Mechanics

Allan Marbaniang

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Linearity and non-linearity

- Suppose there is a function or mapping that does this $f : x \longrightarrow y$. Function f takes x and gives y
- The function is linear if $f(x) = f(x_1) + f(x_2)$. where $x = x_1 + x_2$
- Suppose we know $f()$ and y but not x. Because the function is linear, we can construct solutions x that may be a combination of different solutions x_i. The idea is that f should not depend on x. It can have other parameters, but not the values of the solution itself. In that case, we would have to find the solution x and also the mapping f

Example

Take function or map ()

Structural non-linearity introduction

- What does non-linearity mean ?
- What makes a structure non-linear ?
- How do you model non-linearity and solve it?

- There are different types of non-linearity in a structure
  - Geometrical - $\mathbf{K}(\mathbf{x})\mathbf{x} = \mathbf{F}$ where here $\mathbf{K}$ depends on $\mathbf{x}$ because $\mathbf{K}$ changes as you chagne x due to the change in geometry and the large deformation.
  - Material - $\mathbf{K}(\mathbf{x})\mathbf{x} = \mathbf{F}$ because $\mathbf{K}$ changes due to the changes in the material property. For eg Youngs modulus may be a function of x.
- $\mathbf{K}$ is now a function of (x), it means that it can't be solved by using a linear method (Inverting a matrix).
- But the system of equations now has to be linearised about a point. We keep the system of equations as an equality equation $\mathbf{K}(\mathbf{x})\mathbf{x} - \mathbf{F} = 0$.
- We linearise and try to solve the root. At every point we construct the tangent and find a solution, but when we construct the solution at that point, we find the residual and we then use to iterate until convergence.

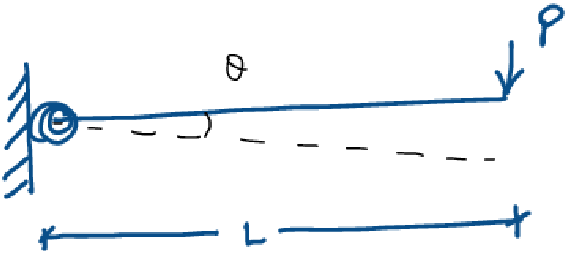


Figure:

Taking a weightless rigid bar with a torsional spring that resists any moment and a load P at the end. Since the bar is rigid so vertical equilibrium is satisfied at the support without any deformation. However, it can rigidly rotate through the spring.

The moment generated at the end support is

$$PLcos\theta = M \tag{1}$$

Taking the equilibrium equation with the spring as $M = K\theta$ we get

$$PLcos\theta = K\theta$$
$$\frac{PL}{K} = \frac{\theta}{cos\theta}$$

- If we take $\theta \rightarrow 0$ then $cos\theta \rightarrow 1$
- So $P = \dfrac{K}{L}\theta$ which is linear wrt θ

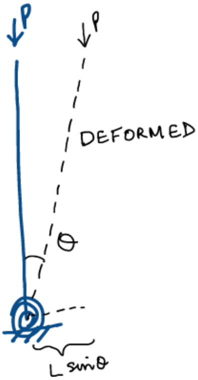


Figure:

Same problem but now the rigid bar is vertical. This is a very common problem for nonlinearity (I think the previos one is better tho!). Nicely represents buckling of columns

Same equilibrium equation but now the lever arm is different (Because the load along the bar)

$$PLsin\theta = M \tag{2}$$

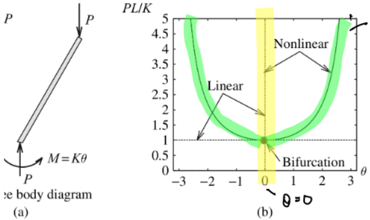
Taking the equilibrium equation with the spring as $M = K\theta$ we get

$$PLsin\theta = K\theta$$

$$\frac{PL}{K} = \frac{\theta}{sin\theta}$$

- If we take $\theta \rightarrow 0$ then $sin\theta \rightarrow 0$ so $M = 0$ (This is one possible equilibrium)
- $\frac{PL}{K} = \frac{\theta}{sin\theta}$: This is the other
- The load $\frac{PL}{K}$ where these two equilibrium equations are possible for the same structure is called the bifurcation point.
- You can imagine that when the load is smaller, it will not buckle and only deform axially so only one equilibrium position ($\theta = 0$) is possible. When $\frac{PL}{K} > 1$ then the other equilibrium comes to play.

Therefore two solutions are there :



imple column.

$\frac{PL}{K} = \frac{\theta}{\sin\theta}$

- Below $\frac{PL}{K} < 1$, Only Equilibrium is $\theta = 0$

Because $\frac{PL}{K} < 1$, $\frac{\theta}{\sin\theta} \not< 1$ so $\theta = 0$

- Above $\frac{PL}{K} > 1$, we get two possible solutions
  - $\theta = 0$: $M = 0$
  - $\theta \neq 0$: $\frac{PL}{K} = \frac{\theta}{\sin\theta}$.

- If we linearise our equilibrium equation for small $\theta \rightarrow sin\theta$
- We get $(Pl - K)\theta = 0$ and we get our linear eigen value problem with a trivial solution of $\theta = 0$ and a nontrivial solution of $PL = K$. Again we get two equilibrium solutions. $PL = K$ being the buckling load.
- So when we reach that load, it means we have reached the bifurcation point, where multiple equilibrium solutions exist and the rod may buckle dependant on imperfection, lateral load etc.

Non-Linear strain measures introduction

- Now suppose of being rigid, the body is deformable that is the relative deformation is introducing strain and stresses
- LARGE DISPLACEMENTS + LARGE STRAINS
- The first deals with displacements that are large, and therefore while finding the strains, we have to use higher orders of the displacement derivatives
- The second deals with ????????????????????????????

- Emphasize its only for One D! But same theory for other dimensions
- A strain measure need not be fixed. Sometimes the strain measure we usually use may not be able to model the correct behaviour. When we choose any strain measure, the proper corresponding stress and the constitutive relationship ($\sigma = \mathbf{C}\varepsilon$) has to be taken.
- The stress and strain have to be "work compatible". That is they are together used in the strain energy density function.

Engineering strain

- Engineering strain $\varepsilon_E = \dfrac{l - L}{L} = \dfrac{\Delta}{L}$. l is deformed length, L is initial undeformed
- We could have also divided Δ by l (Change by deformed length). If $l \approx L$ then it would not matter.
- ε_E is the small infinitesimal strain, where the deformed and undeformed lengths are very similar.

Logarithimic strain

- The instantaneous strain increment can be thought as $\varepsilon_L = \frac{\Delta_1}{L} + \frac{\Delta_2}{l_1}$...
- Or $d\varepsilon_L = \frac{dl}{l}$
- $\varepsilon_L = \int_L^l \dfrac{dl}{l} = ln\frac{l}{L}$
- The integration is done between two configurations $L \longrightarrow l$

These strains are more easily extrapolated to continuum (3d cases)

Green strain

- $\varepsilon_G = \dfrac{l^2 - L^2}{2L^2}$

Almansi strain

- $\varepsilon_A = \dfrac{l^2 - L^2}{2l^2}$

- Suppose $l \approx L$ and therefore Δ is small
- And $l = (L + \Delta)$
- $\varepsilon_G = \dfrac{(L+\Delta)^2 - L^2}{2L^2} = \dfrac{(L^2 + \Delta^2 + 2L\Delta - L^2)}{2L^2} \approx \dfrac{\Delta}{L}$ (As Δ is very small and so Δ^2 vanishes)

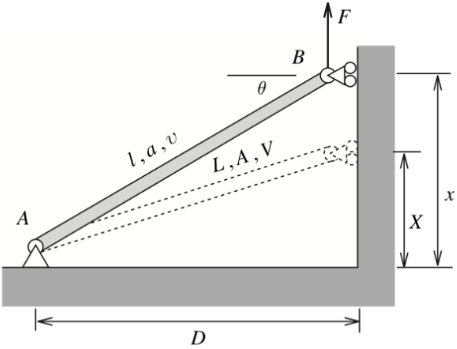


Figure:

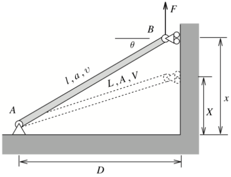
- Initial length L, area A, volume V
- Final length l, area a, volume v

- In defining the equilibrium (Froces = 0 , No moments). We will be defining the internal stress by different strain measures.
- Remember that a proper constitutive law has to be taken for a particularly strain measure
- Here we have chosen the Cauchy stress and E randomly. The cauchy stress is the actual/true stress in the deformed state. (Or it is the stress in the deformed state which is in equilibrium)

Using two strain measures

Green and logarithmic.

- Cauchy stress (True stress) $\sigma = E\varepsilon$ can be :
- $\sigma = E\frac{l^2 - L^2}{L^2}$
- $\sigma = Eln\frac{l}{L}$



- The bar will keep moving up until the vertical equilibrium is reached.
- Vertical equilibrium at B is $F - T(x)sin\theta(x) = 0$, where $T(x)$ is the internal force and depends on x. θ is also dependant on x
- Now we can construct a residual function $R(x) = F - T(x)sin\theta(x)$ where the residual becomes zero for a particular solution of x. [1] So

$$R(x) = \sigma a sin\theta - F = \sigma(x) a \frac{x}{l} - F \tag{3}$$

Stress dependant on different strain measures

- $T(x) = E\frac{l^2 - L^2}{L^2} a \frac{x}{l}$ $\qquad (\sigma = E\varepsilon_G)$
- $T(x) = E ln \frac{l}{L} a \frac{x}{l}$ $\qquad (\sigma = E\varepsilon_L)$

[1] Note that $\frac{dR}{dx}$ is the tangent stifness K_{Bx} or force in direction B due to displacement x.

Stress dependant on different strain measures

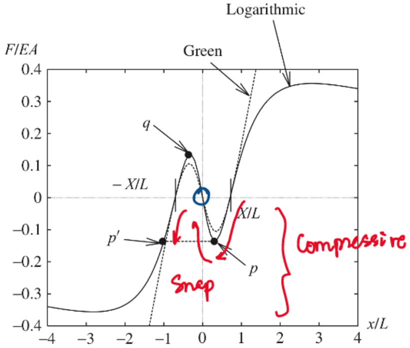
- $T(x) = E\frac{l^2 - L^2}{L^2} a \frac{x}{l}$ $\qquad (\sigma = E\varepsilon_G)$
- $T(x) = Eln\frac{l}{L} a \frac{x}{l}$ $\qquad (\sigma = E\varepsilon_L)$

- l is a function of x, $l^2 = D^2 + x^2$
- R(x) is therefore very nonlinear with respect to x. In R(x), F is not dependant on x. But sometimes it can be the case that the load is also nonlinear.

Solving

We need to solve the nonlinear equation R(x) = 0

- So we use NR, or first order taylor series to linearise R and solve it iteratively
- $R(x_{i+1}) = R(x_i) + \frac{dR}{dx}|_{x_i}(x_{i+1} - xi)$
- We want R = 0 , so the value $R(x_{i+1}) = 0$
- $0 = R(x_i) + \frac{dR}{dx}|_{x_i}(x_{i+1} - xi)$



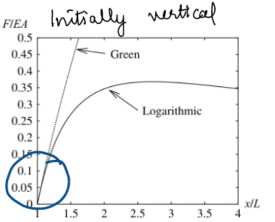


FIGURE 1.6 Large strain rod: load deflection behavior.

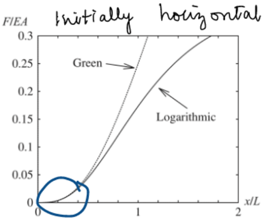


FIGURE 1.7 Horizontal truss: tension stiffening.

- O ← When the strains are small the different measures give similar equilib profile
- As you increase compressive (-ve) you get snap through behaviour from $p \to q \to p'$ (Imagine a wrinkled coca cola bottle you push in).

- Regions where x is small, different mesasures gives okay results
- We see different behaviour behaviours between the strain measures at higher strain
- Snap through behaviour if we increase compressive load too much. Imagine you are pushing the truss down (-x) and suddenly it will roll to the other side.
- If truss is initially vertical (Like a column in tension, Therefore no rotation) : Same E should have not been used for both different strain measures. (It seems that the green strain looks good as we expect it to be linear in axial)
- Initially horizontal : Stiffening due to tension

- A comment was made that E should have not been used for the two