Mechanics

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Updated : Dec 6 2020

Linearity and non-linearity

- Suppose there is a function or mapping that does this $f : x \longrightarrow y$. Function f takes x and gives y
- The function is linear if $f(x) = f(x_1) + f(x_2)$. where $x = x_1 + x_2$
- Suppose we know $f()$ and y but not x. Because the function is linear, we can construct solutions x that may be a combination of different solutions x_i. The idea is that f should not depend on x. It can have other parameters, but not the values of the solution itself. In that case, we would have to find the solution x and also the mapping f

Example

Take function or map ()

Structural non-linearity introduction

- What does non-linearity mean ?
- What makes a structure non-linear ?
- How do you model non-linearity and solve it?

- There are different types of non-linearity in a structure
  - Geometrical - $\mathbf{K}(\mathbf{x})\mathbf{x} = \mathbf{F}$ where here $\mathbf{K}$ depends on $\mathbf{x}$ because $\mathbf{K}$ changes as you chagne x due to the change in geometry and the large deformation.
  - Material - $\mathbf{K}(\mathbf{x})\mathbf{x} = \mathbf{F}$ because $\mathbf{K}$ changes due to the changes in the material property. For eg Youngs modulus may be a function of x.
- $\mathbf{K}$ is now a function of (x), it means that it can't be solved by using a linear method (Inverting a matrix).
- But the system of equations now has to be linearised about a point. We keep the system of equations as an equality equation $\mathbf{K}(\mathbf{x})\mathbf{x} - \mathbf{F} = 0$.
- We linearise and try to solve the root. At every point we construct the tangent and find a solution, but when we construct the solution at that point, we find the residual and we then use to iterate until convergence.

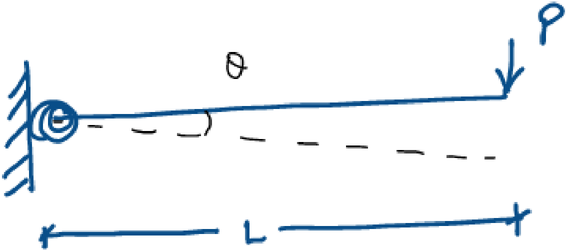


Figure:

Taking a weightless rigid bar with a torsional spring that resists any moment and a load P at the end. Since the bar is rigid so vertical equilibrium is satisfied at the support without any deformation. However, it can rigidly rotate through the spring.

The moment generated at the end support is

$$PLcos\theta = M \tag{1}$$

Taking the equilibrium equation with the spring as $M = K\theta$ we get

$$PLcos\theta = K\theta$$
$$\frac{PL}{K} = \frac{\theta}{cos\theta}$$

- If we take $\theta \to 0$ then $cos\theta \to 1$
- So $P = \dfrac{K}{L}\theta$ which is linear wrt θ

- Linearity : $\frac{PL}{K} = \theta$
- Geometric nonlinearity only : $\frac{PL}{K} = \frac{\theta}{cos\theta}$
- Material nonlinearity only : $\frac{PL}{K(\theta)} = \theta$ [1]
- Geometric + Material nonlinearity : $\frac{PL}{K(\theta)} = \frac{\theta}{cos\theta}$

[1] We can say the spring stiffness as a material parameter K that also depends on θ introducing the material nonlinearity. For eg, we can model $K(\theta) = K_o(1 - c\theta)$

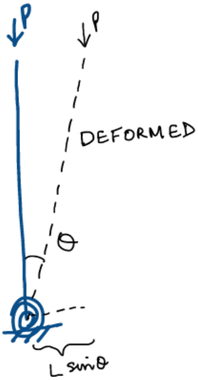


Figure:

Same problem but now the rigid bar is vertical. This is a very common problem for nonlinearity (I think the previos one is better tho!). Nicely represents buckling of columns

Same equilibrium equation but now the lever arm is different (Because the load along the bar)

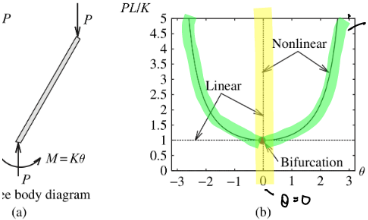
$$PLsin\theta = M \tag{2}$$

Taking the equilibrium equation with the spring as $M = K\theta$ we get

$$PLsin\theta = K\theta$$
$$\frac{PL}{K} = \frac{\theta}{sin\theta}$$

- If we take $\theta \rightarrow 0$ then $sin\theta \rightarrow 0$ so $M = 0$ (This is one possible equilibrium)
- $\frac{PL}{K} = \frac{\theta}{sin\theta}$: This is the other
- The load $\frac{PL}{K}$ where these two equilibrium equations are possible for the same structure is called the bifurcation point.
- You can imagine that when the load is smaller, it will not buckle and only deform axially so only one equilibrium position ($\theta = 0$) is possible. When $\frac{PL}{K} > 1$ then the other equilibrium comes to play.

Therefore two solutions are there :



(a) ...e body diagram

(b)

...imple column.

$\frac{PL}{K} = \frac{\theta}{\sin\theta}$

- Below $\frac{PL}{K} < 1$, Only Equilibrium is $\theta = 0$

  Because $\frac{PL}{K} < 1$, $\frac{\theta}{\sin\theta} \not< 1$ so $\theta = 0$

- Above $\frac{PL}{K} > 1$, we get two possible solutions
  - $\theta = 0$: $M = 0$
  - $\theta \neq 0$: $\frac{PL}{K} = \frac{\theta}{\sin\theta}$.

- If we linearise our equilibrium equation for small $\theta \to sin\theta$
- We get $(Pl - K)\theta = 0$ and we get our linear eigen value problem with a trivial solution of $\theta = 0$ and a nontrivial solution of $PL = K$. Again we get two equilibrium solutions. $PL = K$ being the buckling load.
- So when we reach that load, it means we have reached the bifurcation point, where multiple equilibrium solutions exist and the rod may buckle dependant on imperfection, lateral load etc.

Non-Linear strain measures introduction

- Now suppose of being rigid, the body is deformable that is the relative deformation is introducing strain and stresses
- LARGE DISPLACEMENTS + LARGE STRAINS
- The first deals with displacements that are large, and therefore while finding the strains, we have to use higher orders of the displacement derivatives
- The second deals with ?????????????????????????????

OneD measures

- Emphasize its only for One D! But same theory for other dimensions
- A strain measure need not be fixed. Sometimes the strain measure we usually use may not be able to model the correct behaviour. When we choose any strain measure, the proper corresponding stress and the constitutive relationship ($\sigma = \mathbf{C}\varepsilon$) has to be taken.
- The stress and strain have to be "work compatible". That is they are together used in the strain energy density function.

Engineering strain

- Engineering strain $\varepsilon_E = \dfrac{l - L}{L} = \dfrac{\Delta}{L}$. l is deformed length, L is initial undeformed
- We could have also divided Δ by l (Change by deformed length). If $l \approx L$ then it would not matter.
- ε_E is the small infinitesimal strain, where the deformed and undeformed lengths are very similar.

Logarithimic strain

- The instantaneous strain increment can be thought as $\varepsilon_L = \frac{\Delta_1}{L} + \frac{\Delta_2}{l_1} \ldots$
- Or $d\varepsilon_L = \frac{dl}{l}$
- $\varepsilon_L = \int_L^l \dfrac{dl}{l} = ln\frac{l}{L}$
- The integration is done between two configurations $L \longrightarrow l$

These strains are more easily extrapolated to continuum (3d cases)

Green strain

- $\varepsilon_G = \dfrac{l^2 - L^2}{2L^2}$

Almansi strain

- $\varepsilon_A = \dfrac{l^2 - L^2}{2l^2}$

- Suppose $l \approx L$ and therefore Δ is small
- And $l = (L + \Delta)$
- $\varepsilon_G = \dfrac{(L+\Delta)^2 - L^2}{2L^2} = \dfrac{(L^2 + \Delta^2 + 2L\Delta - L^2)}{2L^2} \approx \dfrac{\Delta}{L}$ (As Δ is very small and so Δ^2 vanishes)

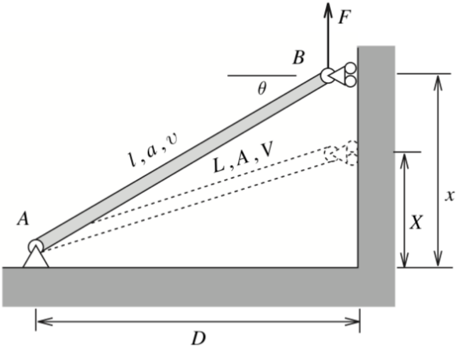


Figure:

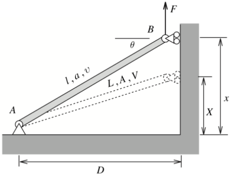
- Initial length L, area A, volume V
- Final length l, area a, volume v

- In defining the equilibrium (Froces = 0 , No moments). We will be defining the internal stress by different strain measures.
- Remember that a proper constitutive law has to be taken for a particularly strain measure
- Here we have chosen the Cauchy stress and E randomly and not dependant on work compatibility. The cauchy stress is the actual/true stress in the deformed state. (Or it is the stress in the deformed state which is in equilibrium)

Using two strain measures

Green and logarithmic.

- Cauchy stress (True stress) $\sigma = E\varepsilon$ can be :
- $\sigma = E\dfrac{l^2 - L^2}{L^2}$
- $\sigma = Eln\dfrac{l}{L}$



- The bar will keep moving up until the vertical equilibrium is reached.
- Vertical equilibrium at B is $F - T(x)sin\theta(x) = 0$, where $T(x)$ is the internal force and depends on x. θ is also dependant on x
- Now we can construct a residual function $R(x) = F - T(x)sin\theta(x)$ where the residual becomes zero for a particular solution of x. [2] So

$$R(x) = \sigma a sin\theta - F = \sigma(x) a \frac{x}{l} - F \tag{3}$$

Stress dependant on different strain measures

- $T(x) = E\frac{l^2 - L^2}{L^2} a \frac{x}{l}$ $\quad(\sigma = E\varepsilon_G)$
- $T(x) = Eln\frac{l}{L} a \frac{x}{l}$ $\quad(\sigma = E\varepsilon_L)$

[2] Note that $\frac{dR}{dx}$ is the tangent stifness K_{Bx} or force in direction B due to displacement x.

Stress dependant on different strain measures

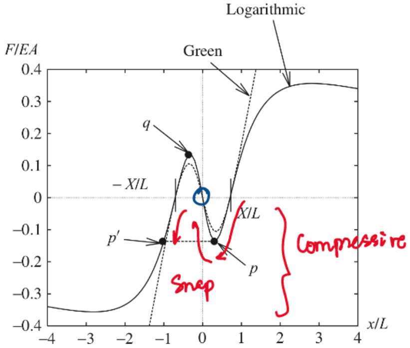
- $T(x) = E\frac{l^2 - L^2}{L^2}a\frac{x}{l}$ $\qquad (\sigma = E\varepsilon_G)$
- $T(x) = Eln\frac{l}{L}a\frac{x}{l}$ $\qquad (\sigma = E\varepsilon_L)$

- l is a function of x, $l^2 = D^2 + x^2$
- R(x) is therefore very nonlinear with respect to x. In R(x), F is not dependant on x. But sometimes it can be the case that the load is also nonlinear.

Solving

We need to solve the nonlinear equation R(x) = 0

- So we use NR, or first order taylor series to linearise R and solve it iteratively
- $R(x_{i+1}) = R(x_i) + \frac{dR}{dx}|_{x_i}(x_{i+1} - xi)$
- We want R = 0 , so the value $R(x_{i+1}) = 0$
- $0 = R(x_i) + \frac{dR}{dx}|_{x_i}(x_{i+1} - xi)$



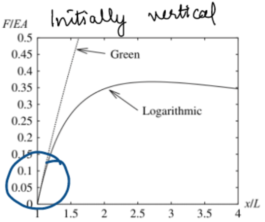


FIGURE 1.6 Large strain rod: load deflection behavior.

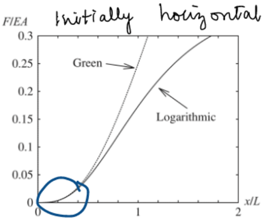


FIGURE 1.7 Horizontal truss: tension stiffening.

- ◯ ← When the strains are small the different measures give similar equilib profile

- As you increase compressive (-ve) you get snap through behaviour from $p \to q \to p'$ (Imagine a wrinkled coca cola bottle you push in).

- Regions where x is small, different mesasures gives okay results
- We see different behaviour behaviours between the strain measures at higher strain
- Snap through behaviour if we increase compressive load too much. Imagine you are pushing the truss down (-x) and suddenly it will roll to the other side.
- If truss is initially vertical (Like a column in tension, Therefore no rotation) : Same E should have not been used for both different strain measures. (It seems that the green strain looks good as we expect it to be linear in axial)
- Initially horizontal : Stiffening due to tension

- A comment was made that E should not have been used.
- The vertical stiffness K_{Bx} is the chagne in equilibrium at B in direction x. $K_{Bx} = \frac{dR}{dx}$. If F is constant then $\frac{dR}{dx} = \frac{dT}{dx}$
- Without the inclusion of the strain measures, internal force is $T(x) = \sigma a \frac{x}{l}$. (All three are a function of x)
- Since both strain measures are function of l we can write $\sigma = f(l)$
- Using the incompressibility codition [3], we can replace a with $a = \frac{V}{l}$, and using chain rule:

$$\frac{dT}{dx} = \frac{d}{dx}\left(\frac{\sigma V x}{l^2}\right) = \frac{Vx}{l^2}\frac{\partial \sigma}{\partial l}\frac{\partial l}{\partial x} - \frac{2\sigma V x}{l^3}\frac{\partial l}{\partial x} + \frac{\sigma V}{l^2}$$

$$\frac{dT}{dx} = \frac{ax}{l}\frac{\partial \sigma}{\partial l}\frac{\partial l}{\partial x} - \frac{2\sigma a x}{l^2}\frac{\partial l}{\partial x} + \frac{\sigma a}{l}$$

[3] The condition states that volume cant change under deformation and so $al = AL$

$$\frac{dT}{dx} = \frac{ax}{l}\frac{\partial\sigma}{\partial l}\frac{\partial l}{\partial x} - \frac{2\sigma ax}{l^2}\frac{\partial l}{\partial x} + \frac{\sigma a}{l}$$

Stress gradient

- So we need to find $\frac{\partial\sigma}{\partial l}$
- Green : $\left(\frac{\partial\sigma}{\partial l}\right)_G = E\frac{\partial\varepsilon_G}{\partial l} = E2l/2L^2 = \frac{El}{L^2}$
- Logarithmic : $\left(\frac{\partial\sigma}{\partial l}\right)_L = E\frac{\partial\varepsilon_L}{\partial l} = E\frac{d}{dl}\left(ln(l) - ln(L)\right) = \frac{E}{l}$

l gradient

- $l^2 = D^2 + x^2$
- $2l\frac{dl}{dx} = 2x$
- $\frac{dl}{dx} = \frac{x}{l}$

- $K_{Bx} = \frac{dR}{dx} = \frac{dT}{dx}$
- Green : $K_G = \frac{A}{L}\left(E - 2\sigma\frac{L^2}{l^2}\right)\frac{x^2}{l^2} + \frac{\sigma a}{l}$ [4]
- Logirthmic : $K_L = \frac{a}{l}\left(E - 2\sigma\right)\frac{x^2}{l^2} + \frac{\sigma a}{l}$

- They look similar but the causal consitutive relation chosen has led to the different results
- We will write K_G as with the idea of getting an insight:

$$K_G = \frac{A}{L}\left(E - 2S\right)\frac{x^2}{l^2} + \frac{SA}{l}$$

$$\text{where } S = \sigma\frac{L^2}{l^2}$$

- Where S is the second-Piola Kirchoff stress which gives the force per unit underformed area transformed by the deformation gradient inverse $(l/L)^{-1}$

[4] V = AL

- S is actually associated with ε_G
- $(x/l)^2$ is the transformation from local to global forces.
- Therefore K_G shows that we can express the stiffness in initial underformed configuration
- If x is close to X and l is close to L then both the stiffness would be the same. The second term contains the change $\frac{\partial l}{\partial x}$, so this term disappears.
- The third term is the initial stress or geometric stiffness. This is unconcerned with the change in cross sectional area and associated only with the change in rigid body rotation. A very negative value can cause instability and singular K. The third term actually came from the derivative of the direction cosines (x/L).

Continuum measures - 2D

- Strain ε has components ε_x, y, xy
- This strain is a measurement at a point!!!
- Infinitesimal strains

$$\varepsilon_x = \frac{\partial u}{\partial x}$$

$$\varepsilon_y = \frac{\partial v}{\partial y}$$

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

- Displacements are small, so only linear orders of displacement gradients are available

- Notation for different configurations undeformed : x, and deformed : X

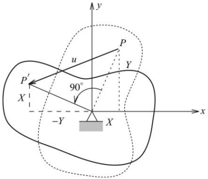


FIGURE 1.8 90° rotation of a two-dimensional body.

- Suppose there is a rotation in any solid by 90 ^{o} , No deformation!. So:

$$u = -X - Y$$
$$v = X - Y$$

- So our infinitesimal strains are :

$$\varepsilon_x = \varepsilon_y = -1$$
$$\varepsilon_{xy} = 0$$

- So we still get strain, when we should have not

- Using the same Green strain, we extend it in some way for 2D
- Taking the differential length dS (Undeformed) and ds (Deformed)
- Take a small element dX initially parallel to *x* axis

$$ds^2 = \left(dX + \frac{\partial u}{\partial X} dX\right)^2 + \left(\frac{\partial v}{\partial X} dX\right)^2$$

$$E_{xx} = \frac{ds^2 - dX^2}{2dX^2} = \frac{1}{2}\left(\left(1 + \frac{\partial u}{\partial X}\right)^2 + \left(\frac{\partial v}{\partial X}\right)^2\right) - 1$$

Similarly we get the Green strains equations :

- Thse strain components = 0 for the rigid rotation case
- Nonlinear strains are better, but they coincide with the infinitesimal strains when x and X are close to each other. [5]

[5] Here x and X are vectors that define the total position of a body in the deformed and undeformed

Stress and equilibrium

- We are dealing with different configurations. One configuration is maybe unstressed and the deformed one is. So at the deformed x we should get an equilibrium of stresses and the external loads
- Now, the actual stresses at the deformed or current configuration is the Cauchy stress : defined as the force in different directions by the area in different planes
- Stresses can also be defined with respect to the initial configuration X

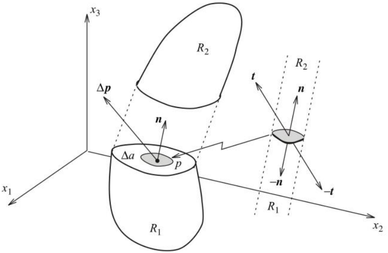


FIGURE 5.1 Traction vector.

At the deformed configuration :

- See two bodies R_1 and R_2 free body with force acting on them
- Imagine the traction vector on a small area element : $t(n) = \frac{\Delta p}{\Delta a}$ as lim $\Delta a \longrightarrow 0$ where Δp is the resultant force
- Obviously t and n will depend on the surface it acts on. Here on the right we can see that based on the surface we get opposite forces. (In the negative normal , we will get negative force which is positive in that direction!)

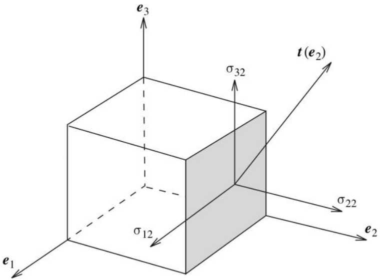
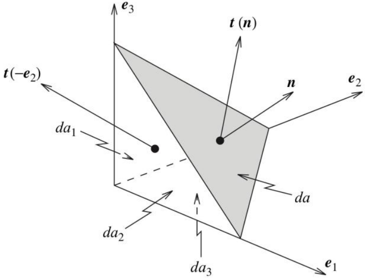


FIGURE 5.2 Stress components.

- Let us denote the traction acting on the surface having normals denoted by e_1, e_2, e_3
- Remember in the other slice we will have an opposite reaction

$$
\begin{aligned}
\mathbf{t}(\mathbf{e_1}) &= \sigma_{\mathbf{1j}}\mathbf{e_j} \\
\mathbf{t}(\mathbf{e_2}) &= \sigma_{\mathbf{2j}}\mathbf{e_j} \\
\mathbf{t}(\mathbf{e_3}) &= \sigma_{\mathbf{3j}}\mathbf{e_j}
\end{aligned}
\tag{4}
$$

- Or $\mathbf{t_i} = \sigma_{\mathbf{ij}}\mathbf{e_j}$



Now let us look if we take a plane cut of that sphere. Again by context of opposite reactions. All the forces should be equal. So we will use here the concept of equilibrium between the traction vector we have defined in the last slide with respect to some basis and the traction vector defined on the angled plane.

Equilibrium

$$\mathbf{t}(\mathbf{n})da + t(-\mathbf{e_i})da_i + \mathbf{f}dv = 0 \tag{5}$$

This states that the force vector on the inclined cut should be in equilibrium with the opposite forces defined on the negative sufraces and the body force

- Now the areas (Because they are with defined respect to the basis vectors) can be written as the projection of the inclined area

$$da_i = da(\mathbf{n}.\mathbf{e_i}) \tag{6}$$

- Diving by da we get

$$\mathbf{t}(\mathbf{n}) + t(-\mathbf{e_i})\frac{da(\mathbf{n}.\mathbf{e_i})}{da} + \mathbf{f}\frac{dv}{da} = 0 \tag{7}$$

- $\frac{dv}{da} \longrightarrow 0$