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Assignment 1

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Instructions: Write your answers in the corresponding `hw1.tex` file, compile it to a pdf, and hand the pdf file to *learn.tsinghua* by the due date. Be sure to add your **student ID** and **full name** in the `stulogin` and `stuname` macros at the top of `hw1.tex`.

1 Propositional Logic

Determine whether each of the following propositional formulas are valid or not. If they are, then prove their validity using the semantic argument method. Otherwise, provide a falsifying interpretation.

1. $(P \land Q) \to (P \to Q)$
2. $(P \to (Q \to R)) \to (\neg R \to (\neg Q \to \neg P))$
3. $\neg(P \land Q) \leftrightarrow (\neg P \lor \neg Q)$

Problem 1.1

Suppose F is not valid, then there exists an interpretation I such that $I \not\models F$:

1. $I \not\models F$	assumption
2. $I \models P \land Q$	1, $\to$
3. $I \not\models P \to Q$	1, $\to$
4. $I \models Q$	2, $\land$
5. $I \not\models Q$	3, $\to$
6. $I \models \perp$	4, 5

Therefore F is valid.

Problem 1.2

Suppose F is not valid, then there exists an interpretation I such that $I \not\models F$:

1. $I \not\models F$	assumption
2. $I \models P \to (Q \to R)$	1, $\to$
3. $I \not\models \neg R \to (\neg Q \to \neg P)$	1, $\to$
4. $I \not\models R$	3, $\to$
5. $I \not\models (\neg Q \to \neg P)$	3, $\to$
6. $I \models \neg Q$	5, $\to$
7. $I \not\models \neg P$	5, $\to$
8. $I \not\models Q$	6, $\neg$
9. $I \models P$	7, $\neg$
10. $I \models Q \to R$	2, 9, and *modus ponens*

There are two cases to consider from 6: either Q is false, or R is true.
In the first case, the table is complete, yet we did not derive a contradiction. In fact, we found the interpretation.

$$I : \{P \mapsto \texttt{true}, Q \mapsto \texttt{false}, R \mapsto \texttt{false}\}$$

Therefore, the original formula F is not valid.

Problem 1.3

(According to the **De Morgan's Law**, $\neg(P \land Q) \Leftrightarrow (\neg P \lor \neg Q)$, so the formula is valid.)
Suppose F is not valid, then there exists an interpretation I such that $I \not\models F$:

1. $I \not\models F$	assumption

There are two cases to consider. In the first case:

2. $I \models \neg(P \land Q) \land \neg(\neg P \lor \neg Q)$	1, $\leftrightarrow$
3. $I \not\models (P \land Q)$	2, $\land$
4. $I \not\models \neg P \lor \neg Q$	2, $\land$
5. $I \models P$	4, $\lor$
6. $I \models Q$	4, $\lor$

There are two cases to consider in 3. In the first case:

7a. $I \not\models P$	3, $\land$
8a. $I \models \perp$	5, 7a

In the second case:

7b. $I \not\models Q$	3, $\land$
8b. $I \models \perp$	5, 7b

In the second case of 1:

2. $I \models (\neg\neg(P \land Q)) \land (\neg P \lor \neg Q)$	1, $\leftrightarrow$
3. $I \models P \land Q$	2, $\land$
4. $I \models P$	3, $\land$
5. $I \models Q$	3, $\land$
6. $I \models \neg P \lor \neg Q$	2, $\land$

There are two cases to consider in 6. In the first case:

7a. $I \not\models P$	6, $\land$
8a. $I \models \perp$	4, 7a

In the second case:

7b. $I \not\models Q$	6, $\land$
8b. $I \models \perp$	5, 7b

In all branches, I derives $\perp$, therefore $\neg F$ is unsatisfiable, i.e. F is valid.

2 Normal Forms

Convert the following propositional formulas into NNF, DNF, and CNF without Tseitin's Transformation. Please show each step of your work.

1. $\neg P \leftrightarrow Q$
2. $\neg(\neg(P \wedge Q) \rightarrow \neg R)$
3. $\neg(Q \rightarrow R) \wedge P \wedge (Q \vee \neg(P \wedge R))$

Convert the following formula into CNF using Tseitin's Transformation. Please show each step of your work.

$$(P \rightarrow (\neg Q \wedge R)) \wedge (P \rightarrow \neg Q)$$

Problem 1.1
First, convert $F : \neg P \leftrightarrow Q$ into NNF:

$$\begin{aligned} F &\Leftrightarrow (\neg P \rightarrow Q) \wedge (Q \rightarrow \neg P) \\ &\Leftrightarrow (P \vee Q) \wedge (\neg Q \vee \neg P) \qquad (*) \end{aligned}$$

(*) is in NNF and also CNF, and it's equivalent to F.
Then apply distributivity twice:

$$\begin{aligned} F &\Leftrightarrow (P \wedge (\neg Q \vee \neg P)) \vee (Q \wedge (\neg Q \vee \neg P)) \\ &\Leftrightarrow (P \wedge \neg Q) \vee (Q \wedge \neg P) \end{aligned}$$

Now F is converted into DNF.

Problem 1.2
First, convert $F : \neg(\neg(P \wedge Q) \rightarrow \neg R)$ into NNF:

$$\begin{aligned} F &\Leftrightarrow \neg(P \wedge Q) \wedge \neg\neg R \\ &\Leftrightarrow (\neg P \vee \neg Q) \wedge R \qquad (*) \end{aligned}$$

(*) is in NNF and also CNF, and it's equivalent to F.
Then we apply distributivity to (*):

$$(*) \Leftrightarrow (\neg P \wedge R) \vee (\neg Q \wedge R)$$

Now F is converted into DNF.

Problem 1.3
First, convert $F : \neg(Q \rightarrow R) \wedge P \wedge (Q \vee \neg(P \wedge R))$

$$\begin{aligned} F &\Leftrightarrow (Q \wedge \neg R) \wedge P \wedge (Q \vee \neg P \vee \neg R) \\ &\Leftrightarrow Q \wedge \neg R \wedge P \wedge (Q \vee \neg P \vee \neg R) \qquad (*) \end{aligned}$$

(*) is in NNF and also CNF, and it's equivalent to F.
Then we apply distributivity to (*):

$$F \Leftrightarrow (Q \land Q \land \neg R \land P) \lor (\neg P \land Q \land \neg R \land P) \lor (\neg R \land Q \land \neg R \land P)$$
$$\Leftrightarrow Q \land \neg R \land P$$

Now F is converted into DNF.

Problem 1.4
According to Tseitin's Transformation, F is equiasatisfiable as:

$$P_F \land (P_F \leftrightarrow (P_{G_1} \land P_{G_2})) \land (P_{G_1} \leftrightarrow (P \to P_{G_3})) \land (P_{G_2} \leftrightarrow (P \to \neg Q)) \land (P_{G_3} \leftrightarrow (\neg Q \land R))$$

Now convert all clauses into CNF :

1. $P_F \leftrightarrow (P_{G_1} \land P_{G_2}) \Leftrightarrow (\neg P_F \lor P_{G_1}) \land (\neg P_F \lor P_{G_2}) \land (\neg P_{G_1} \lor \neg P_{G_2} \lor P_F)$
2. $P_{G_1} \leftrightarrow (P \to P_{G_3}) \Leftrightarrow (P \lor P_{G_1}) \land (P \lor \neg P_{G_3}) \land (\neg P_{G_1} \lor \neg P \lor P_{G_3})$
3. $P_{G_2} \leftrightarrow (P \to \neg Q) \Leftrightarrow (P \lor P_{G_2}) \land (Q \lor P_{G_2}) \land (\neg P_{G_2} \lor \neg P \lor \neg Q)$
4. $\neg P_{G_1} \lor \neg P_{G_2} \lor P_F \Leftrightarrow (\neg P_{G_3} \lor \neg Q) \land (\neg P_{G_3} \lor R) \land (Q \lor \neg R \lor P_{G_3})$

So the CNF of F is:
$P_F \land (\neg P_F \lor P_{G_1}) \land (\neg P_F \lor P_{G_2}) \land (\neg P_{G_1} \lor \neg P_{G_2} \lor P_F) \land (P \lor P_{G_1}) \land (P \lor \neg P_{G_3}) \land (\neg P_{G_1} \lor \neg P \lor P_{G_3}) \land (P \lor P_{G_2}) \land (Q \lor P_{G_2}) \land (\neg P_{G_2} \lor \neg P \lor \neg Q) \land (\neg P_{G_3} \lor \neg Q) \land (\neg P_{G_3} \lor R) \land (Q \lor \neg R \lor P_{G_3})$

3 Modeling

A nondeterministic finite automaton is given by a 5-tuple $(Q, \Sigma, \delta, I, F)$, where:

- Q is a finite set of states
- Σ is a finite alphabet
- $\delta : Q \times \Sigma \times 2^Q$ is a transition function
- $I \subseteq Q$ is a set of initial states
- $F \subseteq Q$ is a set of final (accepting) states

The automaton accepts a finite word $\mathbf{w} = w_0, \ldots, w_n$, where $w_i \in \Sigma$, if and only if there is a sequence of states $q_0, \ldots, q_n$, with $q_i \in Q$, such that:

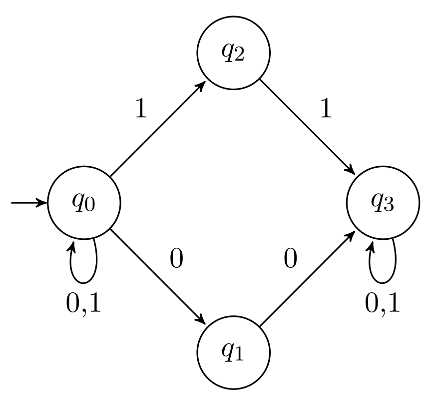
- $q_0 \in I$
- For all $i \in \{1, \ldots, n\}$, $q_i \in \delta(q_{i-1}, w_i)$
- $q_n \in F$

Part 1. Given an NFA $(Q, \Sigma, \delta, I, F)$ and a fixed input string $\mathbf{w} = w_1, \ldots, w_n$, describe how to construct a propositional formula that is satisfiable if and only if the automaton accepts the string.

Hint: Consider defining propositional variables that correspond to the initial states, final states, transition function, and alphabet symbols in $\mathbf{w}$. Then think about "unwinding" the NFA on $\mathbf{w}$. Do you need to define additional variables? How can you encode the fact that $\mathbf{w}$ is accepted?

Part 2. Demonstrate your encoding from part 1 on the following automaton:

$$
\begin{array}{rcl}
Q & = & \{q_0, q_1, q_2, q_3\} \\
\Sigma & = & \{0, 1\} \\
\delta & = & \left[\begin{array}{ccc}
(q_0, 0) & \mapsto & q_0, \\
(q_0, 1) & \mapsto & q_0, \\
(q_0, 0) & \mapsto & q_1, \\
(q_0, 1) & \mapsto & q_2, \\
(q_1, 0) & \mapsto & q_3, \\
(q_2, 1) & \mapsto & q_3, \\
(q_3, 0) & \mapsto & q_3, \\
(q_3, 1) & \mapsto & q_3
\end{array}\right] \\
I & = & \{q_0\} \\
F & = & \{q_3\}
\end{array}
$$



a. Let R_{kq} represents that the current input is w_k in $w = w_1 \dots w_n$ and the current state is $q \in Q$, X_{ijA} represents that $q_j \in \sigma(q_i, A)$.
Specially, to describe arriving the end of the string w, let $R_{n+1q}(w = w_1 \dots w_n)$ represents that all the inputs have been scanned and the current state is q.
Then we can construct the formula as:

$$\begin{aligned}
&\bigvee_{q_i \in I} R_{1q_i} \\
\wedge &\bigwedge_{1<k\leq n} \bigvee_{q_i,q_j \in Q} (X_{ijw_k} \wedge R_{kq_i} \wedge R_{k+1q_j}) \\
\wedge &\bigvee_{q_i \in F} R_{n+1q_i} \\
\wedge &\bigwedge_{\substack{1\leq k\leq n+1,\\ q_i,q_j \in Q}} (\neg R_{kq_i} \vee \neg R_{kq_j})
\end{aligned}$$

- The initial state q of R_{1q} must be one of the state in I.
- w is accepted iff the current state $q_F \in F$ and the string meets the end.
- Two different states can't be arrived at the same time during the whole process.

b Let $w = 11$, then the formula F is:

$$\begin{aligned}
R_{1q_0} &\wedge ((X_{021} \wedge R_{1q_0} \wedge R_{2q_2}) \vee (X_{001} \wedge R_{1q_0} \wedge R_{2q_0}) \vee (X_{231} \wedge R_{1q_2} \wedge R_{2q_3}) \vee (X_{331} \wedge R_{1q_3} \wedge R_{2q_3})) \\
&\wedge ((X_{021} \wedge R_{2q_0} \wedge R_{3q_2}) \vee (X_{001} \wedge R_{2q_0} \wedge R_{3q_0}) \vee (X_{231} \wedge R_{2q_2} \wedge R_{3q_3}) \vee (X_{331} \wedge R_{2q_3} \wedge R_{3q_3})) \\
&\wedge R_{3q_3} \wedge \bigwedge_{\substack{1\leq k\leq 3,\\ q_i,q_j \in Q}} (\neg R_{kq_i} \vee \neg R_{kq_j})
\end{aligned}$$

F is true under the interpretation:

$I : \{R_{1q_0} \mapsto \texttt{true}, R_{2q_2} \mapsto \texttt{true}, R_{3q_3} \mapsto \texttt{true}, X_{021} \mapsto \texttt{true}, X_{231} \mapsto \texttt{true}, R_{1q_1} \mapsto \texttt{false}, \dots \mapsto \texttt{false}\}$

Therefore, the automation accepts w.