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Assignment 4
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Instructions: Write your answers in the corresponding `hw4.tex` file, compile it to a pdf, and hand the pdf file to *Tsinghua Web Learning* by the due date. Be sure to add your **student ID** and **full name** in the `stuid` and `stuname` macros at the top of `hw4.tex`.

Acamedic Honesty: Any kind of plagiarism is strictly prohibited in the full semester for this course. Students who are suspected to copy other's work and is confirmed through investigation will receive no credits (i.e, zero) for this assignment. If you asked other students for help, or your referred to any material that is not provided by us (e.g. websites, blogs, articles, papers, etc., both online and offline), please mention them in your assignment (e.g. writing an acknowledgment, adding a reference).

1 Semantic argument in T_E

Use the semantic method to argue the validity of the following Σ_E-formulae, or identify a counterexample (a falsifying T_E-interpretation).

1. $f(x, y) = f(y, x) \rightarrow f(a, y) = f(y, a)$
2. $f(g(x)) = g(f(x)) \wedge f(g(f(y))) = x \wedge f(y) = x \rightarrow g(f(x)) = x$
3. $f(f(f(a))) = f(f(a)) \wedge f(f(f(f(a)))) = a \rightarrow f(a) = a$

Solution

1. We can easily find a counter-example of the Σ_E-formulae. Let $D = \{0, 1\}$, $I : f(x, y) = x$ for all $x \in D$, $a = 1$ and $\alpha : \{x \mapsto 0, y \mapsto 0\}$. Then $D, I, \alpha \nvDash \neg F$ since $f(x, y) = f(y, x) = 0$, but $f(a, y) = 1 \neq f(y, a) = 0$, i.e, $\neg(f(a, y) = f(y, a))$.Therefore, F is not valid.
2.Suppose F is not valid, $\neg F = f(g(x)) = g(f(x)) \wedge f(g(f(y))) = x \wedge f(y) = x \wedge g(f(x)) \neq x$ is satisfiable. We can construct the congruence relation on the subterm set:

$$\{\{x\}, \{y\}, \{f(g(x))\}, \{g(f(y))\}, \{f(g(f(y)))\}, \{f(y)\}\}$$

Then we merge the congruence classes:

$$\{\{x, f(y), f(g(f(y)))\}, \{f(g(x)), g(f(x))\}, \{y\}\}$$

By $f(y) \sim x$, we get $f(g(x)) \sim f(g(f(y))) \sim x$. Now we get the congruence closure:

$$\{\{x, f(y), f(g(f(y))), f(g(x)), g(f(x))\}, \{y\}\}.$$

$\neg F$ is T_E-unsatisfiable since $g(f(x)) \sim x$ and $\neg F$ asserts that $g(f(x)) \neq x$. Therefore F is valid.
3. Suppose F is not valid, then $F' = \neg F = f(f(f(a))) = f(f(a)) \wedge f(f(f(f(a)))) = a \wedge f(a) \neq a$

is satisfiable. We use the congruence al First, build the subterm set S'_F:

$$\{f^3(a), f^2(a), f^4(a), a, f(a)\}$$

Construct the initial congruence relation on $S_{F'}$:

$$\{\{f^3(a)\}, \{f^2(a)\}, \{f^4(a)\}, \{a\}, \{f(a)\}\}$$

From $f^3(a) = f^2(a)$, merge $\{f^3(a)\}$ and $\{f^2(a)\}$, then, from $f^3(a) \sim f^2(a)$, propagate $f^4(a) \sim f^3(a)$:

$$\{\{f^3(a), f^2(a), f^4(a)\}, \{a\}, \{f(a)\}\}$$

From $f^4(a) = (a)$, merge $\{f^3(a), f^2(a), f^4(a)\}$ and $\{a\}$:

$$\{\{f^3(a), f^2(a), f^4(a), a\}, \{f(a)\}\}$$

From $a \sim f^2(a)$, propagate $f(a) \sim f^3(a)$:

$$\{\{f^3(a), f^2(a), f^4(a), a, f(a)\}\}$$

which is the congruence closure of $S_{F'}$. F' asserts $f(a) \neq a$, while $f(a) \sim a$, so F' is unsatisfiable, therefore, F is valid.

2 Semantic argument in $T_{\mathbb{Z}}$

Use the semantic method to argue the validity of the following $\Sigma_{\mathbb{Z}}$-formulae, or identify a counterexample (a falsifying $T_{\mathbb{Z}}$-interpretation).

1. $x \leq y \land z = x + 1 \rightarrow z \leq y$
2. $3x = 2 \rightarrow x \leq 0$
3. $1 \leq x \land x \leq 2 \rightarrow x = 1 \lor x = 2$

Solution

1. Let $I : \{x \mapsto 0, y \mapsto 0, z \mapsto 1\}$, we have $x \leq y \land z = x + 1 \land \neg(z \leq y)$. I is a falsifying $T_{\mathbb{Z}}$-interpretation of F. Therefore, F is not valid.
2. Suppose F is not valid, there must be a $T_{\mathbb{Z}}$-interpretation such that $I \not\models F$.

1.	$I \not\models F$	assumption
2.	$I \models 3x = 2$	$1, \rightarrow$
3.	$I \models \perp$	$2, T_{\mathbb{Z}}$

Therefore, F is valid.
3.Suppose F is not valid, there must be a $T_{\mathbb{Z}}$-interpretation such that $I \not\models F$.

1.	$I \not\models F$	assumption
2.	$I \models 1 \leq x \land x \leq 2$	$1, \rightarrow$
3.	$I \not\models x = 1 \lor x = 2$	$1, \rightarrow$
4.	$I \not\models x = 1$	$3, \lor$
5.	$I \not\models x = 2$	$3, \lor$
6.	$I \models \perp$	$2, 4, 5, T_{\mathbb{Z}}$

Therefore, F is valid.

3 Semantic argument in T_A

Use the semantic method to argue the validity of the following Σ_A-formulae, or identify a counterexample (a falsifying T_A-interpretation).

1. $a\langle i \triangleleft e\rangle[j] = e \rightarrow i = j$
2. $a\langle i \triangleleft e\rangle[j] = e \rightarrow a[j] = e$
3. $a\langle i \triangleleft e\rangle[j] = e \rightarrow i = j \vee a[j] = e$

Solution

1. We can easily find a falsifying T_A-interpretation of F. Let $a[c_j] = e, a[c_i] = e$ in I and $\alpha : \{i \mapsto c_i, j \mapsto c_j\}$, we can indicate that $I, \alpha \nvDash F$. Therefore, F is not valid.
2. We can find a falsifying T_A-interpretation of F. Let $a[c_i] = k$ where $k \neq e$ in I, and $\alpha : \{i \mapsto c_i, j \mapsto c_j\}$. We can conclude that $I, \alpha \vDash a\langle i \triangleleft e\rangle[j] = e$. and $a[j] = a[c_j] = k \neq e$. $I, \alpha \vDash \neg F$, i.e, $I, \alpha \nvDash F$. Therefore, F is not valid.
3. Suppose F is not valid, there must be a T_A-interpretation such that $I \nvDash F$.

1.	$I \nvDash F$	assumption
2.	$I \vDash a\langle i \triangleleft e\rangle[j] = e$	$1, \rightarrow$
3.	$I \nvDash i = j \vee a[j] = e$	$1, \rightarrow$
4.	$I \nvDash i = j$	$3, \vee$
5.	$I \nvDash a[j] = e$	$3, \vee$
6.	$I \vDash i \neq j$	$4, \neg$
7.	$I \vDash i \neq j \rightarrow a\langle i \triangleleft e\rangle[j] = a[j]$	r-o-w $2, \forall$
8.	$I \vDash a\langle i \triangleleft e\rangle[j] = a[j]$	$6, 7, modusponens$
9.	$I \vDash a[j] = e$	$2, 8$
10.	$I \vDash \perp$	$5, 9$

Therefore, F is valid.