1. Directional Derivative

2. Linearisation

3. Object

4. 1st order tensors: Vectors

5. 2nd order tensors : (Many types of objects!)

Function

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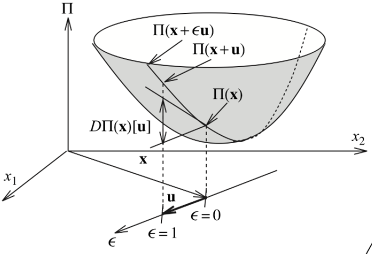
Directional Derivative

- The directional derivative basically states how a function changes along a certain direction
- We can use it to linearise a nonlinear function , which gives us our Newton Rhapson method
- Finding the changes of a functional [1] with respect to its corresponding functions. This is akin to the variational or virtual work theorems
- The directional derivative gives the linear change!!! So at a point in the domain, it gives the linear change (Gradients) in a certain direction

[1] Function of functions



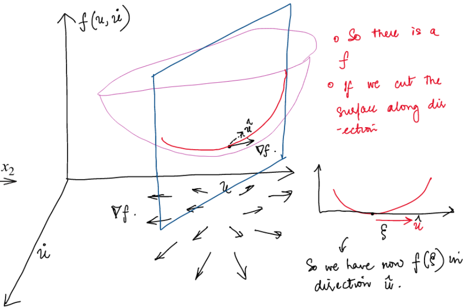
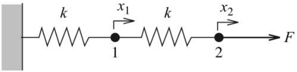


FIGURE 1.11 Directional derivative.

- So we have a funcitonal which depends on differerent functions or **x**
- We cut the function with a plane (Blue) which gives us a curve how the function changes along that direction
- Finding that linear change along the direction u [2] gives the directional derivative. See that the curve is now dependant on ϵ
- It is denoted as ∇_u or $Df(\mathbf{x})[u]$

[2] Remember that u is a unit vector

D(x)



- Potential energy of the structure is

$$f(\mathbf{x}) = \frac{1}{2}kx_1^2 + \frac{1}{2}k(x_2 - x_1)^2 - Fx_2$$
$$f(\mathbf{x}+\mathbf{u}) = \frac{1}{2}k(x_1 + u_1)^2 + \frac{1}{2}k(x_2 + u_2 - x_1 - u_1)^2 - F(x_2 + u_2)$$
$$Df(\mathbf{x})[\mathbf{u}] \approx f(\mathbf{x}+\mathbf{u}) - f(\mathbf{x})$$

- Its approx $\approx$ as we want only the linear change, this is also what we mean when we write δf in variational calculus

- How do we get the linear function? Taylor series!

$$f(\mathbf{x}+\epsilon\mathbf{u}) = \frac{1}{2}k(x_1+\epsilon u_1)^2 + \frac{1}{2}k(x_2+\epsilon u_2 - x_1 - \epsilon u_1)^2 - F(x_2+\epsilon u_2)$$

$$Df(\mathbf{x})[\mathbf{u}] \approx f(\mathbf{x}+\mathbf{u}) - f(\mathbf{x}) \text{ (Approx as only the linear change)}$$

- This is the function on the plane that cuts the surface given in terms of ϵ
- Linearise it about the point we get (And ignoring higher order terms)

$$f(\mathbf{x}+\epsilon\mathbf{u}) = f(\mathbf{x}) + \left(\frac{d}{d\epsilon}|_{\epsilon=0} f(\mathbf{x}+\epsilon\mathbf{u})\right)\epsilon + O(\epsilon^2)$$

- So our potential energy becomes , Take $\epsilon = 1$ for unit direction

$$\begin{aligned} Df(\mathbf{x})[\mathbf{u}] &= \left(\frac{d}{d\epsilon}|_{\epsilon=0} f(\mathbf{x}+\epsilon\mathbf{u})\right) \\ &= \frac{d}{d\epsilon}|_{\epsilon=0}\left(\frac{1}{2}k(x_1+\epsilon u_1)^2 + \frac{1}{2}k(x_2+\epsilon u_2 - x_1 - \epsilon u_1)^2 - F(x_2+\epsilon u_2)\right) \\ &= k_1 x_1 u_1 + k(x_2 - x_1)(u_2 - u_1) - F u_2 \\ &= \mathbf{u}^{\mathbf{T}}(\mathbf{Kx} - \mathbf{F}) \end{aligned}$$

- So we get the form $\mathbf{u}^{\mathbf{T}}(\mathbf{K}\mathbf{x} - \mathbf{F})$ for some direction $\vec{u}$
- Equilbrium is satisfied when the potential is minimum for any $\vec{u}$ So $Df(x)[u] = 0$
- This is exactly like the variational principle where we get something like $Df(x)[\delta u] = 0$
- Where the Equilibrium has to be zero $(\mathbf{K}\mathbf{x} - \mathbf{F})$ and therefore any work done on it by any displacment is zero ("Virtual displcament theory")
- At equilibrium the work done by the external and internal loads is equal to zero
- The functional may be still nonlinear with respect to $\mathbf{x}$ but we are linearising the function with respect to the change or direction $\mathbf{u}$

We can find the directional derivative of different things like the determinant of a matrix etc. Check Bonet Page 16

Linearisation

- In the directional derivative section, we have seen how we can linearise the potential energy or any function, functional
- We get our equilibrium equation $\mathbf{Kx} = \mathbf{F}$
- This equation can be nonlinear with respect to x. So we have a residual function $\mathbf{R}(\mathbf{x}) = \mathbf{Kx} - \mathbf{F}$ which again can be thought as something that we can find the linear change

- Suppose $R = \begin{bmatrix} R_1(x_1 \; x_2) \\ R_2(x_1 \; x_2) \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$\mathbf{R}(\mathbf{x_{i+1}}) \approx \mathbf{R}(\mathbf{x_i}) + \mathbf{DR}(\mathbf{x_k})[\mathbf{u}]$$

$$D\mathbf{R}(x_k)[u] = \frac{d}{d\epsilon} R(\mathbf{x_k} + \epsilon \mathbf{u})$$

$$= \frac{d}{d\epsilon} \begin{bmatrix} R_1(x_1 + \epsilon u_1 \; x_2 + \epsilon u_2) \\ R_2(x_1 + \epsilon u_1 \; x_2 + \epsilon u_2) \end{bmatrix}$$

$$R(x_i) = \mathbf{K_T u} \qquad \textit{Taking} \quad \mathbf{R}(\mathbf{x_{i+1}}) = \mathbf{0}$$

- Where K_T is the tangent stiffness, and then we can find **u** until R is zero [3]
- $K_T = \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix}$ and $u = K_T^{-1} F$

[3] $R_1(x + \epsilon u) = 2k(x_1 + \epsilon u_1) - k(x_2 + \epsilon u_2)$ and $R_2(x + u) = -k(x_1 + \epsilon u_1) + k(x_2 + \epsilon u_2) - F$

Tensors

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5. 2nd order tensors : (Many types of objects!)

Object

- Tensors are objects. An object is a some physical quantity that you can say exists in real life. Examples are forces, stresses etc.
- Each tensor has a co-ordinate frame where you can describe it. For eg : You would need to define a vector by how much it goes in unit basis directions like e^1, e^2, e^3. So we can say that a vector may be defined as (3,4,5) describing how we move in e^1, e^2, e^3 to represent the vector in the basis and so on.

1st order tensors: Vectors

- A vector can be represented by some components. These components are related with a basis.

$$\mathbf{v} = v_i e_i \tag{1}$$

- The basis is defined by the right hand rule
- Main operations in vectors are
  - Scalar product or the dot product
  - Cross product
  - Vector basis transformation

- The dot product is defined as such

$$\begin{aligned} \mathbf{v}.\mathbf{u} = v_i e_i . u_j e_j = v_i u_j \; e_i . e_j = v_i u_i \delta_{ij} = v_i u_i \\ or \\ v = (v_1 e_1 + v_2 e_2 + v_3 e_3)(u_1 e_1 + u_2 e_2 + u_3 e_3) \end{aligned} \tag{2}$$

- e_1, e_2, e_3 are orthogonal to each other therefore we get $v_1 u_1 + v_2 u_2 + v_3 u_3$

- As explained, every vector is a physical object but the way we define it depends on us.
- The way a vector is defined is with respect to the components for any basis. For some basis $\mathbf{e^1}, \mathbf{e^2}, \mathbf{e^3}$, we can write as follows

$$\mathbf{v} = \begin{bmatrix} \mathbf{e_1} & \mathbf{e_2} & \mathbf{e_3} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

- Therefore in some other basis the vector can be defined with different components as :

$$\mathbf{v} = \begin{bmatrix} \mathbf{e'_1} & \mathbf{e'_2} & \mathbf{e'_3} \end{bmatrix} \begin{bmatrix} v'_1 \\ v'_2 \\ v'_3 \end{bmatrix} = \mathbf{Q} \begin{bmatrix} v'_1 \\ v'_2 \\ v'_3 \end{bmatrix}$$

- Therefore $\begin{bmatrix} v'_1 \\ v'_2 \\ v'_3 \end{bmatrix} = \mathbf{Q^{-1}} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$

- $\mathbf{Q}$ is the position of the new basis with respect to the old one. That's why we can operate $\mathbf{Q}$ on $\mathbf{v}$

$$\mathbf{Q} = \begin{bmatrix} e1.e1' & e1.e2' & e1.e3' \\ e2.e1' & e2.e2' & e2.e3' \\ e3.e1' & e3.e2' & e3.e3' \end{bmatrix} \tag{3}$$

- The first column, gives the location of e1' with respect to the old basis. As we can see it gives the direction cosines. And so on
- If we do choose the new basis an orthogonal basis, we get $\mathbf{Q}^{-1} = \mathbf{Q}^{\mathbf{T}}$
- $[v]' = \mathbf{Q}^{\mathbf{T}}[v]$
- $[v] = \mathbf{Q}[v]'$
- [] denotes that we are working with only the components, in basic matrix form
- ?????????????????????????KEEEEEP FIGURE?????????????????????????

Let's look at the transformation $\mathbf{Q}$

- We will show that $\mathbf{e_i'} = \mathbf{Qe_i}$ for $i = 1, 2, 3$

$$\mathbf{Q} = \begin{bmatrix} e1.e1' & e1.e2' & e1.e3' \\ e2.e1' & e2.e2' & e2.e3' \\ e3.e1' & e3.e2' & e3.e3' \end{bmatrix} \tag{4}$$

  We have said that each column represents the location of the new basis to the original basis

- $\mathbf{Qe_1} = \mathbf{Q} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} e1.e1' \\ e2.e1' \\ e3.e1' \end{bmatrix}$ and so on
- Therefore $\mathbf{e_i'} = \mathbf{Qe_i}$
- $\mathbf{Q^T}$ is the opposite transformation. All with respect to and in the same original basis.

2nd order tensors : (Many types of objects!)

- A second order tensor can be a linear map from one vector to another vector
- It can be a mapping that takes two vectors and gives a scalar
- But for now we'll focus mainly on the first thing

Linear map $\mathbf{S}$

$\mathbf{v} = \mathbf{S}\mathbf{u}$

- The map is linear because it is a function that is linear with respect to the vectors it acts on ????FIG????
- $\mathbf{S}(\alpha\mathbf{u_1} + \beta\mathbf{u_2}) = \alpha\mathbf{Su_1} + \beta\mathbf{Su_2}$
- Now **u** could have been defined with respect to any basis. Therefore the linear map as an object is physical.
- The linear map however has it's own components (Think the values in a matrix) that are defined in a certain basis itself.
- Changing the vector basis, means that the linear map should also have its components defined in that basis, for it to do the same transformation. Remember the tensor should do the same thing in any basis!

- $(\mathbf{A} + \mathbf{B})\mathbf{v} = \mathbf{Av} + \mathbf{Bv}$
- $\mathbf{ABv} = \mathbf{A}(\mathbf{Bv})$
- $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$
- $\mathbf{u}.\mathbf{Sv} = \mathbf{v}.\mathbf{S}^\mathbf{T}\mathbf{u}$; Now this can be written as $\mathbf{Sv}$ gives a vector which is then dot producted with $\mathbf{u}$

  $$\mathbf{u}.(\mathbf{Sv}) = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} S_{1j}v_j \\ S_{2j}v_j \\ S_{3j}v_j \end{bmatrix}$$
- $\mathbf{u}.\mathbf{v} = \mathbf{v}.\mathbf{u}$ (Scalar value does not change)
- $\mathbf{S}^\mathbf{T} = \mathbf{S}$ (Symmetric)
- $\mathbf{S}^\mathbf{T} = -\mathbf{S}$ (Skew-symmetric) : Eg $\mathbf{W_w u} = \mathbf{wxu}$ [4]

[4] Check Bonet Page : 30

- $\mathbf{A} = \mathbf{S} + \mathbf{W}$, $\mathbf{S}, \mathbf{W}$ are symmetric and skew symmetric
- $\mathbf{A} = \mathbf{QS}$, $\mathbf{S}, \mathbf{Q}$ are symmetric and orthogonal tensor
  In first one:
  - $\mathbf{S} = \left(\mathbf{A} + \mathbf{A}^{\mathbf{T}}\right)/\mathbf{2}$
  - $\mathbf{W} = \left(\mathbf{A} - \mathbf{A}^{\mathbf{T}}\right)/\mathbf{2}$

  In the second one it's called the polar decomposition

Importance

- as we can see how a stress tensor can be decomposed to its hydrostatic (Only axial) and distortion part(Shearing part)
- The deformation of a body can also be thought about how it rotates (**Q**) and stretches (**S**) the body

- Just like a vector is defined as $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$. These are just components in a basis given as $\mathbf{v} = \mathbf{v_1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \mathbf{v_2} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \mathbf{v_3} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$
- So when we write a tensor in matrix form $\mathbf{S} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ means that these are components in some basis. In choosing this basis we use a product called dyads or tensor product
- $\mathbf{u} \otimes \mathbf{v} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$