N-Fem

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Introduction

- If a response is a function of space and time, we need to move in space and time.
- Partial discretisation, where the differential equation at each node is now already discretised by space and only dependant on time. FDM will do this at a point where at each point we have a differential equation that is now numerically defined
- $\hat{f}(x,t) \approx f(x,t)$
- There are methods developed that are dependant on Minimisation Error = $\hat{f}(x,t) - f(x,t)$
- We can also minimise the differential equation $k\frac{\partial^2 \hat{f}}{\partial^2 x} - f(x) = R(x)$ $\int_{x_a^e} W(x)R(x)dx = 0$

- Galerkin : Babnov, Petrov
- Before dynamics, if we go for statics (Steady state) without partial descritisation
- So we can $\int_{x_a}^{x_b} W(x)R(x)dx = 0$
  It forces the function to be averagely zero. So some nodes will satisfy it exactly!!
  $\int_{x_a}^{x_b} W(x).\left(k\frac{\partial^2\hat{f}}{\partial x^2} - f(x)\right) dx = 0$
- I will have to assume however the function $f(\hat{x})$. Easy to take polynomials, but the order should not vanish. Minimum requirement, but can we reduce this need of order??

- So we can keep the function like this :

$$\int_{x_a}^{x_b} W(x).k\frac{\partial^2 \hat{f}}{\partial x^2}dx - \int_{x_a}^{x_b} W(x).f(x) - dx = 0 \tag{1}$$

- And using integration by parts we get
$\left|\left(W(x).k\frac{\partial \hat{f}}{\partial x}\right)\right|_{x_a}^{x_b} - \int_{x_a}^{x_b} \frac{dW(x)}{dx}.k\frac{\partial \hat{f}}{\partial x}dx - \int_{x_a}^{x_b} W(x).f(x) - dx = 0$
- Note we have reduce the order, and we also get the boundary term. (The flux that we describe at the end). And we have a weak weighted residual statement.
- It is an integral steatement. Not point based in FDM.

- Choosing weight :
    - Any function : Petrov-Galerkin
    - Shape function : Bubnov - Galerkin
- Suppose we have actual $f(x)$ and approxx $f(\hat{x}) = a_0 + a_1 x$. Knowing the boundary conditions at x =x_a and x = x_b, where $\hat{f} = f_a$ and $\hat{f} = f_b$
- We get $f(\hat{x}) = \frac{x_b - x}{x_b - x_a} f_a + \frac{x - x_a}{x_b - x_a} f_b$, which are shape linear functions. Where we get the boundary values if we keep x at boundary.

$$f(\hat{x}) = N_1(x) f_a + N_2(x) f_b \tag{2}$$

given by Ritz, Ritz approximation which gives us a way how to choose $f(\hat{x})$

- Petrov : $W_i = N_i$ and suppose $f(\hat{x}) = \sum_1^3 N_i(x) f_i$
Weak form
$\left|\left(W(x).k\frac{\partial \hat{f}}{\partial x}\right)\right|_{x_a}^{x_b} - \int_{x_a}^{x_b} \frac{dW(x)}{dx}.k\frac{\partial \hat{f}}{\partial x} dx - \int_{x_a}^{x_b} W(x).f(x) - dx = 0$
$B.T - \int_{x_a}^{x_b} \frac{dN_i}{dx}.k\frac{\partial}{\partial x}(N_1 f_1 + N_2 f_2 + N_3 f_3) dx - \int_{x_a}^{x_b} N_i.f(x) - dx = 0$
Weight gives equation at each node. Only is differentiation in partial.
- If we keep the unkown boundary terms as a vector we get

$$\mathbf{0} = \mathbf{B.T} - \mathbf{Kf} - \mathbf{P} \tag{3}$$

where $K_{ij} = \int_{x_a}^{x_b} \frac{dN_i}{dx}.k\frac{\partial N_j}{\partial x} dx$

- FDM writes at every node. Here we minimise the governing differential in the domain (Integral). The weak form is valid over the entire domain. And now we write the equation at some points dependant on the approx funciton.

- For a three parameter approximation

$$\begin{bmatrix} 0-BT \\ 0-BT \\ 0-BT \end{bmatrix} = - \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} - \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \tag{4}$$

where $K_{ij} = \int_{x_a}^{x_b} \frac{dN_i}{dx}.k\frac{\partial N_j}{\partial x}dx$ an $p_i = \int_{x_a}^{x_b} N_i f(x)dx$

- $\mathbf{Kd} = (\mathbf{BT} - \mathbf{p}) = \mathbf{F}$ which is the descretised form.

- Differential system $\rightarrow$ Alegebric system

- We can take N as picewise also (T.Kant)

- B.T will disappear so for a three noded we get :

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} BT-0 \\ 0 \\ BT-0 \end{bmatrix} + \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \tag{5}$$

B.T = $(W.k\frac{du}{dx})|_{x_1}^{x_2}$, once u is known at boundary and prescribed, the weight will be zero. Weight is given only at points where we don't know

- Any gerneral function can be approximated by linear terms in a smaller domain, while in a larger domain the function is more complicated.
- This is an integral method so we can always decretise it

$$0 = \int_{x_a}^{x_b} W(x)R(x)dx = \sum_{e=1}^{n} \int_{x_a^e}^{x_b^e} W^e(x)R^e(x)dx \tag{6}$$

- Now so this is the concept of finite element method form of weighted residual method
  - Continuity of the field variables must be maintained
  - So we can do the computation only on one element
- Boundary volume method : Only at the boundary (Dimension less)

- Again when we keep the full term we get

$$\left|\left(W(x).k\frac{\partial\hat{f}}{\partial x}\right)\right|_{x_a}^{x_b} - \int_{x_a}^{x_b} \frac{dW(x)}{dx}.k\frac{\partial\hat{f}}{\partial x}dx - \int_{x_a}^{x_b} W(x).f(x) - dx = 0 \qquad (7)$$

- In the interior nodes of a 2 element discretisation of 2 noded element, the boundary term from element 1 becomes $BT_{x_2} - BT_{x_1}$ and from element 2 becomes $BT_{x_3} - BT_{x_2}$
- As we Join node 2, we will get at node x2 that the boundary terms will cancel each other.
- The way I like to look at this is that the B.T. is the internal force from each element and corresponding face. Each element will give a boundary term pointing corresponding to the face.

- So in descritised form for a 2 noded element we get :

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} BT_1 \\ BT_2 \end{bmatrix} + \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \tag{8}$$

where $K_{ij} = \int_{x_a^e}^{x_b^e} \frac{\partial N_i^e}{\partial x} k \frac{\partial N_j^e}{\partial x} dx$
$p_i^e = \int_{x_a^e}^{x_b^e} N_i^e f(x) dx$

- And then we can write the equations for each element, and rearange the dof as global dof in the same vector and we get our global matrix!
- The advantage is always that all the integrals are done in the sub domain

- With the local weighted residual method, we can discretise it into simpler domains etc.
- Second order - Axial. Bending is forth order, (If we do second order, then two equations with two unkowns)
- Axial rod subjected to axial deformation

$$\begin{aligned} EA\frac{\partial^2 u}{\partial x^2} &= p(x) \\ EI\frac{\partial^4 u}{\partial x^4} &= p(x)(Bending) \end{aligned} \tag{9}$$

with boundary conditions.

- Unkown per node is 1, but varies over x and y
- $\frac{\partial}{\partial x}\left(k_x \frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y \frac{\partial \phi}{\partial y}\right) + p(x, y) = 0$
- Mixed boundary conditions
  - Dritchlet : $\phi = \hat{\phi}$ in some boundary
  - General Newman :$k_x \frac{\partial \phi}{\partial x} L_x + k_y \frac{\partial \phi}{\partial y}{}_y + q + \alpha(\phi - \phi_a) = 0$: In some portion of the boundary
  - For isotropic material

$$k\frac{\partial \phi}{\partial n} + q + \alpha(\phi - \phi_a) \tag{10}$$

2D QUASI-HARMONIC EQUATION

- 1 dof/node
- Wide range of physical problems governed by Q-H equation.

PHYSICAL PROB.	UNKNOWN, ϕ	k_x, k_y	$p(x,y)$
Heat cond.	Temperature	Conductivity	Int. heat gen.
Gas diffusion	Concentration	Diffusivity	
Seepage	Pressure head	Permeability	
Incompressible ideal flow	Streamfunction	1	
Compressible flow	Velocity	Density	
Magnetostatics	Mag. potential	Reluctivity	Current density
Torsion	Stress function	$(\text{Shear mod})^{-1}$	Twist
Torsion	Warping function	Shear mod.	
Lubrication	Pressure	$(\text{Film thickness})^3/\text{Viscosity}$	Lubricant supply

- So $R(x,y) = k\frac{\partial^2\phi}{\partial x^2} + k\frac{\partial^2\phi}{\partial y^2} + p$
- Global error = $\sum_{e=1}^{N} \int_{\Omega_e} local$
- WRM

$$\int\int_{\Omega} W\left(k\frac{\partial^2\phi}{\partial x^2} + k\frac{\partial^2\phi}{\partial y^2} + p\right) dxdy = 0 \qquad (Strongstatement) \tag{11}$$

- Greens theorem :

$$\int\int_{A}\left(\frac{\partial C}{\partial x}\frac{\partial D}{\partial x} - C\frac{\partial^2 D}{\partial x^2}\right) dA = \int_{S} C\frac{\partial D}{\partial x} L_x ds \tag{12}$$

So we use this we get :

$$\int\int_{A} C\frac{\partial^2 D}{\partial x^2} dA = \int\int_{A}\frac{\partial C}{\partial x}\frac{\partial D}{\partial x} - \int_{S} C\frac{\partial D}{\partial x} L_x ds \tag{13}$$

The first term is like $W(k\frac{\partial^2\phi}{\partial x^2})$

- $\int\int_A \frac{\partial W}{\partial x} k \frac{\partial \hat{\phi}}{\partial x} dA - \int_S W.k \frac{\partial \hat{\phi}}{\partial x} L_x ds + \int\int_A \frac{\partial W}{\partial y} k \frac{\partial \hat{\phi}}{\partial y} dA - \int_S W.k \frac{\partial \hat{\phi}}{\partial y} L_y ds + \int\int WpdA = 0$ (Weak statement)
- Where L_x is direction cosine of s with respect to x
- $\hat{\phi} = \sum_{i=1}^{n} N_i(x, y)\phi_i$ (Ritz, bubonov)
- For j^{th} node $\int\int_A \frac{\partial N_j}{\partial x} k \frac{\partial \sum_{i=1}^{n} N_i\phi_i}{\partial x} dA - \int_S N_j k \frac{\partial \sum_{i=1}^{n} N_i\phi_i}{\partial x} L_x ds + \int\int_A \frac{\partial N_j}{\partial y} k \frac{\partial \sum_{i=1}^{n} N_i\phi_i}{\partial y} dA - \int_S N_j k \frac{\partial \sum_{i=1}^{n} N_i\phi_i}{\partial y} L_y ds + \int\int_A N_j pdA = 0$
- So $K_{ij} = \int\int_A k \left(\frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial y} \right) dA$
- $\mathbf{Kd} = \mathbf{P} + \mathbf{X_i}(\mathbf{Direct}) + \mathbf{Newman} = F$

- Anticlockwise noded. Each node has (x,y)
- $\hat{\phi} = N_1(x, y)\phi_1 + N_2(x, y)\phi_2 + N_3(x, y)\phi_3$
- Let $\hat{\phi} = a + bx + cy$ (Three unknowns, you need three for a plane)
- $\begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$
- Then we can find the shape functions
- Then we can find the element! Stress and strain is constant

- In plane deformation : Plane stress, plane strain, axisymmetric
- Integration over the triangle domain (area) poses problems
- Then they developed area coordinates, which is the area sections that they develop for the shape functions $L_i = N_i$. Some area integration based on these area coordinates.
- The stiffness is constant, and area can be formed while the force can be found

The steps of a finite element method are

- Divide the whole domain ino finite parts
- For each element develop relations between pairs of dual variables (primary and secondary, eg forces and displacements)
- Assemle elements together to get the replationship of the variables for the whole system

We are going to look at second differential equations and how to solve them using fem

Table 3.2.1: List of fields in which the model equation in Eq. (3.2.1) arises, with meaning of various parameters and variables; see the bottom of the table for the meaning of the parameters*.

Field of study	Primary variable u	Coefficient a	Coefficient c	Source term f	Secondary variable Q
Heat transfer	Temperature $T - T_\infty$	Thermal conductance kA	Surface convection $p\beta$	Heat generation f	Heat Q
Flow through porous medium	Fluid head ϕ	Permeability μ	$--$ 0	Infiltration f	Point source Q
Flow through pipes	Pressure P	Pipe resistance $1/R$	$--$ 0	$--$ 0	Point source Q
Flow of viscous fluids	Velocity v_z	Viscosity μ	$--$ 0	Pressure gradient $-dP/dx$	Shear stress σ_{xz}
Elastic cables	Displacement u	Tension T	$--$ 0	Transverse force f	Point force P
Elastic bars	Displacement u	Axial stiffness EA	$--$ 0	Axial force f	Point load P
Torsion of bars	Angle of twist θ	Shear stiffness GJ	$--$ 0	$--$ 0	Torque T
Electrostatics	Electrical potential ϕ	Dielectric constant ϵ	$--$ 0	Charge density ρ	Electric flux E

* k = thermal conductance; β = convective film conductance; p = perimeter; P = pressure or force; T_∞ = ambient temperature of the surrounding fluid medium; $R = 128\mu h/(\pi d^4)$ with μ being the viscosity; h, the length and d the diameter of the pipe; E = Young's modulus; A = area of cross-section; J = polar moment of inertia.

- Consider the equation

$$-\frac{d}{dx}\left(a\frac{u}{x}\right)+cu=f \qquad for \qquad 0<x<L \tag{14}$$

- where $a=a(x), c=c(x), f=f(x)$ are the known quantities and $u(x)$ has to be found

- The domain $\Omega = (0, L)$ is descritised into a set of intervals with $\Omega^e = (x_a^e, x_b^e)$ which denotes the end of the element
- The length of an element $h_e = x_b^e - x_a^e$

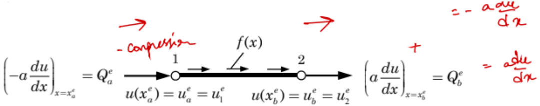


Fig. 3.2.1: A typical finite element in one dimension.

- We find an approx solution over each element Ω^e and then we assemble it all together

$$u(x) \approx u_h^e(x) = \sum_j^n c_j^e N_j^e \tag{15}$$

  where we choose the shape functions and then have to find the coefficients such that oour approx solution is like the real one
- Since there are n unkown parameters (For each dof), we need n linearly independant equations

- Keeping the discretised DE equation we get

$$-\frac{d}{dx}\left(a\frac{u_h^e}{x}\right) + cu_h^e - f(x) = R^e(x, c_1^e, c_2^e, c_3^e, ..., c_n^e) \neq 0 \tag{16}$$

What we want to do is find the coefficients such that the residual is zero. This is again equilibirum at a point!

- One way to make it zero is to set the weighted integral of the residual zero

$$\int_{x_a^e}^{x_b^e} w_i^e(x) R^e(x, c_1^e, c_2^e, c_3^e, ..., c_n^e) dx = 0 \qquad i = 1, 2, ..., n \tag{17}$$

- where w_i^e are different weight functions giving us n equations for the coefficien parameters $(c_1^e, c_2^e, c_3^e, ..., c_n^e)$
- Now the weights however shoul all be independent and invertible. If we took w = 1, we would have only one equation
- These weights feel like the variation, but in the variation we choose also that the variations are the same function like the shape functions hmmmm.

- If we choose w_i^e to be the shape functions. We get the Galerkin method. Which is exactly the same as the virtual work where $\delta v = \sum_i \delta v_i N_i$ where you can say then that δv_i is 1, since the virtual disp magnitude comes out anyways
- Since the residual R_e has the same order derivatives of the dependant unkonw u(x), we need at least quadratic representation of $u_h^e(x)$
- To reduce or weaken the differentiability of the shape functions (node disp are constant) , we distribute the order between the weights and u_h^e

This is the weak form : Reducing the order of the dependant variable to the weight to make the order of the variable leser

- The weights kind of give different component equilib equations

- Note that in usual structural mechanics we derive the virtual work equation from a potentional functional. Most fem methodsa re based on an element wise application of the Ritz method
- The virtual displacement is an integral statement which is he same as the integral weak form found frm the governing differential equations.
- But differential equations are easy to form, and most fem methods are based on de

- After getting the weighted residual statement, the next job is to weaken the differentiability of u_h^e. To make both the orders of u_h^e and w_i^e the same
- The steps are
  - Write weighted residual statement
$$0 = \int_{x_a^e}^{x_b^e} \left[w_i^e \left(-\frac{d}{dx} \left(a \frac{u_h^e}{x} \right) + c u_h^e - f(x) \right) \right] dx \tag{18}$$
    We are taking the summation of the weighted residual over the whole element and saying that it is zero. The weight sort of gets the components
  - Weakening the form using integration by parts
$$(uv)' = uv' + u'v$$
$$\int_a^b uv' = uv|_a^b - \int_a^b u'v$$
$$0 = \int_{x_a^e}^{x_b^e} \left[w_i^e \left(-\frac{d}{dx} \left(a \frac{u_h^e}{x} \right) + c u_h^e - f(x) \right) \right] dx \tag{19}$$
$$0 = \int_{x_a^e}^{x_b^e} \left(a \frac{dw_i^e}{dx} \frac{du_h^e}{dx} + c w_i^e u_h^e - w_i^e f(x) \right) dx - \left[w_i^e a \frac{du_h^e}{dx} \right]_{x_a^e}^{x_b^e}$$

- Very interesting, we actually get the boundary terms too!

$$0 = \int_{x_a^e}^{x_b^e} \left(a\frac{dw_i^e}{dx}\frac{du_h^e}{dx} + cw_i^e u_h^e - w_i^e f(x) \right) dx - \left[w_i^e . a\frac{du_h^e}{dx} \right]_{x_a^e}^{x_b^e} \tag{20}$$

  Direct boundary on the dependant: Dritchlet/essential $u = 0$
  Boundary on the derivatives of dependant : Newman/natural $\frac{du}{dx} = p$

- The coefficient of the weight function which is $a\frac{du}{dx}$ is the second variable

- We state the differenet variables

$$\text{Primary variable : } u \qquad \text{Secondary variable:} n_x(a\frac{du}{dx}) = Q(x) \tag{21}$$

  See that $n_x = -1, 1$ on left and right end. ???WHye

- In the final weak form, we keep the secondary variables at the element ends as

$$Q_a^e = Q(x_a^e) = -\left(a\frac{du}{dx} \right)_{x_a^e} \qquad Q_b^e = Q(x_b^e) = \left(a\frac{du}{dx} \right)_{x_b^e} \tag{22}$$

  In the figure above we can think this of a FBD but in arbitary configruaion. The first one is a compressive, while later is a tensile force. In heat the first would be the heat input and later output

- Althout Q replaced $a(\frac{du}{dx})$, it is not consdiered as a function of u, but a variable dual to u??????

- The final expression for the weak form is

$$0 = \int_{x_a^e}^{x_b^e} \left(a \frac{dw_i^e}{dx} \frac{du_h^e}{dx} + c w_i^e u_h^e - w_i^e f(x) \right) dx - w_i^e(x_a^e) Q_a^e - w_i^e(x_b^e) Q_b^e \tag{23}$$

  But even in the virtual work when you reduce the order of the strains, you get the boundary condition

- The remarks are:
  - Integration by parts (i) reduces the degree of the fem approximation (ii) introduces the secondary variables that are physically meanifull as they can be specified at a point where the primary variable is not specified. If the secondary variable is not a physical quantity, then the integraion by parts should not be carried out even to reduce the order of u_h^e
  - The terms containing both w_h^e and u_h^e are called bilinear functional

$$B(w_i^e, u_h^e) = \int_{x_a^e}^{x_b^e} \left(a \frac{dw_i^e}{dx} \frac{du_h^e}{dx} + c w_i^e u_h^e \right) \tag{24}$$

    but has to be linear with respect to w_i^e and u_i^e. So it has to be billinear map. Like a scalar product with metric tensor where u and v are the input and the metric tensor is the bilinear map. If a or/and c is a fucntion of u. Then B is always linear in w but not u.
  - Terms having only w_i^e are only linear functionals because they are only linear with respect to w_i^e. $l(w_i^e)$

- Therefore the weak form can be expressed as

$$B(w_i^e, u_h^e) = l(w)i^2 \tag{25}$$

which is a vairational problem where we find $u^e \in U$ such that the equation is satisfied for all $w_i^e \in U$ (See Reddy page 103 for hilbert spaces)

- The weak form is nothing but the statement of minimum total potential energy, or the variational minimum

$$\begin{aligned}
&\Pi(u_h^e)\\
\delta\Pi &= B(\delta u_h^e, u_h^e) - l(\delta u_h^e) = 0\\
\Pi(u_h^e) &= \frac{1}{2}B(\delta u_h^e, u_h^e) - l(\delta u_h^e) \qquad (26)\\
&= \int_{x_a^e}^{x_b^e}\left[\frac{a}{2}\left(\frac{du_i^e}{dx}\right)^2 + \frac{c}{2}(u_h^e)^2 - u_h^e f\right]dx - u_h^e(x_a^e)Q_a^e - u_h^e(x_b^e)Q_b^e
\end{aligned}$$

This is when you have reduced the order of the derivative in the virtual work, (We get the Euler lagrange equilibrium).

- This equation $\frac{1}{2}B(w_i^e, u_h^e) = l(w)i^2$, B shoudl be symmetric and the first term is the elastic energy while the later is the work done by the load and point loads.

- We have to satisfy the weak form of the differential equation along with the continuity and boundary conditions. We need to choose a function that satisfies the differntiability requirement and the end conditions $u(x_i) = u_i^e$. Any function with a non zero differentiation of the order of the weak form would be a candidate. We can therefore use interpolation.
- The interpolation is

$$u_h^e(x) = c_1^e + c_2^e x \tag{27}$$

is okay, since the differentiation $\neq 0$, but we only now need to make sure that c_1, c_2 are such that the end displacements match

$$u_h^e(x_a^e) = c_1^e + c_2^e x_a^e = u_a^e \qquad u_h^e(x_b^e) = c_1^e + c_2^e x_b^e = u_b^e \tag{28}$$

or
$\begin{bmatrix} 1 & x_a^e \\ 1 & x_b^e \end{bmatrix} \begin{bmatrix} c_1^e \\ c_2^e \end{bmatrix} = \begin{bmatrix} u_a^e \\ u_b^e \end{bmatrix}$ and we get the interpolating functions $u_h^e(x) = \sum_{j=1}^{2} N_j^e u_j^e$ [1]

- $N_1^e(x) = \frac{x_b^e - x}{x_b^e - x_a^e} \qquad N_2^e(x) = \frac{x - x_a^e}{x_b^e - x_a^e}$

[1] The book makes ϕ for N. But I usually use that for eigen directions?

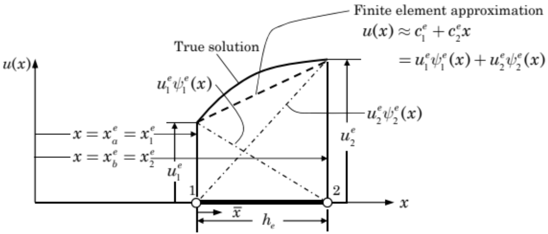


Fig. 3.2.2: Linear approximation over a finite element.

- These N are linear lagrange interpolation functions and $u_1^e = u_a^e \qquad u_2^e = u_b^e$ are the nodal values of the approx function at the ends. U_h belongs to a hilbert subspace spanned by N_1^e, N_2^e
- Remember that $N_i^e(x_j^e) = 1 \quad \text{if} i = j$. They also satisfy the partition of unity where $\sum_{j=1} N_j^e(x) = 1$

For a quadratic approximation we choose

$$u_h^e(x) = c_1^e + c_2^e x + c_3^e x^2 \tag{29}$$

- Since there are three parameters we need to have three nodal points where we can relate the constants to.

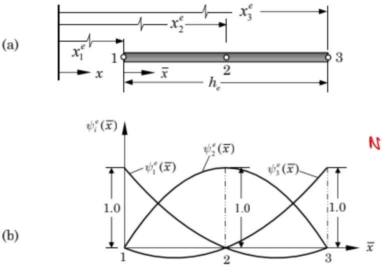
$$x_1^e = x_a^e \qquad x_2^e = x_a^e + \frac{h_e}{2} \qquad x_3^e = x_a^e + h_e = x_b^e \tag{30}$$

And we similary get :

$$u_h^e(x) = \sum_{j=1} N_j^e u_j^e \tag{31}$$

where N are the quadratic lagrange interpolation functions. If they are expressed in local coordinate we get

$$N_1^e(x') = \left(1 - \frac{x'}{h_e}\right)\left(1 - \frac{2x'}{h_e}\right) \qquad N_2^e(x') = 4\frac{x'}{h_e}\left(1 - \frac{x'}{h_e}\right) \qquad N_3^e(x') = -\frac{x'}{h_e}\left(1 - \frac{2x'}{h_e}\right) \tag{32}$$



$N_1 = \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)}$

- This is a quadratic element
- Any higher order lagrange interpolations can be developed. A (n-1)degree can be written as $u_h^e \sum_{j=1}^{n} N_j^e u_j^e$
- Where the interpolating function can be given as

$$N_j^e(x) = \Pi_{i=1, i \neq j} \left(\frac{x - x_i^e}{x_j^e - x_i^e} \right) \tag{33}$$

For example $N_1(x) = \left(\frac{x - x_2}{x_1 - x_2} \right) \left(\frac{x - x_3}{x_1 - x_3} \right)$

- Approx solution should be continous and differentiable as needed by the weak form. This ensures that every term in the differential equaion does not have a zero coefficient.
- It should be a complete polynomial (Pascal's law). To capture all the actual deformation. Lower to higher!
- It should interpolate the primary variables at the nodes of the fem and at the end points. To ensure continuity of the primary variable acorss elements!

- Keeping the approximate solutions in the weak form gives us the algebraic equatins
- The degree of the approx solution has to be decied a priori. If there are more than 2 nodes, then the number of non-zero secondary variables increases at the interior nodes

$$0 = \int_{x_a}^{x_b} \left(a \frac{dw_i^e}{dx} \frac{du_h^e}{dx} + c w_i^e u_h^e \right) dx - \int_{x_a}^{x_b} w_i^e f dx - \sum_{j=1}^{n} w_i^e (x_j^e) Q_j^e \tag{34}$$

- If 1 and n denote the end points then Q_1^e, Q_n^e denote the unknown point sources, while the other $Q_j^e \quad (j = 2, 3 \ldots n-1)$ are the externally applied and known point sources. So at the ends these are internal or external loads??????
- If we keep $w_i^e = N_i^e$ into the weak form, we get n algebraic equations. This is the Galerkin method (Original was weighted of residual and not of the weak form, that would be exactly the same to Ritz method). The *i*th algebraic equation is the one obtained by keeping w_i^e as N_i^e. This is the same as the virtual work method, where each equation of a discretised system comes from the virtual displacement of each node

- So we get

$$0 = \int_{x_a}^{x_b} \left(a \frac{dN_i^e}{dx} \sum_{j=1}^{n} u_j^e \frac{dN_j^e}{dx} + cN_i^e \sum_{j=1}^{n} u_j^e N_j^e \right) dx - \int_{x_a}^{x_b} N_i^e f dx - \sum_{j=1}^{n} N_i^e(x_j^e) Q_j^e \quad (35)$$

so for each equation i there will be a summation on the derivatives of the approx solution due to chain rule! and we get

$$0 = \sum_{j=1}^{n} \left[\int_{x_a}^{x_b} \left(a \frac{dN_i^e}{dx} \frac{dN_j^e}{dx} + cN_i^e N_j^e \right) dx \right] u_j^e - \int_{x_a}^{x_b} N_i^e f dx - Q_i^e \quad (36)$$

where we have taken the summation of u magitude coefficient for, shape functions outside. Also we see that for each shape function at each node we actually only get the boundary load at that variation. Because $N_i^e(x_j^e) = 0$ when $i \neq j$. We get for each node!

$$0 = \sum_{j=1}^{n} K_{ij}^e u_j^e - f_i^e - Q_i^e \quad (37)$$

- Where

$$K_{ij}^e = \int_{x_a}^{x_b} \left(a \frac{dN_i^e}{dx} \frac{dN_j^e}{dx} + cN_i^e N_j^e \right) dx = B(N_i^e, N_j^e)$$
$$f_i^e = \int_{x_a}^{x_b} fN_i^e dx = l(N_i^e) \tag{38}$$

So this is ineresting the coefficient of the stifness says basically states chagne in the shape functions!

Matrix form

$$\mathbf{K^e u^e} = \mathbf{f^e} + \mathbf{Q^e} \approx \mathbf{F^e}$$

where

- K^e is the symmetric coefficient, stiffness matrix
- f^e is the source or force vector
- This method is the weak-form Glaerikin or Ritz finite element method

Matrix form

$$\mathbf{K^e u^e = f^e + Q^e \approx F^e}$$

- But for every element, we have n equations and $n + 2$ unknowns. The 2 unkowns are the secondary nodal values that we don't know Q_a^e, Q_b^e. (u_1^e, u_2^e...u_n^e) are the element primary nodal degrees. Remember these values, we know if they are along the inside of the element as external forces.
- Assembling the elements by imposing the continuity of the elements. U2 of 1 is U1 of 2. We get the same number of equations and unkowns (Primary + Secondary).
- The stiffness and force matrix can be found for a certain value. And if the coefficients (a,c,f) are also functions of x, then we need to do numerical integration.

Linear element (i.e. element with linear approximation)

$$\psi_1^e(\bar{x}) = 1 - \frac{\bar{x}}{h_e}, \qquad \psi_2^e(\bar{x}) = \frac{\bar{x}}{h_e} \tag{3.2.37}$$

$$\left(\frac{a_e}{h_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{c_e h_e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right)\begin{Bmatrix} u_1^e \\ u_2^e \end{Bmatrix} = \frac{f_e h_e}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} + \begin{Bmatrix} Q_1^e \\ Q_2^e \end{Bmatrix} \tag{3.2.38}$$

Quadratic element (i.e. element with quadratic approximation)

$$\psi_1^e(\bar{x}) = \left(1 - \frac{2\bar{x}}{h_e}\right)\left(1 - \frac{\bar{x}}{h_e}\right), \ \psi_2^e(\bar{x}) = \frac{4\bar{x}}{h_e}\left(1 - \frac{\bar{x}}{h_e}\right), \ \psi_3^e(\bar{x}) = -\frac{\bar{x}}{h_e}\left(1 - \frac{2\bar{x}}{h_e}\right) \tag{3.2.39}$$

$$\left(\frac{a_e}{3h_e}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix} + \frac{c_e h_e}{30}\begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix}\right)\begin{Bmatrix} u_1^e \\ u_2^e \\ u_3^e \end{Bmatrix} = \frac{f_e h_e}{6}\begin{Bmatrix} 1 \\ 4 \\ 1 \end{Bmatrix} + \begin{Bmatrix} Q_1^e \\ Q_2^e \\ Q_3^e \end{Bmatrix} \tag{3.2.40}$$

- Where in linear element lets write the expansion for the first linear element equation

$$K_{11} u_1^e + K_{12} u_2^e = f_1 + Q_1^e$$

$$\left[\int_0^{he} a \frac{-1}{he}\frac{-1}{he} + c\left(1 - \frac{x}{he}\right)\left(1 - \frac{x}{he}\right)\right] u_1^e + \left[\int_0^{he} a \frac{-1}{he}\frac{1}{he} + c\left(1 - \frac{x}{he}\right)\left(\frac{x}{he}\right)\right] u_2^e = \int_0^{he} f\left(1 - \frac{x}{he}\right) + Q_1^e \tag{39}$$

2

[2] We note that (i) In quad, the force vector is not just fh/3 but it depends on the work done! Not same with 2 elements combined. (ii) There are also more unkowns thatn the no of equations. When one element is used however we have only n unkowns cause bcs will be applied on Q.

- Consider a homogeneous, isotropic bar of length L(m), cross sectional area (A) and conductivity k (W/($m^o C$))
- Ambient temperature is $T_o(^oC)$
- No heat loss throughout the bar and the right end is exposed to ambient temperature of T_{inf}
- Uniform heat of g_o, heat transfer with fin and air is β
- Check Reddy Page 3.2.1

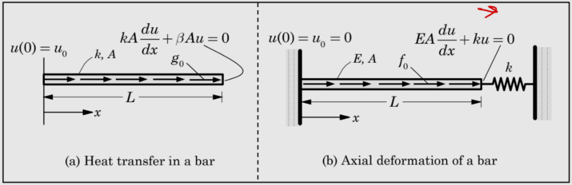
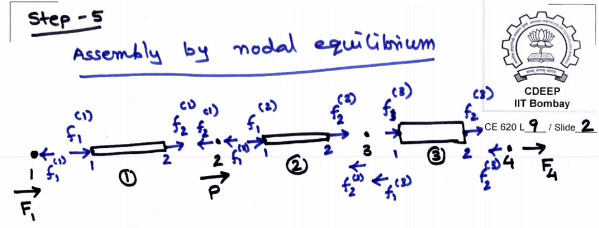


Fig. 3.2.4: (a) Heat transfer in a bar. (b) Axial deformation of a bar.



- Remember when we draw the internal force signs as compressive for a tensile element. This we are drawing for the nodal equilibrium side cut of the fbd. Rembember force, stresses are all defined with respect to the cut and face.
- In the Reddy example you will find that at the B.C., the internal force reaction at the node, Q_2^4 and kU_5 is a force in $\leftarrow$

- Check reddy 114 for origin at left side of element, natural $\bar{x}$
- Origin at center denoted as ξ
  - ξ is -1 and +1 at LHS and RHS. Since your current basis is eucledian. The transfromation is linear.
  - x can be given as a function of ξ and found out
  - This is interesting. Suppose $x = x_a + \frac{h}{2}(1+\xi)$

$$\frac{d}{dx} = \frac{2}{h}\frac{d}{d\xi} \qquad \frac{dN_i}{dx} = \frac{2}{h}\frac{dN_i}{d\xi}$$

$$dx = \frac{h}{2}d\xi$$

$$\int_{x_a^e}^{x_b^e} N_i(x)\,dx = \int_0^h N_i(\bar{x})\,d\bar{x} = \frac{h}{2}\int_{-1}^{1} N_i(\xi)\,d\xi \tag{40}$$

$$\int_{x_a^e}^{x_b^e} \frac{dN_i(x)}{x}\,dx = \int_0^h \frac{N_i(\bar{x})}{\bar{x}}\,d\bar{x} = \int_{-1}^{1} \frac{N_i(\xi)}{d\xi}\,d\xi$$

$$\int_{x_a^e}^{x_b^e} \frac{dN_i(x)}{x}\frac{dN_j(x)}{x}\,dx = \int_0^h \frac{N_i(\bar{x})}{\bar{x}}\frac{N_j(\bar{x})}{\bar{x}}\,d\bar{x} = \frac{2}{h}\int_{-1}^{1} \frac{N_i(\xi)}{d\xi}\frac{N_j(\xi)}{d\xi}\,d\xi$$

- Governing differential equation. Suppose a single field $u(x, y)$ with the following partial differential equation varies over x and y
- $\frac{\partial}{\partial x}\left(k_x \frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y \frac{\partial \phi}{\partial y}\right) + f(x, y) = 0$ where for a heat problem, k is the conucitvity in a orthotropic medium and u is the temperature and f is the internal heat generation.
- Mixed boundary conditions
  - Dritchlet : $\phi = \hat{\phi}$ in some boundary
  - General Newman :$k_x \frac{\partial u}{\partial x} n_x + k_y \frac{\partial u}{\partial y} n_y + q_c = \hat{q}_n$: In some portion of the boundary.
    - q_c represents the convective component of flux (heat problems $q_c = h_c(u - u_c)$)
    - $n_x = cos(x, \mathbf{n})$ and $n_y = cos(y, \mathbf{n})$ which is the angle of the normal of the boundary and the axis
    - u_c is the ambient temperature and h_c is the convective heat coefficient

- In FEM the domain is descritised into subdomains. Any shape qualifies as long as the approximating functions N_i^e can be derived uniquely for the shape. The discretisation may be may not represent the actual boundary though at really curved regions.
- Suppose the dependent unkown u is given by $\hat{u}^e = \sum_{j=1}^{n} u_j^e N_j^e(x, y)$
- The interpolation functions depend not only on the number of nodes but also on the shape of the element
- A triangle will need two points given by $\hat{u}^e = c_1 + c_2 x + c_3 y$
- A triangle with three nodes in each side is given by $\hat{u}^e = c_1 + c_2 x + c_3 y + c_4 xy + c_5 x^2 + c_6 y^2$

- The n nodal values u_j^e must be found such that the approximating solution $u_h^e(x)$ satisfies the governing differential equation in a weak sense. Steps are :

  1. Take non zeros of the G.D.E as R(x,y) and multiply by the weight function w_i^e from a set of linearly independant functions. We get then

  $$\int\int_\Omega w_i^e\left(k\frac{\partial^2 u_h^e}{\partial x^2}+k\frac{\partial^2 u_h^e}{\partial y^2}-f(x,y)\right)dxdy=0 \qquad (Strongstatement) \tag{41}$$

  For n independant choices of w_i^e, we get n independant equations.

  2. Distribute so that for both u_h^e, w_i^e are required to be differentiated once. Using component form of the divergence theorem or Greens theorem :

  $$\begin{gathered}\int\int_A \frac{\partial}{\partial x}(w_i^e F_1)\,dA=\int_S (w_i^e F_i)\,n_x\,ds\\ \int\int_A \frac{\partial}{\partial y}(w_i^e F_2)\,dA=\int_S (w_i^e F_i)\,n_y\,ds\\ F_1=k_x\frac{\partial u_h^e}{\partial x}\qquad F_2=k_y\frac{\partial u_h^e}{\partial y}\end{gathered} \tag{42}$$

  And from product rule we get

  $$-w_i^e\frac{\partial F_1}{\partial x}=-\frac{\partial}{\partial x}(w_i^e F_1)+F_1\frac{\partial w_i^e}{\partial x}\qquad -w_i^e\frac{\partial F_2}{\partial x}=-\frac{\partial}{\partial x}(w_i^e F_2)+F_2\frac{\partial w_i^e}{\partial x} \tag{43}$$

- And we get the weak form as
$0 = \int \int_A \left(k_x \frac{\partial w_i^e}{\partial x} \frac{\partial u_h^e}{\partial x} + k_y \frac{\partial w_i^e}{\partial x} \frac{\partial u_h^e}{\partial y} - w_i^e f(x,y) \right) dA - \int_S w_i^e \left(k_x \frac{\partial u_h^e}{\partial x} n_x + k_y \frac{\partial u_h^e}{\partial x} n_y \right) dS$
- Now the order of the differentiation has been reduced.
- Looking at the boundary terms, we see that u_h^e is the primary variable and essential boundary. $q_n = \left(k_x \frac{\partial u_h^e}{\partial x} n_x + k_y \frac{\partial u_h^e}{\partial x} n_y \right)$ is the secondary variable and the natural boundary condition. It is positive as one travels counterclockwise in the boundary.
- This is wy nodes are counted in counterclockwise and boundary integrals are carried in counter-clockwise sense. $\mathbf{q}$ is the outward flux normal and the flux
$\mathbf{q} = q_x e_1 + q_y e_2 \qquad q_x = k_x \frac{\partial u_h^e}{\partial x} \quad q_y = k_y \frac{\partial u_h^e}{\partial y}$
- The normal flux is given by

$$q_n = \hat{n}.\mathbf{q} = k_x \frac{\partial u_h^e}{\partial x} n_x + k_y \frac{\partial u_h^e}{\partial y} n_y \tag{44}$$

1. So the third step is to use the general Newmann boundary condition and write as

$$0 = \int \int_A \left(k_x \frac{\partial w_i^e}{\partial x} \frac{\partial u_h^e}{\partial x} + k_y \frac{\partial w_i^e}{\partial x} \frac{\partial u_h^e}{\partial y} - w_i^e f(x, y) \right) dA - \int_S w_i^e \left(\hat{q}_n - h_c (u_h^e - u_c) \right) dS \tag{45}$$

2. Rearragning we get

$$0 = \int \int_A \left(k_x \frac{\partial w_i^e}{\partial x} \frac{\partial u_h^e}{\partial x} + k_y \frac{\partial w_i^e}{\partial x} \frac{\partial u_h^e}{\partial y} - w_i^e f(x, y) \right) dA - \int_S h_c w_i^e u_h^e dS - \int_S w_i^e \left(\hat{q}_n + h_c u_c \right) \tag{46}$$

3. So we get the form $B(w_i^e, u_h^e) = l(w_i^e)$

- Note that the variational problem is to find a u_h^e such that $B(w_i^e, u_h^e) = l(w_i^e)$ for all $w_i^e \in U_h$ a subspace span by polynomial basis functions.
- $B(w_i^e, u_h^e)$ is bilinear and symmetric and $l(w_i^e)$ is linear in w. So we can construct a functional $I = \frac{1}{2}B(u_h^e, u_h^e) - l(u_h^e)$ and the minmum is equivalent to solving the variation problem.
- It is not always possible to make a functional whose weak form whose first variation is equivalent to the weak form.

- The weak form in the above equation requires that the approx function to be at least linear in both x and y. Suppose that $u_h^e = N_i u_i$
- The weak form is given as

$$
\begin{aligned}
\left(\int \int_A \frac{\partial N_j}{\partial x} k \frac{\partial \sum_{i=1}^{n} N_i}{\partial x} dA + \int \int_A \frac{\partial N_j}{\partial y} k \frac{\partial \sum_{i=1}^{n} N_i}{\partial y} dA + \int_S h_c w_i^e N_j^e \right) u_j^e = 0 \\
- \int_A N_i^e f dA - \int_S N_i^e (\hat{q} + h_c u_c) dS
\end{aligned}
\tag{47}
$$

- This is when w is taken as the virtual displacemnt of the dependant unkown or $w_i^e = \delta u_h^e$. Each w we get a seperate equation and we get
- $K^e u^e = f^e + q^e$

- u should be continous as required in the weak form that is all the terms are represented as non zero values
- The polynomials must be complete and contain the same order of x and y
- All the terms in the polynomial should be linearly independant. The no o flinearly independent terms in represnetin u dictate the shape and no of nodes. It turns out only triangular and quad elements satisfy this.