N-Fem

Allan Marbaniang

Updated : Dec 18 2020

Introduction

- If a response is a function of space and time, we need to move in space and time.
- Partial discretisation, where the differential equation at each node is now already discretised by space and only dependant on time. FDM will do this at a point where at each point we have a differential equation that is now numerically defined
- $\hat{f}(x,t) \approx f(x,t)$
- There are methods developed that are dependant on Minimisation Error = $\hat{f}(x,t) - f(x,t)$
- We can also minimise the differential equation
  $k\frac{\partial^2 \hat{f}}{\partial^2 x} - f(x) = R(x)$
  $\int_{x_a^e} W(x)R(x)dx = 0$

- Galerkin : Babnov, Petrov
- Before dynamics, if we go for statics (Steady state) without partial descritisation
- So we can $\int_{x_a}^{x_b} W(x)R(x)dx = 0$
  It forces the function to be averagely zero. So some nodes will satisfy it exactly!!
  $\int_{x_a}^{x_b} W(x).\left(k\frac{\partial^2 \hat{f}}{\partial x^2} - f(x)\right)dx = 0$
- I will have to assume however the function $f(\hat{x})$. Easy to take polynomials, but the order should not vanish. Minimum requirement, but can we reduce this need of order??

- So we can keep the function like this :

$$\int_{x_a}^{x_b} W(x).k\frac{\partial^2 \hat{f}}{\partial x^2}dx - \int_{x_a}^{x_b} W(x).f(x) - dx = 0 \tag{1}$$

- And using integration by parts we get
$$\left|\left(W(x).k\frac{\partial \hat{f}}{\partial x}\right)\right|_{x_a}^{x_b} - \int_{x_a}^{x_b} \frac{dW(x)}{dx}.k\frac{\partial \hat{f}}{\partial x}dx - \int_{x_a}^{x_b} W(x).f(x) - dx = 0$$
- Note we have reduce the order, and we also get the boundary term. (The flux that we describe at the end). And we have a weak weighted residual statement.
- It is an integral steatement. Not point based in FDM.

- Choosing weight :
  - Any function : Petrov-Galerkin
  - Shape function : Bubnov - Galerkin
- Suppose we have actual $f(x)$ and approxx $f(\hat{x}) = a_0 + a_1 x$. Knowing the boundary conditions at x =x_a and x = x_b, where $\hat{f} = f_a$ and $\hat{f} = f_b$
- We get $f(\hat{x}) = \frac{x_b - x}{x_b - x_a} f_a + \frac{x - x_a}{x_b - x_a} f_b$, which are shape linear functions. Where we get the boundary values if we keep x at boundary.

$$f(\hat{x}) = N_1(x) f_a + N_2(x) f_b \tag{2}$$

  given by Ritz, Ritz approximation which gives us a way how to choose $f(\hat{x})$
- Petrov : $W_i = N_i$ and suppose $f(\hat{x}) = \sum_1^3 N_i(x) f_i$
  Weak form
  $\left| \left(W(x).k \frac{\partial \hat{f}}{\partial x} \right) \right|_{x_a}^{x_b} - \int_{x_a}^{x_b} \frac{dW(x)}{dx}.k \frac{\partial \hat{f}}{\partial x} dx - \int_{x_a}^{x_b} W(x).f(x) - dx = 0$
  $B.T - \int_{x_a}^{x_b} \frac{dN_i}{dx}.k \frac{\partial}{\partial x}(N_1 f_1 + N_2 f_2 + N_3 f_3) dx - \int_{x_a}^{x_b} N_i.f(x) - dx = 0$
  Weight gives equation at each node. Only is differentiation in partial.
- If we keep the unkown boundary terms as a vector we get

$$\mathbf{0} = \mathbf{B.T} - \mathbf{Kf} - \mathbf{P} \tag{3}$$

  where $K_{ij} = \int_{x_a}^{x_b} \frac{dN_i}{dx}.k \frac{\partial N_j}{\partial x} dx$

- FDM writes at every node. Here we minimise the governing differential in the domain (Integral). The weak form is valid over the entire domain. And now we write the equation at some points dependant on the approx funciton.
- For a three parameter approximation

$$\begin{bmatrix} 0 - BT \\ 0 - BT \\ 0 - BT \end{bmatrix} = - \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} - \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \tag{4}$$

  where $K_{ij} = \int_{x_a}^{x_b} \frac{dN_i}{dx} . k \frac{\partial N_j}{\partial x} dx$ an $p_i = \int_{x_a}^{x_b} N_i f(x) dx$
- **Kd** = (**BT** − **p**) = **F** which is the descretised form.
- Differential system ⟶ Alegebric system
- We can take N as picewise also (T.Kant)
- B.T will disappear so for a three noded we get :

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} BT - 0 \\ 0 \\ BT - 0 \end{bmatrix} + \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \tag{5}$$

  B.T = $(W.k\frac{du}{dx})|_{x_1}^{x_2}$, once u is known at boundary and prescribed, the weight will be zero. Weight is given only at points where we don't know

- Any gerneral function can be approximated by linear terms in a smaller domain, while in a larger domain the function is more complicated.
- This is an integral method so we can always decretise it

$$0 = \int_{x_a}^{x_b} W(x)R(x)dx = \sum_{e=1}^{n} \int_{x_a^e}^{x_b^e} W^e(x)R^e(x)dx \tag{6}$$

- Now so this is the concept of finite element method form of weighted residual method
  - Continuity of the field variables must be maintained
  - So we can do the computation only on one element
- Boundary volume method : Only at the boundary (Dimension less)

- Again when we keep the full term we get

$$\left|\left(W(x).k\frac{\partial\hat{f}}{\partial x}\right)\right|_{x_a}^{x_b} - \int_{x_a}^{x_b}\frac{dW(x)}{dx}.k\frac{\partial\hat{f}}{\partial x}dx - \int_{x_a}^{x_b}W(x).f(x) - dx = 0 \qquad (7)$$

- In the interior nodes of a 2 element discretisation of 2 noded element, the boundary term from element 1 becomes $BT_{x_2} - BT_{x_1}$ and from element 2 becomes $BT_{x_3} - BT_{x_2}$
- As we Join node 2, we will get at node x2 that the boundary terms will cancel each other.
- The way I like to look at this is that the B.T. is the internal force from each element and corresponding face. Each element will give a boundary term pointing corresponding to the face.

- So in descritised form for a 2 noded element we get :

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} BT_1 \\ BT_2 \end{bmatrix} + \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \tag{8}$$

  where $K_{ij} = \int_{x_a^e}^{x_b^e} \frac{\partial N_i^e}{\partial x} k \frac{\partial N_j^e}{\partial x} dx$

  $p_i^e = \int_{x_a^e}^{x_b^e} N_i^e f(x) dx$

- And then we can write the equations for each element, and rearange the dof as global dof in the same vector and we get our global matrix!
- The advantage is always that all the integrals are done in the sub domain

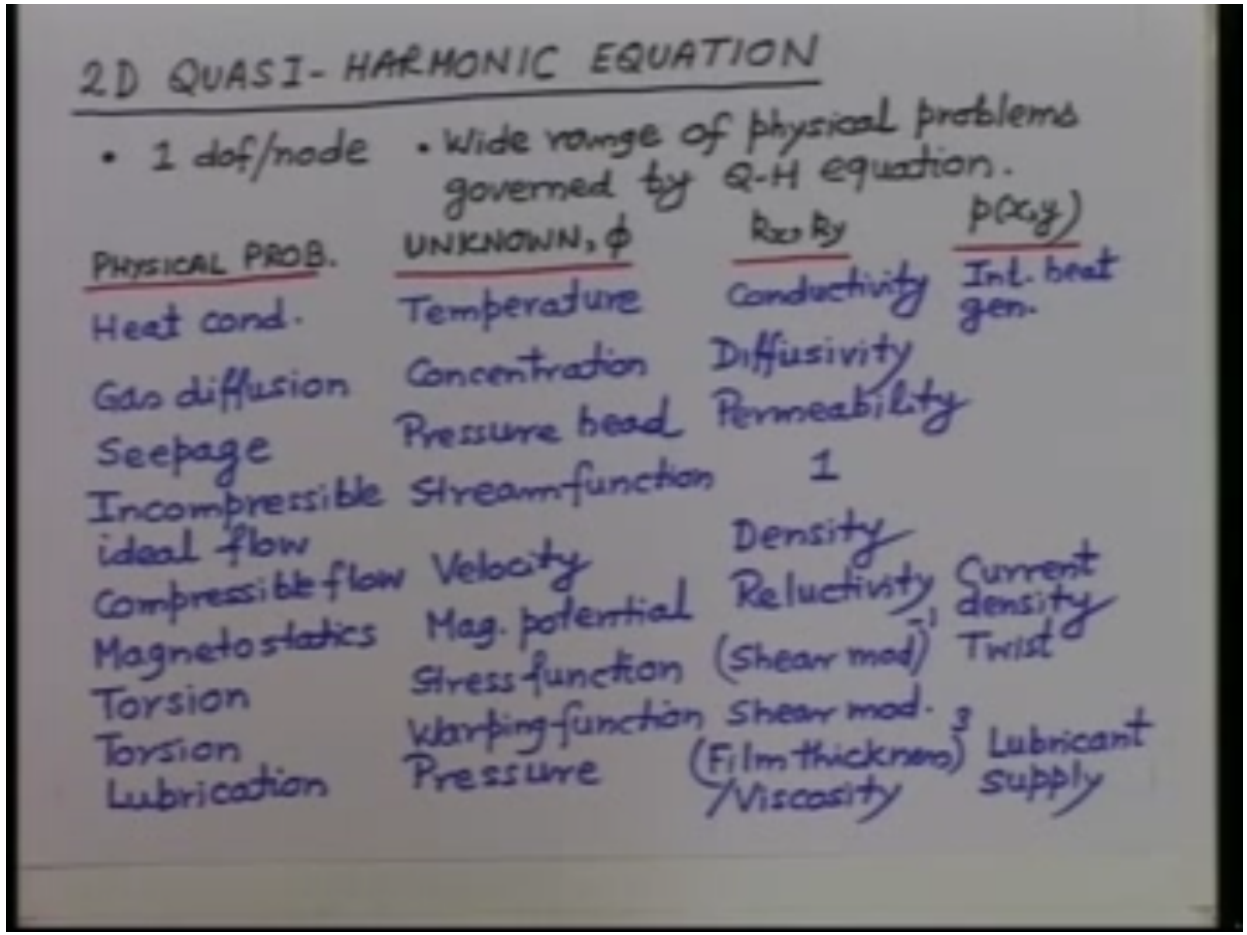
- With the local weighted residual method, we can discretise it into simpler domains etc.
- Second order - Axial. Bending is forth order, (If we do second order, then two equations with two unkowns)
- Axial rod subjected to axial deformation

$$\begin{aligned} EA\frac{\partial^2 u}{\partial x^2} &= p(x) \\ EI\frac{\partial^4 u}{\partial x^4} &= p(x)(Bending) \end{aligned} \tag{9}$$

  with boundary conditions.

- Unkown per node is 1, but varies over x and y
- $\frac{\partial}{\partial x}\left(k_x \frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y \frac{\partial \phi}{\partial y}\right) + p(x, y) = 0$
- Mixed boundary conditions
  - Dritchlet : $\phi = \hat{\phi}$ in some boundary
  - General Newman :$k_x \frac{\partial \phi}{\partial x} L_x + k_y \frac{\partial \phi}{\partial y}_y + q + \alpha(\phi - \phi_a) = 0$: In some portion of the boundary
  - For isotropic material

$$k \frac{\partial \phi}{\partial n} + q + \alpha(\phi - \phi_a) \tag{10}$$


2D QUASI-HARMONIC EQUATION
• 1 dof/node
• Wide range of physical problems governed by Q-H equation.
PHYSICAL PROB.
UNKNOWN, ϕ
Kx, Ky
p(x,y)
Heat cond.
Temperature
Conductivity
Int. heat gen.
Gas diffusion
Concentration
Diffusivity
Seepage
Pressure head
Permeability
Incompressible ideal flow
Streamfunction
1
Compressible flow
Velocity
Density
Magnetostatics
Mag. potential
Reluctivity
Current density
Torsion
Stress function
(Shear mod)$^{-1}$
Twist
Torsion
Warping function
Shear mod.
Lubrication
Pressure
(Film thickness)^3 /Viscosity
Lubricant supply

- So $R(x,y) = k\frac{\partial^2\phi}{\partial x^2} + k\frac{\partial^2\phi}{\partial y^2} + p$
- Global error = $\sum_{e=1}^{N} \int_{\Omega_e} local$
- WRM

$$\int \int_{\Omega} W\left(k\frac{\partial^2\phi}{\partial x^2} + k\frac{\partial^2\phi}{\partial y^2} + p\right) dxdy = 0 \qquad (Strongstatement) \tag{11}$$

- Greens theorem :

$$\int \int_{A} \left(\frac{\partial C}{\partial x}\frac{\partial D}{\partial x} - C\frac{\partial^2 D}{\partial x^2}\right) dA = \int_{S} C\frac{\partial D}{\partial x} L_x ds \tag{12}$$

So we use this we get :

$$\int \int_{A} C\frac{\partial^2 D}{\partial x^2} dA = \int \int_{A} \frac{\partial C}{\partial x}\frac{\partial D}{\partial x} - \int_{S} C\frac{\partial D}{\partial x} L_x ds \tag{13}$$

The first term is like $W(k\frac{\partial^2\phi}{\partial x^2})$

- $\int \int_A \frac{\partial W}{\partial x} k \frac{\partial \hat{\phi}}{\partial x} dA - \int_S W.k \frac{\partial \hat{\phi}}{\partial x} L_x ds + \int \int_A \frac{\partial W}{\partial y} k \frac{\partial \hat{\phi}}{\partial y} dA - \int_S W.k \frac{\partial \hat{\phi}}{\partial y} L_y ds + \int \int WpdA = 0$ (Weak statement)
- Where L_x is direction cosine of s with respect to x
- $\hat{\phi} = \sum_{i=1}^{n} N_i(x, y)\phi_i$ (Ritz, bubonov)
- For j^{th} node
  $\int \int_A \frac{\partial N_j}{\partial x} k \frac{\partial \sum_{i=1}^{n} N_i \phi_i}{\partial x} dA - \int_S N_j k \frac{\partial \sum_{i=1}^{n} N_i \phi_i}{\partial x} L_x ds + \int \int_A \frac{\partial N_j}{\partial y} k \frac{\partial \sum_{i=1}^{n} N_i \phi_i}{\partial y} dA -$
  $\int_S N_j k \frac{\partial \sum_{i=1}^{n} N_i \phi_i}{\partial y} L_y ds + \int \int_A N_j pdA = 0$
- So $K_{ij} = \int \int_A k \left(\frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial y} \right) dA$
- $\mathbf{Kd} = \mathbf{P} + \mathbf{X_i}(\mathbf{Direct}) + \mathbf{Newman} = F$

- Anticlockwise noded. Each node has (x,y)
- $\hat{\phi} = N_1(x, y)\phi_1 + N_2(x, y)\phi_2 + N_3(x, y)\phi_3$
- Let $\hat{\phi} = a + bx + cy$ (Three unknowns, you need three for a plane)
- $$\begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$
- Then we can find the shape functions
- Then we can find the element! Stress and strain is constant

- In plane deformation : Plane stress, plane strain, axisymmetric
- Integration over the triangle domain (area) poses problems
- Then they developed area coordinates, which is the area sections that they develop for the shape functions $L_i = N_i$. Some area integration based on these area coordinates.
- The stiffness is constant, and area can be formed while the force can be found

The steps of a finite element method are

- Divide the whole domain ino finite parts
- For each element develop relations between pairs of dual variables (primary and secondary, eg forces and displacements)
- Assemle elements together to get the replationship of the variables for the whole system

We are going to look at second differential equations and how to solve them using fem

Table 3.2.1: List of fields in which the model equation in Eq. (3.2.1) arises, with meaning of various parameters and variables; see the bottom of the table for the meaning of the parameters*.

Field of study	Primary variable u	Coefficient a	Coefficient c	Source term f	Secondary variable Q
Heat transfer	Temperature $T - T_\infty$	Thermal conductance kA	Surface convection $p\beta$	Heat generation f	Heat Q
Flow through porous medium	Fluid head ϕ	Permeability μ	$--$ 0	Infiltration f	Point source Q
Flow through pipes	Pressure P	Pipe resistance $1/R$	$--$ 0	$--$ 0	Point source Q
Flow of viscous fluids	Velocity v_z	Viscosity μ	$--$ 0	Pressure gradient $-dP/dx$	Shear stress σ_{xz}
Elastic cables	Displacement u	Tension T	$--$ 0	Transverse force f	Point force P
Elastic bars	Displacement u	Axial stiffness EA	$--$ 0	Axial force f	Point load P
Torsion of bars	Angle of twist θ	Shear stiffness GJ	$--$ 0	$--$ 0	Torque T
Electro-statics	Electrical potential ϕ	Dielectric constant ϵ	$--$ 0	Charge density ρ	Electric flux E

* k = thermal conductance; β = convective film conductance; p = perimeter; P = pressure or force; T_∞ = ambient temperature of the surrounding fluid medium; $R = 128\mu h/(\pi d^4)$ with μ being the viscosity; h, the length and d the diameter of the pipe; E = Young's modulus; A = area of cross-section; J = polar moment of inertia.

- Consider the equation

$$-\frac{d}{dx}\left(a\frac{u}{x}\right)+cu=f \qquad for \qquad 0<x<L \tag{14}$$

- where $a=a(x), c=c(x), f=f(x)$ are the known quantities and $u(x)$ has to be found

- The domain $\Omega = (0, L)$ is descritised into a set of intervals with $\Omega^e = (x_a^e, x_b^e)$ which denotes the end of the element
- The length of an element $h_e = x_b^e - x_a^e$

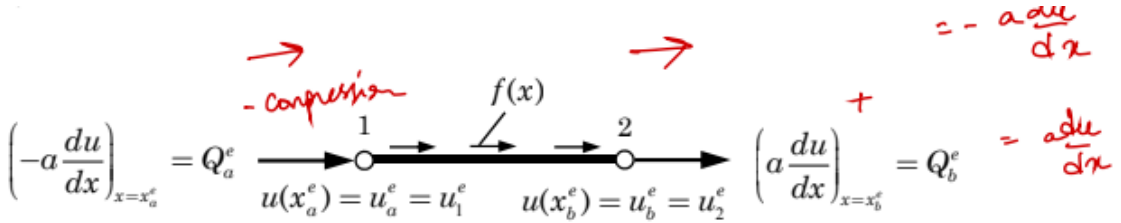


Fig. 3.2.1: A typical finite element in one dimension.

- We find an approx solution over each element Ω^e and then we assemble it all together

$$u(x) \approx u_h^e(x) = \sum_j^n c_j^e N_j^e \tag{15}$$

  where we choose the shape functions and then have to find the coefficients such that oour approx solution is like the real one

- Since there are n unkown parameters (For each dof), we need n linearly independant equations

- Keeping the discretised DE equation we get

$$-\frac{d}{dx}\left(a\frac{u_h^e}{x}\right)+cu_h^e-f(x)=R^e(x,c_1^e,c_2^e,c_3^e,...,c_n^e)\neq 0 \tag{16}$$

  What we want to do is find the coefficients such that the residual is zero. This is again equilibirum at a point!

- One way to make it zero is to set the weighted integral of the residual zero

$$\int_{x_a^e}^{x_b^e} w_i^e(x)R^e(x,c_1^e,c_2^e,c_3^e,...,c_n^e)dx=0 \qquad i=1,2,...,n \tag{17}$$

- where w_i^e are different weight functions giving us n equations for the coefficien parameters $(c_1^e,c_2^e,c_3^e,...,c_n^e)$
- Now the weights however shoul all be independent and invertible. If we took w = 1, we would have only one equation
- These weights feel like the variation, but in the variation we choose also that the variations are the same function like the shape functions hmmmm.

- If we choose w_i^e to be the shape functions. We get the Galerkin method. Which is exactly the same as the virtual work where $\delta v = \sum_i \delta v_i N_i$ where you can say then that δv_i is 1, since the virtual disp magnitude comes out anyways
- Since the residual R_e has the same order derivatives of the dependant unkonw u(x), we need at least quadratic representation of $u_h^e(x)$
- To reduce or weaken the differentiability of the shape functions (node disp are constant) , we distribute the order between the weights and u_h^e

This is the weak form : Reducing the order of the dependant variable to the weight to make the order of the variable leser

- The weights kind of give different component equilib equations

Note

- Note that in usual structural mechanics we derive the virtual work equation from a potentional functional. Most fem methodsa re based on an element wise application of the Ritz method
- The virtual displacement is an integral statement which is he same as the integral weak form found frm the governing differential equations.
- But differential equations are easy to form, and most fem methods are based on de

- After getting the weighted residual statement, the next job is to weaken the differentiability of u_h^e. To make both the orders of u_h^e and w_i^e the same
- The steps are
  - Write weighted residual statement

$$0 = \int_{x_a^e}^{x_b^e} \left[w_i^e \left(-\frac{d}{dx}\left(a\frac{u_h^e}{x} \right) + cu_h^e - f(x) \right) \right] dx \tag{18}$$

    We are taking the summation of the weighted residual over the whole element and saying that it is zero. The weight sort of gets the components

  - Weakening the form using integration by parts

$$(uv)' = uv' + u'v$$

$$\int_a^b uv' = uv|_a^b - \int_a^b u'v$$

$$0 = \int_{x_a^e}^{x_b^e} \left[w_i^e \left(-\frac{d}{dx}\left(a\frac{u_h^e}{x} \right) + cu_h^e - f(x) \right) \right] dx \tag{19}$$

$$0 = \int_{x_a^e}^{x_b^e} \left(a\frac{dw_i^e}{dx}\frac{du_h^e}{dx} + cw_i^e u_h^e - w_i^e f(x) \right) dx - \left[w_i^e a\frac{du_h^e}{dx} \right]_{x_a^e}^{x_b^e}$$

- Very interesting, we actually get the boundary terms too!

$$0 = \int_{x_a^e}^{x_b^e} \left(a \frac{dw_i^e}{dx} \frac{du_h^e}{dx} + c w_i^e u_h^e - w_i^e f(x) \right) dx - \left[w_i^e . a \frac{du_h^e}{dx} \right]_{x_a^e}^{x_b^e} \tag{20}$$

  Direct boundary on the dependant: Dritchlet/essential $u = 0$
  Boundary on the derivatives of dependant : Newman/natural $\frac{du}{dx} = p$

- The coefficient of the weight function which is $a\frac{du}{dx}$ is the second variable
- We state the differenet variables

$$\text{Primary variable : } u \qquad \text{Secondary variable:} n_x (a \frac{du}{dx}) = Q(x) \tag{21}$$

  See that $n_x = -1, 1$ on left and right end. ???WHye

- In the final weak form, we keep the secondary variables at the element ends as

$$Q_a^e = Q(x_a^e) = -\left(a \frac{du}{dx} \right)_{x_a^e} \qquad Q_b^e = Q(x_b^e) = \left(a \frac{du}{dx} \right)_{x_b^e} \tag{22}$$

  In the figure above we can think this of a FBD but in arbitary configruaion. The first one is a compressive, while later is a tensile force. In heat the first would be the heat input and later output

- Althout Q replaced $a(\frac{du}{dx})$, it is not consdiered as a function of u, but a variable dual to u??????

- The final expression for the weak form is

$$0 = \int_{x_a^e}^{x_b^e} \left(a \frac{dw_i^e}{dx} \frac{du_h^e}{dx} + c w_i^e u_h^e - w_i^e f(x) \right) dx - w_i^e(x_a^e) Q_a^e - w_i^e(x_b^e) Q_b^e \tag{23}$$

  But even in the virtual work when you reduce the order of the strains, you get the boundary condition

- The remarks are:
  - Integration by parts (i) reduces the degree of the fem approximation (ii) introduces the secondary variables that are physically meanifull as they can be specified at a point where the primary variable is not specified. If the secondary variable is not a physical quantity, then the integraion by parts should not be carried out even to reduce the order of u_h^e
  - The terms containing both w_h^e and u_h^e are called bilinear functional

$$B(w_i^e, u_h^e) = \int_{x_a^e}^{x_b^e} \left(a \frac{dw_i^e}{dx} \frac{du_h^e}{dx} + c w_i^e u_h^e \right) \tag{24}$$

    but has to be linear with respect to w_i^e and u_i^e. So it has to be billinear map. Like a scalar product with metric tensor where u and v are the input and the metric tensor is the bilinear map. If a or/and c is a fucntion of u. Then B is always linear in w but not u.
  - Terms having only w_i^e are only linear functionals because they are only linear with respect to w_i^e. $l(w_i^e)$

- Therefore the weak form can be expressed as

$$B(w_i^e, u_h^e) = l(w)i^2 \tag{25}$$

  which is a vairational problem where we find $u^e \in U$ such that the equation is satisfied for all $w_i^e \in U$ (See Reddy page 103 for hilbert spaces)

- The weak form is nothing but the statement of minimum total potential energy, or the variational minimum

$$\begin{aligned} \Pi(u_h^e) \\ \delta\Pi = B(\delta u_h^e, u_h^e) - l(\delta u_h^e) = 0 \\ \Pi(u_h^e) = \frac{1}{2} B(\delta u_h^e, u_h^e) - l(\delta u_h^e) \qquad (26) \\ = \int_{x_a^e}^{x_b^e} \left[\frac{a}{2} \left(\frac{du_i^e}{dx} \right)^2 + \frac{c}{2} (u_h^e)^2 - u_h^e f \right] dx - u_h^e(x_a^e) Q_a^e - u_h^e(x_b^e) Q_b^e \end{aligned}$$

  This is when you have reduced the order of the derivative in the virtual work, (We get the Euler lagrange equilibrium).

- This equation $\frac{1}{2} B(w_i^e, u_h^e) = l(w)i^2$, B shoudl be symmetric and the first term is the elastic energy while the later is the work done by the load and point loads.

- We have to satisfy the weak form of the differential equation along with the continuity and boundary conditions. We need to choose a function that satisfies the differntiability requirement and the end conditions $u(x_i) = u_i^e$. Any function with a non zero differentiation of the order of the weak form would be a candidate. We can therefore use interpolation.
- The interpolation is

$$u_h^e(x) = c_1^e + c_2^e x \tag{27}$$

is okay, since the differentiation $\neq 0$, but we only now need to make sure that c_1, c_2 are such that the end displacements match

$$u_h^e(x_a^e) = c_1^e + c_2^e x_a^e = u_a^e \qquad u_h^e(x_b^e) = c_1^e + c_2^e x_b^e = u_b^e \tag{28}$$

or $\begin{bmatrix} 1 & x_a^e \\ 1 & x_b^e \end{bmatrix} \begin{bmatrix} c_1^e \\ c_2^e \end{bmatrix} = \begin{bmatrix} u_a^e \\ u_b^e \end{bmatrix}$ and we get the interpolating functions $u_h^e(x) = \sum_{j=1}^{2} N_j^e u_j^e$ [1]

- $N_1^e(x) = \frac{x_b^e - x}{x_b^e - x_a^e} \qquad N_2^e(x) = \frac{x - x_a^e}{x_b^e - x_a^e}$

[1] The book makes ϕ for N. But I usually use that for eigen directions?

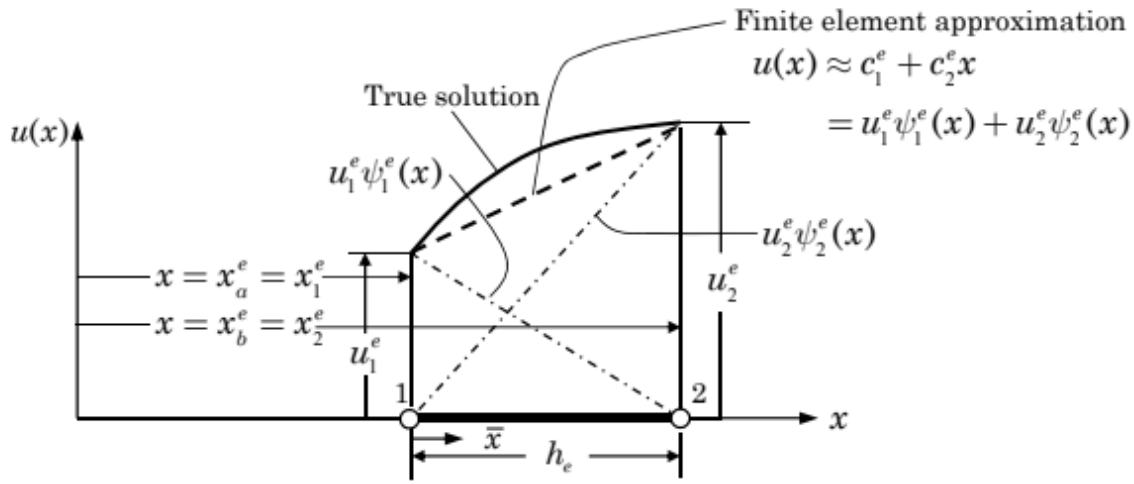


Fig. 3.2.2: Linear approximation over a finite element.

- These N are linear lagrange interpolation functions and $u_1^e = u_a^e \qquad u_2^e = u_b^e$ are the nodal values of the approx function at the ends. U_h belongs to a hilbert subspace spanned by N_1^e, N_2^e
- Remember that $N_i^e(x_j^e) = 1 \quad \text{if} i = j$. They also satisfy the partition of unity where $\sum_{j=1} N_j^e(x) = 1$

For a quadratic approximation we choose

$$u_h^e(x) = c_1^e + c_2^e x + c_3^e x^2 \tag{29}$$

- Since there are three parameters we need to have three nodal points where we can relate the constants to.

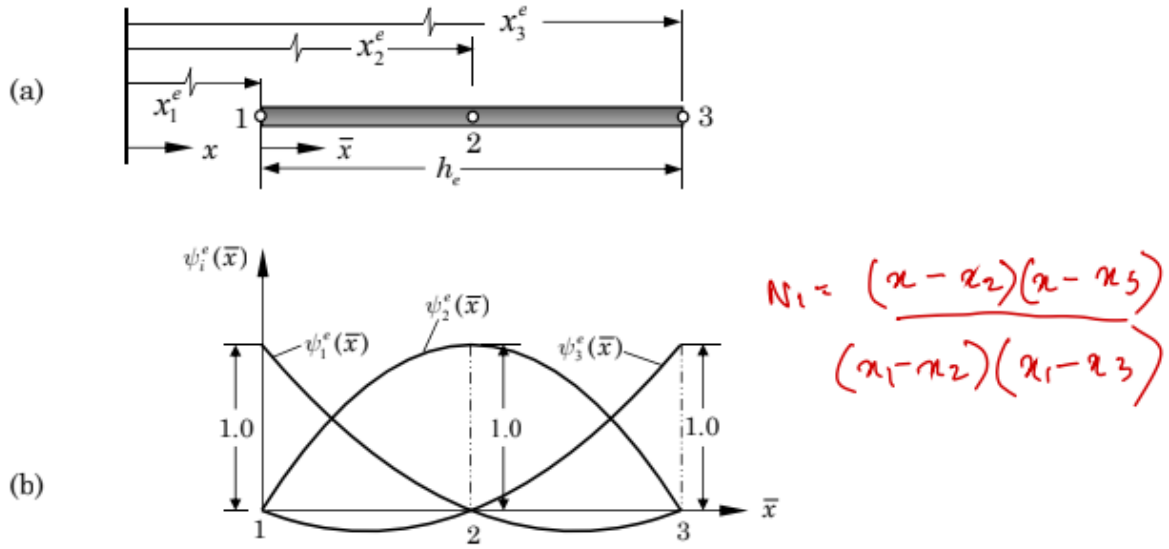
$$x_1^e = x_a^e \qquad x_2^e = x_a^e + \frac{h_e}{2} \qquad x_3^e = x_a^e + h_e = x_b^e \tag{30}$$

And we similary get :

$$u_h^e(x) = \sum_{j=1} N_j^e u_j^e \tag{31}$$

where N are the quadratic lagrange interpolation functions. If they are expressed in local coordinate we get

$$N_1^e(x') = \left(1 - \frac{x'}{h_e}\right)\left(1 - \frac{2x'}{h_e}\right) \qquad N_2^e(x') = 4\frac{x'}{h_e}\left(1 - \frac{x'}{h_e}\right) \qquad N_3^e(x') = -\frac{x'}{h_e}\left(1 - \frac{2x'}{h_e}\right) \tag{32}$$



$$N_1 = \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)}$$

- This is a quadratic element
- Any higher order lagrange interpolations can be developed. A (n-1)degree can be written as $u_h^e \sum_{j=1}^{n} N_j^e u_j^e$
- Where the interpolating function can be given as

$$N_j^e(x) = \Pi_{i=1, i\neq j} \left(\frac{x - x_i^e}{x_j^e - x_i^e} \right) \tag{33}$$

For example $N_1(x) = \left(\frac{x-x_2}{x_1-x_2}\right)\left(\frac{x-x_3}{x_1-x_3}\right)$

- Approx solution should be continous and differentiable as needed by the weak form. This ensures that every term in the differential equaion does not have a zero coefficient.
- It should be a complete polynomial (Pascal's law). To capture all the actual deformation. Lower to higher!
- It should interpolate the primary variables at the nodes of the fem and at the end points. To ensure continuity of the primary variable acorss elements!

- Keeping the approximate solutions in the weak form gives us the algebraic equatins
- The degree of the approx solution has to be decied a priori. If there are more than 2 nodes, then the number of non-zero secondary variables increases at the interior nodes

$$0 = \int_{x_a}^{x_b} \left(a \frac{dw_i^e}{dx} \frac{du_h^e}{dx} + c w_i^e u_h^e \right) dx - \int_{x_a}^{x_b} w_i^e f dx - \sum_{j=1}^{n} w_i^e(x_j^e) Q_j^e \qquad (34)$$

- If 1 and n denote the end points then Q_1^e, Q_n^e denote the unknown point sources, while the other $Q_j^e \quad (j = 2, 3 \ldots n-1)$ are the externally applied and known point sources. So at the ends these are internal or external loads??????
- If we keep $w_i^e = N_i^e$ into the weak form, we get n algebraic equations. This is the Galerkin method (Original was weighted of residual and not of the weak form, that would be exactly the same to Ritz method). The ith algebraic equation is the one obtained by keeping w_i^e as N_i^e. This is the same as the virtual work method, where each equation of a discretised system comes from the virtual displacement of each node

- So we get

$$0 = \int_{x_a}^{x_b} \left(a \frac{dN_i^e}{dx} \sum_{j=1}^{n} u_j^e \frac{dN_j^e}{dx} + cN_i^e \sum_{j=1}^{n} u_j^e N_j^e \right) dx - \int_{x_a}^{x_b} N_i^e f dx - \sum_{j=1}^{n} N_i^e(x_j^e) Q_j^e \quad (35)$$

so for each equation i there will be a summation on the derivatives of the approx solution due to chain rule! and we get

$$0 = \sum_{j=1}^{n} \left[\int_{x_a}^{x_b} \left(a \frac{dN_i^e}{dx} \frac{dN_j^e}{dx} + cN_i^e N_j^e \right) dx \right] u_j^e - \int_{x_a}^{x_b} N_i^e f dx - Q_i^e \quad (36)$$

where we have taken the summation of u magitude coefficient for, shape functions outside. Also we see that for each shape function at each node we actually only get the boundary load at that variation. Because $N_i^e(x_j^e) = 0$ when $i \neq j$. We get for each node!

$$0 = \sum_{j=1}^{n} K_{ij}^e u_j^e - f_i^e - Q_i^e \quad (37)$$

- Where

$$K_{ij}^e = \int_{x_a}^{x_b} \left(a \frac{dN_i^e}{dx} \frac{dN_j^e}{dx} + cN_i^e N_j^e \right) dx = B(N_i^e, N_j^e)$$
$$f_i^e = \int_{x_a}^{x_b} fN_i^e dx = l(N_i^e) \tag{38}$$

  So this is ineresting the coefficient of the stifness says basically states chagne in the shape functions!

Matrix form

$$\mathbf{K^e u^e} = \mathbf{f^e} + \mathbf{Q^e} \approx \mathbf{F^e}$$

where

- K^e is the symmetric coefficient, stiffness matrix
- f^e is the source or force vector
- This method is the weak-form Glaerikin or Ritz finite element method

Matrix form

$$\mathbf{K^e u^e} = \mathbf{f^e} + \mathbf{Q^e} \approx \mathbf{F^e}$$

- But for every element, we have n equations and $n+2$ unknowns. The 2 unkowns are the secondary nodal values that we don't know Q_a^e, Q_b^e. (u_1^e, u_2^e...u_n^e) are the element primary nodal degrees. Remember these values, we know if they are along the inside of the element as external forces.
- Assembling the elements by imposing the continuity of the elements. U2 of 1 is U1 of 2. We get the same number of equations and unkowns (Primary + Secondary).
- The stiffness and force matrix can be found for a certain value. And if the coefficients (a,c,f) are also functions of x, then we need to do numerical integration.

Linear element (i.e. element with linear approximation)

$$\psi_1^e(\bar{x}) = 1 - \frac{\bar{x}}{h_e}, \qquad \psi_2^e(\bar{x}) = \frac{\bar{x}}{h_e} \tag{3.2.37}$$

$$\left(\frac{a_e}{h_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{c_e h_e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right)\begin{Bmatrix} u_1^e \\ u_2^e \end{Bmatrix} = \frac{f_e h_e}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} + \begin{Bmatrix} Q_1^e \\ Q_2^e \end{Bmatrix} \tag{3.2.38}$$

Quadratic element (i.e. element with quadratic approximation)

$$\psi_1^e(\bar{x}) = \left(1 - \frac{2\bar{x}}{h_e}\right)\left(1 - \frac{\bar{x}}{h_e}\right), \ \psi_2^e(\bar{x}) = \frac{4\bar{x}}{h_e}\left(1 - \frac{\bar{x}}{h_e}\right), \ \psi_3^e(\bar{x}) = -\frac{\bar{x}}{h_e}\left(1 - \frac{2\bar{x}}{h_e}\right) \tag{3.2.39}$$

$$\left(\frac{a_e}{3h_e}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix} + \frac{c_e h_e}{30}\begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix}\right)\begin{Bmatrix} u_1^e \\ u_2^e \\ u_3^e \end{Bmatrix} = \frac{f_e h_e}{6}\begin{Bmatrix} 1 \\ 4 \\ 1 \end{Bmatrix} + \begin{Bmatrix} Q_1^e \\ Q_2^e \\ Q_3^e \end{Bmatrix} \tag{3.2.40}$$

- Where in linear element lets write the expansion for the first linear element equation

$$K_{11}u_1^e + K_{12}u_2^e = f_1 + Q_1^e$$

$$\left[\int_0^{h_e} a\frac{-1}{h_e}\frac{-1}{h_e} + c\left(1 - \frac{x}{h_e}\right)\left(1 - \frac{x}{h_e}\right)\right]u_1^e + \left[\int_0^{h_e} a\frac{-1}{h_e}\frac{1}{h_e} + c\left(1 - \frac{x}{h_e}\right)\left(\frac{x}{h_e}\right)\right]u_2^e = \int_0^{h_e} f\left(1 - \frac{x}{h_e}\right) + Q_1^e \tag{39}$$

2

[2] We note that (i) In quad, the force vector is not just fh/3 but it depends on the work done! Not same with 2 elements combined. (ii) There are also more unkowns thatn the no of equations. When one element is used however we have only n unkowns cause bcs will be applied on Q.

- Consider a homogeneous, isotropic bar of length L(m), cross sectional area (A) and conductivity k (W/(m^oC))
- Ambient temperature is $T_o(^oC)$
- No heat loss throughout the bar and the right end is exposed to ambient temperature of T_{inf}
- Uniform heat of g_o, heat transfer with fin and air is β
- Check Reddy Page 3.2.1

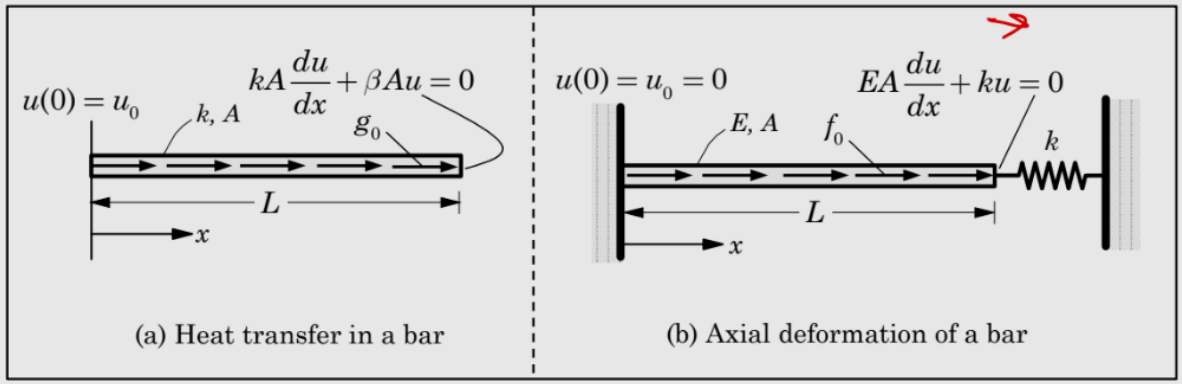
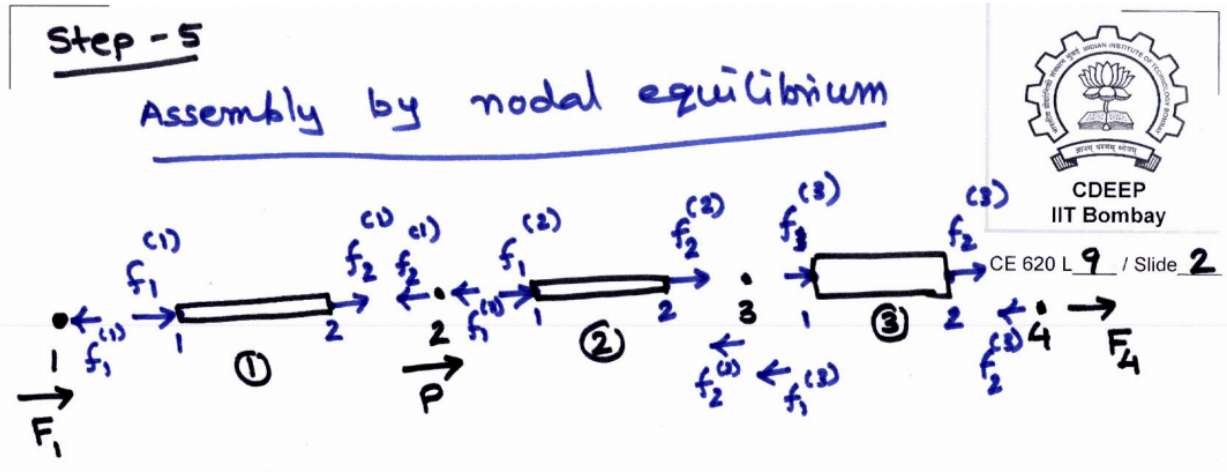


Fig. 3.2.4: (a) Heat transfer in a bar. (b) Axial deformation of a bar.



- Remember when we draw the internal force signs as compressive for a tensile element. This we are drawing for the nodal equilibrium side cut of the fbd. Rembember force, stresses are all defined with respect to the cut and face.
- In the Reddy example you will find that at the B.C., the internal force reaction at the node, Q_2^4 and kU_5 is a force in $\leftarrow$

- Check reddy 114 for origin at left side of element, natural $\bar{x}$
- Origin at center denoted as ξ
  - ξ is -1 and +1 at LHS and RHS. Since your current basis is eucledian. The transfromation is linear.
  - x can be given as a function of ξ and found out
  - This is interesting. Suppose $x = x_a + \frac{h}{2}(1+\xi)$

$$\frac{d}{dx} = \frac{2}{h}\frac{d}{d\xi} \qquad \frac{dN_i}{dx} = \frac{2}{h}\frac{dN_i}{d\xi}$$

$$dx = \frac{h}{2}d\xi$$

$$\int_{x_a^e}^{x_b^e} N_i(x)\,dx = \int_0^h N_i(\bar{x})\,d\bar{x} = \frac{h}{2}\int_{-1}^{1} N_i(\xi)\,d\xi \tag{40}$$

$$\int_{x_a^e}^{x_b^e} \frac{dN_i(x)}{x}\,dx = \int_0^h \frac{N_i(\bar{x})}{\bar{x}}\,d\bar{x} = \int_{-1}^{1} \frac{N_i(\xi)}{d\xi}\,d\xi$$

$$\int_{x_a^e}^{x_b^e} \frac{dN_i(x)}{x}\frac{dN_j(x)}{x}\,dx = \int_0^h \frac{N_i(\bar{x})}{\bar{x}}\frac{N_j(\bar{x})}{\bar{x}}\,d\bar{x} = \frac{2}{h}\int_{-1}^{1} \frac{N_i(\xi)}{d\xi}\frac{N_j(\xi)}{d\xi}\,d\xi$$

- Governing differential equation. Suppose a single field $u(x, y)$ with the following partial differential equation varies over x and y
- $\frac{\partial}{\partial x}\left(k_x \frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y \frac{\partial \phi}{\partial y}\right) + f(x, y) = 0$ where for a heat problem, k is the conucitvity in a orthotropic medium and u is the temperature and f is the internal heat generation.
- Mixed boundary conditions
  - Dritchlet : $\phi = \hat{\phi}$ in some boundary
  - General Newman :$k_x \frac{\partial u}{\partial x} n_x + k_y \frac{\partial u}{\partial y} n_y + q_c = \hat{q}_n$: In some portion of the boundary.
    - q_c represents the convective component of flux (heat problems $q_c = h_c(u - u_c)$)
    - $n_x = cos(x, \mathbf{n})$ and $n_y = cos(y, \mathbf{n})$ which is the angle of the normal of the boundary and the axis
    - u_c is the ambient temperature and h_c is the convective heat coefficient

- In FEM the domain is descritised into subdomains. Any shape qualifies as long as the approximating functions N_i^e can be derived uniquely for the shape. The discretisation may be may not represent the actual boundary though at really curved regions.
- Suppose the dependent unkown u is given by $\hat{u^e} = \sum_{j=1}^{n} u_j^e N_j^e(x, y)$
- The interpolation functions depend not only on the number of nodes but also on the shape of the element
- A triangle will need two points given by $\hat{u^e} = c_1 + c_2 x + c_3 y$
- A triangle with three nodes in each side is given by $\hat{u^e} = c_1 + c_2 x + c_3 y + c_4 xy + c_5 x^2 + c_6 y^2$

- The n nodal values u_j^e must be found such that the approximating solution $u_h^e(x)$ satisfies the governing differential equation in a weak sense. Steps are :

  1. Take non zeros of the G.D.E as R(x,y) and multiply by the weight function w_i^e from a set of linearly independant functions. We get then

$$\int \int_\Omega w_i^e \left(k \frac{\partial^2 u_h^e}{\partial x^2} + k \frac{\partial^2 u_h^e}{\partial y^2} - f(x,y) \right) dxdy = 0 \qquad (Strongstatement) \tag{41}$$

     For n independant choices of w_i^e, we get n independant equations.

  2. Distribute so that for both u_h^e, w_i^e are required to be differentiated once. Using component form of the divergence theorem or Greens theorem :

$$\begin{aligned} \int \int_A \frac{\partial}{\partial x} (w_i^e F_1) \, dA &= \int_S (w_i^e F_i) \, n_x \, ds \\ \int \int_A \frac{\partial}{\partial y} (w_i^e F_2) \, dA &= \int_S (w_i^e F_i) \, n_y \, ds \end{aligned} \tag{42}$$

$$F_1 = k_x \frac{\partial u_h^e}{\partial x} \qquad F_2 = k_y \frac{\partial u_h^e}{\partial y}$$

     And from product rule we get

$$-w_i^e \frac{\partial F_1}{\partial x} = -\frac{\partial}{\partial x}(w_i^e F_1) + F_1 \frac{\partial w_i^e}{\partial x} \qquad - w_i^e \frac{\partial F_2}{\partial x} = -\frac{\partial}{\partial x}(w_i^e F_2) + F_2 \frac{\partial w_i^e}{\partial x} \tag{43}$$

- And we get the weak form as
$$0 = \int\int_A \left(k_x \frac{\partial w_i^e}{\partial x} \frac{\partial u_h^e}{\partial x} + k_y \frac{\partial w_i^e}{\partial x} \frac{\partial u_h^e}{\partial y} - w_i^e f(x,y) \right) dA - \int_S w_i^e \left(k_x \frac{\partial u_h^e}{\partial x} n_x + k_y \frac{\partial u_h^e}{\partial x} n_y \right) dS$$
- Now the order of the differentiation has been reduced.
- Looking at the boundary terms, we see that u_h^e is the primary variable and essential boundary. $q_n = \left(k_x \frac{\partial u_h^e}{\partial x} n_x + k_y \frac{\partial u_h^e}{\partial x} n_y \right)$ is the secondary variable and the natural boundary condition. It is positive as one travels counterclockwise in the boundary.
- This is wy nodes are counted in counterclockwise and boundary integrals are carried in counter-clockwise sense. $\mathbf{q}$ is the outward flux normal and the flux
$$\mathbf{q} = q_x e_1 + q_y e_2 \qquad q_x = k_x \frac{\partial u_h^e}{\partial x} \quad q_y = k_y \frac{\partial u_h^e}{\partial y}$$
- The normal flux is given by
$$q_n = \hat{n}.\mathbf{q} = k_x \frac{\partial u_h^e}{\partial x} n_x + k_y \frac{\partial u_h^e}{\partial y} n_y \tag{44}$$

1. So the third step is to use the general Newmann boundary condition and write as

$$0 = \int \int_A \left(k_x \frac{\partial w_i^e}{\partial x} \frac{\partial u_h^e}{\partial x} + k_y \frac{\partial w_i^e}{\partial x} \frac{\partial u_h^e}{\partial y} - w_i^e f(x,y) \right) dA - \int_S w_i^e \left(\hat{q}_n - h_c (u_h^e - u_c) \right) dS \tag{45}$$

2. Rearragning we get

$$0 = \int \int_A \left(k_x \frac{\partial w_i^e}{\partial x} \frac{\partial u_h^e}{\partial x} + k_y \frac{\partial w_i^e}{\partial x} \frac{\partial u_h^e}{\partial y} - w_i^e f(x,y) \right) dA - \int_S h_c w_i^e u_h^e dS - \int_S w_i^e \left(\hat{q}_n + h_c u_c \right) \tag{46}$$

3. So we get the form $B(w_i^e, u_h^e) = l(w_i^e)$

- Note that the variational problem is to find a u_h^e such that $B(w_i^e, u_h^e) = l(w_i^e)$ for all $w_i^e \in U_h$ a subspace span by polynomial basis functions.
- $B(w_i^e, u_h^e)$ is bilinear and symmetric and $l(w_i^e)$ is linear in w. So we can construct a functional $I = \frac{1}{2}B(u_h^e, u_h^e) - l(u_h^e)$ and the minmum is equivalent to solving the variation problem.
- It is not always possible to make a functional whose weak form whose first variation is equivalent to the weak form.

- The weak form in the above equation requires that the approx function to be at least linear in both x and y. Suppose that $u_h^e = N_i u_i$
- The weak form is given as

$$\left(\int \int_A \frac{\partial N_j}{\partial x} k \frac{\partial \sum_{i=1}^{n} N_i}{\partial x} dA + \int \int_A \frac{\partial N_j}{\partial y} k \frac{\partial \sum_{i=1}^{n} N_i}{\partial y} dA + \int_S h_c w_i^e N_j^e \right) u_j^e = 0 \\ - \int_A N_i^e f dA - \int_S N_i^e (\hat{q} + h_c u_c) dS \tag{47}$$

- This is when w is taken as the virtual displacemnt of the dependant unkown or $w_i^e = \delta u_h^e$. Each w we get a seperate equation and we get
- $K^e u^e = f^e + q^e$

- u should be continous as required in the weak form that is all the terms are represented as non zero values
- The polynomials must be complete and contain the same order of x and y
- All the terms in the polynomial should be linearly independant. The no o flinearly independent terms in represnetin u dictate the shape and no of nodes. It turns out only triangular and quad elements satisfy this.

- The lowest order polynomial that we can come up with is

$$u_h^e(x, y) = c_1^e + c_2^e x + c_3^e y \tag{48}$$

- The set 1,x,y is also linearly independant forming the basis for the subspace of the H1 hilbert space
- Now we need to find out a geometry that we can also use the continuity cnditions. And we get a triangle
- Check reddy 123 for shape functions
- The shape functions are lagrange interpolation functions, with the sum =1
- If you mess up the aspect ratio, you will mess up the underlying physics. An intuition is as sometimes the stiffness can be decomposed to different matrixes that are dependant on its aspect ratio and the material properties.
- The boundary conditions q are not found for elements connected on all sides, but with nodes on the boundary, q_n^e is known and found by $q_i^e = \int_S \hat{q}_n N_i^e(s) dS$. The internal boundary forces are canceled, like the balance of internal forces.

- The next polynomial that meets the requirements on the approx solution is

$$u_h^e(x, y) = c_1^e + c_2^e x + c_3^e y + c_4 xy \tag{49}$$

- Th geometry is a quad element with a linear variation along two points in the element. We usually use isoparametric elements to represent the element.
- Also node is named counterclockwise.
- Reddy 126 for shape functions. Again we see that the aspect ration mess up the stiffness matrix.

- For triangular elements, we can construct natural coordinates L_i
- $N_i = L_i = \frac{A_i}{A}$
- $\mathbf{N} = \begin{bmatrix} L_1(2L_1 - 1) \\ L_2(2L_2 - 1) \\ L_3(2L_3 - 1) \\ 4L_1L_2 \\ 4L_2L_3 \\ 4L_3L_1 \end{bmatrix}$
- Formula to find the area also given using fromula (Reddy 128)

- By multiplying from single linear we get four node
- Tensor product two quadratic one d we get quadratic quad element.
- Serendipity have no interior node. Not complete. They don't have the dual^2 term.

- Continuity of primary variable : Same nodes from different elements
- Equilibrium of secondary variables : At the interface between two elements, the flux or internal force fro the two elements is equal and opposite in sign.

- The same second order equation in the polar coordinates are given as

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rk_{rr}\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \theta}\left(k_{\theta\theta}\frac{\partial u}{\partial \theta}\right)+\frac{\partial}{\partial z}\left(k_{zz}\frac{\partial u}{\partial z}\right)+f(r,\theta,z)=0 \tag{50}$$

- We can remove the terms of z and θ as if the problem is independant of these parameters.
- Reddy Page 137, One D and two D equations

- We use numerical integration. We use some parametric form to also do the integration on a square

- We do it on a square region dimension (2x2) with respect to (ξ, η) in domain -1< <1.
- The element of the fem mesh is transformed, only for the purpose of numerical integration.
- We use a coordinate transformation of the form

$$x = \sum_{j=1}^{m} x_j^e N_j^e(\xi, \eta) \qquad y = \sum_{j=1}^{m} y_j^e N_j^e(\xi, \eta) \tag{51}$$

- The shape functions are of the master element in the -1 to 1 coordinate.
- The master element is transformed in the linear transfmration to the quadrilateral element.
- The dependant variable is also approximated by the same

$$u_h^e = \sum_{j=1}^{n} u_j^e \phi_j^e(\xi, \eta) \tag{52}$$

- Superparametric : If m > n. Geometry approx is higher
- Isoparametric : If m = n
- Subparametric : If m < n. If dependant variable is higher.

Transfomration of the quad to the master is for numerical evaluation. The final equations are alwayas in terms of the nodal values of the physical domain.

1. Different elements can be generated from the same master element. A master elemeent can have differnet order.
2. A quad master can genreate quad curvilinear elements.
3. But elements should not overlap each other.

- Consider the stiffness matrix

$$K_{ji}^{e} = \int \int_{A} \frac{\partial N_j}{\partial x} k_x \frac{\partial N_i}{\partial x} dA + \int \int_{A} \frac{\partial N_j}{\partial y} k_y \frac{\partial N_i}{\partial y} dA \tag{53}$$

- Now this is in the global coordinates and we want to write it in terms of ξ and η.
- We can write using chain rule

$$\begin{bmatrix} \frac{\partial N_i^e}{\partial \xi} \\ \frac{\partial N_i^e}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{bmatrix} \frac{\partial N_i^e}{\partial x} \\ \frac{\partial N_i^e}{\partial y} \end{bmatrix} \tag{54}$$

  With the inner matrix the Jacobian $\mathbf{J}$, whose determinant should be >0

- We can then find the inverese transformation for the derivatives as $\mathbf{N}_{,(\mathbf{x},\mathbf{y})} = \mathbf{J}^{-1}\mathbf{N}_{,(\xi,\eta)}$
- The geometry can also be easily differentiated

$$\frac{\partial x}{\partial \xi} = \sum_{j=1}^{m} x_j^e \frac{\partial N_j}{\partial \xi} \quad \text{and so on} \tag{55}$$

- Obviously we also get $dA = det(J)\xi\eta$
- When all or some of the variables are approximated using hermite interpolating functions, linear approx of geometry and so subparametric or isoparametric forms are adopted.
- Full stiffness matrix, we keep the terms in intergration in the master coodrinate. See Page 147
- Integration over a master element can also be looked at page 148 and 149. Itis basically $\int_{-1}^{1}\int_{-1}^{1} F(\xi,\eta)d\xi d\eta = \sum_{I=1}^{M}\int_{J=1}^{N} F(\xi_I,\eta_J)W_iW_j$

One D problem : Single variable

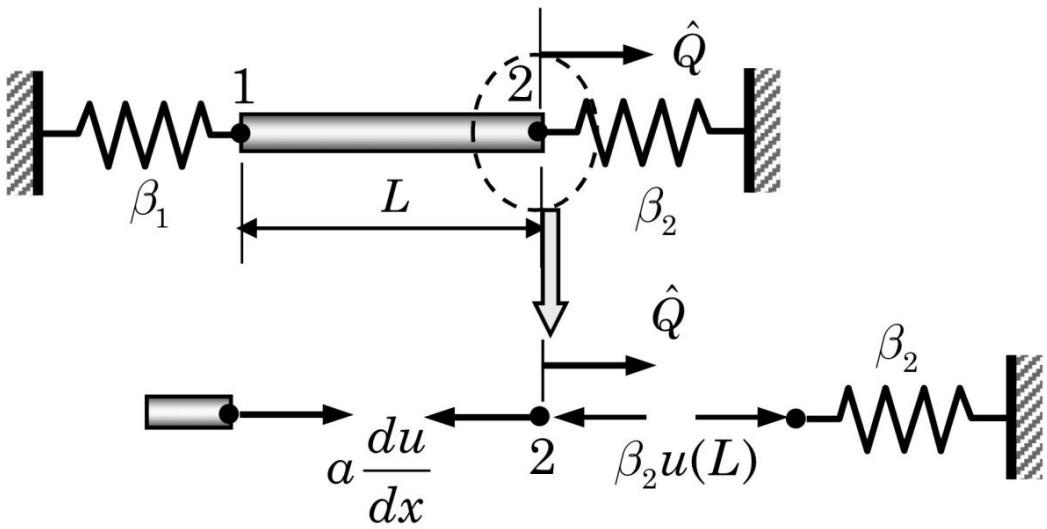
- A(u(x)) = f(x) in interval 0 <x<L B(u) = g
- Consider the differential equation

$$-\frac{d}{dx}\left(k(x,u)\frac{du}{dx}\right)+b(x,u)\frac{du}{dx}+c(x,u)u=f(x) \quad 0<x<L$$

Boundary conditions (56)

$$n_x k\frac{du}{dx}+\beta(x,u)(u-u_\infty)=\hat{Q} \quad or \quad u=\hat{u}$$

- Note that $n_x=-1, \beta=\beta_1$ at $x=x_a$ and $n_x=1, \beta=\beta_2$ at $x=x_b$
- For a bar with a spring, $u_\infty=0$ and we get the equation that the bar should be equal to the spring force βu. Or $-k\frac{du}{dx}-\beta_2 u=Q_2$ where Q2 is the extrenal force.



- Therefore we can keep it generally and nicely as:

$$A(u) = f \quad in \quad 0 < x < L \qquad B(u) = g \quad at \quad x = 0 or L$$
$$A = -\frac{d}{dx}\left(a\frac{d}{dx}\right) + \frac{d}{dx} + c. \quad B = n_x a\frac{d}{dx} + \beta, \quad g = \beta u_\infty + \hat{Q} \tag{57}$$

- If a,b,c are functions of u then A and B become nonlienear.
- In heat a = kA, b = 0 and c = perimeter .β

4.2 Weak Formulation

Suppose that the domain $\Omega = (0, L)$ is divided into N line elements. A typical element from the collection of N elements is denoted as $\Omega^e = (x_a, x_b)$, where x_a and x_b denote the global coordinates of the end nodes of the line element. The weak form of Eq. (4.1.1) over the element can be developed as follows (see Section 3.2 for details):

$$\begin{aligned} 0 &= \int_{x_a}^{x_b} \left(a\frac{dw_i^e}{dx}\frac{du_h^e}{dx} + bw_i^e\frac{du_h^e}{dx} + cw_i^e u_h^e - w_i^e f \right) dx - \left[w_i^e \left(a\frac{du_h^e}{dx} \right) \right]_{x_a}^{x_b} \\ &= \int_{x_a}^{x_b} \left[a(x,u)\frac{dw_i^e}{dx}\frac{du_h^e}{dx} + b(x,u)w_i^e\frac{du_h^e}{dx} + c(x,u)w_i^e u_h^e - w_i^e f(x) \right] dx \\ &\quad - \left\{ Q_a^e - \beta_a \left[u_h^e(x_a) - u_\infty^a \right] \right\} w_i^e(x_a) - \left\{ Q_b^e - \beta_b \left[u_h^e(x_b) - u_\infty^b \right] \right\} w_i^e(x_b) \end{aligned} \tag{4.2.1}$$

where $w_i^e(x)$ is the ith weight function. The number of weight functions is equal to the number of unknowns in the approximation of u_h. The first line of Eq. (4.2.1) suggests that u is the primary variable and $Q = a(du/dx)$ is the secondary variable of the formulation. Using the mixed boundary condition in Eq. (4.1.2), we can express $a(du/dx)$ in terms of (Q_a^e, Q_b^e) and (u_∞^a, u_∞^b) as

$$\begin{aligned} -\left[a\frac{du_h^e}{dx} \right]_{x=x_a} &= Q_a^e - \beta_a \left[u_h^e(x_a) - u_\infty^a \right] \\ \left[a\frac{du_h^e}{dx} \right]_{x=x_b} &= Q_b^e - \beta_b \left[u_h^e(x_b) - u_\infty^b \right] \end{aligned} \tag{4.2.2}$$

where (Q_a^e, Q_b^e) are the nodal values, (u_∞^a, u_∞^b) denote the values of the variable u_∞, and (β_a, β_b) denote certain physical parameters (e.g. film conductances) at the left and right ends of the element, respectively. When a node is in the

- If we use the discretisation then we will get

$$K(U)U = F \tag{58}$$

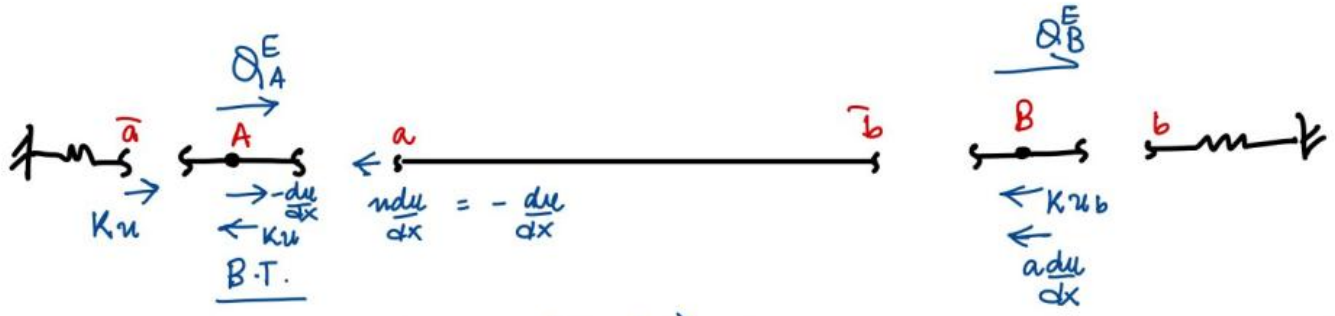
where

$$K_{ij}^e = \int_{x_a}^{x_b} \left[a(x, u_h^e) \frac{d\psi_i^e}{dx} \frac{d\psi_j^e}{dx} + b(x, u_h^e)\psi_i^e \frac{d\psi_j^e}{dx} + c(x, u_h^e)\psi_i^e \psi_j^e \right] dx + \beta_a \psi_i^e(x_a)\psi_j^e(x_a) + \beta_b \psi_i^e(x_b)\psi_j^e(x_b) \tag{4.3.3}$$

$$F_i^e = \int_{x_a}^{x_b} f(x)\psi_i^e \, dx + \beta_a u_\infty^a \psi_i^e(x_a) + \beta_b u_\infty^b \psi_i^e(x_b) + Q_a \psi_i^e(x_a) + Q_b \psi_i^e(x_b)$$

Note that the coefficient matrix $\mathbf{K}^e$ is a function of the unknown nodal values u_j^e, and it is an unsymmetric matrix when $b \neq 0$; when $b = 0$, $\mathbf{K}^e$ is a symmetric matrix. The term involving c is symmetric, independent of whether it depends on u and/or du/dx. Therefore, it is advisable to include nonlinear terms of the type $u\,(du/dx)$ in a differential equation as the c-term in the equation by writing it as $u\,(du/dx) = cu$, with $c = du/dx$; otherwise, it will be unsymmetric and convergence of the solution may become a problem. The coefficients involving β in $\mathbf{K}^e$ and $\mathbf{F}^e$ should be included only in elements that have end nodes with the convection type boundary condition. Example 4.3.1 provides more insight into the make-up of the coefficient matrix $\mathbf{K}^e$.

- The Boundary terms still confuses me. Especially the sign part. So here it is! Qe is the external force. F is the nodal force with equivalent parts. We actually get the external force from the internal Boundary terms!



- Suppose $\frac{du}{dx} = +ve$ (Tension) so

At a

Internal force $= n a \frac{du}{dx} = -a\frac{du}{dx}$

At $\bar{a}$ Internal force $= Ku$

At A: $Q_A^E - Ku = -a\frac{du}{dx}$

At B: $Q_B^E - Ku = a\frac{du}{dx}$.

$$\left[w_i^e \left(a \frac{du_h^e}{dx} \right) \right]_{x_a}^{x_b}$$

$$= w_1^e(x_b) \left[a \frac{du_h^e(x_b)}{dx} \right]$$

$$+ w_1^e(x_a) \left[-a \frac{du_h^e(x_a)}{dx} \right]$$

$$= \left(Q_A^e - Ku_a \right) + \left(Q_A^B - Ku_b \right)$$

- Check problem from Reddy for nonlinear constraints etc.

- Direct iteration procedure
- Newton rhapson method

- We solve this system of equations using direct iteration, Picard iteration or method of successive substitutions

$$\mathbf{K}(\mathbf{U}^{(\mathbf{r-1})})\mathbf{U}^{\mathbf{r}} = \mathbf{F}(\mathbf{U}^{(\mathbf{r-1})}) \tag{59}$$

Box 4.4.1: Steps involved in the direct iteration scheme.

1. **Initial solution vector.** Assume an initial solution vector $\mathbf{U}^{(0)}$ such that it (a) satisfies the specified boundary conditions on $\mathbf{U}$ and (b) does not make $\mathbf{K}^e$ singular.
2. **Computation of K and F.** Use the latest known vector $\mathbf{U}^{(r-1)}$ ($\mathbf{U}^{(0)}$ during the first iteration) to evaluate $\mathbf{K}^e$ and $\mathbf{F}^e$, assemble them to obtain global $\mathbf{K}$ and $\mathbf{F}$, and apply the specified boundary conditions on the assembled system.
3. **Computation of $\mathbf{U}^{(r)}$.** Compute the solution at the rth iteration

$$\mathbf{U}^{(r)} = [\mathbf{K}(\mathbf{U}^{(r-1)})]^{-1}\mathbf{F}^{(r-1)}$$

4. **Convergence check.** Compute the residual

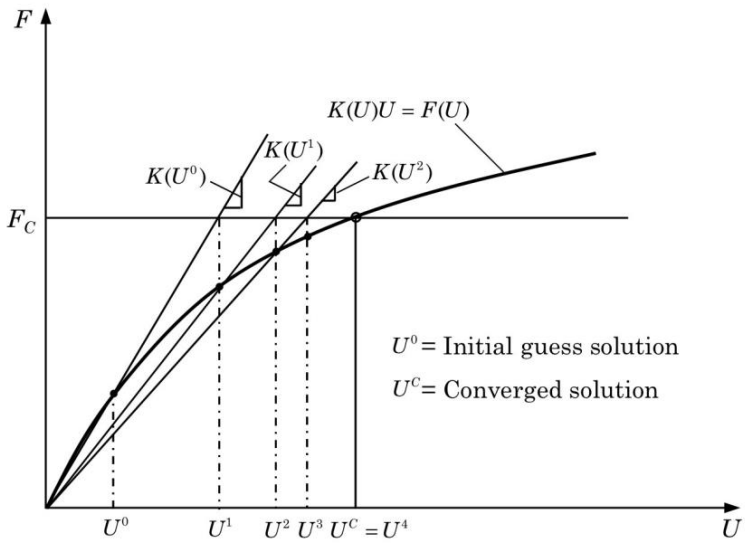
$$\mathbf{R}^{(r)} = \mathbf{K}(\mathbf{U}^{(r)})\mathbf{U}^{(r)} - \mathbf{F}^{(r)}$$

with the latest known solution and check if

$$\|\mathbf{R}^{(r)}\| \leq \epsilon \|\mathbf{F}^{(r)}\| \quad \text{or} \quad \|\mathbf{U}^{(r)} - \mathbf{U}^{(r-1)}\| \leq \epsilon \|\mathbf{U}^{(r)}\|$$

where $\|\cdot\|$ denotes the euclidean norm and ϵ is the convergence tolerance (read as an input). If the solution has converged, print the solution and move to the next "load" level or quit if it is the only or final load; otherwise, continue.

5. **Maximum iteration check.** Check if $r < itmax$, where $itmax$ is the maximum number of iterations allowed (read as an input). If *yes*, set $r \rightarrow r + 1$ and go to Step 2; if *no*, print a message that the iteration scheme did not converge and quit.



- The method can be accelerated by using a weighted average of solutions from the last two iterations

$$\begin{aligned} \mathbf{U}^{\mathbf{r}} &= \mathbf{K}(\mathbf{\breve{U}})^{-1}\mathbf{F}(\mathbf{\breve{U}}) \\ \mathbf{\breve{U}} &= \beta\mathbf{U}^{(\mathbf{r-2})} + (\mathbf{1}-\beta)\mathbf{U}^{(\mathbf{r-1})} \end{aligned} \tag{60}$$

- Check Reddy 184 for some pages

- In NM we expand he residual vector $\mathbf{R^{(r)}}$ in Taylor series about a kown solution $\mathbf{U^{(r-1)}}$ to get

$$\mathbf{R^r} = \mathbf{R^{(r-1)}} + \left(\frac{\partial \mathbf{R}}{\partial \mathbf{U}}\right)^{\mathbf{(r-1)}} \mathbf{'U} + \mathbf{O(h^2)} \tag{61}$$

  where $\Delta U = U^r - U^{(r-1)}$

- And saying that R in the next iteration should be zero we get

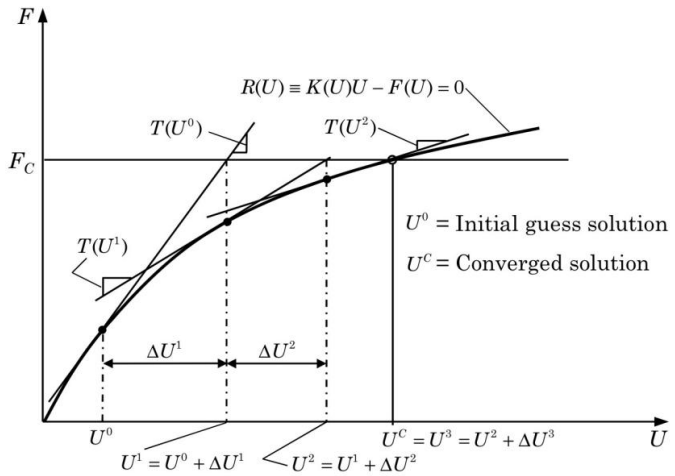
$$\left(\frac{\partial R}{\partial U}\right)^{(r-1)} \Delta U = -R^{(r-1)}$$
$$T^{(r-1)}\, U^r = -R^{(r-1)} + T^{(r-1)}\, U^{r-1} \tag{62}$$

  where T is the tangent matrix can be found at element level given as

$$\left(T_{IJ} = \frac{\partial R_I}{\partial U_J} = K_{IJ} + \sum_{m=1}^{N} \left(\frac{\partial K_{Im}}{\partial U_j} U_m\right) - \frac{\partial F_I}{\partial U_J}\right)^e \tag{63}$$

  The force derivative is zero if it is not a function of the load

1. **Initial solution vector.** Assume an initial solution vector $\mathbf{U}^{(0)}$ such that: (a) it satisfies the specified boundary conditions on $\mathbf{U}$ and (b) it does not make $\mathbf{T}^e$ singular.
2. **Computation of T and R.** Use the latest known vector $\mathbf{U}^{(r-1)}$ ($\mathbf{U}^{(0)}$ during the first iteration) to: (a) evaluate $\mathbf{K}^e$, $\mathbf{F}^e$, $\mathbf{T}^e$, and $-\mathbf{R}^e = \mathbf{F}^e - \mathbf{K}^e\mathbf{U}^e$, (b) assemble $\mathbf{T}^e$ and $\mathbf{R}^e$ to obtain global $\mathbf{T}$ and $\mathbf{R}$, and (c) apply the specified *homogeneous* boundary conditions (since $\mathbf{U}^{(0)}$ already satisfies the actual boundary conditions) on the assembled system, $\mathbf{TU} = -\mathbf{R}$.
3. **Computation of $\mathbf{U}^{(r)}$.** Compute the solution increment at the rth iteration
$$\Delta\mathbf{U} = -[\mathbf{T}(\mathbf{U}^{(r-1)})]^{-1}\mathbf{R}^{(r-1)}$$
and update the total solution
$$\mathbf{U}^{(r)} = \mathbf{U}^{(r-1)} + \Delta\mathbf{U}$$
4. **Convergence check.** Compute the residual
$$\mathbf{R}^{(r)} = \mathbf{K}(\mathbf{U}^{(r)})\mathbf{U}^{(r)} - \mathbf{F}^{(r)}$$
and check if
$$\|\mathbf{R}^{(r)}\| \leq \epsilon\|\mathbf{F}^{(r)}\| \quad \text{or} \quad \|\Delta\mathbf{U}\| \leq \epsilon\|\mathbf{U}^{(r)}\|$$
where ϵ is the convergence tolerance (read as an input). If the solution has converged, print the solution and move to the next "load" level or quit. Otherwise, continue.
5. **Maximum iteration check.** Check if $r < itmax$, where $itmax$ is the maximum number of iterations allowed (read as an input). If *yes*, set $r \to r + 1$ and go to Step 2; if *no*, print a message that the iteration scheme did not converge and quit.



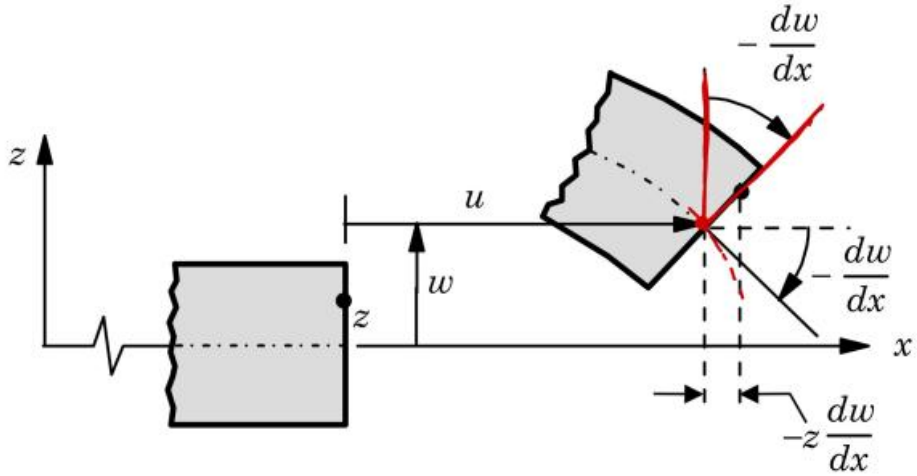
- In the direct iteration method, the actual bc are applied at each iteration. In NR we find the increment to the known solution. If previous displ satisfies, then the increment should be zero and satisfy the boundary condition.
- The symmetrry of K and T depends on the weak form. even if K is symmetric, T may not be symmetric.
- T can be approximate, and convergance is only when the residual is small. If it is only updated once then it is called the modified Netwons method.
- See the problem in Reddy -188 . Finding T

Nonlinear Bending of straight beams

- Assuming he geomery does no change significanty, allows he principle of virtual work o be written over the underformed body. So stress is force per unit undeforemed are, strain measure of change in length w.r.t original length and shear as change in length from $\pi/2$. No distincion between Piola kirchoff and cauchy stress
- Nonlinearity comes solely from inplane forces proportional to he square of the rotation of a transverse normal line in the beam
- There are two theories
  - Euler bernoulli beam (EBT)
  - Timoshenko beam (TBT)

Assumptions:

- Plane sections perpendicular to he axis of the beam remain (a) plane (b) rigid (not deform) (c) rotate such that they remain plane to the deformed axis after deformation
- Assumptions amount to neglection poissons effect and transverse normal and shear strains



■

The displacement is given as : $u_1 = u(x) - z\frac{dw}{dx} \qquad u_2 = 0 \qquad u_3 = w(x)$
u_1, u_2, u_3 is the disp of any point while u, w are the disp of the centroidal axis

- So the total disp can be found from the neutral axis values.

- The greens theorem is

$$E_{ij} = \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}$$
$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1} + \frac{1}{2}\left[\left(\frac{\partial u_1}{\partial x_1}\right)^2 + \left(\frac{\partial u_3}{\partial x_1}\right)^2\right] \tag{64}$$

- Now the axial strains of higher orders we can igore, bu the rotation of the line perpendicular to the beam is pretty large so we have to retain it.
- Non-linear strains where only squares of the rotations are included are known as von Karman nonlinearity
- The zero strains are $\varepsilon_{33} = \varepsilon_{13} = 0$ so only non zero strain is

$$\varepsilon_{11} = \frac{du}{dx} - z\frac{d^2w}{dx^2} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 \tag{65}$$

- So we have two longitudinal von karman strains:

$$\varepsilon^0_{11} = \frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 \quad , \quad \varepsilon^1_{11} = -z\frac{d^2w}{dx^2} \tag{66}$$

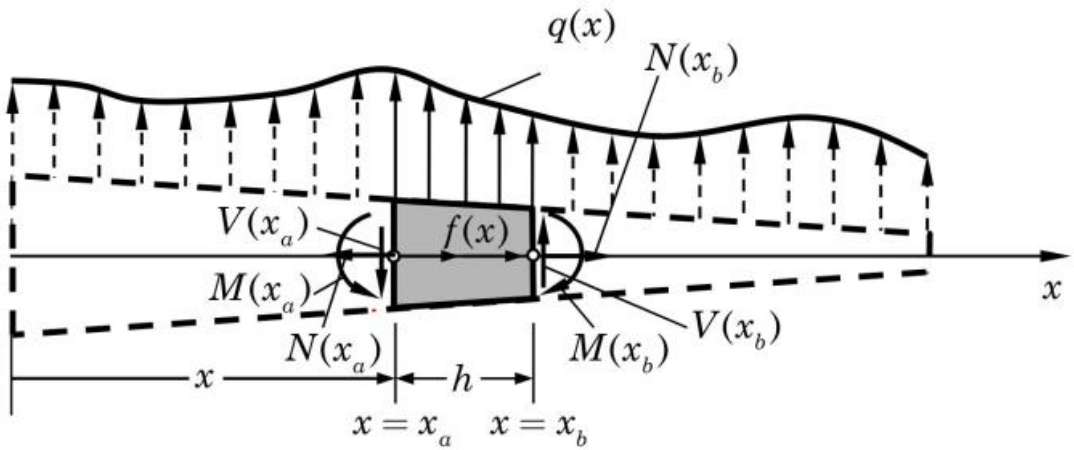
- The weak form can be found withou knowing the governing d.e
- If a body is in equilibrium, the toal virtual work done in moving through the repsective displacements is zero

$$\delta W^e = \delta W_I^e + \delta W_E^e \tag{67}$$

δW_I^e denotes the virtual strain stored due to the actual cauchy or second piolay stresses (Same geom) σ_{ij}
δW_E^e is the work done by external applied loads

- So we have

$$\delta W_I^e = \int_V^e \delta\varepsilon_{xx}\sigma_{xx}dV = \int_{x_a}^{x_b}\int_A^e \left(\frac{d\delta u}{dx} + \frac{dw}{dx}\frac{d\delta w}{dx} - z\frac{d^2\delta w}{dx^2}\right) dAdx$$
$$\delta W_E^e = -\left[\int_{x_a}^{x_b} q\delta w dx + \int_{x_a}^{x_b} f\delta u dx + \sum_{i=1}^{6} Q_i^e \delta\Delta_i^e\right] \tag{68}$$

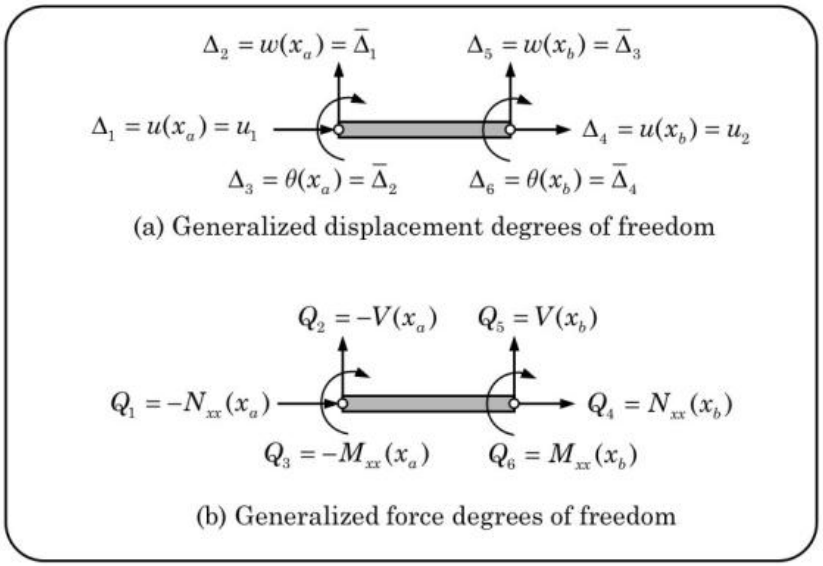


$$Q_1 = -N(x_a),\quad Q_2 = -V(x_a),\quad Q_3 = -M(x_a)$$
$$Q_4 = N(x_b),\quad Q_5 = V(x_b),\quad Q_6 = M(x_b)$$

Fig. 5.2.2: A typical beam element, $\Omega^e = (x_a, x_b)$.

$$\Delta_1^e = u(x_a), \quad \Delta_2^e = w(x_a), \quad \Delta_3^e = \left[-\frac{dw}{dx}\right]_{x_a} \equiv \theta(x_a)$$
$$\Delta_4^e = u(x_b), \quad \Delta_5^e = w(x_b), \quad \Delta_6^e = \left[-\frac{dw}{dx}\right]_{x_b} \equiv \theta(x_b) \tag{5.2.6}$$

$$Q_1^e = -N_{xx}(x_a), \quad Q_4^e = N_{xx}(x_b), \quad Q_2^e = -\left[\frac{dw}{dx}N_{xx} + \frac{dM_{xx}}{dx}\right]_{x_a} \equiv -V_x(x_a)$$
$$Q_5^e = \left[\frac{dw}{dx}N_{xx} + \frac{dM_{xx}}{dx}\right]_{x_b} \equiv V_x(x_b), \quad Q_3^e = -M_{xx}(x_a), \quad Q_6^e = M_{xx}(x_b) \tag{5.2.7}$$



- The main thing is that for V_x, the component of N_x is $sin(\frac{dw}{dx}) \approx \frac{dw}{dx}$, and also in moment equilibirum we get $N_{xx}\frac{dw}{dx}\Delta x = N_{xx}\Delta w$

- The free body diagram is what it is. But we can take our generalised displacement and force dof signs based on what we want.
- Generalised is used to show that rotationsa and moments are treated as displacements and forces
- By integrating in the area we get the virtual work as

$$\delta W_I^e = \int_{x_a}^{x_b} \left[\left(\frac{d\delta u}{dx} + \frac{dw}{dx}\frac{d\delta w}{dx} \right) N_{xx} - \frac{d^2\delta w}{dx^2} M_{xx} \right] - \int_{x_a}^{x_b} \delta w q dx - \int_{x_a}^{x_b} \delta u f dx - \sum_{i=1}^{6} \delta\Delta_i^e Q_i^e \tag{69}$$

where $N_{xx} = \int_A \sigma_{xx} dA \qquad M_{xx} = \int_A \sigma_{xx} z dA$

By taking the individual virtual terms, as the coefficients have to seperately equal to zero we get

$$\int_{x_a}^{x_b} \left(\frac{d\delta u}{dx} N_{xx} - \delta u f \right) dx - \delta \Delta_1^e Q_1^e - \delta \Delta_4^e Q_4^e = 0$$

$$\int_{x_a}^{x_b} \left(\frac{d\delta w}{dx} \left(\frac{dw}{dx} N_{xx} \right) - \frac{d^2 \delta w}{dx^2} M_{xx} - \delta w q \right) dx - \delta \Delta_2^e Q_2^e - \delta \Delta_3^e Q_3^e - \delta \Delta_5^e Q_5^e - \delta \Delta_6^e Q_6^e = 0$$

- By integration by parts we get the euler lagrange equilibrium equations : given as

$$\int_{x_a}^{x_b} \delta u \left(-\frac{d\delta N_{xx}}{dx} - f \right) dx - \delta \Delta_1^e Q_1^e + [\delta u N_{xx}]_{x_b}^{x_a} - \delta \Delta_4^e Q_4^e = 0$$

$$\int_{x_a}^{x_b} -\delta w \left(\frac{d}{dx} \left(\frac{dw}{dx} N_{xx} \right) + \frac{d^2 M_{xx}}{dx^2} + q \right) dx - \left[\frac{d\delta w}{dx} M_{xx} \right]_{x_b}^{x_a} \tag{70}$$

$$+ \left[\delta w \left(\frac{dw}{dx} N_{xx} + \frac{dM_{xx}}{dx} \right) \right]_{x_b}^{x_a} - \delta \Delta_2^e Q_2^e - \delta \Delta_3^e Q_3^e - \delta \Delta_5^e Q_5^e - \delta \Delta_6^e Q_6^e = 0$$

- We get our euler equations from coefficients of the two variations

$$\begin{aligned} -\frac{dN_{xx}}{dx} &= f(x) \\ -\frac{d}{dx}\left(\frac{dw}{dx}N_{xx}\right) - \frac{d^2M_{xx}}{dx^2} &= q(x) \end{aligned} \tag{71}$$

- We get our boundary conditions as follows :

$$\begin{aligned} Q_1^e &= -N_{xx}(x_a) & Q_4^e &= N_{xx}(x_b) \\ Q_2^e &= -\left[\frac{dw}{dx}N_{xx} + \frac{dM_{xx}}{dx}\right]_{x_a} & Q_5^e &= \left[\frac{dw}{dx}N_{xx} + \frac{dM_{xx}}{dx}\right]_{x_b} \\ Q_3^e &= M_{xx}(x_a) & Q_6^e &= M_{xx}(x_b) \end{aligned} \tag{72}$$

Again the - sign makes all the Qs positive in the equation

- We can also find the differential equations by writing the equilibirum equaitons (Fx,Fy,Mz = 0) for a small element and then take $\Delta x \longrightarrow 0$. But we don't get the boundary conditions by using this.
- After getting the euler lagrange differnetial equations, we can again use the weighted residual method to find the stiffness forms

$$
\begin{aligned}
0 &= \int_{x_a}^{x_b} v_1 \left(-\frac{dN_{xx}}{dx} - f \right) dx = \int_{x_a}^{x_b} \left(\frac{dv_1}{dx} N_{xx} - v_1 f \right) dx - [v_1 N_{xx}]_{x_a}^{x_b} \\
&= \int_{x_a}^{x_b} \left(\frac{dv_1}{dx} N_{xx} - v_1 f \right) dx - v_1(x_a)[-N_{xx}(x_a)] - v_1(x_b) N_{xx}(x_b) \qquad (5.2.16) \\
0 &= \int_{x_a}^{x_b} v_2 \left[-\frac{d}{dx} \left(\frac{dw}{dx} N_{xx} \right) - \frac{d^2 M_{xx}}{dx^2} - q \right] dx \\
&= \int_{x_a}^{x_b} \left[\frac{dv_2}{dx} \left(\frac{dw}{dx} N_{xx} \right) - \frac{d^2 v_2}{dx^2} M_{xx} - v_2 q \right] dx \\
&\quad - \left[v_2 \left(\frac{dw}{dx} N_{xx} + \frac{dM_{xx}}{dx} \right) \right]_{x_a}^{x_b} - \left[\left(-\frac{dv_2}{dx} \right) M_{xx} \right]_{x_a}^{x_b} \\
&= \int_{x_a}^{x_b} \left[\frac{dv_2}{dx} \left(\frac{dw}{dx} N_{xx} \right) - \frac{d^2 v_2}{dx^2} M_{xx} - v_2 q \right] dx \\
&\quad - v_2(x_a) \left[- \left(\frac{dw}{dx} N_{xx} + \frac{dM_{xx}}{dx} \right) \right]_{x_a} - v_2(x_b) \left[\frac{dw}{dx} N_{xx} + \frac{dM_{xx}}{dx} \right]_{x_b} \\
&\quad - \left[-\frac{dv_2}{dx} \right]_{x_a} [-M_{xx}(x_a)] - \left[-\frac{dv_2}{dx} \right]_{x_b} M_{xx}(x_b) \qquad (5.2.17)
\end{aligned}
$$

- We clearly see that the weights are nothing but the variations and the external load work has been replaced by the internal force equivalents!!!!

- Now the funny thing is that we kept as N and M and now we shall replace them by

$$N_{xx} = \int_A \sigma_{xx} dA = \int_{A^e} E^e \left[\frac{du}{dx} + \frac{1}{2}(\frac{dw}{dx})^2 - z\frac{d^2w}{dx^2} \right] dA$$
$$= A^e \left[\frac{du}{dx} + \frac{1}{2}(\frac{dw}{dx})^2 \right] - B^e z \frac{d^2w}{dx^2} \tag{73}$$
$$M_{xx} = \int_{A^e} \sigma_{xx} z dA = B^e \left[\frac{du}{dx} + \frac{1}{2}(\frac{dw}{dx})^2 \right] - D^e \frac{d^2w}{dx^2}$$

where $A = EA, \qquad B = 0 \left(\int zdA = 0 \right), \qquad D = EI \left(\int z^2 dA \right)$

- Therefore the virtual work equations can be written as

$$0 = \int_{x_a^e}^{x_b^e} \left(A\frac{d\delta u}{x} \left[\frac{du}{dx} + \frac{1}{2} \left(\frac{dw}{dx} \right)^2 \right] - f\delta u \right) dx - \delta u(x_a) Q_1 - \delta u(x_b) Q_4$$
$$0 = \int_{x_a^e}^{x_b^e} \left(\frac{d\delta w}{dx} \frac{dw}{dx} A \left(\frac{du}{dx} + \frac{1}{2} \left(\frac{dw}{dx} \right)^2 \right) + D\frac{d^2\delta w}{dx^2} \frac{d^2w}{dx^2} - q\delta w \right) dx \tag{74}$$
$$-\delta w(x_a) Q_2 - \delta\theta(x_a) Q_3 - \delta w(x_b) Q_5 - \delta\theta(x_b) Q_6$$

- We take the rotation as $\theta = -\frac{dw}{dx}$
- We approximate the axial disp as linear lagrange and the transverse with hermite cubic interpolation functions

$$0 = \sum_{j=1}^{2} K_{ij}^{11} u_j + \sum_{J=1}^{4} K_{iJ}^{12} \bar{\Delta}_J - F_i^1 \quad (i = 1, 2) \quad \text{- Axial} \qquad u = [u_1 \; u_2]^T$$

$$0 = \sum_{j=1}^{2} K_{Ij}^{21} u_j + \sum_{J=1}^{4} K_{IJ}^{22} \bar{\Delta}_J - F_I^2 \quad (I = 1, 2, 3, 4) \quad \text{- Bending} \qquad \Delta = [w_1 \; \theta_1 \; w_2 \; \theta_2] \tag{5.2.30}$$

where, for the case in which $B_{xx} = 0$, we have

$$K_{ij}^{11} = \int_{x_a}^{x_b} A_{xx} \frac{d\psi_i}{dx} \frac{d\psi_j}{dx} \, dx \,, \qquad K_{iJ}^{12} = \frac{1}{2} \int_{x_a}^{x_b} \left(A_{xx} \frac{dw}{dx} \right) \frac{d\psi_i}{dx} \frac{d\varphi_J}{dx} \, dx$$

$$K_{Ij}^{21} = \int_{x_a}^{x_b} A_{xx} \frac{dw}{dx} \frac{d\varphi_I}{dx} \frac{d\psi_j}{dx} \, dx \,, \quad K_{Ij}^{21} = 2K_{jI}^{12}$$

$$K_{IJ}^{22} = \int_{x_a}^{x_b} D_{xx} \frac{d^2\varphi_I}{dx^2} \frac{d^2\varphi_J}{dx^2} \, dx + \frac{1}{2} \int_{x_a}^{x_b} \left[A_{xx} \left(\frac{dw}{dx} \right)^2 \right] \frac{d\varphi_I}{dx} \frac{d\varphi_J}{dx} dx$$

$$F_i^1 = \int_{x_a}^{x_b} f \, \psi_i \, dx + \hat{Q}_i \,, \qquad F_I^2 = \int_{x_a}^{x_b} q \, \varphi_I \, dx + \bar{Q}_I \tag{5.2.31}$$

for $(i, j = 1, 2)$ and $(I, J = 1, 2, 3, 4)$, where $\hat{Q}_1 = Q_1$, $\hat{Q}_2 = Q_4$, $\bar{Q}_1 = Q_2$, $\bar{Q}_2 = Q_3$, $\bar{Q}_3 = Q_5$, and $\bar{Q}_4 = Q_6$, and the definition of Q_i is given Eq. (5.2.15). The pair of equations in Eq. (5.2.30) can be expressed in matrix form as

$$\begin{bmatrix} \mathbf{K}^{11} & \mathbf{K}^{12} \\ \mathbf{K}^{21} & \mathbf{K}^{22} \end{bmatrix} \begin{Bmatrix} \boldsymbol{\Delta}^1 \\ \boldsymbol{\Delta}^2 \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}^1 \\ \mathbf{F}^2 \end{Bmatrix} \quad \text{or} \quad \mathbf{K}^e \boldsymbol{\Delta}^e = \mathbf{F}^e \tag{5.2.32}$$

where

$$\Delta_i^1 = u_i, \quad i = 1, 2; \quad \Delta_I^2 = \bar{\Delta}_I, \quad I = 1, 2, 3, 4 \tag{5.2.33}$$

We also note that $(\mathbf{K}^{12})^T \neq \mathbf{K}^{21}$.

- Note that the Stiffness is not symmetric. When nonlinearity is not there then 12 and 21 (which depend on w) become 0 and the system is uncoupled. That is axial is only dependant on u and bending only dependant on Δ.
- See that we have tried to keep the shape functions symmetric
- When we find these conefficients from the previous iteration and then we say that the equations are linearised
- The coefficients can be different if we had accounted for the terms differently, and uncouple then to a system of equations . Read reddy page 223 for this (Did not understand as of now)
- We can also decompose the matrix because the 12 and 21 terms are very similar (Reddy 224)
- We obviously solve using direct iterative or NR method. Check reddy for the derivations of T

- For the von Karman nonlinearity, for two beams : roler roler and pined pined under transverse loading. Will not have $u = 0$ because of the coupling and the solution will not be thesame.
- Suppose the roler roler beam has a constraint on u in the middle to remove rigid body movemenl, and transverse load does not make axial strain because the beam can slide without making stresses. This will have higher transverse deflection because it can stretch.
- The pined pined however has constraints on x = 0 and x = L so it will develop axial strains.
- To make sure that the roler roler does not have any axial strain
  $\varepsilon^o_{xx} = \frac{du}{dx} + \frac{1}{2}(\frac{dw}{dx})^2 = 0$
  And basically both shoud be of the same order $-\frac{du}{dx} = \left(\frac{dw}{dx}\right)^2$
- So when w is cubic $\frac{dw}{dx}$ is square and the power makes it quad. So u should be atleast order fifth. Taking u with any polynomial less , will make the constraint not satisfied and stiff, giving zero displacement field. (Membrane locking)
- We can also treat the axial strain as constant. Since $\frac{du}{dx}$ is const, we can make $(\frac{dw}{dx})^2$ also constant. We can do this by reduced integration of all nonlinear stiffness coefficients (A,B,D).

- Strains and stresses are close when found at the Gauss points
- u is aprrox with linear lagrange and w is approx with hermite cubic polynomials.
- Membrane strain ε^0_{xx} is assumed constant is evaluated using one point G.Q, and the bending strain ε^1_{xx} is also linear and done by one point G.Q.
-

$$\varepsilon^0_{xx} = \frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^w \qquad (75)$$

$$\varepsilon^1_{xx} = -\frac{d^2w}{dx^2}$$

And the stresses are given as follows :

- $\sigma_{xx} = \sigma^0_{xx} + z\sigma^1_{xx} = E\varepsilon^0_{xx} + zE\varepsilon^1_{xx}$

- The euler bernoulli beam is based on fact that a straight line transverse to axis before defomration remains (1) straight (2) inextensible (3) normal to mid plane after deformation. In TBT, we say the last assumption, the rotation is independant of the slope
- So we get the displacement field with an independant slope

$$u_1 = u(x) + z\phi_x(x) \qquad u2 = 0 \qquad u_3 = w(x) \tag{76}$$

- The non zero strains are :

$$\begin{aligned} \varepsilon_{xx} = \frac{du_1}{dx} + \frac{1}{2}(\frac{du_3}{dx})^2 = \frac{du}{dx} + \frac{1}{2}(\frac{dw}{dx})^2 + z\frac{d\phi}{dx} = \varepsilon^0_{xx} + z\varepsilon^1_{xx} \\ \gamma_{xz} = \frac{du_1}{dz}\frac{du_3}{dx} = \phi + -\left(-\frac{dw}{dx}\right) = \gamma^0_{xz} \end{aligned} \tag{77}$$

The last one we get because we remove the rotation due to w

- The virtual strains are :

$$\delta\varepsilon^0_{xx} = \frac{d\delta u}{dx} + \frac{dw}{dx}\frac{d\delta w}{dx}$$
$$\delta\varepsilon^1_{xx} = z\frac{d\phi}{dx} \tag{78}$$
$$\delta\gamma_{xz} = \delta\phi + \frac{d\delta w}{dx}$$

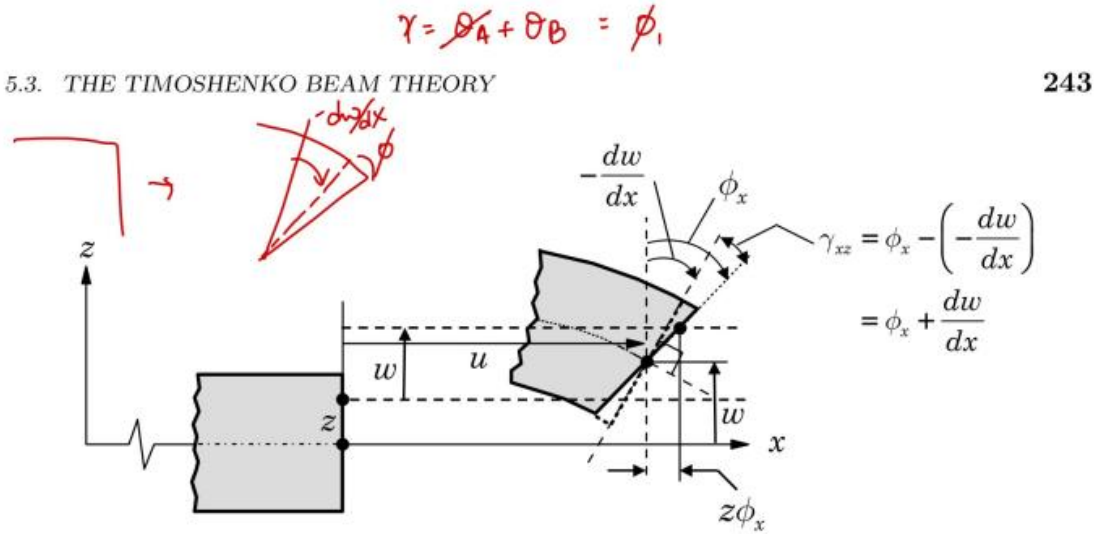


Fig. 5.3.1: Kinematics of a beam in the Timoshenko beam theory.

- The weak form is similaly developed (Reddy 243)

$$\delta W = \int_{x_a^e}^{x_b^e} \int_{A^e} \left(N_{xx}\delta\varepsilon_{xx}^0 + M_{xx}\delta\varepsilon_{xx}^1 + Q_x\delta\varepsilon_{xz}^0 \right) - \delta W_E^e \tag{79}$$

Note that :
$N = \int \sigma dA, \quad M = \int \sigma z dA \quad Q = \int \sigma_{xz} dA$
$\sigma = E\varepsilon \quad \sigma_{xz} = KG\gamma_{xz}$
The K thing is cause we assume a constant shear over the cross section when we just say stress is G x strain. So it is a correcting factor! We compare the two energies and then we find K.

- Trying to keep all the variations in the same derivative order, we get the euler equilibrium equations, taking each coefficient as zero

$$\begin{aligned} \delta u : \quad & -\frac{dN}{dx} = f(x) \\ \delta w : \quad & -\frac{dQ}{dx} - \frac{d}{dx}\left(N\frac{dw}{dx} \right) = q(x) \\ \delta\phi : \quad & -\frac{dM}{dx} + Q_x = 0 \end{aligned} \tag{80}$$

u,w, ϕ are the primary and N,Q,M are the secondary variables

- We keep the secondary variables in terms of the independant primary variables

$$N = A\left[\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right] + B\frac{d\phi}{dx}$$
$$M = B\left[\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right] + D\frac{d\phi}{dx} \tag{81}$$
$$Q = S\left(\frac{dw}{dx} + \phi\right)$$

  A,B,D are the previous terms moments of area giving the integral only in the x direction in the virtual work. S is the shear stiffness given as
  $S = K\int GdA = KGA = \frac{KEA}{2(1+\nu)}$

- Obviously we can derive the governing differential equilibrium equations (Page 245). For homogeneous beams , we have B = 0. which are the equilibrium equations given by the variational principle

- The virtual work statement is then given as

$$0 = \int_{x_a}^{x_b} \left\{ A_{xx} \frac{d\delta u}{dx} \left[\frac{du}{dx} + \frac{1}{2} \left(\frac{dw}{dx} \right)^2 \right] - f\delta u \right\} dx - Q_1^e \, \delta u(x_a) - Q_4^e \, \delta u(x_b) \tag{5.3.18}$$

$$0 = \int_{x_a}^{x_b} \frac{d\delta w}{dx} \left\{ S_{xx} \left(\frac{dw}{dx} + \phi_x \right) + A_{xx} \frac{dw}{dx} \left[\frac{du}{dx} + \frac{1}{2} \left(\frac{dw}{dx} \right)^2 \right] \right\} dx - \int_{x_a}^{x_b} \delta w q \, dx - Q_2^e \, \delta w(x_a) - Q_5^e \, \delta w(x_b) \tag{5.3.19}$$

$$0 = \int_{x_a}^{x_b} \left[D_{xx} \frac{d\delta\phi_x}{dx} \frac{d\phi_x}{dx} + S_{xx} \, \delta\phi_x \left(\frac{dw}{dx} + \phi_x \right) \right] dx - Q_3^e \, \delta\phi_x(x_a) - Q_6^e \, \delta\phi_x(x_b) \tag{5.3.20}$$

- The Q's have the same meaning as the euler bernoulli element

ϕ_x may be used. Substitution of Eq. (5.3.22) for u, w, and ϕ_x, and $\delta u = \psi_i^{(1)}$, $\delta w = \psi_i^{(2)}$, and $\delta\phi_x = \psi_i^{(3)}$ into Eqs. (5.3.18)–(5.3.20) yields the finite element model

$$0 = \sum_{j=1}^{m} K_{ij}^{11} u_j^e + \sum_{j=1}^{n} K_{ij}^{12} w_j^e + \sum_{j=1}^{p} K_{ij}^{13} s_j^e - F_i^1 \tag{5.3.23}$$

$$0 = \sum_{j=1}^{m} K_{ij}^{21} u_j^e + \sum_{j=1}^{n} K_{ij}^{22} w_j^e + \sum_{j=1}^{p} K_{ij}^{23} s_j^e - F_i^2 \tag{5.3.24}$$

$$0 = \sum_{j=1}^{m} K_{ij}^{31} u_j^e + \sum_{j=1}^{n} K_{ij}^{32} w_j^e + \sum_{j=1}^{p} K_{ij}^{33} s_j^e - F_i^3 \tag{5.3.25}$$

The stiffness and force coefficients are

$$K_{ij}^{11} = \int_{x_a}^{x_b} A_{xx} \frac{d\psi_i^{(1)}}{dx} \frac{d\psi_j^{(1)}}{dx}\, dx, \quad K_{ij}^{12} = \frac{1}{2} \int_{x_a}^{x_b} A_{xx} \frac{dw}{dx} \frac{d\psi_i^{(1)}}{dx} \frac{d\psi_j^{(2)}}{dx}\, dx$$

$$K_{ij}^{21} = \int_{x_a}^{x_b} A_{xx} \frac{dw}{dx} \frac{d\psi_i^{(2)}}{dx} \frac{d\psi_j^{(1)}}{dx}\, dx, \quad K_{ij}^{13} = 0, \quad K_{ij}^{31} = 0$$

$$K_{ij}^{22} = \int_{x_a}^{x_b} S_{xx} \frac{d\psi_i^{(2)}}{dx} \frac{d\psi_j^{(2)}}{dx}\, dx + \frac{1}{2} \int_{x_a}^{x_b} A_{xx} \left(\frac{dw}{dx}\right)^2 \frac{d\psi_i^{(2)}}{dx} \frac{d\psi_j^{(2)}}{dx}\, dx$$

$$K_{ij}^{23} = \int_{x_a}^{x_b} S_{xx} \frac{d\psi_i^{(2)}}{dx}\, \psi_j^{(3)}\, dx = K_{ji}^{32} \tag{5.3.26}$$

$$K_{ij}^{33} = \int_{x_a}^{x_b} \left(D_{xx} \frac{d\psi_i^{(3)}}{dx} \frac{d\psi_j^{(3)}}{dx} + S_{xx} \psi_i^{(3)} \psi_j^{(3)} \right) dx$$

$$F_i^1 = \int_{x_a}^{x_b} \psi_i^{(1)} f\, dx + Q_1^e \psi_i^{(1)}(x_a) + Q_4^e \psi_i^{(1)}(x_b)$$

$$F_i^2 = \int_{x_a}^{x_b} \psi_i^{(2)} q\, dx + Q_2^e \psi_i^{(2)}(x_a) + Q_5^e \psi_i^{(2)}(x_b)$$

$$F_i^3 = Q_3^e \psi_i^{(3)}(x_a) + Q_6^e \psi_i^{(3)}(x_b)$$

- We then get $\mathbf{Ku} = \mathbf{F}$ where $\mathbf{u} = \begin{bmatrix} u & w & \phi \end{bmatrix}^{\mathbf{T}}$. Check Reddy 249 for T stiffness derivation

- Timoshenko beam without von Karman nonlinearity differ from each other in the choice of the approx function for w and ϕ. Some are equal and others different
- Linear interpolation of both w and ϕ is the easisest. This makes the slope $\frac{dw}{dx}$ constant. In a think beam, as the length to thickness ratio becomes large (100), the slope would be equal to $-\phi$ which is linear instead of constant.
- On the other hand a constant ϕ leads to zero bending energy while the transverse shear is nonzero.
- Check Reddy 248 (Have not understood fully Locking issue)
- The primary variables may not be approximated by the same shape functions :

$$u(x) = N_i^1 u_i \qquad w(x) = N_i^2 w_i \qquad \phi(x) = N_i^3 \phi_i \tag{82}$$

Check reddy

Two dimensional problems having a single variable

- Suppose we are trying to find the solution $u(x, y)$ of the following partial differential equation

$$-\frac{d}{dx}\left(a_{xx}\frac{\partial du}{\partial dx}+a_{xy}\frac{\partial u}{\partial y}\right)-\frac{d}{dy}\left(a_{yx}\frac{\partial du}{\partial dx}+a_{yy}\frac{\partial u}{\partial y}\right)+a_{00}\,u=f(x,y)\quad \text{in}\quad \Omega \tag{83}$$

  The coefficients are also a function o u : eg $a_{xx}=f(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y})$

- When we descritize it with $\bar{\Omega}$ with a boundary $\bar{\Gamma}$, we get a residual given as :

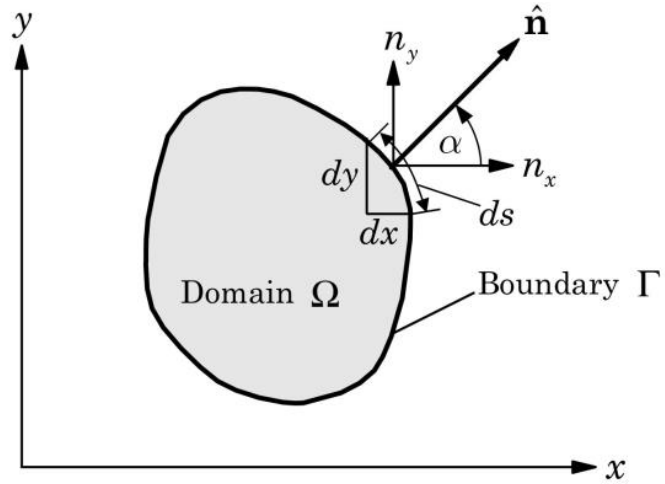
$$R(u_h)=-\frac{d}{dx}\left(a_{xx}\frac{\partial du_h}{\partial dx}+a_{xy}\frac{\partial u_h}{\partial y}\right)-\frac{d}{dy}\left(a_{yx}\frac{\partial du_h}{\partial dx}+a_{yy}\frac{\partial u_h}{\partial y}\right)+a_{00}\,u=f(x,y) \tag{84}$$

The step is to multiply the residual with the ith weight function $w_i(x, y)$ which should be differentiable too. We then set $w_i R$ over the element domain $\Omega^e = 0$

- $0 = \int_{\Omega_e} w_i \left[-\frac{d}{dx}\left(a_{xx}\frac{\partial u_h}{\partial x} + a_{xy}\frac{\partial u_h}{\partial y}\right) - \frac{d}{dy}\left(a_{yx}\frac{\partial u_h}{\partial x} + a_{yy}\frac{\partial u_h}{\partial y}\right) + a_{00}u - f(x,y)\right] dxdy$
- Now we know : $\frac{d}{dx}\left(w\frac{du}{dx}\right) = w\frac{d^2u}{dx^2} + \frac{dw}{dx}\frac{du}{dx}$
  and $\int_A \frac{d}{dx}\left(w\frac{du}{dx}\right) dA = \int_S \left(w\frac{du}{dx}\right) dS$
- We get

$$0 = \int_A \left[\frac{\partial w_i}{\partial x}\left(a_{xx}\frac{\partial u_h}{\partial x} + a_{xy}\frac{\partial u_h}{\partial y}\right) + \int_A \frac{\partial w_i}{\partial y}\left(a_{yx}\frac{\partial u_h}{\partial x} + a_{yy}\frac{\partial u_h}{\partial y}\right) + a_{00}w_i u_h - w_i f\right] dxdy - \int_S w_i \left[\left(a_{xx}\frac{\partial u_h}{\partial x} + a_{xy}\frac{\partial u_h}{\partial y}\right) n_x + \left(a_{yx}\frac{\partial u_h}{\partial x} + a_{yy}\frac{\partial u_h}{\partial y}\right) n_y \right] dS \tag{85}$$

  where $\mathbf{n} = \mathbf{n_x e_1} + \mathbf{n_y e_2}$ which gives the direction consies of the boundary Γ^e. The second term can also be written as $-\int_S q_n dS$ which is the external flux normal as we move counter clockwise.



- In the case of heat transfer through an anisotropic medium, aij denotes the conductivity and qn is the normal heat flux

- The weak form states that u should be atleast linear in both x and y
- $u_h^e = \sum u_i N_i(x, y)$ with $N_i(x_j, y_j) = \delta_{ij}$ and $\sum_j N_j(x, y) = 1$

Substituting the finite element approximation in Eq. (6.3.1) for u_h into the weak form, Eq. (6.2.5), we obtain

$$0 = \sum_{j=1}^{n} u_j^e \int_{\Omega^e} \left[\frac{\partial w_i}{\partial x} \left(a_{xx} \frac{\partial \psi_j^e}{\partial x} + a_{xy} \frac{\partial \psi_j^e}{\partial y} \right) + \frac{\partial w_i}{\partial y} \left(a_{yx} \frac{\partial \psi_j^e}{\partial x} + a_{yy} \frac{\partial \psi_j^e}{\partial y} \right) + a_{00} w_i \psi_j^e \right] dx dy - \int_{\Omega^e} w_i f \, dx dy - \oint_{\Gamma^e} w_i \, q_n \, ds \tag{6.3.3}$$

For the weak-form Galerkin model, we replace the weight function with ψ_i^e and obtain

$$\sum_{j=1}^{n} K_{ij}^e u_j^e - f_i^e - Q_i^e = 0 \quad \text{or} \quad \mathbf{K}^e \mathbf{u}^e = \mathbf{f}^e + \mathbf{Q}^e \tag{6.3.4}$$

where

$$K_{ij}^e = \int_{\Omega^e} \left[\frac{\partial \psi_i^e}{\partial x} \left(a_{xx} \frac{\partial \psi_j^e}{\partial x} + a_{xy} \frac{\partial \psi_j^e}{\partial y} \right) + \frac{\partial \psi_i^e}{\partial y} \left(a_{yx} \frac{\partial \psi_j^e}{\partial x} + a_{yy} \frac{\partial \psi_j^e}{\partial y} \right) + a_{00} \psi_i^e \psi_j^e \right] dx dy \tag{6.3.5}$$

$$f_i^e = \int_{\Omega^e} \psi_i^e f \, dx \, dy, \quad Q^e = \oint_{\Gamma^e} \psi_i^e q_n \, ds$$

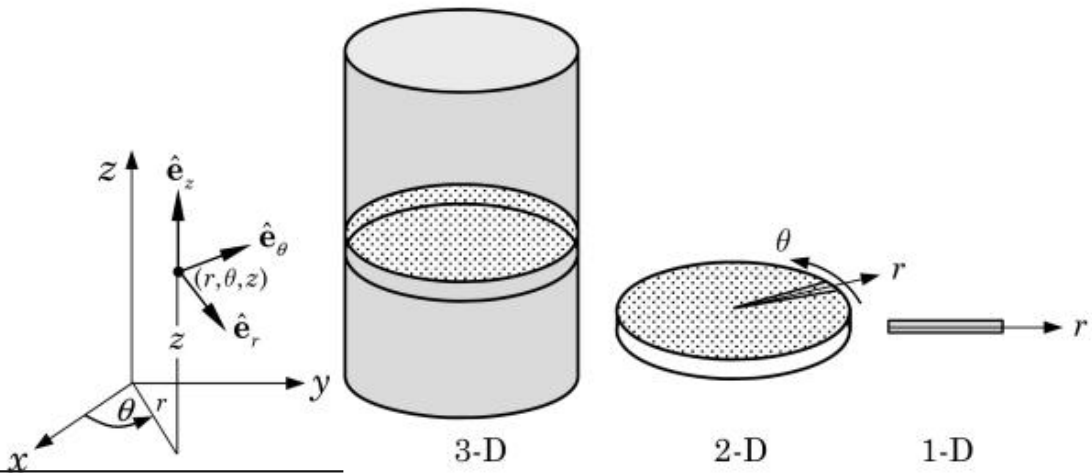
Note that $K_{ij}^e \neq K_{ji}^e$ (i.e. $\mathbf{K}^e$ is not symmetric) unless $a_{xy} = a_{yx}$. Equation (6.3.4) represents a set of n nonlinear algebraic equations.

- The equations have to be solved by nonlinear methods
- The tangent T is given in page 271

- The differential equation in cylindrical coordinate system (r,θ, z)

$$-\frac{1}{r}\frac{\partial}{\partial r}\left(ra_{rr}\frac{\partial u}{\partial r}\right)-\frac{1}{r^2}\frac{\partial}{\partial \theta}\left(a_{\theta\theta}\frac{\partial u}{\partial \theta}\right)-\frac{\partial}{\partial z}\left(a_{zz}\frac{\partial u}{\partial z}\right)=f \tag{86}$$

- where u,f and the coefficients are a function of (r,θ, z)
- Dependant on the coefficents, boundary conditions and load f the problem can be made to 2D or even 1D
- If the cylinder is very long and stuff dont vary and depend on z, then we can assume a disk. No also if there is independance from θ then we can even just do a radial line [3]



[3] We just remove the derivatives in the equations where the change will be zero

- Suppose all the variables are not dependant on θ, therefore we would get something like a plane and the Governing differential equation will be

$$-\frac{1}{r}\frac{\partial}{\partial r}\left(ra_{rr}\frac{\partial u}{\partial r}\right)-\frac{\partial}{\partial z}\left(a_{zz}\frac{\partial u}{\partial z}\right)=f(r,z) \tag{87}$$

- The weighted statement and weak form (Using greens theorem) will be given as :

$$0=\int_A w_i\left[-\frac{1}{r}\frac{\partial}{\partial r}\left(ra_{rr}\frac{\partial u}{\partial r}\right)-\frac{\partial}{\partial z}\left(a_{zz}\frac{\partial u}{\partial z}\right)-f(r,z)\right]rdrdz$$

$$=\int_A\left[a_{rr}(r,z,u_h)\frac{\partial w_i}{\partial r}\frac{\partial u_h}{\partial r}+a_{zz}(r,z,u_h)\frac{\partial w_i}{\partial z}\frac{\partial u_h}{\partial z}\right]rdrdz-\int_A w_i f(r,z)rdrdz-\int_S w_i q_n ds \tag{88}$$

where $q_n=r\left[a_{rr}\frac{\partial u_h}{\partial r}n_r+a_{zz}\frac{\partial u_h}{\partial z}n_z\right]$

- And we get $\mathbf{Ku}=\mathbf{f}+\mathbf{Q}=\mathbf{F}$
- Remember the shape functions are functions of N(r,z)

The transformation between Ω^e and $\hat{\Omega}$ is accomplished by a coordinate transformation of the form [see Eqs. (3.6.1) and (3.6.2)]

$$x = \sum_{j=1}^{m} x_j^e \hat{\psi}_j^e(\xi, \eta) \ , \qquad y = \sum_{j=1}^{m} y_j^e \hat{\psi}_j^e(\xi, \eta) \tag{6.6.1}$$

while a typical dependent variable $u(x, y)$ is approximated by

$$u(x, y) = \sum_{j=1}^{n} u_j^e \psi_j^e(x, y) = \sum_{j=1}^{n} u_j^e \psi_j^e(x(\xi, \eta), y(\xi, \eta)) \tag{6.6.2}$$

where $\hat{\psi}_j^e$ denote the interpolation functions of the master element $\hat{\Omega}$ and ψ_j^e are the interpolation functions of a typical element Ω^e over which u is approximated. The transformation in Eq. (6.6.1) maps a point (x, y) in a typical element Ω^e of the mesh to a point (ξ, η) in the master element $\hat{\Omega}$ and vice versa, if the Jacobian J_e of the transformation is positive-definite [see Eqs. (3.6.5)–(3.6.8)].

■

- Remember that we have to we have to transform the integral domain to the master element so that Gauss quadrature can be usd. The derivatives in the original geometry are also expressed with respect to ξ, η given by

$$\begin{bmatrix} \frac{\partial N_i}{\partial dx} \\ \\ \frac{\partial N_i}{\partial dy} \end{bmatrix} = \mathbf{J^{-1}} \begin{bmatrix} \frac{\partial N_i}{\partial d\xi} \\ \\ \frac{\partial N_i}{\partial d\eta} \end{bmatrix} \tag{89}$$

$$\text{where } \mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \sum x_i \frac{\partial N_i}{\partial \xi} & \sum y_i \frac{\partial N_i}{\partial \xi} \\ \\ \sum x_i \frac{\partial N_i}{\partial \eta} & \sum y_i \frac{\partial N_i}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & .... & \frac{\partial N_m}{\partial \xi} \\ \\ \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & .... & \frac{\partial N_m}{\partial \eta} \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ \\ x_2 & y_2 \\ \vdots & \vdots \\ x_m & y_m \end{bmatrix}$$

- If the same shape functionsa re used for the geometry and field variable, we say its isoparametric

Returning to the coefficients K_{ij}^e in Eq. (6.6.3), we can write it now in terms of the natural coordinates (ξ, η) as

$$\begin{aligned} K_{ij}^e = \int_{\hat{\Omega}} \Bigg\{ & \hat{a}_{xx}(\xi,\eta)\left(J_{11}^\star \frac{\partial \psi_i^e}{\partial \xi} + J_{12}^\star \frac{\partial \psi_i^e}{\partial \eta}\right)\left(J_{11}^\star \frac{\partial \psi_j^e}{\partial \xi} + J_{12}^\star \frac{\partial \psi_j^e}{\partial \eta}\right) \\ & + \hat{a}_{yy}(\xi,\eta)\left(J_{21}^\star \frac{\partial \psi_i^e}{\partial \xi} + J_{22}^\star \frac{\partial \psi_i^e}{\partial \eta}\right)\left(J_{21}^\star \frac{\partial \psi_j^e}{\partial \xi} + J_{22}^\star \frac{\partial \psi_j^e}{\partial \eta}\right) \\ & + \hat{a}_{00}(\xi,\eta)\psi_i^e \psi_j^e \Bigg\} J_e \, d\xi \, d\eta \\ \equiv & \int_{\hat{\Omega}} F_{ij}^e(\xi,\eta) \, d\xi \, d\eta \end{aligned} \tag{6.6.9}$$

where the element area $dA = dxdy$ in element Ω^e is transformed to $J_e \, d\xi \, d\eta$ in the master element $\hat{\Omega}$, and $\hat{a}_{xx} = a_{xx}(x(\xi,\eta), y(\xi,\eta), u(\xi,\eta))$, and so on.

Using the $M \times N$ Gauss quadrature to evaluate integrals defined over a rectangular master element $\hat{\Omega}$, we obtain

$$\begin{aligned} \int_{\hat{\Omega}} F_{ij}^e(\xi,\eta) \, d\xi \, d\eta &= \int_{-1}^{1} \left[\int_{-1}^{1} F_{ij}^e(\xi,\eta) \, d\eta \right] d\xi \approx \int_{-1}^{1} \left[\sum_{J=1}^{N} F_{ij}^e(\xi,\eta_J) \, W_J \right] d\xi \\ &\approx \sum_{I=1}^{M} \sum_{J=1}^{N} F_{ij}^e(\xi_I,\eta_J) \, W_I W_J \end{aligned} \tag{6.6.10}$$

where M and N denote the number of Gauss quadrature points in the ξ and η directions, respectively, (ξ_I, η_J) denote the Gauss points, and W_I and W_J denote the corresponding Gauss weights, as listed in Table 3.6.1.

As already discussed, if the integrand is a polynomial of degree p in a coordinate direction, it is integrated exactly by employing $NGP \equiv N = \text{int}[\frac{1}{2}(p+1)]$ (the nearest equal or larger integer number) Gauss points in that direction. In most cases, the interpolation functions are of the same degree in both ξ and η, and we take $N = M$. For example, consider the expression involving a_{00} in K_{ij}^e of Eq. (6.6.3). When a_{00} is a linear function of ξ and η, it requires a 2×2 Gauss rule when ψ_i^e are linear and a 3×3 Gauss rule when ψ_i^e are quadratic to be evaluated exactly. When a_{00} is quadratic or cubic, it requires 3×3 and 4×4 Gauss rules for linear and quadratic *rectangular* elements, respectively.

where J* are the components of the inverse jacobian

Time dependant problems

Will have to read later!!