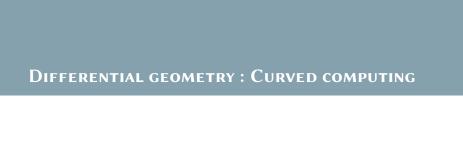
# **DIFFERENTIAL GEOMETRY**

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UPDATED: FEBRUARY 18, 2021



- A shell is comprised of its reference surface, thickness and edges. The thickness of the shell at a point is found as the distance between the bounding surface (Top and bottom surfaces) as measured along a normal to the reference surface that passes through the point.
- FIX ME????????? New things also have to be added

### Space curves: Parameterisation of curve

■ A 3d curve in a rectangular coordinate system  $(x_1, x_2, x_3)$  can be represented by the locus of the end point of the position vector

$$\mathbf{x} = x_1(t)\mathbf{e_1} + x_2(t)\mathbf{e_2} + x_3(t)\mathbf{e_3}$$
 (1)

for all the values of the parameter t that are in the interval  $t_1 < t < t_2$ . Suppose that  $\mathbf{x}$  takes only a single value of the parameter t, then we insure it's uniqueness.

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#### Unit tangent vector

■ We take *s* as the variable of the arc length along the space curve. The derivative of the position vector *x* with respect to *s* is given as

$$\frac{d\mathbf{x}}{ds} = \frac{dx_1}{ds} \mathbf{e}_1 + \frac{dx_2}{ds} \mathbf{e}_2 + \frac{dx_3}{ds} \mathbf{e}_3 \tag{2}$$

If we form a scalar product with itself we get

$$\frac{d\mathbf{x}}{ds} \cdot \frac{d\mathbf{x}}{ds} = \left(\frac{dx_1}{ds}\right)^2 + \left(\frac{dx_2}{ds}\right)^2 + \left(\frac{dx_3}{ds}\right)^2 \tag{3}$$

■ From differential calculus we know that

$$(ds)^{2} = (dx_{1})^{2} + (dx_{2})^{2} + (dx_{3})^{2}$$
(4)

Hence

$$\frac{d\mathbf{x}}{ds} \cdot \frac{d\mathbf{x}}{ds} = 1 \tag{5}$$

This confuese me how the magnitude of the tangent becomes 1, because sometimes in calculus some mainpulations seem very devious

- We can think of a vector  $\Delta \mathbf{x}$  joining two points Q and Q' on a curve. The vector  $\Delta \mathbf{x}/\Delta s$  has the same directiona as  $\Delta \mathbf{x}$  and as  $\Delta s$  approaches 0,  $\Delta \mathbf{x}/\Delta s$  becomes the tangent vector to the curve at he poin Q.
- The vector **T** is as the limit of  $\Delta s \rightarrow 0$  we get

$$\mathbf{T} = \frac{d\mathbf{x}}{ds} \tag{6}$$

We also note that

$$x' = \frac{d\mathbf{x}}{dt} = \frac{d\mathbf{x}}{ds}\frac{ds}{dt} \tag{7}$$

This comes from chain rule by the way the parametrisation is done orginally. This is als a tangent vector but it is not of unit length

## OSCULATING PLANE, PRINCIPLA NORMAL

- The tangent to the Curve at Q is found to be the limiting position of the line connecting the points Q and Q'. Now if we consider the limiting position of a plane passing through three consequitve places. We will get the osculating plane joining the normal and the tangent of a curve.
- The tripple scalar produc of three coplanar vectors is zero and the expression of the osculating plane is as

$$(\mathbf{X} - \mathbf{x}) \cdot (\mathbf{x}' \times \mathbf{x}'') \tag{8}$$

#### CURVATURE

■ By taking the dot product with itself  $\mathbf{T} \cdot \mathbf{T} = \mathbf{1}$ . If we differentiate this scalar product with respect to the arc length we get

$$\frac{d}{ds}(\mathbf{T} \cdot \mathbf{T}) = 2\mathbf{T} \cdot \mathbf{T}^o = 0 \tag{9}$$

where  $T^o$  is the derivative with respect to s. Now this tells us that the tangent is prependicular to the curvature. We can also then write

$$\mathbf{T} = \frac{d\mathbf{x}}{ds} = \frac{d\mathbf{x}}{dt}\frac{dt}{ds} = \mathbf{x}'t^o \tag{10}$$

where this comes from chain rule but the parameterisation is messed up, originally it is  $\frac{d\mathbf{x}}{dt} = \frac{d\mathbf{x}}{ds}\frac{ds}{dt}$ . I don't know when things can be used like fractions!! But it checks out here. Also note that s and t will be different functions for different components of  $\mathbf{x}$ 

■ Therefore we can say

$$\mathbf{T}^{\mathbf{o}} = \mathbf{x}' t^{oo} + \mathbf{x}'' (t^{o})^2 \tag{11}$$

Which shoes that the curvature also lies in the plane of  $\mathbf{x}'$  and  $\mathbf{x}''$ , or the osculating plane.

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■ Since the curvature has been shown to be perpendicular to the tangent **T** we can say that the curvature is parallel to the principal normal given by

$$\mathbf{T}^{\mathbf{o}} = k\mathbf{N} = \mathbf{k} \tag{12}$$

where **N** is a unit normal vector in the direction of the principal normal to the curve at a point (1/R). We can see that, k is the curvature and k**N** is the curvature vector.

■ Although the sense of  $T^o$  is determined solely by the curve, the sense of the principal normal N is arbitary. k therefore depends on the sense of N.