

1 Discretisation

MECHANICS

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DISCRETISATION

- Using either configuration, the resulting linearised quantity will be the same
- Spatial is easier though

- Isoparametric elements, discretise the initial geometry X , deformed geometry x , virtual velocity v , δv and linearised displacement u

$$\begin{aligned}
 X &= N_a X_a \\
 x &= N_a x_a \\
 v &= N_a v_a \\
 \delta v &= N_a \delta v_a \\
 u &= N_a u_a
 \end{aligned} \tag{1}$$

where $a = 1 \dots n$ Number of nodes in element, and $N(\xi_1, \xi_2, \xi_3)$ are the standard shape parametric shape function

- The deformation gradiend \mathbf{F} is interpolated over an element

$$\mathbf{F} = \sum_i^n \mathbf{x}_a \otimes \nabla_o \mathbf{N}_a \quad (2)$$

$$\mathbf{F} = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix} \quad \mathbf{x} = \mathbf{N}_a \mathbf{x}_a, \text{ so } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{N}_a \begin{bmatrix} x_{i_1} \\ x_{i_2} \\ x_{i_3} \end{bmatrix}$$

and $\frac{\partial x_1}{\partial X_1} = \frac{\partial N_a x_{i_1}}{\partial X_1}$

- $F_{ij} = \sum_a^n x_{a,i} \frac{\partial N_a}{\partial X_j}$

- where $\nabla_o N_a = \frac{\partial N_a}{\partial \mathbf{X}}$

- We can relate it to $\nabla_\xi N_a = \frac{\partial N_a}{\partial \xi}$ using chain rule as

$$\frac{\partial N_a}{\partial \mathbf{X}} = \left(\frac{\partial \mathbf{X}}{\partial \xi} \right)^{-T} \frac{\partial N_a}{\partial \xi} \quad \left(\frac{\partial N}{\partial \xi} = \frac{\partial N}{\partial \mathbf{X}} \frac{\partial \mathbf{X}}{\partial \xi} \right) \quad \left(\frac{\partial N_1}{\partial \xi_1} = \frac{\partial N_1}{\partial X_1} \frac{\partial X_1}{\partial \xi_1} + \frac{\partial N_1}{\partial X_3} \frac{\partial X_3}{\partial \xi_1} + \frac{\partial N_1}{\partial X_2} \frac{\partial X_2}{\partial \xi_1} \right)$$

$$\frac{\partial \mathbf{X}}{\partial \xi} = \sum_i^n X_a \otimes \nabla_\xi N_a \quad \frac{\partial \mathbf{X}}{\partial \xi} = \begin{bmatrix} \frac{\partial X_1}{\partial \xi_1} & \frac{\partial X_1}{\partial \xi_2} & \frac{\partial X_1}{\partial \xi_3} \\ \frac{\partial X_2}{\partial \xi_1} & \frac{\partial X_2}{\partial \xi_2} & \frac{\partial X_2}{\partial \xi_3} \\ \frac{\partial X_3}{\partial \xi_1} & \frac{\partial X_3}{\partial \xi_2} & \frac{\partial X_3}{\partial \xi_3} \end{bmatrix}$$

- We can work also with the right and left Cauchy tensors \mathbf{C} and \mathbf{b}

$$\begin{aligned}\mathbf{C} &= \mathbf{F}^T \mathbf{F} = \sum_{\mathbf{a}, \mathbf{b}} (\mathbf{x}_{\mathbf{a}} \cdot \mathbf{x}_{\mathbf{b}}) \nabla_{\mathbf{o}} \mathbf{N}_{\mathbf{a}} \otimes \nabla_{\mathbf{o}} \mathbf{N}_{\mathbf{b}} \\ \mathbf{b} &= \mathbf{F} \mathbf{F}^T = \sum_{\mathbf{a}, \mathbf{b}} (\nabla_{\mathbf{o}} \mathbf{N}_{\mathbf{a}} \cdot \nabla_{\mathbf{o}} \mathbf{N}_{\mathbf{b}}) \mathbf{x}_{\mathbf{a}} \otimes \mathbf{x}_{\mathbf{b}}\end{aligned}\tag{3}$$

- We also know that the velocity gradient $\mathbf{d} = \frac{1}{2} (\mathbf{I} + \mathbf{I}^T)$ can be given as

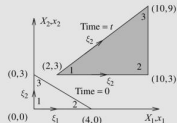
$$\begin{aligned}\mathbf{d} &= \frac{1}{2} (\mathbf{v}_{\mathbf{a}} \otimes \nabla \mathbf{N}_{\mathbf{a}} + \nabla \mathbf{N}_{\mathbf{a}} \otimes \mathbf{v}_{\mathbf{a}}) \\ \delta \mathbf{d} &= \frac{1}{2} (\delta \mathbf{v}_{\mathbf{a}} \otimes \nabla \mathbf{N}_{\mathbf{a}} + \nabla \mathbf{N}_{\mathbf{a}} \otimes \delta \mathbf{v}_{\mathbf{a}})^1 \\ \varepsilon &= \frac{1}{2} (u_a \otimes \nabla N_a + \nabla N_a \otimes u_a)\end{aligned}\tag{4}$$

So basically whenever we are finding the derivative of the continuous function, we replace it with the discretised one! Representing the derivative of N in parametric form, as:

$$\frac{\partial N_a}{\partial \mathbf{x}} = \left(\frac{\partial \mathbf{x}}{\partial \xi} \right)^{-T} ; \frac{\partial N_a}{\partial \xi} = \sum_a^n x_a \otimes \nabla_{\xi} N_a ; \frac{\partial x_i}{\partial \xi_{\alpha}} = \sum_a^n x_{a,i} \frac{\partial N_a}{\partial \xi_{\alpha}}\tag{5}$$

PROBLEM #1

This simple example illustrates the discretization and subsequent calculation of key shape function derivatives. Because the initial and current geometries comprise right-angled triangles, these are easily checked.



The initial \mathbf{X} and current \mathbf{x} nodal coordinates are

$$\begin{aligned} X_{1,1} &= 0; & X_{2,1} &= 4; & X_{3,1} &= 0; \\ X_{1,2} &= 0; & X_{2,2} &= 0; & X_{3,2} &= 3; \\ x_{1,1} &= 2; & x_{2,1} &= 10; & x_{3,1} &= 10; \\ x_{1,2} &= 3; & x_{2,2} &= 3; & x_{3,2} &= 9. \end{aligned}$$

(continued)

■ The shape function are

$$\begin{aligned} N_1 &= 1 - \xi_1 + \xi_2; & N_2 &= \xi_2; & N_3 &= \xi_2 \\ \frac{\partial N_1}{\partial \xi} &= \begin{bmatrix} -1 \\ -1 \end{bmatrix} & \frac{\partial N_2}{\partial \xi} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \frac{\partial N_3}{\partial \xi} &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{aligned} \quad (6)$$

Also

$$\mathbf{X} = N_a(\xi) \mathbf{X}_a \quad X_1 = 4\xi_1 \quad X_2 = 3\xi_2 \quad (7)$$

Think of $N(\xi)$ like some scaling that takes \mathbf{X} to the actual value \mathbf{X}_a . Like a mapper (maybe linear or not) from 0 to 1 that makes you actually move in the \mathbf{X} space.

$$\blacksquare \frac{\partial \mathbf{X}}{\partial \xi} = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix} \quad \left(\frac{\partial \mathbf{X}}{\partial \xi} \right)^{-T} = \frac{1}{12} \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$$

$$\blacksquare \frac{\partial N_a}{\partial \mathbf{X}} = \left(\frac{\partial \mathbf{X}}{\partial \xi} \right)^{-T} \frac{\partial N_a}{\partial \xi}$$

$$\frac{\partial N_1}{\partial \mathbf{X}} = \frac{1}{12} \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \frac{-1}{12} \begin{bmatrix} 3 \\ 4 \end{bmatrix} \quad \frac{\partial N_2}{\partial \mathbf{X}} = \frac{1}{12} \begin{bmatrix} 3 \\ 0 \end{bmatrix} \quad \frac{\partial N_3}{\partial \mathbf{X}} = \frac{1}{12} \begin{bmatrix} 0 \\ 4 \end{bmatrix} \quad (8)$$

■ We can do the same with respect to the spatial coordinates giving us

$$\frac{\partial N_1}{\partial \mathbf{x}} = \frac{1}{24} \begin{bmatrix} 3 & 0 \\ -4 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \frac{-1}{24} \begin{bmatrix} 3 \\ 0 \end{bmatrix} \quad \frac{\partial N_2}{\partial \mathbf{x}} = \frac{1}{24} \begin{bmatrix} 3 \\ -4 \end{bmatrix} \quad \frac{\partial N_3}{\partial \mathbf{x}} = \frac{1}{24} \begin{bmatrix} 0 \\ 4 \end{bmatrix} \quad (9)$$

$$\blacksquare F_{ij} = \sum_a^n x_{a,i} \frac{\partial N_a}{\partial X_j}$$

$$\blacksquare F_{11} = x_{1,1} \frac{\partial N_1}{\partial X_1} + x_{2,1} \frac{\partial N_2}{\partial X_1} + x_{3,1} \frac{\partial N_3}{\partial X_1} = \frac{1}{12} (-2.3 + 10.3 + 10.0) = \frac{6}{3}$$

$$\blacksquare \mathbf{F} = \frac{1}{3} \begin{bmatrix} 6 & 8 \\ 0 & 6 \end{bmatrix}$$

$$\blacksquare \mathbf{C} = \mathbf{F}^T \mathbf{F} \quad \mathbf{b} = \mathbf{F} \mathbf{F}^T \quad J = \det \mathbf{F}$$

Spatial description

$$\delta W(\phi, \delta v) = \int_v \sigma : \delta d dv - \int_v f \cdot \delta v dv - \int_\Gamma t \cdot \delta v da \quad (10)$$

which is the virtual work done by the residual \mathbf{r} .

A pretty neat thing is that we can consider a single virtual nodal velocity δv_a occurring at node a in element e .

$$\delta W^e(\phi, N_a \delta v_a) = \int_{v^e} \sigma : (\delta v_a \otimes \nabla N_a) dv - \int_{v^e} f \cdot (N_a \delta v_a) dv - \int_{\Gamma^e} t \cdot (N_a \delta v_a) da \quad (11)$$

* The first term because σ is symmetric so the 1/2 and all disappears
But what about nodal connectivities! ??

Insight :

$$d = \frac{1}{2} (l + l^T) = \frac{1}{2} (\nabla v + (\nabla v)^T) \quad (12)$$

$$\nabla v = n_a \otimes \nabla N_a$$

We know that $\sigma : (\mathbf{u} \otimes \mathbf{v}) = \mathbf{u} \cdot \sigma \mathbf{v}$ for any vectors \mathbf{v}, \mathbf{u}

So we get

$$\delta W^e(\phi, N_a \delta v_a) = \delta v_a \cdot \left(\int_{v^e} \sigma \nabla N_a dv - \int_{v^e} N_a f dv - \int_{\Gamma^e} N_a t da \right) \quad (13)$$

$$\delta v_a = \begin{bmatrix} \delta v_{a,1} \\ \delta v_{a,2} \\ \delta v_{a,3} \end{bmatrix}$$

- So per element the virtual work is expressed in terms of the internal and external nodal forces \mathbf{T}_a^e and \mathbf{F}_a^e
- $\delta W^e(\phi, N_a \delta v_a) = \delta \mathbf{v} \cdot (\mathbf{T}_a^e - \mathbf{F}_a^e)$
 \mathbf{T}_a^e is the internal force with , $T_{a,i}^e = \sum_j^3 \int_{v^e} \sigma_{ij} \frac{\partial N_a}{\partial x_j} dv$ ($\sigma \nabla N_a$ is a linear map where ∇N_a is a vector)
- The cauchy stress is found from the constitutive relationship and the left cauchy tensor

■ Same example as last where $b = \frac{1}{9} \begin{bmatrix} 100 & 48 & 0 \\ 48 & 36 & 0 \\ 0 & 0 & 9 \end{bmatrix}$ and $J = 4$

■ $\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix} = \frac{\mu}{J}(b - I) + \frac{\lambda}{J}(\ln J)I = \begin{bmatrix} 8 & 4 & 0 \\ 4 & 3 & 0 \\ 0 & 0 & 0.8 \end{bmatrix}$

■ $T_{a,i} = \int_{V^e} \left(\sigma_{i1} \frac{\partial N_a}{\partial x_1} + \sigma_{i2} \frac{\partial N_a}{\partial x_2} \right) dv$

■ $T_{1,1} = -24t \quad T_{1,1} = 8t \quad T_{1,1} = 16t$
 $T_{1,2} = -12t \quad T_{2,2} = 0 \quad T_{3,2} = 12t$

Equilibrium at a global level :

- From all elements e (1 to m_1) containing node a

$$\delta W(\phi, N_a \delta v) = \sum_{e=1, e \ni a}^{max} \delta W^e(\phi, N_a \delta \mathbf{v}_a) = \delta \mathbf{v}_a \cdot (\mathbf{T}_a - \mathbf{F}_a)$$

- where assembled equivalent nodal forces are:

$$\mathbf{T}_a = \sum_{e=1, e \ni a}^m \mathbf{T}_a^e \quad \mathbf{F}_a = \sum_{e=1, e \ni a}^m \mathbf{F}_a^e$$

- And then for all nodes we get $\delta W(\phi, \delta v) = \sum_a^n \delta \mathbf{v}_a \cdot (\mathbf{T}_a - \mathbf{F}_a)$
- Since the virtual work should be satisfied for any virtual nodal velocity, we get the Residual force with respect to the whole system $\mathbf{R}_a = \mathbf{T}_a - \mathbf{F}_a$

- Organise R in an array $\mathbf{T} = [\mathbf{T}_1 \ \mathbf{T}_2 \dots \mathbf{T}_N]$, $\mathbf{F} = [...]$, $\mathbf{R} = [...]$ (I think each T_i contains three components)
- Virtual work equation is : $\delta \mathbf{v}^T \mathbf{R} = \delta \mathbf{v}^T (\mathbf{T} - \mathbf{F}) = 0$
where $\delta \mathbf{v}^T = [\delta \mathbf{v}_1^T \ \delta \mathbf{v}_2^T \dots]$
- Since the internal forces are nonlinear functions of the current nodal positions $\mathbf{x} = [\mathbf{x}_1 \ \mathbf{x}_2 \ \mathbf{x}_3 \dots]$
- In matrix notation, we keep the symmetric tensor as $\sigma' = [\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{13}, \sigma_{23}]^T$ and \mathbf{d} as $\mathbf{d} = [d_{11}, d_{22}, d_{33}, 2d_{12}, 2d_{13}, 2d_{23}]^T$
(Where off diagonals is twice to make sure $\mathbf{d}^T \sigma$ gives the correct internal energy)
- $\int \sigma : \mathbf{d} = \int \mathbf{d}^T \sigma dv$

$$\mathbf{d} = \sum_a^n \mathbf{B}_a \mathbf{v}_a \text{ where } B_a = \begin{bmatrix} \frac{\partial N_a}{\partial x_1} & 0 & 0 \\ 0 & \frac{\partial N_a}{\partial x_2} & 0 \\ 0 & 0 & \frac{\partial N_a}{\partial x_3} \\ \frac{\partial N_a}{\partial x_2} & \frac{\partial N_a}{\partial x_1} & 0 \\ \frac{\partial N_a}{\partial x_3} & 0 & \frac{\partial N_a}{\partial x_1} \\ 0 & \frac{\partial N_a}{\partial x_3} & \frac{\partial N_a}{\partial x_2} \end{bmatrix}$$

- So : $\delta W = \int (B_a \delta v_a)^T \sigma dv - \int f \cdot (N_a \delta v_a) dv - \int t \cdot (N_a \delta v_a) da$
- We can also write the internal force as : $T_a^e = \int_{v^e} B_a^T \sigma' dv$

- The equilibrium equations are still nonlinear with respect to the nodal positions. A NR is used to solve it
- The linear virtual work components is found using the directional derivative as

$$D\delta W(\phi, \delta \mathbf{v})[u] = D\delta W_{int}(\phi, \delta \mathbf{v})[u] - D\delta W_{ext}(\phi, \delta \mathbf{v})[u] \quad (14)$$

- The internal work linearisation can be decomposed into the constitutive and initial stress components

$$D\delta W_{int}(\phi, \delta \mathbf{v})[u] = D\delta W_C(\phi, \delta \mathbf{v})[u] + D\delta W_\sigma(\phi, \delta \mathbf{v})[u] \quad (15)$$

$$= \int_{\mathbf{v}} \delta \mathbf{d} : \mathbf{c} : \varepsilon d\mathbf{v} + \int_{\mathbf{v}} \sigma : \left((\nabla \mathbf{u})^T (\nabla \delta \mathbf{v}) \right) d\mathbf{v} \quad (16)$$

which is the tangent stiffness matrix

- Remember that at each node, we get the residual due to the nodal equivalent forces at a due to the whole equilibrium of node a ($R = T - F$)
- F may be dependant on a and so linearisation in the direction of u_b or $N_b u_b$ with $N_a v_a$ constant, gives only the change of the residual force at node a due to the change u_b in the current position of node b

$$D\delta W^e(\phi, N_a \delta v_a)[N_b u_b] = D(\delta v_a \cdot (T_a^e - F_a^e))[N_b u_b] \quad (17)$$

$$= \delta v_a \cdot D(T_a^e - F_a^e)[N_b u_b] = \delta v_a \cdot K_{ab}^e u_b \quad (18)$$

- Change in force at node a due to change in current position of node b
- This is not the full stiffness matrix, but each component. When we do the whole thing we get the full linear map

- Check bonet page 248

■

$$D\delta W_c^e(\phi, N_a \delta v_a)[N_b u_b] = \delta v_a K_{c,ab}^e u_b \quad (19)$$

- where

$$[K_{c,ab}]_{jk} = \int_{v^e} \sum_{k,l=1}^3 \frac{\partial N_a}{\partial x_k} C_{ikjl} \frac{\partial N_b}{\partial x_l} dv \quad i, j = 1, 2, 3 \quad (20)$$