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Tangent matrix : February 19, 2021

- Linearity allows us to know the solution at any value. What I mean is if we have a function f(x), we can find the slope at any point of f and then get the solution
- In nonlinear cases, the slope at one point may not represent the actual solution at another so we have an issue. So we linearise the solution at some point and then find how far off is the linear solution from the nonlinear one.
- We will focus here on different derivations of the tangent stiffness matrix for different domains

Euler beam elements

- For the euler beam element we get the stiffness matrix as follows $\begin{bmatrix} K^{11} & K^{12} \\ K^{21} & K^{22} \end{bmatrix} \begin{bmatrix} \Delta^1 \\ \Delta^2 \end{bmatrix} = \begin{bmatrix} F^1 \\ F^2 \end{bmatrix}$ where we see a coupling between the two dof here (Note there are more dof inside each one)
- Sometimes we can derive the stiffness matrix terms involving the von karman nonlinearity $\frac{\partial dw}{\partial dx}$ insuch a way that $K^{12} = 0$ and so the nonlinear equations become uncoupled and we can solve them iteratively. In this case u and w will become nonlinear terms. However the tangent matrix will remain the same.

- In direct method, we just keep looping through the process till the nonlinear terms or the residual stabilise.
- Here we talk about the NR method:
- There are two things which are a bit complicated. One is the position of a body and the other is the displacement whereby it moves towards the equilibrium state.
- We see from the directional derivative notes, that the directional derivative is used to linearise two things, one is the functional of the potential energy which gives us our equilibrium equations
- The other linearisation is of these equilibrium equations giving us a linearised version of the equations at a location.

- The main problem lies I guess in the assumption of small deformations. FIX ME!!!!!!!!!!!!!!!!!!
- We have a functional which is has both internal and external work . $\Pi = d^T Kd - Fd$
- This energy functional is therefore dependant on the intial undeformed geometry. Based on it if we move by some displacement d, we change the internal energy and potential of the external work.
- In the Updated weight method, we however keep the energy in terms of the location in space. Once we found the first directional derivative, invariably we became intersted in displacements. But these displacements were of an energy functional that depended only on the position.
- What is then the difference between du and dx??? I think that u is the way to think about it.
- The first directional derivative gives you the direction that you need to move your displacement such that your potential energy gets minimized. Again note that the potential is ususally referenced to an undeformed shape.

- We have the residual equation given as $\mathbf{R} = \mathbf{F} - \mathbf{K}\Delta$
- Linearising this we get $\mathbf{R^{i+1}} = \mathbf{R^i} + \frac{dR}{d\Delta}|_i \delta\Delta$
  Interesting thing to note is that the derivative of the residual is with respect to the displacements, and the small amount we move is in a small direction in that displacement we're actually on. That is the displacement increment !!! ΔU

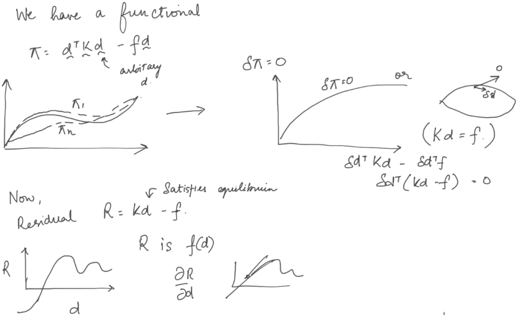

Figure:

- In Reddy 5.2.43 we get our displacement increment, which is $\Delta U = -\left(T(U^{r-1})\right)^{-1} R^{r-1}$
- Where the next solution is then found as $\mathbf{U^r} = \mathbf{U^{r-1}} + \Delta\mathbf{U}$
- The coefficients of the tangent stiffness amtrix has a similar structure to **K** so we can have the form $\begin{bmatrix} T^{11} & T^{12} \\ T^{21} & T^{22} \end{bmatrix} \begin{bmatrix} \delta\Delta^1 \\ \delta\Delta^2 \end{bmatrix} = -\begin{bmatrix} R^1 \\ R^2 \end{bmatrix}$
- Note that each coeffcients in both **T** and Δ are matrices and vectors.
- So lets say that $T^{11}\delta\Delta^1 + T^{12}\delta\Delta^2 = -R^1$ or $T^{11}_{ij}\delta\Delta^1_j + T^{12}_{ij}\delta\Delta^2_j = -R^1_i$
- Here we note that each component of T and Δ are matrices.

where symbol δ is used in place of Δ, for obvious reason, to denote the increment of the displacements. Also, note that $\mathbf{\Delta}^1 = \mathbf{u}^e$ and $\mathbf{\Delta}^2 = \bar{\mathbf{\Delta}}^e$. Then we can compute $\mathbf{T}^{\alpha\beta}$ from the definition

$$T_{ij}^{\alpha\beta} = \frac{\partial R_i^\alpha}{\partial \Delta_j^\beta}, \quad \alpha, \beta = 1, 2 \tag{5.2.46}$$

$T^{11} = \frac{\partial R^1}{\partial \Delta^1}$ $T^{12} = \frac{\partial R^2}{\partial \Delta^2}$

The components R_i^α of the residual vector $\mathbf{R}$ can be expressed as

$$\begin{aligned} R_i^\alpha &= \sum_{\gamma=1}^{2} \sum_{p=1}^{n(\gamma)} K_{ip}^{\alpha\gamma} \Delta_p^\gamma - F_i^\alpha = \sum_{p=1}^{2} K_{ip}^{\alpha 1} \Delta_p^1 + \sum_{P=1}^{4} K_{iP}^{\alpha 2} \Delta_P^2 - F_i^\alpha \\ &= \sum_{p=1}^{2} K_{ip}^{\alpha 1} u_p + \sum_{P=1}^{4} K_{iP}^{\alpha 2} \bar{\Delta}_P - F_i^\alpha \end{aligned} \tag{5.2.47}$$

where $n(\gamma)$ denotes the number of element degrees of freedom [$n(1) = 2$ and $n(2) = 4$]. We have

← Big DOF

small DOF

$$T_{ij}^{\alpha\beta} = \frac{\partial R_i^\alpha}{\partial \Delta_j^\beta} = \frac{\partial}{\partial \Delta_j^\beta} \left(\sum_{\gamma=1}^{2} \sum_{p=1}^{n(\gamma)} K_{ip}^{\alpha\gamma} \Delta_p^\gamma - F_i^\alpha \right)$$

- Here we see that the top indices are for the smaller condensed matrix while the lower one takes care of every degree of freedom.
- Note that $n(\gamma)$ means the summation differs depending on whether its u(2 dof per node) and $\bar{\Delta}$ having 4 dof per node.
- Pretty interesting

$$= \sum_{\gamma=1}^{2} \sum_{p=1}^{n(\gamma)} \left(K_{ip}^{\alpha\gamma} \frac{\partial \Delta_p^\gamma}{\partial \Delta_j^\beta} + \frac{\partial K_{ip}^{\alpha\gamma}}{\partial \Delta_j^\beta} \Delta_p^\gamma \right) - \frac{\partial F_i^\alpha}{\partial \Delta_j^\beta}$$

$$= K_{ij}^{\alpha\beta} + \sum_{p=1}^{2} \frac{\partial}{\partial \Delta_j^\beta} \left(K_{ip}^{\alpha 1} \right) u_p + \sum_{P=1}^{4} \frac{\partial}{\partial \Delta_j^\beta} \left(K_{iP}^{\alpha 2} \right) \bar{\Delta}_P - \frac{\partial F_i^\alpha}{\partial \Delta_j^\beta} \quad (5.2.48)$$

- We get the tangent matrix as above.
- Since the derivative is not summed, we can take it inside for every summation of γ, p, α
- Since β, j are not summed over, we get our $K_{ij}^{\alpha\beta}$ out. The other terms are the actuall summed terms of the condensed dof γ and because of that each γ dof corresponds to the actual one of u(p=2) or $\bar{\Delta}$ (p=4)

- In finding the tangent stiffness components, the stiffness components have to be recalled.

$$
\begin{aligned}
T_{iJ}^{12} &= K_{iJ}^{12} + \sum_{P=1}^{4} \frac{\partial}{\partial \bar{\Delta}_J} \left(K_{iP}^{12}\right) \bar{\Delta}_P \\
&= K_{iJ}^{12} + \sum_{P=1}^{4} \left[\int_{x_a}^{x_b} \frac{1}{2} A_{xx}^{e} \frac{\partial}{\partial \bar{\Delta}_J} \left(\sum_{K=1}^{4} \bar{\Delta}_K \frac{d\varphi_K}{dx} \right) \frac{d\psi_i}{dx} \frac{d\varphi_P}{dx}\, dx \right] \bar{\Delta}_P \\
&= K_{iJ}^{12} + \sum_{P=1}^{4} \left[\int_{x_a}^{x_b} \frac{1}{2} A_{xx}^{e} \frac{d\varphi_J}{dx} \frac{d\psi_i}{dx} \frac{d\varphi_P}{dx}\, dx \right] \bar{\Delta}_P \\
&= K_{iJ}^{12} + \int_{x_a}^{x_b} \frac{1}{2} A_{xx}^{e} \frac{d\psi_i}{dx} \frac{d\varphi_J}{dx} \left(\sum_{P=1}^{4} \frac{d\varphi_P}{dx} \bar{\Delta}_P \right) dx \\
&= K_{iJ}^{12} + \int_{x_a}^{x_b} \left(\frac{1}{2} A_{xx}^{e} \frac{dw}{dx} \right) \frac{d\psi_i}{dx} \frac{d\varphi_J}{dx}\, dx = 2K_{iJ}^{12} = K_{Ji}^{21} \qquad (5.2.50)
\end{aligned}
$$

- T^{12} is for the coupling of the membrane and bending dof. Now we know that it's rate of change of membrane residual by the bending dof. For each row i of the dof p, we see that term due to u is 0 as the derivative is $\bar{\Delta}$. In the stiffness term, note that the nonlienar term is $\frac{dw}{dx} = \sum \bar{\Delta} \frac{d\phi}{dx}$ which is getting differentiated.

$$
\begin{aligned}
T_{IJ}^{22} &= K_{IJ}^{22} + \sum_{p=1}^{2} \frac{\partial}{\partial \bar{\Delta}_J} \left(K_{Ip}^{21}\right) u_p + \sum_{P=1}^{4} \frac{\partial}{\partial \bar{\Delta}_J} \left(K_{IP}^{22}\right) \bar{\Delta}_P \\
&= K_{IJ}^{22} + \sum_{p=1}^{2} \left[\int_{x_a}^{x_b} A_{xx}^{e} \frac{\partial}{\partial \bar{\Delta}_J} \left(\sum_{K=1}^{4} \bar{\Delta}_K \frac{d\varphi_K}{dx} \right) \frac{d\varphi_I}{dx} \frac{d\psi_p}{dx}\, dx \right] u_p \\
&\quad + \sum_{P=1}^{4} \left[\int_{x_a}^{x_b} \frac{1}{2} A_{xx}^{e} \frac{\partial}{\partial \bar{\Delta}_J} \left(\frac{dw}{dx} \right)^2 \frac{d\varphi_I}{dx} \frac{d\varphi_P}{dx}\, dx \right] \bar{\Delta}_P \\
&= K_{IJ}^{22} + \int_{x_a}^{x_b} A_{xx}^{e} \frac{d\varphi_I}{dx} \frac{d\varphi_J}{dx} \left(\sum_{p=1}^{2} \frac{d\psi_p}{dx} u_p \right) dx \\
&\quad + \int_{x_a}^{x_b} A_{xx}^{e} \left(\frac{dw}{dx} \right) \frac{d\varphi_I}{dx} \frac{d\varphi_J}{dx} \left(\sum_{P=1}^{4} \bar{\Delta}_P \frac{d\varphi_P}{dx} \right) dx \\
&= K_{IJ}^{22} + \int_{x_a}^{x_b} A_{xx}^{e} \left(\frac{du}{dx} + \frac{dw}{dx} \frac{dw}{dx} \right) \frac{d\varphi_I}{dx} \frac{d\varphi_J}{dx}\, dx \qquad (5.2.51)
\end{aligned}
$$

■

- For T^{22}, all the dof are included. Interesting!

Plates

- In terms of plates we have three condensed degrees of freedom $\Delta^1, \Delta^2, \Delta^3$ which correspond to (For each node) 4 dof for u, v and 12 or 16 dof for w, $\frac{dw}{dx}$, $\frac{dw}{dy}$.

  Remember in conforming plates we will also have $\frac{d^2 w}{dxdy}$ as a dof

$$\begin{bmatrix} \mathbf{T}^{11} & \mathbf{T}^{12} & \mathbf{T}^{13} \\ \mathbf{T}^{21} & \mathbf{T}^{22} & \mathbf{T}^{23} \\ \mathbf{T}^{31} & \mathbf{T}^{32} & \mathbf{T}^{33} \end{bmatrix} \begin{Bmatrix} \delta\mathbf{\Delta}^1 \\ \delta\mathbf{\Delta}^2 \\ \delta\mathbf{\Delta}^3 \end{Bmatrix} = - \begin{Bmatrix} \mathbf{R}^1 \\ \mathbf{R}^2 \\ \mathbf{R}^3 \end{Bmatrix} \quad \text{or} \quad \mathbf{T}\,\delta\mathbf{\Delta} = -\mathbf{R} \tag{7.4.7}$$

where

$$\Delta_i^1 = u_i, \quad \Delta_i^2 = v_i, \quad \Delta_i^3 = \bar{\Delta}_i^3 \tag{7.4.8}$$

The coefficients of the submatrices $\mathbf{T}^{\alpha\beta}$ of the tangent stiffness matrix $\mathbf{T}$ and the components of the residual vector $\mathbf{R}^\alpha$ are defined by

$$T_{ij}^{\alpha\beta} = \frac{\partial R_i^\alpha}{\partial \Delta_j^\beta}, \quad R_i^\alpha = \sum_{\gamma=1}^{3} \sum_{k=1}^{n(\gamma)} K_{ik}^{\alpha\gamma} \Delta_k^\gamma - F_i^\alpha \tag{7.4.9}$$

where $n(\gamma)$ denotes m or n, depending on γ value $[n(1) = n(2) = m = 4$ and $n(3) = n]$. We obtain

- We can see a similar arrangement to how it was done previously. Note that here n will depend on the dof type, u,v, and transverse dofs. The type is again represented with γ

$$T_{ij}^{\alpha\beta} = \frac{\partial}{\partial \Delta_j^\beta}\left(\sum_{\gamma=1}^{3}\sum_{k=1}^{n(\gamma)} K_{ik}^{\alpha\gamma}\Delta_k^\gamma - F_i^\alpha\right) = \sum_{\gamma=1}^{3}\sum_{k=1}^{n(\gamma)} \frac{\partial K_{ik}^{\alpha\gamma}}{\partial \Delta_j^\beta}\Delta_k^\gamma + K_{ij}^{\alpha\beta} \qquad (7.4.10)$$

- Since the nonlinearity involved in the problem is only a function of w (or, equivalently, only a function of the nodal values $\bar{\mathbf{\Delta}} = \mathbf{\Delta}_3$) and the only coefficients that depend on w are K_{ij}^{13}, K_{ij}^{23}, K_{ij}^{31}, K_{ij}^{32}, and K_{ij}^{33}, the derivatives of all submatrices $\mathbf{K}^{\alpha\beta}$ with respect to $\Delta_j^1 = u_j$ and $\Delta_j^2 = v_j$ are zero. Hence, we have (assuming that $\mathbf{F}^\alpha$ are independent of Δ)

- Note that just in the beam in case of K^{11}, here all the nonlinearity is only due to the presence of $\frac{dw}{dx}$, therefore when we take the derivative with respect to Δ_j^β we know that all the terms will be zero when β is 1,2.

T^{13}

$$
\begin{aligned}
T_{ij}^{13} &= \sum_{\gamma=1}^{3}\sum_{k=1}^{n(\gamma)} \frac{\partial K_{ik}^{1\gamma}}{\partial \bar{\Delta}_j^3}\Delta_k^{\gamma} + K_{ij}^{13} = \sum_{k=1}^{n} \frac{\partial K_{ik}^{13}}{\partial \bar{\Delta}_j^3}\bar{\Delta}_k^3 + K_{ij}^{13} \\
&= \frac{1}{2}\sum_{k=1}^{n} \bar{\Delta}_k^3 \frac{\partial}{\partial \bar{\Delta}_j^3}\left\{\int_{\Omega^e}\left[\frac{\partial \psi_i^e}{\partial x}\left(A_{11}\frac{\partial w}{\partial x}\frac{\partial \varphi_k^e}{\partial x} + A_{12}\frac{\partial w}{\partial y}\frac{\partial \varphi_k^e}{\partial y}\right)\right.\right. \\
&\quad \left.\left. + A_{66}\frac{\partial \psi_i^e}{\partial y}\left(\frac{\partial w}{\partial x}\frac{\partial \varphi_k^e}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial \varphi_k^e}{\partial x}\right)\right] dx\, dy\right\} + K_{ij}^{13} \\
&= \frac{1}{2}\sum_{k=1}^{n} \bar{\Delta}_k^3 \left\{\int_{\Omega^e}\left[\frac{\partial \psi_i^e}{\partial x}\left(A_{11}\frac{\partial \varphi_j^e}{\partial x}\frac{\partial \varphi_k^e}{\partial x} + A_{12}\frac{\partial \varphi_j^e}{\partial y}\frac{\partial \varphi_k^e}{\partial y}\right)\right.\right. \\
&\quad \left.\left. + A_{66}\frac{\partial \psi_i^e}{\partial y}\left(\frac{\partial \varphi_j^e}{\partial x}\frac{\partial \varphi_k^e}{\partial y} + \frac{\partial \varphi_j^e}{\partial y}\frac{\partial \varphi_k^e}{\partial x}\right)\right] dx\, dy\right\} + K_{ij}^{13}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2}\int_{\Omega^e}\left[\frac{\partial \psi_i^e}{\partial x}\left(A_{11}\frac{\partial w}{\partial x}\frac{\partial \varphi_j^e}{\partial x} + A_{12}\frac{\partial w}{\partial y}\frac{\partial \varphi_j^e}{\partial y}\right)\right. \\
&\quad \left. + A_{66}\frac{\partial \psi_i^e}{\partial y}\left(\frac{\partial w}{\partial x}\frac{\partial \varphi_j^e}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial \varphi_j^e}{\partial x}\right)\right] dx\, dy + K_{ij}^{13} \\
&= K_{ij}^{13} + K_{ij}^{13} = 2K_{ij}^{13}\ (= T_{ji}^{31});\quad \left(\mathbf{T}^{13}\right)^{\mathrm{T}} = \mathbf{T}^{31} = \mathbf{K}^{31} \qquad (7.4.12)
\end{aligned}
$$

- We see that the derivative is inside and now only concerened with the p = 12 or 16, as the other p derivatives terms are zero, i.e $\gamma = 1, 2$.
- Remember again that the shape functions are constant. It’s their coefficients (or dofs) that we are finding the derivative of.

T^{23}

$$
\begin{aligned}
T_{ij}^{23} &= \sum_{\gamma=1}^{3}\sum_{k=1}^{n(\gamma)} \frac{\partial K_{ik}^{2\gamma}}{\partial \bar{\Delta}_j^3}\Delta_k^\gamma + K_{ij}^{23} = \sum_{k=1}^{n} \frac{\partial K_{ik}^{23}}{\partial \bar{\Delta}_j^3}\bar{\Delta}_k^3 + K_{ij}^{23} \\
&= \frac{1}{2}\sum_{k=1}^{n} \bar{\Delta}_k^3 \frac{\partial}{\partial \bar{\Delta}_j^3}\left\{\int_{\Omega^e}\left[\frac{\partial \psi_i^e}{\partial y}\left(A_{12}\frac{\partial w}{\partial x}\frac{\partial \varphi_k^e}{\partial x} + A_{22}\frac{\partial w}{\partial y}\frac{\partial \varphi_k^e}{\partial y}\right)\right.\right. \\
&\qquad \left.\left. + A_{66}\frac{\partial \psi_i^e}{\partial x}\left(\frac{\partial w}{\partial x}\frac{\partial \varphi_k^e}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial \varphi_k^e}{\partial x}\right)\right] dx\,dy\right\} + K_{ij}^{23} \\
&= \frac{1}{2}\sum_{k=1}^{n} \bar{\Delta}_k^3 \left\{\int_{\Omega^e}\left[\frac{\partial \psi_i^e}{\partial y}\left(A_{12}\frac{\partial \varphi_j^e}{\partial x}\frac{\partial \varphi_k^e}{\partial x} + A_{22}\frac{\partial \varphi_j^e}{\partial y}\frac{\partial \varphi_k^e}{\partial y}\right)\right.\right. \\
&\qquad \left.\left. + A_{66}\frac{\partial \psi_i^e}{\partial x}\left(\frac{\partial \varphi_j^e}{\partial x}\frac{\partial \varphi_k^e}{\partial y} + \frac{\partial \varphi_j^e}{\partial y}\frac{\partial \varphi_k^e}{\partial x}\right)\right] dx\,dy\right\} + K_{ij}^{23} \\
&= \frac{1}{2}\int_{\Omega^e}\left[\frac{\partial \psi_i^e}{\partial y}\left(A_{12}\frac{\partial w}{\partial x}\frac{\partial \varphi_j^e}{\partial x} + A_{22}\frac{\partial w}{\partial y}\frac{\partial \varphi_j^e}{\partial y}\right)\right. \\
&\qquad \left. + A_{66}\frac{\partial \psi_i^e}{\partial x}\left(\frac{\partial w}{\partial y}\frac{\partial \varphi_j^e}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial \varphi_j^e}{\partial y}\right)\right] dx\,dy + K_{ij}^{23} \\
&= K_{ij}^{23} + K_{ij}^{23} = 2K_{ij}^{23}\ (= T_{ji}^{32});\quad \left(\mathbf{T}^{23}\right)^{\mathrm{T}} = \mathbf{T}^{32} = \mathbf{K}^{32} \qquad (7.4.13)
\end{aligned}
$$

- Here again we see the terms due to $\gamma = 1, 2$ become zero.

The computation of T_{ij}^{33} requires the calculation of three parts:

$$T_{ij}^{33} = K_{ij}^{33} + \sum_{k=1}^{m=4} \frac{\partial K_{ik}^{31}}{\partial \bar{\Delta}_j^3} u_k + \sum_{k=1}^{m=4} \frac{\partial K_{ik}^{32}}{\partial \bar{\Delta}_j^3} v_k + \sum_{k=1}^{n} \frac{\partial K_{ik}^{33}}{\partial \bar{\Delta}_j^3} \bar{\Delta}_k^3 \qquad (7.4.14)$$

- Here we see that all stiffness terms have the nonlinear $\frac{dw}{dx}$
- These derivations are found in page329 Reddy.
- Note that $\frac{dw}{dx}$ is the term getting differentiated and $\frac{1}{\partial \Delta_j^3}(\frac{dw}{dx})^2 = 2\frac{dw}{dx}\frac{d\phi_j}{dx}$