1 Two dimensional problems having a single variable

2 Time dependant problems

Two dimensional problems having a single variable : February 18, 2021

- Suppose we are trying to find the solution $u(x, y)$ of the following partial differential equation

$$-\frac{d}{dx}\left(a_{xx}\frac{\partial du}{\partial dx} + a_{xy}\frac{\partial u}{\partial y}\right) - \frac{d}{dy}\left(a_{yx}\frac{\partial du}{\partial dx} + a_{yy}\frac{\partial u}{\partial y}\right) + a_{00}u = f(x, y) \quad \text{in} \quad \Omega \tag{1}$$

The coefficients are also a function o u : eg $a_{xx} = f(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y})$

- When we descritize it with $\bar{\Omega}$ with a boundary $\bar{\Gamma}$, we get a residual given as :

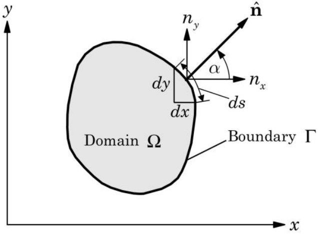
$$R(u_h) = -\frac{d}{dx}\left(a_{xx}\frac{\partial du_h}{\partial dx} + a_{xy}\frac{\partial u_h}{\partial y}\right) - \frac{d}{dy}\left(a_{yx}\frac{\partial du_h}{\partial dx} + a_{yy}\frac{\partial u_h}{\partial y}\right) + a_{00}u = f(x, y) \tag{2}$$

The step is to multiply the residual with the ith weight function $w_i(x, y)$ which should be differentiable too. We then set $w_i R$ over the element domain $\Omega^e = 0$

- $0 = \int_{\Omega_e} w_i \left[-\frac{d}{dx}\left(a_{xx}\frac{\partial u_h}{\partial x} + a_{xy}\frac{\partial u_h}{\partial y} \right) - \frac{d}{dy}\left(a_{yx}\frac{\partial u_h}{\partial x} + a_{yy}\frac{\partial u_h}{\partial y} \right) + a_{00}\, u - f(x, y) \right] dxdy$
- Now we know : $\frac{d}{dx}\left(w\frac{du}{dx} \right) = w\frac{d^2u}{dx^2} + \frac{dw}{dx}\frac{du}{dx}$
  and $\int_A \frac{d}{dx}\left(w\frac{du}{dx} \right) dA = \int_S \left(w\frac{du}{dx} \right) dS$
- We get

$$0 = \int_A \left[\frac{\partial w_i}{\partial x}\left(a_{xx}\frac{\partial u_h}{\partial x} + a_{xy}\frac{\partial u_h}{\partial y} \right) + \int_A \frac{\partial w_i}{\partial y}\left(a_{yx}\frac{\partial u_h}{\partial x} + a_{yy}\frac{\partial u_h}{\partial y} \right) + a_{00}\, w_i u_h - w_i f \right] dxdy \\ - \int_S w_i \left[\left(a_{xx}\frac{\partial u_h}{\partial x} + a_{xy}\frac{\partial u_h}{\partial y} \right) n_x + \left(a_{yx}\frac{\partial u_h}{\partial x} + a_{yy}\frac{\partial u_h}{\partial y} \right) n_y \right] dS \tag{3}$$

where $\mathbf{n} = \mathbf{n_x e_1} + \mathbf{n_y e_2}$ which gives the direction consies of the boundary Γ^e. The second term can also be written as $-\int_S q_n dS$ which is the external flux normal as we move counter clockwise.



- In the case of heat transfer through an anisotropic medium, aij denotes the conductivity and qn is the normal heat flux

- The weak form states that u should be atleast linear in both x and y
- $u_h^e = \sum u_i N_i(x,y)$ with $N_i(x_j, y_j) = \delta_{ij}$ and $\sum_j N_j(x,y) = 1$

Substituting the finite element approximation in Eq. (6.3.1) for u_h into the weak form, Eq. (6.2.5), we obtain

$$0 = \sum_{j=1}^{n} u_j^e \int_{\Omega^e} \left[\frac{\partial w_i}{\partial x} \left(a_{xx} \frac{\partial \psi_j^e}{\partial x} + a_{xy} \frac{\partial \psi_j^e}{\partial y} \right) + \frac{\partial w_i}{\partial y} \left(a_{yx} \frac{\partial \psi_j^e}{\partial x} + a_{yy} \frac{\partial \psi_j^e}{\partial y} \right) + a_{00} w_i \psi_j^e \right] dxdy - \int_{\Omega^e} w_i f \, dxdy - \oint_{\Gamma^e} w_i \, q_n \, ds \tag{6.3.3}$$

For the weak-form Galerkin model, we replace the weight function with ψ_i^e and obtain

$$\sum_{j=1}^{n} K_{ij}^e u_j^e - f_i^e - Q_i^e = 0 \quad \text{or} \quad \mathbf{K}^e \mathbf{u}^e = \mathbf{f}^e + \mathbf{Q}^e \tag{6.3.4}$$

where

$$K_{ij}^e = \int_{\Omega^e} \left[\frac{\partial \psi_i^e}{\partial x} \left(a_{xx} \frac{\partial \psi_j^e}{\partial x} + a_{xy} \frac{\partial \psi_j^e}{\partial y} \right) + \frac{\partial \psi_i^e}{\partial y} \left(a_{yx} \frac{\partial \psi_j^e}{\partial x} + a_{yy} \frac{\partial \psi_j^e}{\partial y} \right) + a_{00} \psi_i^e \psi_j^e \right] dxdy \tag{6.3.5}$$

$$f_i^e = \int_{\Omega^e} \psi_i^e f \, dx\, dy, \quad Q^e = \oint_{\Gamma^e} \psi_i^e q_n \, ds$$

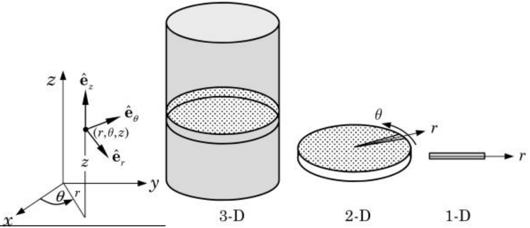
Note that $K_{ij}^e \neq K_{ji}^e$ (i.e. $\mathbf{K}^e$ is not symmetric) unless $a_{xy} = a_{yx}$. Equation (6.3.4) represents a set of n nonlinear algebraic equations.

- The equations have to be solved by nonlinear methods
- The tangent T is given in page 271

- The differential equation in cylindrical coordinate system (r,θ, z)

$$-\frac{1}{r}\frac{\partial}{\partial r}\left(ra_{rr}\frac{\partial u}{\partial r}\right)-\frac{1}{r^2}\frac{\partial}{\partial \theta}\left(a_{\theta\theta}\frac{\partial u}{\partial \theta}\right)-\frac{\partial}{\partial z}\left(a_{zz}\frac{\partial u}{\partial z}\right)=f \quad (4)$$

- where u,f and the coefficients are a function of (r,θ, z)
- Dependant on the coefficents, boundary conditions and load f the problem can be made to 2D or even 1D
- If the cylinder is very long and stuff dont vary and depend on z, then we can assume a disk. No also if there is independance from θ then we can even just do a radial line [1]



[1] We just remove the derivatives in the equations where the change will be zero

- Suppose all the variables are not dependant on θ, therefore we would get something like a plane and the Governing differential equation will be

$$-\frac{1}{r}\frac{\partial}{\partial r}\left(ra_{rr}\frac{\partial u}{\partial r}\right)-\frac{\partial}{\partial z}\left(a_{zz}\frac{\partial u}{\partial z}\right)=f(r,z) \tag{5}$$

- The weighted statement and weak form (Using greens theorem) will be given as :

$$\begin{aligned} 0 &= \int_A w_i\left[-\frac{1}{r}\frac{\partial}{\partial r}\left(ra_{rr}\frac{\partial u}{\partial r}\right)-\frac{\partial}{\partial z}\left(a_{zz}\frac{\partial u}{\partial z}\right)-f(r,z)\right]rdrdz \\ &= \int_A\left[a_{rr}(r,z,u_h)\frac{\partial w_i}{\partial r}\frac{\partial u_h}{\partial r}+a_{zz}(r,z,u_h)\frac{\partial w_i}{\partial z}\frac{\partial u_h}{\partial z}\right]rdrdz-\int_A w_i f(r,z)rdrdz-\int_S w_i q_n ds \end{aligned} \tag{6}$$

where $q_n = r\left[a_{rr}\frac{\partial u_h}{\partial r}n_r + a_{zz}\frac{\partial u_h}{\partial z}n_z\right]$

- And we get $\mathbf{Ku} = \mathbf{f} + \mathbf{Q} = \mathbf{F}$
- Remember the shape functions are functions of N(r,z)

The transformation between Ω^e and $\hat{\Omega}$ is accomplished by a coordinate transformation of the form [see Eqs. (3.6.1) and (3.6.2)]

$$x = \sum_{j=1}^{m} x_j^e \hat{\psi}_j^e(\xi, \eta) , \qquad y = \sum_{j=1}^{m} y_j^e \hat{\psi}_j^e(\xi, \eta) \tag{6.6.1}$$

while a typical dependent variable $u(x, y)$ is approximated by

$$u(x, y) = \sum_{j=1}^{n} u_j^e \psi_j^e(x, y) = \sum_{j=1}^{n} u_j^e \psi_j^e(x(\xi, \eta), y(\xi, \eta)) \tag{6.6.2}$$

where $\hat{\psi}_j^e$ denote the interpolation functions of the master element $\hat{\Omega}$ and ψ_j^e are the interpolation functions of a typical element Ω^e over which u is approximated. The transformation in Eq. (6.6.1) maps a point (x, y) in a typical element Ω^e of the mesh to a point (ξ, η) in the master element $\hat{\Omega}$ and vice versa, if the Jacobian J_e of the transformation is positive-definite [see Eqs. (3.6.5)–(3.6.8)].

- Remember that we have to we have to transform the integral domain to the master element so that Gauss quadrature can be usd. The derivatives in the original geometry are also expressed with respect to ξ, η given by

$$\begin{bmatrix} \frac{\partial N_i}{\partial dx} \\ \\ \frac{\partial N_i}{\partial dy} \end{bmatrix} = \mathbf{J}^{-1} \begin{bmatrix} \frac{\partial N_i}{\partial d\xi} \\ \\ \frac{\partial N_i}{\partial d\eta} \end{bmatrix} \tag{7}$$

where $\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \sum x_i \frac{\partial N_i}{\partial \xi} & \sum y_i \frac{\partial N_i}{\partial \xi} \\ \\ \sum x_i \frac{\partial N_i}{\partial \eta} & \sum y_i \frac{\partial N_i}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & .... & \frac{\partial N_m}{\partial \xi} \\ \\ \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & .... & \frac{\partial N_m}{\partial \eta} \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ \\ x_2 & y_2 \\ \vdots & \vdots \\ x_m & y_m \end{bmatrix}$

- If the same shape functionsa re used for the geometry and field variable, we say its isoparametric

Returning to the coefficients K_{ij}^e in Eq. (6.6.3), we can write it now in terms of the natural coordinates (ξ, η) as

$$
\begin{aligned}
K_{ij}^e = \int_{\hat{\Omega}} \Big\{ &\hat{a}_{xx}(\xi,\eta) \left(J_{11}^* \frac{\partial \psi_i^e}{\partial \xi} + J_{12}^* \frac{\partial \psi_i^e}{\partial \eta} \right) \left(J_{11}^* \frac{\partial \psi_j^e}{\partial \xi} + J_{12}^* \frac{\partial \psi_j^e}{\partial \eta} \right) \\
&+ \hat{a}_{yy}(\xi,\eta) \left(J_{21}^* \frac{\partial \psi_i^e}{\partial \xi} + J_{22}^* \frac{\partial \psi_i^e}{\partial \eta} \right) \left(J_{21}^* \frac{\partial \psi_j^e}{\partial \xi} + J_{22}^* \frac{\partial \psi_j^e}{\partial \eta} \right) \\
&+ \hat{a}_{00}(\xi,\eta) \psi_i^e \psi_j^e \Big\} J_e \, d\xi \, d\eta \\
\equiv \int_{\hat{\Omega}} & F_{ij}^e(\xi,\eta) \, d\xi \, d\eta
\end{aligned}
\tag{6.6.9}
$$

where the element area $dA = dxdy$ in element Ω^e is transformed to $J_e \, d\xi \, d\eta$ in the master element $\hat{\Omega}$, and $\hat{a}_{xx} = a_{xx}(x(\xi,\eta), y(\xi,\eta), u(\xi,\eta))$, and so on.

Using the $M \times N$ Gauss quadrature to evaluate integrals defined over a rectangular master element $\hat{\Omega}$, we obtain

$$
\begin{aligned}
\int_{\hat{\Omega}} F_{ij}^e(\xi,\eta) \, d\xi \, d\eta &= \int_{-1}^{1} \left[\int_{-1}^{1} F_{ij}^e(\xi,\eta) \, d\eta \right] d\xi \approx \int_{-1}^{1} \left[\sum_{J=1}^{N} F_{ij}^e(\xi,\eta_J) \, W_J \right] d\xi \\
&\approx \sum_{I=1}^{M} \sum_{J=1}^{N} F_{ij}^e(\xi_I,\eta_J) \, W_I W_J
\end{aligned}
\tag{6.6.10}
$$

where M and N denote the number of Gauss quadrature points in the ξ and η directions, respectively, (ξ_I, η_J) denote the Gauss points, and W_I and W_J denote the corresponding Gauss weights, as listed in Table 3.6.1.

As already discussed, if the integrand is a polynomial of degree p in a coordinate direction, it is integrated exactly by employing $NGP \equiv N = \mathrm{int}[\frac{1}{2}(p+1)]$ (the nearest equal or larger integer number) Gauss points in that direction. In most cases, the interpolation functions are of the same degree in both ξ and η, and we take $N = M$. For example, consider the expression involving a_{00} in K_{ij}^e of Eq. (6.6.3). When a_{00} is a linear function of ξ and η, it requires a 2×2 Gauss rule when ψ_i^e are linear and a 3×3 Gauss rule when ψ_i^e are quadratic to be evaluated exactly. When a_{00} is quadratic or cubic, it requires 3×3 and 4×4 Gauss rules for linear and quadratic *rectangular* elements, respectively.

where J* are the components of the inverse jacobian

Time dependant problems

Will have to read later!!