Discrete Structures: CMPSC 102

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Week 3

Suppose that a car dealer offers to sell you a car for $18,000 or for payments of $375 per month for five years. You would like to know what monthly interest rate the dealer is, in effect, charging you.

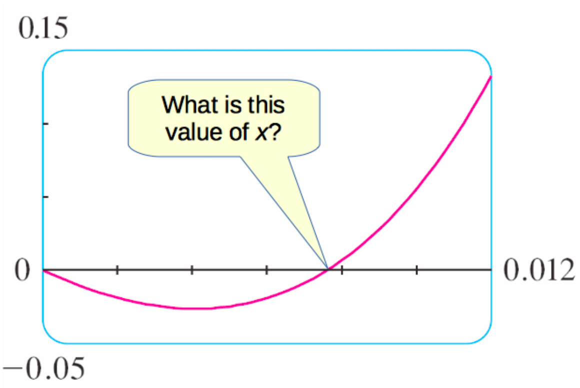
- To find the answer, there is an equation from the bank

Equation to Solve : Find the Roots

$$f(x) = 48x(1+x)^{60}(1+x)^{60} + 1 = 0$$

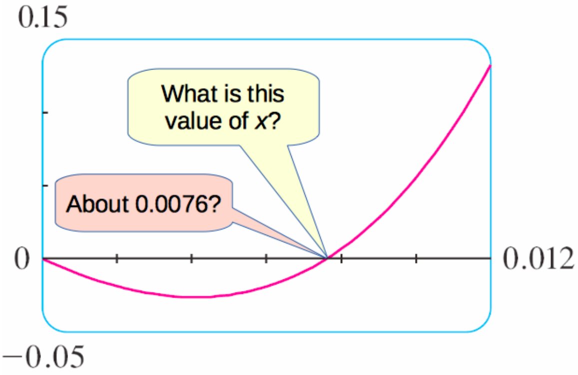
- An approximate solution
- Let's plot the equation to see where it crosses the x axis
- Ask: for what value of x does this x-axis intersection happen?

Plot the Equation

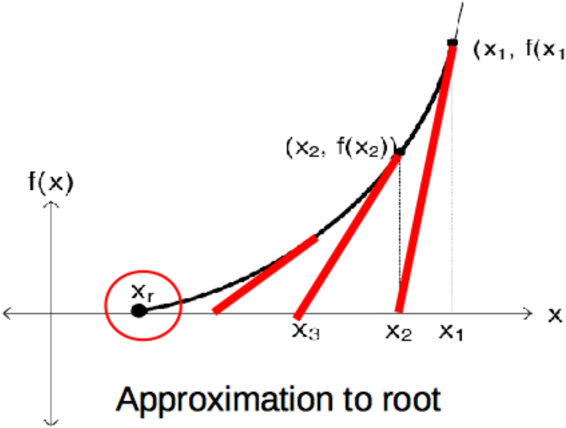


- Two roots to find to solve equation: $x = 0$ and some other x value.
- We want the *other*, non-zero intersection point!

Plot the Equation

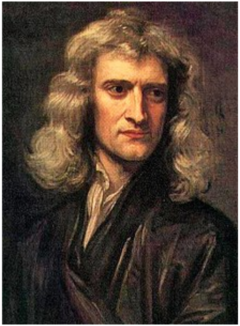


- A solution between 0.007 and 0.008 (approx 0.0076)
- Want to be able to calculate this value to out own level of accuracy.
- How to find the *roots*?



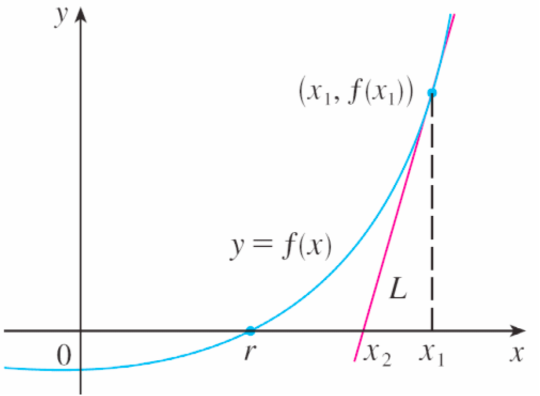
- Approximate the root (x_r) using *Newton's Method*

Isaac Newton

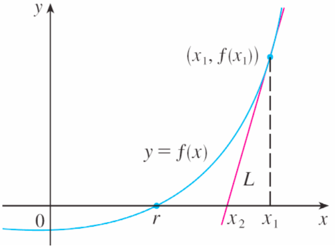


- Time line: 25 December 1642 20 March 1726 or 1727)
- English mathematician, astronomer, theologian, author and physicist
- One of the most influential scientists of all time
- A key figure in the scientific revolution.

Guess a root



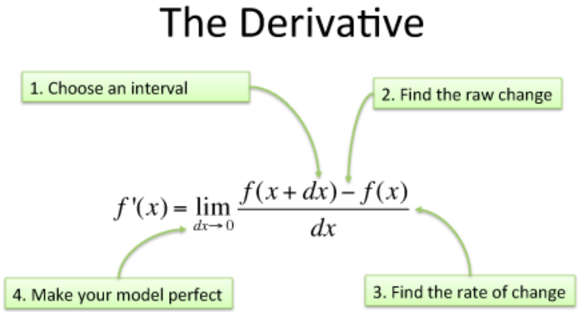
- The relevant root is labeled r
- First approximation for x_1 is a simple *guess* made by understanding the plot



- Consider the tangent line, L to the curve $y = f(x)$ at the point $(x_1, f(x_1))$ and look at the x-intercept of L, labeled x_2.
- Main idea: the tangent line is close to the curve and its x-intercept (an intersection point at x_2), is close to the x-intercept of the curve (the root r).
- This point root r that want to find!

Derivatives

Defined Mathematically (somewhat ...)



Roughly speaking...

- The derivative is an equation extracted from the original $f(x)$ used to find the x values of where the $y = 0$.

Derivatives

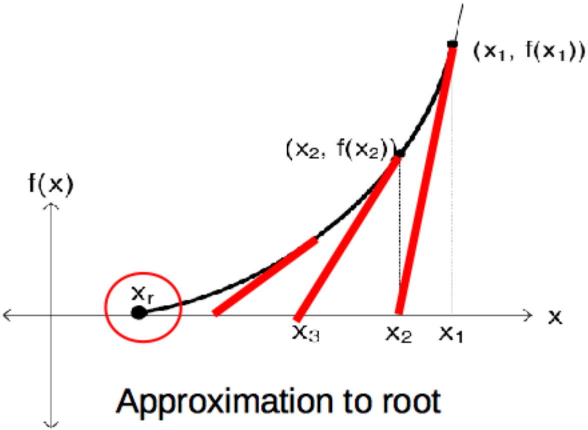
Roughly speaking...

- We want to find where line L passes x-intercept
- Slope of line L: $f'(x)$
- Line formula (from algebra): $y = m * x + b$
- To find a formula for x_2 in terms of x_1

$$y - f(x_1) = f'(x_1)(x_2 - x_1) \tag{1}$$

Thinking Recursively

Use derivatives to find lines crossing x-axis, converging on root



We want to insert initial approximation values of x back into the line equation (recursively) to find the next approximation (and converge on the root, x_r).

Find a General Equation for Finding Roots

We are solving for x_2, or a closer approx. of root!

$$
\begin{aligned}
y - f(x_1) &= f'(x_1)(x_2 - x_1) \\
0 - f(x_1) &= f'(x_1)(x_2 - x_1) \\
f(x_1) &= -f'(x_1)(x_2 - x_1) \\
f(x_1) &= x_1 * f'(x_1) - x_2 * f'(x_1) \\
f(x_1) - x_1 * f'(x_1) &= x_2 * f'(x_1) \\
x_2 * f'(x_1) &= f(x_1) - x_1 * f'(x_1) \\
\frac{x_2 * f'(x_1)}{f'(x_1)} &= \frac{f(x_1)}{f'(x_1)} - \frac{x_1 * f'(x_1)}{f'(x_1)}
\end{aligned}
$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

Now what?!

We use this equation to approx values of roots

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

General approx. for root x_{n_1} from approx. root x_n

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(n)}$$

Let's find the square root for a number

Recall that Newtons method finds an approximate root of $f(x) = 0$

Define f(x)

$$f(x) = x^2 - a$$
$$x^2 = a$$
$$x = \sqrt{a} \qquad \text{(find positive root, } a\text{)}$$
$$x = -\sqrt{a}$$

Define the derivative of $f(x)$, $f'(x)$, using calculus

$$f'(x) = 2x$$

Establish the Approximation Equation

Note: a in $f(x)$ is the initial guess!

$f(x) = x^2 - a$
$f'(x) = 2x$

The root to find

$a = x_r$

The initial guess of root (to start the method)

$x_1 = 1.0$

General approx. for root x_{n_1} from approx. root x_n

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(n)}$$

Substitute into Equation

Finding square root of a

Knowns

$a = 2$ (find sqrt of a)	$f(x) = x^2 - 2$ (function)
$x_1 = 1.0$ (guess)	$f'(x) = 2x$ (derivative)

$$\begin{aligned} x_1 &= 1.0 - \frac{f(1.0)}{f'(1.0)} \\ &= 1.0 - \frac{(1.0)^2 - 2}{2 * (1.0)} \\ &= 1.0 - \frac{1.0 - 2}{2} \\ &= 1.0 - \frac{-1.0}{2} \\ &= \frac{3.0}{2} \\ &= 1.5 \end{aligned}$$

Table of Iterations

Finding square root of a

Calculations

Guess			Approx. root
x_n	$f(x) = x_n^2 - 2$	$f'(x_n) = 2x$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	-1	2	$1 - \frac{-1}{2} = \frac{3}{2} = 1.5$
$\frac{3}{2}$	$\frac{1}{4} = 0.25$	3.0	$\frac{3}{2} - \frac{(\frac{1}{4})}{3} = \frac{17}{12} = 1.4167$
$\frac{17}{12}$	$\frac{1}{144}$	$\frac{17}{6}$	$\frac{17}{6} - \frac{\frac{1}{144}}{\frac{17}{6}} = \frac{577}{408} = 1.4142$

Python to the rescue

```
>>> math.sqrt(2)
1.4142135623730951
```

vim newtonsMethod.py

```
n = 2.0 # the number from which to find square root.
guess = 1.0 # initial value for approx

print("  Initial values:  n = ",n, "guess = ",guess)

while abs(n - guess*guess) > .0001:
   #find  x_n - \frac{f(x_n)}{f'(x_n)}
   guess = guess - (guess*guess - n)/(2*guess)
   print("   *Current guess:   ",guess)
root = guess

print("   Result :",root)
```

Put This Script Into a Function

Finding square root of a

vim newtonsMethodFunction.py

```
def NM(n, guess):
    print("  Initial values:  n = ",n, "guess = ",guess)

    while abs(n - guess*guess) > .0001:
        #find  x_n - \frac{f(x_n)}{f'(x_n)}
        guess = guess - (guess*guess - n)/(2*guess)
        print("   *Current guess:   ",guess)
        root = guess
    return root
#end of NM()

#get parameters to call function NM()
n = 2 # the number from which to find square root.
guess = 1.0 # initial value for approx
print(" Finding root : ",n)
print(" Approx guess : ", guess)
print("  Result : ",NM(n, guess))
```