1 Directional Derivative : March 2, 2021

2 Potential energy

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Directional derivative

- The directional derivative basically states how a function changes along a certain direction
- We can use it to linearise a nonlinear function , which gives us our Newton Rhapson method
- Finding the changes of a functional [1] with respect to its corresponding functions. This is akin to the variational or virtual work theorems
- The directional derivative gives the linear change!!! So at a point in the domain, it gives the linear change (Gradients) in a certain direction

[1] Function of functions

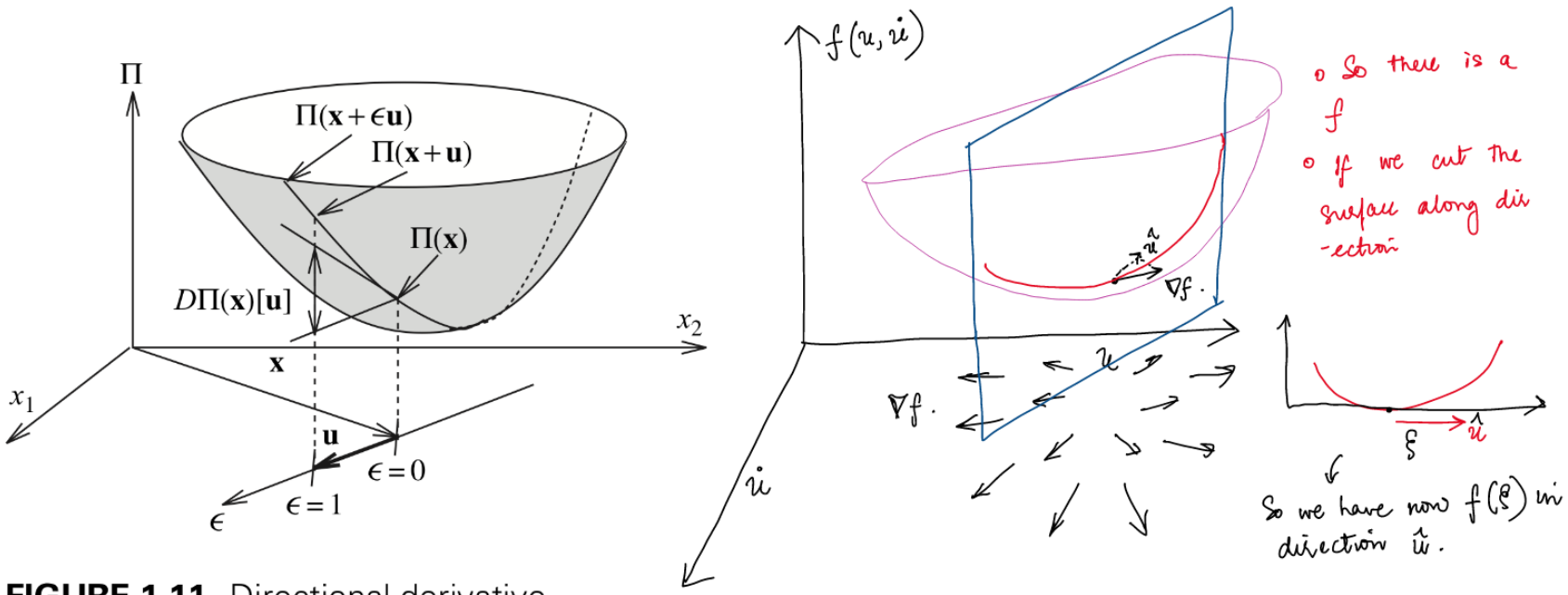
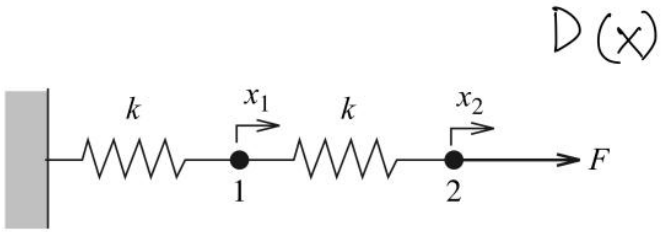


FIGURE 1.11 Directional derivative.

- So we have a funcitonal which depends on differerent functions or **x**
- We cut the function with a plane (Blue) which gives us a curve how the function changes along that direction
- Finding that linear change along the direction u [2] gives the directional derivative. See that the curve is now dependant on ϵ
- It is denoted as ∇_u or $Df(\mathbf{x})[u]$

[2] Remember that u is a unit vector

Example #1



- Potential energy of the structure is

$$f(\mathbf{x}) = \frac{1}{2}kx_1^2 + \frac{1}{2}k(x_2 - x_1)^2 - Fx_2$$

$$f(\mathbf{x}+\mathbf{u}) = \frac{1}{2}k(x_1 + u_1)^2 + \frac{1}{2}k(x_2 + u_2 - x_1 - u_1)^2 - F(x_2 + u_2)$$

$$Df(\mathbf{x})[\mathbf{u}] \approx f(\mathbf{x}+\mathbf{u}) - f(\mathbf{x})$$

- Its approx $\approx$ as we want only the linear change, this is also what we mean when we write δf in variational calculus

- How do we get the linear function? Taylor series!

$$f(\mathbf{x}+\epsilon\mathbf{u}) = \frac{1}{2}k(x_1+\epsilon u_1)^2 + \frac{1}{2}k(x_2+\epsilon u_2 - x_1 - \epsilon u_1)^2 - F(x_2+\epsilon u_2)$$

$$Df(\mathbf{x})[\mathbf{u}] \approx f(\mathbf{x}+\mathbf{u}) - f(\mathbf{x}) \text{ (Approx as only the linear change)}$$

- This is the function on the plane that cuts the surface given in terms of ϵ
- Linearise it about the point we get (And ignoring higher order terms)

$$f(\mathbf{x}+\epsilon\mathbf{u}) = f(\mathbf{x}) + \left(\frac{d}{d\epsilon}|_{\epsilon=0} f(\mathbf{x}+\epsilon\mathbf{u})\right)\epsilon + O(\epsilon^2)$$

- So our potential energy becomes , Take $\epsilon = 1$ for unit direction

$$Df(\mathbf{x})[\mathbf{u}] = \left(\frac{d}{d\epsilon}|_{\epsilon=0} f(\mathbf{x}+\epsilon\mathbf{u})\right)$$

$$= \frac{d}{d\epsilon}|_{\epsilon=0}\left(\frac{1}{2}k(x_1+\epsilon u_1)^2 + \frac{1}{2}k(x_2+\epsilon u_2 - x_1 - \epsilon u_1)^2 - F(x_2+\epsilon u_2)\right)$$

$$= k_1 x_1 u_1 + k(x_2 - x_1)(u_2 - u_1) - F u_2$$

$$= \mathbf{u}^{\mathbf{T}}(\mathbf{K}\mathbf{x} - \mathbf{F})$$

- So we get the form $\mathbf{u}^{\mathbf{T}}(\mathbf{Kx} - \mathbf{F})$ for some direction $\vec{u}$
- Equilbrium is satisfied when the potential is minimum for any $\vec{u}$ So $Df(x)[u] = 0$
- This is exactly like the variational principle where we get something like $Df(x)[\delta u] = 0$
- Where the Equilibrium has to be zero $(\mathbf{Kx} - \mathbf{F})$ and therefore any work done on it by any displacment is zero ("Virtual displcament theory")
- At equilibrium the work done by the external and internal loads is equal to zero
- The functional may be still nonlinear with respect to **x** but we are linearising the function with respect to the change or direction **u**

Potential energy

- I have always had a confusion whether the potential energy is defined with respect to the position or with respect to displacements
- The answer is more nuanced. It can be with both but the times when it is used has to be understood properly in terms of what it actually represents.
- The Residual is the derivative of the potential energy with respect to its degrees of freedom

$$\Pi() = \text{Internal - External potential} \tag{1}$$

Now suppose **x** is the position and **u** are the nodal dispalcements. Can write

$$\Pi(\mathbf{x}) = C(x_1 - x_2) - Px_1 - Px_2 \tag{2}$$

where we say that C is a material parameter, and P are external loads. Now there is a problem with this equation because the internal energy is defined with respect to the relative displacements/deformation/displacement gradients of the continuum.

A more correct way is to write

$$\Pi(\mathbf{u}) = C(u_1 - u_2) - Pu_1 - Pu_2 \tag{3}$$

So now the difference in the displacements gives the deformation that results in the strain energy.

However what about the external work. The external work can be thought of as the potential of the load due it's movement in u. It can also be sometimes thought of as the potential due to its' position, in the case of the well known potential of a ball = mgh, where h is it's position. But here the difference is P is dependant on h.

- One way in which that can be made clear is if we represent the potential with respect to the position, but we include the initial undeformed position to give the displacement. It would look something like this:

$$\Pi(\mathbf{x}) = C((x_1 - x_{1ref}) - (x_2 - x_{2ref})) - P(x_1 - x_{1ref}) - P(x_2 - x_{2ref}) \qquad (4)$$

  where the ref are the initial coordinates. So we can see that in the external work the coefficients Px_{1ref} are constants and so are not important when you only want to minimize the potential energy. So Pu and Px are okay!

We can find the directional derivative of different things like the determinant of a matrix etc. Check Bonet Page 16