Lecture Notes in Mathematics

An Introduction to Gaussian Geometry

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Preface

These lecture notes grew out of a course on elementary differential geometry which I have given at Lund University for a number of years.

The purpose is to introduce the most beautiful theory of Gaussian geometry i.e. the theory of curves and surfaces in three dimensional Euclidean space. This is a subject with no lack of interesting examples. They are indeed the key to a good understanding of it and will therefore play a major role throughout this work.

The text is written for students with a good understanding of linear algebra, real analysis of several variables, and basic knowledge of the classical theory of ordinary differential equations and some topology.

I am grateful to my enthusiastic students who have contributed to the text by finding numerous typing errors and giving many helpful comments on the presentation.

Norra Nöbbelöv, 8 September 2014

Sigmundur Gudmundsson

Contents

CHAPTER 1

Introduction

Around 300 BC Euclid wrote "The Thirteen Books of the Elements". These were used as the basic text on geometry throughout the Western world for about 2000 years. Euclidean geometry is the theory one yields when assuming Euclid's five axioms, including the parallel postulate.

Gaussian geometry is the study of curves and surfaces in three dimensional Euclidean space. This theory was initiated by the ingenious Carl Friedrich Gauss (1777-1855). The work of Gauss, János Bolyai (1802-1860) and Nikolai Ivanovich Lobachevsky (1792-1856) lead to their independent discovery of non-Euclidean geometry. This solved the best known mathematical problem ever and proved that the parallel postulate is indeed independent of the other four axioms that Euclid used for his theory.

CHAPTER 2

Curves in the Euclidean Plane $\mathbb{R}^2$

In this chapter we study regular curves in the two dimensional Euclidean plane. We define their curvature and show that this determines the curves up to orientation preserving Euclidean motions. We then prove the isoperimetric inequality for plane curves.

Let the n-dimensional real vector space $\mathbb{R}^n$ be equipped with its standard Euclidean **scalar product** $\langle \cdot , \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$. This is given by

$$\langle x, y \rangle = x_1 y_1 + \cdots + x_n y_n$$

and induces the norm $| \cdot | : \mathbb{R}^n \to \mathbb{R}_0^+$ on $\mathbb{R}^n$ with

$$|x| = \sqrt{x_1^2 + \cdots + x_n^2}.$$

Definition 2.1. A map $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ is said to be a **Euclidean motion** of $\mathbb{R}^n$ if it is given by $\Phi : x \mapsto Ax + b$ where $b \in \mathbb{R}^n$ and

$$A \in \mathbf{O}(n) = \{X \in \mathbb{R}^{n \times n} | \ X^t X = I\}.$$

A Euclidean motion Φ is said to be **rigid** or **orientation preserving** if

$$A \in \mathbf{SO}(n) = \{X \in \mathbf{O}(n) | \ \det X = 1\}.$$

Definition 2.2. A **parametrized curve** in $\mathbb{R}^n$ is a C^1-map $\gamma : I \to \mathbb{R}^n$ from an open interval I on the real line $\mathbb{R}$. The image $\gamma(I)$ in $\mathbb{R}^n$ is the corresponding **geometric curve**. We say that the map $\gamma : I \to \mathbb{R}^n$ parametrizes $\gamma(I)$. The derivative $\gamma'(t)$ is called the **tangent** of γ at the point $\gamma(t)$ and

$$L(\gamma) = \int_I |\gamma'(t)| \ dt \leq \infty$$

is the **arclength** of γ. The curve γ is said to be **regular** if $\gamma'(t) \neq 0$ for all $t \in I$.

Example 2.3. If p and q are two distinct points in $\mathbb{R}^2$ then $\gamma : \mathbb{R} \to \mathbb{R}^2$ with

$$\gamma : t \mapsto (1 - t) \cdot p + t \cdot q$$

parametrizes the **straight line** through $p = \gamma(0)$ and $q = \gamma(1)$.

Example 2.4. If $r \in \mathbb{R}^+$ and $p \in \mathbb{R}^2$ then $\gamma : \mathbb{R} \to \mathbb{R}^2$ with

$$\gamma : t \mapsto p + r \cdot (\cos t, \sin t)$$

parametrizes a **circle** with **center** p and **radius** r. The arclength of the curve $\gamma|_{(0,2\pi)}$ is

$$L(\gamma|_{(0,2\pi)}) = \int_0^{2\pi} |\gamma'(t)| dt = 2\pi r.$$

Definition 2.5. A regular curve $\gamma : I \to \mathbb{R}^n$ is said to **parametrize** $\gamma(I)$ **by arclength** if $|\dot{\gamma}(s)| = 1$ for all $s \in I$ i.e. the tangents $\dot{\gamma}(s)$ are elements of the **unit sphere** S^{n-1} in $\mathbb{R}^n$.

Theorem 2.6. *Let $\gamma : I = (a, b) \to \mathbb{R}^n$ be a regular curve in $\mathbb{R}^n$. Then the image $\gamma(I)$ of γ can be parametrized by arclength.*

Proof. Define the arclength function $\sigma : (a, b) \to \mathbb{R}^+$ by

$$\sigma(t) = \int_a^t |\gamma'(u)| du.$$

Then $\sigma'(t) = |\gamma'(t)| > 0$ so σ is strictly increasing and

$$\sigma((a, b)) = (0, L(\gamma)).$$

Let $\tau : (0, L(\gamma)) \to (a, b)$ be the inverse of σ such that $\sigma(\tau(s)) = s$ for all $s \in (0, L(\gamma))$. By differentiating we get

$$\frac{d}{ds}(\sigma(\tau(s)) = \sigma'(\tau(s)) \cdot \dot{\tau}(s) = 1.$$

If we define the curve $\alpha : (0, L(\gamma)) \to \mathbb{R}^n$ by $\alpha = \gamma \circ \tau$ then the chain rule gives $\dot{\alpha}(s) = \gamma'(\tau(s)) \cdot \dot{\tau}(s)$. Hence

$$\begin{aligned} |\dot{\alpha}(s)| &= |\gamma'(\tau(s))| \cdot \dot{\tau}(s) \\ &= \sigma'(\tau(s)) \cdot \dot{\tau}(s) \\ &= 1. \end{aligned}$$

The function τ is bijective so α parametrizes $\gamma(I)$ by arclength. $\square$

For a regular curve $\gamma : I \to \mathbb{R}^2$, parametrized by arclength, we define the **tangent** $T : I \to S^1$ along γ by

$$T(s) = \dot{\gamma}(s)$$

and the **normal** $N : I \to S^1$ with

$$N(s) = R \circ T(s).$$

Here $R : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear rotation by the angle $+\pi/2$ satisfying

$$R : \begin{bmatrix} a \\ b \end{bmatrix} \mapsto \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.$$

It follows that for each $s \in I$ the set $\{T(s), N(s)\}$ is an orthonormal basis for $\mathbb{R}^2$. It is called the **Frenet frame** along the curve.

Definition 2.7. Let $\gamma : I \to \mathbb{R}^2$ be a regular C^2-curve parametrized by arclength. Then we define its **curvature** $\kappa : I \to \mathbb{R}$ by

$$\kappa(s) = \langle \dot{T}(s), N(s) \rangle.$$

Note that the curvature is a measure of how fast the unit tangent $T(s) = \dot{\gamma}(s)$ is bending in the direction of the normal $N(s)$, or equivalently, out of the line generated by $T(s)$.

Theorem 2.8. *Let $\gamma : I \to \mathbb{R}^2$ be a C^2-curve parametrized by arclength. Then the Frenet frame satisfies the following system of ordinary differential equations.*

$$\begin{bmatrix} \dot{T}(s) \\ \dot{N}(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) \\ -\kappa(s) & 0 \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \end{bmatrix}.$$

Proof. The curve $\gamma : I \to \mathbb{R}^2$ is parametrized by arclength so

$$2\langle \dot{T}(s), T(s) \rangle = \frac{d}{ds}(\langle T(s), T(s) \rangle) = 0$$

and

$$2\langle \dot{N}(s), N(s) \rangle = \frac{d}{ds}(\langle N(s), N(s) \rangle) = 0.$$

As a direct consequence we have

$$\begin{aligned} \dot{T}(s) &= \langle \dot{T}(s), N(s) \rangle N(s) = \kappa(s) N(s) \\ \dot{N}(s) &= \langle \dot{N}(s), T(s) \rangle T(s) = -\kappa(s) T(s), \end{aligned}$$

since

$$\langle \dot{T}(s), N(s) \rangle + \langle T(s), \dot{N}(s) \rangle = \frac{d}{ds}(\langle T(s), N(s) \rangle) = 0.$$

□

Theorem 2.9. *Let $\gamma : I \to \mathbb{R}^2$ be a C^2-curve parametrized by arclength. Then its curvature $\kappa : I \to \mathbb{R}$ vanishes identically if and only if the geometric curve $\gamma(I)$ is contained in a line.*

Proof. If follows from Theorem 2.8 that the curvature $\kappa(s)$ vanishes identically if and only if the tangent is constant i.e. there exist a unit vector $Z \in S^1$ and a point $p \in \mathbb{R}^2$ such that

$$\gamma(s) = p + s \cdot Z.$$

□

The following result tells us that a planar curve is, up to orientation preserving Euclidean motions, completely determined by its curvature.

Theorem 2.10. *Let $\kappa : I \to \mathbb{R}$ be a continuous function. Then there exists a C^2-curve $\gamma : I \to \mathbb{R}^2$ parametrized by arclength with curvature κ. If $\tilde{\gamma} : I \to \mathbb{R}^2$ is another such curve, then there exists a matrix $A \in \mathbf{SO}(2)$ and an element $b \in \mathbb{R}^2$ such that*

$$\tilde{\gamma}(s) = A \cdot \gamma(s) + b.$$

Proof. See the proof of Theorem 3.11. $\square$

In differential geometry we are interested in properties of geometric objects which are independent of how these objects are parametrized. The curvature of a geometric curve should therefore not depend on its parametrization.

Definition 2.11. Let $\gamma : I \to \mathbb{R}^2$ be a regular C^2-curve in $\mathbb{R}^2$ not necessarily parametrized by arclength. Let $t : J \to I$ be a strictly increasing C^2-function such that the composition $\alpha = \gamma \circ t : J \to \mathbb{R}^2$ is a curve parametrized by arclength. Then we define the **curvature** $\kappa : I \to \mathbb{R}$ of $\gamma : I \to \mathbb{R}^2$ by

$$\kappa(t(s)) = \tilde{\kappa}(s),$$

where $\tilde{\kappa} : J \to \mathbb{R}$ is the curvature of α.

Proposition 2.12. *Let $\gamma : I \to \mathbb{R}^2$ be a regular C^2-curve in $\mathbb{R}^2$. Then its curvature κ satisfies*

$$\kappa(t) = \frac{\det[\gamma'(t), \gamma''(t)]}{|\gamma'(t)|^3}.$$

Proof. See Exercise 2.5. $\square$

Corollary 2.13. *Let $\gamma : I \to \mathbb{R}^2$ be a regular C^2-curve in $\mathbb{R}^2$. Then the geometric curve $\gamma(I)$ is contained in a line if and only if $\gamma'(t)$ and $\gamma''(t)$ are linearly dependent for all $t \in I$.*

Proof. The statement is a direct consequence of Theorem 2.9 and Proposition 2.12. $\square$

We complete this chapter by proving the isoperimetric inequality. But let us first remind ourselves of the following topological facts.

Definition 2.14. A continuous map $\gamma : \mathbb{R} \to \mathbb{R}^2$ is said to parametrize a **simple closed curve** if it is periodic with period $L \in \mathbb{R}^+$ and the restriction

$$\gamma|_{[0,L)} : [0, L) \to \mathbb{R}^2$$

is injective.

The following result is called the **Jordan curve theorem**.

Fact 2.15. *Let the continuous map $\gamma : \mathbb{R} \to \mathbb{R}^2$ parametrize a simple closed curve. Then the subset $\mathbb{R}^2 \setminus \gamma(\mathbb{R})$ of the plane has exactly two connected components. The interior $Int(\gamma)$ of γ is bounded and the exterior $Ext(\gamma)$ is unbounded.*

Definition 2.16. A regular map $\gamma : \mathbb{R} \to \mathbb{R}^2$, parametrizing a simple closed curve, is said to be **positively oriented** if its normal

$$N(t) = R \cdot \gamma'(t) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \gamma'(t)$$

is an inner normal to the interior $\mathrm{Int}(\gamma)$ for all $t \in \mathbb{R}$. It is said to be **negatively oriented** otherwise.

We are now ready for the **isoperimetric inequality**.

Theorem 2.17. *Let C be a regular simple closed curve in the plane with arclength L and let A be the area of the region enclosed by C. Then*

$$4\pi \cdot A \leq L^2$$

with equality if and only if C is a circle.

Proof. Let l_1 and l_2 be two parallel lines touching the curve C such that C is contained in the strip between them. Introduce a coordinate system in the plane such that l_1 and l_2 are orthogonal to the x-axis and given by

$$l_1 = \{(x, y) \in \mathbb{R}^2 |\ x = -r\} \quad \text{and} \quad l_2 = \{(x, y) \in \mathbb{R}^2 |\ x = r\}.$$

Let $\gamma = (x, y) : \mathbb{R} \to \mathbb{R}^2$ be a positively oriented curve parameterizing C by arclength, such that $x(0) = r$ and $x(s_1) = -r$ for some $s_1 \in (0, L)$.

Define the curve $\alpha : \mathbb{R} \to \mathbb{R}^2$ by $\alpha(s) = (x(s), \tilde{y}(s))$ where

$$\tilde{y}(s) = \begin{cases} +\sqrt{r^2 - x^2(s)} & \text{if } s \in [0, s_1), \\ -\sqrt{r^2 - x^2(s)} & \text{if } s \in [s_1, L). \end{cases}$$

Then this new curve parameterizes the circle given by $x^2 + y^2 = r^2$.

As an immediate consequence of Lemma 2.18 we have

$$A = \int_0^L x(s) \cdot y'(s) ds \quad \text{and} \quad \pi \cdot r^2 = -\int_0^L \tilde{y}(s) \cdot x'(s) ds.$$

Employing the Cauchy-Schwartz inequality we then get

$$\begin{aligned} A + \pi \cdot r^2 &= \int_0^L (x(s) \cdot y'(s) - \tilde{y}(s) \cdot x'(s)) ds \\ &\leq \int_0^L \sqrt{(x(s) \cdot y'(s) - \tilde{y}(s) \cdot x'(s))^2} ds \end{aligned}$$

$$\begin{aligned} &\leq \int_0^L \sqrt{(x(s)^2 + \tilde{y}(s)^2) \cdot ((x'(s))^2 + (y'(s))^2)} ds \\ &= L \cdot r. \end{aligned}$$

The inequality

$$0 \leq (\sqrt{A} - r\sqrt{\pi})^2 = A - 2r\sqrt{A}\sqrt{\pi} + \pi r^2$$

implies that

$$2r\sqrt{A}\sqrt{\pi} \leq A + \pi r^2 \leq Lr$$

so

$$4A\pi r^2 \leq L^2 r^2$$

or equivalently

$$4\pi A \leq L^2.$$

It follows from our construction above that the positive real number r depends on the direction of the two parallel lines l_1 and l_2 chosen. In the case of equality $4\pi A = L^2$ we get $A = \pi r^2$. Since A is independent of the direction of the two lines, we see that so is r. This implies that in that case the curve C must be a circle. □

Lemma 2.18. *Let the regular, positively oriented map $\gamma : \mathbb{R} \to \mathbb{R}^2$ parameterize a simple closed curve in the plane. If A is the area of the interior Int(γ) of γ then*

$$\begin{aligned} A &= \frac{1}{2} \int_{\gamma(\mathbb{R})} (x(t)y'(t) - y(t)x'(t)) dt \\ &= \int_{\gamma(\mathbb{R})} x(t)y'(t) dt \\ &= -\int_{\gamma(\mathbb{R})} x'(t)y(t) dt. \end{aligned}$$

Exercises

Exercise 2.1. A **cycloid** is a curve in the plane parametrized by a map $\gamma : \mathbb{R} \to \mathbb{R}^2$ of the form

$$\gamma(t) = r(t, 1) - r(\sin(-t), \cos(-t)),$$

where $r \in \mathbb{R}^+$. Describe the curve geometrically and calculate the arclength

$$\sigma(2\pi) = \int_0^{2\pi} |\gamma'(t)| dt.$$

Is the curve regular ?

Exercise 2.2. An **astroid** is a curve in the plane parametrized by a map $\gamma : \mathbb{R} \to \mathbb{R}^2$ of the form

$$\gamma(t) = (4r\cos^3 t, 4r\sin^3 t) = 3r(\cos t, \sin t) + r(\cos(-3t), \sin(-3t)),$$

where $r \in \mathbb{R}^+$. Describe the curve geometrically and calculate the arclength

$$\sigma(2\pi) = \int_0^{2\pi} |\gamma'(t)| dt.$$

Is the curve regular ?

Exercise 2.3. Let the curves $\gamma_1, \gamma_2 : \mathbb{R} \to \mathbb{R}^2$ be given by

$$\gamma_1(t) = r(\cos(at), \sin(at)) \quad \text{and} \quad \gamma_2(t) = r(\cos(-at), \sin(-at)).$$

Calculate the curvatures κ_1, κ_2 of γ_1 and γ_2, respectively. Find a Euclidean motion $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ of $\mathbb{R}^2$ such that $\gamma_2 = \Phi \circ \gamma_1$. Is Φ orientation preserving ?

Exercise 2.4. Let $\gamma : I \to \mathbb{R}^2$ be a regular C^2-curve, parametrized by arclength, with Frenet frame $\{T(s), N(s)\}$. For $\lambda \in \mathbb{R}$ we define the parallel curve $\gamma_\lambda : I \to \mathbb{R}^2$ by

$$\gamma_\lambda(t) = \gamma(t) + \lambda N(t).$$

Calculate the curvature κ_λ of those curves γ_λ which are regular.

Exercise 2.5. Prove the curvature formula in Proposition 2.12.

Exercise 2.6. Let $\gamma : \mathbb{R} \to \mathbb{R}^2$ be the parametrized curve in $\mathbb{R}^2$ given by $\gamma(t) = (\sin t, \sin 2t)$. Is γ regular, closed and simple ?

Exercise 2.7. Let the positively oriented $\gamma : \mathbb{R} \to \mathbb{R}^2$ parametrize a simple closed C^2-curve by arclength. Show that if the period of γ is $L \in \mathbb{R}^+$ then the total curvature satisfies

$$\int_0^L \kappa(s)ds = 2\pi.$$

CHAPTER 3

Curves in the Euclidean Space $\mathbb{R}^3$

In this chapter we study regular curves in the three dimensional Euclidean space. We define their curvature and torsion and show that these determine the curves up to orientation preserving Euclidean motions.

Let the 3-dimensional real vector space $\mathbb{R}^3$ be equipped with its standard Euclidean **scalar product** $\langle \cdot, \cdot \rangle : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$. This is given by

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3$$

and induces the norm $| \cdot | : \mathbb{R}^3 \to \mathbb{R}_0^+$ on $\mathbb{R}^3$ with

$$|x| = \sqrt{x_1^2 + x_2^2 + x_3^2}.$$

Further we equip the three dimensional real vector space $\mathbb{R}^3$ with the standard **cross product** $\times : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ satisfying

$$(x_1, y_1, z_1) \times (x_2, y_2, z_2) = (y_1 z_2 - y_2 z_1, z_1 x_2 - z_2 x_1, x_1 y_2 - x_2 y_1).$$

Definition 3.1. A map $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ is said to be a **Euclidean motion** of $\mathbb{R}^3$ if it is given by $\Phi : x \mapsto Ax + b$ where $b \in \mathbb{R}^3$ and

$$A \in \mathbf{O}(3) = \{X \in \mathbb{R}^{3\times 3} |\ X^t X = I\}.$$

A Euclidean motion Φ is said to be **rigid** or **orientation preserving** if

$$A \in \mathbf{SO}(3) = \{X \in \mathbf{O}(3) |\ \det X = 1\}.$$

Example 3.2. If p and q are two distinct points in $\mathbb{R}^3$ then $\gamma : \mathbb{R} \to \mathbb{R}^3$ with

$$\gamma : t \mapsto (1 - t) \cdot p + t \cdot q$$

parametrizes the **straight line** through $p = \gamma(0)$ and $q = \gamma(1)$.

Example 3.3. Let $\{Z, W\}$ be an orthonormal basis for a 2-plane V in $\mathbb{R}^3$, $r \in \mathbb{R}^+$ and $p \in \mathbb{R}^3$. Then $\gamma : \mathbb{R} \to \mathbb{R}^3$ with

$$\gamma : t \mapsto p + r \cdot (\cos t \cdot Z + \sin t \cdot W)$$

parametrizes a **circle** in the affine 2-plane $p + V$ with **center** p and **radius** r.

Example 3.4. If $r, b \in \mathbb{R}^+$ then $\gamma : \mathbb{R} \to \mathbb{R}^3$ with

$$\gamma = (x, y, z) : t \mapsto (r \cdot \cos(t), r \cdot \sin(t), bt)$$

parametrizes a **helix**. It is easy to see that $x^2 + y^2 = r^2$ so the image $\gamma(\mathbb{R})$ is contained in the **circular cylinder**

$$\{(x, y, z) \in \mathbb{R}^3 |\ x^2 + y^2 = r^2\}$$

of radius r.

Definition 3.5. Let $\gamma : I \to \mathbb{R}^3$ be a regular C^2-curve parametrized by arclength. Then the **curvature** $\kappa : I \to \mathbb{R}_0^+$ of γ is defined by

$$\kappa(s) = |\ddot{\gamma}(s)|.$$

Theorem 3.6. *Let $\gamma : I \to \mathbb{R}^3$ be a regular C^2-curve parametrized by arclength. Then its curvature $\kappa : I \to \mathbb{R}_0^+$ vanishes identically if and only if the geometric curve $\gamma(I)$ is contained in a line.*

Proof. The curvature $\kappa(s) = |\ddot{\gamma}(s)|$ vanishes identically if and only if there exist a unit vector $Z \in S^2$ and a point $p \in \mathbb{R}^3$ such that

$$\gamma(s) = p + s \cdot Z$$

i.e. the geometric curve $\gamma(I)$ is contained in a straight line. $\square$

Definition 3.7. A regular C^3-curve $\gamma : I \to \mathbb{R}^3$, parametrized by arclength, is said to be a **Frenet curve** if its curvature κ is non-vanishing i.e. $\kappa(s) \neq 0$ for all $s \in I$.

For a Frenet curve $\gamma : I \to \mathbb{R}^3$ we define the **tangent** $T : I \to S^2$ along γ by

$$T(s) = \dot{\gamma}(s),$$

the **principal normal** $N : I \to S^2$ with

$$N(s) = \frac{\ddot{\gamma}(s)}{|\ddot{\gamma}(s)|} = \frac{\ddot{\gamma}(s)}{\kappa(s)}$$

and the **binormal** $B : I \to S^2$ as the cross product

$$B(s) = T(s) \times N(s).$$

The curve $\gamma : I \to \mathbb{R}^3$ is parametrized by arclength so

$$0 = \frac{d}{ds}\langle\dot{\gamma}(s), \dot{\gamma}(s)\rangle = 2\ \langle\ddot{\gamma}(s), \dot{\gamma}(s)\rangle.$$

This means that for each $s \in I$ the set $\{T(s), N(s), B(s)\}$ is an orthonormal basis for $\mathbb{R}^3$. It is called the **Frenet frame** along the curve.

Definition 3.8. Let $\gamma : I \to \mathbb{R}^3$ be a Frenet curve. Then we define the **torsion** $\tau : I \to \mathbb{R}$ of γ by

$$\tau(s) = \langle \dot{N}(s), B(s)\rangle.$$

Note that the torsion is a measure of how fast the principal normal $N(s) = \ddot{\gamma}(s)/|\ddot{\gamma}(s)|$ is bending in the direction of the binormal $B(s)$, or equivalently, out of the plane generated by $T(s)$ and $N(s)$.

Theorem 3.9. *Let $\gamma : I \to \mathbb{R}^3$ be a Frenet curve. Then the Frenet frame satisfies the following system of ordinary differential equations.*

$$\begin{bmatrix} \dot{T}(s) \\ \dot{N}(s) \\ \dot{B}(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix}.$$

Proof. The first equation is a direct consequence of the definition of the curvature

$$\dot{T}(s) = \ddot{\gamma}(s) = |\ddot{\gamma}(s)| \cdot N = \kappa(s) \cdot N(s).$$

We get the second equation from

$$\begin{aligned} \langle \dot{N}(s), T(s)\rangle &= \frac{d}{ds}\langle N(s), T(s)\rangle - \langle N(s), \dot{T}(s)\rangle \\ &= -\langle \frac{\ddot{\gamma}(s)}{|\ddot{\gamma}(s)|}, \ddot{\gamma}(s)\rangle \\ &= -\kappa(s), \end{aligned}$$

$$2\langle \dot{N}(s), N(s)\rangle = \frac{d}{ds}\langle N(s), N(s)\rangle = 0$$

and

$$\langle \dot{N}(s), B(s)\rangle = \frac{d}{ds}\langle N(s), B(s)\rangle - \langle N(s), \dot{B}(s)\rangle = \tau(s).$$

When differentiating $B(s) = T(s) \times N(s)$ we obtain

$$\begin{aligned} \dot{B}(s) &= \dot{T}(s) \times N(s) + T(s) \times \dot{N}(s) \\ &= \kappa(s) \cdot N(s) \times N(s) + T(s) \times \dot{N}(s) \\ &= T(s) \times \dot{N}(s), \end{aligned}$$

hence $\langle \dot{B}(s), T(s)\rangle = 0$. The definition of the torsion

$$\langle \dot{B}(s), N(s)\rangle = -\langle B(s), \dot{N}(s)\rangle = -\tau(s)$$

and the fact

$$2\ \langle \dot{B}(s), B(s)\rangle = \frac{d}{ds}\langle B(s), B(s)\rangle = 0$$

give us the third and last equation. $\square$

Theorem 3.10. *Let $\gamma : I \to \mathbb{R}^3$ be a Frenet curve. Then its torsion $\tau : I \to \mathbb{R}$ vanishes identically if and only if the geometric curve $\gamma(I)$ is contained in a plane.*

Proof. It follows from the third Frenet equation that if the torsion vanishes identically then

$$\frac{d}{ds}\langle \gamma(s) - \gamma(0), B(s)\rangle = \langle T(s), B(s)\rangle = 0.$$

Because $\langle \gamma(0) - \gamma(0), B(0)\rangle = 0$ it follows that $\langle \gamma(s) - \gamma(0), B(s)\rangle = 0$ for all $s \in I$. This means that $\gamma(s)$ lies in a plane containing $\gamma(0)$ with constant normal $B(s)$.

Let us now assume that the geometric curve $\gamma(I)$ is contained in a plane i.e. there exists a point $p \in \mathbb{R}^3$ and a normal $n \in \mathbb{R}^3 \setminus \{0\}$ to the plane such that

$$\langle \gamma(s) - p, n\rangle = 0$$

for all $s \in I$. When differentiating we get

$$\langle T(s), n\rangle = \langle \dot\gamma(s), n\rangle = 0$$

and

$$\langle \ddot\gamma(s), n\rangle = 0$$

so $\langle N(s), n\rangle = 0$. This means that $B(s)$ is a constant multiple of n, so $\dot B(s) = 0$ and hence $\tau \equiv 0$. □

The next result is called the **fundamental theorem of curve theory**. It tells us that a Frenet curve is, up to orientation preserving Euclidean motions, completely determined by its curvature and torsion.

Theorem 3.11. *Let $\kappa : I \to \mathbb{R}^+$ and $\tau : I \to \mathbb{R}$ be two continuous functions. Then there exists a Frenet curve $\gamma : I \to \mathbb{R}^3$ with curvature κ and torsion τ. If $\tilde\gamma : I \to \mathbb{R}^3$ is another such curve, then there exists a matrix $A \in \mathbf{SO}(3)$ and an element $b \in \mathbb{R}^3$ such that*

$$\tilde\gamma(s) = A \cdot \gamma(s) + b.$$

Proof. The proof is based on Theorem 3.9 and a well-known result of **Picard-Lindelöf** formulated here as Fact 3.12, see Exercise 3.6. □

Fact 3.12. *Let $f : U \to \mathbb{R}^n$ be a continuous map defined on an open subset U of $\mathbb{R} \times \mathbb{R}^n$ and $L \in \mathbb{R}^+$ such that*

$$|f(t, x) - f(t, y)| \leq L \cdot |x - y|$$

for all $(t, x), (t, y) \in U$. If $(t_0, x_0) \in U$ then there exists a unique local solution $x : I \to \mathbb{R}^n$ to the following initial value problem

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0.$$

In differential geometry we are interested in properties of geometric objects which are independent of how these objects are parametrized. The curvature and the torsion of a geometric curve should therefore not depend on its parametrization.

Definition 3.13. Let $\gamma : I \to \mathbb{R}^3$ be a regular C^2-curve in $\mathbb{R}^3$ not necessarily parametrized by arclength. Let $t : J \to I$ be a strictly increasing C^2-function such that the composition $\alpha = \gamma \circ t : J \to \mathbb{R}^3$ is a curve parametrized by arclength. Then we define the **curvature** $\kappa : I \to \mathbb{R}^+$ of $\gamma : I \to \mathbb{R}^3$ by

$$\kappa(t(s)) = \tilde{\kappa}(s),$$

where $\tilde{\kappa} : J \to \mathbb{R}^+$ is the curvature of α. If further $\gamma : I \to \mathbb{R}^3$ is a regular C^3-curve with non-vanishing curvature and $t : J \to I$ is C^3, then we define the **torsion** $\tau : I \to \mathbb{R}$ of γ by

$$\tau(t(s)) = \tilde{\tau}(s),$$

where $\tilde{\tau} : J \to \mathbb{R}$ is the torsion of α.

We are now interested in deriving formulae for the curvature κ and the torsion τ in terms of γ, under the above mentioned conditions.

Proposition 3.14. *If $\gamma : I \to \mathbb{R}^3$ be a regular C^2-curve in $\mathbb{R}^3$ then its curvature satisfies*

$$\kappa(t) = \frac{|\gamma'(t) \times \gamma''(t)|}{|\gamma'(t)|^3}.$$

Proof. By differentiating $\gamma(t) = \alpha(s(t))$ we get

$$\gamma'(t) = \dot{\alpha}(s(t)) \cdot s'(t),$$

$$\langle \gamma'(t), \gamma'(t) \rangle = s'(t)^2 \langle \dot{\alpha}(s(t)), \dot{\alpha}(s(t)) \rangle = s'(t)^2$$

and

$$2\langle \gamma''(t), \gamma'(t) \rangle = \frac{d}{dt}(s'(t)^2) = 2 \cdot s'(t) \cdot s''(t).$$

When differentiating once more we get

$$s'(t) \cdot \ddot{\alpha}(s(t)) = \frac{s'(t) \cdot \gamma''(t) - s''(t) \cdot \gamma'(t)}{s'(t)^2}$$

and

$$\begin{aligned} \ddot{\alpha}(s(t)) &= \frac{s'(t)^2 \cdot \gamma''(t) - s'(t) \cdot s''(t) \cdot \gamma'(t)}{s'(t)^4} \\ &= \frac{\gamma''(t)\langle \gamma'(t), \gamma'(t) \rangle - \gamma'(t)\langle \gamma''(t), \gamma'(t) \rangle}{|\gamma'(t)|^4} \end{aligned}$$

$$= \frac{\gamma'(t) \times (\gamma''(t) \times \gamma'(t))}{|\gamma'(t)|^4}.$$

Finally we get a formula for the curvature of $\gamma : I \to \mathbb{R}^3$ by

$$\begin{aligned}
\kappa(t) &= \tilde{\kappa}(s(t)) \\
&= |\ddot{\alpha}(s(t))| \\
&= \frac{|\gamma'(t)| \cdot |\gamma''(t) \times \gamma'(t)|}{|\gamma'(t)|^4} \\
&= \frac{|\gamma'(t) \times \gamma''(t)|}{|\gamma'(t)|^3}.
\end{aligned}$$

□

Corollary 3.15. *If $\gamma : I \to \mathbb{R}^3$ is a regular C^2-curve in $\mathbb{R}^3$ then the geometric curve $\gamma(I)$ is contained in a line if and only if $\gamma'(t)$ and $\gamma''(t)$ are linearly dependent for all $t \in I$.*

PROOF. The statement is a direct consequence of Theorem 3.6 and Proposition 3.14. □

Proposition 3.16. *Let $\gamma : I \to \mathbb{R}^3$ be a regular C^3-curve with non-vanishing curvature. Then its torsion τ satisfies*

$$\tau(t) = \frac{\det[\gamma'(t), \gamma''(t), \gamma'''(t)]}{|\gamma'(t) \times \gamma''(t))|^2}.$$

PROOF. See Exercise 3.5. □

Corollary 3.17. *Let $\gamma : I \to \mathbb{R}^3$ be a regular C^3-curve with non-vanishing curvature. Then the geometric curve $\gamma(I)$ is contained in a plane if and only if $\gamma'(t)$, $\gamma''(t)$ and $\gamma'''(t)$ are linearly dependent for all $t \in I$.*

PROOF. The statement is a direct consequence of Theorem 3.10 and Proposition 3.16. □

Exercises

Exercise 3.1. For $r, a, b \in \mathbb{R}^+$ parametrize the helixes $\gamma_1, \gamma_2 : \mathbb{R} \to \mathbb{R}^3$ by

$$\gamma_1 : t \mapsto (r \cdot \cos(at), r \cdot \sin(at), abt),$$
$$\gamma_2 : t \mapsto (r \cdot \cos(-at), r \cdot \sin(-at), abt).$$

Calculate their curvatures κ_1, κ_2 and torsions τ_1, τ_2, respectively. Find a Euclidean motion $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ of $\mathbb{R}^3$ such that $\gamma_2 = \Phi \circ \gamma_1$. Is Φ orientation preserving ?

Exercise 3.2. For any $\kappa \in \mathbb{R}^+$ and $\tau \in \mathbb{R}$ construct a regular curve $\gamma : \mathbb{R} \to \mathbb{R}^3$ with constant curvature κ and constant torsion τ.

Exercise 3.3. Prove that the curve $\gamma : (-\pi/2, \pi/2) \to \mathbb{R}^3$ with

$$\gamma : t \mapsto (2\cos^2 t - 3, \sin t - 8, 3\sin^2 t + 4)$$

is regular. Determine whether the image of γ is contained in

ii) a straight line in $\mathbb{R}^3$ or not,
i) a plane in $\mathbb{R}^3$ or not.

Exercise 3.4. Show that the curve $\gamma : \mathbb{R} \to \mathbb{R}^3$ given by

$$\gamma(t) = (t^3 + t^2 + 3, t^3 - t + 1, t^2 + t + 1)$$

is regular. Determine whether the image of γ is contained in

ii) a straight line in $\mathbb{R}^3$ or not,
i) a plane in $\mathbb{R}^3$ or not.

Exercise 3.5. Prove the torsion formula in Proposition 3.16.

Exercise 3.6. Use your local library to find a proof of Theorem 3.11.

Exercise 3.7. Let $\gamma : \mathbb{R} \to \mathbb{R}^3$ be a regular C^2-map parametrizing a closed curve in $\mathbb{R}^3$ by arclength. Use your local library to find a proof of Fenchel's theorem i.e.

$$L(\dot{\gamma}) = \int_0^P \kappa(s)ds \geq 2\pi,$$

where P is the period of γ.

CHAPTER 4

Surfaces in the Euclidean Space $\mathbb{R}^3$

In this chapter we introduce the notion of a regular surface in three dimensional Euclidean space. We give several examples and study differentiable maps between them. We define the tangent space at a point of a regular surface and show that this is a two dimensional vector space. Further we introduce the first fundamental form which enables us to measure angles between tangent vectors, lengths of curves and even distances between points on the surface.

Definition 4.1. A non-empty subset M of $\mathbb{R}^3$ is said to be a **regular surface** if for each point $p \in M$ there exist open, connected and simply connected neighbourhoods V in $\mathbb{R}^3$ and U in $\mathbb{R}^2$ and a bijective C^1-map $X : U \to V \cap M$, such that X is a homeomorphism and

$$X_u(q) \times X_v(q) \neq 0.$$

for all $q \in U$. The map $X : U \to V \cap M$ is said to be a **local parametrization** of M and the inverse $X^{-1} : V \cap M \to U$ a **local chart** or **local coordinates** on M. An **atlas** on M is a collection

$$\mathcal{A} = \{(V_\alpha \cap M, X_\alpha^{-1}) |\ \alpha \in I\}$$

of local charts on M such that $\mathcal{A}$ covers the whole of M i.e.

$$M = \bigcup_\alpha (V_\alpha \cap M).$$

Example 4.2. Let $f : U \to \mathbb{R}$ be a differentiable function from an open subset U of $\mathbb{R}^2$. Then $X : U \to M$ with

$$X : (u, v) \mapsto (u, v, f(u, v))$$

is a local parametrization of the graph

$$M = \{(u, v, f(u, v)) |\ (u, v) \in U\}$$

of f. The corresponding local chart $X^{-1} : M \to U$ is given by

$$X^{-1} : (x, y, z) \mapsto (x, y).$$

Example 4.3. Let S^2 denote the unit sphere in $\mathbb{R}^3$ given by

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 |\ x^2 + y^2 + z^2 = 1\}.$$

Let $N = (0, 0, 1)$ be the north pole and $S = (0, 0, -1)$ be the south pole of S^2, respectively. Put $U_N = S^2 \setminus \{N\}$, $U_S = S^2 \setminus \{S\}$ and define the stereographic projection from the north pole $x_N : U_N \to \mathbb{R}^2$ by

$$x_N : (x, y, z) \mapsto \frac{1}{1-z}(x, y)$$

and the stereographic projection from the south pole $x_S : U_S \to \mathbb{R}^2$

$$x_S : (x, y, z) \mapsto \frac{1}{1+z}(x, y).$$

Then $\mathcal{A} = \{(U_N, x_N), (U_S, x_S)\}$ is an atlas on S^2. The inverses

$$F_N = x_N^{-1} : \mathbb{R}^2 \to U_N \quad \text{and} \quad F_S = x_S^{-1} : \mathbb{R}^2 \to U_S$$

are local parametrizations of the sphere S^2 given by

$$F_N : (u, v) \mapsto \frac{1}{1+u^2+v^2}(2u, 2v, u^2 + v^2 - 1),$$

$$F_S : (u, v) \mapsto \frac{1}{1+u^2+v^2}(2u, 2v, 1 - u^2 - v^2).$$

Our next step is to prove the implicit function theorem which is a useful tool for constructing surfaces in $\mathbb{R}^3$. For this we use the classical inverse mapping theorem stated below. Remember that if $F : U \to \mathbb{R}^m$ is a differentiable map defined on an open subset U of $\mathbb{R}^n$ then its differential

$$dF(p) : \mathbb{R}^n \to \mathbb{R}^m$$

at a point $p \in U$ is a linear map given by the $m \times n$ matrix

$$dF(p) = \begin{bmatrix} \partial F_1/\partial x_1(p) & \dots & \partial F_1/\partial x_n(p) \\ \vdots & & \vdots \\ \partial F_m/\partial x_1(p) & \dots & \partial F_m/\partial x_n(p) \end{bmatrix}.$$

The **classical inverse mapping theorem** can be formulated as follows.

Theorem 4.4. *Let U be an open subset of $\mathbb{R}^n$ and $F : U \to \mathbb{R}^n$ be a differentiable map. If $p \in U$ and the differential*

$$dF(p) : \mathbb{R}^n \to \mathbb{R}^n$$

of F at p is invertible then there exist open neighbourhoods U_p around p and U_q around $q = F(p)$ such that $f = F|_{U_p} : U_p \to U_q$ is bijective *and the inverse $f^{-1} : U_q \to U_p$ is a differentiable map. The differential $df^{-1}(q)$ of f^{-1} at q satisfies*

$$df^{-1}(q) = (dF(p))^{-1}$$

i.e. it is the inverse of the differential $dF(p)$ of F at p.

Before stating the implicit function theorem we remind the reader of the following notions.

Definition 4.5. Let m, n be positive integers, U be an open subset of $\mathbb{R}^n$ and $F : U \to \mathbb{R}^m$ be a differentiable map. A point $p \in U$ is said to be **critical** for F if the differential

$$dF(p) : \mathbb{R}^n \to \mathbb{R}^m$$

is not of full rank, and **regular** if it is not critical. A point $q \in F(U)$ is said to be a **regular value** of F if every point of the pre-image $F^{-1}(\{q\})$ of q is regular and a **critical value** otherwise.

Note that if $m \leq n$ then $p \in U$ is a regular point of

$$F = (F_1, \dots, F_m) : U \to \mathbb{R}^m$$

if and only if the gradients $\nabla F_1, \dots, \nabla F_m$ of the coordinate functions $F_1, \dots, F_m : U \to \mathbb{R}$ are linearly independent at p, or equivalently, the differential $dF(p)$ of F at p satisfies the following condition

$$\det(dF(p) \cdot dF(p)^t) \neq 0.$$

In differential geometry, the following important result is often called the **implicit function theorem**.

Theorem 4.6. *Let $f : U \to \mathbb{R}$ be a differentiable function defined on an open subset U of $\mathbb{R}^3$ and q be a regular value of f i.e.*

$$(\nabla f)(p) \neq 0$$

for all p in $M = f^{-1}(\{q\})$. Then M is a regular surface in $\mathbb{R}^3$.

Proof. Let p be an arbitrary element of M. The gradient $\nabla f(p)$ at p is non-zero so we can, without loss of generality, assume that $f_z(p) \neq 0$. Then define the map $F : U \to \mathbb{R}^3$ by

$$F(x, y, z) \mapsto (x, y, f(x, y, z)).$$

Its differential $dF(p)$ at p satisfies

$$dF(p) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ f_x & f_y & f_z \end{bmatrix},$$

so the determinant $\det dF(p) = f_z$ is non-zero. Following the inverse mapping theorem there exist open neighbourhoods V around p and W around $F(p)$ such that the restriction $F|_V : V \to W$ of F to V is invertible. The inverse $(F|_V)^{-1} : W \to V$ is differentiable of the form

$$(u, v, t) \mapsto (u, v, g(u, v, t)),$$

where g is a real-valued function on W. It follows that the restriction

$$X = F^{-1}|_{\hat{W}} : \hat{W} \to \mathbb{R}^3$$

to the planar set

$$\hat{W} = \{(u, v, t) \in W | \ t = q\}$$

is differentiable, so $X : \hat{W} \to V \cap M$ is a local parametrization of the open neighbourhood $V \cap M$ around p. Since p was chosen arbitrarily we have shown that M is a regular surface in $\mathbb{R}^3$. □

We shall now apply the implicit function theorem to construct examples of regular surfaces in $\mathbb{R}^3$.

Example 4.7. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be the differentiable function given by

$$f(x, y, z) = x^2 + y^2 + z^2.$$

The gradient $\nabla f(p)$ of f at p satisfies $\nabla f(p) = 2p$, so each positive real number is a regular value for f. This means that the **sphere**

$$S^2_r = \{(x, y, z) \in \mathbb{R}^3 | \ x^2 + y^2 + z^2 = r^2\} = f^{-1}(\{r^2\})$$

of radius r is a regular surface in $\mathbb{R}^3$.

Example 4.8. Let r, R be real numbers such that $0 < r < R$ and define the differentiable function

$$f : U = \{(x, y, z) \in \mathbb{R}^3 | \ x^2 + y^2 \neq 0\} \to \mathbb{R}$$

by

$$f(x, y, z) = z^2 + (\sqrt{x^2 + y^2} - R)^2$$

and let T^2 be the pre-image

$$f^{-1}(\{r^2\}) = \{(x, y, z) \in U | \ z^2 + (\sqrt{x^2 + y^2} - R)^2 = r^2\}.$$

The gradient ∇f of f at $p = (x, y, z)$ satisfies

$$\nabla f(p) = \frac{2}{\sqrt{x^2 + y^2}}(x(\sqrt{x^2 + y^2} - R), y(\sqrt{x^2 + y^2} - R), z\sqrt{x^2 + y^2}).$$

If $p \in T^2$ and $\nabla f(p) = 0$ then $z = 0$ and

$$\nabla f(p) = \frac{2r}{\sqrt{x^2 + y^2}}(x, y, 0) \neq 0.$$

This contradiction shows that r^2 is a regular value for f and that the **torus** T^2 is a regular surface in $\mathbb{R}^3$.

We now introduce the useful notion of a regular parametrized surface in $\mathbb{R}^3$. This is a map and should not be confused with a regular surface as a subset of $\mathbb{R}^3$, introduced in Definition 4.1.

Definition 4.9. A differentiable map $X : U \to \mathbb{R}^3$ from an open subset U of $\mathbb{R}^2$ is said to be a **regular parametrized surface** in $\mathbb{R}^3$ if for each point $q \in U$

$$X_u(q) \times X_v(q) \neq 0.$$

Definition 4.10. Let M be a regular surface in $\mathbb{R}^3$. A differentiable map $X : U \to M$ defined on an open subset of $\mathbb{R}^2$ is said to **parametrize** M if X is surjective and for each p in U there exists an open neighbourhood U_p of p such that $X|_{U_p} : U_p \to X(U_p)$ is a local parametrization of M.

Example 4.11. We know that for two real numbers r, R satisfying $0 < r < R$ the torus

$$T^2 = \{(x, y, z) \in \mathbb{R}^2 |\ z^2 + (\sqrt{x^2 + y^2} - R)^2 = r^2\}$$

is a regular surface in $\mathbb{R}^2$. It is easily seen that T^2 is obtained by rotating the circle

$$\{(x, 0, z) \in \mathbb{R}^3 |\ z^2 + (x - R)^2 = r^2\}$$

in the (x, z)-plane around the z-axes. This rotation naturally induces the regular parametrized surface $X : \mathbb{R}^2 \to T^2$ with

$$X : (u, v) \mapsto \begin{pmatrix} \cos v & -\sin v & 0 \\ \sin v & \cos v & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} R + r \cos u \\ 0 \\ r \sin u \end{pmatrix}.$$

This parametrizes the regular surface T^2 as a subset of $\mathbb{R}^2$.

Example 4.12. Let $\gamma = (r, 0, z) : I \to \mathbb{R}^3$ be differentiable curve in the (x, z)-plane such that $r(s) > 0$ and $\dot{r}(s)^2 + \dot{z}(s)^2 = 1$ for all $s \in I$. By rotating the curve around the z-axes we obtain a regular **surface of revolution** parametrized by $X : I \times \mathbb{R} \to \mathbb{R}^3$ with

$$X(u, v) = \begin{pmatrix} \cos v & -\sin v & 0 \\ \sin v & \cos v & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} r(u) \\ 0 \\ z(u) \end{pmatrix} = \begin{pmatrix} r(u) \cos v \\ r(u) \sin v \\ z(u) \end{pmatrix}.$$

The surface is regular because the vectors

$$X_u = \begin{pmatrix} \dot{r}(u)\cos v \\ \dot{r}(u)\sin v \\ \dot{z}(u) \end{pmatrix}, \quad X_v = \begin{pmatrix} -r(u)\sin v \\ r(u)\cos v \\ 0 \end{pmatrix}$$

are not only linearly independent but even orthogonal.

We now introduce the tangent space at a point of a regular surface and show that this is a two dimensional vector space.

Definition 4.13. Let M be a regular surface in $\mathbb{R}^3$. A continuous map $\gamma : I \to M$, defined on an open interval I of the real line, is said to be a **differentiable curve** on M if it is differentiable as a map into $\mathbb{R}^3$.

Definition 4.14. Let M be a regular surface in $\mathbb{R}^3$ and p be a point on M. Then the **tangent space** T_pM of M at p is the set of all tangents $\dot{\gamma}(0)$ to C^1-curves $\gamma : I \to M$ such that $\gamma(0) = p$.

Proposition 4.15. *Let M be a regular surface in $\mathbb{R}^3$ and p be a point on M. Then the tangent space T_pM of M at p is a 2-dimensional real vector space.*

Proof. Let M be a regular surface in $\mathbb{R}^3$, $p \in M$ and $X : U \to M$ be a local parametrization of M such that $0 \in U$ and $X(0) = p$. Let $\alpha : I \to U$ be a C^1-curve in U such that $0 \in I$ and $\alpha(0) = 0 \in U$. Then the composition $\gamma = X \circ \alpha : I \to X(U)$ is a C^1-curve in $X(U)$ such that $\gamma(0) = p$. Since $X : U \to X(U)$ is a homeomorphism it is clear that any curve in $X(U)$ with $\gamma(0) = p$ can be obtained this way.

It follows from the chain rule that the tangent $\dot{\gamma}(0)$ of $\gamma : I \to M$ at p satisfies

$$\dot{\gamma}(0) = dX(0) \cdot \dot{\alpha}(0),$$

where $dX(0) : \mathbb{R}^2 \to \mathbb{R}^3$ is the differential of the local parametrization $X : U \to M$. The differential is a linear map and the condition

$$X_u \times X_v \neq 0$$

implies that $dX(0)$ is of full rank i.e. the vectors

$$X_u = dX(0) \cdot e_1 \quad \text{and} \quad X_v = dX(0) \cdot e_2$$

are linearly independent. This shows that the image

$$\{dX(0) \cdot Z |\ Z \in \mathbb{R}^2\}$$

of $dX(0)$ is a two dimensional subspace of $\mathbb{R}^3$. If $(a, b) \in \mathbb{R}^2$ then

$$\begin{aligned} dX(0) \cdot (a, b) &= dX(0) \cdot (ae_1 + be_2) \\ &= a\ dX(0) \cdot e_1 + b\ dX(0) \cdot e_2 \end{aligned}$$

$$= \quad aX_u + bX_v.$$

It is clear that T_pM is the space of all tangents $\dot{\gamma}(0)$ to C^1-curves $\gamma : I \to M$ in M such that $\gamma(0) = p$. $\square$

Example 4.16. Let $\gamma : I \to S^2$ be a curve into the unit sphere in $\mathbb{R}^3$ with $\gamma(0) = p$ and $\dot{\gamma}(0) = Z$. The curve satisfies

$$\langle \gamma(t), \gamma(t) \rangle = 1$$

and differentiation yields

$$\langle \dot{\gamma}(t), \gamma(t) \rangle + \langle \gamma(t), \dot{\gamma}(t) \rangle = 0.$$

This means that $\langle Z, p \rangle = 0$ so every tangent vector $Z \in T_pS^m$ must be orthogonal to p. On the other hand if $Z \neq 0$ satisfies $\langle Z, p \rangle = 0$ then $\gamma : \mathbb{R} \to S^2$ with

$$\gamma : t \mapsto \cos(t|Z|) \cdot p + \sin(t|Z|) \cdot Z/|Z|$$

is a curve into S^2 with $\gamma(0) = p$ and $\dot{\gamma}(0) = Z$. This shows that the tangent space T_pS^2 is given by

$$T_pS^2 = \{Z \in \mathbb{R}^3 |\ \langle p, Z \rangle = 0\}.$$

Example 4.17. Let us parametrize the torus

$$T^2 = \{(x, y, z) \in U |\ z^2 + (\sqrt{x^2 + y^2} - R)^2 = r^2\}$$

by $X : \mathbb{R}^2 \to T^2$ with

$$X : (u, v) \mapsto \begin{pmatrix} \cos v & -\sin v & 0 \\ \sin v & \cos v & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} R + r\cos u \\ 0 \\ r \sin u \end{pmatrix}.$$

By differentiating we get a basis $\{X_v, X_u\}$ for the tangent space T_pT^2 at $p = X(u, v)$ with

$$X_u = r \begin{pmatrix} -\sin u \cos v \\ -\sin u \sin v \\ \cos u \end{pmatrix}, \quad X_v = (R + r\cos u) \begin{pmatrix} -\sin v \\ \cos v \\ 0 \end{pmatrix}.$$

Definition 4.18. Let M be a regular surface in $\mathbb{R}^3$. A real valued **function** $f : M \to \mathbb{R}$ on M is said to be **differentiable** if for each local parametrization $X : U \to M$ of M the composition $f \circ X : U \to \mathbb{R}$ is differentiable.

Example 4.19. Let $f : T^2 \to \mathbb{R}$ be the real-valued function on the torus $T^2 = \{(x, y, z) \in \mathbb{R}^2 |\ z^2 + (\sqrt{x^2 + y^2} - R)^2 = r^2\}$ given by

$$f : (x, y, z) \mapsto x.$$

For the natural parametrization $X : \mathbb{R}^2 \to T^2$ with

$$X : (u, v) \mapsto \begin{pmatrix} \cos v & -\sin v & 0 \\ \sin v & \cos v & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} R + r\cos u \\ 0 \\ r \sin u \end{pmatrix}.$$

of T^2 we see that $f \circ X : \mathbb{R}^2 \to \mathbb{R}$ is given by

$$f \circ X : (u, v) \mapsto (R + r\cos u)\cos v.$$

This is clearly differentiable.

Definition 4.20. A **map** $\phi : M_1 \to M_2$ between two regular surfaces in $\mathbb{R}^3$ is said to be **differentiable** if for all local parametrizations (U_1, X_1) on M_1 and (U_2, X_2) on M_2 the map

$$X_2^{-1} \circ \phi \circ X_1|_U : U \to \mathbb{R}^2,$$

defined on the open subset $U = X_1^{-1}(X_1(U_1) \cap \phi^{-1}(X_2(U_2)))$ of $\mathbb{R}^2$, is differentiable.

The next very useful proposition generalizes a result from classical real analysis of several variables.

Proposition 4.21. *Let M_1 and M_2 be two regular surfaces in $\mathbb{R}^3$. Let $\phi : U \to \mathbb{R}^3$ be a differentiable map defined on an open subset of $\mathbb{R}^3$ such that M_1 is contained in U and the image $\phi(M_1)$ is contained in M_2. Then the restriction $\phi|_{M_1} : M_1 \to M_2$ is differentiable map from M_1 to M_2.*

Proof. See Exercise 4.2. □

Example 4.22. We have earlier parametrized the torus

$$T^2 = \{(x, y, z) \in U |\ z^2 + (\sqrt{x^2 + y^2} - R)^2 = r^2\}$$

with the map $X : \mathbb{R}^2 \to T^2$ defined by

$$X : (u, v) \mapsto \begin{pmatrix} \cos v & -\sin v & 0 \\ \sin v & \cos v & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} R + r\cos u \\ 0 \\ r \sin u \end{pmatrix}.$$

We can now map the torus T^2 into $\mathbb{R}^3$ with the following formula

$$N : \begin{pmatrix} \cos v & -\sin v & 0 \\ \sin v & \cos v & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} R + r\cos u \\ 0 \\ r \sin u \end{pmatrix} \mapsto \begin{pmatrix} \cos u \cos v \\ \cos u \sin v \\ \sin u \end{pmatrix}.$$

It is easy to see that this gives a well-defined map $N : T^2 \to S^2$ from the torus to the unit sphere

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 |\ x^2 + y^2 + z^2 = 1\}.$$

In the local coordinates (u, v) on the torus the map N is given by

$$N(u, v) = \begin{pmatrix} \cos u \cos v \\ \cos u \sin v \\ \sin u \end{pmatrix}.$$

It now follows from Proposition 4.21 that $N : T^2 \to S^2$ is differentiable.

Proposition 4.23. *Let $\phi_1 : M_1 \to M_2$ and $\phi_2 : M_2 \to M_3$ be differentiable maps between regular surfaces in $\mathbb{R}^3$. Then the composition $\phi_2 \circ \phi_1 : M_1 \to M_3$ is differentiable.*

PROOF. See Exercise 4.4. □

Definition 4.24. Two regular surfaces M_1 and M_2 in $\mathbb{R}^3$ are said to be **diffeomorphic** if there exists a bijective differentiable map $\phi : M_1 \to M_2$ such that the inverse $\phi^{-1} : M_2 \to M_1$ is differentiable. In that case the map ϕ is said to be a **diffeomorphism** between M_1 and M_2.

Remember that if $F : U \to \mathbb{R}^m$ is a differentiable map defined on an open subset U of $\mathbb{R}^n$ then its differential

$$dF(p) : \mathbb{R}^n \to \mathbb{R}^m$$

at a point $p \in U$ is a linear map given by the $m \times n$ matrix

$$dF(p) = \begin{bmatrix} \partial F_1/\partial x_1(p) & \dots & \partial F_1/\partial x_n(p) \\ \vdots & & \vdots \\ \partial F_m/\partial x_1(p) & \dots & \partial F_m/\partial x_n(p) \end{bmatrix}.$$

If $\gamma : \mathbb{R} \to U$ is a curve in U such that $\gamma(0) = p$ and $\dot\gamma(0) = Z$ then the composition $F \circ \gamma : \mathbb{R} \to \mathbb{R}^m$ is a curve in $\mathbb{R}^m$ and according to the chain rule we have

$$dF(p) \cdot Z = \frac{d}{dt}(F \circ \gamma(t))|_{t=0},$$

which is the tangent vector of the curve $F \circ \gamma$ at the image point $F(p) \in \mathbb{R}^m$. This shows that the **differential** $dF(p)$ of F at p is the linear map given by the formula

$$dF(p) : \dot\gamma(0) = Z \mapsto dF(p) \cdot Z = \frac{d}{dt}(F \circ \gamma(t))|_{t=0}$$

mapping the tangent vectors at $p \in U$ to tangent vectors at the image point $F(p) \in \mathbb{R}^m$. This formula will now be generalized to the surface setting.

Proposition 4.25. *Let M_1 and M_2 be two regular surfaces in $\mathbb{R}^3$, $p \in M_1$, $q \in M_2$ and $\phi : M_1 \to M_2$ be a differentiable map with $\phi(p) = q$. Then the formula*

$$d\phi_p : \dot\gamma(0) \mapsto \frac{d}{dt}\big(\phi \circ \gamma(t)\big)_{|t=0}$$

determines a well-defined linear map $d\phi_p : T_pM_1 \to T_qM_2$. Here $\gamma : I \to M_1$ is any C^1-curve satisfying $\gamma(0) = p$,

Proof. Let $X : U \to M_1$ and $Y : V \to M_2$ be local parametrizations such that $X(0) = p$, $Y(0) = q$ and $\phi(X(U))$ contained in $Y(V)$. Then define

$$F = Y^{-1} \circ \phi \circ X : U \to \mathbb{R}^2$$

and let $\alpha : I \to U$ be a C^1-curve with $\alpha(0) = 0$ and $\dot\alpha(0) = (a, b) \in \mathbb{R}^2$. If

$$\gamma = X \circ \alpha : I \to X(U)$$

then $\gamma(0) = p$ and

$$\dot\gamma(0) = dX(0) \cdot (a, b) = aX_u + bX_v.$$

The image curve $\phi \circ \gamma : I \to Y(V)$ is given by $\phi \circ \gamma = Y \circ F \circ \alpha$ so the chain rule implies that

$$\begin{aligned}\frac{d}{dt}\big(\phi \circ \gamma(t)\big)_{|t=0} &= dY(F(0)) \cdot dF(0) \cdot \dot\alpha(0)\\ &= dY(0) \cdot \frac{d}{dt}\big(F \circ \alpha(t)\big)_{|t=0}.\end{aligned}$$

This means that $d\phi_p : T_pM_1 \to T_qM_2$ is given by

$$d\phi_p : (aX_u + bX_v) \mapsto dY(F(0)) \cdot dF(0) \cdot (a, b)$$

and hence clearly linear. □

Definition 4.26. Let M_1 and M_2 be two regular surfaces in $\mathbb{R}^3$, $p \in M_1$, $q \in M_2$ and $\phi : M_1 \to M_2$ be a differentiable map such that $\phi(p) = q$. The map $d\phi_p : T_pM_1 \to T_qM_2$ is called the **differential** or the **tangent map** of ϕ at p.

The classical **inverse mapping theorem** generalizes as follows.

Theorem 4.27. *Let $\phi : M_1 \to M_2$ be a differentiable map between surfaces in $\mathbb{R}^3$. If p is a point in M, $q = \phi(p)$ and the differential*

$$d\phi_p : T_pM_1 \to T_qM_2$$

is bijective then there exist open neighborhoods U_p around p and U_q around q such that $\phi|_{U_p} : U_p \to U_q$ is bijective and the inverse $(\phi|_{U_p})^{-1} : U_q \to U_p$ is differentiable.

PROOF. See Exercise 4.7 □

We now introduce the first fundamental form of a regular surface. This enables us to measure angles between tangent vectors, lengths of curves and even distances between points on the surface.

Definition 4.28. Let M be a regular surface in $\mathbb{R}^3$ and $p \in M$. Then we define the **first fundamental form** $I_p : T_pM \times T_pM \to \mathbb{R}$ of M at p by

$$I_p(Z, W) = \langle Z, W \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean scalar product in $\mathbb{R}^3$ restricted to the tangent space T_pM of M at p. Properties of the surface which only depend on its first fundamental form are called **inner properties**.

Definition 4.29. Let M be a regular surface in $\mathbb{R}^3$ and $\gamma : I \to M$ be a C^1-curve on M. Then the **length** $L(\gamma)$ of γ is defined by

$$L(\gamma) = \int_I \sqrt{\langle \dot\gamma(t), \dot\gamma(t) \rangle}\, dt.$$

As we shall now see a regular surface in $\mathbb{R}^3$ has a natural distance function d. This gives (M, d) the structure of a **metric space**.

Proposition 4.30. *Let M be a regular surface in $\mathbb{R}^3$. For two points $p, q \in M$ let C_{pq} denote the set of C^1-curves $\gamma : [0, 1] \to M$ such that $\gamma(0) = p$ and $\gamma(1) = q$ and define the function $d : M \times M \to \mathbb{R}_0^+$ by*

$$d(p, q) = \inf\{L(\gamma) |\ \gamma \in C_{pq}\}.$$

Then (M, d) is a metric space i.e. for all $p, q, r \in M$ we have

(i) $d(p, q) \geq 0$,
(ii) $d(p, q) = 0$ *if and only if* $p = q$,
(iii) $d(p, q) = d(q, p)$,
(iv) $d(p, q) \leq d(p, r) + d(r, q)$.

PROOF. See for example: Peter Petersen, *Riemannian Geometry*, Graduate Texts in Mathematics **171**, Springer (1998). □

Definition 4.31. A differentiable map $\phi : M_1 \to M_2$ between two regular surfaces in $\mathbb{R}^3$ is said to be **isometric** if for each $p \in M_1$ the differential $d\phi(p) : T_pM_1 \to T_{\phi(p)}M_2$ preserves the first fundamental forms of the surfaces involved i.e.

$$\langle d\phi(p) \cdot Z, d\phi(p) \cdot W \rangle = \langle Z, W \rangle$$

for all $Z, W \in T_pM_1$. An isometric diffeomorphism $\phi : M_1 \to M_2$ is called an **isometry**. Two regular surfaces M_1 and M_2 are said to be **isometric** if there exists an isometry $\phi : M_1 \to M_2$ between them.

Definition 4.32. A differentiable map $\phi : M_1 \to M_2$ between two regular surfaces in $\mathbb{R}^3$ is said to be **conformal** if there exists a differentiable function $\lambda : M_1 \to \mathbb{R}$ such that for each $p \in M$ the differential $d\phi(p) : T_pM \to T_{\phi(p)}M$ satisfies

$$\langle d\phi(p) \cdot Z, d\phi(p) \cdot W\rangle = e^{2\lambda}\langle Z, W\rangle$$

for all $Z, W \in T_pM$. Two regular surfaces M_1 and M_2 are said to be **conformally equivalent** if there exists a conformal diffeomorphism $\phi : M_1 \to M_2$ between them.

Let M be a regular surface in $\mathbb{R}^3$ and $X : U \to M$ be a local parametrization of M. At each point $X(u, v)$ in $X(U)$ the tangent space is generated by the vectors $X_u(u, v)$ and $X_v(u, v)$. For these we define the matrix-valued map $[DX] : U \to \mathbb{R}^{2\times 3}$ by

$$[DX] = [X_u, X_v]^t$$

and the real-valued functions $E, F, G : U \to \mathbb{R}$ by the symmetric matrix

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = [DX] \cdot [DX]^t$$

containing the scalar products

$$E = \langle X_u, X_u\rangle, \quad F = \langle X_u, X_v\rangle = \langle X_v, X_u\rangle \ \text{ and } \ G = \langle X_v, X_v\rangle.$$

This induces a so called **metric**

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2$$

in the parameter region U as follows: For each point $q \in U$ we have a scalar product $ds^2_q : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by

$$ds^2_q(z, w) = z^t \begin{pmatrix} E(q) & F(q) \\ F(q) & G(q) \end{pmatrix} w.$$

The following shows that the diffeomorphism X preserves the scalar products so it is actually an isometry.

Let $\alpha_1 = (u_1, v_1) : I \to U$ and $\alpha_2 = (u_2, v_2) : I \to U$ be two curves in U meeting at $\alpha_1(0) = q = \alpha_2(0)$. Further let $\gamma_1 = X \circ \alpha_1$ and $\gamma_2 = X \circ \alpha_2$ be the image curves in $X(U)$ meeting at $\gamma_1(0) = p = \gamma_2(0)$. Then the differential $dX(q)$ is given by

$$dX(q) : (a, b) = (a \cdot e_1 + b \cdot e_2) \mapsto aX_u(q) + bX_v(q)$$

so at q we have

$$\begin{aligned} ds^2_q(\dot\alpha_1, \dot\alpha_2) &= \dot\alpha_1^t \begin{pmatrix} E & F \\ F & G \end{pmatrix} \dot\alpha_2 \\ &= E\dot u_1 \dot u_2 + F(\dot u_1 \dot v_2 + \dot u_2 \dot v_1) + G\dot v_1 \dot v_2 \end{aligned}$$

$$\begin{aligned} &= \langle \dot{u}_1 X_u + \dot{v}_1 X_v, \dot{u}_2 X_u + \dot{v}_2 X_v \rangle \\ &= \langle dX \cdot \dot{\alpha}_1, dX \cdot \dot{\alpha}_2 \rangle \\ &= \langle \dot{\gamma}_1, \dot{\gamma}_2 \rangle. \end{aligned}$$

It now follows that the length of a curve $\alpha : I \to U$ in U is exactly the same as the length of the corresponding curve $X \circ \alpha$ in $X(U)$. We have "pulled back" the first fundamental form on the surface $X(U)$ to a metric on U.

Deep Result 4.33. *Every regular surface M in $\mathbb{R}^3$ can locally be parametrized by* **isothermal coordinates** *i.e. for each point $p \in M$ there exists a local parametrization $X : U \to M$ such that $p \in X(U)$*

$$E(u, v) = G(u, v) \quad \textit{and} \quad F(u, v) = 0$$

for all $(u, v) \in U$.

Proof. A complete twelve page proof can be found in the standard text: M. Spivak, *A Comprehensive Introduction to Differential Geometry*, Publish or Perish (1979). □

Definition 4.34. Let M be a regular surface in $\mathbb{R}^3$ and $X : U \to M$ be a local parametrization of M where U is a measurable subset of the plane $\mathbb{R}^2$. Then we define the **area** of $X(U)$ by

$$A(X(U)) = \int_U \sqrt{EG - F^2} dudv.$$

Exercises

Exercise 4.1. Determine whether the following subsets of $\mathbb{R}^3$ are regular surfaces or not.

$$\begin{aligned}
M_1 &= \{(x,y,z) \in \mathbb{R}^3 |\ x^2 + y^2 = z\},\\
M_2 &= \{(x,y,z) \in \mathbb{R}^3 |\ x^2 + y^2 = z^2\},\\
M_3 &= \{(x,y,z) \in \mathbb{R}^3 |\ x^2 + y^2 - z^2 = 1\},\\
M_4 &= \{(x,y,z) \in \mathbb{R}^3 |\ x \sin z = y \cos z\}.
\end{aligned}$$

Find a parametrization for those which are regular surfaces in $\mathbb{R}^3$.

Exercise 4.2. Prove Proposition 4.21.

Exercise 4.3. Prove that the map $\phi : T^2 \to S^2$ in Example 4.22 is differentiable.

Exercise 4.4. Prove Proposition 4.23.

Exercise 4.5. Construct a diffeomorphism $\phi : S^2 \to M$ between the unit sphere S^2 and the ellipsoid

$$M = \{(x,y,z) \in \mathbb{R}^3 |\ x^2 + 2y^2 + 3z^2 = 1\}.$$

Exercise 4.6. Let $U = \{(u,v) \in \mathbb{R}^2 |\ -\pi < u < \pi,\ 0 < v < 1\}$, define $X : U \to \mathbb{R}^3$ by $X(u,v) = (\sin u, \sin 2u, v)$ and set $M = X(U)$ Sketch M and show that X is differentiable, regular and injective but X^{-1} is not continuous. Is M a regular surface in $\mathbb{R}^3$?

Exercise 4.7. Find a proof for Theorem 4.27

Exercise 4.8. For $\alpha \in (0, \pi/2)$ define the parametrized surface M_α by $X_\alpha : \mathbb{R}^+ \times \mathbb{R} \to M$ by

$$X_\alpha(r,\theta) = (r \sin\alpha \cos(\frac{\theta}{\sin\alpha}), r\sin\alpha\sin(\frac{\theta}{\sin\alpha}), r\cos\alpha).$$

Calculate the first fundamental form of M_α and find an equation of the form $f_\alpha(x,y,z) = 0$ describing the surface.

Exercise 4.9. Find an isometric parametrization $X : \mathbb{R}^2 \to M$ of the circular cylinder

$$M = \{(x,y,z) \in \mathbb{R}^3 |\ x^2 + y^2 = 1\}.$$

Exercise 4.10. Let M be the unit sphere S^2 with the two poles removed. Prove that Mercator's parametrization $X : \mathbb{R}^2 \to M$ of M with

$$X(u,v) = (\frac{\cos v}{\cosh u}, \frac{\sin v}{\cosh u}, \frac{\sinh u}{\cosh u})$$

is conformal.

Exercise 4.11. Prove that the first fundamental form of a regular surface M in $\mathbb{R}^3$ is invariant under Euclidean motions.

Exercise 4.12. Let $X, Y : \mathbb{R}^2 \to \mathbb{R}^3$ be the regular parametrized surfaces given by

$$X(u, v) = (\cosh u \cos v, \cosh u \sin v, u),$$
$$Y(u, v) = (\sinh u \cos v, \sinh u \sin v, v).$$

Calculate the first fundamental forms of X and Y. Find equations of the form $f(x, y, z) = 0$ describing the surfaces X and Y. Compare with Exercise 4.1.

Exercise 4.13. Calculate the area $A(T^2)$ of the torus

$$T^2 = \{(x, y, z) \in U | \ z^2 + (\sqrt{x^2 + y^2} - R)^2 = r^2\}.$$

CHAPTER 5

Curvature

In this chapter we define the shape operator of an oriented surface and its second fundamental form. These measure the behaviour of the normal of the surface and lead us to the notions of normal curvature, Gaussian curvature and mean curvature.

Definition 5.1. Let M be a regular surface in $\mathbb{R}^3$. A differentiable map $N : M \to S^2$ with values in the unit sphere is said to be a **Gauss map** for M if for each point $p \in M$ the image $N(p)$ is perpendicular to the tangent plane T_pM. The surface M is said to be **orientable** if such a Gauss map exists. A surface M equipped with a Gauss map is said to be **oriented**.

Let M be an oriented regular surface in $\mathbb{R}^3$ with Gauss map $N : M \to S^2$. Let $p \in M$ and $\gamma : I \to M$ be a regular curve parametrized by arclength such that $\gamma(0) = p$ and $\dot\gamma(0) = Z \in T_pM$. Then the composition $N \circ \gamma : I \to S^2$ is a regular curve on the unit sphere and the differential $dN_p : T_pM \to T_{N(p)}S^2$ of N at p is given by the formula

$$dN_p : Z = \dot\gamma(0) \mapsto \frac{d}{dt}(N(\gamma(t)))_{|t=0} = dN_p \cdot Z.$$

At the point p the second derivative $\ddot\gamma(0)$ has a natural decomposition

$$\ddot\gamma(0) = \ddot\gamma(0)^{\tan} + \ddot\gamma(0)^{\text{norm}}$$

into its tangential part, contained in T_pM, and its normal part in the orthogonal complement $(T_pM)^\perp$ of T_pM. Along the curve $\gamma : I \to M$ the normal $N(\gamma(s))$ is perpendicular to the tangent $\dot\gamma(s)$. This implies that

$$\begin{aligned} 0 &= \frac{d}{ds}(\langle \dot\gamma(s), N(\gamma(s))\rangle) \\ &= \langle \ddot\gamma(s), N(\gamma(s))\rangle + \langle \dot\gamma(s), dN_{\gamma(s)} \cdot \dot\gamma(s)\rangle. \end{aligned}$$

Hence the normal part of the second derivative $\ddot\gamma(0)$ is given by

$$\begin{aligned} \ddot\gamma(0)^{\text{norm}} &= \langle \ddot\gamma(0), N(p)\rangle N(p) \\ &= -\langle \dot\gamma(0), dN_p \cdot \dot\gamma(0)\rangle N(p) \end{aligned}$$

$$= \quad -\langle Z, dN_p \cdot Z\rangle N(p).$$

This shows that the normal component of $\ddot{\gamma}(0)$ is completely determined by the value of $\dot{\gamma}(0)$ and the values of the Gauss map along any curve through p with tangent $\dot{\gamma}(0) = Z$ at p.

Since $N : M \to S^2$ is a Gauss map for the surface M and $p \in M$ we see that $N(p)$ is a unit normal to both the tangent planes T_pM and $T_{N(p)}S^2$ so we can make the identification $T_pM \cong T_{N(p)}S^2$.

Definition 5.2. Let M be an oriented regular surface in $\mathbb{R}^3$ with Gauss map $N : M \to S^2$ and $p \in M$. Then the **shape operator**

$$S_p : T_pM \to T_pM$$

of M at p is the linear endomorphism given by

$$S_p(Z) = -dN_p \cdot Z$$

for all $Z \in T_pM$.

Proposition 5.3. *Let M be an oriented regular surface with Gauss map $N : M \to S^2$ and $p \in M$. Then the shape operator $S_p : T_pM \to T_pM$ is symmetric i.e.*

$$\langle S_p(Z), W\rangle = \langle Z, S_p(W)\rangle$$

for all $Z, W \in T_pM$.

Proof. Let $X : U \to M$ be a local parametrization of M such that $X(0) = p$ and let $N : X(U) \to S^2$ be the Gauss map on $X(U)$ given by

$$N(u, v) = \pm \frac{X_u(u, v) \times X_v(u, v)}{|X_u(u, v) \times X_v(u, v)|}.$$

Then the vector $N \circ X(u, v)$ is orthogonal to the tangent plane T_pM so

$$0 = \frac{d}{dv}\langle N \circ X, X_u\rangle = \langle dN_p \cdot X_v, X_u\rangle + \langle N \circ X, X_{vu}\rangle$$

and

$$0 = \frac{d}{du}\langle N \circ X, X_v\rangle = \langle dN_p \cdot X_u, X_v\rangle + \langle N \circ X, X_{uv}\rangle$$

By subtracting the second equation from the first one and employing the fact that $X_{uv} = X_{vu}$ we obtain

$$\langle dN_p \cdot X_v, X_u\rangle = \langle X_v, dN_p \cdot X_u\rangle.$$

The symmetry of the linear endomorphism $dN_p : T_pM \to T_pM$ is a direct consequence of this last equation and the following obvious relations

$$\langle dN_p \cdot X_u, X_u\rangle = \langle X_u, dN_p \cdot X_u\rangle,$$

$$\langle dN_p \cdot X_v, X_v \rangle = \langle X_v, dN_p \cdot X_v \rangle.$$

The statement now follows from the fact that $S_p = -dN_p$. $\square$

Corollary 5.4. *Let M be an oriented regular surface in $\mathbb{R}^3$ with Gauss map $N : M \to S^2$ and $p \in M$. Then there exists an orthonormal basis $\{Z_1, Z_2\}$ for the tangent plane T_pM such that*

$$S_p(Z_1) = \lambda_1 Z_1 \quad \text{and} \quad S_p(Z_2) = \lambda_2 Z_2,$$

for some $\lambda_1, \lambda_2 \in \mathbb{R}$.

Definition 5.5. Let M be an oriented regular surface in $\mathbb{R}^3$ with Gauss map $N : M \to S^2$ and $p \in M$. Then we define the **second fundamental form** $\mathrm{II}_p : T_pM \times T_pM \to \mathbb{R}$ of M at p by

$$\mathrm{II}_p(Z, W) = \langle S_p(Z), W \rangle.$$

It is an immediate consequence of Proposition 5.3 that the second fundamental form is symmetric and bilinear.

Definition 5.6. Let M be an oriented regular surface in $\mathbb{R}^3$ with Gauss map $N : M \to S^2$, $p \in M$ and $Z \in T_pM$ with $|Z| = 1$. Then the **normal curvature** $\kappa_n(Z)$ of M at p in the direction of Z is defined by

$$\kappa_n(Z) = \langle \ddot{\gamma}(0), N(p) \rangle,$$

where $\gamma : I \to M$ is any curve parametrized by arclength such that $\gamma(0) = p$ and $\dot{\gamma}(0) = Z$.

Proposition 5.7. *Let M be an oriented regular surface in $\mathbb{R}^3$ with Gauss map $N : M \to S^2$, $p \in M$ and $Z \in T_pM$ with $|Z| = 1$. Then the normal curvature $\kappa_n(Z)$ of M at p in the direction of Z satisfies*

$$\kappa_n(Z) = \langle S_p(Z), Z \rangle = \mathit{II}_p(Z, Z).$$

PROOF. Let $\gamma : I \to M$ be a curve parametrized by arclength such that $\gamma(0) = p$ and $\dot{\gamma}(0) = Z$. Along the curve the normal $N(\gamma(s))$ is perpendicular to the tangent $\dot{\gamma}(s)$. This means that

$$\begin{aligned} 0 &= \frac{d}{ds}(\langle \dot{\gamma}(s), N(\gamma(s)) \rangle) \\ &= \langle \ddot{\gamma}(s), N(\gamma(s)) \rangle + \langle \dot{\gamma}(s), dN_{\gamma(s)} \cdot \dot{\gamma}(s) \rangle. \end{aligned}$$

As a direct consequence we get

$$\begin{aligned} \kappa_n(Z) &= \langle \ddot{\gamma}(0), N(p) \rangle \\ &= -\langle Z, dN_p \cdot Z \rangle \\ &= \langle S_p(Z), Z \rangle \\ &= \mathrm{II}_p(Z, Z). \end{aligned}$$

$\square$

For an oriented regular surface M with Gauss map $N : M \to S^2$ and $p \in M$ let $T_p^1 M$ denote the unit circle in the tangent plane $T_p M$ i.e.

$$T_p^1 M = \{Z \in T_p M |\ |Z| = 1\}.$$

Then the real-valued function $\kappa_n : T_p^1 M \to \mathbb{R}$ is defined by

$$\kappa_n : Z \mapsto \kappa_n(Z).$$

The unit circle is compact and κ_n is continuous so there exist two directions $Z_1, Z_2 \in T_p^1 M$ such that

$$\kappa_1(p) = \kappa_n(Z_1) = \max_{Z \in T_p^1 M} \kappa_n(Z)$$

and

$$\kappa_2(p) = \kappa_n(Z_2) = \min_{Z \in T_p^1 M} \kappa_n(Z).$$

These are called **principal directions** at p and $\kappa_1(p), \kappa_2(p)$ the corresponding **principal curvatures**. A point $p \in M$ is said to be **umbilic** if $\kappa_1(p) = \kappa_2(p)$.

Theorem 5.8. *Let M be an oriented regular surface in $\mathbb{R}^3$ with Gauss map $N : M \to S^2$ and $p \in M$. Then $Z \in T_p^1 M$ is a principal direction at p if and only if it is an eigenvector for the shape operator $S_p : T_p M \to T_p M$.*

Proof. Let $\{Z_1, Z_2\}$ be an orthonormal basis for the tangent space $T_p M$ of eigenvectors to S_p i.e.

$$S_p(Z_1) = \lambda_1 Z_1 \quad \text{and} \quad S_p(Z_2) = \lambda_2 Z_2$$

for some $\lambda_1, \lambda_2 \in \mathbb{R}$. Then every unit vector $Z \in T_p^1 M$ can be written as

$$Z(\theta) = \cos\theta Z_1 + \sin\theta Z_2$$

and

$$\begin{aligned}
\kappa_n(Z(\theta)) &= \langle S_p(\cos\theta Z_1 + \sin\theta Z_2), \cos\theta Z_1 + \sin\theta Z_2\rangle \\
&= \cos^2\theta \langle S_p(Z_1), Z_1\rangle + \sin^2\theta \langle S_p(Z_2), Z_2\rangle \\
&\quad + \cos\theta \sin\theta (\langle S_p(Z_1), Z_2\rangle + \langle S_p(Z_2), Z_1\rangle) \\
&= \lambda_1 \cos^2\theta + \lambda_2 \sin^2\theta.
\end{aligned}$$

If $\lambda_1 = \lambda_2$ then $\kappa_n(Z(\theta)) = \lambda_1$ for all $\theta \in \mathbb{R}$ so any direction is both principal and an eigenvector for the shape operator S_p.

If $\lambda_1 \neq \lambda_2$, then we can assume, without loss of generality, that $\lambda_1 > \lambda_2$. Then $Z(\theta)$ is a maximal principal direction if and only if $\cos^2\theta = 1$ i.e. $Z = \pm Z_1$ and clearly a minimal principal direction if and only if $\sin^2\theta = 1$ i.e. $Z = \pm Z_2$. □

Definition 5.9. Let M be an oriented regular surface in $\mathbb{R}^3$ with Gauss map $N : M \to S^2$. Then we define the **Gaussian curvature** $K : M \to \mathbb{R}$ and the **mean curvature** $H : M \to \mathbb{R}$ by

$$K(p) = \det S_p \quad \text{and} \quad H(p) = \frac{1}{2} \operatorname{trace} S_p,$$

respectively. The surface M is said to be **flat** if $K(p) = 0$ for all $p \in M$ and **minimal** if $H(p) = 0$ for all $p \in M$.

Let M be a regular surface in $\mathbb{R}^3$, $p \in M$ and $\{Z_1, Z_2\}$ be an orthonormal basis for the tangent plane T_pM at p such that

$$S_p(Z_1) = \lambda_1 Z_1 \quad \text{and} \quad S_p(Z_2) = \lambda_1 Z_2.$$

Further let $\alpha_1, \alpha_2 : I \to M$ be two curves, parametrized by arclength, meeting at p i.e. $\alpha_1(0) = p = \alpha_2(0)$ such that

$$\dot\alpha_1(0) = Z_1, \quad \text{and} \quad \dot\alpha_2(0) = Z_2.$$

Then the eigenvalues of the shape operator S_p satisfy

$$\lambda_1 = \langle S_p(Z_1), Z_1\rangle = \langle \ddot\alpha_1(0), N(p)\rangle$$

and

$$\lambda_2 = \langle S_p(Z_2), Z_2\rangle = \langle \ddot\alpha_2(0), N(p)\rangle.$$

If $K(p) = \lambda_1\lambda_2 > 0$ then λ_1 and λ_2 have the same sign so the curves $\alpha_1, \alpha_2 : I \to M$ stay locally on the same side of the tangent plane. This means that the normal curvature $\kappa_n(Z)$ has the same sign independent of the direction $Z \in T_pM$ at p so any curve through the point p stays on the same side of the tangent plane.

If $K(p) = \lambda_1\lambda_2 < 0$ then λ_1 and λ_2 have different signs so the curves $\alpha_1, \alpha_2 : I \to M$ stay locally on different sides of the tangent plane T_pM at p.

Theorem 5.10. *Let M be a path connected, oriented regular surface in $\mathbb{R}^3$ with Gauss map $N : M \to S^2$. Then the shape operator $S_p : T_pM \to T_pM$ vanishes for all $p \in M$ if and only if M is contained in a plane.*

PROOF. If M is contained in a plane, then the Gauss map is constant so $S_p = -dN_p = 0$ at any point $p \in M$.

Fix a point $p \in M$, let q be an arbitrary point on M and $\gamma : I \to M$ be a curve such that $\gamma(0) = q$ and $\gamma(1) = p$. Then the real-valued function $f_q : I \to \mathbb{R}$ with

$$f_q(t) = \langle q - \gamma(t), N(\gamma(t))\rangle$$

satisfies $f_q(0) = 0$ and

$$\dot f_q(t) = -\langle \dot\gamma, N(\gamma(t))\rangle + \langle q - \gamma(t), dN_p \cdot \dot\gamma(t)\rangle = 0.$$

This implies that $\langle q - \gamma(t), N(\gamma(t))\rangle = 0$ for all $t \in I$ in particular

$$f_q(1) = \langle (q-p), N(p)\rangle = 0$$

for all $q \in M$. This shows that the surface is contained in the plane through p with normal $N(p)$. $\square$

We will now calculate the Gaussian curvature K and the mean curvature H of a surface in terms on its local parametrization. Let M be an oriented surface in $\mathbb{R}^3$ with Gauss map $N : M \to S^2$. Let $X : U \to M$ be a local parametrization such that $X(0) = p \in M$. Then the tangent space T_pM is generated by X_u and X_v so there exists a 2×2 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

such that the shape operator $S_p : T_pM \to T_pM$ satisfies

$$\begin{aligned} S_p(aX_u + bX_v) &= aS_p(X_u) + bS_p(X_v) \\ &= a(a_{11}X_u + a_{21}X_v) + b(a_{12}X_u + a_{22}X_v) \\ &= (a_{11}a + a_{12}b)X_u + (a_{21}a + a_{22}b)X_v. \end{aligned}$$

This means that, with respect to the basis $\{X_u, X_v\}$, the shape operator S_p at p is given by

$$S_p : \begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix}.$$

If we define the matrix-valued maps $[DX], [DN] : U \to \mathbb{R}^{2\times 3}$ by

$$[DX] = [X_u, X_v]^t \quad \text{and} \quad [DN] = [N_u, N_v]^t$$

then it follows from the definition $S_p = -dN_p$ of the shape operator that

$$-[DN] = A^t \cdot [DX].$$

To the local parametrization $X : U \to M$ we now associate the functions $e, f, g : U \to \mathbb{R}$ given by

$$\begin{aligned} \begin{pmatrix} e & f \\ f & g \end{pmatrix} &= -[DN] \cdot [DX]^t \\ &= A^t \cdot [DX] \cdot [DX]^t \\ &= A^t \cdot \begin{pmatrix} E & F \\ F & G \end{pmatrix}. \end{aligned}$$

We now obtain the shape operator $S_p : T_pM \to T_pM$ at $p \in M$ with $S_p(Z) = A \cdot Z$ where the matrix A is given by

$$A^t = \begin{pmatrix} e & f \\ f & g \end{pmatrix} \cdot \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}$$

$$= \frac{1}{EG - F^2} \begin{pmatrix} e & f \\ f & g \end{pmatrix} \cdot \begin{pmatrix} G & -F \\ -F & E \end{pmatrix}.$$

This implies that the Gaussian curvature K and the mean curvature H satisfy

$$K = \frac{eg - f^2}{EG - F^2} \quad \text{and} \quad H = \frac{1}{2}\, \frac{eG - 2fF + gE}{EG - F^2}.$$

Example 5.11. Let $\gamma = (r, 0, z) : I \to \mathbb{R}^3$ be a differentiable curve in the (x, z)-plane such that $r(s) > 0$ and $\dot{r}(s)^2 + \dot{z}(s)^2 = 1$ for all $s \in I$. Then $X : I \times \mathbb{R} \to \mathbb{R}^3$ with

$$X(u, v) = \begin{pmatrix} \cos v & -\sin v & 0 \\ \sin v & \cos v & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} r(u) \\ 0 \\ z(u) \end{pmatrix} = \begin{pmatrix} r(u)\cos v \\ r(u)\sin v \\ z(u) \end{pmatrix}$$

parametrizes a regular surface of revolution M. The linearly independent and orthogonal tangent vectors

$$X_u = \begin{pmatrix} \dot{r}(u)\cos v \\ \dot{r}(u)\sin v \\ \dot{z}(u) \end{pmatrix}, \quad X_v = \begin{pmatrix} -r(u)\sin v \\ r(u)\cos v \\ 0 \end{pmatrix}$$

generate a Gauss map

$$N(u, v) = \begin{pmatrix} \cos v & -\sin v & 0 \\ \sin v & \cos v & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\dot{z}(u) \\ 0 \\ \dot{r}(u) \end{pmatrix} = \begin{pmatrix} -\dot{z}(u)\cos v \\ -\dot{z}(u)\sin v \\ \dot{r}(u) \end{pmatrix}.$$

Furthermore

$$[DX] = \begin{pmatrix} \dot{r}(u)\cos v & \dot{r}(u)\sin v & \dot{z}(u) \\ -r(u)\sin v & r(u)\cos v & 0 \end{pmatrix},$$

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = [DX] \cdot [DX]^t = \begin{pmatrix} 1 & 0 \\ 0 & r(u)^2 \end{pmatrix}$$

and

$$\begin{aligned} \begin{pmatrix} e & f \\ f & g \end{pmatrix} &= -[DN] \cdot [DX]^t \\ &= \begin{pmatrix} -\ddot{z}(u)\cos v & -\ddot{z}(u)\sin v & \ddot{r}(u) \\ \dot{z}(u)\sin v & -\dot{z}(u)\cos v & 0 \end{pmatrix} \\ &\qquad \cdot \begin{pmatrix} \dot{r}(u)\cos v & -r(u)\sin v \\ \dot{r}(u)\sin v & r(u)\cos v \\ \dot{z}(u) & 0 \end{pmatrix} \\ &= \begin{pmatrix} \ddot{r}(u)\dot{z}(u) - \ddot{z}(u)\dot{r}(u) & 0 \\ 0 & -\dot{z}(u)r(u) \end{pmatrix}. \end{aligned}$$

Using the fact that the curve $(r, 0, z)$ is parametrized by arclength we get the following remarkably simple expression for the Gaussian curvature

$$\begin{aligned} K &= \frac{eg - f^2}{EG - F^2} \\ &= \frac{\dot{z}(u)r(u)(\ddot{z}(u)\dot{r}(u) - \ddot{r}(u)\dot{z}(u))}{r(u)^2} \\ &= \frac{\dot{r}(u)\dot{z}(u)\ddot{z}(u) - \ddot{r}(u)\dot{z}(u)^2}{r(u)} \\ &= \frac{\dot{r}(u)(-\dot{r}(u)\ddot{r}(u)) - \ddot{r}(u)(1 - \dot{r}(u)^2)}{r(u)} \\ &= -\frac{\ddot{r}(u)}{r(u)}. \end{aligned}$$

This shows that the function $r : I \to \mathbb{R}$ satisfies the following second order linear ordinary differential equation

$$\ddot{r}(s) + K(s) \cdot r(s) = 0.$$

Theorem 5.12. *Let M be a connected oriented regular surface in $\mathbb{R}^3$ with Gauss map $N : M \to S^2$. If every point $p \in M$ is an umbilic point, then M is either contained in a plane or in a sphere.*

Proof. Let $X : U \to M$ be a local parametrization such that U is connected. Since each point in $X(U)$ is umbilic there exists a differentiable function $\kappa : U \to \mathbb{R}$ such that the shape operator is given by

$$S_p : (aX_u + bX_v) \mapsto \kappa(u, v)(aX_u + bX_v)$$

so in particular

$$(N \circ X)_u = -\kappa X_u \quad \text{and} \quad (N \circ X)_v = -\kappa X_v.$$

Furthermore

$$\begin{aligned} 0 &= (N \circ X)_{uv} - (N \circ X)_{vu} \\ &= (-\kappa X_u)_v - (-\kappa X_v)_u \\ &= -\kappa_v X_u - \kappa X_{uv} + \kappa_u X_v + \kappa X_{uv} \\ &= -\kappa_v X_u + \kappa_u X_v. \end{aligned}$$

The vectors X_u and X_v are linearly independent so $\kappa_u = \kappa_v = 0$. The domain U is connected which means that κ is constant on U and hence on the whole of M since M is connected.

If $\kappa = 0$ then the shape operator vanishes and Theorem 5.10 tells us that the surface is contained in a plane.

If $\kappa \neq 0$ then we define $Y : U \to \mathbb{R}^3$ by
$$Y(u, v) = X(u, v) + \frac{1}{\kappa} N(u, v).$$
Then
$$dY = dX + \frac{1}{\kappa} dN = dX - \frac{1}{\kappa} \kappa dX = 0$$
so Y is constant and
$$|X - Y|^2 = \frac{1}{\kappa^2}.$$
This shows that $X(U)$ is contained in a sphere with centre Y and radius $1/\kappa$. Since M is connected the whole of M is contained in the same sphere. $\square$

Theorem 5.13. *Let M be a compact regular surface in $\mathbb{R}^3$. Then there exists at least one point $p \in M$ such that the Gaussian curvature $K(p)$ is positive.*

Proof. See Exercise 5.7. $\square$

Exercises

Exercise 5.1. Let U be an open subset of $\mathbb{R}^3$ and $q \in \mathbb{R}$ be a regular value of the differentiable function $f : U \to \mathbb{R}$. Prove that the regular surface $M = f^{-1}(\{q\})$ in $\mathbb{R}^3$ is orientable.

Exercise 5.2. Determine the Gaussian curvature and the mean curvature of the sphere $S^2_r = \{(x, y, z) \in \mathbb{R}^2 |\ x^2 + y^2 + z^2 = r^2\}$.

Exercise 5.3. Determine the Gaussian curvature and the mean curvature of the parametrized Enneper surface $X : \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$X(u, v) : (u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 - v^2).$$

Exercise 5.4. Determine the Gaussian curvature and the mean curvature of the cateniod M parametrized by $X : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^3$ with

$$X : (\theta, r) \mapsto (\frac{1 + r^2}{2r} \cos\theta, \frac{1 + r^2}{2r} \sin\theta, \log r).$$

Find an equation of the form $f(x, y, z) = 0$ describing the surface M. Compare with Exercise 4.12.

Exercise 5.5. Prove that the second fundamental form of an oriented regular surface M in $\mathbb{R}^3$ is invariant under rigid Euclidean motions.

Exercise 5.6. Let $X, Y : \mathbb{R}^2 \to \mathbb{R}^3$ be the regular parametrized surfaces given by

$$X(u, v) = (\cosh u \cos v, \cosh u \sin v, u),$$

$$Y(u, v) = (\sinh u \cos v, \sinh u \sin v, v).$$

Calculate the principal curvatures κ_1, κ_2 of X and Y. Compare with Exercise 4.12.

Exercise 5.7. Prove Theorem 5.13.

Exercise 5.8. Let $\gamma : \mathbb{R} \to \mathbb{R}^3$ be a regular curve, parametrized by arclength, with non-vanishing curvature and n, b denote the principal normal and the binormal of γ, respectively. Let r be a positive real number and assume that the r-tube M around γ given by $X : \mathbb{R}^2 \to \mathbb{R}^3$ with

$$X(s, \theta) \mapsto \gamma(s) + r(\cos\theta \cdot n(s) + \sin\theta \cdot b(s))$$

is a regular surface in $\mathbb{R}^3$. Determine the Gaussian curvature K of M in terms of $s, \theta, r, \kappa(s)$ and $\tau(s)$.

Exercise 5.9. Let M be a regular surface in $\mathbb{R}^3$, $p \in M$ and $\{Z, W\}$ be an orthonormal basis for T_pM. Let $\kappa_n(\theta)$ be the normal curvature of M at p in the direction of $\cos\theta Z + \sin\theta W$. Prove that the mean curvature H satisfies

$$H(p) = \frac{1}{2\pi}\int_0^{2\pi} \kappa_n(\theta) d\theta.$$

Exercise 5.10. Let M be an oriented regular surface in $\mathbb{R}^3$ with Gauss map $N : M \to S^2$. Let $X : U \to M$ be a local parametrization of M and $A(N \circ X(U))$ be the area of the image $N \circ X(U)$ on the unit sphere S^2. Prove that

$$A(N \circ X(U)) = \int_{X(U)} |K| dA,$$

where K is the Gaussian curvature of M. Compare with Exercise 3.7.

Exercise 5.11. Let a be a positive real number and U be the open set

$$U = \{(x, y, z) \in \mathbb{R}^3 |\ a(y^2 + x^2) < z\}.$$

Prove that there does not exist a complete regular minimal surface M in $\mathbb{R}^3$ which is contained in U.

CHAPTER 6

Theorema Egregium

In this chapter we prove the remarkable Theorema Egregium which tells us that the Gaussian curvature, of a regular surface, is actually completely determined by its first fundamental form.

Theorem 6.1. *Let M be a regular surface in $\mathbb{R}^3$. Then the Gaussian curvature K of M is determined by its first fundamental form.*

This result has a highly interesting consequence.

Corollary 6.2. *It is impossible to construct a distance preserving planar chart of the unit sphere S^2.*

Proof. If there existed a local parametrization $X : U \to S^2$ of the unit sphere which was an isometry then the Gaussian curvature of the flat plane and the unit sphere would be the same. But we know that S^2 has constant curvature $K = 1$. $\square$

We shall now prove Theorem 6.1.

Proof. Let M be a surface and $X : U \to M$ be a local parametrization of M with first fundamental form determined by

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = [DX] \cdot [DX]^t.$$

The set $\{X_u, X_v\}$ is a basis for the tangent space at each point $X(u, v)$ in $X(U)$. Applying the Gram-Schmidt process on this basis we get an orthonormal basis $\{Z, W\}$ for the tangent space as follows:

$$Z = \frac{X_u}{\sqrt{E}},$$

$$\begin{aligned} \tilde{W} &= X_v - \langle X_v, Z\rangle Z \\ &= X_v - \frac{\langle X_v, X_u\rangle X_u}{\langle X_u, X_u\rangle} \\ &= X_v - \frac{F}{E} X_u \end{aligned}$$

and finally

$$W = \frac{\tilde{W}}{|\tilde{W}|} = \frac{\sqrt{E}}{\sqrt{EG - F^2}}(X_v - \frac{F}{E}X_u).$$

This means that there exist functions $a, b, c : U \to \mathbb{R}$ only depending on E, F, G such that

$$Z = aX_u \quad \text{and} \quad W = bX_u + cX_v.$$

If we define a local Gauss map $N : X(U) \to S^2$ by

$$N = \frac{X_u \times X_v}{|X_u \times X_v|} = Z \times W$$

then $\{Z, W, N\}$ is a positively oriented orthonormal basis for $\mathbb{R}^3$ along the open subset $X(U)$ of M. This means that the derivatives

$$Z_u, Z_v, W_u, W_v$$

satisfy the following system of equations

$$\begin{aligned}
Z_u &= \langle Z_u, Z\rangle Z + \langle Z_u, W\rangle W + \langle Z_u, N\rangle N,\\
Z_v &= \langle Z_v, Z\rangle Z + \langle Z_v, W\rangle W + \langle Z_v, N\rangle N,\\
W_u &= \langle W_u, Z\rangle Z + \langle W_u, W\rangle W + \langle W_u, N\rangle N,\\
W_v &= \langle W_v, Z\rangle Z + \langle W_v, W\rangle W + \langle W_v, N\rangle N.
\end{aligned}$$

Using the fact that $\{Z, W\}$ is orthonormal we can simplify to

$$\begin{aligned}
Z_u &= \langle Z_u, W\rangle W + \langle Z_u, N\rangle N,\\
Z_v &= \langle Z_v, W\rangle W + \langle Z_v, N\rangle N,\\
W_u &= \langle W_u, Z\rangle Z + \langle W_u, N\rangle N,\\
W_v &= \langle W_v, Z\rangle Z + \langle W_v, N\rangle N.
\end{aligned}$$

The following shows that $\langle Z_u, W\rangle$ is a function of $E, F, G : U \to \mathbb{R}$ and their first order derivatives.

$$\begin{aligned}
\langle Z_u, W\rangle &= \langle (aX_u)_u, W\rangle\\
&= \langle a_u X_u + aX_{uu}, bX_u + cX_v\rangle\\
&= a_u bE + a_u cF + ab\langle X_{uu}, X_u\rangle + ac\langle X_{uu}, X_v\rangle\\
&= a_u bE + a_u cF + \frac{1}{2}abE_u + ac(F_u - \frac{1}{2}E_v)
\end{aligned}$$

It is easily seen that the same applies to $\langle Z_v, W\rangle$.

Employing Lemma 6.3 we now obtain

$$\begin{aligned}
&\langle Z_u, W\rangle_v - \langle Z_v, W\rangle_u\\
= &\langle Z_{uv}, W\rangle + \langle Z_u, W_v\rangle - \langle Z_{vu}, W\rangle - \langle Z_v, W_u\rangle\\
= &\langle Z_u, W_v\rangle - \langle Z_v, W_u\rangle
\end{aligned}$$

$$= K\sqrt{EG - F^2}.$$

Hence the Gaussian curvature K of M is given by

$$K = \frac{\langle Z_u, W\rangle_v - \langle Z_v, W\rangle_u}{\sqrt{EG - F^2}}$$

As an immediate consequence we see that K only depends on the functions E, F, G and their first and second order derivatives. Hence it is completely determined by the first fundamental form of M. □

Lemma 6.3. *For the above situation we have*

$$\langle Z_u, W_v\rangle - \langle Z_v, W_u\rangle = K\sqrt{EG - F^2}.$$

Proof. If A is the matrix for the shape operator in the basis $\{X_u, X_v\}$ then

$$-N_u = a_{11}X_u + a_{21}X_v \quad \text{and} \quad -N_v = a_{12}X_u + a_{22}X_v.$$

This means that

$$\begin{aligned} \langle N_u \times N_v, N\rangle &= \langle (a_{11}X_u + a_{21}X_v) \times (a_{12}X_u + a_{22}X_v), N\rangle \\ &= (a_{11}a_{22} - a_{12}a_{21})\langle X_u \times X_v, N\rangle \\ &= K\langle (\sqrt{EG - F^2})N, N\rangle \\ &= K\sqrt{EG - F^2}. \end{aligned}$$

We also have

$$\begin{aligned} \langle N_u \times N_v, N\rangle &= \langle N_u \times N_v, Z \times W\rangle \\ &= \langle N_u, Z\rangle\langle N_v, W\rangle - \langle N_u, W\rangle\langle N_v, Z\rangle \\ &= \langle Z_u, N\rangle\langle N, W_v\rangle - \langle W_u, N\rangle\langle N, Z_v\rangle \\ &= \langle Z_u, W_v\rangle - \langle Z_v, W_u\rangle. \end{aligned}$$

This proves the statement. □

Deep Result 6.4. *Let M_1 and M_2 be two regular surfaces in $\mathbb{R}^3$ and $\phi : M_1 \to M_2$ be a diffeomorphism respecting their first and second fundamental forms, i.e.*

$$I_p(X, Y) = I_{\phi(p)}(d\phi(X), d\phi(Y))$$

and

$$I\!I_p(X, Y) = I\!I_{\phi(p)}(d\phi(X), d\phi(Y)),$$

for all $p \in M_1$ and $X, Y \in T_pM_1$. Then $\phi : M_1 \to M_2$ is the restriction $\phi = \Phi|_{M_1} : M_1 \to M_2$ of a Euclidean motion $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ of $\mathbb{R}^3$ to the surface M_1.

The proof of the last result is beyond the scope of these lecture notes. Here we need arguments from the theory of partial differential equations.

Exercises

Exercise 6.1. For $\alpha \in (0, \pi/2)$ define the parametrized surface M_α by $X_\alpha : \mathbb{R}^+ \times \mathbb{R} \to M$ by

$$X_\alpha(r, \theta) = (r \sin\alpha \cos(\frac{\theta}{\sin\alpha}), r \sin\alpha \sin(\frac{\theta}{\sin\alpha}), r\cos\alpha).$$

Calculate its Gaussian curvature K.

Exercise 6.2. Equip $\mathbb{R}^2$ and $\mathbb{R}^4$ with their standard Euclidean scalar products. Prove that the parametrization $X : \mathbb{R}^2 \to \mathbb{R}^4$,

$$X(u, v) = (\cos u, \sin u, \cos v, \sin v)$$

of the compact torus M in $\mathbb{R}^4$ is isometric. What does this tell us about the Gaussian curvature of M. Compare the result with Theorem 5.13.

Exercise 6.3. Let M be a regular surface in $\mathbb{R}^3$ and $X : U \to M$ be an orthogonal parametrization i.e. $F = 0$. Prove that the Gaussian curvature satisfies

$$K = -\frac{1}{2\sqrt{EG}}\Big((\frac{E_v}{\sqrt{EG}})_v + (\frac{G_u}{\sqrt{EG}})_u\Big).$$

Exercise 6.4. Let M be a regular surface in $\mathbb{R}^3$ and $X : U \to M$ be an isothermal parametrization i.e. $F = 0$ and $E = G$. Prove that the Gaussian curvature satisfies

$$K = -\frac{1}{2E}((\log E)_{uu} + (\log E)_{vv}),$$

Determine the Gaussian curvature K in the cases when

$$E = \frac{4}{(1 + u^2 + v^2)^2}, \quad E = \frac{4}{(1 - u^2 - v^2)^2} \quad \text{or} \quad E = \frac{1}{u^2}.$$

CHAPTER 7

Geodesics

In this chapter we introduce the notion of a geodesic on a regular surface in $\mathbb{R}^3$. We show that locally they are the shortest paths between their endpoints. Geodesics generalize the straight lines in the Euclidean plane.

Let M be a regular surface in $\mathbb{R}^3$ and $\gamma : I \to M$ be a C^2-curve on M such that $\gamma(0) = p$. As we have seen earlier the second derivative $\ddot{\gamma}(0)$ at p has a natural decomposition

$$\ddot{\gamma}(0) = \ddot{\gamma}(0)^{\tan} + \ddot{\gamma}(0)^{\text{norm}}$$

into its tangential part, contained in T_pM, and its normal part in the orthogonal complement $T_pM^{\perp}$.

Definition 7.1. Let M be a regular surface in $\mathbb{R}^3$. A C^2-curve $\gamma : I \to M$ on M is said to be a **geodesic** if for all $t \in I$ the tangential part of the second derivative $\ddot{\gamma}(t)$ vanishes i.e.

$$\ddot{\gamma}(t)^{\tan} = 0.$$

Example 7.2. Let $p \in S^2$ be a point on the unit sphere and $Z \in T_pS^2$ be a unit tangent vector. Then $\langle p, Z\rangle = 0$ so $\{p, Z\}$ is an orthonormal basis for a plane in $\mathbb{R}^3$, through the origin, which intersects the sphere in a great circle. This circle is parametrized by the curve $\gamma : \mathbb{R} \to S^2$

$$\gamma(s) = \cos s \cdot p + \sin s \cdot Z.$$

Then the second derivative $\ddot{\gamma}(s)$ satisfies $\ddot{\gamma}(s) = -\gamma(s)$ for all $s \in I$. This means that the tangential part $\ddot{\gamma}(s)^{\tan}$ vanishes so the curve is a geodesic on S^2.

Proposition 7.3. *Let M be a regular surface in $\mathbb{R}^3$ and $\gamma : I \to M$ be a geodesic on M. Then the norm $|\dot{\gamma}| : I \to \mathbb{R}$ of the tangent $\dot{\gamma}$ of γ is constant i.e. the curve is parametrized proportional to arclength.*

Proof. The statement is an immediate consequence of the following calculation

$$\frac{d}{dt}|\dot{\gamma}(t)|^2 \;=\; \frac{d}{dt}\langle \dot{\gamma}(t), \dot{\gamma}(t)\rangle$$

$$
\begin{aligned}
&= 2\,\langle \ddot\gamma(t), \dot\gamma(t)\rangle \\
&= 2\,\langle \ddot\gamma(t)^{\tan} + \ddot\gamma(t)^{\text{norm}}, \dot\gamma(t)\rangle \\
&= 2\,\langle \ddot\gamma(t)^{\tan}, \dot\gamma(t)\rangle \\
&= 0.
\end{aligned}
$$

□

Let M be an oriented regular surface in $\mathbb{R}^3$ with Gauss map $N : M \to S^2$. Along a C^2-curve $\gamma : I \to M$, parametrized by arclength, the two vectors $\dot\gamma$ and N are orthogonal and both of unit length, so the set

$$\{\dot\gamma(s), N(\gamma(s)), N(\gamma(s)) \times \dot\gamma(s)\}$$

is an orthonormal basis for $\mathbb{R}^3$. This implies that the second derivative $\ddot\gamma : I \to \mathbb{R}^3$ has the decomposition

$$
\begin{aligned}
\ddot\gamma &= \langle \ddot\gamma, \dot\gamma\rangle\dot\gamma + \langle \ddot\gamma, N \times \dot\gamma\rangle(N \times \dot\gamma) + \langle \ddot\gamma, N\rangle N \\
&= \langle \ddot\gamma, N \times \dot\gamma\rangle(N \times \dot\gamma) + \langle \ddot\gamma, N\rangle N \\
&= \ddot\gamma(t)^{\tan} + \ddot\gamma(t)^{\text{norm}}.
\end{aligned}
$$

Definition 7.4. Let M be an oriented regular surface in $\mathbb{R}^3$ with Gauss map $N : M \to S^2$ and $\gamma : I \to M$ be a curve on M parametrized by arclength. Then we define the **geodesic curvature** $k_g : I \to \mathbb{R}$ of γ by

$$\kappa_g(s) = \langle \ddot\gamma(s), N(\gamma(s)) \times \dot\gamma(s)\rangle.$$

The set $\{\dot\gamma(s), N(\gamma(s)) \times \dot\gamma(s)\}$ is an orthonormal basis for the tangent plane $T_{\gamma(s)}M$ of M at $\gamma(s)$. The curve $\gamma : I \to M$ is parametrized by arclength so the second derivative is perpendicular to $\dot\gamma$. This means that

$$\kappa_g(s)^2 = |\ddot\gamma(s)^{\tan}|^2,$$

so the geodesic curvature is therefore a measure of how far the curve is from being a geodesic.

Corollary 7.5. *Let M be an oriented regular surface in $\mathbb{R}^3$ with Gauss map $N : M \to S^2$ and $\gamma : I \to M$ be a curve on M parametrized by arclength. Let $\kappa : I \to \mathbb{R}$ be the curvature of γ as a curve in $\mathbb{R}^3$ and $\kappa_n, \kappa_g : I \to \mathbb{R}$ be the normal and geodesic curvatures, respectively. Then*

$$\kappa(s)^2 = \kappa_g(s)^2 + \kappa_n(s)^2.$$

Proof. This is a direct consequence of the orthogonal decomposition

$$\ddot\gamma(s) = \ddot\gamma(s)^{\tan} + \ddot\gamma(s)^{\text{norm}}.$$

□

Example 7.6. Let $\gamma = (r, 0, z) : I \to \mathbb{R}^3$ be a differentiable curve in the (x, z)-plane such that $r(s) > 0$ and $\dot{r}(s)^2 + \dot{z}(s)^2 = 1$ for all $s \in I$. Then $X : I \times \mathbb{R} \to \mathbb{R}^3$ with

$$X(u, v) = \begin{pmatrix} \cos v & -\sin v & 0 \\ \sin v & \cos v & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} r(u) \\ 0 \\ z(u) \end{pmatrix} = \begin{pmatrix} r(u) \cos v \\ r(u) \sin v \\ z(u) \end{pmatrix}$$

parametrizes a surface of revolution M. The tangent space is generated by the vectors

$$X_u = \begin{pmatrix} \dot{r}(u) \cos v \\ \dot{r}(u) \sin v \\ \dot{z}(u) \end{pmatrix}, \quad X_v = \begin{pmatrix} -r(u) \sin v \\ r(u) \cos v \\ 0 \end{pmatrix}.$$

For a fixed $v \in \mathbb{R}$ the curve $\gamma_1 : I \to M$ with

$$\gamma_1(u) = \begin{pmatrix} r(u) \cos v \\ r(u) \sin v \\ z(u) \end{pmatrix}$$

parametrizes a **meridian** on M by arclength. It is easily seen that

$$\langle \ddot{\gamma}_1, X_u \rangle = \langle \ddot{\gamma}_1, X_v \rangle = 0.$$

This means that $(\ddot{\gamma}_1)^{\tan} = 0$, so $\gamma_1 : I \to M$ is a geodesic.

For a fixed $u \in \mathbb{R}$ the curve $\gamma_2 : I \to M$ with

$$\gamma_2(v) = \begin{pmatrix} r(u) \cos v \\ r(u) \sin v \\ z(u) \end{pmatrix}$$

parametrizes a **parallel** on M. A simple calculation yields

$$\langle \ddot{\gamma}_2, X_u \rangle = -\dot{r}(u) r(u) \quad \text{and} \quad \langle \ddot{\gamma}_2, X_v \rangle = 0.$$

This means that $\gamma_2 : I \to M$ is a geodesic if and only if $\dot{r}(u) = 0$.

The next result states the important **geodesic equations**.

Theorem 7.7. *Let M be a regular surface in $\mathbb{R}^3$ and $X : U \to M$ be a local parametrization of M with*

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = [DX] \cdot [DX]^t.$$

If $(u, v) : I \to U$ is a C^2-curve in U then the composition

$$\gamma = X \circ (u, v) : I \to X(U)$$

is a geodesic on M if and only if

$$\frac{d}{dt}(E\dot{u} + F\dot{v}) \quad = \quad \frac{1}{2}(E_u \dot{u}^2 + 2F_u \dot{u}\dot{v} + G_u \dot{v}^2)$$

$$\frac{d}{dt}(F\dot{u} + G\dot{v}) \;=\; \frac{1}{2}(E_v\dot{u}^2 + 2F_v\dot{u}\dot{v} + G_v\dot{v}^2).$$

PROOF. The tangent vector of the curve $(u, v) : I \to U$ is given by $(\dot{u}, \dot{v}) = \dot{u}e_1 + \dot{v}e_2$ so for the tangent $\dot{\gamma}$ of γ we have

$$\begin{aligned}\dot{\gamma} &= dX \cdot (\dot{u}e_1 + \dot{v}e_2)\\ &= \dot{u}\; dX \cdot e_1 + \dot{v}\; dX \cdot e_2\\ &= \dot{u}X_u + \dot{v}X_v.\end{aligned}$$

Following the definition we see that $\gamma : I \to X(U)$ is a geodesic if and only if

$$\langle \ddot{\gamma}, X_u\rangle = 0 \quad \text{and} \quad \langle \ddot{\gamma}, X_v\rangle = 0$$

The first equation gives

$$\begin{aligned}0 &= \langle \frac{d}{dt}(\dot{u}X_u + \dot{v}X_v), X_u\rangle\\ &= \frac{d}{dt}\langle \dot{u}X_u + \dot{v}X_v, X_u\rangle - \langle \dot{u}X_u + \dot{v}X_v, \frac{d}{dt}X_u\rangle.\end{aligned}$$

This implies that

$$\begin{aligned}&\frac{d}{dt}(E\dot{u} + F\dot{v})\\ =\;& \frac{d}{dt}\langle \dot{u}X_u + \dot{v}X_v, X_u\rangle\\ =\;& \langle \dot{u}X_u + \dot{v}X_v, \frac{d}{dt}X_u\rangle\\ =\;& \langle \dot{u}X_u + \dot{v}X_v, \dot{u}X_{uu} + \dot{v}X_{uv}\rangle\\ =\;& \dot{u}^2\langle X_u, X_{uu}\rangle + \dot{u}\dot{v}(\langle X_u, X_{uv}\rangle + \langle X_v, X_{uu}\rangle) + \dot{v}^2\langle X_v, X_{uv}\rangle\\ =\;& \frac{1}{2}E_u\dot{u}^2 + F_u\dot{u}\dot{v} + \frac{1}{2}G_u\dot{v}^2.\end{aligned}$$

This gives us the first geodesic equation. The second one is obtained in exactly the same way. □

Theorem 7.7 characterizes geodesics as solutions to a second order non-linear system of ordinary differential equations. For this we have the following existence result.

Theorem 7.8. *Let M be a regular surface in $\mathbb{R}^3$, $p \in M$ and $Z \in T_pM$. Then there exists a unique, locally defined, geodesic*

$$\gamma : (-\epsilon, \epsilon) \to M$$

satisfying the initial conditions $\gamma(0) = p$ and $\dot{\gamma}(0) = Z$.

PROOF. The proof is based on a second order consequence of the well-known theorem of Picard-Lindelöf formulated here as Fact 7.9. □

Fact 7.9. *Let $f : U \to \mathbb{R}^n$ be a continuous map defined on an open subset U of $\mathbb{R} \times \mathbb{R}^n$ and $L \in \mathbb{R}^+$ such that*

$$|f(t,x) - f(t,y)| \leq L \cdot |x - y|$$

for all $(t,x),(t,y) \in U$. If $(t_0, x_0) \in U$ and $x_1 \in \mathbb{R}^n$ then there exists a unique local solution $x : I \to \mathbb{R}^n$ to the following initial value problem

$$x''(t) = f(t, x(t)), \quad x(t_0) = x_0, \quad x'(t_0) = x_1.$$

Proposition 7.10. *Let M_1 and M_2 be two regular surfaces in $\mathbb{R}^3$ and $\phi : M_1 \to M_2$ be an isometric differentiable map. Then $\gamma_1 : I \to M_1$ is a geodesic on M_1 if and only if the composition $\gamma_2 = \phi \circ \gamma_1 : I \to M_2$ is a geodesic on M_2*

Proof. See Exercise 7.7 □

The next result is the famous Theorem of Clairaut.

Theorem 7.11. *Let M be a regular surface of revolution and $\gamma : I \to M$ be a geodesic on M parametrized by arclength. Let $r : I \to \mathbb{R}^+$ be the function describing the distance between a point $\gamma(s)$ and the axis of rotation and $\theta : I \to \mathbb{R}$ be the angle between $\dot{\gamma}(s)$ and the meridian through $\gamma(s)$. Then the product $r(s)\sin\theta(s)$ is constant along the geodesic.*

Proof. Let the surface M be parametrized by $X : I \times \mathbb{R} \to \mathbb{R}^3$ with

$$X(u,v) = \begin{pmatrix} \cos v & -\sin v & 0 \\ \sin v & \cos v & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} r(u) \\ 0 \\ z(u) \end{pmatrix} = \begin{pmatrix} r(u)\cos v \\ r(u)\sin v \\ z(u) \end{pmatrix},$$

where $(r, 0, z) : I \to \mathbb{R}^3$ is a differentiable curve in the (x,z)-plane such that $r(s) > 0$ and $\dot{r}(s)^2 + \dot{z}(s)^2 = 1$ for all $s \in I$. Then

$$X_u = \begin{pmatrix} \dot{r}(u)\cos v \\ \dot{r}(u)\sin v \\ \dot{z}(u) \end{pmatrix}, \quad X_v = \begin{pmatrix} -r(u)\sin v \\ r(u)\cos v \\ 0 \end{pmatrix}$$

give

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = [DX] \cdot [DX]^t = \begin{pmatrix} 1 & 0 \\ 0 & r(u)^2 \end{pmatrix}$$

so the set

$$\{X_u, \frac{1}{r(u)} X_v\}$$

is an orthonormal basis for the tangent space of M at $X(u, v)$. This means that the tangent $\dot{\gamma}(s)$ of the geodesic $\gamma : I \to M$ can be written as

$$\dot{\gamma}(s) = \cos\theta(s) X_u(u(s), v(s)) + \sin\theta(s)\frac{1}{r(s)} X_v(u(s), v(s)),$$

where $r(s)$ is the distance to the axes of revolution and $\theta(s)$ the angle between $\dot{\gamma}(s)$ and the tangent $X_u(u(s), v(s))$ to the meridian. It follows that

$$\begin{aligned} X_u \times \dot{\gamma} &= X_u \times (\cos\theta X_u + \frac{\sin\theta}{r} X_v) \\ &= \frac{\sin\theta}{r}(X_u \times X_v) \end{aligned}$$

but also

$$\begin{aligned} X_u \times \dot{\gamma} &= X_u \times (\dot{u} X_u + \dot{v} X_v) \\ &= \dot{v}(X_u \times X_v). \end{aligned}$$

Hence

$$r(s)^2 \dot{v}(s) = r(s) \sin\theta(s).$$

It now follows from the second geodesic equation that

$$\frac{d}{ds}(r(s)\sin\theta(s)) = \frac{d}{ds}(r(s)^2 \dot{v}(s)) = 0.$$

□

Example 7.12. Let M be a surface of revolution parametrized by $X : I \times \mathbb{R} \to M$,

$$X(s, v) = \begin{pmatrix} \cos v & -\sin v & 0 \\ \sin v & \cos v & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} r(s) \\ 0 \\ z(s) \end{pmatrix} = \begin{pmatrix} r(s)\cos v \\ r(s)\sin v \\ z(s) \end{pmatrix},$$

where $(r, 0, z) : I \to \mathbb{R}^3$ is a differentiable curve in the (x, z)-plane such that $r(s) > 0$ and $\dot{r}(s)^2 + \dot{z}(s)^2 = 1$ for all $s \in I$.

In Example 5.11 we have proved that the Gaussian curvature K of M satisfies the equation

$$\ddot{r}(s) + K(s) \cdot r(s) = 0.$$

If we put $K \equiv -1$ and solve this linear ordinary differential equation we get the general solution $r(s) = ae^s + be^{-s}$. By the particular choice of $r, z : \mathbb{R}^+ \to \mathbb{R}$ satisfying

$$r(s) = e^{-s} \quad \text{and} \quad z(s) = \int_0^s \sqrt{1 - e^{-2t}} dt$$

we get a parametrization $X : \mathbb{R}^+ \times \mathbb{R} \to M$ of the famous **pseudosphere**. The corresponding first fundamental form is

$$\begin{pmatrix} E_X & F_X \\ F_X & G_X \end{pmatrix} = [DX] \cdot [DX]^t = \begin{pmatrix} 1 & 0 \\ 0 & e^{-2s} \end{pmatrix}.$$

For convenience we introduce a new variable u satisfying

$$s(u) = \log u, \text{ or equivalently, } u(s) = e^s.$$

This gives us a new parametrization $Y : (1, \infty) \times \mathbb{R} \to M$ of the pseudosphere where $Y(u, v) = X(s(u), v)$. Then the chain rule gives

$$Y_u = s_u X_s = \frac{1}{u} X_s, \quad Y_v = X_v$$

and we get the following first fundamental form for Y

$$\begin{pmatrix} E_Y & F_Y \\ F_Y & G_Y \end{pmatrix} = [DY] \cdot [DY]^t = \frac{1}{u^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The corresponding metric satisfies

$$ds^2 = \frac{1}{u^2}(dv^2 + du^2).$$

It is clear that this extends to a metric defined in the upper half plane

$$H^2 = \{(v, u) \in \mathbb{R}^2 |\ u > 0\}.$$

This is called the **hyperbolic metric**. The hyperbolic plane (H^2, ds^2) is very interesting both for its rich geometry but also for its historic importance. It is a model for the famous **non-Euclidean geometry**.

We shall now determine the geodesics in the hyperbolic plane. Let

$$\gamma = (v, u) : I \to H^2$$

be a geodesic parametrized by arclength. Then $\dot{\gamma} = (\dot{v}, \dot{u})$ and

$$|\dot{\gamma}|^2_{H^2} = ds^2(\dot{\gamma}, \dot{\gamma}) = \frac{1}{u^2}(\dot{v}^2 + \dot{u}^2) = 1$$

or equivalently $\dot{v}^2 + \dot{u}^2 = u^2$. Following the proof of Clairaut's theorem we know that

$$r(s) \sin \theta(s) = \frac{1}{u^2(s)} \dot{v}(s) = R$$

is a real constant along the geodesic. This implies that

$$\frac{dv}{ds} = \dot{v} = u^2 R.$$

If $R = 0$ we see that $\dot{v} = 0$ so the function v is constant. This means that the vertical lines $v = v_0$ in the upper half plane H^2 are geodesics. It this case, they are parametrized by $\gamma(s) = (e^s, v_0)$.

If $R \neq 0$ then we have

$$u^4R^2 + \dot{u}^2 = u^2$$

or equivalently

$$\frac{du}{ds} = \dot{u} = \pm u\sqrt{1 - R^2u^2}$$

This means that

$$\frac{dv}{du} = \dot{v}/\dot{u} = \pm\frac{Ru}{\sqrt{1 - R^2u^2}}$$

or equivalently

$$dv = \pm\frac{Ru}{\sqrt{1 - R^2u^2}}du.$$

We integrate this to

$$R(v - v_0) = \pm\sqrt{1 - R^2u^2}$$

which implies

$$(v - v_0)^2 + u^2 = \frac{1}{R^2}.$$

This means that the geodesic is a half circle in H^2 with centre at $(v_0, 0)$ and radius $1/R$.

Deep Result 7.13 (David Hilbert 1901)**.** *There does not exist an isometric embedding of the hyperbolic plane H^2 into the standard Euclidean $\mathbb{R}^3$.*

Definition 7.14. Let M be a regular surface in $\mathbb{R}^3$ and $\gamma : I \to M$ be a C^2-curve on M. A **variation** of γ is a C^2-map

$$\Phi : (-\epsilon, \epsilon) \times I \to M$$

such that for all $s \in I$, $\Phi_0(s) = \Phi(0, s) = \gamma(s)$. If the interval is compact i.e. of the form $I = [a, b]$, then the variation Φ is said to be **proper** if for all $t \in (-\epsilon, \epsilon)$, $\Phi_t(a) = \gamma(a)$ and $\Phi_t(b) = \gamma(b)$.

Definition 7.15. Let M be a regular surface in $\mathbb{R}^3$ and $\gamma : I \to M$ be a C^2-curve on M. For every compact subinterval $[a, b]$ of I we define the **length functional** $L_{[a,b]}$ by

$$L_{[a,b]}(\gamma) = \int_a^b |\dot{\gamma}(t)|dt.$$

A C^2-curve $\gamma : I \to M$ is said to be a **critical point** for the length functional if every **proper variation** Φ of $\gamma|_{[a,b]}$ satisfies

$$\frac{d}{dt}(L_{[a,b]}(\Phi_t))|_{t=0} = 0.$$

We shall now prove that geodesics can be characterized as the critical points of the length functional.

Theorem 7.16. *Let $\gamma : I = [a, b] \to M$ be a C^2-curve parametrized by arclength. Then γ is a critical point for the length functional if and only if it is a geodesic.*

Proof. Let $\Phi : (-\epsilon, \epsilon) \times I \to M$ with $\Phi : (t, s) \mapsto \Phi(t, s)$ be a proper variation of $\gamma : I \to M$. Then

$$\begin{aligned}
&\frac{d}{dt}(L_{[a,b]}(\Phi_t))|_{t=0}\\
= \ & \frac{d}{dt}\Big(\int_a^b |\dot{\gamma}_t(s)|ds\Big)|_{t=0}\\
= \ & \int_a^b \frac{d}{dt}\sqrt{\langle\frac{\partial\Phi}{\partial s}, \frac{\partial\Phi}{\partial s}\rangle} ds|_{t=0}\\
= \ & \int_a^b \Big(\langle\frac{\partial^2\Phi}{\partial t\partial s}, \frac{\partial\Phi}{\partial s}\rangle/\sqrt{\langle\frac{\partial\Phi}{\partial s}, \frac{\partial\Phi}{\partial s}\rangle}\Big) ds|_{t=0}\\
= \ & \int_a^b \langle\frac{\partial^2\Phi}{\partial s\partial t}, \frac{\partial\Phi}{\partial s}\rangle ds|_{t=0}\\
= \ & \int_a^b \Big(\frac{d}{ds}(\langle\frac{\partial\Phi}{\partial t}, \frac{\partial\Phi}{\partial s}\rangle) - \langle\frac{\partial\Phi}{\partial t}, \frac{\partial^2\Phi}{\partial s^2}\rangle\Big) ds|_{t=0}\\
= \ & [\langle\frac{\partial\Phi}{\partial t}(0, s), \frac{\partial\Phi}{\partial s}(0, s)\rangle]_a^b - \int_a^b \langle\frac{\partial\Phi}{\partial t}(0, s), \frac{\partial^2\Phi}{\partial s^2}(0, s)\rangle ds.
\end{aligned}$$

The variation is proper, so

$$\frac{\partial\Phi}{\partial t}(0, a) = \frac{\partial\Phi}{\partial t}(0, b) = 0.$$

Furthermore

$$\frac{\partial^2\Phi}{\partial s^2}(0, s) = \ddot{\gamma}(s),$$

so

$$\frac{d}{dt}(L_{[a,b]}(\Phi_t))|_{t=0} = -\int_a^b \langle\frac{\partial\Phi}{\partial t}(0, s), \ddot{\gamma}(s)^{\tan}\rangle) ds.$$

The last integral vanishes for every proper variation Φ of γ if and only if γ is a geodesic. □

Let M be a regular surface in $\mathbb{R}^3$, $p \in M$ and

$$T_p^1 M = \{e \in T_p M|\ |e| = 1\}$$

be the unit circle in the tangent plane T_pM. Then every non-zero tangent vector $Z \in T_pM$ can be written as

$$Z = r_Z \cdot e_Z,$$

where $r_Z = |Z|$ and $e_Z = Z/|Z| \in T_p^1M$. For $e \in T_p^1M$ let

$$\gamma_e : (-a_e, b_e) \to M$$

be the maximal geodesic such that $a_e, b_e \in \mathbb{R}^+ \cup \{\infty\}$, $\gamma_e(0) = p$ and $\dot{\gamma}_e(0) = e$. It can be shown that the real number

$$\epsilon_p = \inf\{a_e, b_e | \ e \in T_p^1M\}.$$

is positive so the open ball

$$B^2_{\epsilon_p}(0) = \{Z \in T_pM | \ |Z| < \epsilon_p\}$$

is non-empty. The **exponential map** $\exp_p : B^2_{\epsilon_p}(0) \to M$ at p is defined by

$$\exp_p : Z \mapsto \begin{cases} p & \text{if } Z = 0 \\ \gamma_{e_Z}(r_Z) & \text{if } Z \neq 0. \end{cases}$$

Note that for $e \in T_p^1M$ the line segment $\lambda_e : (-\epsilon_p, \epsilon_p) \to T_pM$ with $\lambda_e : t \mapsto t \cdot e$ is mapped onto the geodesic γ_e i.e. locally we have $\gamma_e = \exp_p \circ \lambda_e$.

One can prove that the map $\exp_p$ is smooth and it follows from its definition that the differential

$$d(\exp_p)_0 : T_pM \to T_pM$$

is the identity map for the tangent space T_pM. Then the inverse mapping theorem tells us that there exists an $r_p \in \mathbb{R}^+$ such that if $U_p = B^2_{r_p}(0)$ and $V_p = \exp_p(U_p)$ then

$$\exp_p|_{U_p} : U_p \to V_p$$

is a diffeomorphism parametrizing the open subset V_p of M.

Example 7.17. Let S^2 be the unit sphere in $\mathbb{R}^3$ and $p = (1, 0, 0)$ be the north pole. Then the unit circle in the tangent plane T_pS^2 is given by

$$T_p^1S^2 = \{(0, \cos\theta, \sin\theta) | \ \theta \in \mathbb{R}\}.$$

The exponential map $\exp_p : T_pS^2 \to S^2$ of S^2 at p is defined by

$$\exp_p : s(0, \cos\theta, \sin\theta) \mapsto \cos s \cdot (1, 0, 0) + \sin s \cdot (0, \cos\theta, \sin\theta).$$

This is clearly injective on the open ball

$$B_\pi(0) = \{Z \in T_pS^2 | \ |Z| < \pi\}$$

and the geodesic

$$\gamma : s \mapsto \exp_p(s(0, \cos\theta, \sin\theta))$$

is the shortest path between p and $\gamma(r)$ as long as $r < \pi$.

Theorem 7.18. *Let M be a regular surface in $\mathbb{R}^3$. Then the geodesics are locally the shortest paths between their end points.*

Proof. For $p \in M$, choose $r > 0$ such that the restriction

$$\phi = \exp_p |_U : U \to \exp_p(U)$$

of the exponential map at p, to the open disc $U = B_r^2(0)$ in the tangent plane T_pM, is a diffeomorphism onto the image $\exp_p(U)$. Then define the metric ds^2 on U such that ϕ is an isometry, i.e. for any two vector fields Z, W on U we have

$$ds^2(X, Y) = \langle d\phi(X), d\phi(Y)\rangle.$$

It then follows from the construction of the exponential map, that the geodesics in U through the point $0 = \phi^{-1}(p)$ are exactly the lines

$$\lambda_Z : t \mapsto t \cdot Z$$

where $Z \in T_pM$.

Now let $q \in U \setminus \{0\}$ and $\lambda_q : [0, 1] \to U$ be the curve $\lambda_q : t \mapsto t \cdot q$. Further let $\sigma : [0, 1] \to U$ be any curve in U such that $\sigma(0) = 0$ and $\sigma(1) = q$. Along the curve σ we define two vector fields $\hat{\sigma}$ and $\dot{\sigma}_{\text{rad}}$ by

$$\hat{\sigma} : t \mapsto \sigma(t) \quad \text{and} \quad \dot{\sigma}_{\text{rad}} : t \mapsto \frac{ds^2(\dot{\sigma}(t), \sigma(t))}{ds^2(\sigma(t), \sigma(t))} \cdot \sigma(t).$$

Note that $\dot{\sigma}_{\text{rad}}(t)$ is the radial projection of the tangent $\dot{\sigma}(t)$ of the curve σ onto the line generated by the vector $\sigma(t)$.

Then it is easily checked that

$$|\dot{\sigma}_{\text{rad}}(t)| = \frac{|ds^2(\dot{\sigma}(t), \sigma(t))|}{|\sigma(t)|}$$

and

$$\frac{d}{dt}|\sigma(t)| = \frac{d}{dt}\sqrt{ds^2(\sigma(t), \sigma(t))} = \frac{ds^2(\dot{\sigma}(t), \sigma(t))}{|\sigma(t)|}.$$

Combining these two relations we obtain

$$|\dot{\sigma}_{\text{rad}}(t)| \geq \frac{d}{dt}|\sigma(t)|.$$

This means that

$$L(\sigma) = \int_0^1 |\dot{\sigma}(t)| dt$$

$$\begin{aligned} &\geq \int_0^1 |\dot{\sigma}_{\mathrm{rad}}(t)| dt \\ &\geq \int_0^1 \frac{d}{dt} |\sigma(t)| dt \\ &= |\sigma(1)| - |\sigma(0)| \\ &= |q| \\ &= L(\lambda_q). \end{aligned}$$

This proves that in fact λ_q is the shortest path connecting p and q. $\square$

Exercises

Exercise 7.1. Describe the geodesics on the circular cylinder

$$M = \{(x, y, z) \in \mathbb{R}^3 |\ x^2 + y^2 = 1\}.$$

Exercise 7.2. Find four different geodesics passing through the point $p = (1, 0, 0)$ on the one-sheeted hyperboloid

$$M = \{(x, y, z) \in \mathbb{R}^3 |\ x^2 + y^2 - z^2 = 1\}.$$

Exercise 7.3. Find four different geodesics passing through the point $p = (0, 0, 0)$ on the surface

$$M = \{(x, y, z) \in \mathbb{R}^3 |\ xy(x^2 - y^2) = z\}.$$

Exercise 7.4. Let $X : \mathbb{R}^2 \to \mathbb{R}^3$ be the parametrized surface in $\mathbb{R}^3$ given by

$$X(u, v) = (u \cos v, u \sin v, v).$$

Determine for which values of $\alpha \in \mathbb{R}$ the curve $\gamma_\alpha : \mathbb{R} \to M$ with

$$\gamma_\alpha(t) = X(t, \alpha t) = (t \cos(\alpha t), t \sin(\alpha t), \alpha t)$$

is a geodesic on M

Exercise 7.5. Let $X : \mathbb{R}^2 \to \mathbb{R}^3$ be the parametrized surface in $\mathbb{R}^3$ given by

$$X(u, v) = (u, v, \sin u \cdot \sin v).$$

Determine for which values of $\theta \in \mathbb{R}$ the curve $\gamma_\theta : \mathbb{R} \to M$ with

$$\gamma_\theta(t) = X(t \cdot \cos\theta, t \cdot \sin\theta)$$

is a geodesic on M

Exercise 7.6. Let $\gamma : \mathbb{R} \to \mathbb{R}^3$ be a regular curve, parametrized by arclength, with non-vanishing curvature and n, b denote the principal normal and the binormal of γ, respectively. Let $r \in \mathbb{R}^+$ such that the r-tube M around γ given by $X : \mathbb{R}^2 \to \mathbb{R}^3$ with

$$X(s, \theta) \mapsto \gamma(s) + r(\cos\theta \cdot n(s) + \sin\theta \cdot b(s))$$

is a regular surface. Show that the circles $\gamma_s = X(s, \cdot) : \mathbb{R} \to \mathbb{R}^3$ are geodesics on the surface.

Exercise 7.7. Find a proof of Proposition 7.10.

Exercise 7.8. Let M be the regular surface in $\mathbb{R}^3$ parametrized by $X : \mathbb{R} \times (-1, 1) \to \mathbb{R}^3$ with

$$\begin{aligned} X(u,v) &= 2(\cos u, \sin u, 0) + v\sin(u/2)(0,0,1) \\ &\quad +v\cos(u/2)(\cos u, \sin u, 0). \end{aligned}$$

Determine whether the curve $\gamma : \mathbb{R} \to M$ defined by

$$\gamma : t \mapsto X(t, 0)$$

is a geodesic or not. Is the surface M orientable ?

Exercise 7.9. Let M be the regular surface in $\mathbb{R}^3$ given by

$$M = \{(x, y, z) \in \mathbb{R}^3 |\ x^2 + y^2 - z^2 = 1\}.$$

Show that $v = (-1, 3, -\sqrt{2})$ is a tangent vector to M at $p = (\sqrt{2}, 0, 1)$. Let $\gamma = (\gamma_1, \gamma_2, \gamma_3) : \mathbb{R} \to M$ be the geodesic which is uniquely determined by $\gamma(0) = p$ and $\dot{\gamma}(0) = v$. Determine the value

$$\inf_{s\in\mathbb{R}} \gamma_3(s).$$

Exercise 7.10. Let M be a regular surface in $\mathbb{R}^3$ such that every geodesic $\gamma : I \to M$ is contained in a plane. Show that M is either contained in a plane or in a sphere.

Exercise 7.11. The regular surface Σ in $\mathbb{R}^3$ is parametrized by $X : \mathbb{R}^2 \to \mathbb{R}^3$ with

$$X : (u, v) = ((2 + \cos u) \cos v, (2 + \cos u) \sin v, \sin u).$$

Let $\gamma = (x, y, z) : \mathbb{R} \to \Sigma$ be the geodesic on Σ satisfying

$$\gamma(0) = (3, 0, 0) \quad \text{and} \quad \gamma'(0) = (0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}).$$

Determine the value

$$\inf_{s\in\mathbb{R}} (x^2(s) + y^2(s)).$$

CHAPTER 8

The Gauss-Bonnet Theorems

In this chapter we prove three different versions of the famous Gauss-Bonnet theorem.

Theorem 8.1. *Let M be an oriented regular surface in $\mathbb{R}^3$ with Gauss map $N : M \to S^2$. Let $X : U \to M$ be a local parametrization of M such that $X(U)$ is connected and simply connected. Let $\gamma : \mathbb{R} \to X(U)$ parametrize a regular, simple, closed and positively oriented curve on $X(U)$ by arclength. Let $Int(\gamma)$ be the interior of γ and $\kappa_g : \mathbb{R} \to \mathbb{R}$ be its geodesic curvature. If $L \in \mathbb{R}^+$ is the period of γ then*

$$\int_0^L \kappa_g(s)ds = 2\pi - \int_{Int(\gamma)} KdA,$$

where K is the Gaussian curvature of M.

Proof. Let $\{Z, W\}$ be the orthonormal basis which we obtain by applying the Gram-Schmidt process on the basis $\{X_u, X_v\}$. Along the curve $\gamma : \mathbb{R} \to X(U)$ we define the angle $\theta : \mathbb{R} \to \mathbb{R}$ such that the unit tangent vector $\dot{\gamma}$ satisfies

$$\dot{\gamma}(s) = \cos\theta(s)Z(s) + \sin\theta(s)W(s).$$

Then

$$\begin{aligned} N \times \dot{\gamma} &= N \times (\cos\theta Z + \sin\theta W) \\ &= \cos\theta(N \times Z) + \sin\theta(N \times W) \\ &= \cos\theta W - \sin\theta Z. \end{aligned}$$

and for the second derivative $\ddot{\gamma}$ we have

$$\ddot{\gamma} = \dot{\theta}(-\sin\theta Z + \cos\theta W) + \cos\theta\dot{Z} + \sin\theta\dot{W}$$

so the geodesic curvature satisfies

$$\begin{aligned} \kappa_g &= \langle N \times \dot{\gamma}, \ddot{\gamma}\rangle \\ &= \dot{\theta}\langle -\sin\theta Z + \cos\theta W, -\sin\theta Z + \cos\theta W\rangle \\ &\quad + \langle -\sin\theta Z + \cos\theta W, \cos\theta\dot{Z} + \sin\theta\dot{W}\rangle \\ &= \dot{\theta} - \langle Z, \dot{W}\rangle. \end{aligned}$$

If we integrate the geodesic curvature $\kappa_g : \mathbb{R} \to \mathbb{R}$ over one period we get

$$\begin{aligned}\int_0^L \kappa_g(s)ds &= \int_0^L \dot{\theta}(s)ds - \int_0^L \langle Z(s), \dot{W}(s)\rangle ds \\ &= \theta(L) - \theta(0) - \int_0^L \langle Z(s), \dot{W}(s)\rangle ds \\ &= 2\pi - \int_0^L \langle Z(s), \dot{W}(s)\rangle ds.\end{aligned}$$

Let $\alpha = X^{-1} \circ \gamma : \mathbb{R} \to U$ be the inverse image of the curve γ in the simply connected parameter region U. The curve α is simple, closed and positively oriented. Utilizing Lemma 6.3 and Green's theorem we now get

$$\begin{aligned}\int_0^L \langle Z(s), \dot{W}(s)\rangle ds &= \int_\gamma \langle Z, \dot{u}W_u + \dot{v}W_v\rangle ds \\ &= \int_\alpha \langle Z, W_u\rangle du + \langle Z, W_v\rangle dv \\ &= \int_{\mathrm{Int}(\alpha)} \big(\langle Z, W_v\rangle_u - \langle Z, W_u\rangle_v\big) dudv \\ &= \int_{\mathrm{Int}(\alpha)} \big(\langle Z_u, W_v\rangle + \langle Z, W_{uv}\rangle \\ &\qquad\qquad -\langle Z_v, W_u\rangle - \langle Z, W_{vu}\rangle\big) dudv \\ &= \int_{\mathrm{Int}(\alpha)} \big(\langle Z_u, W_v\rangle - \langle Z_v, W_u\rangle\big) dudv \\ &= \int_{\mathrm{Int}(\alpha)} K\sqrt{EG - F^2} dudv \\ &= \int_{\mathrm{Int}(\gamma)} K dA.\end{aligned}$$

This proves the statement. □

Corollary 8.2. *Let $\gamma : \mathbb{R} \to \mathbb{R}^2$ parametrize a regular, simple, closed and positively oriented curve by arclength. If $L \in \mathbb{R}^+$ is the period of γ then*

$$\int_0^L \kappa_g(s)ds = 2\pi,$$

where $\kappa_g : \mathbb{R} \to \mathbb{R}$ is the geodesic curvature of γ.

The reader should compare the result of Corollary 8.2 with Exercise 2.7 and Exercise 3.7.

Definition 8.3. Let M be a regular surface in $\mathbb{R}^3$. A periodic continuous curve $\gamma : \mathbb{R} \to M$ of period $L \in \mathbb{R}^+$ is said to parametrize a **simple piecewise regular polygon** on M if

(1) $\gamma(t) = \gamma(t^*)$ if and only if $(t - t^*) \in L \cdot \mathbb{Z}$,
(2) there exists a subdivision
$$0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = L$$
of the interval $[0, L]$ such that $\gamma|_{(t_i,t_{i+1})} : (t_i, t_{i+1}) \to M$ is smooth for $i = 0, \dots, n-1$, and
(3) the one-sided derivatives
$$\dot\gamma^-(t_i) = \lim_{t\to t_i^-} \frac{\gamma(t_i) - \gamma(t)}{t_i - t}, \quad \dot\gamma^+(t_i) = \lim_{t\to t_i^+} \frac{\gamma(t_i) - \gamma(t)}{t_i - t}$$
exist, are non-zero and not parallel.

The next result generalizes Theorem 8.1

Theorem 8.4. *Let M be an oriented regular surface in $\mathbb{R}^3$ with Gauss map $N : M \to S^2$. Let $X : U \to M$ be a local parametrization of M such that $X(U)$ is connected and simply connected. Let $\gamma : \mathbb{R} \to X(U)$ parametrize a positively oriented, simple piecewise regular polygon on M by arclength. Let $Int(\gamma)$ be the interior of γ and $\kappa_g : \mathbb{R} \to \mathbb{R}$ be its geodesic curvature on each regular piece. If $L \in \mathbb{R}^+$ is the period of γ then*
$$\int_0^L \kappa_g(s)ds = \sum_{i=1}^n \alpha_i - (n-2)\pi - \int_{Int(\gamma)} KdA$$
where K is the Gaussian curvature of M and $\alpha_1, \dots, \alpha_n$ are the inner angles at the n corner points.

Proof. Let $\{Z, W\}$ the orthonormal basis which we obtain by applying the Gram-Schmidt process on the basis $\{X_u, X_v\}$. Let $\mathcal{D}$ be the discrete subset of $\mathbb{R}$ corresponding to the corner points of $\gamma(\mathbb{R})$. Along the regular arcs of $\gamma : \mathbb{R} \to X(U)$ we define an angle $\theta : \mathbb{R} \setminus \mathcal{D} \to \mathbb{R}$ such that the unit tangent vector $\dot\gamma$ satisfies
$$\dot\gamma(s) = \cos\theta(s)Z(s) + \sin\theta(s)W(s).$$
We have seen earlier that in this case the geodesic curvature is given by $\kappa_g = \dot\theta - \langle Z, \dot W\rangle$ and integration over one period gives
$$\int_0^L \kappa_g(s)ds = \int_0^L \dot\theta(s)ds - \int_0^L \langle Z(s), \dot W(s)\rangle ds.$$

As a consequence of Green's theorem we have

$$\int_0^L \langle Z(s), \dot{W}(s)\rangle ds = \int_{\mathrm{Int}(\gamma)} KdA.$$

The integral over the derivative $\dot{\theta}$ splits up into integrals over each regular arc

$$\int_0^L \dot{\theta}(s)ds = \sum_{i=1}^{n} \int_{s_{i-1}}^{s_i} \dot{\theta}(s)ds$$

which measures the change of angle with respect to the orthonormal basis $\{Z, W\}$ along each arc. At each corner point the tangent jumps by the angle $(\pi - \alpha_i)$ where α_i is the corresponding inner angle. When moving around the curve once the changes along the arcs and the jumps at the corner points add up to 2π. Hence

$$2\pi = \int_0^L \dot{\theta}(s)ds + \sum_{i=1}^{n}(\pi - \alpha_i).$$

This proves the statement □

Theorem 8.5. *Let M be an orientable and compact regular surface in $\mathbb{R}^3$. If K is the Gaussian curvature of M then*

$$\int_M KdA = 2\pi\chi(M),$$

where $\chi(M)$ is the Euler characteristic of the surface.

Proof. Let $\mathcal{T} = \{T_1, \dots, T_F\}$ be a triangulation of the surface M such that each T_k is a geodesic triangle contained in the image $X_k(U_k)$ of a local parametrization $X_k : U_k \to M$. Then the integral of the Gaussian curvature K over M splits

$$\int_M KdA = \sum_{k=1}^{F} \int_{T_k} KdA$$

into the sum of integrals over each triangle $T_k \in \mathcal{T}$.

According to Theorem 8.4 we now have

$$\int_{T_k} KdA = \sum_{i=1}^{n_k} \alpha_{ki} + (2 - n_k)\pi$$

for each triangle T_k. By adding these relations we obtain

$$\int_M KdA = \sum_{k=1}^{F}\Big((2 - n_k)\pi + \sum_{i=1}^{n_k} \alpha_{ki}\Big)$$

$$
\begin{aligned}
&= 2\pi F - 2\pi E + \sum_{k=1}^{F} \sum_{i=1}^{n_k} \alpha_{ki} \\
&= 2\pi(F - E + V).
\end{aligned}
$$

This proves the statement. □

Exercises

Exercise 8.1. Let M be a regular surfaces in $\mathbb{R}^3$ diffeomorphic to the torus. Show that there exists a point $p \in M$ where the Gaussian curvature $K(p)$ is negative.

Exercise 8.2. The regular surface M in $\mathbb{R}^3$ is given by

$$M = \{(x, y, z) \in \mathbb{R}^3 |\ x^2 + y^2 - z^2 = 1 \ \text{ and } \ -1 < z < 1\}.$$

Determine the value of the integral

$$\int_M K dA,$$

where K is the Gaussian curvature of M.

Exercise 8.3. For $r \in \mathbb{R}^+$ let the surface Σ_r be given by

$$\Sigma_r = \{(x, y, z) \in \mathbb{R}^3 |\ z = \cos\sqrt{x^2 + y^2},\ x^2 + y^2 < r^2,\ x, y > 0\}.$$

Determine the value of the integral

$$\int_{\Sigma_r} K dA,$$

where K is the Gaussian curvature of Σ_r.

Exercise 8.4. For $n \geq 1$ let M_n be the regular surface in $\mathbb{R}^3$ given by

$$M_n = \{(x, y, z) \in \mathbb{R}^3 |\ x^2 + y^2 = (1 + z^{2n})^2,\ 0 < z < 1\}.$$

Determine the value of the integral

$$\int_{M_n} K dA,$$

where K is the Gaussian curvature of M_n.