Chapter 1

Preliminaries

1.1 $n!$, ${}_nP_k$, ${}_nC_k$

The quantity

$$n! := 1 \times 2 \times \cdots \times (n-1) \times n$$

is read n *factorial* and equals the number of ways to arrange (or permutate) n items in order from left to right. The number of ways to choose and arrange k items from n items is

$${}_nP_k := \frac{n!}{(n-k)!}$$

and read as n *permutate* k. The number of distinct subsets of size k that can be taken from a set of n items is

$${}_nC_k = \binom{n}{k} := \frac{n!}{k!(n-k)!}$$

and read as n *choose* k.

Example 1

1.2 Binomial Theorem

For real numbers a and b and non-negative integer n,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}. \tag{1.1}$$

Example 2

1.3 Differentiation

Let f be a 'nice' function, mapping a subset of the real numbers to a subset of the real numbers. The *derivative* of f with respect to x is

$$\frac{d}{dx} f(x) := \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{1.2}$$

and equals the slope of the line tangent to f at horizontal coordinate x:

A function is *differentiable* if taking its derivative 'makes sense':

Proposition 3 *Power rule. If $f(x) = ax^n + c$ where a and c are constants and $n \neq 0$ then*

$$\frac{d}{dx} f(x) = anx^{n-1}.$$

Proof

$\blacksquare$

Proposition 4 *Power rule for polynomials. Suppose* $f(x) = \sum_{k=0}^{n} a_k x^k$*. Then*

$$\frac{d}{dx} f(x) = \sum_{k=1}^{n} a_k x^{k-1}.$$

Proof The result follows since the derivative is a limit and the limit of a sum of a finite number of terms where each terms has limit in $(-\infty, \infty)$ is the sum of the limits. ∎

Example 5

Some other derivatives:

$$\begin{aligned}
\frac{d}{dx} e^x &= e^x \\
\frac{d}{dx} \log x &= \frac{1}{x} \\
\frac{d}{dx} f(g(x)) &= \frac{df(x)}{dg(x)} \times \frac{dg(x)}{dx} \qquad \text{(chain rule)} && (1.3) \\
\frac{d}{dx} f(x)g(x) &= g(x)\frac{d}{dx} f(x) + f(x)\frac{d}{dx} g(x) \qquad \text{(multiplication rule)} && (1.4)
\end{aligned}$$

Example 6 Differentiate $e^{5x^3-x+100}$ with respect to x.

Example 7 Calculate the derivative of $\log(5x^3 - x + 100)$ with respect to x.

Example 8 Calculate the derivative of $\log(5x^3 - x + 100)^{x^5-3x}$ with respect to x.

1.4 Integration

Suppose f maps some reals to reals. The *integral* of the function $f(x)$ between two points a and b is the area bounded by the function f, the horizontal line $y = 0$, and the vertical lines $x = a$ and $x = b$ and *could* be defined to be

$$\int_a^b f(x)dx := \lim_{n\to\infty} \sum_{k=1}^{n} \frac{b-a}{n} f\left(a + k\left(\frac{b-a}{n}\right)\right)$$

Theorem 9 *Fundamental Theorem of Calculus. If f and F are continuous on $[a, b]$ and $dF/dx = f(x)$ for every $x \in (a, b)$ then*

$$\int_a^b f(x)dx = F(b) - F(a).$$

Example 10 Integrate the following:
$\int_1^3 2e^{-2x}dx =$

$\int_0^\infty \lambda e^{-\lambda x}dx =$

$\int_0^3 (x^3 - 5x^2)dx =$

1.5 Probabilities

If X is a discrete random variable,

$$P(a \leq X \leq b) = \sum_{k=a}^{b} P(X = k).$$

If X is a continuous random variable,

$$P(a \leq X \leq b) = \int_a^b f_X(x)dx.$$

1.6 Moments of Random Variables

The kth moment of the random variable X about the point a is

$$E[(X-a)^k] \tag{1.5}$$

and is a sort-of average distance measurement between X and the location a.

Some moments of particular interest to us are the *expected value* and *variance* of X:

$$E[X] := \begin{cases} \sum_x xP(X=x) & \text{(discrete)} \\ \int_{\mathbb{R}} xf(x)dx & \text{(continuous)} \end{cases} \tag{1.6}$$

$$\mathrm{Var}(X) := E[(X-E[X])^2]. \tag{1.7}$$

So the mean is really the first moment about 0 and the variance is the second moment about the mean. The mean is the same as the center of mass calculation in physics. The variance is the same as the moment of inertia calculation in physics, but without any mass units attached.

If $\hat{\theta}$ is a random estimator for a population parameter θ, then

$$E[(\hat{\theta}-\theta)^2]$$

is the second moment of $\hat{\theta}$ about θ. It's also called the *mean squared error*. Note that mean squared error and variance are different because θ (the thing you do not know, but are trying to estimate) is not necessarily equal to $E[\hat{\theta}]$:

1.7 Homework

1. Differentiate the following functions with respect to x.

   (a) $f(x) = 2x^6 - 3x - 29.3$

   (b) $g(x) = \log[(4x^2 + x - 16)^{2x}]$

   (c) $h(x) = 3e^{-5x}$

   (d) $p(x) = 3x^2e^{-5x}$

2. Consider the normal density

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

   (a) Graph (by hand or technology) $f(x)$.

   (b) Differentiate $f(x)$ with respect to x.

   (c) Set this derivative equal to 0 and solve for x. What do you observe and why?

   (d) Differentiate the derivative of $f(x)$ (called the *second derivative*) and set it equal to zero. This will give the so-called *inflection points* of the $f(x)$ curve (where the curve changes from concave-up to concave-down, or vice-versa). Where are the inflection points?

3. X is a *continuous uniform* random variable on (a, b) if its density function is

$$f(x) = \frac{1}{b-a}\mathbb{1}\{a < x < b\}.$$

   (a) Graph this density function.

   (b) Calculate $P(a < X < a + .2(b - a))$.

   (c) Calculate $P(a + .2(b - a) < X < b)$.

   (d) Calculate the expected value of X. That is, find $E[X] = \int_a^b xf(x)dx$.

4. Show that $\text{Var}(X) = E[X^2] - (E[X])^2$. Begin with the definition in (1.7) and apply FOIL to the inside.