# *CS202 - Algorithm Analysis*

# Tree Algorithms - Module 1

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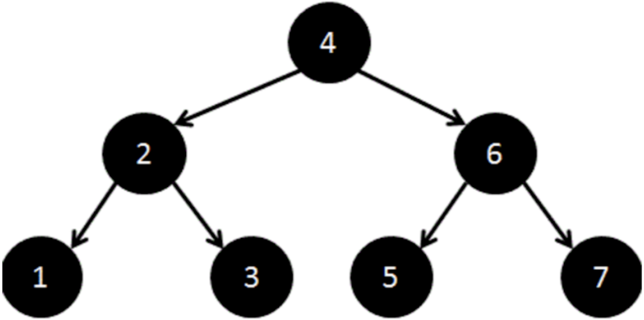
# Discussion Based On ...

**Sedgewick 2.4 Heap Sort**

# Data Structures - An overview

- So far we have seen linear structures:
  - linear: before and after relationship
  - Arrays, Stacks, and Queues
- Non-linear structure: **Trees**
  - probably the most fundamental structure in computing
  - hierarchical structure
  - Terminology: from family trees (genealogy)

# Trees



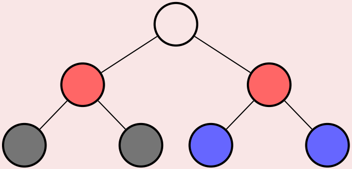
# Trees More Formally

- **Definition:** A tree T is a set of nodes storing elements such that the nodes have a parent-child relationship that satisfies the following properties:
  - If T is nonempty, it has a special node, called the root of T, that has no parent.
  - Each node v of T different than the root has a unique **parent node** w; every node with parent w is a **child** of w

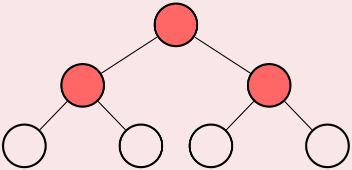
# Trees More Formally

- **Recursive Definition:**
  - T is either empty
  - or consists of a node r, called the root of T, and a (possibly empty) set of trees whose roots are the children of r.

- **Siblings:** Two nodes that have the same parent are called siblings.

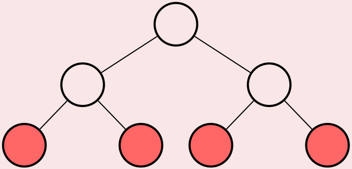


- **Internal nodes:** Nodes that have one or more children(s).



# Trees - Some basic terminologies

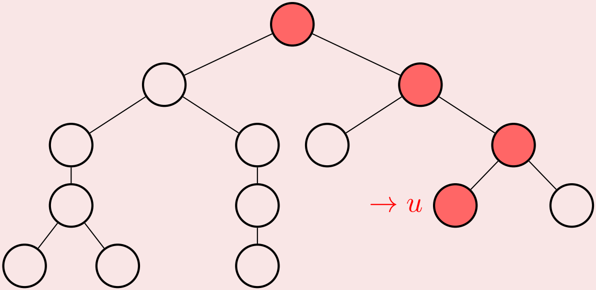
- **External nodes:** Nodes that don't have any children.



- **Ancestors:**

  Ancestors of a node u are u itself and the ancestors of its parent.

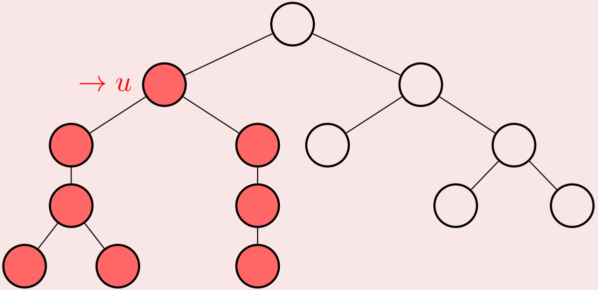
  (INCLUSIVE)



- **Descendants:**

  v is a descendants of u if u is an ancestor of v.

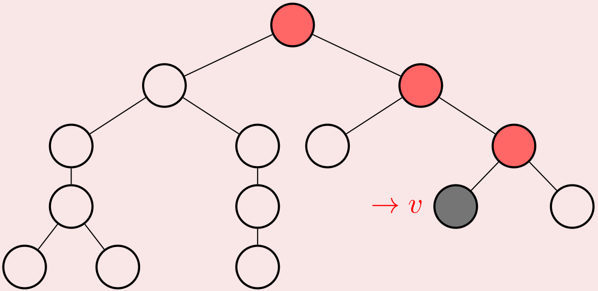
  (INCLUSIVE)



# Trees - Some basic terminologies

- **Depth(T, v):**

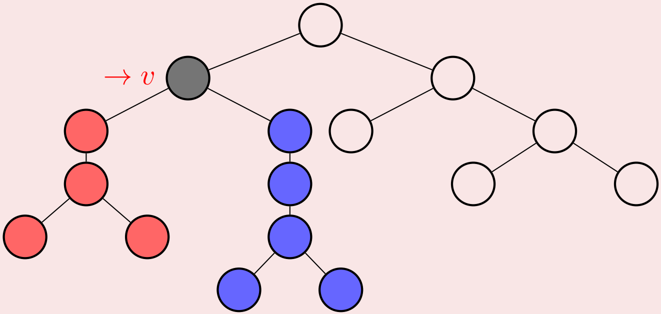
  Number of ancestors of v, excluding v itself.




**Depth(T, v) = 3**

- **Height(T, v):**

  Number of nodes in the longest path from v to any leaf, excluding v itself.




**Height(T, v) = 4**

# Trees - Some basic terminologies

- What is the height of the leaf node(s)?

- The height of a tree is the height of its root.
- Height and Depth are symmetrical.

**Proposition:** The **height** of a tree T is the **maximum depth** of one of its leaves.

# Trees - Applications

- Scheduling and Priority Queue (Heap)
- Class Hierarchy in Java
- File System
- Storing hierarchies in organizations

# Binary Trees More Formally

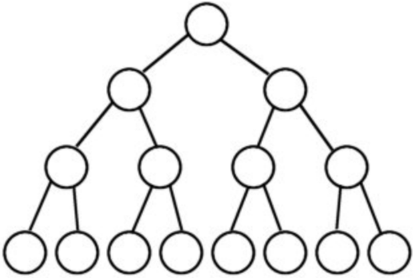
- **Definition:** A binary tree is a tree such that:
  - every node has at most 2 children
  - each node is labeled as being either a left childd or a right child

# Binary Trees More Formally

- **Recursive Definition:**
  - a binary tree is empty;
  - or it consists of
    - a node (the root) that stores an element
    - another binary tree, called the left subtree of T
    - another binary tree, called the right subtree of T

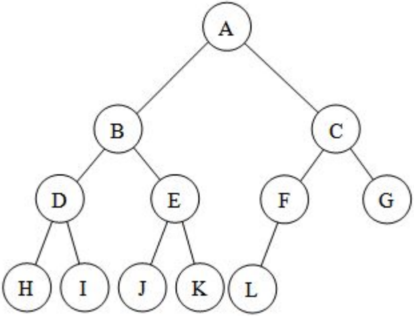
# Binary Tree Examples

- A full binary tree (sometimes complete or proper binary tree or 2-tree) is a tree in which every node other than the leaves has two children.



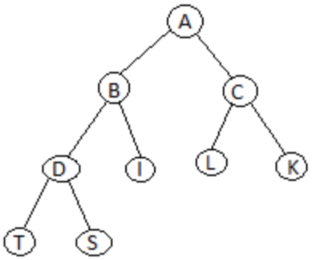
# Binary Tree Examples

- An atmost complete binary tree is a binary tree in which every level, except possibly the last, is completely filled, and all nodes are as far left as possible.

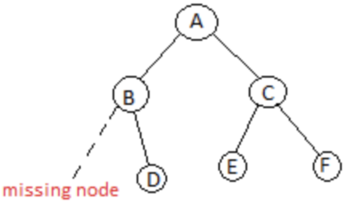


# Binary Tree Examples

- An In-Complete binary tree is a binary tree in which the properties of the complete binary tree is not true.




Complete Binary Tree



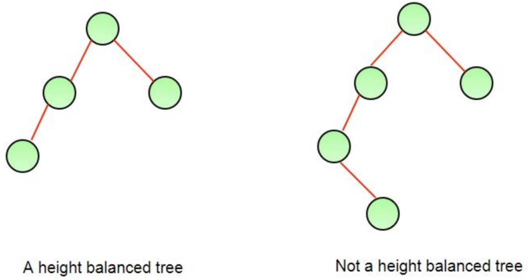

In-Complete Binary Tree

# Binary Tree Examples

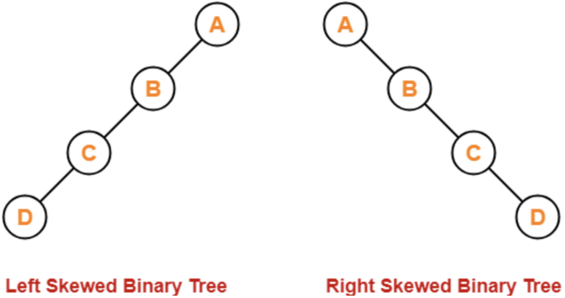
- **Balanced** : Difference between the height of the left and right subtree is atmost 1.
- **Unbalanced** : Difference between the height of the left and right subtree is greater than 1.

**Depend on the balancing scheme. Later.**

- **Balanced Vs Unbalanced**



- A skewed binary tree is a binary tree that satisfies the following 2 properties:
  1. All the nodes except one node has one and only one child.
  2. The remaining node has no child.

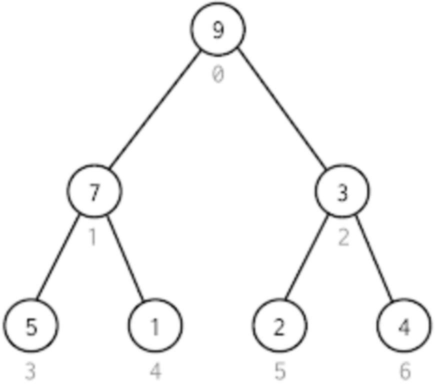


# Properties of Binary Trees

- In a binary tree
  - level 0 has $<= 1$ node
  - level 1 has $<= 2$ nodes
  - level 2 has $<= 4$ nodes
  - $\cdots$
  - level i has $<= 2^{\mathrm{i}}$ nodes

# How to store a binary tree in a program?

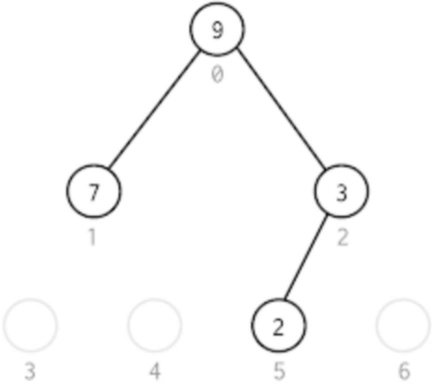
- An array can be used to represent the binary tree structure.




| 9 | 7 | 3 | 5 | 1 | 2 | 4 |
|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 |

# How to store a binary tree in a program?

- An array can be used to represent the binary tree structure.



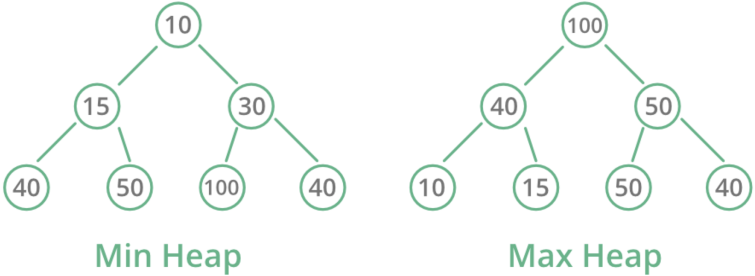

| 9 | 7 | 3 | null | null | 2 | null |
|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 |

# What is a Binary Heap?

- Each node has atmost two children.
- Complete binary tree or atmost complete binary tree qualified as binary heap.
- Node with no children is also qualified as heap.
- Left skewed or right skewed tree is not a heap.
- There are two types of heap, namely:

  Max heap and Min heap.

# What is a Binary Heap?



# Binary Heap Properties

- Binary Heap has two main properties:
  1. Order property
  2. Shape property

# Binary Heap Properties

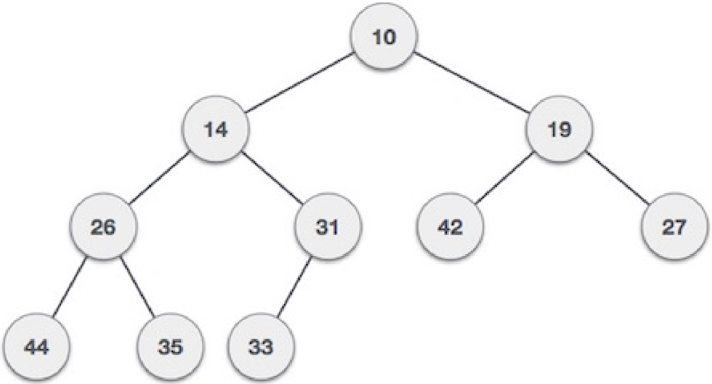
- **Order** property: The value in node $n$ is $\geq$ the values in its children, for every node $n$ (MAX heap).
- How about MIN heap?

# Binary Heap Properties

**Shape Property:**

- All leaves are either at depth $d$ or $d-1$ for some $d$
- All of the leaves at depth $d-1$ are to the right of the leaves at depth $d$
- And the following:
    1. There is at most 1 node with just 1 child $v$.
    2. $v$ is the left child of its parent.
    3. $v$ is the rightmost leaf at depth $d$.

# Binary Heap Example



# Heap Sort

- Phase 1: convert the array into an $n$-element heap
- Phase 2: repeatedly remove maximum element from the heap, and place that element in its proper position in the array
  - swap element at $0$th position with element at $(n-1)$th position and then "reheapify" considering only the first $n-1$ elements
  - repeat this process until heap size is reduced to $1$ (minimum element remains, at $0$th position)

# Heap Sort: Phase 1 - build the heap

```
for i = 1 to n-1 do
  insert element s[i] into the heap consisting
   of the elements s[0]...s[i-1]    [heapify]
```

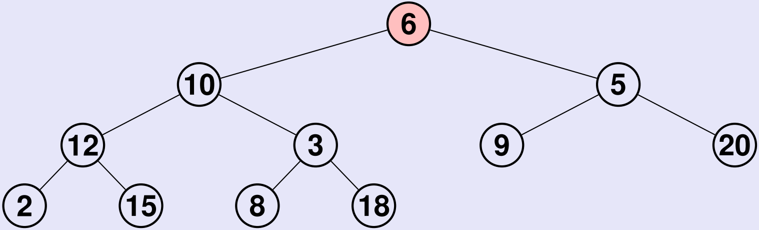
Once the heap is built, $s[0]$ will contain the maximum element

# Heap Sort Example

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 6 | 10 | 5 | 12 | 3 | 9 | 20 | 2 | 15 | 8 | 18 |

**Step 1: Initial Heapify**

**Fix-1**



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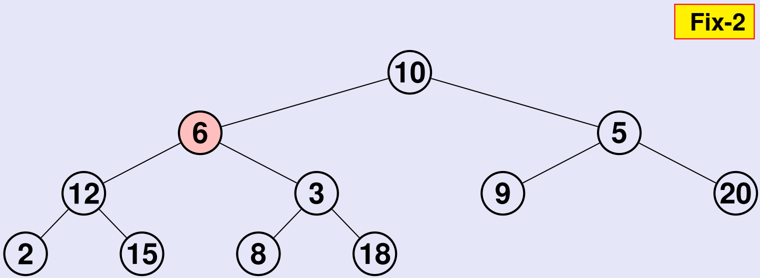


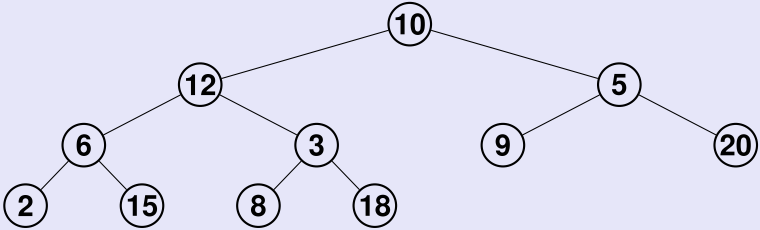

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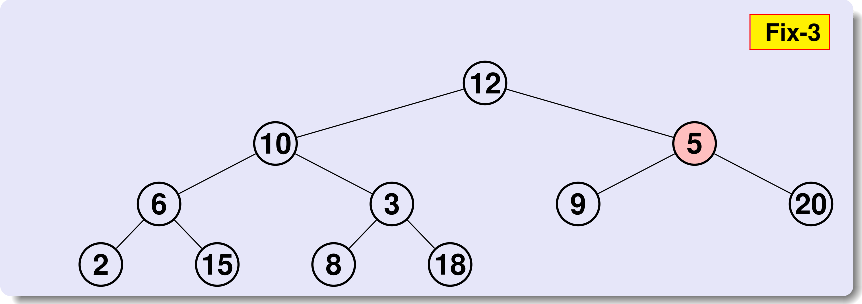


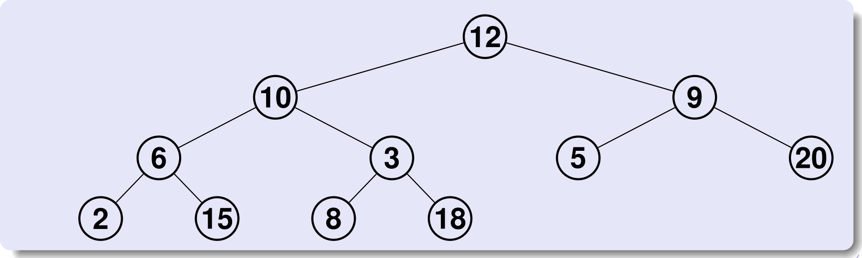

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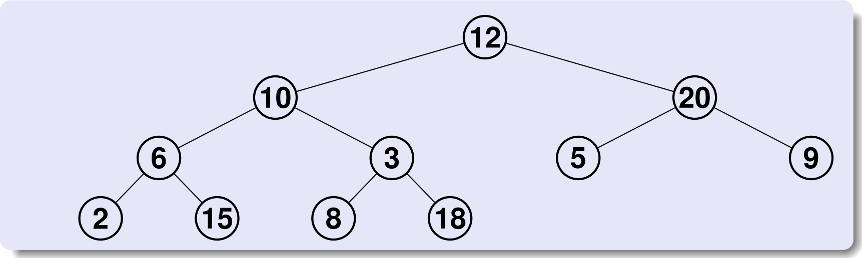


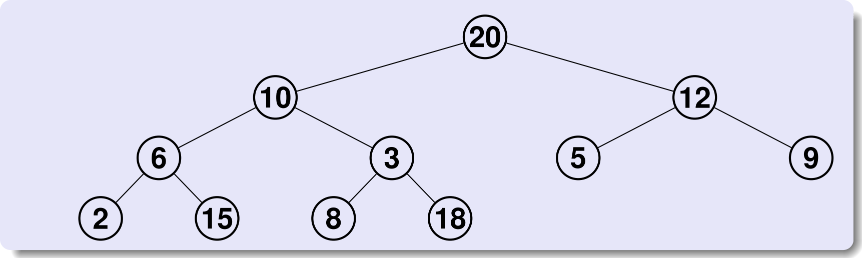

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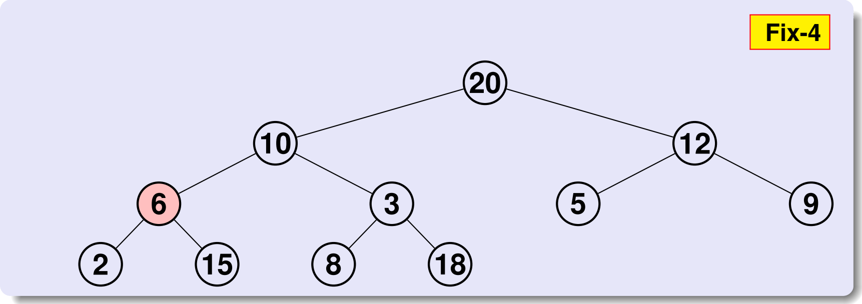


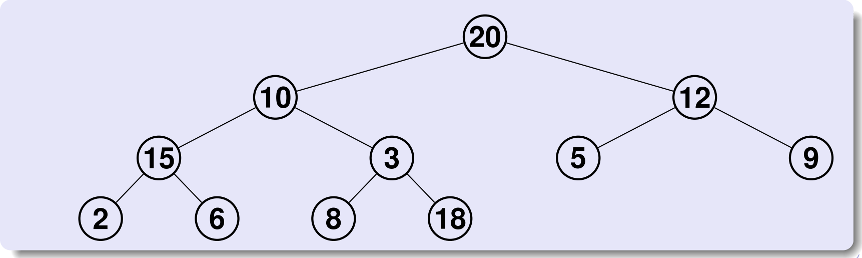

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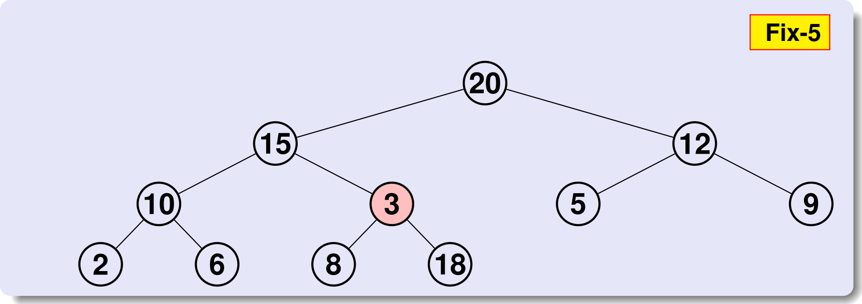


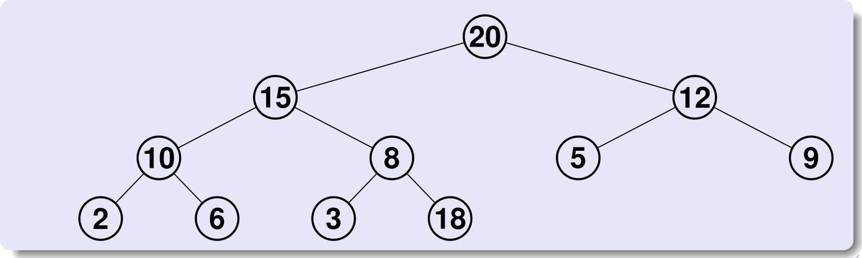

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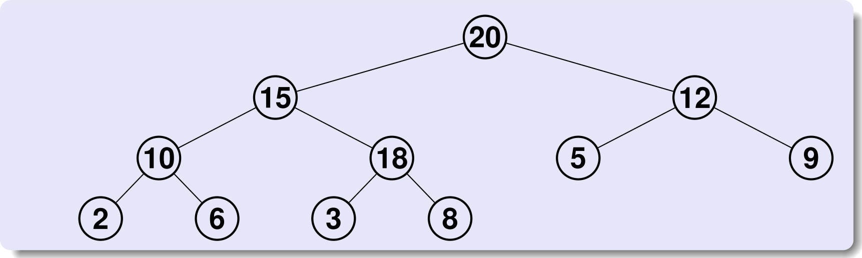


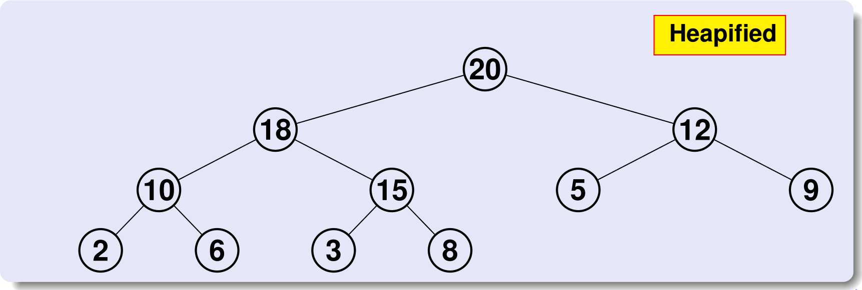

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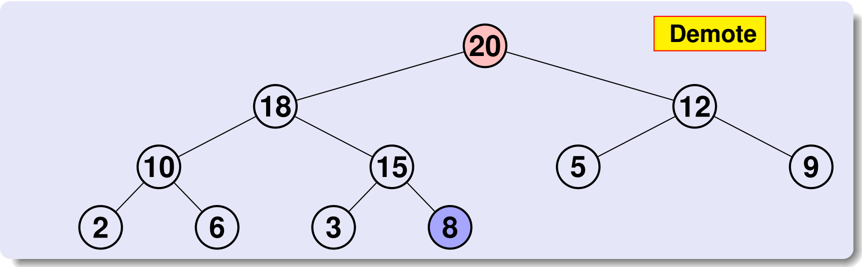


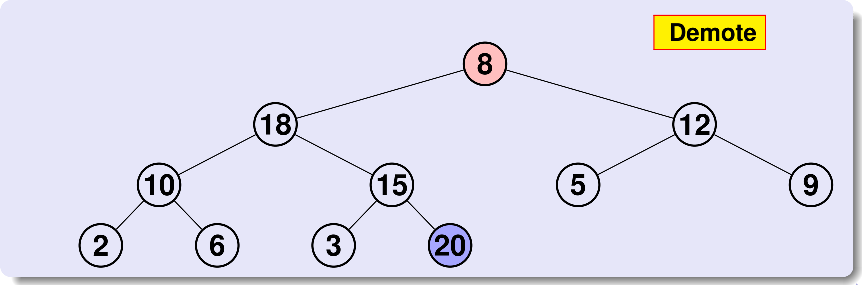

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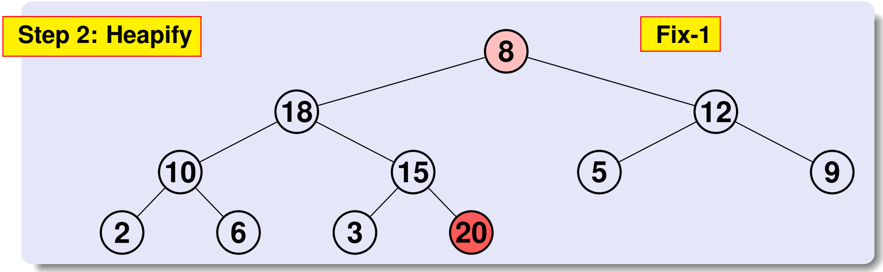


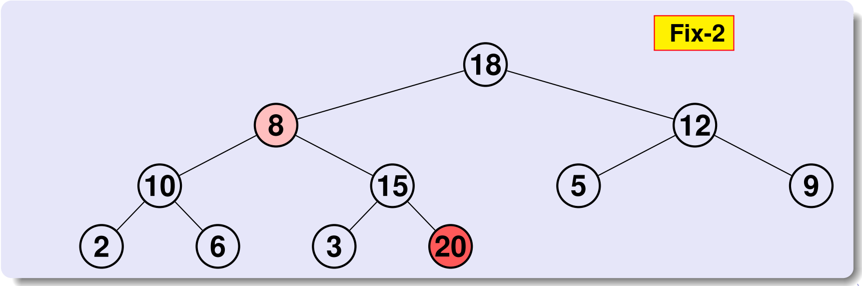

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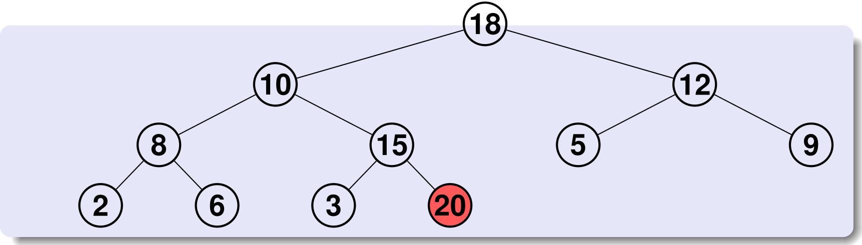


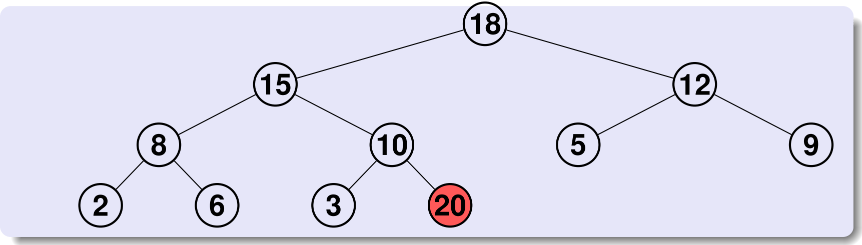

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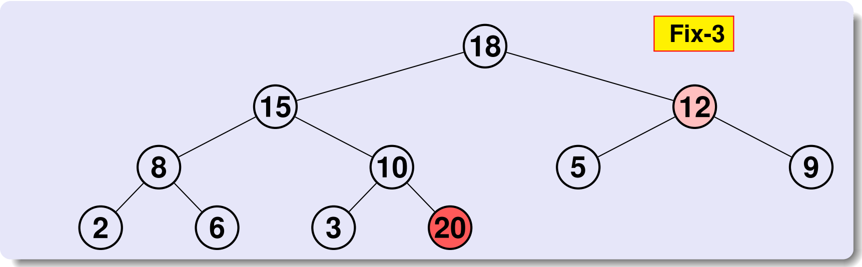


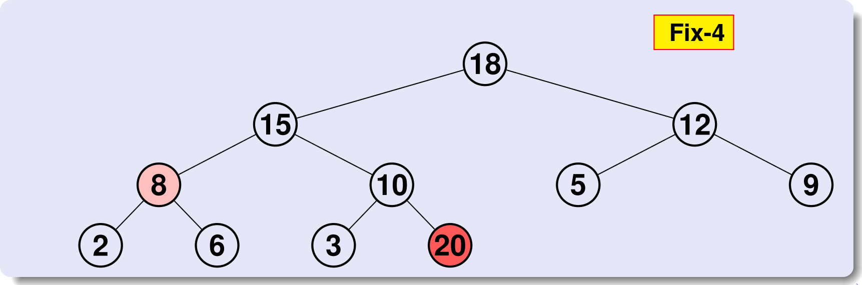

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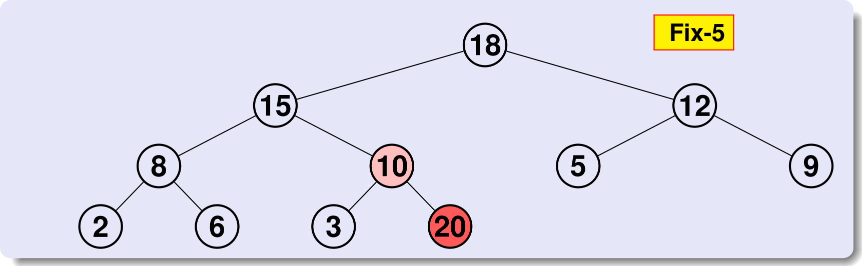


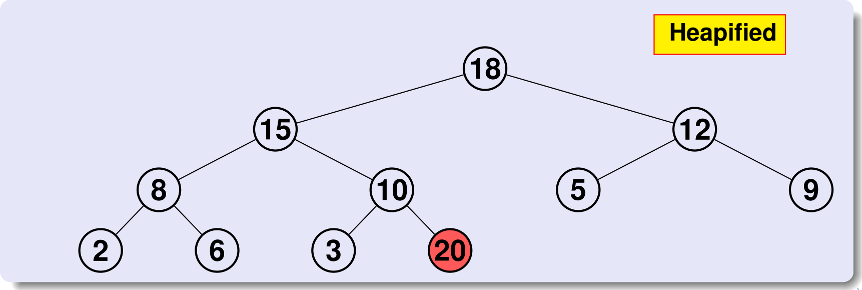

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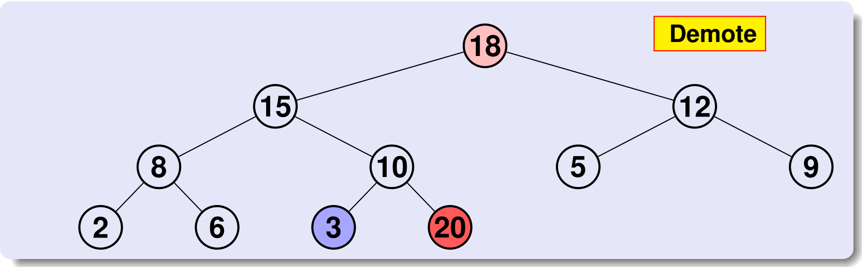


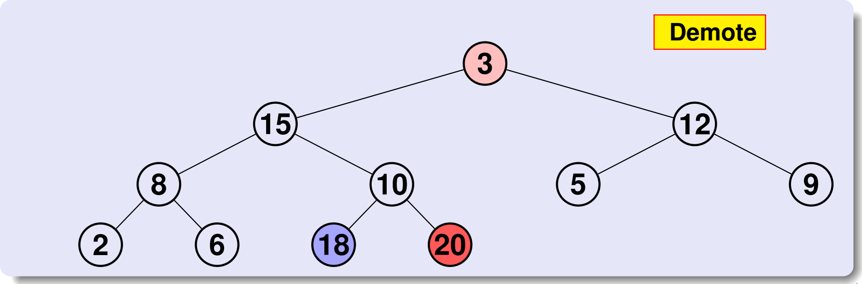

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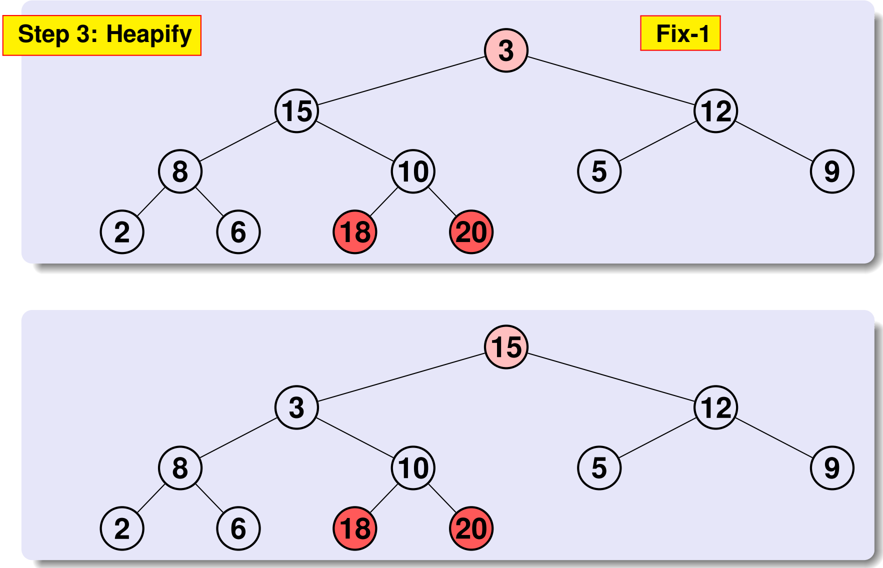

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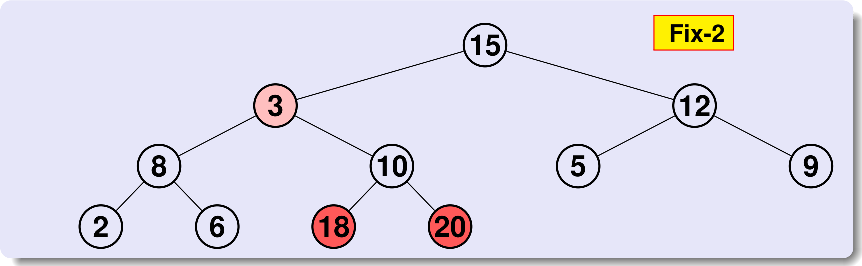


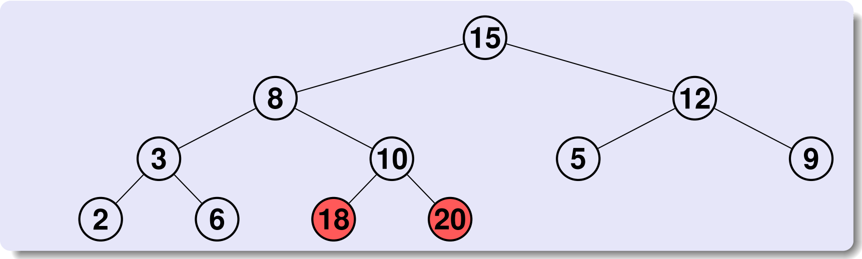

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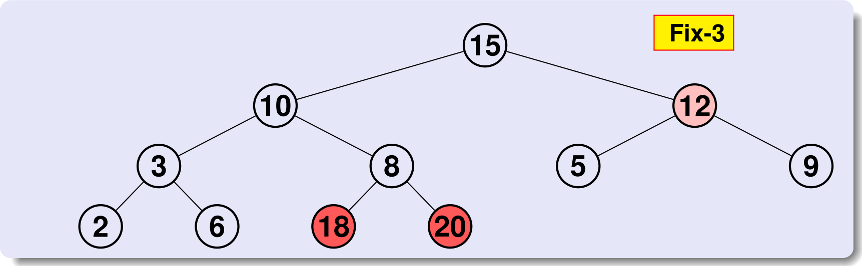


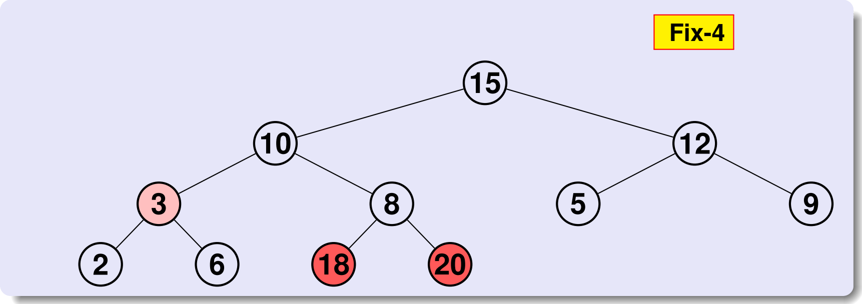

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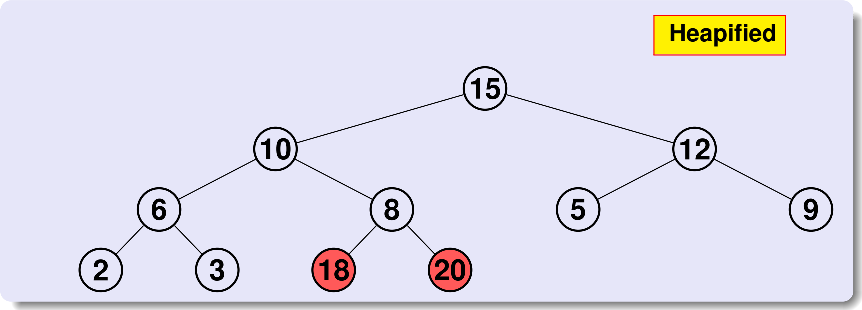

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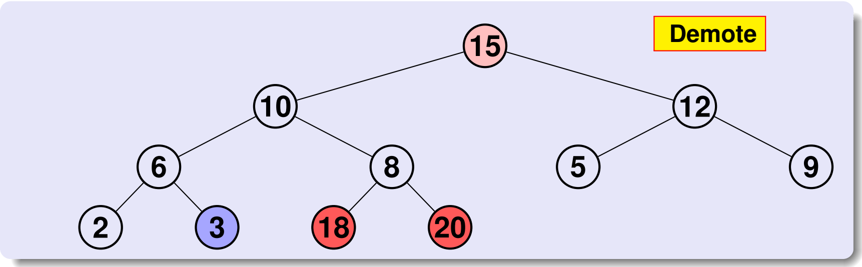


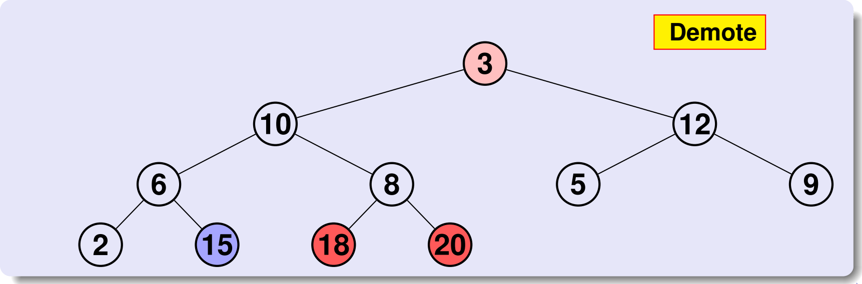

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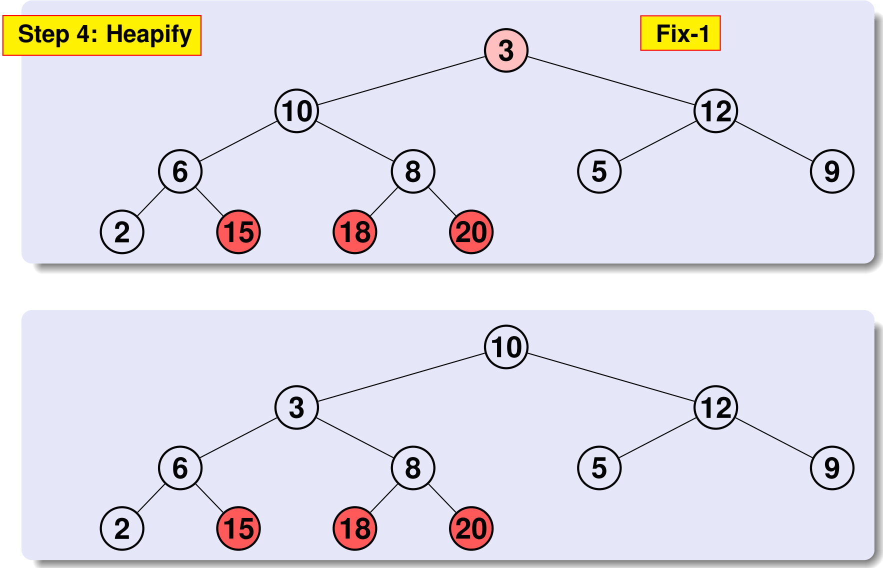

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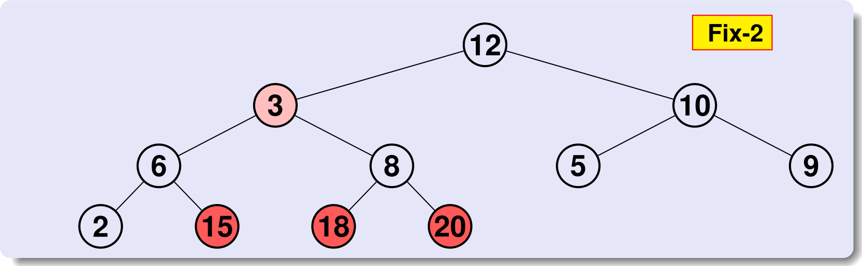


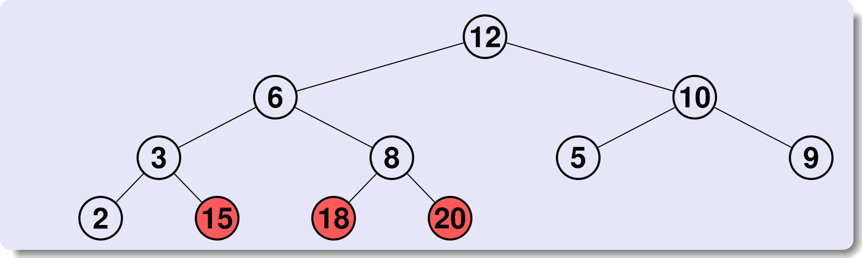

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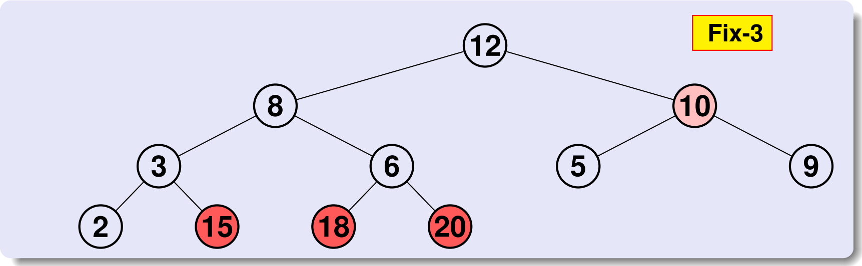


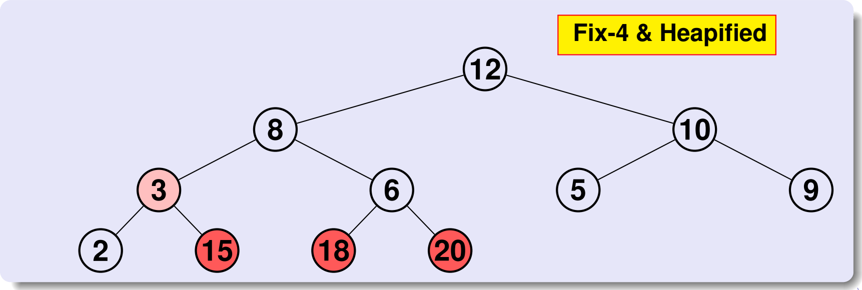

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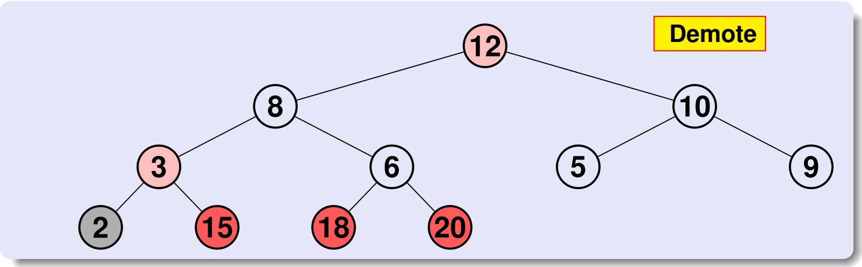


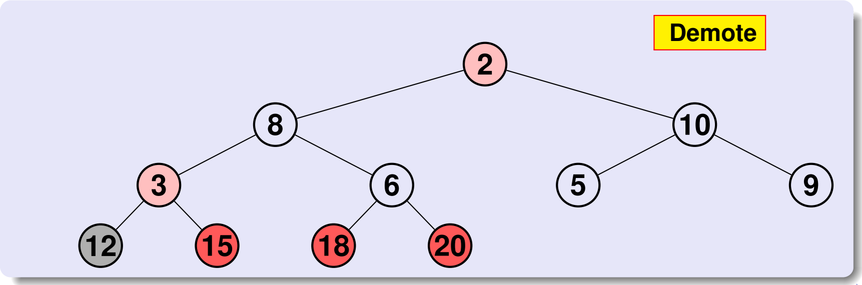

Complete other steps.
Repeat steps till all nodes becomes **Red**
At the end of all steps, given array is **Sorted**.

# Heap Sort Complexity

```
for i ← 1 to n-1 do
    insert element s[i] into the
      heap consisting of the
      elements s[0]...s[i-1]
```

O( n log n )

O( log n ) operations

```
for i ← n-1 down to 1 do
    swap s[0] and s[i]
    "demote" s[0] to its proper place
      in the heap consisting of the
      elements s[0]...s[i-1]
```

O( n log n )

# Heap Sort

Note that heap sort is just a more clever version of selection sort since a maximum is repeatedly selected and placed in its proper position

# Sorting Algorithms - Comparison

| Algorithm | Time | Notes |
|---|---|---|
| selection-sort | $O(n^2)$ | ◆ slow<br>◆ in-place<br>◆ for small data sets (< 1K) |
| insertion-sort | $O(n^2)$ | ◆ slow<br>◆ in-place<br>◆ for small data sets (< 1K) |
| heap-sort | $O(n \log n)$ | ◆ fast<br>◆ in-place<br>◆ for large data sets (1K — 1M) |
| merge-sort | $O(n \log n)$ | ◆ fast<br>◆ sequential data access<br>◆ for huge data sets (> 1M) |

# Reading Assignment

**Sedgewick 2.4 Heap Sort**

# Questions?

**Please ask if there are any Questions!**