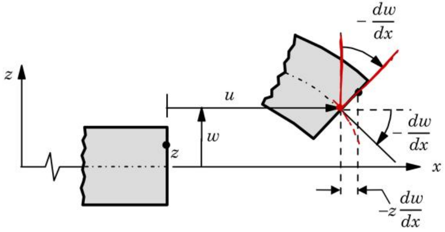
Nonlinear Bending of straight beams : February 18, 2021

- Assuming he geomery does no change significanty, allows he principle of virtual work o be written over the underformed body. So stress is force per unit undeforemed are, strain measure of change in length w.r.t original length and shear as change in length from $\pi/2$. No distincion between Piola kirchoff and cauchy stress
- Nonlinearity comes solely from inplane forces proportional to he square of the rotation of a transverse normal line in the beam
- There are two theories
  - Euler bernoulli beam (EBT)
  - Timoshenko beam (TBT)

Assumptions:

- Plane sections perpendicular to he axis of the beam remain (a) plane (b) rigid (not deform) (c) rotate such that they remain plane to the deformed axis after deformation
- Assumptions amount to neglection poissons effect and transverse normal and shear strains



- The displacement is given as : $u_1 = u(x) - z\frac{dw}{dx}$ $\quad u_2 = 0 \quad u_3 = w(x)$
  u_1, u_2, u_3 is the disp of any point while u, w are the disp of the centroidal axis
- So the total disp can be found from the neutral axis values.

- The greens theorem is

$$\begin{aligned} E_{ij} = \varepsilon_{ij} &= \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j} \\ \varepsilon_{11} &= \frac{\partial u_1}{\partial x_1} + \frac{1}{2}\left[\left(\frac{\partial u_1}{\partial x_1}\right)^2 + \left(\frac{\partial u_3}{\partial x_1}\right)^2\right] \end{aligned} \tag{1}$$

- Now the axial strains of higher orders we can igore, bu the rotation of the line perpendicular to the beam is pretty large so we have to retain it.
- Non-linear strains where only squares of the rotations are included are known as von Karman nonlinearity
- The zero strains are $\varepsilon_{33} = \varepsilon_{13} = 0$ so only non zero strain is

$$\varepsilon_{11} = \frac{du}{dx} - z\frac{d^2w}{dx^2} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 \tag{2}$$

- So we have two longitudinal von karman strains:

$$\varepsilon_{11}^0 = \frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 \qquad , \quad \varepsilon_{11}^1 = -z\frac{d^2w}{dx^2} \tag{3}$$

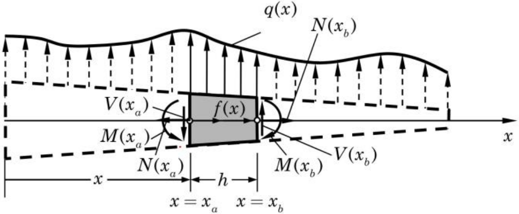
- The weak form can be found withou knowing the governing d.e
- If a body is in equilibrium, the toal virtual work done in moving through the repsective displacements is zero

$$\delta W^e = \delta W_I^e + \delta W_E^e \tag{4}$$

δW_I^e denotes the virtual strain stored due to the actual cauchy or second piolay stresses (Same geom) σ_{ij}
δW_E^e is the work done by external applied loads

- So we have

$$\begin{aligned} \delta W_I^e &= \int_V^e \delta\varepsilon_{xx}\sigma_{xx} dV = \int_{x_a}^{x_b}\int_A^e \left(\frac{d\delta u}{dx} + \frac{dw}{dx}\frac{d\delta w}{dx} - z\frac{d^2\delta w}{dx^2}\right) dAdx \\ \delta W_E^e &= -\left[\int_{x_a}^{x_b} q\delta w dx + \int_{x_a}^{x_b} f\delta u dx + \sum_{i=1}^{6} Q_i^e \delta\Delta_i^e\right] \end{aligned} \tag{5}$$



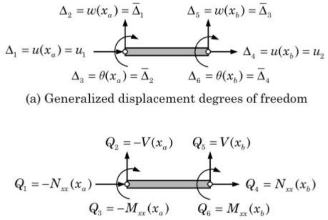
$Q_1 = -N(x_a),\ Q_2 = -V(x_a),\ Q_3 = -M(x_a)$

$Q_4 = N(x_b),\ Q_5 = V(x_b),\ Q_6 = M(x_b)$

Fig. 5.2.2: A typical beam element, $\Omega^e = (x_a, x_b)$.

$$\Delta_1^e = u(x_a), \quad \Delta_2^e = w(x_a), \quad \Delta_3^e = \left[-\frac{dw}{dx}\right]_{x_a} \equiv \theta(x_a)$$
$$\Delta_4^e = u(x_b), \quad \Delta_5^e = w(x_b), \quad \Delta_6^e = \left[-\frac{dw}{dx}\right]_{x_b} \equiv \theta(x_b) \tag{5.2.6}$$

$$Q_1^e = -N_{xx}(x_a), \quad Q_4^e = N_{xx}(x_b), \quad Q_2^e = -\left[\frac{dw}{dx}N_{xx} + \frac{dM_{xx}}{dx}\right]_{x_a} \equiv -V_x(x_a)$$
$$Q_5^e = \left[\frac{dw}{dx}N_{xx} + \frac{dM_{xx}}{dx}\right]_{x_b} \equiv V_x(x_b), \quad Q_3^e = -M_{xx}(x_a), \quad Q_6^e = M_{xx}(x_b) \tag{5.2.7}$$



(a) Generalized displacement degrees of freedom

(b) Generalized force degrees of freedom

- The main thing is that for V_x, the component of N_x is $sin(\frac{dw}{dx}) \approx \frac{dw}{dx}$, and also in moment equilibirum we get $N_{xx}\frac{dw}{dx}\Delta x = N_{xx}\Delta w$

- The free body diagram is what it is. But we can take our generalised displacement and force dof signs based on what we want.
- Generalised is used to show that rotationsa and moments are treated as displacements and forces
- By integrating in the area we get the virtual work as

$$\delta W_I^e = \int_{x_a}^{x_b} \left[\left(\frac{d\delta u}{dx} + \frac{dw}{dx}\frac{d\delta w}{dx} \right) N_{xx} - \frac{d^2\delta w}{dx^2} M_{xx} \right] - \int_{x_a}^{x_b} \delta w q dx - \int_{x_a}^{x_b} \delta u f dx - \sum_{i=1}^{6} \delta\Delta_i^e Q_i^e \quad (6)$$

where $N_{xx} = \int_A \sigma_{xx} dA \qquad M_{xx} = \int_A \sigma_{xx} z dA$

By taking the individual virtual terms, as the coefficients have to seperately equal to zero we get

$\int_{x_a}^{x_b} \left(\frac{d\delta u}{dx} N_{xx} - \delta u f \right) dx - \delta\Delta_1^e Q_1^e - \delta\Delta_4^e Q_4^e = 0$

$\int_{x_a}^{x_b} \left(\frac{d\delta w}{dx} \left(\frac{dw}{dx} N_{xx} \right) - \frac{d^2\delta w}{dx^2} M_{xx} - \delta w q \right) dx - \delta\Delta_2^e Q_2^e - \delta\Delta_3^e Q_3^e - \delta\Delta_5^e Q_5^e - \delta\Delta_6^e Q_6^e = 0$

- By integration by parts we get the euler lagrange equilibrium equations : given as

$$\int_{x_a}^{x_b} \delta u \left(-\frac{d\delta N_{xx}}{dx} - f \right) dx - \delta\Delta_1^e Q_1^e + [\delta u N_{xx}]_{x_b}^{x_a} - \delta\Delta_4^e Q_4^e = 0$$

$$\int_{x_a}^{x_b} -\delta w \left(\frac{d}{dx} \left(\frac{dw}{dx} N_{xx} \right) + \frac{d^2 M_{xx}}{dx^2} + q \right) dx - \left[\frac{d\delta w}{dx} M_{xx} \right]_{x_b}^{x_a} \tag{7}$$

$$+ \left[\delta w \left(\frac{dw}{dx} N_{xx} + \frac{dM_{xx}}{dx} \right) \right]_{x_b}^{x_a} - \delta\Delta_2^e Q_2^e - \delta\Delta_3^e Q_3^e - \delta\Delta_5^e Q_5^e - \delta\Delta_6^e Q_6^e = 0$$

- We get our euler equations from coefficients of the two variations

$$
\begin{aligned}
-\frac{dN_{xx}}{dx} &= f(x) \\
-\frac{d}{dx}\left(\frac{dw}{dx}N_{xx}\right) - \frac{d^2 M_{xx}}{dx^2} &= q(x)
\end{aligned} \tag{8}
$$

- We get our boundary conditions as follows :

$$
\begin{aligned}
Q_1^e &= -N_{xx}(x_a) & Q_4^e &= N_{xx}(x_b) \\
Q_2^e &= -\left[\frac{dw}{dx}N_{xx} + \frac{dM_{xx}}{dx}\right]_{x_a} & Q_5^e &= \left[\frac{dw}{dx}N_{xx} + \frac{dM_{xx}}{dx}\right]_{x_b} \\
Q_3^e &= M_{xx}(x_a) & Q_6^e &= M_{xx}(x_b)
\end{aligned} \tag{9}
$$

Again the - sign makes all the Qs positive in the equation

- We can also find the differential equations by writing the equilibirum equaitons (Fx,Fy,Mz = 0) for a small element and then take $\Delta x \to 0$. But we don't get the boundary conditions by using this.
- After getting the euler lagrange differnetial equations, we can again use the weighted residual method to find the stiffness forms

$$
\begin{aligned}
0 &= \int_{x_a}^{x_b} v_1 \left(-\frac{dN_{xx}}{dx} - f \right) dx = \int_{x_a}^{x_b} \left(\frac{dv_1}{dx} N_{xx} - v_1 f \right) dx - [v_1 N_{xx}]_{x_a}^{x_b} \\
&= \int_{x_a}^{x_b} \left(\frac{dv_1}{dx} N_{xx} - v_1 f \right) dx - v_1(x_a)[-N_{xx}(x_a)] - v_1(x_b) N_{xx}(x_b) \quad (5.2.16)
\end{aligned}
$$

$$
\begin{aligned}
0 &= \int_{x_a}^{x_b} v_2 \left[-\frac{d}{dx}\left(\frac{dw}{dx} N_{xx} \right) - \frac{d^2 M_{xx}}{dx^2} - q \right] dx \\
&= \int_{x_a}^{x_b} \left[\frac{dv_2}{dx}\left(\frac{dw}{dx} N_{xx} \right) - \frac{d^2 v_2}{dx^2} M_{xx} - v_2 q \right] dx \\
&\quad - \left[v_2 \left(\frac{dw}{dx} N_{xx} + \frac{dM_{xx}}{dx} \right) \right]_{x_a}^{x_b} - \left[\left(-\frac{dv_2}{dx} \right) M_{xx} \right]_{x_a}^{x_b} \\
&= \int_{x_a}^{x_b} \left[\frac{dv_2}{dx}\left(\frac{dw}{dx} N_{xx} \right) - \frac{d^2 v_2}{dx^2} M_{xx} - v_2 q \right] dx \\
&\quad - v_2(x_a) \left[-\left(\frac{dw}{dx} N_{xx} + \frac{dM_{xx}}{dx} \right) \right]_{x_a} - v_2(x_b) \left[\frac{dw}{dx} N_{xx} + \frac{dM_{xx}}{dx} \right]_{x_b} \\
&\quad - \left[-\frac{dv_2}{dx} \right]_{x_a} [-M_{xx}(x_a)] - \left[-\frac{dv_2}{dx} \right]_{x_b} M_{xx}(x_b) \quad (5.2.17)
\end{aligned}
$$

- We clearly see that the weights are nothing but the variations and the external load work has been replaced by the internal force equivalents!!!!

- Now the funny thing is that we kept as N and M and now we shall replace them by

$$
\begin{aligned}
N_{xx} = \int_A \sigma_{xx} dA = \int_{A^e} E^e \left[\frac{du}{dx} + \frac{1}{2}(\frac{dw}{dx})^2 - z\frac{d^2 w}{dx^2} \right] dA \\
= A^e \left[\frac{du}{dx} + \frac{1}{2}(\frac{dw}{dx})^2 \right] - B^e z \frac{d^2 w}{dx^2} \qquad (10) \\
M_{xx} = \int_{A^e} \sigma_{xx} z dA = B^e \left[\frac{du}{dx} + \frac{1}{2}(\frac{dw}{dx})^2 \right] - D^e \frac{d^2 w}{dx^2}
\end{aligned}
$$

where $A = EA, \qquad B = 0 \left(\int z dA = 0 \right), \qquad D = EI \left(\int z^2 dA \right)$

- Therefore the virtual work equations can be written as

$$
\begin{aligned}
0 = \int_{x_a^e}^{x_b^e} \left(A\frac{d\delta u}{x} \left[\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 \right] - f\delta u \right) dx - \delta u(x_a)Q_1 - \delta u(x_b)Q_4 \\
0 = \int_{x_a^e}^{x_b^e} \left(\frac{d\delta w}{dx}\frac{dw}{dx} A \left(\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 \right) + D\frac{d^2\delta w}{dx^2}\frac{d^2 w}{dx^2} - q\delta w \right) dx \qquad (11) \\
-\delta w(x_a)Q_2 - \delta\theta(x_a)Q_3 - \delta w(x_b)Q_5 - \delta\theta(x_b)Q_6
\end{aligned}
$$

- We take the rotation as $\theta = -\frac{dw}{dx}$
- We approximate the axial disp as linear lagrange and the transverse with hermite cubic interpolation functions

$$0 = \sum_{j=1}^{2} K_{ij}^{11} u_j + \sum_{J=1}^{4} K_{iJ}^{12} \bar{\Delta}_J - F_i^1 \quad (i = 1, 2)$$ - Axial

$u = [u_1 \ u_2]^T$

(5.2.30)

$$0 = \sum_{j=1}^{2} K_{Ij}^{21} u_j + \sum_{J=1}^{4} K_{IJ}^{22} \bar{\Delta}_J - F_I^2 \quad (I = 1, 2, 3, 4)$$ - Bending

$$\Delta = \begin{bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{bmatrix}$$

where, for the case in which $B_{xx} = 0$, we have

$$K_{ij}^{11} = \int_{x_a}^{x_b} A_{xx} \frac{d\psi_i}{dx} \frac{d\psi_j}{dx}\, dx\ , \qquad K_{iJ}^{12} = \frac{1}{2} \int_{x_a}^{x_b} \left(A_{xx} \frac{dw}{dx} \right) \frac{d\psi_i}{dx} \frac{d\varphi_J}{dx}\, dx$$

$$K_{Ij}^{21} = \int_{x_a}^{x_b} A_{xx} \frac{dw}{dx} \frac{d\varphi_I}{dx} \frac{d\psi_j}{dx}\, dx\ , \quad K_{Ij}^{21} = 2K_{jI}^{12}$$

$$K_{IJ}^{22} = \int_{x_a}^{x_b} D_{xx} \frac{d^2\varphi_I}{dx^2} \frac{d^2\varphi_J}{dx^2}\, dx + \frac{1}{2} \int_{x_a}^{x_b} \left[A_{xx} \left(\frac{dw}{dx} \right)^2 \right] \frac{d\varphi_I}{dx} \frac{d\varphi_J}{dx} dx$$

$$F_i^1 = \int_{x_a}^{x_b} f\, \psi_i\, dx + \hat{Q}_i\ , \qquad F_I^2 = \int_{x_a}^{x_b} q\, \varphi_I\, dx + \bar{Q}_I \tag{5.2.31}$$

for $(i, j = 1, 2)$ and $(I, J = 1, 2, 3, 4)$, where $\hat{Q}_1 = Q_1$, $\hat{Q}_2 = Q_4$, $\bar{Q}_1 = Q_2$, $\bar{Q}_2 = Q_3$, $\bar{Q}_3 = Q_5$, and $\bar{Q}_4 = Q_6$, and the definition of Q_i is given Eq. (5.2.15). The pair of equations in Eq. (5.2.30) can be expressed in matrix form as

$$\begin{bmatrix} \mathbf{K}^{11} & \mathbf{K}^{12} \\ \mathbf{K}^{21} & \mathbf{K}^{22} \end{bmatrix} \begin{Bmatrix} \mathbf{\Delta}^1 \\ \mathbf{\Delta}^2 \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}^1 \\ \mathbf{F}^2 \end{Bmatrix} \quad \text{or} \quad \mathbf{K}^e \mathbf{\Delta}^e = \mathbf{F}^e \tag{5.2.32}$$

where

$$\Delta_i^1 = u_i, \quad i = 1, 2; \quad \Delta_I^2 = \bar{\Delta}_I, \quad I = 1, 2, 3, 4 \tag{5.2.33}$$

We also note that $(\mathbf{K}^{12})^T \neq \mathbf{K}^{21}$.

- Note that the Stiffness is not symmetric. When nonlinearity is not there then 12 and 21 (which depend on w) become 0 and the system is uncoupled. That is axial is only dependant on u and bending only dependant on Δ.
- See that we have tried to keep the shape functions symmetric
- When we find these conefficients from the previous iteration and then we say that the equations are linearised
- The coefficients can be different if we had accounted for the terms differently, and uncouple then to a system of equations . In this case we have to assume that u(x) is also known and so u and w both contribute towards the nonlienarity. Read reddy page 223 for this (Did not understand as of now)
- We can also decompose the matrix because the 12 and 21 terms are very similar (Reddy 224)
- We obviously solve using direct iterative or NR method. Check reddy for the derivations of T

- For the von Karman nonlinearity, for two beams : roler roler and pined pined under transverse loading. Will not have $u = 0$ because of the coupling and the solution will not be thesame.
- Suppose the roler roler beam has a constraint on u in the middle to remove rigid body movemenl, and transverse load does not make axial strain because the beam can slide without making stresses. This will have higher transverse deflection because it can stretch.
- The pined pined however has constraints on x = 0 and x = L so it will develop axial strains.
- To make sure that the roler roler does not have any axial strain $\varepsilon^{o}_{xx} = \frac{du}{dx} + \frac{1}{2}(\frac{dw}{dx})^2 = 0$

  And basically both shoud be of the same order $-\frac{du}{dx} = \left(\frac{dw}{dx}\right)^2$
- So when w is cubic $\frac{dw}{dx}$ is square and the power makes it quad. So u should be atleast order fifth. Taking u with any polynomial less , will make the constraint not satisfied and stiff, giving zero displacement field. (Membrane locking)
- We can also treat the axial strain as constant. Since $\frac{du}{dx}$ is const, we can make $(\frac{dw}{dx})^2$ also constant. We can do this by reduced integration of all nonlinear stiffness coefficients (A,B,D).

- Strains and stresses are close when found at the Gauss points
- u is aprrox with linear lagrange and w is approx with hermite cubic polynomials.
- Membrane strain ε_{xx}^{0} is assumed constant is evaluated using one point G.Q, and the bending strain ε_{xx}^{1} is also linear and done by one point G.Q.
- $$\begin{aligned} \varepsilon_{xx}^{0} &= \frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^{w} \\ \varepsilon_{xx}^{1} &= -\frac{d^2 w}{dx^2} \end{aligned} \tag{12}$$

  And the stresses are given as follows :
- $\sigma_{xx} = \sigma_{xx}^{0} + z\sigma_{xx}^{1} = E\varepsilon_{xx}^{0} + zE\varepsilon_{xx}^{1}$

- The euler bernoulli beam is based on fact that a straight line transverse to axis before defomration remains (1) straight (2) inextensible (3) normal to mid plane after deformation. In TBT, we say the last assumption, the rotation is independant of the slope
- So we get the displacement field with an independant slope

$$u_1 = u(x) + z\phi_x(x) \qquad u2 = 0 \qquad u_3 = w(x) \tag{13}$$

- The non zero strains are :

$$\begin{aligned}\varepsilon_{xx} = \frac{du_1}{dx} + \frac{1}{2}(\frac{du_3}{dx})^2 = \frac{du}{dx} + \frac{1}{2}(\frac{dw}{dx})^2 + z\frac{d\phi}{dx} = \varepsilon_{xx}^0 + z\varepsilon_{xx}^1 \\ \gamma_{xz} = \frac{du_1}{dz}\frac{du_3}{dx} = \phi + -\left(-\frac{dw}{dx}\right) = \gamma_{xz}^0\end{aligned} \tag{14}$$

The last one we get because we remove the rotation due to w

- The virtual strains are :

$$
\begin{aligned}
\delta\varepsilon_{xx}^{0} &= \frac{d\delta u}{dx} + \frac{dw}{dx}\frac{d\delta w}{dx} \\
\delta\varepsilon_{xx}^{1} &= z\frac{d\phi}{dx} \\
\delta\gamma_{xz} &= \delta\phi + \frac{d\delta w}{dx}
\end{aligned}
\tag{15}
$$

$\gamma = \cancel{\theta_A} + \theta_B = \phi,$

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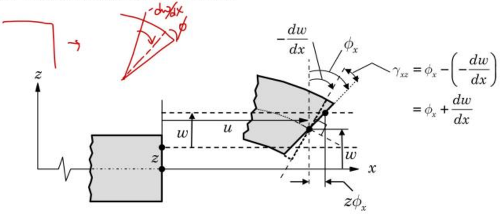

Fig. 5.3.1: Kinematics of a beam in the Timoshenko beam theory.

- The weak form is similaly developed (Reddy 243)

$$\delta W = \int_{x_a^e}^{x_b^e} \int_{A^e} \left(N_{xx} \delta \varepsilon_{xx}^0 + M_{xx} \delta \varepsilon_{xx}^1 + Q_x \delta \varepsilon_{xz}^0 \right) - \delta W_E^e \tag{16}$$

  Note that :
  $N = \int \sigma dA, \quad M = \int \sigma z dA \quad Q = \int \sigma_{xz} dA$
  $\sigma = E\varepsilon \quad \sigma_{xz} = KG\gamma_{xz}$
  The K thing is cause we assume a constant shear over the cross section when we just say stress is G x strain. So it is a correcting factor! We compare the two energies and then we find K.

- Trying to keep all the variations in the same derivative order, we get the euler equilibrium equations, taking each coefficient as zero

$$\begin{aligned} \delta u : \quad -\frac{dN}{dx} &= f(x) \\ \delta w : \quad -\frac{dQ}{dx} - \frac{d}{dx}\left(N\frac{dw}{dx}\right) &= q(x) \\ \delta \phi : -\frac{dM}{dx} + Q_x &= 0 \end{aligned} \tag{17}$$

  u,w, ϕ are the primary and N,Q,M are the secondary variables

- We keep the secondary variables in terms of the independant primary variables

$$
\begin{aligned}
N &= A\left[\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right] + B\frac{d\phi}{dx} \\
M &= B\left[\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right] + D\frac{d\phi}{dx} \\
Q &= S\left(\frac{dw}{dx} + \phi\right)
\end{aligned}
\tag{18}
$$

  A,B,D are the previous terms moments of area giving the integral only in the x direction in the virtual work. S is the shear stiffness given as $S = K\int GdA = KGA = \frac{KEA}{2(1+\nu)}$

- Obviously we can derive the governing differential equilibrium equations (Page 245). For homogeneous beams , we have B = 0. which are the equilibrium equations given by the variational principle

- The virtual work statement is then given as

$$0 = \int_{x_a}^{x_b} \left\{ A_{xx} \frac{d\delta u}{dx} \left[\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 \right] - f\,\delta u \right\} dx - Q_1^e\,\delta u(x_a) - Q_4^e\,\delta u(x_b) \qquad (5.3.18)$$

$$0 = \int_{x_a}^{x_b} \frac{d\delta w}{dx} \left\{ S_{xx}\left(\frac{dw}{dx} + \phi_x\right) + A_{xx}\frac{dw}{dx}\left[\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 \right] \right\} dx - \int_{x_a}^{x_b} \delta w q\, dx - Q_2^e\,\delta w(x_a) - Q_5^e\,\delta w(x_b) \qquad (5.3.19)$$

$$0 = \int_{x_a}^{x_b} \left[D_{xx}\frac{d\delta\phi_x}{dx}\frac{d\phi_x}{dx} + S_{xx}\,\delta\phi_x\left(\frac{dw}{dx} + \phi_x\right) \right] dx - Q_3^e\,\delta\phi_x(x_a) - Q_6^e\,\delta\phi_x(x_b) \qquad (5.3.20)$$

- The Q's have the same meaning as the euler bernoulli element

ϕ_x may be used. Substitution of Eq. (5.3.22) for u, w, and ϕ_x, and $\delta u = \psi_i^{(1)}$, $\delta w = \psi_i^{(2)}$, and $\delta\phi_x = \psi_i^{(3)}$ into Eqs. (5.3.18)–(5.3.20) yields the finite element model

$$0 = \sum_{j=1}^{m} K_{ij}^{11} u_j^e + \sum_{j=1}^{n} K_{ij}^{12} w_j^e + \sum_{j=1}^{p} K_{ij}^{13} s_j^e - F_i^1 \tag{5.3.23}$$

$$0 = \sum_{j=1}^{m} K_{ij}^{21} u_j^e + \sum_{j=1}^{n} K_{ij}^{22} w_j^e + \sum_{j=1}^{p} K_{ij}^{23} s_j^e - F_i^2 \tag{5.3.24}$$

$$0 = \sum_{j=1}^{m} K_{ij}^{31} u_j^e + \sum_{j=1}^{n} K_{ij}^{32} w_j^e + \sum_{j=1}^{p} K_{ij}^{33} s_j^e - F_i^3 \tag{5.3.25}$$

The stiffness and force coefficients are

$$K_{ij}^{11} = \int_{x_a}^{x_b} A_{xx} \frac{d\psi_i^{(1)}}{dx}\frac{d\psi_j^{(1)}}{dx}\, dx, \quad K_{ij}^{12} = \frac{1}{2}\int_{x_a}^{x_b} A_{xx} \frac{dw}{dx}\frac{d\psi_i^{(1)}}{dx}\frac{d\psi_j^{(2)}}{dx}\, dx$$

$$K_{ij}^{21} = \int_{x_a}^{x_b} A_{xx} \frac{dw}{dx}\frac{d\psi_i^{(2)}}{dx}\frac{d\psi_j^{(1)}}{dx}\, dx, \quad K_{ij}^{13} = 0, \quad K_{ij}^{31} = 0$$

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$$\begin{aligned}
K_{ij}^{22} &= \int_{x_a}^{x_b} S_{xx} \frac{d\psi_i^{(2)}}{dx}\frac{d\psi_j^{(2)}}{dx}\, dx + \frac{1}{2}\int_{x_a}^{x_b} A_{xx}\left(\frac{dw}{dx}\right)^2 \frac{d\psi_i^{(2)}}{dx}\frac{d\psi_j^{(2)}}{dx}\, dx \\
K_{ij}^{23} &= \int_{x_a}^{x_b} S_{xx} \frac{d\psi_i^{(2)}}{dx}\psi_j^{(3)}\, dx = K_{ji}^{32} \\
K_{ij}^{33} &= \int_{x_a}^{x_b} \left(D_{xx}\frac{d\psi_i^{(3)}}{dx}\frac{d\psi_j^{(3)}}{dx} + S_{xx}\psi_i^{(3)}\psi_j^{(3)}\right) dx \\
F_i^1 &= \int_{x_a}^{x_b} \psi_i^{(1)} f\, dx + Q_1^e \psi_i^{(1)}(x_a) + Q_4^e \psi_i^{(1)}(x_b) \\
F_i^2 &= \int_{x_a}^{x_b} \psi_i^{(2)} q\, dx + Q_2^e \psi_i^{(2)}(x_a) + Q_5^e \psi_i^{(2)}(x_b) \\
F_i^3 &= Q_3^e \psi_i^{(3)}(x_a) + Q_6^e \psi_i^{(3)}(x_b)
\end{aligned} \tag{5.3.26}$$

- We then get $\mathbf{Ku} = \mathbf{F}$ where $\mathbf{u} = \begin{bmatrix} u & w & \phi \end{bmatrix}^{\mathbf{T}}$. Check Reddy 249 for T stiffness derivation

- Timoshenko beam without von Karman nonlinearity differ from each other in the choice of the approx function for w and ϕ. Some are equal and others different
- Linear interpolation of both w and ϕ is the easisest. This makes the slope $\frac{dw}{dx}$ constant. In a think beam, as the length to thickness ratio becomes large (100), the slope would be equal to $-\phi$ which is linear instead of constant.
- On the other hand a constant ϕ leads to zero bending energy while the transverse shear is nonzero.
- Check Reddy 248 (Have not understood fully Locking issue)
- The primary variables may not be approximated by the same shape functions :

$$u(x) = N_i^1 u_i \qquad w(x) = N_i^2 w_i \qquad \phi(x) = N_i^3 \phi_i \tag{19}$$

Check reddy