$Q_{11} = \frac{E_1}{1 - \nu_{12}\nu_{21}} \qquad Q_{12} = \frac{\nu_{12} E_2}{1 - \nu_{12}\nu_{21}} = \frac{\nu_{21} E_1}{1 - \nu_{12}\nu_{21}}$$
$$Q_{22} = \frac{E_2}{1 - \nu_{12}\nu_{21}} \qquad Q_{66} = G_{12} \tag{22}$$

- The temperature increment is froma reference state.

- Now we can relate the forces, moments per unit length to the strains. Integrating the stresses over the cross section gives the required axial force and moments (With the lever arm z).
- For plates that are laminated with multiple orthotropoic layers, whose material axes are **arbitrarily** oriented with respect to the plate eaxes, the constitutive relations couple the inplane and out of plane displacements even for linear prolbems
- For a single orthortropic layer, the constitutive relations are simplified as :

$$\begin{aligned}\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} &= \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^0 + z\varepsilon_{xx}^1 - \alpha_1 \Delta T \\ \varepsilon_{yy}^0 + z\varepsilon_{yy}^1 - \alpha_2 \Delta T \\ \gamma_{xy}^0 + z\gamma_{xy}^1 \end{Bmatrix} dz \\ &= \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^0 \\ \varepsilon_{yy}^0 \\ \gamma_{xy}^0 \end{Bmatrix} - \begin{Bmatrix} N_{xx}^T \\ N_{yy}^T \\ 0 \end{Bmatrix} \end{aligned} \qquad (7.3.32)$$

$$\begin{aligned}\begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} &= \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^0 + z\varepsilon_{xx}^1 - \alpha_1 \Delta T \\ \varepsilon_{yy}^0 + z\varepsilon_{yy}^1 - \alpha_2 \Delta T \\ \gamma_{xy}^0 + z\gamma_{xy}^1 \end{Bmatrix} z\, dz \\ &= \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^1 \\ \varepsilon_{yy}^1 \\ \gamma_{xy}^1 \end{Bmatrix} - \begin{Bmatrix} M_{xx}^T \\ M_{yy}^T \\ 0 \end{Bmatrix} \end{aligned} \qquad (7.3.33)$$

where A_{ij} are *extensional stiffnesses* and D_{ij} are *bending stiffnesses*, which are defined in terms of the elastic stiffnesses Q_{ij} as

$$(A_{ij}, D_{ij}) = \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{ij}\,(1, z^2)\, dz \quad \text{or} \quad A_{ij} = Q_{ij}\, h, \quad D_{ij} = Q_{ij}\,\frac{h^3}{12} \qquad (7.3.34)$$

and $\mathbf{N}^T$ and $\mathbf{M}^T$ are thermal stress resultants

$$\begin{Bmatrix} N_{xx}^T \\ N_{yy}^T \end{Bmatrix} = \begin{Bmatrix} Q_{11}\alpha_1 + Q_{12}\alpha_2 \\ Q_{12}\alpha_1 + Q_{22}\alpha_2 \end{Bmatrix} \int_{-\frac{h}{2}}^{\frac{h}{2}} \Delta T(x, y, z)\, dz \qquad (7.3.35a)$$

$$\begin{Bmatrix} M_{xx}^T \\ M_{yy}^T \end{Bmatrix} = \begin{Bmatrix} Q_{11}\alpha_1 + Q_{12}\alpha_2 \\ Q_{12}\alpha_1 + Q_{22}\alpha_2 \end{Bmatrix} \int_{-\frac{h}{2}}^{\frac{h}{2}} \Delta T(x, y, z)\, z\, dz \qquad (7.3.35b)$$

where α_1 and α_2 are the thermal coefficients of expansion, and ΔT is the temperature change (above a stress-free temperature), which is a known function of position. For isotropic plates, Eqs. (7.3.35a,b) simplify to $N_{xx}^T = N_{yy}^T = N^T$ and $M_{xx}^T = M_{yy}^T = M^T$, where

$$(N^T, M^T) = \frac{E\alpha}{1-\nu} \int_{-\frac{h}{2}}^{\frac{h}{2}} \Delta T(1, z)\, dz \qquad (7.3.36)$$

$$0 = \int_{\Omega^e} \left(\frac{\partial \delta u}{\partial x} N_{xx} + \frac{\partial \delta u}{\partial y} N_{xy} \right) dx\, dy - \oint_{\Gamma^e} N_{nn} \delta u_n \, ds \tag{7.3.10}$$

$$0 = \int_{\Omega^e} \left(\frac{\partial \delta v}{\partial x} N_{xy} + \frac{\partial \delta v}{\partial y} N_{yy} \right) dx\, dy - \oint_{\Gamma^e} N_{ns} \delta u_s \, ds \tag{7.3.11}$$

$$\begin{aligned} 0 = \int_{\Omega^e} \Bigg[& \frac{\partial \delta w}{\partial x} \left(\frac{\partial w}{\partial x} N_{xx} + \frac{\partial w}{\partial y} N_{xy} \right) + \frac{\partial \delta w}{\partial y} \left(\frac{\partial w}{\partial x} N_{xy} + \frac{\partial w}{\partial y} N_{yy} \right) \\ & - \frac{\partial^2 \delta w}{\partial x^2} M_{xx} - \frac{\partial^2 \delta w}{\partial y^2} M_{yy} - 2 \frac{\partial^2 \delta w}{\partial x \partial y} M_{xy} + k \delta w w - \delta w q \Bigg] dx\, dy \\ & - \oint_{\Gamma^e} \left(-M_{nn} \frac{\partial \delta w}{\partial n} - M_{ns} \frac{\partial \delta w}{\partial s} + Q_n \delta w \right) ds \end{aligned} \tag{7.3.12}$$

$$\begin{aligned} 0 = \int_{\Omega^e} & \left(\frac{\partial \delta u}{\partial x} \left\{ \underbrace{A_{11} \left[\frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \right] + A_{12} \left[\frac{\partial v}{\partial y} + \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2 \right]}_{} \right\} \right. \\ & \left. + \underbrace{A_{66} \frac{\partial \delta u}{\partial y} \left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right]}_{N_{xy}} \right) dx\, dy - \int_{\Omega^e} \frac{\partial \delta u}{\partial x} \underbrace{N_{xx}^T}_{\text{Temp}} dx dy \\ & - \oint_{\Gamma^e} N_{nn} \delta u_n \, ds \end{aligned} \tag{7.4.1}$$

N_{xx}

$$\begin{aligned} 0 = \int_{\Omega^e} & \left(\frac{\partial \delta v}{\partial y} \left\{ A_{12} \left[\frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \right] + A_{22} \left[\frac{\partial v}{\partial y} + \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2 \right] \right\} \right. \\ & \left. + A_{66} \frac{\partial \delta v}{\partial x} \left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right] \right) dx\, dy - \int_{\Omega^e} \frac{\partial \delta v}{\partial y} N_{yy}^T dx dy \\ & - \oint_{\Gamma^e} N_{ns} \delta u_s \, ds \end{aligned} \tag{7.4.2}$$

$$\begin{aligned} 0 = \int_{\Omega^e} \Bigg\{ & \frac{\partial \delta w}{\partial x} \left[\frac{\partial w}{\partial x} \left\{ A_{11} \left[\frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \right] + A_{12} \left[\frac{\partial v}{\partial y} + \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2 \right] \right\} \right. \\ & \left. + A_{66} \frac{\partial w}{\partial y} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right) \right] + \frac{\partial \delta w}{\partial y} \left[\frac{\partial w}{\partial y} \left\{ A_{12} \left[\frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \right] \right. \right. \\ & \left. \left. + A_{22} \left[\frac{\partial v}{\partial y} + \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2 \right] \right\} + A_{66} \frac{\partial w}{\partial x} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right) \right] \end{aligned}$$

- We know that u_n, u_s, w, $\frac{\partial w}{\partial n}$ are used as primary variables (or gerneralised displacements)
- $\hat{N_{nn}}, \hat{N_{ns}}, \hat{V_n}, \hat{M_{nn}}$ as secondary degrees of greedom (generalised forces)
- The finite elements based on the transverse deflection w and its derivative acroos element boundary (C1 conitunity). In completeness, it should be a full quadratic.
- u_n and u_s need only be C0. We shall use $u, v, w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}$ as the generalised displacements. We assume as

$$u(x,y) = N_i^1 u_i \qquad v(x,y) = N_i^1 u_i \qquad w(x,y) = N_i^2 u_i \tag{23}$$

- In case of a rectangular element wecan take two sets of dof at each node : $u,, v, w, w_{,x}, w_{,y}$ and $u,, v, w, w_{,x}, w_{,y}, w_{,xy}$ called non conforming and conforming element.
- In substituting we get

$$\begin{bmatrix} K_{11} & K_{11} & K_{11} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix} \begin{bmatrix} u \\ v \\ \Delta \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} + \begin{bmatrix} F_{1T} \\ F_{2T} \\ F_{3T} \end{bmatrix} \tag{24}$$

(a)

(b)

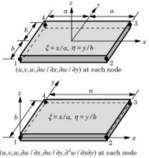


Fig. 7.4.1: (a) The non-conforming and (b) the conforming rectangular elements.

The stiffness matrix coefficients $K_{ij}^{\alpha\beta}$ and force vector components F_i^{α} and $F_i^{\alpha T}$ $(\alpha, \beta = 1, 2, 3)$ are defined as follows:

$$
\begin{aligned}
K_{ij}^{11} &= \int_{\Omega^e} \left(A_{11} \frac{\partial \psi_i^e}{\partial x} \frac{\partial \psi_j^e}{\partial x} + A_{66} \frac{\partial \psi_i^e}{\partial y} \frac{\partial \psi_j^e}{\partial y} \right) dx\, dy \\
K_{ij}^{12} &= \int_{\Omega^e} \left(A_{12} \frac{\partial \psi_i^e}{\partial x} \frac{\partial \psi_j^e}{\partial y} + A_{66} \frac{\partial \psi_i^e}{\partial y} \frac{\partial \psi_j^e}{\partial x} \right) dx\, dy = K_{ji}^{21} \\
K_{ij}^{13} &= \frac{1}{2} \int_{\Omega^e} \left[\frac{\partial \psi_i^e}{\partial x} \left(A_{11} \frac{\partial w}{\partial x} \frac{\partial \varphi_j^e}{\partial x} + A_{12} \frac{\partial w}{\partial y} \frac{\partial \varphi_j^e}{\partial y} \right) \right. \\
&\quad \left. + A_{66} \frac{\partial \psi_i^e}{\partial y} \left(\frac{\partial w}{\partial x} \frac{\partial \varphi_j^e}{\partial y} + \frac{\partial w}{\partial y} \frac{\partial \varphi_j^e}{\partial x} \right) \right] dx\, dy \\
K_{ij}^{22} &= \int_{\Omega^e} \left(A_{66} \frac{\partial \psi_i^e}{\partial x} \frac{\partial \psi_j^e}{\partial x} + A_{22} \frac{\partial \psi_i^e}{\partial y} \frac{\partial \psi_j^e}{\partial y} \right) dx\, dy \\
K_{ij}^{23} &= \frac{1}{2} \int_{\Omega^e} \left[\frac{\partial \psi_i^e}{\partial y} \left(A_{12} \frac{\partial w}{\partial x} \frac{\partial \varphi_j^e}{\partial x} + A_{22} \frac{\partial w}{\partial y} \frac{\partial \varphi_j^e}{\partial y} \right) \right. \\
&\quad \left. + A_{66} \frac{\partial \psi_i^e}{\partial x} \left(\frac{\partial w}{\partial x} \frac{\partial \varphi_j^e}{\partial y} + \frac{\partial w}{\partial y} \frac{\partial \varphi_j^e}{\partial x} \right) \right] dx\, dy \\
K_{ij}^{31} &= \int_{\Omega^e} \left[\frac{\partial \varphi_i^e}{\partial x} \left(A_{11} \frac{\partial w}{\partial x} \frac{\partial \psi_j^e}{\partial x} + A_{66} \frac{\partial w}{\partial y} \frac{\partial \psi_j^e}{\partial y} \right) \right. \\
&\quad \left. + \frac{\partial \varphi_i^e}{\partial y} \left(A_{66} \frac{\partial w}{\partial x} \frac{\partial \psi_j^e}{\partial y} + A_{12} \frac{\partial w}{\partial y} \frac{\partial \psi_j^e}{\partial x} \right) \right] dx\, dy \\
K_{ij}^{32} &= \int_{\Omega^e} \left[\frac{\partial \varphi_i^e}{\partial x} \left(A_{12} \frac{\partial w}{\partial x} \frac{\partial \psi_j^e}{\partial y} + A_{66} \frac{\partial w}{\partial y} \frac{\partial \psi_j^e}{\partial x} \right) \right. \\
&\quad \left. + \frac{\partial \varphi_i^e}{\partial y} \left(A_{66} \frac{\partial w}{\partial x} \frac{\partial \psi_j^e}{\partial x} + A_{22} \frac{\partial w}{\partial y} \frac{\partial \psi_j^e}{\partial y} \right) \right] dx\, dy \\
K_{ij}^{33} &= \int_{\Omega^e} \left[D_{11} \frac{\partial^2 \varphi_i^e}{\partial x^2} \frac{\partial^2 \varphi_j^e}{\partial x^2} + D_{22} \frac{\partial^2 \varphi_i^e}{\partial y^2} \frac{\partial^2 \varphi_j^e}{\partial y^2} \right. \\
&\quad + D_{12} \left(\frac{\partial^2 \varphi_i^e}{\partial x^2} \frac{\partial^2 \varphi_j^e}{\partial y^2} + \frac{\partial^2 \varphi_i^e}{\partial y^2} \frac{\partial^2 \varphi_j^e}{\partial x^2} \right) \\
&\quad \left. + 4D_{66} \frac{\partial^2 \varphi_i^e}{\partial x \partial y} \frac{\partial^2 \varphi_j^e}{\partial x \partial y} + k \varphi_i^e \varphi_j^e \right] dx\, dy \\
&\quad + \frac{1}{2} \int_{\Omega^e} \left\{ \left[A_{11} \left(\frac{\partial w}{\partial x} \right)^2 + A_{66} \left(\frac{\partial w}{\partial y} \right)^2 \right] \frac{\partial \varphi_i^e}{\partial x} \frac{\partial \varphi_j^e}{\partial x} \right. \\
&\quad + \left[A_{66} \left(\frac{\partial w}{\partial x} \right)^2 + A_{22} \left(\frac{\partial w}{\partial y} \right)^2 \right] \frac{\partial \varphi_i^e}{\partial y} \frac{\partial \varphi_j^e}{\partial y} \\
&\quad \left. + (A_{12} + A_{66}) \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \left(\frac{\partial \varphi_i^e}{\partial x} \frac{\partial \varphi_j^e}{\partial y} + \frac{\partial \varphi_i^e}{\partial y} \frac{\partial \varphi_j^e}{\partial x} \right) \right\} dx\, dy \\
&\quad - \int_{\Omega^e} \left(N_{xx}^T \frac{\partial \varphi_i^e}{\partial x} \frac{\partial \varphi_j^e}{\partial x} + N_{yy}^T \frac{\partial \varphi_i^e}{\partial y} \frac{\partial \varphi_j^e}{\partial y} \right) dx\, dy \\
F_i^1 &= \oint_{\Gamma^e} N_{nn}\, \psi_i^e\, ds, \quad F_i^2 = \oint_{\Gamma^e} N_{ns}\, \psi_i^e\, ds \\
F_i^3 &= \int_{\Omega^e} q\, \varphi_i^e\, dx\, dy + \oint_{\Gamma^e} \left(V_n \varphi_i^e - M_{nn} \frac{\partial \varphi_i^e}{\partial n} \right) ds \\
F_i^{1T} &= \int_{\Omega^e} N_{xx}^T \frac{\partial \psi_i^e}{\partial x}\, dx\, dy, \quad F_i^{2T} = \int_{\Omega^e} N_{yy}^T \frac{\partial \psi_i^e}{\partial y}\, dx\, dy \\
F_i^{3T} &= - \int_{\Omega^e} \left(\frac{\partial^2 \varphi_i^e}{\partial x^2} M_{xx}^T + \frac{\partial^2 \varphi_i^e}{\partial y^2} M_{yy}^T \right) dx\, dy
\end{aligned} \tag{7.4.6}
$$

where N_{xx}^T, M_{xx}^T, etc., are the thermal forces and moments in Eqs. (7.3.35a,b). Hats ˆ on the stress resultants N_{nn}, N_{ns}, M_{nn}, and V_n are removed because they are now defined on the element boundary. Note that the thermal resultant term included in K_{ij}^{33} is due to the von Kármán nonlinearity; alternatively, it could have been included in F_i^{3T} as a nonlinear term. The finite element model in Eq. (7.4.5) is called a *displacement finite element model*.

- Check Tangent in Reddy 328

- A non conforming element also has nodal variable of $w_{,x}$, $w_{,y}$
- Using the parametric form we see that

$$u = u_h = u_i N_i^1(\xi, \eta) \qquad v = v_h = v_i N_i^1(\xi, \eta)$$
$$w_h = \Delta N_i^2(\xi, \eta) \text{ Cubic to associate with dof } w, w_{,x}, w_{,y} \tag{25}$$

- The variation of the normal slope $w_{,n}$ is cubic while there are only two values of it available on the edge ()??????????.
- Therefore, cubic polynomials for the normal derivatives of w are not the same on an edge common to two elements, hence non-conforming. I guess corner one is not the same
- Conforming elemen has w approximated by 16 term complete polyunomial with dof w, $w_{,x}$, $w_{,y}$, $w_{,xy}$
- Here the normal slope continuity between elements is satisfied.

- Here we use bilinear interpolation of u, v and hermite of w
- The geometry is also represented using bilinear interpolating functions

$$x = N_i(\xi, \eta) x_i \qquad y = N_i(\xi, \eta) y_i \tag{26}$$

- The derivatives of the inerpolating funcction with respect to global is given by

$$\begin{bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{bmatrix} \tag{27}$$

- For the second order derivatives, we can use again chain rule

$$\begin{aligned} \frac{\partial N_i}{\partial \xi} = \frac{\partial N_i}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial N_i}{\partial y}\frac{\partial y}{\partial \xi} \qquad \frac{\partial N_i}{\partial \eta} = \frac{\partial N_i}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial N_i}{\partial y}\frac{\partial y}{\partial \eta} \\ \frac{\partial^2 N_i}{\partial \xi^2} = \frac{\partial}{\partial \xi}\left(\frac{\partial N_i}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial N_i}{\partial y}\frac{\partial y}{\partial \xi}\right) \qquad \frac{\partial^2 N_i}{\partial \eta^2} = \frac{\partial}{\partial \eta}\left(\frac{\partial N_i}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial N_i}{\partial y}\frac{\partial y}{\partial \eta}\right) \end{aligned} \tag{28}$$

- The stresses $\sigma_{xx}, \sigma_{yy}, \sigma_{xy}$ are computed using stress-strain relations and computed in global coordinates using

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} \tag{7.5.6a}$$

where the material stiffnesses Q_{ij} are defined in Eqs. (7.3.31a,b)

$$Q_{11} = \frac{E_1}{1-\nu_{12}\nu_{21}}, \quad Q_{12} = \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}}, \quad Q_{22} = \frac{E_2}{1-\nu_{12}\nu_{21}}, \quad Q_{66} = G_{12} \tag{7.5.6b}$$

and

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \\ \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \end{Bmatrix} - z \begin{Bmatrix} \frac{\partial^2 w}{\partial x^2} \\ \frac{\partial^2 w}{\partial y^2} \\ \frac{\partial^2 w}{\partial x \partial y} \end{Bmatrix} \tag{7.5.7}$$

Typically the strains and stresses are computed at points (x, y) corresponding to the reduced Gauss points of the element, called *Barlow points*, as they are found to be more accurate there (see Barlow [62, 63] for a discussion). The stresses can be evaluated for any desired value of z, say at the top ($z = +h/2$) and bottom ($z = -h/2$) of the element. For example, the values of σ_{xx} at the top and bottom of the element at a point (x_c, y_c) corresponding to a Gauss point are computed using

$$\begin{aligned} \sigma_{xx}^{\text{top}} = \sigma_{xx}(x_c, y_c, h/2) &= Q_{11}\left\{\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] - \frac{h}{2}\frac{\partial^2 w}{\partial x^2}\right\}_{(x_c,y_c)} \\ &+ Q_{12}\left\{\left[\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right] - \frac{h}{2}\frac{\partial^2 w}{\partial y^2}\right\}_{(x_c,y_c)} \end{aligned} \tag{7.5.8}$$

$$\begin{aligned} \sigma_{xx}^{\text{bottom}} = \sigma_{xx}(x_c, y_c, -h/2) &= Q_{11}\left\{\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] + \frac{h}{2}\frac{\partial^2 w}{\partial x^2}\right\}_{(x_c,y_c)} \\ &+ Q_{12}\left\{\left[\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right] + \frac{h}{2}\frac{\partial^2 w}{\partial y^2}\right\}_{(x_c,y_c)} \end{aligned} \tag{7.5.9}$$

Similar expressions hold for σ_{yy} and σ_{xy}.

- Transverese normal and shear are not neglected. This formulation requires only C0 interpolation of all gerneralized displacements!
- Displacement field : Same assumptions of classical theroy but relaxing the normality condition

$$
\begin{aligned}
u_1(x, y, z) &= u(x, y) + z\phi_x(x, y) \\
u_2(x, y, z) &= v(x, y) + z\phi_y(x, y) \\
u_3(x, y, z) &= w(x, y)
\end{aligned} \tag{29}
$$

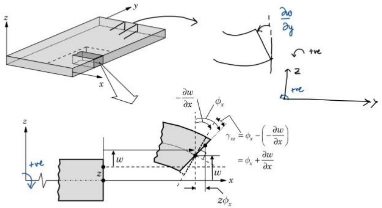


Fig. 7.7.1: Undeformed and deformed geometries of an edge of a plate under the assumptions of the FSDT.

- Where $(u, v, w, \phi_x, \phi_y)$ are unkown functions. ϕ_x, ϕ_y denote the rotations of transverser normal line about y and x axes. These are called generalized displacements

- Notation that ϕ_x, ϕ_y denoe rotation about a transverse normal about y
- We then get

$$\beta_x = -\phi_y \qquad \beta_y = \phi_x$$

$$\phi_x = -\frac{\partial w}{\partial x} \qquad \phi_y = -\frac{\partial w}{\partial y} \quad \text{For thin plates of thickness ratio of order Õ(50)} \tag{30}$$

- However this equality is not achieved in the discrete fem model, resulting in shear locking as in Timoshenko beam, when the same lower order approx are used for transverse deflection w and rotation ϕ

- The von karman strains are:

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{yz} \\ \varepsilon_{xz} \\ \varepsilon_{xy} \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx}^0 \\ \varepsilon_{yy}^0 \\ \varepsilon_{yz}^0 \\ \varepsilon_{xz}^0 \\ \varepsilon_{xy}^0 \end{bmatrix} + z \begin{bmatrix} \varepsilon_{xx}^1 \\ \varepsilon_{yy}^1 \\ 0 \\ 0 \\ \varepsilon_{xy}^1 \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \\ \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \\ \frac{\partial w}{\partial y} + \phi_y \\ \frac{\partial w}{\partial x} + \phi_x \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \end{bmatrix} + z \begin{bmatrix} \frac{\partial \phi_x}{\partial x} \\ \frac{\partial \phi_y}{\partial y} \\ 0 \\ 0 \\ \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \end{bmatrix} \tag{31}$$

- Note that the strains ($\varepsilon_{xx,yy,xy}$) are linear through the plate thickness while the transverse strains (γ_{xz}, γ_{yz}) are constant.

- The weak form for δW_I and δW_E is :

$$\delta W_I^e = \int_{\Omega^e} \left\{ \int_{-\frac{h}{2}}^{\frac{h}{2}} \left[\sigma_{xx} \left(\delta\varepsilon_{xx}^0 + z\delta\varepsilon_{xx}^1 \right) + \sigma_{yy} \left(\delta\varepsilon_{yy}^0 + z\delta\varepsilon_{yy}^1 \right) \right. \right.$$
$$\left. \left. + \sigma_{xy} \left(\delta\gamma_{xy}^0 + z\delta\gamma_{xy}^1 \right) + K_s\sigma_{xz}\delta\gamma_{xz}^0 + K_s\sigma_{yz}\delta\gamma_{yz}^0 \right] dz \right\} dx\, dy \quad (7.7.6)$$

$$\delta W_E^e = -\left\{ \oint_{\Gamma^e} \int_{-\frac{h}{2}}^{\frac{h}{2}} \left[\sigma_{nn} \left(\delta u_n + z\delta\phi_n \right) + \sigma_{ns} \left(\delta u_s + z\delta\phi_s \right) + \sigma_{nz}\delta w \right] dz\, ds \right.$$
$$\left. + \int_{\Omega^e} (q - kw)\, \delta w\ dx\, dy \right\} \quad (7.7.7)$$

where Ω^e denotes the undeformed mid-plane of a typical plate element, h is the total thickness, k is the modulus of the elastic foundation (if any), (σ_{nn}, σ_{ns}, σ_{nz}) are the edge stresses along the (n, s, z) coordinates, and K_s is the shear correction coefficient ($K_s = 5/6$).

Substituting for δW_I^e and δW_E^e from Eqs. (7.7.6) and (7.7.7) into the virtual work statement in Eq. (7.7.5) and integrating through the thickness, we obtain

$$0 = \int_{\Omega^e} \left[N_{xx}\delta\varepsilon_{xx}^0 + M_{xx}\delta\varepsilon_{xx}^1 + N_{yy}\delta\varepsilon_{yy}^0 + M_{yy}\delta\varepsilon_{yy}^1 + N_{xy}\delta\gamma_{xy}^0 \right.$$
$$\left. + M_{xy}\delta\gamma_{xy}^1 + Q_x\delta\gamma_{xz}^0 + Q_y\delta\gamma_{yz}^0 + kw\delta w - q\delta w \right] dx\, dy$$
$$- \oint_{\Gamma^e} \left(N_{nn}\delta u_n + N_{ns}\delta u_s + M_{nn}\delta\phi_n + M_{ns}\delta\phi_s + Q_n\delta w \right) ds \quad (7.7.8)$$

where ϕ_n and ϕ_s are the rotations of a transverse normal about s and $-n$ coordinates, respectively, (N_{xx}, N_{yy}, N_{xy}, M_{xx}, M_{yy}, M_{xy}) are defined in Eq. (7.3.4), and (N_{nn}, N_{ns}, M_{nn}, M_{ns}, Q_n) are defined in Eq. (7.3.7). The *transverse shear forces per unit length* (Q_x, Q_y) are defined by

$$\left\{ \begin{array}{c} Q_x \\ Q_y \end{array} \right\} = K_s \int_{-\frac{h}{2}}^{\frac{h}{2}} \left\{ \begin{array}{c} \sigma_{xz} \\ \sigma_{yz} \end{array} \right\} dz \quad (7.7.9)$$

- The virtual work equation contains five different statements associated with five virtual displacements ($\delta u, \delta v, \delta w, \delta\phi_x, \delta\phi_y$) forming the basis of the fem model
- The governing equations not required for fem, are shown as done previously by removing all the virtual displacments of differentiation. We then get:

$$
\begin{aligned}
0 = \int_{\Omega^e} \Big[& -(N_{xx,x} + N_{xy,y})\,\delta u - (N_{xy,x} + N_{yy,y})\,\delta v \\
& - (M_{xx,x} + M_{xy,y} - Q_x)\,\delta\phi_x - (M_{xy,x} + M_{yy,y} - Q_y)\,\delta\phi_y \\
& - (Q_{x,x} + Q_{y,y} - kw + \mathcal{N} + q)\,\delta w \Big] dx\,dy \\
+ \oint_{\Gamma^e} \Big[& (N_{xx}n_x + N_{xy}n_y)\,\delta u + (N_{xy}n_x + N_{yy}n_y)\,\delta v \\
& + (M_{xx}n_x + M_{xy}n_y)\,\delta\phi_x + (M_{xy}n_x + M_{yy}n_y)\,\delta\phi_y \\
& + \Big(Q_x n_x + Q_y n_y + \mathcal{P}\Big)\delta w \Big] ds \\
- \oint_{\Gamma^e} & (N_{nn}\delta u_n + N_{ns}\delta u_s + M_{nn}\delta\phi_n + M_{ns}\delta\phi_s + Q_n\delta w)\,ds
\end{aligned}
\tag{7.7.10}
$$

where $\mathcal{N}$ and $\mathcal{P}$ are defined by Eq. (7.3.14). The boundary terms can be expressed in terms of the normal and tangential components u_n, u_s, ϕ_n, and ϕ_s using Eqs. (7.3.18a) and

$$
\phi_x = n_x\phi_n - n_y\phi_s\ , \quad \phi_y = n_y\delta\phi_n + n_x\delta\phi_s \tag{7.7.11}
$$

This will yield the natural boundary conditions given in Eqs. (7.3.20a, b), which relate the forces and moments on an arbitrary edge to those on edges parallel to the coordinates (x, y, z).

The Euler–Lagrange equations are

$$
\delta u: \quad \frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 \tag{7.7.12}
$$

$$
\delta v: \quad \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} = 0 \tag{7.7.13}
$$

$$
\delta w: \quad \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} - kw + \mathcal{N}(u, v, w, \phi_x, \phi_y) + q = 0 \tag{7.7.14}
$$

$$
\delta\phi_x: \quad \frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = 0 \tag{7.7.15}
$$

$$
\delta\phi_y: \quad \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y} - Q_y = 0 \tag{7.7.16}
$$

The primary and secondary variables of the theory are

$$
\begin{aligned}
&\text{primary variables:} \quad u_n,\ u_s,\ w,\ \phi_n,\ \phi_s \\
&\text{secondary variables:} \quad N_{nn},\ N_{ns},\ Q_n,\ M_{nn},\ M_{ns}
\end{aligned}
\tag{7.7.17}
$$

- Since the transverse shear strains are constant, the shear stresses will also be constant. But the transverse shear stress is actually parabolic, whihc is taken care using a shape factor on the shear stiffness.

$$\begin{Bmatrix} Q_y \\ Q_x \end{Bmatrix} = K_s \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_{yz} \\ \sigma_{xz} \end{Bmatrix} dz = \begin{bmatrix} A_{44} & 0 \\ 0 & A_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} \tag{7.7.18}$$

where the transverse shear stiffnesses A_{44} and A_{55} are defined by

$$(A_{44}, A_{55}) = K_s \int_{-\frac{h}{2}}^{\frac{h}{2}} (Q_{44}, Q_{55})\, dz, \quad Q_{44} = G_{23}, \quad Q_{55} = G_{13} \tag{7.7.19}$$

In summary, the stress resultants in an orthotropic plate are related to the generalized displacements $(u, v, w, \phi_x, \phi_y)$ by [see Eqs. (7.3.32) and (7.3.33)]

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^0 \\ \varepsilon_{yy}^0 \\ \gamma_{xy}^0 \end{Bmatrix} - \begin{Bmatrix} N_{xx}^T \\ N_{yy}^T \\ 0 \end{Bmatrix} \tag{7.7.20}$$

$$\begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^1 \\ \varepsilon_{yy}^1 \\ \gamma_{xy}^1 \end{Bmatrix} - \begin{Bmatrix} M_{xx}^T \\ M_{yy}^T \\ 0 \end{Bmatrix} \tag{7.7.21}$$

$$\begin{Bmatrix} Q_y \\ Q_x \end{Bmatrix} = \begin{bmatrix} A_{44} & 0 \\ 0 & A_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{yz}^0 \\ \gamma_{xz}^0 \end{Bmatrix} \tag{7.7.22}$$

where $(i, j = 1, 2, 6)$

$$A_{ij} = Q_{ij}\, h, \quad D_{ij} = Q_{ij}\frac{h^3}{12}; \quad A_{44} = K_s G_{23} h, \quad A_{55} = K_s G_{13} h \tag{7.7.23}$$

$$Q_{11} = \frac{E_1}{1 - \nu_{12}\nu_{21}}, \quad Q_{22} = Q_{11}\frac{E_2}{E_1}, \quad Q_{12} = \nu_{12} Q_{22}, \quad Q_{66} = G_{12} \tag{7.7.24}$$

- As you can see that once you have the shear deformation, we change how G (shear modulus is)

- FEM model
- The rotationsa (ϕ) are independant of w. So no derivatives appear and all the generalized displafcements can be interpolated using Largrange interpolating functions.
- Tangent matrix : See Reddy

- The lienarised interpolation of the generalized displcamnets is used, making the element very stiff in the thin plate limit. This is called shear locking. A common technique is to use selective integration. Reduced integration is used to evalueate all the transeverse shear stiffnesses.
- **Transient:**Check Reddy