Surfaces : February 18, 2021

- Every surface S can be written as a function of two points ξ_1 and ξ_2, where we get

$$x_1 = f_1(\xi_1, \xi_2) \qquad y = f_2(\xi_1, \xi_2) \qquad z = f_3(\xi_1, \xi_2) \tag{1}$$

  where all the functionsa re single valued continuous functions of ξ_1, ξ_2 (Which are called curvilinear coordinates of the surface). By finxing one and changing the other, we get a family of curves called parameteric curves on the surface.

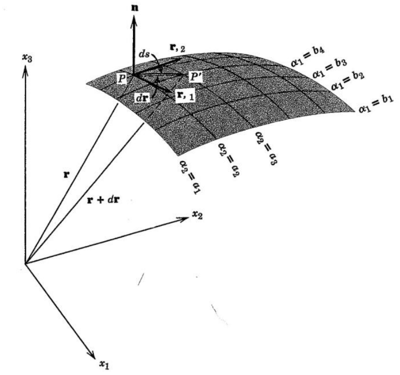

Figure: PATH NOT PROPER

- Now the position of a point on the curve is given as $\mathbf{r}(\xi_1, \xi_2) = \mathbf{f_1}(\xi_1, \xi_2)\mathbf{e_1} + \mathbf{f_2}(\xi_1, \xi_2)\mathbf{e_2} + \mathbf{f_3}(\xi_1, \xi_2)\mathbf{e_3}$
- Now a small differential change d**r** in the vector **r** as we move from a point P to avery close point P' on the surface S can be found as

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial \xi_1} d\xi_1 + \frac{\partial \mathbf{r}}{\partial \xi_2} d\xi_2 \tag{2}$$

where we can also write $\mathbf{r}_{,i} = \frac{\partial \mathbf{r}}{\partial \xi_i}$ with i =1,2
for partial derivatives of vectors. The square of the magnitude of the differential change vector $d\mathbf{r}$ is found by taking the scalar product of d**r** with itself

$$\begin{aligned}(ds)^2 = d\mathbf{r} \cdot d\mathbf{r} = \mathbf{r}_{,1} \cdot \mathbf{r}_{,1}(d\xi_1)^2 + \mathbf{r}_{,1} \cdot \mathbf{r}_{,2}(d\xi_1)(d\xi_2) + \mathbf{r}_{,2} \cdot \mathbf{r}_{,2}(d\xi_2)^2 \\ = E(d\xi_1)^2 + F(d\xi_1)(d\xi_2) + G(d\xi_2)^2\end{aligned} \tag{3}$$

which is the fundamental form of the surface by knowing the fundamental magnitudes E,F,G.

- We note that $\mathbf{r}_{,i}$ are the tangents to the curves of constant ξ_1, ξ_2. F will be zero if the paremetric curves form an orthogonal net. We can write it also as

$$(ds)^2 = A_1^2(d\xi_1)^2 + A_2^2(d\xi_2)^2 \tag{4}$$

where $A_1 = \sqrt{E}, \quad A_2 = \sqrt{G}$

- At every point P, there exists a unit normal vector $\mathbf{n}(\xi_1, \xi_2)$ which is perpendicular to $\mathbf{r}_{,1}, \mathbf{r}_{,2}$ and hence to the tangent plane at P. We get n by the familiar expression

$$\mathbf{n}(\xi_1, \xi_2) = \mathbf{r}_{,1} \times \mathbf{r}_{,2} / |\mathbf{r}_{,1} \times \mathbf{r}_{,2}| \tag{5}$$

  where we have

$$\begin{aligned} |\mathbf{r}_{,1} \times \mathbf{r}_{,2}| &= |\mathbf{r}_{,1}||\mathbf{r}_{,2}| sin\theta \\ \mathbf{r}_{,1} \cdot \mathbf{r}_{,2} &= |\mathbf{r}_{,1}||\mathbf{r}_{,2}| cos\theta \end{aligned} \tag{6}$$

  and we have expressions by manipulation :
  $cos\theta = F/\sqrt{EG}, \quad sin\theta = \sqrt{(EG - F^2)/EG} = H/\sqrt{G}$. We can also have $\mathbf{n}(\xi_1, \xi_2) = \mathbf{r}_{,1} \times \mathbf{r}_{,2}/H$ provided H does not vanish

- The principal normal N does not need to be normal to the surface ($\mathbf{N}.\mathbf{n} \neq 1$)

We have described the curvature vector $\mathbf{k}$ of a space curve. We shall consider a curve on a surface and use the properties of the curvature vector to derive an important features of surfaces of the second fundamental form

- Recall that the curvature vector $\mathbf{k}$ is given by $\frac{d\mathbf{T}}{ds}$ where $\mathbf{T}$ is the unit vector tangent to the curve.
- The curvature vector $\mathbf{k}$ of the curve into its normal and tangenetial components to the surface. Thus

$$\mathbf{k} = \frac{d\mathbf{T}}{ds} = \mathbf{k_n} + \mathbf{k_t} \tag{7}$$

  k_n and k_t are the normal and tangential curvature vector.
- Since $\mathbf{k_n}$ is in the direction of the normal to the surface, it is proportional to $\mathbf{n}$ and can be expressed in terms as

$$\mathbf{k_n} = -K_n\mathbf{n} \tag{8}$$

  The minus sign takes into account the fact that the senese of the curvature vector $\mathbf{k}$ is opposite to that of the normal vector $\mathbf{n}$

- Since **n** is perpendicular to **T**, differentiation with respect to s along the surface gives

$$\frac{d\mathbf{n}}{ds} \cdot \mathbf{T} = -\mathbf{n} \cdot \frac{d\mathbf{T}}{s} \tag{9}$$

- If we form the scalar of $\mathbf{k_n} = - K_n \mathbf{n}$, where K_n , we get

$$-(\mathbf{k_n} \cdot n) = K_n \tag{10}$$

- And finally we will get

$$K_n = \frac{d\mathbf{r} \cdot d\mathbf{n}}{d\mathbf{r} \cdot d\mathbf{r}} \tag{11}$$

where we use
$(ds)^2 = d\mathbf{r} \cdot d\mathbf{r}$, now if we keep
$d\mathbf{n} = \mathbf{n},_\mathbf{1} d\xi_1 + \mathbf{n},_\mathbf{2} d\xi_2$
$d\mathbf{r} = \mathbf{r},_\mathbf{1} d\xi_1 + \mathbf{r},_\mathbf{2} d\xi_2$

$$K_n = \frac{II}{I} = \frac{L(d\xi_1)^2 + 2M(d\xi_1 d\xi_2) + N(d\xi_2)^2}{E(d\xi_1)^2 + 2F(d\xi_1 d\xi_2) + G(d\xi_2)^2} \tag{12}$$

where $L = \mathbf{r},_\mathbf{1} \cdot \mathbf{n},_\mathbf{1}$ $\quad 2M = \mathbf{r},_\mathbf{1} \cdot \mathbf{n},_\mathbf{2} + \mathbf{r},_\mathbf{2} \cdot \mathbf{n},_\mathbf{1}$ $\quad L = \mathbf{r},_\mathbf{2} \cdot \mathbf{n},_\mathbf{2}$

- Now since we have the principal curvatures, we want to find the direction ($\alpha = d\xi_1/d\xi_2$) for which the normal curvature K_n has a maximum or minimum force.
- We then get the curvature as

$$K(\alpha) = \frac{L + 2M\lambda + N\lambda^2}{E + 2F\lambda + G\lambda^2} \tag{13}$$

- The curvature attains an extremum in some direction λ if $\frac{dK}{d\lambda} = 0$
- Solving the quadratic equation will result into two roos corresonding to the two directions $(d\xi_1/d\xi_2)_1$ and $(d\xi_1/d\xi_2)_2$ of

$$(MG - NF)\lambda^2 + (LG - NE)\lambda + (LF - ME) = 0 \tag{14}$$

One solution is the maximum while the other is the minimum curvature

- Now $K_1 = 1/R_1 \quad K_2 = 1/R_2$ are called principal curces.
- These two are actually orthogonal and it can be proven (Check Harry Kraus)
- Suppose the lines of curvature are taken as the parametric lines. Therefore $d\xi_1/d\xi_2 = 0$ and $d\xi_2/d\xi_1 = 0$ should statisfy the eigen value problem
- It will come out that F = M = 0 and we can get

$$K_1 = 1/R_1 = L/E \qquad K_2 = 1/R_2 = N/G \tag{15}$$

- Consider a triplet of orthogonal unit vectors $(\mathbf{t_1}, \mathbf{t_2}, \mathbf{n})$ so that they are thangent to ξ_1, ξ_2 and normal to the surface respectively.
- As we move on the surfcace, we end up changing the orientation of these unit vector, bu they remain constant at unity and orthogonal
- Let us firstly define these vectors

$$\begin{aligned} \mathbf{t_1} &= \frac{\mathbf{r},_\mathbf{1}}{|\mathbf{r},_\mathbf{1}|} \\ \mathbf{t_2} &= \frac{\mathbf{r},_\mathbf{2}}{|\mathbf{r},_\mathbf{2}|} \\ \mathbf{n} &= \mathbf{t_1} \times \mathbf{t_2} \end{aligned} \tag{16}$$

- Interesting thing is that since $\mathbf{n}_{,1}$ and $\mathbf{n}_{,2}$ are prependicular to $\mathbf{n}$, they lie in the plane formed by $\mathbf{t_1}$ and $\mathbf{t_2}$. Therefore we can decompose like $\mathbf{n}_{,1} = a\mathbf{t_1} + b\mathbf{t_2}$
- To find a, b we dot product with $\mathbf{t_1}, \mathbf{t_2}$ to get two equations
- Since the system is orthogoanl $\mathrm{M} = \mathbf{t_1} \cdot \mathbf{t_2} = 0$

$$a = \frac{L}{A_1} \qquad b = 0 \tag{17}$$

- And interstingly we get that it is in the direction of the tangents itself given as

$$\begin{aligned} \mathbf{n}_{,1} &= \frac{A_1}{R_1}\mathbf{t_1} \\ \mathbf{n}_{,2} &= \frac{A_2}{R_2}\mathbf{t_1} \end{aligned} \tag{18}$$

So we have found the derivatives of the normal along the parametric lines.

- To find the derivatives of $\mathbf{t_1}, \mathbf{t_2}$ along the parmetric lines we proceed as we did for the case of the derivatives of $\mathbf{n}$
- For continuous with continuous second derivatives we get $\mathbf{r}_{,\mathbf{12}} = \mathbf{r}_{,\mathbf{21}}$ allowing us to write

$$(A_1 t_1)_{,2} = (A_2 t_2)_{,1}$$

$$\mathbf{t_2}_{,\mathbf{1}} = \frac{1}{A_2}\left[A_1 \mathbf{t_1}_{,\mathbf{2}} + A_{1,2}\, \mathbf{t_1} - A_{2,1}\, \mathbf{t_2}\right] \tag{19}$$

- As state we want to find $\mathbf{t_1}_{,\mathbf{1}} \quad \mathbf{t_1}_{,\mathbf{2}}$
- We observe that this derivative will be perpendicular to $\mathbf{t_1}$ and will lie in the plane formed by $\mathbf{t_2}$ and $\mathbf{n}$. We can therefore express it as

$$\mathbf{t_1}_{,\mathbf{1}} = c\mathbf{n} + d\mathbf{t_2} \tag{20}$$

  where we can find c and d by finding the dot product with $\mathbf{n}$ and $\mathbf{t_2}$
- We then get

$$\mathbf{t_1}_{,\mathbf{1}} = -\frac{A_1}{R_1}\mathbf{n} - \frac{1}{A_2}A_{1,2}\, \mathbf{t_2} \tag{21}$$

  Other directions, you can find from Harry Kraus

- The first condition is

$$\frac{1}{R_1}A_{1,2} = \left(\frac{A_1}{R_1}\right)_{,2} \qquad , \frac{1}{R_2}A_{2,2} = \left(\frac{A_2}{R_2}\right)_{,1} \tag{22}$$

- The other is

$$\left(\frac{1}{A_1}A_{2,1}\right)_{,1} + \left(\frac{1}{A_2}A_{1,2}\right)_{,2} = -\frac{A_1 A_2}{R_1 R_2} \tag{23}$$

- If E,G,L and N are given as funcions of the real curvilinear coordinates ξ_1, ξ_2 and are sufficiently differentialbe and satisfy Gauss codazzi equations wile E>0 and G>0. Then there exist a real surface with 1 and 11 form as

$$I = E(d\xi_1)^2 + G(d\xi_2)^2 \qquad II = L(d\xi_1)^2 + N(d\xi_2)^2 \tag{24}$$

- This is for the simplified cases where the lines of principle curvature are the parametric lines (F=M=0). Now more general Gauss Codazzi equations also exist.

- Suppose that there is a srface which is given as

$$\mathbf{r}(x_3, \theta) = R_o(x_3)cos\theta\mathbf{e_1} + R_o(x_3)sin\theta\mathbf{e_2} + x_3\mathbf{e_3} \tag{25}$$

- Now in finding g we get, keeping x_3, θ as ξ_1 and θ as ξ_2, we get:

$$\begin{aligned} \frac{d\mathbf{r}}{d\xi_1} &= \mathbf{r}_{,1} = R_o'(x_3)cos\theta\mathbf{e_1} + R_o'(x_3)sin\theta\mathbf{e_2} + \mathbf{e_3} \\ \frac{d\mathbf{r}}{d\xi_2} &= \mathbf{r}_{,2} = -R_o(x_3)sin\theta\mathbf{e_1} + R_o(x_3)cos\theta\mathbf{e_2} + 0\mathbf{e_3} \end{aligned} \tag{26}$$

- E = $\mathbf{r}_{,1} \cdot \mathbf{r}_{,1} = R_o^{2'} cos^2\theta + R_o^{2'} sin^2\theta + 1 = 1 + R_o^{2'}$
- F = $\mathbf{r}_{,1} \cdot \mathbf{r}_{,2} = -R_o^2 cos\theta sin\theta + R_o^2 cos\theta sin\theta = 0$
- G = $\mathbf{r}_{,2} \cdot \mathbf{r}_{,2} = R_o^2 sin^2\theta + R_o^2 cos^2\theta = R_o^2$
- H = $\sqrt{EG - F^2} = R_o\sqrt{1 + R_o^{2'}}$
- $A_1 = \sqrt{E} \qquad A_2 = \sqrt{G}$
- First form : $(ds)^2 = \left(1 + R_o^{2'}\right) dx_3^2 + R_o^2 d\theta^2$
- Normal to surface $\mathbf{n} = \frac{\mathbf{r}_{,1} \times \mathbf{r}_{,2}}{H} = \frac{-R_o}{H}\left(cos\theta\mathbf{e_1} + sin\theta\mathbf{e_1} - R_o'\mathbf{e_3}\right)$
  It acts as assumed from conncave to convex side

- $L = -\mathbf{r}_{,11} \cdot \mathbf{n} = \frac{-R_o}{H} \begin{bmatrix} R_o'' cos\theta & R_o'' sin\theta & 0 \end{bmatrix} \cdot \begin{bmatrix} cos\theta \\ sin\theta \\ -R_o' \end{bmatrix} = -R_o R_o H''/H$
- $\mathrm{M} = -\mathbf{r}_{,12} = 0$
- $\mathrm{N} = -\mathbf{r}_{,22} = R_o^2/H$
- Principal curvatures $R_1 = E/L \quad R_2 = G/N$
- We can check the gauss codazzi to test the surface
- An alternate description is if we use ϕ, θ where the first is the angle between the axis revolution of the surface and a normal to the surface of the point. (See Kraus)