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Mechanics

Allan Marbaniang

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Stress and equilibrium

- We are dealing with different configurations. One configuration is maybe unstressed and the deformed one is. So at the deformed x we should get an equilibrium of stresses and the external loads
- Now, the actual stresses at the deformed or current configuration is the Cauchy stress : defined as the force in different directions by the area in different planes
- Stresses can also be defined with respect to the initial configuration X

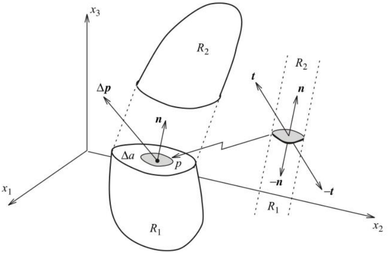


FIGURE 5.1 Traction vector.

At the deformed configuration :

- See two bodies R_1 and R_2 free body with force acting on them
- Imagine the traction vector on a small area element : $t(n) = \frac{\Delta p}{\Delta a}$ as $\lim \Delta a \longrightarrow 0$ where Δp is the resultant force
- Obviously t and n will depend on the surface it acts on. Here on the right we can see that based on the surface we get opposite forces. (In the negative normal , we will get negative force which is positive in that direction!)

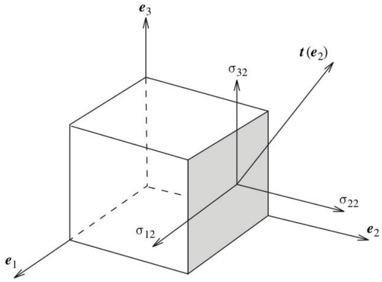
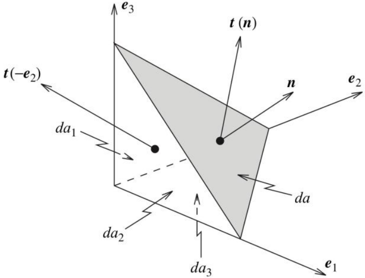


FIGURE 5.2 Stress components.

- Let us denote the traction acting on the surface having normals denoted by e_1, e_2, e_3
- Remember in the other slice we will have an opposite reaction

$$\begin{aligned} \mathbf{t}(\mathbf{e_1}) &= \sigma_{\mathbf{j1}}\mathbf{e_j} \\ \mathbf{t}(\mathbf{e_2}) &= \sigma_{\mathbf{j2}}\mathbf{e_j} \\ \mathbf{t}(\mathbf{e_3}) &= \sigma_{\mathbf{j3}}\mathbf{e_j} \end{aligned} \tag{1}$$

- Or $\mathbf{t_i} = \sigma_{\mathbf{ji}}\mathbf{e_j}$ or $\mathbf{t} = \sigma^{\mathbf{T}}\mathbf{e}$



Now let us look if we take a plane cut of that sphere. Again by context of opposite reactions. All the forces should be equal. So we will use here the concept of equilibrium between the traction vector we have defined in the last slide with respect to some basis and the traction vector defined on the angled plane.

Equilibrium

$$\mathbf{t}(\mathbf{n})da + t(-\mathbf{e_i})da_i + \mathbf{f}dv = 0 \tag{2}$$

This states that the force vector on the inclined cut should be in equilibrium with the opposite forces defined on the negative sufraces and the body force

- Now the areas (Because they are with defined respect to the basis vectors) can be written as the projection of the inclined area

$$da_i = da(\mathbf{n}.\mathbf{e_i}) \tag{3}$$

- Diving by da we get

$$\mathbf{t}(\mathbf{n}) + t(-\mathbf{e_i})\frac{da(\mathbf{n}.\mathbf{e_i})}{da} + \mathbf{f}\frac{dv}{da} = 0 \tag{4}$$

- $\frac{dv}{da} \rightarrow 0$ (I don't know why????????????????)
- We get :

$$\begin{aligned} \mathbf{t}(\mathbf{n}) = -t(-\mathbf{e_i})(\mathbf{n}.\mathbf{e_i}) = t(\mathbf{e_i})(\mathbf{n}.\mathbf{e_i}) \\ \mathbf{t}(\mathbf{n}) = (\sigma_{\mathbf{ji}}\mathbf{e_j})(\mathbf{n}.\mathbf{e_i}) \\ \mathbf{t}(\mathbf{n}) = (\sigma_{\mathbf{ij}}\mathbf{e_i})(\mathbf{n}.\mathbf{e_j})\text{Replacing indexs} \end{aligned} \tag{5}$$

- Very interesting, we started off with a statement that the resultant force on the plane is equal to the summation of the opposite forces
- Then we got the traction vector is equal to the traction vectors multiplied by some scalar product (Think of ratio)
- $t(\mathbf{e_i})(\mathbf{n}.\mathbf{e_i})$ states the traction in e_i direction multiplied by the projection of planar area for i =1,2,3
- Now we can replace the traction vector by the components of stress vectors in the basis direction $\mathbf{t}(\mathbf{n}) = (\sigma_{\mathbf{ij}}\mathbf{e_i})(\mathbf{n}.\mathbf{e_j})$
- Here we have to point out that σ_{ij} has not been described as a tensor yet

- If we look at $\mathbf{t}(\mathbf{n}) = (\sigma_{ij}\mathbf{e_i})(\mathbf{n}.\mathbf{e_j})$, we can see that σ_{ij} is just a component and $\mathbf{n}.\mathbf{e_j}$ is a scalar (Or a projection).
- That scalar value then becomes the component of e_i. Lets see what that means

$$\sigma_{12}\mathbf{e_1}(\mathbf{n}.\mathbf{e_2}) = \sigma_{12}\begin{bmatrix}1\\0\\0\end{bmatrix}\left(\begin{bmatrix}n_1 & n_2 & n_3\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \sigma_{12}n_2\begin{bmatrix}1\\0\\0\end{bmatrix} \tag{6}$$

- So we took the second component of n and added to the result of linear map in e_1.This is what we do when we multiply the first row of a matrix and a vector. We add all the components of the vector to keep in the first component of the output
- So we can write it therefore as

$$\mathbf{e_i}(\mathbf{n}.\mathbf{e_j}) = (\mathbf{e_i} \otimes \mathbf{e_j}).\mathbf{n} \tag{7}$$

  which states that the tensor takes the projection of e_j in n and maps as components of e_i
- This then allows us to understand that $\sigma_{ij}(\mathbf{e_i} \otimes \mathbf{e_j})$ is a tensor σ

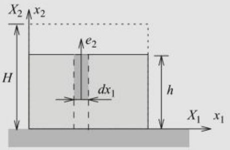
- In simplicity $\mathbf{t}(\mathbf{n}) = (\sigma_{ij}\mathbf{e_i})(\mathbf{n}.\mathbf{e_j})$ says that for every cut, $\mathbf{n}$

$$\begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \begin{bmatrix} \sigma_{11}\mathbf{e_1}(\mathbf{n}.\mathbf{e_1}) + \sigma_{12}\mathbf{e_1}(\mathbf{n}.\mathbf{e_2}) + \sigma_{13}\mathbf{e_1}(\mathbf{n}.\mathbf{e_3}) \\ \sigma_{21}\mathbf{e_2}(\mathbf{n}.\mathbf{e_1}) + \sigma_{22}\mathbf{e_2}(\mathbf{n}.\mathbf{e_2}) + \sigma_{23}\mathbf{e_2}(\mathbf{n}.\mathbf{e_3}) \\ \sigma_{31}\mathbf{e_3}(\mathbf{n}.\mathbf{e_1}) + \sigma_{32}\mathbf{e_3}(\mathbf{n}.\mathbf{e_2}) + \sigma_{33}\mathbf{e_3}(\mathbf{n}.\mathbf{e_3}) \end{bmatrix} \tag{8}$$

  FIXXX. These are not components yet!

- And now you can see that $\mathbf{e_i} \otimes \mathbf{e_j}$ just makes the tensor take the right projection and keeps in the component in the output

- $\sigma_{ij} n_j$ is therefore something like taking the projection of j for every component i



A simple example of a two-dimensional stress tensor results from the self-weight of a block of uniform initial density ρ_0 resting on a frictionless surface as shown in the figure above. For simplicity we will assume that there is no lateral deformation (in linear elasticity this would imply that the Poisson ratio $\nu = 0$).

- $t(\mathbf{e_2}) = \dfrac{\left(-\int_y^h \rho g dx_2\right)\mathbf{e_2}\, dx_1}{dx_1}$
- Mass conservation $\rho dx_1 dx_2 = \rho_o dX_1 dX_2$ + poisson = 0 give :

$$t(\mathbf{e_2}) = \rho_o g(H - X_2)\mathbf{e_2} \tag{9}$$

$$t(\mathbf{e_2}) = \sigma_{12}\mathbf{e_1} + \sigma_{22}\mathbf{e_2} \tag{10}$$

so $\sigma_{12} = 0$, so you can construct σ. Check Bonet Pge 138

- Obviously the Cauchy stress components can be described with respect to its principal directions ϕ_1, ϕ_2, ϕ_3 with principal stresses $\sigma_{\lambda_1}, \sigma_{\lambda_2}, \sigma_{\lambda_3}$
- So we can write in tensor notation : $\sigma = \sigma_{\lambda_\mathbf{i}} (\lambda_\mathbf{i} \otimes \lambda_\mathbf{i})$ (Only diagonals, the tensor is $i \otimes i$ which only is for the diagonal components)
- The cauchy stress is a spatial tensor (In the deformed configuration), and is symmetric because of the rotational equilibrium

- The idea is that the same stress should be the same as measured by different observers
- The same problem is equivalent to if we applied a rigid body motion
- So the stress tensor should not change it's property when there is a rigid body motion etc.
- ????????????????

Equilibrium

- The spatial configuration of the body has to be in equilibrium having volume v and boundary Γ
- At equilibrium, the body is under forces f and traction forces t
- Looking at the translational equilibrium of the structure, we get :

$$\int_{\delta\Gamma} \mathbf{t} da + \int_{v} \mathbf{d} dv = 0 \tag{11}$$

- In terms of The Cauchy stress we get

$$\int_{\delta\Gamma} \sigma \mathbf{n} da + \int_{v} \mathbf{d} dv = 0 \tag{12}$$

- If we use the Gauss theorem to convert the area to volume integral we get

$$\int_{\delta v} (DIV\sigma + \mathbf{d})\, dv = 0 \tag{13}$$

- As the above region can be applied to any closed region, the integrand must vanish to get $DIV\sigma + \mathbf{f} = \mathbf{0}$

- This is the equilibrium equation at a very smalllll level :

$$\frac{\partial \sigma_{ij}}{\partial xj} + f_i = 0 \tag{14}$$

- This equation is the *local* or spatial (deformed) equilibrium.
- While solving this equation may not be satisfied, and we have a pointwise out of balance or residual given as

$$r = DIV\sigma + f \tag{15}$$

- Not gonna explain. The rotational equilibrium gives you the fact that the Cauchy stress is symmetric. $\sigma^T = \sigma$
- See Bonet Page 142

Principle of virtual work

- FEM is usually based in terms of a weak form of the differential equations
- Let δv denote an arbitrary virtual velocity
- Virtual work, δw per unit volume and time by a residual force r during the virtual work as $r.\delta v$

$$\delta w = r.\delta v = 0 \tag{16}$$

- The equation above is equivalent to the equation $r = 0$
- Since δv is arbitary, we can get the seperate components of $\mathbf{r}$ if we take $\delta v = \begin{bmatrix}1 & 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0 & 1\end{bmatrix}$

The weak statement of static equilibrium of a body over it's volume is given then as

$$\delta W = \int_v \left(DIV\sigma + f\right).\delta v dv = 0 \tag{17}$$

- $DIV\,(\sigma\delta v) = (DIV\sigma).\delta v + \sigma : \nabla\delta v$

$$DIV(\sigma\delta v) = DIV\left(\delta v_1 \begin{bmatrix}\sigma_{11}\\ \sigma_{21}\\ \sigma_{31}\end{bmatrix} + \delta v_2 \begin{bmatrix}\sigma_{12}\\ \sigma_{22}\\ \sigma_{32}\end{bmatrix} + \delta v_3 \begin{bmatrix}\sigma_{13}\\ \sigma_{23}\\ \sigma_{33}\end{bmatrix}\right) \tag{18}$$

- We get the equilibrium therefore as :

$$\int_{\Gamma} \mathbf{n}.\sigma\delta\mathbf{v}\mathbf{da} - \int_{\mathbf{v}} \sigma : \nabla\delta\mathbf{v}\mathbf{dv} + \int_{\mathbf{v}} \mathbf{f}.\delta\mathbf{v}\mathbf{dv} = \mathbf{0} \tag{19}$$

- Check Bonet page 143 for symmetric velocity defined equilibrium $\delta W = \int_{\mathbf{v}} \sigma : \delta\mathbf{d}\mathbf{dv} - \int_{\mathbf{v}} \mathbf{f}.\delta\mathbf{v}\mathbf{dv} - \int_{\Gamma} \mathbf{t}.\delta\mathbf{v}\mathbf{da} = \mathbf{0}$

Alternate stress definitions

Internal virtual work given as : $\delta W = \int_{\mathbf{v}} \sigma : \delta \mathbf{d} dv$

- Now σ and d are said to be work conjugate with respect to the current deformed volume. Product gives the work per unit current volume
- If we defined the stress with respect to the undeformed(material) coordinates, alternative work conjugete pairs are needed
- The virtual work with respect to the intial volume and area by transforming the integrals is given as

$$\int_V J\sigma : \delta d \; dV = \int_V f_o.\delta v dV + \int_{\Gamma_o} t_o.\delta v dA \tag{20}$$

where $f_0 = J f$ and $t_o = t\frac{da}{dA}$ and $\frac{da}{dA} = J\sqrt{N.C^{-1}N}$

The internal virtual work then expressed as the Kirchoff tensor is

$$\delta W_{int} = \int_V \tau : \delta d dV \tag{21}$$

where $\tau = J\sigma$, we can see that τ is work conjugate to the rate of the defomration tensor with respsect to the initial volume (Rember same work density per unit mass should be invariant or $\frac{1}{\rho}\sigma : d = \frac{1}{\rho_o}\tau : d$ [J takes care of the density])

The previous internal work definition still relied on the spatial (current configuration) quantities τ and d

- Remember in the internal virtual work, the stress is conjugate with the gradient of the virtual displacements : $\int \sigma : \nabla \delta v dv = \int \sigma : \delta l dv$
- Writing as

$$\begin{aligned}
\int_V J\sigma : \delta l dv \\
\int_V J\sigma : (\delta \dot{F} F^{-1}) dV \\
\int_V J \, tr(\sigma(F^{-T}\delta \dot{F})) dV \\
\int_V J\sigma F^{-T} : \delta \dot{F} dV \\
\int_V P : \delta \dot{F} dV
\end{aligned} \tag{22}$$

where P is the first Piola-Kirchhoff stress conjugate with the deformation rate

- Unsymmetric two-point tensor with components

$$
\begin{aligned}
P &= P_{iJ}\mathbf{e_i} \otimes \mathbf{E_j} \\
P_{iJ} &= J\sigma_{ik}(F^{-1})_{Jk}
\end{aligned}
\tag{23}
$$

Virtual work is then

$$\int_V P : \delta\dot{F}dV \int_V f_o.\delta v dV + \int_\Gamma t_o.\delta v dA \tag{24}$$

If we had reversed the virtual displacement to get the equilrium differential equations we get

$$r_o = DIVP + f_o = Jr \tag{25}$$

where DIV P is withre repect to the intial configuration given as DIV P = $\nabla_o P : I$ (Diagonal elemnts basically!)

- In the current configuration a force vector dp acting on an element area $\mathbf{da} = \mathbf{n}.da$

$$d\mathbf{p} = \mathbf{t}da = \sigma da \tag{26}$$

- The Cauchy stress gives the current force per unit deformed area
- Now d**p** can be written in erms of the undedformed area dA giving

$$d\mathbf{p} = \sigma \mathbf{F}^{-\mathbf{T}} \mathbf{dA} J = Pd\mathbf{A} \tag{27}$$

so P relates an area vector in the intial configuration to a force vector in the current one

- Unsymmetric two-point tensor that is not at all related to the material (intial) configuration
- We need to pull back the force from the spatial to material configuration $d\mathbf{p} \rightarrow d\mathbf{P}$

$$d\mathbf{P} = \phi^{-1}[d\mathbf{p}] = F^{-1}d\mathbf{p} \tag{28}$$

- Now we define the second piola kirchoff as

$$\begin{aligned} d\mathbf{P} &= \mathbf{SdA} \\ \mathbf{F^{-1}}\sigma\mathbf{da} &= S\frac{1}{J}\mathbf{F^T}\mathbf{da} \\ S &= J\mathbf{F^{-1}}\sigma\mathbf{F^{-T}} \end{aligned} \tag{29}$$

- The spatial virtual rate of deformation is related to the material as

$$\delta d = F^{-T}\delta\dot{E}F^{-1} \tag{30}$$

- Keeping it in the internal virtual work

$$\begin{aligned}
\delta W_{int} &= \int_v \sigma : d dv \\
&\int_V J\sigma : d dV \\
&\int_V J\sigma : F^{-T}\delta\dot{E}F^{-1} dV \\
&\int_V J\mathrm{tr}(\sigma(F^{-T}\delta\dot{E}F^{-1})^T) dV \\
\int_V J\mathrm{tr}(\sigma F^{-T}\delta\dot{E}F^{-1}) dV = \int_V J\mathrm{tr}(\sigma F^{-1}\delta\dot{E}F^{-T}) dV &\quad [\mathrm{tr(ABCD) = tr(DABC)}] \\
&= \int_V S : \delta\dot{E} dV
\end{aligned} \tag{31}$$

- Threfore S is work conjugate to $\dot{E}$ and we have to total material description

$$\int_V S : \delta\dot{E}dV = \int_V f_o.\delta v dV + \int_{\Gamma_o} t_o.\delta v dA \tag{32}$$

We get then the relation ship between the two piola stresses and cauchy stress

$$\begin{aligned} \sigma &= J^{-1}PF^T \\ \sigma &= J^{-1}FSF^T \end{aligned} \tag{33}$$

And we also get push forward and pull back operations

- $S = J\phi_*^{-1}[\sigma]$
- $\sigma = J^{-1}\phi_*[S]$

- In case of rigid body motion, the polar decomposition of the defomration gradient gives F = R and J = 1
- S = $\mathbf{R^T Q R}$
- Second piola kirchoff components coincide with the Cauchy components given in a different basis rotated by R!!
- S is also objective and independant from reimposed rotations Q
- Check bonet Page 151 for biot stress

- It is practical to decompose the stress tensor to its deviatoric and pressure components
- This is useful as both these tensors play a different role in failure theory
- $\sigma = \sigma_{\mathbf{D}} + \sigma_{\mathbf{H}}$ whre p = 1/3 tr(σ) and tr($\sigma_D = 0$)
- Also we can do

$$\begin{aligned} P = P_D + pJF^{-T} \quad P_D = J\sigma_D F^{-T} \\ S = S_D + pJC^{-T} \quad S_D = JF^{-1}\sigma_D F^{-T} \end{aligned} \tag{34}$$

  where trace of S_D and P_D need not be zero
- Check Bonet page 151 for other relations

Check Bonet Page 152