Preliminaries Homework Solutions

1. (a) $$\frac{d}{dx} f(x) = 6 \cdot 2x^5 - 1 \cdot 3x^0 = 12x^5 - 3$$

(b) $$\frac{d}{dx} g(x) = \frac{d}{dx} 2x \log(4x^2+x-16) = \left(\frac{d}{dx} 2x\right) * \log(4x^2+x-16) + 2x \frac{d}{dx} \log(4x^2+x-16)$$

multiplication rule and rule for natural logs.

chain rule

$$= 2 \log(4x^2+x-16) + 2x \cdot \frac{8x+1}{4x^2+x-16}$$

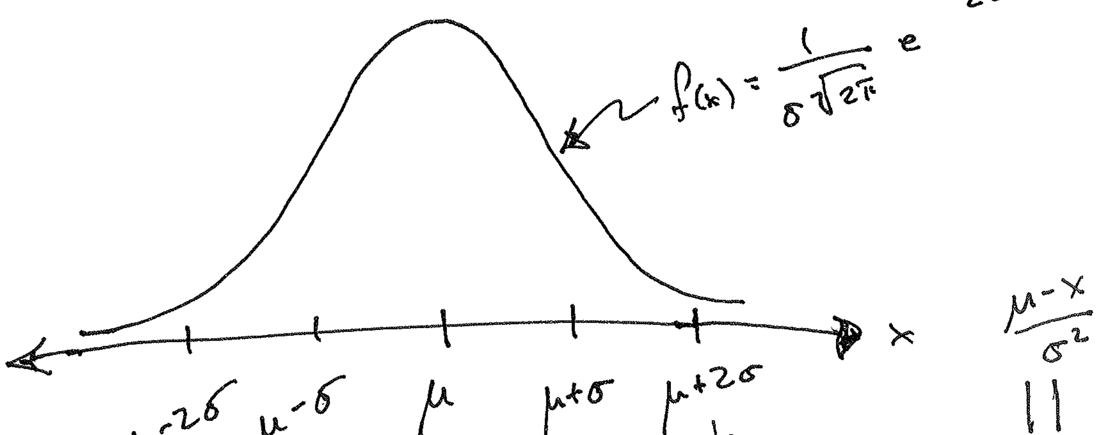
(c) $$\frac{d}{dx} h(x) = (-5) \cdot 3e^{-5x} = -15e^{-5x}$$

(d) $$\frac{d}{dx} p(x) = \left[\frac{d}{dx}(3x^2)\right] e^{-5x} + 3x^2 \frac{d}{dx} e^{-5x} = 6xe^{-5x} + (-5) \cdot 3x^2 e^{-5x}$$

multiplication rule

$$= 6xe^{-5x} - 15x^2 e^{-5x}$$

2. (a)



(b) $$\frac{d}{dx} f(x) = \frac{1}{\sigma\sqrt{2\pi}} \frac{d}{dx} e^{\frac{-(x-\mu)^2}{2\sigma^2}} = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} \cdot \underbrace{\frac{d}{dx}\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)}_{= \frac{\mu - x}{\sigma^2}}$$

chain rule

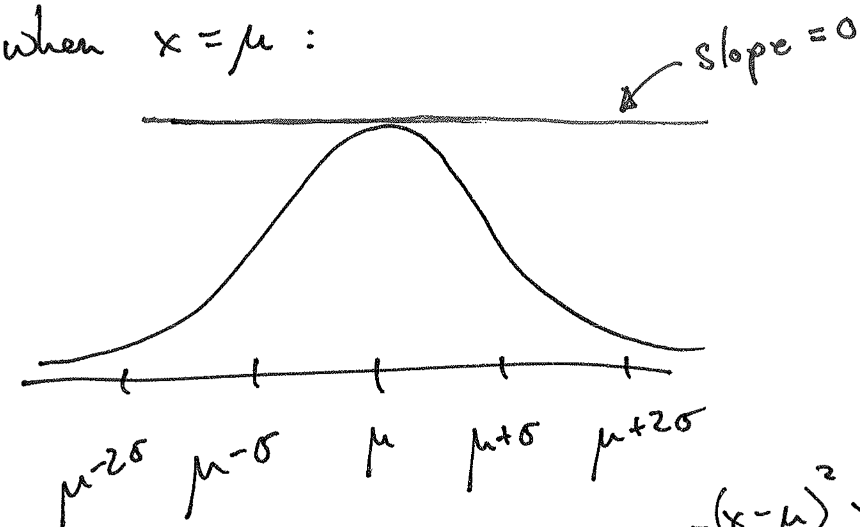
$$= \frac{1}{\sigma\sqrt{2\pi}} \cdot \frac{\mu-x}{\sigma^2} \cdot e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

2. (c) Setting $\frac{\delta}{\delta x} f(x) = 0$ means

$$\frac{\mu - x}{\sigma^2} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}} = 0$$

$$\Longrightarrow \mu = x.$$

This means the slope of a line tangent to $f(x)$ is 0 when $x = \mu$:



(d) $$\frac{\delta^2}{dx^2} f(x) = \frac{\delta}{dx}\left(\frac{1}{\sigma\sqrt{2\pi}} \cdot \frac{\mu - x}{\sigma^2} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}\right)$$

using the multiplication rule

$$= \frac{1}{\sigma\sqrt{2\pi}} \cdot \frac{\mu - x}{\sigma^2} \cdot \frac{\delta}{dx}\left(e^{-\frac{(x-\mu)^2}{2\sigma^2}}\right) + \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}} \cdot \frac{\delta}{dx}\left(\frac{\mu - x}{\sigma^2}\right)$$

$$= \frac{1}{\sigma\sqrt{2\pi}}\left(\frac{\mu - x}{\sigma^2}\right)^2 e^{-\frac{(x-\mu)^2}{2\sigma^2}} + \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \cdot \frac{-1}{\sigma^2}$$

2(d) continued ...

Setting the second derivative $\frac{d^2}{dx} f(x)$ equal to zero implies

$$\left(\frac{\mu - x}{\sigma^2}\right)^2 \cancel{e^{\frac{-(x-\mu)^2}{2\sigma^2}}} = \frac{1}{\sigma^2} \cancel{e^{\frac{-(x-\mu)^3}{2\sigma^2}}}$$

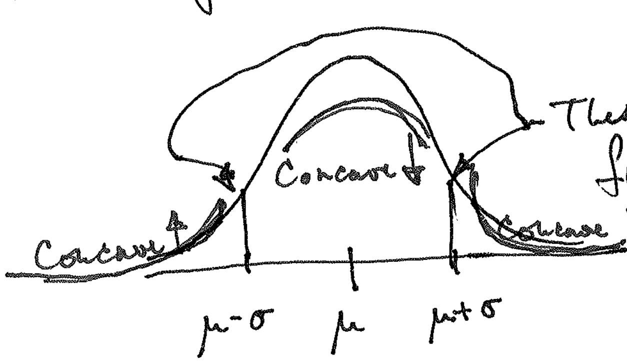
$$\Rightarrow \frac{(\mu - x)^2}{\sigma^4} = \frac{1}{\sigma^2}$$

$$\Rightarrow \sigma^2 = (\mu - x)^2$$

$$\Rightarrow \pm\sigma = \mu - x$$

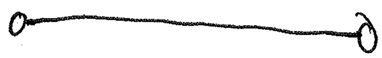
$$\Rightarrow \boxed{x = \mu \pm \sigma}$$

So the two inflection points occur at $x = \mu - \sigma$ and $x = \mu + \sigma$:



These are the places where $f(x)$ changes concavity.

3. (a)



little x

$$f(x) = \frac{1}{b-a}\,\mathbb{1}\{a < \cancel{X} < b\}$$

This is called an indicator function:

$$\mathbb{1}\{\text{stuff}\} = \begin{cases} 1 & \text{if "stuff" is true} \\ 0 & \text{if "stuff" is false} \end{cases}$$

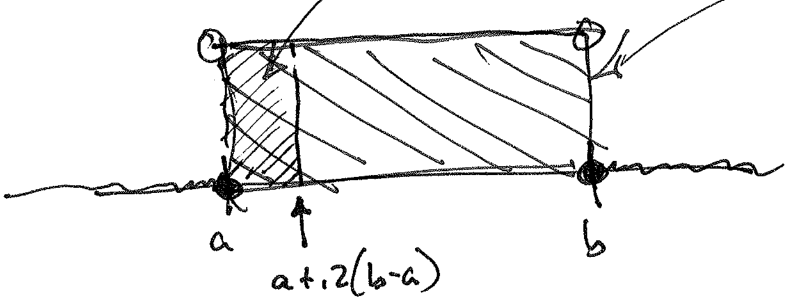
(b)
$$P\big(a < X < a + .2(b-a)\big) = \int_a^{a+.2(b-a)} \frac{1}{b-a}\,dx$$

$$= \left.\frac{x}{b-a}\right|_a^{a+.2(b-a)}$$

$$= \frac{a + .2(b-a) - a}{b-a}$$

$$= \frac{.2\cancel{(b-a)}}{\cancel{b-a}} = .2$$

This makes perfect sense: This area is 1/5 the total area here.



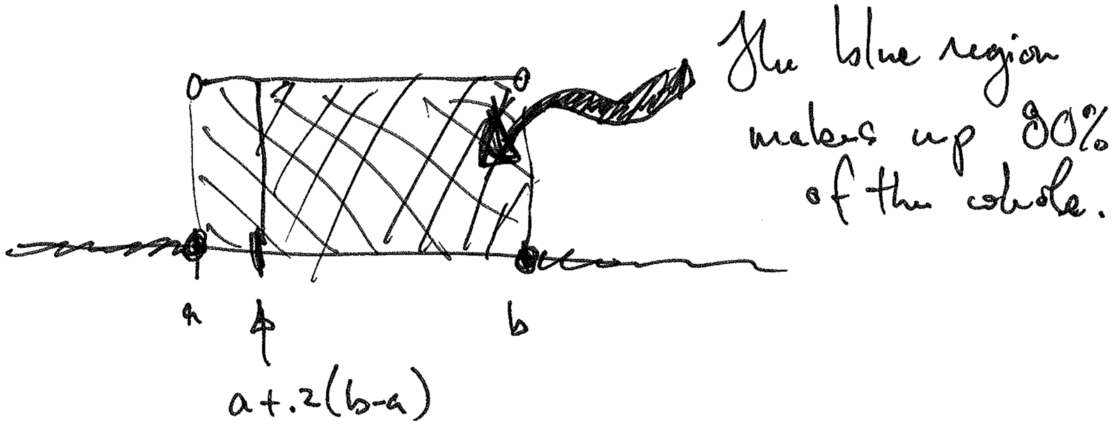
3. (c) $P(a+.2(b-a) < X < b) = \int_{a+.2(b-a)}^{b} \frac{1}{b-a}\,dx$

$$= \frac{b}{b-a} - \frac{a+.2(b-a)}{b-a}$$

$$= \frac{b-a-.2b+.2a}{b-a}$$

$$= \frac{.8(b-a)}{b-a} = .80$$

This also makes sense when you examine the density function:



4. $\mathrm{Var}(X) := E\left[\left(X - E[X]\right)^2\right]$

F.O.I.L.

$$= E\left[X^2 + \left(E[X]\right)^2 - 2XE[X]\right]$$

$$= E\left[X^2\right] + E\left[\left(E[X]\right)^2\right] - E\left[2XE[X]\right]$$

The mean of a sum of random (or non-random) terms is the sum of the means.

$$= E\left[X^2\right] + \left(E[X]\right)^2 - 2E[X]E[X]$$

$$= E\left[X^2\right] - \left(E[X]\right)^2$$

You might call this the "<u>inside-outside</u>" formula. It comes in handy when computing variance.