Linearised equilibrium :March 2, 2021

- The virtual work gives the linear variation in work. Meaning it is linear with respect to the variations
- The equilibrium equation however will still be nonlinear with respect to the geometry and material
- Now this equilibrium equation will then need to be linearised

- The princliple of virtual work is as follows

$$\delta W(\phi, \delta v) = \int_v \sigma : \delta d dv - \int_v f.\delta v dv - \int_\Gamma t.\delta v da = 0 \tag{1}$$

where ϕ is the trial solution

- Linaerising f agains meaning f + D f

$$\delta W(\phi, \delta v) + D\delta W(\phi, \delta v)[u] = 0 \tag{2}$$

- So we are finding the directional derivative of the virtual work equation, i.e at ϕ at a direction u
- Remember to derive the equilibrium, we set up a virtual work equation about a position x. Here we are then trying to find that position x, making the non-linear equilibrium equations linear

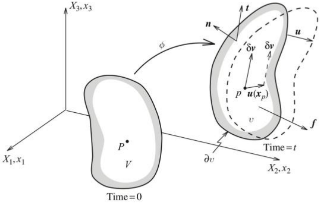

FIGURE 8.1 Linearized equilibrium.

- As shown in the figure, the virtual displacement/velocity is still the same but now the actual quilibrium configuration is changing
- At a trial solution ϕ_k, the virtual work $\delta W(\phi, \delta v) \neq 0$
- Therefore $D\delta W(\phi_k, \delta v)[u]$ is the change in δW due to ϕ_k change to $\phi_k + u$.
- Since δv is not changing, therefore it is r or the internal forces that are changing as the solution changes due to u
- NR therfore makes the internal and external forces in equilibrium by changing the configuration to the equilibrium configuration

$$D\delta W(\phi, \delta v)[u] = D\delta W_{int}(\phi, \delta v)[u] - D\delta W_{ext}(\phi, \delta v)[u] \tag{3}$$

$$D\delta W(\phi, \delta v)[u] = D\left(\int_v \sigma : \delta d \ dv\right)[u] - D\left(\int_\Gamma t.\delta v \ da\right)[u] \tag{4}$$

- Linearisation of the equilibrium equations with respect to the material description
- Simpler as we have the material description , and we know the dV integral to integrate over
- We can push forward the equations to the spatial description

Lagrangian linearised internal virtual work

- Internal virtual work is given as

$$\delta W_{int}(\phi, \delta v) = \int_V S : \delta\dot{E} dV \tag{5}$$

- Using product rule for directional derivatives, we get

$$\begin{aligned} D\delta W_{int}(\phi, \delta v)[u] = D\left(\int_V S : \delta\dot{E}\right)[u]dV \\ \left(\int_V DS : \delta\dot{E}\right)[u]dV \\ \left(\int_V \delta\dot{E} : DS[u]\right)dV + \left(\int_V S : D\delta\dot{E}[u]\right)dV \\ \left(\int_V \delta\dot{E} : C : DS[u]\right)dV + \left(\int_V S : D\delta\dot{E}[u]\right)dV \end{aligned} \tag{6}$$

(Have to check why can we take the derivative inside the integral?????????????/)

- Where we can then find $D\dot{E}[u]$ Bonet

- $D\delta\dot{E}[u]$ is a function of δv and also of configuration ϕ

$$\delta\dot{E} = \frac{1}{2}\left(\delta\dot{F}^T F + F^T \delta\dot{F}\right) \tag{7}$$

and $\delta\dot{F} = \frac{\partial \delta v}{\partial X} = \nabla_o \delta v$ and $DF[u] = \nabla_o u$

- So

$$D\delta\dot{E}[u] = \frac{1}{2}\left(\nabla(\delta v)^T \nabla_o u + \nabla(\delta u)^T \nabla_o v\right) \tag{8}$$

Here $\nabla_o v$ remains constant, as they are not functions of the configuration

$$D\delta W_{int}(\phi, \delta v)[u] = \int_V \delta\dot{E} : C : DE[u] dV + \int_V S : [(\nabla_o u)^T \nabla_o \delta v] dV \tag{9}$$

and $\delta\dot{E}$ can be written as $DE[\delta v]$ which gives a symmetric form:

$$D\delta W_{int}(\phi, \delta v)[u] = \int_V DE[\delta v] : C : DE[u] dV + \int_V S : [(\nabla_o u)^T \nabla_o \delta v] dV \tag{10}$$

- The formulations are pushed forward, check Bonet page 219

- Body forces
- Surface forces

- The advantage of finding a stationary problem with respect to displacements, is the advantage that such a treatment gives a uniform framework to find
  - Incompressibility, contact boundary conditions and finite element methods
  - Done by use of lagrangian multipliers or penalty methods where the variational principle incorporates eg internal pressure

- The potential energy whose directional derivative gives the virtual work is

$$\Pi(\phi) = \int_V \Psi(C)dV - \int_V f_o.\phi dV - \int_{\Gamma_o} t.\phi dA \tag{11}$$

- Assuming that the body force and traction not a function of ϕ (Not actuall for traction). The directional derivative is

$$\Pi(\phi)[\delta v] = \int_V S : DC[\delta v]dV - \int_V f_o.\delta v dV - \int_{\Gamma_o} t.\delta v dA \tag{12}$$

which is similar to the theory of virtual work that is $D\Pi(\phi)[\delta v] = \delta W(\phi, \delta v)$

- The stationary conition of $\Pi(\phi)$ gives the equilibirum and known as the varitational statement of equilibrium. The lienarised equation (NR) can be taken as the second derivative of the energy functional

$$D\delta W(\phi, \delta v)[u] = D^2\Pi(\phi)[\delta v, u] \tag{13}$$

- We need to keep a incompressibility constraint J =1
- Lagrange multiplier term

$$\Pi_L(\phi, p) = \hat{\Pi}(\phi) + \int_V p(J-1)\,dV \tag{14}$$

where p is the lagrange multiplier with knowing that it will be the internal pressure. $\hat{\Pi}$ is the strain energy given as a function of the distrotion component of the right Cauchy tensor

- We will obviously have to find $\Pi_L(\phi, p)[\delta v]$ and $\Pi_L(\phi, p)\delta p$
- And linearise the equilibirum equations again with respect to p and u
- Check bonet page 226 for details

- Different alternative to Lagrangian methods
  - Eliminates pressure as an independant variable keeping a large value of the bulk modulus
  - Perturb the lagrangian functional with a penalty allowing the pressure to be associated with the deformation
    - The perturbed lagrangian is

$$\Pi_P(\phi, p) = \Pi_L(\phi, p) - \int_V \frac{1}{2k} p^2 dV \tag{15}$$

as $k \longrightarrow \infty : \Pi_P \longrightarrow \Pi_L$
with the stationary condition as :

$$D\Pi_P(\phi, p)[\delta p] = \int_V \delta p \left((J-1) - \frac{p}{k}\right) dV = 0 \tag{16}$$

The (J-1) comes from the Π_L term. Now this euilibrium equation gives us a relationship between p and J as p = k(J-1).

    - This represents a nearly incompresible material with he penalty number as the bulk modulus (So same thing like increasing bulk modulus)

Check bonet page 229

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