1 Directional Derivative

2 Linearisation

Function

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Updated : Dec 7 2020

Directional Derivative

- The directional derivative basically states how a function changes along a certain direction
- We can use it to linearise a nonlinear function , which gives us our Newton Rhapson method
- Finding the changes of a functional [1] with respect to its corresponding functions. This is akin to the variational or virtual work theorems
- The directional derivative gives the linear change!!! So at a point in the domain, it gives the linear change (Gradients) in a certain direction

[1] Function of functions

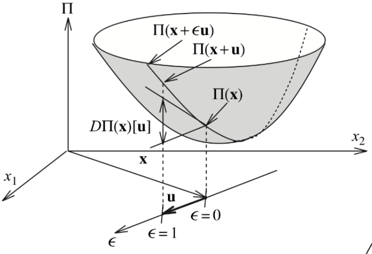
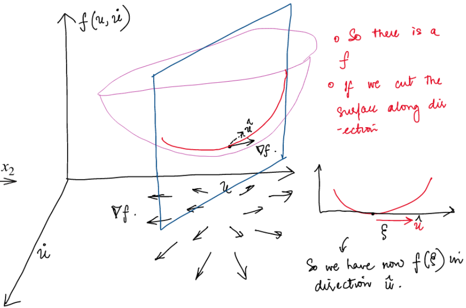


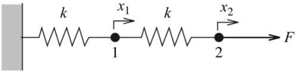
FIGURE 1.11 Directional derivative.



- So we have a funcitonal which depends on differerent functions or $\mathbf{x}$
- We cut the function with a plane (Blue) which gives us a curve how the function changes along that direction
- Finding that linear change along the direction u [2] gives the directional derivative. See that the curve is now dependant on ϵ
- It is denoted as ∇_u or $Df(\mathbf{x})[u]$

[2] Remember that u is a unit vector

D(x)



- Potential energy of the structure is

$$f(\mathbf{x}) = \frac{1}{2}kx_1^2 + \frac{1}{2}k(x_2 - x_1)^2 - Fx_2$$

$$f(\mathbf{x}+\mathbf{u}) = \frac{1}{2}k(x_1 + u_1)^2 + \frac{1}{2}k(x_2 + u_2 - x_1 - u_1)^2 - F(x_2 + u_2)$$

$$Df(\mathbf{x})[\mathbf{u}] \approx f(\mathbf{x}+\mathbf{u}) - f(\mathbf{x})$$

- Its approx $\approx$ as we want only the linear change, this is also what we mean when we write δf in variational calculus

- How do we get the linear function? Taylor series!

$$f(\mathbf{x}+\epsilon\mathbf{u}) = \frac{1}{2}k(x_1+\epsilon u_1)^2 + \frac{1}{2}k(x_2+\epsilon u_2 - x_1 - \epsilon u_1)^2 - F(x_2+\epsilon u_2)$$

$$Df(\mathbf{x})[\mathbf{u}] \approx f(\mathbf{x}+\mathbf{u}) - f(\mathbf{x}) \text{ (Approx as only the linear change)}$$

- This is the function on the plane that cuts the surface given in terms of ϵ
- Linearise it about the point we get (And ignoring higher order terms)

$$f(\mathbf{x}+\epsilon\mathbf{u}) = f(\mathbf{x}) + \left(\frac{d}{d\epsilon}|_{\epsilon=0} f(\mathbf{x}+\epsilon\mathbf{u})\right)\epsilon + O(\epsilon^2)$$

- So our potential energy becomes , Take $\epsilon = 1$ for unit direction

$$\begin{aligned} Df(\mathbf{x})[\mathbf{u}] &= \left(\frac{d}{d\epsilon}|_{\epsilon=0} f(\mathbf{x}+\epsilon\mathbf{u})\right) \\ &= \frac{d}{d\epsilon}|_{\epsilon=0}\left(\frac{1}{2}k(x_1+\epsilon u_1)^2 + \frac{1}{2}k(x_2+\epsilon u_2 - x_1 - \epsilon u_1)^2 - F(x_2+\epsilon u_2)\right) \\ &= k_1 x_1 u_1 + k(x_2 - x_1)(u_2 - u_1) - F u_2 \\ &= \mathbf{u}^{\mathbf{T}}(\mathbf{Kx} - \mathbf{F}) \end{aligned}$$

- So we get the form $\mathbf{u}^{\mathbf{T}}(\mathbf{Kx} - \mathbf{F})$ for some direction $\vec{u}$
- Equilbrium is satisfied when the potential is minimum for any $\vec{u}$ So $Df(x)[u] = 0$
- This is exactly like the variational principle where we get something like $Df(x)[\delta u] = 0$
- Where the Equilibrium has to be zero $(\mathbf{Kx} - \mathbf{F})$ and therefore any work done on it by any displacment is zero ("Virtual displcament theory")
- At equilibrium the work done by the external and internal loads is equal to zero
- The functional may be still nonlinear with respect to $\mathbf{x}$ but we are linearising the function with respect to the change or direction $\mathbf{u}$

We can find the directional derivative of different things like the determinant of a matrix etc. Check Bonet Page 16

Linearisation

- In the directional derivative section, we have seen how we can linearise the potential energy or any function, functional
- We get our equilibrium equation $\mathbf{Kx} = \mathbf{F}$
- This equation can be nonlinear with respect to x. So we have a residual function $\mathbf{R}(\mathbf{x}) = \mathbf{Kx} - \mathbf{F}$ which again can be thought as something that we can find the linear change

- Suppose $R = \begin{bmatrix} R_1(x_1\ x_2) \\ R_2(x_1\ x_2) \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$\mathbf{R}(\mathbf{x_{i+1}}) \approx \mathbf{R}(\mathbf{x_i}) + \mathbf{DR}(\mathbf{x_k})[\mathbf{u}]$$

$$D\mathbf{R}(x_k)[u] = \frac{d}{d\epsilon} R(\mathbf{x_k} + \epsilon \mathbf{u})$$

$$= \frac{d}{d\epsilon} \begin{bmatrix} R_1(x_1 + \epsilon u_1\ x_2 + \epsilon u_2) \\ R_2(x_1 + \epsilon u_1\ x_2 + \epsilon u_2) \end{bmatrix}$$

$$R(x_i) = \mathbf{K_T u} \qquad \textit{Taking} \quad \mathbf{R}(\mathbf{x_{i+1}}) = \mathbf{0}$$

- Where K_T is the tangent stiffness, and then we can find **u** until R is zero [3]
- $K_T = \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix}$ and $u = K_T^{-1} F$

[3] $R_1(x + \epsilon u) = 2k(x_1 + \epsilon u_1) - k(x_2 + \epsilon u_2)$ and $R_2(x + u) = -k(x_1 + \epsilon u_1) + k(x_2 + \epsilon u_2) - F$