An Introduction to Trigonometric Functions

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CHAPTER CONNECTIONS

While the beauty and color of rainbows have been admired for centuries, understanding the physics of rainbows is of fairly recent origin. Answers to questions regarding their seven-color constitution, the order the colors appear, the circular shape of the bow, and their relationship to moisture are all deeply rooted in mathematics. The relationship between light and color can be understood in terms of trigonometry, with questions regarding the apparent height of the rainbow, as well as the height of other natural and man-made phenomena, found using the trigonometry of right triangles.

▶ This application appears as Exercise 87 in Section 6.6

Connections to Calculus

The trigonometry of right triangles plays an important role in a study of calculus, as we seek to simplify certain expressions, or convert from one system of graphing to another. The *Connections to Calculus* feature following Chapter 6 offers additional practice and insight in preparation for a study of calculus.

6.1 Angle Measure, Special Triangles, and Special Angles

LEARNING OBJECTIVES

In Section 6.1 you will see how we can:

- ❑ **A.** Use the vocabulary associated with a study of angles and triangles
- ❑ **B.** Find fixed ratios of the sides of special triangles
- ❑ **C.** Use radians for angle measure and compute circular arc length and area using radians
- ❑ **D.** Convert between degrees and radians for nonstandard angles
- ❑ **E.** Solve applications involving angular velocity and linear velocity

Trigonometry, like her sister science geometry, has its origins deeply rooted in the practical use of measurement and proportion. In this section, we'll look at the fundamental concepts on which trigonometry is based, which we hope will lead to a better understanding and a greater appreciation of the wonderful study that trigonometry has become.

A. Angle Measure in Degrees

Beginning with the common notion of a straight line, a **ray** is a half line, or all points extending from a single point, in a single direction. An **angle** is the joining of two rays at a common endpoint called the **vertex.** Arrowheads are used to indicate the half lines continue forever and can be extended if necessary. Angles can be named using a single letter at the vertex, the letters from the rays forming the sides, or by a single Greek letter, with the favorites being **alpha** α, **beta** β, **gamma** γ, and **theta** θ. The symbol $\angle$ is often used to designate an angle (see Figure 6.1).

Figure 6.1

Euclid (325–265 B.C.), often thought of as the father of geometry, described an angle as "the inclination of one to another of two lines which meet in a plane." This *amount of inclination* gives rise to the common notion of angle measure in degrees, often measured with a semicircular **protractor** like the one shown in Figure 6.2. The notation for degrees is the ° symbol. By definition 1° is $\frac{1}{360}$ of a full rotation, so this protractor can be used to measure any angle from 0° (where the two rays are coincident), to 180° (where they form a straight line). An angle measuring 180° is called a **straight angle,** while an angle that measures 90° is called a **right angle.** An angle greater than 0° but less than 90° is called an **acute** angle (Figure 6.3). An angle greater than 90° but less than 180° is called an **obtuse** angle (Figure 6.4). Two angles that sum to 90° are said to be **complementary** (Figure 6.5), while two that sum to 180° are **supplementary** angles (Figure 6.6). Recall the "¬" symbol represents a 90° angle.

Figure 6.2

Figure 6.3

Figure 6.4

Figure 6.5

$\alpha + \beta = 90°$

α and β are complementary
or
α and β are complements

Figure 6.6

$\theta + \gamma = 180°$

θ and γ are supplementary
or
θ and γ are supplements

EXAMPLE 1 ▶ **Finding the Complement and Supplement of an Angle**

Determine the measure of each angle described.

a. the complement of a 57° angle **b.** the supplement of a 132° angle

c. the measure of angle θ shown in the figure

Solution ▶

a. The complement of 57° is 33° since
$90° - 57° = 33° \Rightarrow 33° + 57° = 90°$.

b. The supplement of 132° is 48° since
$180° - 132° = 48° \Rightarrow 48° + 132° = 180°$.

c. Since θ and 39° are complements,
$\theta = 90° - 39° = 51°$.

Now try Exercises 7 through 10 ▶

In the study of trigonometry and its applications, degree measure alone does not provide the precision we often need. For very fine measurements, each degree is divided into 60 smaller parts called **minutes,** and each minute into 60 smaller parts called **seconds.** This means that a minute is $\frac{1}{60}$ of a degree, while a second is $\frac{1}{60}$ of a minute or $\frac{1}{3600}$ of a degree. The angle whose measure is "sixty-one degrees, eighteen minutes, and forty-five seconds" is written as $61° \, 18' \, 45''$. The degrees-minutes-seconds (DMS) method of measuring angles is commonly used in aviation and navigation, while in other areas **decimal degrees** such as 61.3125° are preferred. You will sometimes be asked to convert between the two.

EXAMPLE 2 ▶ **Converting Between Decimal Degrees and Degrees/Minutes/Seconds**

Convert as indicated.

a. $61° \, 18' \, 45''$ to decimal degrees

b. 142.2075° to DMS

Solution ▶

a. Since $1' = \frac{1}{60}$ of a degree and $1'' = \frac{1}{3600}$ of a degree, we have

$$61°18'45'' = \left[61 + 18\left(\frac{1}{60}\right) + 45\left(\frac{1}{3600}\right)\right]^{\circ}$$
$$= 61.3125°$$

b. For the conversion to DMS we write the fractional part separate from the whole number part to compute the number of degrees and minutes represented, then repeat the process to find the number of degrees, minutes, and seconds.

$142.2075°$	$= 142° + 0.2075°$	separate fractional part from the whole
	$= 142° + 0.2075(60)'$	$0.2075° = 0.2075 \cdot 1°$; substitute $60'$ for $1°$
	$= 142° \, 12.45'$	result in degrees and minutes
	$= 142° \, 12' + 0.45'$	separate fractional part from the whole
	$= 142° \, 12' + 0.45(60)''$	$0.45' = 0.45 \cdot 1'$; substitute $60''$ for $1'$
	$= 142° \, 12' \, 27''$	result in degrees, minutes, and seconds

Now try Exercises 11 through 26 ▶

The results from Example 2 can be quickly verified using a graphing calculator. Make sure the calculator is in degree MODE, as shown in Figure 6.7. For part (a), the degree and minute symbols can be found in the 2nd APPS **(ANGLE)** menu, while the seconds symbol is accessed with the keystrokes ALPHA + **(″).** Pressing ENTER will give

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the decimal form of the angle (see Figure 6.8). With the keystrokes 142.2075 2nd APPS 4 ENTER, the results for part (b) are likewise confirmed in Figure 6.8.

Figure 6.7

Figure 6.8

A. You've just seen how we can use the vocabulary associated with a study of angles and triangles

B. Triangles and Properties of Triangles

A triangle is a closed plane figure with three straight sides and three angles. It is customary to name each angle using a capital letter and the side opposite the angle using the corresponding lowercase letter. Regardless of their size or orientation, triangles have the following properties.

Properties of Triangles

Given triangle ABC with sides a, b, and c respectively,

I. The sum of the angles is 180°:
$A + B + C = 180°$

II. The combined length of any two sides exceeds that of the third side:
$a + b > c$, $a + c > b$, and $b + c > a$.

III. Larger angles are opposite larger sides:
If $C > B$, then $c > b$.

Two triangles are **similar triangles** if corresponding angles are equal, meaning for $\triangle ABC$ and $\triangle DEF$, $A = D$, $B = E$, and $C = F$. Since antiquity it's been known that *if two triangles are similar, corresponding sides are proportional* (corresponding sides are those opposite the equal angles from each triangle). This relationship, used extensively by the engineers of virtually all ancient civilizations, is very important to our study of trigonometry. Example 3 illustrates how proportions and similar triangles are often used.

EXAMPLE 3 ▶ **Using Similar Triangles to Find Heights**

To estimate the height of a flagpole, Carrie reasons that $\triangle ABC$ formed by her height and shadow must be similar to $\triangle DEF$ formed by the flagpole. She is 5 ft 6 in. tall and casts an 8-ft shadow, while the shadow of the flagpole measures 44 ft. How tall is the pole?

Solution ▶ Let H represent the height of the flagpole.

$$\frac{\text{Height}}{\text{Shadow Length}}: \frac{5.5}{8} = \frac{H}{44} \quad \text{original proportion, } 5'\,6'' = 5.5 \text{ ft}$$

$$8H = 242 \quad \text{cross multiply}$$

$$H = 30.25 \quad \text{result}$$

The flagpole is $30\frac{1}{4}$ ft tall (30 ft 3 in.).

Now try Exercises 27 through 34 ▶

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Figure 6.9 shows Carrie standing along the shadow of the flagpole, again illustrating the proportional relationships that exist. Early mathematicians recognized the power of these relationships, realizing that if the triangles were similar and the related fixed proportions were known, they had the ability to find mountain heights, the widths of lakes, and even the ability to estimate the distance to the Sun and Moon. In support of this search, two special triangles were used. These triangles, commonly called **45-45-90** and **30-60-90** triangles, are *special* because no estimation or interpolation is needed to find the relationships between their sides. For the first, consider an isosceles right triangle—a right triangle with two equal sides and two 45° angles (Figure 6.10). After naming the equal sides x and the hypotenuse h, we can apply the Pythagorean theorem to find a relationship between the sides and the hypotenuse in terms of x.

Figure 6.9

Figure 6.10

$$c^2 = a^2 + b^2 \quad \text{Pythagorean theorem}$$
$$h^2 = x^2 + x^2 \quad \text{substitute } x \text{ for } a, x \text{ for } b, \text{ and } h \text{ for } c$$
$$= 2x^2 \quad \text{combine like terms}$$
$$h = \sqrt{2}x \quad \text{solve for } h\,(h > 0)$$

WORTHY OF NOTE

Recall that the Pythagorean theorem states that for any right triangle, the sum of the squares of the two legs, is equal to the square of the hypotenuse: $a^2 + b^2 = c^2$.

This important result is summarized in the following box.

45-45-90 Triangles

Given a 45-45-90 triangle with one leg of length x, the relationship between the side lengths is:

$$1x : 1x : \sqrt{2}\,x$$

(1) The two legs are equal;

(2) The hypotenuse is $\sqrt{2}$ times the length of either leg;

$$a = b \quad c = \sqrt{2}\,a \quad c = \sqrt{2}\,b$$

The proportional relationship for a 30-60-90 triangle is developed in **Exercise 107,** and the result is stated here.

30-60-90 Triangles

Given a 30-60-90 triangle with the shortest side of length x, the relationship between the side lengths is:

$$1x : \sqrt{3}\,x : 2x$$

(1) The hypotenuse is 2 times the length of the shorter leg;

(2) The longer leg is $\sqrt{3}$ times the length of the shorter leg;

$$b = \sqrt{3}\,a \qquad c = 2a$$

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EXAMPLE 4 ▶ **Applications of 45-45-90 Triangles: The Height of a Kite**

Seeking to reduce energy consumption and carbon emissions, many shipping companies are harnessing wind power using specialized kites. If such a kite is flying at a 45° angle, and 216 m of cable has been let out, what is the height h of the kite's bridle point B (see illustration)?

Solution ▶ In the 45-45-90 triangle, the hypotenuse is formed by the cable: $c = 216$ m. The height of the kite will be the length of one of the legs.

$$c = \sqrt{2}\,a \quad \text{45-45-90 triangle property (2)}$$
$$216 = \sqrt{2}\,a \quad \text{substitute 216 for } c$$
$$\frac{216}{\sqrt{2}} = a \quad \text{solve for } a$$
$$a = \frac{216\sqrt{2}}{2} \quad \text{rewrite equation and rationalize denominator}$$
$$= 108\sqrt{2} \quad \text{simplify: exact form}$$
$$\approx 152.7 \text{ m} \quad \text{approximate form (rounded to tenths)}$$

The bridle point of the kite is $108\sqrt{2}$ m (about 152.7 m) higher than the deck of the ship.

Now try Exercises 35 and 36 ▶

☑ **B.** You've just seen how we can find fixed ratios of the sides of special triangles

C. Angle Measure in Radians; Arc Length and Area

As an alternative to viewing angles as "the amount of inclination" between two rays, angle measure can also be considered as the *amount of rotation* from a fixed ray called the **initial side,** to a rotated ray called the **terminal side.** This enables angle measure to be free from the context of a triangle, and allows for positive or negative angles, depending on the direction of rotation. Angles formed by a counterclockwise rotation are considered **positive angles,** and angles formed by a clockwise rotation are **negative angles** (see Figure 6.11). We can then name an angle of any size, including those greater than 360° where the amount of rotation exceeds one revolution. See Figures 6.12 through 6.16.

Figure 6.11

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Figure 6.12

Figure 6.13

Figure 6.14

Figure 6.15

Figure 6.16

Figure 6.17

Note in Figure 6.17 that angle $\theta = 330°$ and angle $\alpha = -30°$ share the same initial and terminal sides and are called **coterminal angles.** Coterminal angles will always differ by multiples of 360°, meaning that for any integer k, angles θ and $\theta + 360°k$ will be coterminal.

EXAMPLE 5 ▶ **Finding Coterminal Angles**

Find two positive angles and two negative angles that are coterminal with 60°.

▼ Algebraic Solution

For $k = -2$,
$$60° + 360°(-2) = -660°.$$
For $k = -1$,
$$60° + 360°(-1) = -300°.$$
For $k = 1$,
$$60° + 360°(1) = 420°.$$
For $k = 2$,
$$60° + 360°(2) = 780°.$$

▼ Graphical Solution

With $\theta = 60°$, we enter $60 + 360X$ as Y_1 on the Y= screen of a graphing calculator and use the **TABLE** feature. The table shown starts at -3 with a step size of 1 and includes all the coterminal angles found in the algebraic solution.

As the table indicates, many other answers are possible.

Now try Exercises 37 through 40 ▶

An angle is said to be in **standard position** in the rectangular coordinate system if its vertex is at the origin and the initial side is along the positive x-axis. In standard position, the terminal sides of 90°, 180°, 270°, and 360° angles coincide with one of the axes and are called **quadrantal angles.** To further develop ideas related to angles, we use a **central circle,** which is a circle in the xy-plane with its center at the origin. A **central angle** is an angle whose vertex is at the center of the circle. For a central angle θ intersecting the circle at points B and C, we say circular arc BC (denoted $\widehat{BC}$), **subtends**

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Figure 6.18

Figure 6.19

$\angle BAC$, as shown in Figure 6.18. The relationship between an arc and the angle it subtends leads us to a new unit for measuring angles that will simplify many formulas and procedures. The letter s is commonly used to represent arc length, and if we define **1 radian** (abbreviated *rad*) to be the measure of an angle subtended by an arc equal in length to the radius, then $\theta = 1$ rad when $s = r$ (see Figure 6.19).

Radians

If central angle θ is subtended by an arc that is equal in length to the radius, then

$$\theta = 1 \textit{ radian}.$$

WORTHY OF NOTE

Using the properties of ratios, we note that since both r (radius) and s (arc length) are measured in like units, the units actually "cancel" making radians a unitless measure: $\theta = \frac{s \text{ units}}{r \text{ units}} = \frac{s}{r}$.

We can then find the radian measure of any central angle by dividing the length of the subtended arc by r: $\frac{s}{r} = \theta$ radians. Multiplying both sides by r gives a formula for the length of any arc subtended on a circle of radius r: $s = r\theta$ if θ is in radians.

Arc Length

If θ is a central angle in a circle of radius r, then the length of the subtended arc s is

$$s = r\theta,$$

provided θ is expressed in radians.

EXAMPLE 6 ▶ **Using the Formula for Arc Length**

If the circle in Figure 6.19 has a radius of $r = 10$ cm, what is the length of the arc subtended by an angle of 3.5 rad?

Solution ▶ Using the formula $s = r\theta$ with $r = 10$ and $\theta = 3.5$ gives

$$s = 10(3.5) \quad \text{substitute 10 for } r \text{ and 3.5 for } \theta$$
$$s = 35 \quad \text{result}$$

The subtended arc has a length of 35 cm.

Now try Exercises 41 through 50 ▶

Using a central angle θ measured in radians, we can also develop a formula for the **area of a circular sector** (a pie slice) using a proportion. Recall the circumference of a circle is $C = 2\pi r$. While you may not have considered this before, note the formula can be written as $C = 2\pi \cdot r$, which implies that the radius can be wrapped around

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the circumference of the circle $2\pi \approx 6.28$ times, as illustrated in Figure 6.20. This shows the radian measure of a full 360° rotation is 2π: $2\pi\ rad = 360°$. This can be verified as before, using the relation θ radians $= \dfrac{s}{r} = \dfrac{2\pi r}{r} = 2\pi$. The ratio of the area of a sector to the total area will be identical to the ratio of the subtended angle to one full rotation. Using $\mathcal{A}$ to represent the area of the sector, we have $\dfrac{\mathcal{A}}{\pi r^2} = \dfrac{\theta}{2\pi}$ and solving for $\mathcal{A}$ gives $\mathcal{A} = \dfrac{1}{2}r^2\theta$.

Figure 6.20

Area of a Sector

If θ is a central angle in a circle of radius r, the area of the sector formed is

$$\mathcal{A} = \frac{1}{2}r^2\theta,$$

provided θ is expressed in radians.

EXAMPLE 7 ▶ **Using the Formula for the Area of a Sector**

What is the area of the circular sector formed by a central angle of $\dfrac{3\pi}{4}$, if the radius of the circle is 72 ft? Round to tenths.

Solution ▶ Using the formula $\mathcal{A} = \dfrac{1}{2}r^2\theta$ we have

$$\mathcal{A} = \left(\frac{1}{2}\right)(72)^2\left(\frac{3\pi}{4}\right) \quad \text{substitute 72 for } r\text{, and } \frac{3\pi}{4} \text{ for } \theta$$

$$= 1944\pi \text{ ft}^2 \quad \text{result}$$

The area of this sector is approximately 6107.3 ft^2, as shown in the figure.

Now try Exercises 51 through 62 ▶

C. You've just seen how we can use radians for angle measure and compute circular arc length and area using radians

D. Converting Between Degrees and Radians

In addition to its use in developing formulas for arc length and the area of a sector, the relation 2π rad $= 360°$ enables us to state the radian measure of standard angles using a simple division. For π rad $= 180°$ we have

division by 2: $\dfrac{\pi}{2} = 90°$ division by 3: $\dfrac{\pi}{3} = 60°$

division by 4: $\dfrac{\pi}{4} = 45°$ division by 6: $\dfrac{\pi}{6} = 30°$.

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WORTHY OF NOTE

We will often use the convention that unless degree measure is explicitly implied or noted with the ° symbol, radian measure is being used. In other words,

$\theta = \frac{\pi}{2}$, $\theta = 2$, and $\theta = 32.76$

all indicate angles measured in radians.

The radian measure of these standard angles play a major role in this chapter, and you are encouraged to become very familiar with them (see Figure 6.21). Additional conversions can quickly be found using multiples of these four. For example, multiplying both sides of $\frac{\pi}{3} = 60°$ by two gives $\frac{2\pi}{3} = 120°$. The relationship $\pi = 180°$ also gives the factors needed for converting from degrees to radians or from radians to degrees, even if θ is a nonstandard angle. Dividing by π we have $1 = \frac{180°}{\pi}$, while division by 180° shows $1 = \frac{\pi}{180°}$. Multiplying a given angle by the appropriate conversion factor gives the equivalent measure.

Figure 6.21

Degrees/Radians Conversion Factors

To convert from radians to degrees: multiply by $\frac{180°}{\pi}$.

To convert from degrees to radians: multiply by $\frac{\pi}{180°}$.

Most graphing calculators are programmed to compute conversions involving angle measure. This is accomplished using the "°" feature for degree-to-radian conversions *while in radian* MODE, and the "*r*" feature for radian-to-degree conversions *while in degree* MODE. Both are found on the **ANGLE** submenu located at 2nd APPS.

EXAMPLE 8 ▶ **Converting Between Radians and Degrees**

Convert each angle as indicated:

a. $-75°$ to radians. **b.** $\frac{\pi}{24}$ to degrees.

Algebraic Solution ▶ **a.** For degrees to radians, use the conversion factor $\frac{\pi}{180°}$.

$$-75° = -75° \cdot \frac{\pi}{180°} = -\frac{5\pi}{12} \qquad \frac{75}{180} = \frac{5}{12}$$

b. For radians to degrees, use the conversion factor $\frac{180°}{\pi}$.

$$\frac{\pi}{24} = \frac{\pi}{24} \cdot \left(\frac{180°}{\pi}\right) = 7.5° \qquad \frac{\pi}{\pi} = 1, \frac{180}{24} = 7.5$$

Graphical Solution ▶ **a.** While in radian MODE, enter -75 on the calculator's home screen and use the keystrokes 2nd APPS ENTER ENTER. The result is shown in the figure, along with a calculator approximation of $-\frac{5\pi}{12}$ [from the algebraic solution to part (a)].

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b. After changing the calculator to degree MODE, entering $(\pi/24)$ 2nd APPS 3 ENTER will perform the radians-to-degrees conversion, as seen in the third entry of the figure. Note that here the parentheses around $\pi/24$ are necessary as both the "r" and "°" operators follow the order of operations for exponents.

Change to degree mode

Now try Exercises 62 through 86 ▶

One example where these conversions are useful is in applications involving longitude and latitude (see Figure 6.22). The **latitude** of a fixed point on the Earth's surface tells how many degrees north or south of the equator the point is, as measured from the center of the Earth. The **longitude** of a fixed point on the Earth's surface tells how many degrees east or west of the Prime Meridian (through Greenwich, England) the point is, as measured along the equator to the longitude line going through the point. For example, the city of New Orleans, Louisiana, is located at 30°N latitude, 90°W longitude (see Figure 6.22).

Figure 6.22

EXAMPLE 9 ▶ **Applying the Arc Length Formula: Distances Between Cities**

The cities of Quito, Ecuador, and Macapá, Brazil, both lie very near the equator, at a latitude of 0°. However, Quito is at approximately 78° west longitude, while Macapá is at 51° west longitude. Assuming the Earth has a radius of 3960 mi, how far apart are these cities?

Solution ▶ First we note that $(78 - 51)° = 27°$ of longitude separate the two cities. Using the conversion factor $1 = \dfrac{\pi}{180°}$ we find the equivalent radian measure

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WORTHY OF NOTE

Note that $r = 3960$ mi was used because Quito and Macapá *are both on the equator.* For other cities sharing the same latitude but not on the equator, the radius of the Earth *at that latitude* must be used. **See Section 6.6, Exercise 93.**

is $27°\left(\frac{\pi}{180°}\right) = \frac{3\pi}{20}$. The arc length formula gives

$$\begin{aligned} s &= r\theta && \text{arc length formula; } \theta \text{ in radians} \\ &= 3960\left(\frac{3\pi}{20}\right) && \text{substitute 3960 for } r \text{ and } \frac{3\pi}{20} \text{ for } \theta \\ &= 594\pi && \text{result} \end{aligned}$$

Quito and Macapá are approximately 1866 mi apart (see *Worthy of Note* in the margin).

D. You've just seen how we can convert between degrees and radians for nonstandard angles

Now try Exercises 89 and 90 ▶

E. Angular and Linear Velocity

The **angular velocity** of an object is defined as the *amount of rotation* per unit time. Here, we often use the symbol ω (omega) to represent the angular velocity, and θ to represent the angle through which the terminal side has rotated, measured in radians: $\omega = \frac{\theta}{t}$. For instance, a Ferris wheel turning at 10 revolutions per minute has an angular velocity of

$$\begin{aligned} \omega &= \frac{10 \text{ revolutions}}{1 \text{ min}} && \omega = \frac{\theta}{t} \\ &= \frac{10(2\pi)}{1 \text{ min}} && \text{substitute } 2\pi \text{ for 1 revolution} \\ &= \frac{20\pi \text{ rad}}{1 \text{ min}} && 10(2) = 20 \end{aligned}$$

WORTHY OF NOTE

Generally speaking, the *velocity* of an object is its change in position per unit time, and can be either positive or negative. The *rate* or *speed* of an object is the magnitude of the velocity, regardless of direction.

The **linear velocity** of an object is defined as a *change of position* or *distance traveled* per unit time: $V = \frac{s}{t}$. In the context of angular motion, we consider the distance traveled by a point on the circumference of the Ferris wheel, *which is equivalent to the length of the resulting arc s*. This formula can be written directly in terms of the angular velocity since $s = r\theta$: $V = \frac{r\theta}{t} = r\left(\frac{\theta}{t}\right) = r\omega$.

Angular and Linear Velocity

Given a circle of radius r with point P on the circumference, and central angle θ in radians with P on the terminal side. If P moves along the circumference at a uniform rate:

1. The rate at which θ changes is called the *angular velocity* ω,
$$\omega = \frac{\theta}{t}.$$
2. The rate at which the position of P changes is called the *linear velocity* V,
$$V = \frac{r\theta}{t} \quad \Rightarrow \quad V = r\omega.$$

EXAMPLE 10 ▶ **Using Angular Velocity to Determine Linear Velocity**

The wheels on a racing bicycle have a radius of 13 in. How fast is the cyclist traveling in miles per hour, if the wheels are turning at 300 rpm?

Solution ▶ Note that $\omega = \dfrac{300 \text{ rev}}{1 \text{ min}} = \dfrac{300(2\pi)}{1 \text{ min}} = \dfrac{600\pi}{1 \text{ min}}$.

Using the formula $V = r\omega$ gives a linear velocity of

$$V = (13 \text{ in.})\frac{600\pi}{1 \text{ min}} \approx \frac{24{,}504.4 \text{ in.}}{1 \text{ min}}.$$

To convert this to miles per hour we convert minutes to hours (1 hr = 60 min) and inches to miles (1 mi = 5280 × 12 in.):

$$\left(\frac{24{,}504.4 \cancel{\text{in.}}}{1 \cancel{\text{min}}}\right)\left(\frac{60 \cancel{\text{min}}}{1 \text{ hr}}\right)\left(\frac{1 \text{ mi}}{63{,}360 \cancel{\text{in.}}}\right) \approx 23.2 \text{ mph.}$$

The bicycle is traveling about 23.2 mph.

Now try Exercises 93 through 98 ▶

To help understand the relationship between angular velocity and linear velocity, consider two large rollers, both with a radius of 1.6 ft, used to move an industrial conveyor belt. The rollers both have circumferences of $C = 2\pi(1.6 \text{ ft}) \approx 10.05$ ft, meaning that for each revolution of the rollers, an object on the belt will move 10.05 ft (from P_1 to P_2).

E. You've just seen how we can solve applications involving angular velocity and linear velocity using radians

If the rollers are rotating at 20 rpm, an object on the belt (or a point on the circumference of a roller), will be moving at a rate of $20 \cdot 10.05 = 201$ ft/min (about 2.3 miles per hour). Take the time to verify this using the formula $V = r\omega$.

6.1 EXERCISES

▶ CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section if needed.

1. _____ angles sum to 90°. Supplementary angles sum to ____°. Acute angles are _____ than 90°. Obtuse angles are _____ than 90°.

2. The expression "theta equals two degrees" is written _____ using the "°" notation. The expression, "theta equals two radians" is simply written _____.

3. The formula for arc length is $s =$ ____. The area of a sector is $\mathcal{A} =$ _____. For both formulas, θ must be in _____.

4. If θ is not a special angle, multiply by _____ to convert radians to degrees. To convert degrees to radians, multiply by _____.

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5. Discuss/Explain the difference between angular velocity and linear velocity. In particular, why does one depend on the radius while the other does not?

6. Discuss/Explain the difference between 1° and 1 radian. Exactly what is a radian? Without any conversions, explain why an angle of 4 rad terminates in QIII.

▶ DEVELOPING YOUR SKILLS

Determine the measure of the angle described.

7. **a.** The complement of a 12.5° angle
   b. The supplement of a 149.2° angle

8. **a.** The complement of a 62.4° angle
   b. The supplement of a 74.7° angle

9. The measure of angle α

10. The measure of angle β

Convert from DMS (degree/minute/seconds) notation to decimal degrees. Use a graphing calculator to verify your results.

11. 42° 30′
12. 125° 45′
13. 67° 33′ 18″
14. 9° 15′ 36″
15. 285° 00′ 09″
16. 312° 00′ 54″
17. 45° 45′ 45″
18. 30° 30′ 27″

Convert the angles from decimal degrees to DMS (degree/minute/sec) notation. Use a graphing calculator to verify your results.

19. 20.25°
20. 40.75°
21. 67.307°
22. 83.516°
23. 275.33°
24. 330.45°
25. 5.4525°
26. 12.3275°

Are the triangles shown possible? Why/why not?

27.

28.

Determine the measure of the angle indicated.

29. angle α

30. angle β

31. $\angle A$

32. $\angle B$

33. **Similar triangles:** A helicopter is hovering over a crowd of people watching a police standoff in a parking garage across the street. Stewart notices the shadow of the helicopter is lagging approximately 50 m behind a point directly below the helicopter. If he is 2 m tall and casts a shadow of 1.6 m at this time, what is the altitude of the helicopter?

34. **Similar triangles:** Near Fort Macloud, Alberta (Canada), there is a famous cliff known as *Head Smashed in Buffalo Jump*. The area is now a Canadian National Park, but at one time the Native Americans hunted buffalo by steering a part of the herd over the cliff. While visiting the park late one afternoon, Denise notices that its shadow reaches 201 ft from the foot of the cliff, at the same time she is casting a shadow of 12′1″. If Denise is 5′4″. tall, what is the height of the cliff?

Solve using special triangles. Answer in both exact and approximate form.

35. **Special triangles:** A ladder-truck arrives at a high-rise apartment complex where a fire has broken out. If the maximum length the ladder extends is 82 ft and the angle of inclination is 45°, how high up the side of the building does the ladder reach? Assume the ladder is mounted atop a 10 ft high truck.

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36. Special triangles: A heavy-duty ramp is used to winch heavy appliances from street level up to a warehouse loading dock. If the ramp is 7.5 ft high and the incline is 15 ft long, (a) what angle α does the dock make with the street? (b) How long is the base of the ramp?

Use the TABLE feature on a graphing calculator to find two positive angles and two negative angles that are coterminal with the angle given. Answers may vary.

37. $\theta = 75°$ **38.** $\theta = 225°$

39. $\theta = -45°$ **40.** $\theta = -60°$

Use the formula for arc length to find the value of the unknown quantity: $s = r\theta$. Round to the nearest hundredth when necessary.

41. $\theta = 3.5$; $r = 280$ m

42. $\theta = 2.3$; $r = 129$ cm

43. $s = 2007$ mi; $r = 2676$ mi

44. $s = 4435.2$ km; $r = 12{,}320$ km

45. $\theta = \dfrac{3\pi}{4}$; $s = 4146.9$ yd

46. $\theta = \dfrac{11\pi}{6}$; $s = 28.8$ nautical miles

47. $\theta = \dfrac{4\pi}{3}$; $r = 2$ mi

48. $\theta = \dfrac{3\pi}{2}$; $r = 424$ in.

49. $\theta = \dfrac{16\pi}{9}$; $s = 52.5$ km

50. $\theta = 1.225\pi$; $s = 7627$ m

Use the formula for area of a circular sector to find the value of the unknown quantity: $\mathcal{A} = \frac{1}{2}r^2\theta$. Round to the nearest tenth when necessary.

51. $\theta = 5$; $r = 6.8$ km

52. $\theta = 3$; $r = 45$ mi

53. $\mathcal{A} = 1080$ mi^2; $r = 60$ mi

54. $\mathcal{A} = 437.5$ cm^2; $r = 12.5$ cm

55. $\theta = \dfrac{7\pi}{6}$; $\mathcal{A} = 16.5$ m^2

56. $\theta = \dfrac{19\pi}{12}$; $\mathcal{A} = 753$ cm^2

Find the angle, radius, arc length, and/or area as needed, until all values are known.

57.

58.

59.

60.

61.

62.

Convert the following degree measures to radians in exact form, without the use of a calculator.

63. $\theta = 360°$ **64.** $\theta = 180°$

65. $\theta = 45°$ **66.** $\theta = 30°$

67. $\theta = 210°$ **68.** $\theta = 330°$

69. $\theta = -120°$ **70.** $\theta = -225°$

Convert each degree measure to radians. Round to the nearest ten-thousandth.

71. $\theta = 27°$ **72.** $\theta = 52°$

73. $\theta = 227.9°$ **74.** $\theta = 154.4°$

Convert each radian measure to degrees, without the use of a calculator.

75. $\theta = \dfrac{\pi}{3}$ **76.** $\theta = \dfrac{\pi}{4}$

77. $\theta = \dfrac{\pi}{6}$ **78.** $\theta = \dfrac{\pi}{2}$

79. $\theta = \dfrac{2\pi}{3}$ **80.** $\theta = \dfrac{5\pi}{6}$

81. $\theta = 4\pi$ **82.** $\theta = 6\pi$

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Convert each radian measure to degrees. Round to the nearest tenth.

83. $\theta = \dfrac{11\pi}{12}$ **84.** $\theta = \dfrac{17\pi}{36}$ **85.** $\theta = 3.2541$ **86.** $\theta = 1.0257$

▶ WORKING WITH FORMULAS

87. Relationships in a right triangle: $h = \dfrac{ab}{c}$, $m = \dfrac{b^2}{c}$, **and** $n = \dfrac{a^2}{c}$

Given $\angle C$ is a right angle, and h is the altitude of $\triangle ABC$, then h, m, and n can all be expressed directly in terms of a, b, and c by the relationships shown here. Compute the value of h, m, and n for a right triangle with sides of 8, 15, and 17 cm.

88. The height of an equilateral triangle: $H = \dfrac{\sqrt{3}}{2}S$

Given an equilateral triangle with sides of length S, the height of the triangle is given by the formula shown. Once the height is known the area of the triangle can easily be found (also see Exercise 87). The Gateway Arch in St. Louis, Missouri, is actually composed of stainless steel sections that are equilateral triangles. At the base of the arch the length of the sides is 54 ft. The smallest cross section at the top of the arch has sides of 17 ft. Find the areas of these cross sections.

▶ APPLICATIONS

89. Distance between cities: The city of Pittsburgh, Pennsylvania, is directly north of West Palm Beach, Florida. Pittsburg is at 40.3° north latitude, while West Palm Beach is at 26.4° north latitude. Assuming the Earth has a radius of 3960 mi, how far apart are these cities?

90. Width of Africa: Both Libreville, Gabon, and Jamame, Somalia, lie near the equator, but on opposite ends of the African continent. If Libreville is at 9.3° east longitude and Jamame is 42.5° east longitude, how wide is the continent of Africa at the equator?

91. Watering a lawn: A water sprinkler is set to shoot a stream of water a distance of 12 m and rotate through an angle of 40°. (a) What is the area of the lawn it waters? (b) For $r = 12$ m, what angle is required to water twice as much area? (c) For $\theta = 40°$, what range for the water stream is required to water twice as much area?

92. Motion detectors: A motion detector can detect movement up to 25 m away through an angle of 75°. (a) What area can the motion detector monitor? (b) For $r = 25$ m, what angle is required to monitor 50% more area? (c) For $\theta = 75°$, what range is required for the detector to monitor 50% more area?

93. Riding a round-a-bout: At the park two blocks from our home, the kids' round-a-bout has a radius of 56 in.

About the time the kids stop screaming, "Faster, Daddy, faster!" I estimate the round-a-bout is turning at $\frac{3}{4}$ revolutions per second. (a) What is the related angular velocity in radians per second? (b) What is the linear velocity (in miles per hour) of Eli and Reno, who are "hanging on for dear life" at the rim of the round-a-bout?

94. Carnival rides: At carnivals and fairs, the *Gravity Drum* is a popular ride. People stand along the wall of a circular drum with radius 12 ft, which begins spinning very fast, pinning them against the wall. The drum is then turned on its side by an armature, with the riders screaming and squealing with delight. As the drum is raised to a near-vertical position, it is spinning at a rate of 35 rpm. (a) What is the angular velocity in radians per minute? (b) What is the linear velocity (in miles per hour) of a person on this ride?

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95. Speed of a winch: A winch is being used to lift a turbine off the ground so that a tractor-trailer can back under it and load it up for transport. The winch drum has a radius of 3 in. and is turning at 20 rpm. Find (a) the angular velocity of the drum in radians per minute, (b) the linear velocity of the turbine in feet per second as it is being raised, and (c) how long it will take to get the load to the desired height of 6 ft (ignore the fact that the cable may wind over itself on the drum).

96. Speed of a current: An instrument called a flowmeter is used to measure the speed of flowing water, like that in a river or stream. A cruder method involves placing a paddle wheel in the current, and using the wheel's radius and angular velocity to calculate the speed of water flow. If the paddle wheel has a radius of 5.6 ft and is turning at 30 rpm, find (a) the angular velocity of the wheel in radians per minute and (b) the linear velocity of the water current in miles per hour.

97. Baking: So long rolling pins! While flattening pie crust dough, the large rollers on the assembly line at Shay's bakery are continuously turning at 500 rpm. Given the radius of each roller is 2 in., find (a) the angular velocity of the rollers in radians per minute, (b) the linear velocity of the dough-supplying conveyor belt in feet per minute, and (c) how many seconds it would take to roll out a football field's length of dough (300 ft). Round to the nearest integer when necessary.

98. Unicycle racing: In 2008, the four man team "German Speeders" won the 800 km "Ride the Lobster" unicycle race in Nova Scotia, Canada. During the five day race, their one wheel was spinning at an average of 154 rpm. If each of the riders only used unicycles with a 38 cm radius wheel, find (a) the average angular velocity of the wheel in radians per minute, (b) their average linear velocity in kph, and (c) how long it would take them to complete the final 160 km of the race. Round to the nearest integer and second when necessary.

Topographical maps are two-dimensional representations of three-dimensional terrains. Each closed figure on the map represents a fixed elevation, according to a given *contour interval*. By carefully counting these closed figures, we can determine the vertical change between two points *A* and *B*. To find the horizontal change between point *A* and a location directly beneath point *B*, we measure the distance between the two points on the map and use the given *scale of distances*.

Exercises 99 and 100

99. Special triangles: In the figure shown, the *contour interval* is 1:250 (each figure indicates a change of 250 m in elevation) and the scale of distances is 1 cm = 625 m. (a) Find the change of elevation from *A* to *B*; (b) use a proportion to find the horizontal distance between points *A* and *B* if the measured distance on the map is 1.6 cm; and (c) draw the corresponding right triangle and use a special triangle relationship to find the length of the trail up the mountainside that connects *A* and *B*.

Exercise 99

100. Special triangles: As part of park maintenance, the 2 by 4 handrail alongside a mountain trail leading to the summit of Mount Marilyn must be replaced. In the figure, the *contour interval* is 1:200 (each figure indicates a change of 200 m in elevation) and the scale of distances is 1 cm = 400 m. (a) Find the change of elevation from *A* to *B*; (b) use a proportion to find the horizontal distance between *A* and *B* if the measured distance on the map is 4.33 cm; and (c) draw the corresponding right triangle and use a special triangle relationship to find the length needed to replace the handrail (recall that $\sqrt{3} \approx 1.732$).

Exercise 100

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101. Special triangles: Two light planes are flying in formation at 100 mph, doing some reconnaissance work. At a designated instant, one pilot breaks to the left at an angle of 90° to the other plane. Assuming they keep the same altitude and continue to fly at 100 mph, use a special triangle to find the distance between them after 0.5 hr.

102. Special triangles: Two ships are cruising together on the open ocean at 10 nautical miles per hour. One of them turns to make a 90° angle with the first and increases speed, heading for port. Assuming the first ship continues traveling at 10 knots, use a special triangle to find the speed of the other ship if they are 20 mi apart after 1 hr.

103. Angular and linear velocity: The planet Jupiter's largest moon, Ganymede, rotates around the planet at a distance of about 656,000 miles, in an orbit that is perfectly circular. If the moon completes one rotation about Jupiter in 7.15 days, (a) find the angle θ that the moon moves through in 1 day, in both degrees and radians, (b) find the angular velocity of the moon in radians per hour, and (c) find the moon's linear velocity in miles per second as it orbits Jupiter.

104. Angular and linear velocity: The planet Neptune has an orbit that is nearly circular. It orbits the Sun at a distance of 4497 million kilometers and completes one revolution every 165 yr. (a) Find the angle θ that the planet moves through in one year in both degrees and radians and (b) find the linear velocity (km/hr) as it orbits the Sun.

105. Center-pivot irrigation: A standard ¼-mi irrigation system has a radius of 400 m and rotates once every 3 days. Find the linear velocity of the outside set of wheels to the nearest tenth of a meter per hour.

106. Area of crop field: If a center-pivot irrigation system with radius of 500 m rotates once every 100 hr, find the area of a crop field irrigated in 1 day. Round to the nearest square meter.

▶ EXTENDING THE CONCEPT

107. Ancient geometers knew that a hexagon (six sides) could be inscribed in a circle by laying out six consecutive chords equal in length to the radius ($r = 10$ cm for illustration). After connecting the diagonals of the hexagon, six equilateral triangles are formed with sides of 10 cm. Use the diagram given to develop the fixed ratios for the sides of a 30-60-90 triangle. (*Hint:* Use a perpendicular bisector.)

108. The Duvall family is out on a family bicycle ride around Creve Coeur Lake. The adult bikes have a pedal sprocket with a 4-in. radius, wheel sprocket with 2-in. radius, and tires with a 13-in. radius. The kids' bikes have pedal sprockets with a 2.5-in. radius, wheel sprockets with 1.5-in. radius, and tires with a 9-in. radius. (a) If adults and kids all pedal at 50 rpm, how far ahead (in yards) are the adults after 2 min? (b) If adults pedal at 50 rpm, how fast do the kids have to pedal to keep up?

▶ MAINTAINING YOUR SKILLS

109. **(2.2)** Describe how the graph of $g(x) = -2\sqrt{x+3} - 1$ can be obtained from transformations of $y = \sqrt{x}$.

110. **(5.6)** Find the interest rate required for \$1000 to grow to \$1500 if the money is compounded monthly and remains on deposit for 5 yr.

111. **(1.4)** Given a line segment with endpoints $(-2, 3)$ and $(6, -1)$, find the equation of the line that bisects and is perpendicular to this segment.

112. **(2.2)** Find the equation of the function whose graph is shown.

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6.2 Unit Circles and the Trigonometry of Real Numbers

LEARNING OBJECTIVES

In Section 6.2 you will see how we can:

- ❑ **A.** Locate points on a unit circle and use symmetry to locate other points
- ❑ **B.** Use special triangles to find points on a unit circle and locate other points using symmetry
- ❑ **C.** Define the six trig functions in terms of a point on the unit circle
- ❑ **D.** Define the six trig functions in terms of a real number t
- ❑ **E.** Find the real number t corresponding to given values of $\sin t$, $\cos t$, and $\tan t$

In this section, we introduce the **trigonometry of real numbers,** a view of trigonometry that can exist free of its historical roots in a study of right triangles. In fact, the ultimate value of these functions is not in their classical study, but in the implications and applications that follow from understanding them as functions of a real number, rather than simply as functions of a given angle.

A. The Unit Circle

Recall that a circle is defined as the set of all points in a plane that are a *fixed distance*, called the radius, from a *fixed point* called the center. In Chapter 1, we used the distance formula to find the equation of a circle of radius r with center at (h, k): $(x - h)^2 + (y - k)^2 = r^2$. For central circles both h and k are zero, and the result is the equation for a central circle of radius r: $x^2 + y^2 = r^2\,(r > 0)$. The **unit circle** is defined as a central circle with radius 1 unit: $x^2 + y^2 = 1$. As such, the figure can easily be graphed by drawing a circle through the four **quadrantal points** $(1, 0)$, $(-1, 0)$, $(0, 1)$, and $(0, -1)$ as in Figure 6.23. To algebraically find other points on the circle, we simply select any value of x, where $|x| < 1$, then substitute and solve for y; or any value of y, where $|y| < 1$, then solve for x.

Figure 6.23

Circles and Their Equations

General Circle	Central Circle	Unit Circle
$(x - h)^2 + (y - k)^2 = r^2$	$x^2 + y^2 = r^2$	$x^2 + y^2 = 1$
radius $= r$, center at (h, k)	radius $= r$, center at $(0, 0)$	radius $= 1$, center at $(0, 0)$

EXAMPLE 1 ▶ **Finding Points on the Unit Circle**

Find a point on the unit circle given $x = -\frac{1}{2}$ with (x, y) in QII.

Algebraic Solution ▶ Using the equation of a unit circle, we have

$$x^2 + y^2 = 1 \qquad \text{unit circle equation}$$

$$\left(-\frac{1}{2}\right)^2 + y^2 = 1 \qquad \text{substitute } -\frac{1}{2} \text{ for } x$$

$$\frac{1}{4} + y^2 = 1 \qquad \left(-\frac{1}{2}\right)^2 = \frac{1}{4}$$

$$y^2 = \frac{3}{4} \qquad \text{subtract } \frac{1}{4}$$

$$y = \pm\frac{\sqrt{3}}{2} \qquad \text{result}$$

Since (x, y) is in QII, $y = \frac{\sqrt{3}}{2}$. The point is $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

Graphical Solution ▶ Recall in Section 1.1 we graphed circles on a calculator by separating the equation into two parts. For the unit circle, we enter the equation of the "upper half" as $Y_1 = \sqrt{1 - X^2}$ and the "lower half" as $Y_2 = -\sqrt{1 - X^2}$. With the unit circle displayed in the WINDOW shown, we TRACE Y_1 (the upper half-circle) since (x, y) is in QII. After moving the cursor to $x = -0.5$, we observe $y \approx 0.866$, the decimal equivalent of $\frac{\sqrt{3}}{2}$ (see Figure 6.24).

Figure 6.24

Now try Exercises 7 through 18 ▶

Additional points on the unit circle can be found using symmetry. The simplest examples come from the quadrantal points, where (1, 0) and (−1, 0) are on opposite sides of the y-axis, and (0, 1) and (0, −1) are on opposite sides of the x-axis. In general, if a and b are positive real numbers and (a, b) is on the unit circle, then $(-a, b)$, $(a, -b)$, and $(-a, -b)$ are also on the circle *because a circle is symmetric to both axes and the origin*! For the point $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ from Example 1, three other points are $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ in QIII, $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ in QIV, and $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ in QI. See Figure 6.25.

Figure 6.25

EXAMPLE 2 ▶ **Using Symmetry to Locate Points on the Unit Circle**

Name the quadrant containing $\left(-\frac{3}{5}, -\frac{4}{5}\right)$ and verify it's on the unit circle. Then use symmetry to find three other points on the circle.

Solution ▶ Since both coordinates are negative, $\left(-\frac{3}{5}, -\frac{4}{5}\right)$ is in QIII. Substituting into the equation for the unit circle yields

$$x^2 + y^2 = 1 \quad \text{unit circle equation}$$

$$\left(\frac{-3}{5}\right)^2 + \left(\frac{-4}{5}\right)^2 \stackrel{?}{=} 1 \quad \text{substitute } \tfrac{-3}{5} \text{ for } x \text{ and } \tfrac{-4}{5} \text{ for } y$$

$$\frac{9}{25} + \frac{16}{25} \stackrel{?}{=} 1 \quad \text{simplify}$$

$$\frac{25}{25} = 1 \quad \text{result checks}$$

Since $\left(\frac{-3}{5}, \frac{-4}{5}\right)$ is on the unit circle, $\left(\frac{3}{5}, \frac{-4}{5}\right)$, $\left(\frac{-3}{5}, \frac{4}{5}\right)$, and $\left(\frac{3}{5}, \frac{4}{5}\right)$ are also on the circle due to symmetry (see Figure 6.26).

Figure 6.26

✓ **A.** You've just seen how we can locate points on a unit circle and use symmetry to locate other points

Now try Exercises 19 through 26 ▶

B. Special Triangles and the Unit Circle

The special triangles from Section 6.1 can also be used to find points on the unit circle. As usually written, the triangles state a proportional relationship between their sides after assigning a value of 1 to the shortest side. However, precisely due to this proportional relationship, *we can divide all sides by the length of the hypotenuse,* giving *it* a length of 1 unit (see Figures 6.27 and 6.28).

Figure 6.27

Figure 6.28

We then place the triangle within the unit circle, and reflect it from quadrant to quadrant to find additional points. The *sides* of the triangle are used to determine the absolute value of each coordinate, and the *quadrant* to give each coordinate the appropriate sign. Note the angles in these special triangles are now expressed in radians.

EXAMPLE 3 ▶ Using a Special Triangle and Symmetry to Locate Points on a Unit Circle

Use the $\frac{\pi}{4}:\frac{\pi}{4}:\frac{\pi}{2}$ triangle from Figure 6.28 to find four points on the unit circle.

Solution ▶ Begin by superimposing the triangle in QI, noting it gives the point $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ shown in Figure 6.29. By reflecting the triangle into QII, we find the additional point $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ on this circle. Realizing we can simply apply the circle's remaining symmetries, we obtain the two additional points $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ and $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ shown in Figure 6.30.

WORTHY OF NOTE

Applying the same idea to a $\frac{\pi}{6}:\frac{\pi}{3}:\frac{\pi}{2}$ triangle would give the points $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right), \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right), \left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ and $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$.

Figure 6.29

Figure 6.30

Now try Exercises 27 and 28 ▶

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To graphically verify the results of Example 3, we begin by noting that the slope of the line coincident with the hypotenuse of the triangle in Figure 6.29 must be $m = \dfrac{\sqrt{2}/2}{\sqrt{2}/2} = 1$. The equation of this line is then $y = 1x$, and similarly the line coincident with the hypotenuse of the triangle's QII reflection is $y = -1x$ (see Figure 6.30). Using a graphing calculator, enter $Y_1 = \sqrt{1 - X^2}$ (the upper half of the unit circle), $Y_2 = 1X$, $Y_3 = -\sqrt{1 - X^2}$ (the lower half of the unit circle), and $Y_4 = -1X$ (see Figure 6.31). Graphing the functions using the ZOOM **4:ZDecimal** feature will produce a "friendly" and square viewing window, but with a relatively small unit circle. Using the ZOOM **2:Zoom In** feature will give a better view, with the final window size depending on the settings of your zoom factors. Our settings were at XFactor = 2 and YFactor = 2, which produced the screen shown in Figure 6.32 (to access the zoom factors, press ZOOM › **4:SetFactors).** To find a point on the unit circle using the $\frac{\pi}{4}:\frac{\pi}{4}:\frac{\pi}{2}$ triangle, we need only find where the line $Y_2 = 1X$ intersects the unit circle in QI. This is accomplished using the keystrokes 2nd TRACE **(CALC) 5:Intersect,** and identifying the graphs we're interested in. After doing so, the calculator returns the values shown in Figure 6.33, which are indeed the equivalent of $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$. Using the down arrow ˅ at this point will "jump the cursor" to the point where $Y_3 = -\sqrt{1 - X^2}$ and $Y_4 = -1X$ intersect, where we note the output values remain the same except that in QIV, the y-coordinate is negative (see Figure 6.34). In addition, since the calculator stores the last used x-value in the temporary location X,T,θ,n, we can find the point of intersection in QIII by simply using the keystrokes (–) X,T,θ,n, and then the point of intersection in QII by pressing the up arrow ˄ twice.

Figure 6.31

Figure 6.32

Figure 6.33

Figure 6.34

Figure 6.35

When a central angle θ is viewed as a rotation, each rotation can be associated with a unique point (x, y) on the terminal side, where it intersects the unit circle (see Figure 6.35). For the quadrantal angles $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$, and 2π, we associate the points (0, 1), (−1, 0), (0, −1), and (1, 0), respectively. When this rotation results in a special angle θ, the association can be found using a special triangle in a manner similar to Example 3.

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Figure 6.36

Figure 6.36 shows we associate the point $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ with $\theta = \frac{\pi}{6}$, $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ with $\theta = \frac{\pi}{4}$, and by reorienting the $\frac{\pi}{6}$: $\frac{\pi}{3}$: $\frac{\pi}{2}$ triangle, $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ is associated with a rotation of $\theta = \frac{\pi}{3}$.

For standard rotations from $\theta = 0$ to $\theta = \frac{\pi}{2}$ we have the following:

Rotation θ	**0**	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
Associated point (x, y)	$(1, 0)$	$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$	$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$	$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$	$(0, 1)$

Each of these points give rise to three others using the symmetry of the circle. By defining a reference angle θ_r, we can associate these points with the related rotation $\theta > \frac{\pi}{2}$.

Reference Angles

For any angle θ in standard position, the acute angle θ_r formed by the terminal side and the x-axis is called the *reference angle* for θ.

Several examples of the reference angle concept are shown in Figure 6.37 for $\theta > 0$ in radians.

Figure 6.37

$0 < \theta < \frac{\pi}{2}$, $\theta_r = \theta$

$\frac{\pi}{2} < \theta < \pi$, $\theta_r = \pi - \theta$

$\pi < \theta < \frac{3\pi}{2}$, $\theta_r = \theta - \pi$

$\frac{3\pi}{2} < \theta < 2\pi$, $\theta_r = 2\pi - \theta$

Due to the symmetries of the circle, reference angles of $\frac{\pi}{6}$, $\frac{\pi}{4}$, and $\frac{\pi}{3}$ serve to fix the absolute values of the coordinates for x and y, and we simply *use the appropriate sign for each coordinate* to name the point. As before this depends solely on the quadrant of the terminal side.

EXAMPLE 4 ▶ **Finding Points on a Unit Circle Associated with a Rotation θ**

Determine the reference angle for each rotation given, then find the associated point (x, y) on the unit circle.

a. $\theta = \frac{5\pi}{6}$ **b.** $\theta = \frac{7\pi}{4}$

Solution ▶ **a.** A rotation of $\frac{5\pi}{6}$ terminates in QII:

$\theta_r = \pi - \frac{5\pi}{6} = \frac{\pi}{6}$. The associated point is $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ since $x < 0$ in QII. See Figure 6.38.

Figure 6.38

b. A rotation of $\frac{7\pi}{4}$ terminates in QIV:

$\theta_r = 2\pi - \frac{7\pi}{4} = \frac{\pi}{4}$. The associated point is $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ since $y < 0$ in QIV. See Figure 6.39.

Figure 6.39

Now try Exercises 29 through 36 ▶

✓ **B.** You've just seen how we can use special triangles to find points on a unit circle and locate other points using symmetry

C. Trigonometric Functions and Points on the Unit Circle

While this focus on the relationship between an angle θ and points on the unit circle may seem arbitrary, the connection actually leads to an entirely new branch of mathematics called **trigonometry**, with applications beyond any of those presented in earlier chapters. Since the terminal side of any central angle θ intersects the unit circle at a *unique* point (x, y), a function relationship exists between θ and the coordinates of that point. To capitalize on these relationships, each is given a name as follows:

$$\text{cosine } \theta = x \qquad \text{sine } \theta = y \qquad \text{tangent } \theta = \frac{y}{x}$$

The reciprocals of these functions also play a significant role in the development of trigonometry, and are likewise given names:

$$\text{secant } \theta = \frac{1}{x} \qquad \text{cosecant } \theta = \frac{1}{y} \qquad \text{cotangent } \theta = \frac{x}{y}$$

In actual use, each function name is written in abbreviated form as $\cos\theta$, $\sin\theta$, $\tan\theta$, $\sec\theta$, $\csc\theta$, and $\cot\theta$, respectively, and are collectively called the **trigonometric functions** (or simply the **trig functions**). Over the course of this study, you will see many connections between this view of trigonometry and the alternative approaches that follow. In summary, we have

The Trigonometric Functions

For any rotation θ and point $P(x, y)$ on the unit circle associated with θ,

$$\cos\theta = x \qquad \sin\theta = y \qquad \tan\theta = \frac{y}{x};\ x \neq 0$$

$$\sec\theta = \frac{1}{x};\ x \neq 0 \qquad \csc\theta = \frac{1}{y};\ y \neq 0 \qquad \cot\theta = \frac{x}{y};\ y \neq 0$$

In a study of trigonometry, a number of useful relationships result directly from the way these six functions are defined. For instance, since $\cos\theta = x$ and $\sec\theta = \frac{1}{x}$, the relationship $\sec\theta = \frac{1}{\cos\theta}$ follows immediately from a direct substitution (substitute $\cos\theta$ for x). These reciprocal relationships are summarized in the following box.

The Reciprocal Relationships

For any rotation θ where both functions are defined,

$$\sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$\cos\theta = \frac{1}{\sec\theta} \qquad \sin\theta = \frac{1}{\csc\theta} \qquad \tan\theta = \frac{1}{\cot\theta}$$

Note that once $\sin\theta$, $\cos\theta$, and $\tan\theta$ are known, the values of $\csc\theta$, $\sec\theta$, and $\cot\theta$ follow automatically since a number and its reciprocal always have the same sign. See Figure 6.40.

Figure 6.40

EXAMPLE 5 ▶ **Evaluating Trig Functions for a Rotation θ**

Evaluate the six trig functions for $\theta = \frac{5\pi}{4}$.

Solution ▶ A rotation of $\frac{5\pi}{4}$ terminates in QIII, so $\theta_r = \frac{5\pi}{4} - \pi = \frac{\pi}{4}$. The associated point is $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ since $x < 0$ and $y < 0$ in QIII.

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This yields

$$\cos\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2} \qquad \sin\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2} \qquad \tan\left(\frac{5\pi}{4}\right) = 1$$

Noting the reciprocal of $-\frac{\sqrt{2}}{2}$ is $-\sqrt{2}$ after rationalizing, we have

$$\sec\left(\frac{5\pi}{4}\right) = -\sqrt{2} \qquad \csc\left(\frac{5\pi}{4}\right) = -\sqrt{2} \qquad \cot\left(\frac{5\pi}{4}\right) = 1$$

C. You've just seen how we can define the six trig functions in terms of a point on the unit circle

Now try Exercises 37 through 40 ▶

D. The Trigonometry of Real Numbers

Defining the trig functions in terms of a point on the unit circle is precisely what we needed to work with them as functions of real numbers. This is because when $r = 1$ and θ is in radians, *the length of the subtended arc is numerically the same as the measure of the angle:* $s = (1)\theta \Rightarrow s = \theta$! This means we can view any function of θ as a like function of arc length s, where $s \in \mathbb{R}$ (see the *Reinforcing Basic Concepts* feature following Section 6.4). As a compromise the variable t is commonly used, with t representing *either* the amount of rotation *or* the length of the arc. As such we will assume t is a unitless quantity, although there are other reasons for this assumption. In Figure 6.41, a rotation of $\theta = \frac{3\pi}{4}$ is subtended by an arc length of $s = \frac{3\pi}{4}$ (about 2.356 units). The reference angle for θ is $\frac{\pi}{4}$, which we will now refer to as a **reference arc.** As you work through the remaining examples and the exercises that follow, it will often help to draw a quick sketch similar to that in Figure 6.41 to determine the quadrant of the terminal side, the reference arc, and the sign of each function.

Figure 6.41

EXAMPLE 6 ▶ **Evaluating Trig Functions for a Real Number t**

Evaluate the six trig functions for the given value of t.

a. $t = \frac{11\pi}{6}$ **b.** $t = \frac{3\pi}{2}$

Solution ▶ **a.** For $t = \frac{11\pi}{6}$, the arc terminates in QIV where $x > 0$ and $y < 0$. The reference arc is $2\pi - \frac{11/\pi}{6} = \frac{\pi}{6}$ and from our previous work we know the corresponding point (x, y) is $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$. See Figure 6.42. This gives

$$\cos\left(\frac{11\pi}{6}\right) = \frac{\sqrt{3}}{2} \qquad \sin\left(\frac{11\pi}{6}\right) = -\frac{1}{2} \qquad \tan\left(\frac{11\pi}{6}\right) = -\frac{\sqrt{3}}{3}$$

$$\sec\left(\frac{11\pi}{6}\right) = \frac{2\sqrt{3}}{3} \qquad \csc\left(\frac{11\pi}{6}\right) = -2 \qquad \cot\left(\frac{11\pi}{6}\right) = -\sqrt{3}$$

Figure 6.42

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Figure 6.43

b. $t = \dfrac{3\pi}{2}$ is a quadrantal angle and the associated point is $(0, -1)$.

See Figure 6.43. This yields

$$\cos\left(\frac{3\pi}{2}\right) = 0 \qquad \sin\left(\frac{3\pi}{2}\right) = -1 \qquad \tan\left(\frac{3\pi}{2}\right) = \text{undefined}$$

$$\sec\left(\frac{3\pi}{2}\right) = \text{undefined} \qquad \csc\left(\frac{3\pi}{2}\right) = -1 \qquad \cot\left(\frac{3\pi}{2}\right) = 0$$

Now try Exercises 41 through 44 ▶

As Example 6(b) indicates, as functions of a real number the concept of domain comes into play. From their definition it is apparent there are no restrictions on the domains of cosine and sine, but the domains of the other functions must be restricted to exclude division by zero. For functions with x in the denominator, we cast out the odd multiples of $\dfrac{\pi}{2}$, since the x-coordinate of the related quadrantal points is zero: $\dfrac{\pi}{2} \to (0, 1)$, $\dfrac{3\pi}{2} \to (0, -1)$, and so on. The excluded values can be stated as $t \neq \dfrac{\pi}{2} + \pi k$ for all integers k. For functions with y in the denominator, we cast out all multiples of π ($t \neq \pi k$ for all integers k) since the y-coordinate of these points is zero: $0 \to (1, 0)$, $\pi \to (-1, 0)$, $2\pi \to (1, 0)$, and so on.

The Domains of the Trig Functions as Functions of a Real Number

For $t \in \mathbb{R}$ and $k \in \mathbb{Z}$, the domains of the trig functions are:

$\cos t = x$, $t \in \mathbb{R}$	$\sin t = y$, $t \in \mathbb{R}$	$\tan t = \dfrac{y}{x}$; $x \neq 0$, $t \neq \dfrac{\pi}{2} + \pi k$
$\sec t = \dfrac{1}{x}$; $x \neq 0$, $t \neq \dfrac{\pi}{2} + \pi k$	$\csc t = \dfrac{1}{y}$; $y \neq 0$, $t \neq \pi k$	$\cot t = \dfrac{x}{y}$; $y \neq 0$, $t \neq \pi k$

For a given point (x, y) on the unit circle associated with the real number t, the value of each function at t can still be determined even if t is unknown.

EXAMPLE 7 ▶ **Finding Function Values Given a Point on the Unit Circle**

Given $\left(\frac{-7}{25}, \frac{24}{25}\right)$ is a point on the unit circle corresponding to a real number t, find the value of all six trig functions of t.

Solution ▶ Using the definitions from the previous box we have $\cos t = \frac{-7}{25}$, $\sin t = \frac{24}{25}$, and $\tan t = \frac{\sin t}{\cos t} = \frac{24}{-7}$. The values of the reciprocal functions are then $\sec t = \frac{25}{-7}$, $\csc t = \frac{25}{24}$, and $\cot t = \frac{-7}{24}$.

Now try Exercises 45 through 60 ▶

☑ **D.** You've just seen how we can define the six trig functions in terms of a real number t

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E. Finding a Real Number *t* Whose Function Value Is Known

In Example 7, we were able to determine the values of the trig functions even though t was unknown. In many cases, however, we need to *find* the value of t. For instance, what is the value of t given $\cos t = -\frac{\sqrt{3}}{2}$ with t in QII? Exercises of this type fall into two broad categories: (1) you recognize the given number as one of the special values: $\pm\left\{0, \frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{3}, \sqrt{3}, 1\right\}$; or (2) you don't. If you recognize a special value, you can often name the real number t after a careful consideration of the related quadrant and required sign. The diagram in Figure 6.44 reviews these special values for $0 \le t \le \frac{\pi}{2}$ but remember—all other special values can be found using reference arcs and the symmetry of the circle.

Figure 6.44

EXAMPLE 8 ▶ **Finding *t* for Given Values and Conditions**

Find the value of t that corresponds to the given function values.

a. $\cos t = -\frac{\sqrt{2}}{2}$; t in QII **b.** $\tan t = \sqrt{3}$; t in QIII

Solution ▶

a. The cosine function is negative in QII and QIII, where $x < 0$. We recognize $-\frac{\sqrt{2}}{2}$ as a standard value for sine and cosine, related to certain multiples of $t = \frac{\pi}{4}$. In QII, we have $t = \frac{3\pi}{4}$.

b. The tangent function is positive in QI and QIII, where x and y have like signs. We recognize $\sqrt{3}$ as a standard value for tangent and cotangent, related to certain multiples of $t = \frac{\pi}{6}$. For tangent in QIII, we have $t = \frac{4\pi}{3}\left(= \frac{8\pi}{6}\right)$.

Now try Exercises 61 through 84 ▶

If the given function value is not one of the special values, properties of the inverse trigonometric functions must be used to find the associated value of t. The inverse functions are fully developed in Section 7.5.

Using radian measure and the unit circle is much more than a simple convenience to trigonometry and its applications. Whether the unit is 1 cm, 1 m, 1 km, or even 1 light-year, using 1 unit designations serves to simplify a great many practical applications, including those involving the arc length formula, $s = r\theta$. **See Exercises 95 through 102.**

The following table summarizes the relationship between a special arc t (t in QI) and the value of each trig function at t. Due to the frequent use of these relationships, students are encouraged to commit them to memory.

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t	$\sin t$	$\cos t$	$\tan t$	$\csc t$	$\sec t$	$\cot t$
0	0	1	0	undefined	1	undefined
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$	2	$\frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$	$\sqrt{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$	2	$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$
$\frac{\pi}{2}$	1	0	undefined	1	undefined	0

E. You've just seen how we can find the real number t corresponding to given values of $\sin t$, $\cos t$, and $\tan t$

6.2 EXERCISES

CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section if needed.

1. A central circle is symmetric to the _____ axis, the _____ axis and to the _____.

2. Since $(\frac{5}{13}, -\frac{12}{13})$ is on the unit circle, the point _____ in QII is also on the circle.

3. On the unit circle, $\cos t =$ _____, $\sin t =$ _____, and $\tan t =$ __; while $\frac{1}{x} =$ _____, $\frac{1}{y} =$ _____, and $\frac{x}{y} =$ _____.

4. On a unit circle with θ in radians, the length of a(n) _____ is numerically the same as the measure of the _____, since for $s = r\theta$, $s = \theta$ when $r = 1$.

5. Discuss/Explain how knowing only one point on the unit circle, actually gives the location of four points. Why is this helpful to a study of the circular functions?

6. A student is asked to find t using a calculator, given $\sin t \approx 0.5592$ with t in QII. The answer submitted is $t = \sin^{-1} 0.5592 \approx 34°$. Discuss/Explain why this answer is not correct. What is the correct response?

DEVELOPING YOUR SKILLS

Given the point is on the unit circle, complete the ordered pair (x, y) for the quadrant indicated. Answer in radical form as needed.

7. $(x, -0.8)$; QIII

8. $(-0.6, y)$; QII

9. $\left(\frac{5}{13}, y\right)$; QIV

10. $\left(x, -\frac{8}{17}\right)$; QIV

11. $\left(\frac{\sqrt{11}}{6}, y\right)$; QI

12. $\left(x, -\frac{\sqrt{13}}{7}\right)$; QIII

13. $\left(-\frac{\sqrt{11}}{4}, y\right)$; QII

14. $\left(x, \frac{\sqrt{6}}{5}\right)$; QI

Given the point is on the unit circle, complete the ordered pair (x, y) for the quadrant indicated. Round results to four decimal places. Check your answer using 2nd TRACE (CALC) with a graph of the unit circle.

15. $(x, -0.2137)$; QIII

16. $(0.9909, y)$; QIV

17. $(x, 0.1198)$; QII

18. $(0.5449, y)$; QI

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Verify the point given is on the unit circle, then use symmetry to find three more points on the circle. Coordinates for Exercises 19 to 22 are exact, coordinates for Exercises 23 to 26 are approximate.

19. $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ **20.** $\left(\frac{\sqrt{7}}{4}, -\frac{3}{4}\right)$

21. $\left(\frac{\sqrt{11}}{6}, -\frac{5}{6}\right)$ **22.** $\left(-\frac{\sqrt{6}}{3}, -\frac{\sqrt{3}}{3}\right)$

23. (0.3325, 0.9431) **24.** (0.7707, −0.6372)

25. (0.9937, −0.1121) **26.** (−0.2029, 0.9792)

27. Use a $\frac{\pi}{6}$: $\frac{\pi}{3}$: $\frac{\pi}{2}$ triangle with a hypotenuse of length 1 to verify that $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ is a point on the unit circle.

28. Use the results from Exercise 27 to find three additional points on the circle and name the quadrant of each point.

Find the reference angle associated with each rotation, then find the associated point (*x*, *y*) on the unit circle.

29. $\theta = \frac{5\pi}{4}$ **30.** $\theta = \frac{5\pi}{3}$

31. $\theta = -\frac{5\pi}{6}$ **32.** $\theta = -\frac{7\pi}{4}$

33. $\theta = \frac{11\pi}{4}$ **34.** $\theta = \frac{11\pi}{3}$

35. $\theta = \frac{25\pi}{6}$ **36.** $\theta = \frac{39\pi}{4}$

Without the use of a calculator, state the exact value of the trig functions for the given angles. A diagram may help.

37. a. $\sin\left(\frac{\pi}{4}\right)$ **b.** $\sin\left(\frac{3\pi}{4}\right)$
c. $\sin\left(\frac{5\pi}{4}\right)$ **d.** $\sin\left(\frac{7\pi}{4}\right)$
e. $\sin\left(\frac{9\pi}{4}\right)$ **f.** $\sin\left(-\frac{\pi}{4}\right)$
g. $\sin\left(-\frac{5\pi}{4}\right)$ **h.** $\sin\left(-\frac{11\pi}{4}\right)$

38. a. $\tan\left(\frac{\pi}{3}\right)$ **b.** $\tan\left(\frac{2\pi}{3}\right)$
c. $\tan\left(\frac{4\pi}{3}\right)$ **d.** $\tan\left(\frac{5\pi}{3}\right)$
e. $\tan\left(\frac{7\pi}{3}\right)$ **f.** $\tan\left(-\frac{\pi}{3}\right)$
g. $\tan\left(-\frac{4\pi}{3}\right)$ **h.** $\tan\left(-\frac{10\pi}{3}\right)$

39. a. $\cos \pi$ **b.** $\cos 0$
c. $\cos\left(\frac{\pi}{2}\right)$ **d.** $\cos\left(\frac{3\pi}{2}\right)$

40. a. $\sin \pi$ **b.** $\sin 0$
c. $\sin\left(\frac{\pi}{2}\right)$ **d.** $\sin\left(\frac{3\pi}{2}\right)$

Use the symmetry of the circle and reference arcs as needed to state the exact value of the trig functions for the given real numbers without the use of a calculator. A diagram may help.

41. a. $\cos\left(\frac{\pi}{6}\right)$ **b.** $\cos\left(\frac{5\pi}{6}\right)$
c. $\cos\left(\frac{7\pi}{6}\right)$ **d.** $\cos\left(\frac{11\pi}{6}\right)$
e. $\cos\left(\frac{13\pi}{6}\right)$ **f.** $\cos\left(-\frac{\pi}{6}\right)$
g. $\cos\left(-\frac{5\pi}{6}\right)$ **h.** $\cos\left(-\frac{23\pi}{6}\right)$

42. a. $\csc\left(\frac{\pi}{6}\right)$ **b.** $\csc\left(\frac{5\pi}{6}\right)$
c. $\csc\left(\frac{7\pi}{6}\right)$ **d.** $\csc\left(\frac{11\pi}{6}\right)$
e. $\csc\left(\frac{13\pi}{6}\right)$ **f.** $\csc\left(-\frac{\pi}{6}\right)$
g. $\csc\left(-\frac{11\pi}{6}\right)$ **h.** $\csc\left(-\frac{17\pi}{6}\right)$

43. a. $\tan \pi$ **b.** $\tan 0$
c. $\tan\left(\frac{\pi}{2}\right)$ **d.** $\tan\left(\frac{3\pi}{2}\right)$

44. a. $\cot \pi$ **b.** $\cot 0$
c. $\cot\left(\frac{\pi}{2}\right)$ **d.** $\cot\left(\frac{3\pi}{2}\right)$

Given (*x*, *y*) is a point on the unit circle corresponding to *t*, find the value of all six trig functions of *t*.

45.

46.

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47.

48.

49. $\left(-\frac{2}{5}, \frac{\sqrt{21}}{5}\right)$

50. $\left(\frac{\sqrt{7}}{4}, -\frac{3}{4}\right)$

51. $\left(-\frac{1}{3}, -\frac{2\sqrt{2}}{3}\right)$

52. $\left(-\frac{2\sqrt{6}}{5}, -\frac{1}{5}\right)$

On the unit circle, the real number *t* can represent either the amount of rotation or the *length of the arc* when we associate *t* with a point (*x*, *y*) on the circle. In the circle diagram shown, the real number *t* in radians is marked off along the circumference. For Exercises 53 through 60, name the quadrant in which *t* terminates and use the figure to estimate function values to one decimal place (use a straightedge). Check results using a calculator.

Exercises 53 to 60

53. sin 0.75

54. cos 2.75

55. cos 5.5

56. sin 4.0

57. tan 0.8

58. sec 3.75

59. csc 2.0

60. cot 0.5

Without using a calculator, find the value of *t* in [0, 2π) that corresponds to the following functions.

61. $\sin t = \frac{\sqrt{3}}{2}$; *t* in QII

62. $\cos t = \frac{1}{2}$; *t* in QIV

63. $\cos t = -\frac{\sqrt{3}}{2}$; *t* in QIII

64. $\sin t = -\frac{1}{2}$; *t* in QIV

65. $\tan t = -\sqrt{3}$; *t* in QII

66. $\sec t = -2$; *t* in QIII

67. $\sin t = 1$

68. $\cos t = -1$

Without using a calculator, find the two values of *t* (where possible) in [0, 2π) that make each equation true.

69. $\sec t = -\sqrt{2}$

70. $\csc t = -\frac{2}{\sqrt{3}}$

71. tan *t* undefined

72. csc *t* undefined

73. $\cos t = -\frac{\sqrt{2}}{2}$

74. $\sin t = \frac{\sqrt{2}}{2}$

75. $\sin t = 0$

76. $\cos t = -1$

77. Given $\left(\frac{3}{5}, -\frac{4}{5}\right)$ is a point on the unit circle that corresponds to *t*. Find the coordinates of the point corresponding to (a) $-t$ and (b) $t + \pi$.

78. Given $\left(-\frac{7}{25}, \frac{24}{25}\right)$ is a point on the unit circle that corresponds to *t*. Find the coordinates of the point corresponding to (a) $-t + \pi$ and (b) $t - \pi$.

Find an additional value of *t* in [0, 2π) that makes the equation true.

79. $\sin 0.8 \approx 0.7174$

80. $\cos 2.12 \approx -0.5220$

81. $\cos 4.5 \approx -0.2108$

82. $\sin 5.23 \approx -0.8690$

83. $\tan 0.4 \approx 0.4228$

84. $\sec 5.7 \approx 1.1980$

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▶ WORKING WITH FORMULAS

85. From Pythagorean triples to points on the unit circle: $(x, y, r) \to \left(\frac{x}{r}, \frac{y}{r}, 1\right)$

While not strictly a "formula," dividing a Pythagorean triple by r is a simple algorithm for rewriting any Pythagorean triple as a triple with hypotenuse 1. This enables us to identify certain points on the unit circle, and to evaluate the six trig functions of the related acute angle. Rewrite each triple as a triple with hypotenuse 1, verify $\left(\frac{x}{r}, \frac{y}{r}\right)$ is a point on the unit circle, and evaluate the six trig functions using this point.

a. (5, 12, 13) **b.** (7, 24, 25)
c. (12, 35, 37) **d.** (9, 40, 41)

86. The sine and cosine of $(2k + 1)\frac{\pi}{4}$; $k \in \mathbb{Z}$

In the solution to Example 8(a), we mentioned $\pm\frac{\sqrt{2}}{2}$ were standard values for sine and cosine, "related to certain multiples of $\frac{\pi}{4}$." Actually, we meant "odd multiples of $\frac{\pi}{4}$." The odd multiples of $\frac{\pi}{4}$ are given by the "formula" shown, where k is any integer. (a) What multiples of $\frac{\pi}{4}$ are generated by $k = -3, -2, -1, 0, 1, 2, 3$? (b) Find similar formulas for Example 8(b), where $\sqrt{3}$ is a standard value for tangent and cotangent, "related to certain multiples of $\frac{\pi}{6}$."

▶ APPLICATIONS

Unit circle points: In Exercises 23 through 26, four decimal approximations of unit circle points were given. Find such unit circle points that are on the terminal side of the following angles in standard position. (*Hint:* Use the definitions of the trig functions.)

87. $\theta = 2$ **88.** $\theta = 1$ **89.** $\theta = 5$ **90.** $\theta = 4$

Mosaic design: The floor of a local museum contains a mosaic zodiac circle as shown. The circle is 1 m (= 100 cm) in radius, with all 12 zodiac signs represented by small, evenly spaced bronze disks on the circumference.

Exercises 91 to 94

91. If the disk representing Libra is 96.6 cm to the right of and 25.9 cm above the center of the zodiac, what is the distance d (see illustration) from Libra's disk to Pisces' disk? What is the distance from Libra's disk to Virgo's disk?

92. If the disk representing Taurus is 70.7 cm to the left of and 70.7 cm below the center of the zodiac, what is the distance from Taurus' disk to Leo's disk? What is the distance from Taurus' disk to Scorpio's disk?

93. Using a special triangle, identify (in relation to the center of the zodiac) the location of the point P (see illustration) where the steel rod separating Scorpio and Sagittarius intersects the circumference. Round to the nearest tenth of a centimeter.

94. Using a special triangle, identify (in relation to the center of the zodiac) the location of the point where the steel rod separating Aquarius and Pisces intersects the circumference. Round to the nearest tenth of a centimeter.

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95. Laying new sod: When new sod is laid, a heavy roller is used to press the sod down to ensure good contact with the ground beneath. The radius of the roller is 1 ft. (a) Through what angle (in radians) has the roller turned after being pulled across 5 ft of yard? (b) What angle must the roller turn through to press a length of 30 ft?

96. Cable winch: A large winch with a radius of 1 ft winds in 3 ft of cable. (a) Through what angle (in radians) has it turned? (b) What angle must it turn through in order to winch in 12.5 ft of cable?

Exercise 96

97. Wiring an apartment: In the wiring of an apartment complex, electrical wire is being pulled from a spool with radius 1 decimeter (1 dm = 10 cm). (a) What length (in decimeters) is removed as the spool turns through 5 rad? (b) How many decimeters are removed in one complete turn ($t = 2\pi$) of the spool?

98. Barrel races: In the barrel races popular at some family reunions, contestants stand on a hard rubber barrel with a radius of 1 cubit (1 cubit = 18 in.), and try to "walk the barrel" from the start line to the finish line without falling. (a) What distance (in cubits) is traveled as the barrel is walked through an angle of 4.5 rad? (b) If the race is 25 cubits long, through what angle will the winning barrel walker walk the barrel?

Interplanetary measurement: In the year 1905, astronomers began using astronomical units or AU to study the distances between the celestial bodies of our solar system. One AU represents the average distance between the Earth and the Sun, which is about 93 million miles. Pluto is roughly 39.24 AU from the Sun.

99. If the Earth travels through an angle of 2.5 rad about the Sun, (a) what distance in astronomical units (AU) has it traveled? (b) How many AU does it take for one complete orbit around the Sun?

100. If you include the dwarf planet Pluto, Jupiter is the middle (fifth of nine) planet from the Sun. Suppose astronomers had decided to use *its* average distance from the Sun as 1 AU. In this case, 1 AU would be 480 million miles. If Jupiter travels through an angle of 4 rad about the Sun, (a) what distance in the "new" astronomical units (AU) has it traveled? (b) How many of the new AU does it take to complete one-half an orbit about the Sun? (c) What distance in the new AU is the dwarf planet Pluto from the Sun?

101. Compact disk circumference: A standard compact disk has a radius of 6 cm. Call this length "1 unit." Mark a starting point on any large surface, then carefully roll the compact disk along this line without slippage, through one full revolution (2π rad) and mark this spot. Take an accurate measurement of the resulting line segment. Is the result close to 2π "units" ($2\pi \times 6$ cm)?

102. Verifying $s = r\theta$: On a protractor, carefully measure the distance from the middle of the protractor's eye to the edge of the protractor along the 0° mark, to the nearest half-millimeter. Call this length "1 unit." Then use a ruler to draw a straight line on a blank sheet of paper, and with the protractor on edge, start the zero degree mark at one end of the line, carefully roll the protractor until it reaches 1 radian (57.3°), and mark this spot. Now measure the length of the line segment created. Is it very close to 1 "unit" long?

Exercise 102

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▶ EXTENDING THE CONCEPT

103. In this section, we discussed the *domain* of the circular functions, but said very little about their *range*. Review the concepts presented here and determine the range of $y = \cos t$ and $y = \sin t$. In other words, what are the smallest and largest output values we can expect?

104. Since $\tan t = \dfrac{\sin t}{\cos t}$, what can you say about the range of the tangent function?

Use the radian grid given with Exercises 53–60 to answer Exercises 105 and 106.

105. Given $\cos(2t) = -0.6$ with the terminal side of the arc in QII, (a) what is the value of $2t$? (b) What quadrant is t in? (c) What is the value of $\cos t$? (d) Does $\cos(2t) = 2\cos t$?

106. Given $\sin(2t) = -0.8$ with the terminal side of the arc in QIII, (a) what is the value of $2t$? (b) What quadrant is t in? (c) What is the value of $\sin t$? (d) Does $\sin(2t) = 2\sin t$?

▶ MAINTAINING YOUR SKILLS

107. **(1.3)** Given the points $(-3, -4)$ and $(5, 2)$ find

a. the distance between them

b. the midpoint between them

c. the slope of the line through them.

108. **(5.4)** Use a calculator to find the value of each expression, then explain the results.

a. $\log 2 + \log 5 =$ _____

b. $\log 20 - \log 2 =$ _____

109. **(2.3, R.6)** Solve each equation:

a. $2|x + 1| - 3 = 7$

b. $2\sqrt{x + 1} - 3 = 7$

110. **(4.2)** Use the rational zeroes theorem to solve the equation completely, given $x = -3$ is one root.

$$x^4 + x^3 - 3x^2 + 3x - 18 = 0$$

6.3 Graphs of the Sine and Cosine Functions

LEARNING OBJECTIVES

In Section 6.3 you will see how we can:

- ❑ **A.** Graph $f(t) = \sin t$ using special values and symmetry
- ❑ **B.** Graph $f(t) = \cos t$ using special values and symmetry
- ❑ **C.** Graph sine and cosine functions with various amplitudes and periods
- ❑ **D.** Write the equation for a given graph

As with the graphs of other functions, trigonometric graphs contribute a great deal toward the understanding of each function and its applications. For now, our primary interest is the general shape of each basic graph and some of the transformations that can be applied. We will also learn to analyze each graph, and to capitalize on the features that enable us to apply the functions as real-world models.

A. Graphing $f(t) = \sin t$

Consider the following table of values (Table 6.1) for $\sin t$ and the special angles in QI.

Table 6.1

t	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin t$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1

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Observe that in this interval, sine values are increasing from 0 to 1. From $\frac{\pi}{2}$ to π (QII), special values taken from the unit circle show sine values are decreasing from 1 to 0, *but through the same output values as in* QI. See Figures 6.45 through 6.47.

Figure 6.45

$\sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$

Figure 6.46

$\sin\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2}$

Figure 6.47

$\sin\left(\frac{5\pi}{6}\right) = \frac{1}{2}$

With this information we can extend our table of values through π, noting that $\sin \pi = 0$ (see Table 6.2).

Table 6.2

t	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\sin t$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

Using the symmetry of the circle and the fact that y is negative in QIII and QIV, we can complete the table for values between π and 2π.

EXAMPLE 1 ▶ **Finding Function Values Using Symmetry**

Use the symmetry of the unit circle and reference arcs of special values to complete Table 6.3. Recall that y is negative in QIII and QIV.

Table 6.3

t	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
$\sin t$									

Solution ▶ Symmetry shows that for any odd multiple of $t = \frac{\pi}{4}$, $\sin t = \pm\frac{\sqrt{2}}{2}$ depending on the quadrant of the terminal side. Similarly, for any reference arc of $\frac{\pi}{6}$, $\sin t = \pm\frac{1}{2}$, while any reference arc of $\frac{\pi}{3}$ will give $\sin t = \pm\frac{\sqrt{3}}{2}$. The completed table is shown in Table 6.4.

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Table 6.4

t	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
$\sin t$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0

Now try Exercises 7 and 8 ▶

Noting that $\frac{1}{2} = 0.5$, $\frac{\sqrt{2}}{2} \approx 0.71$, and $\frac{\sqrt{3}}{2} \approx 0.87$, we plot these points and connect them with a smooth curve to graph $y = \sin t$ in the interval $[0, 2\pi]$. The first five plotted points are labeled in Figure 6.48.

Figure 6.48

These approximate values can be quickly generated using the table feature of a graphing calculator set in radian MODE. Begin by noticing the least common denominator of the standard values $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$, and $\frac{\pi}{2}$ is 12. After entering $Y_1 = \sin X$, we use this observation and the keystrokes 2nd WINDOW **(TBLSET)** to set the table as shown in Figure 6.49. Note after pressing ENTER, the calculator automatically replaces the exact value $\frac{\pi}{12}$ with a decimal approximation (0.261...). Pressing the keys 2nd GRAPH **(TABLE)** results in the table shown in Figure 6.50, and we quickly recognize the Figure 6.48 decimal approximations in the Y_1 column. **Exercises 9 and 10** explore the two additional values that appear in this column.

Figure 6.49

Figure 6.50

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Expanding the table from 2π to 4π using reference arcs and the unit circle shows that function values begin to repeat. For example, $\sin\left(\frac{13\pi}{6}\right) = \sin\left(\frac{\pi}{6}\right)$ since $\theta_r = \frac{\pi}{6}$, $\sin\left(\frac{9\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right)$ since $\theta_r = \frac{\pi}{4}$, and so on. Functions that cycle through a set pattern of values are said to be **periodic functions.** Using the down arrow to scroll through the table in Figure 6.50, *while observing the graph in Figure 6.48,* helps highlight these concepts.

Periodic Functions

A function f is said to be periodic if there is a positive number P such that

$$f(t + P) = f(t)$$

for all t in the domain. The smallest number P for which this occurs is called the **period** of f.

For the sine function note $\sin t = \sin(t + 2\pi)$, as in $\sin\left(\frac{13\pi}{6}\right) = \sin\left(\frac{\pi}{6} + 2\pi\right)$ and $\sin\left(\frac{9\pi}{4}\right) = \sin\left(\frac{\pi}{4} + 2\pi\right)$, with the idea extending to all other real numbers t: $\sin t = \sin(t + 2\pi k)$ for all integers k. The sine function is periodic with period $P = 2\pi$.

Although we initially focused on positive values of t in $[0, 2\pi]$, $t < 0$ and $k < 0$ are certainly possibilities and we note the graph of $f(t) = \sin t$ extends infinitely in both directions (see Figure 6.51).

Figure 6.51

To see even more of this graph, we can use a graphing calculator and a preset ZOOM option that automatically sets a window size convenient to many trigonometric graphs. The resulting WINDOW after pressing ZOOM **7:ZTrig** is shown in Figure 6.52 for a calculator set in radian MODE. Pressing GRAPH with $Y_1 = \sin X$ displays the graph of $f(t) = \sin t$ in this predefined window (see Figure 6.53).

Figure 6.52

Figure 6.53

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Finally, the table, graph, and unit circle all confirm that the range of $f(t) = \sin t$ is $[-1, 1]$, and that $f(t) = \sin t$ is an odd function. In particular, the graph shows $\sin\left(-\frac{\pi}{2}\right) = -\sin\left(\frac{\pi}{2}\right)$, and the unit circle shows $\sin t = y$, and $\sin(-t) = -y$ (see Figure 6.54), from which we obtain $\sin(-t) = -\sin t$ by substitution. As a handy reference, the following box summarizes the main characteristics of $f(t) = \sin t$.

Figure 6.54

Characteristics of $f(t) = \sin t$

For all real numbers t and integers k,

Domain	Range	Period
$(-\infty, \infty)$	$[-1, 1]$	2π
Symmetry	**Maximum value**	**Minimum value**
odd	$\sin t = 1$	$\sin t = -1$
$\sin(-t) = -\sin t$	at $t = \frac{\pi}{2} + 2\pi k$	at $t = \frac{3\pi}{2} + 2\pi k$
Increasing	**Decreasing**	**Zeroes**
$\left(0, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right)$	$\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$	$t = k\pi$

EXAMPLE 2 ▶ **Using the Period of sin t to Find Function Values**

Use the characteristics of $f(t) = \sin t$ to match the given value of t to the correct value of $\sin t$.

a. $t = \frac{\pi}{4} + 8\pi$ **b.** $t = -\frac{\pi}{6}$ **c.** $t = \frac{17\pi}{2}$ **d.** $t = 21\pi$ **e.** $t = \frac{11\pi}{2}$

I. $\sin t = 1$ **II.** $\sin t = -\frac{1}{2}$ **III.** $\sin t = -1$ **IV.** $\sin t = \frac{\sqrt{2}}{2}$ **V.** $\sin t = 0$

Solution ▶

a. Since $\sin\left(\frac{\pi}{4} + 8\pi\right) = \sin\frac{\pi}{4}$, the correct match is (IV).

b. Since $\sin\left(-\frac{\pi}{6}\right) = -\sin\frac{\pi}{6}$, the correct match is (II).

c. Since $\sin\left(\frac{17\pi}{2}\right) = \sin\left(\frac{\pi}{2} + 8\pi\right) = \sin\frac{\pi}{2}$, the correct match is (I).

d. Since $\sin(21\pi) = \sin(\pi + 20\pi) = \sin\pi$, the correct match is (V).

e. Since $\sin\left(\frac{11\pi}{2}\right) = \sin\left(\frac{3\pi}{2} + 4\pi\right) = \sin\left(\frac{3\pi}{2}\right)$, the correct match is (III).

Now try Exercises 11 and 12 ▶

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Many of the transformations applied to algebraic graphs can also be applied to trigonometric graphs. These transformations may stretch, reflect, or translate the graph, but it will still retain its basic shape. In numerous applications, it will help if you're able to draw a quick, accurate sketch of the transformations involving $f(t) = \sin t$. To assist this effort, we'll begin with the interval $[0, 2\pi]$, combine the characteristics just listed with some simple geometry, and offer the following four-step process.

Step I: Draw the y-axis, mark zero halfway up, with -1 and 1 an equal distance from this zero. Then draw an extended t-axis and tick mark 2π to the extreme right (Figure 6.55).

Figure 6.55

Step II: On the t-axis, mark halfway between 0 and 2π and label it "π," mark halfway between π on either side and label the marks $\frac{\pi}{2}$ and $\frac{3\pi}{2}$. Halfway between these you can draw additional tick marks to represent the remaining multiples of $\frac{\pi}{4}$ (Figure 6.56).

Figure 6.56

Step III: Next, lightly draw a rectangular frame, which we'll call the **reference rectangle,** $P = 2\pi$ units wide and 2 units tall, centered on the t-axis and with the y-axis along the left side (Figure 6.57).

Figure 6.57

Step IV: Knowing $y = \sin t$ is positive and increasing in QI, that the range is $[-1, 1]$, that the zeroes are 0, π, and 2π, and that maximum and minimum values *occur halfway between the zeroes* (since there is no horizontal shift), we can draw a reliable graph of $y = \sin t$ by partitioning the rectangle into four equal parts to locate these values (note **bold** tick-marks). We will call this partitioning of the reference rectangle the **rule of fourths,** since we are then scaling the t-axis in increments of $\frac{P}{4}$ (Figure 6.58).

Figure 6.58

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EXAMPLE 3 ▶ **Graphing $y = \sin t$ Using a Reference Rectangle**

Use steps I through IV to draw a sketch of $y = \sin t$ for the interval $\left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$.

Solution ▶ Start by completing steps I and II, then extend the t-axis to include -2π. Beginning at $t = 0$, draw a reference rectangle 2π units wide and 2 units tall, centered on the t-axis. After applying the rule of fourths, we note the zeroes occur at $t = 0$, $t = \pi$, and $t = 2\pi$, while the max/min values fall halfway between them at $t = \frac{\pi}{2}$ and $t = \frac{3\pi}{2}$ (see Figure 6.59). Plot these points and connect them with a smooth, dashed curve. This is the primary period of the sine curve.

Figure 6.59

Using the periodic nature of the sine function, we can also graph the sine curve on the interval $[-2\pi, 0]$, as shown in Figure 6.60. The rule of fourths again helps to locate the zeroes and max/min values (note the bold tick-marks) over this interval.

Figure 6.60

For the graph of $y = \sin t$ in $\left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$, we simply highlight the graph in this interval using a solid curve, as shown in Figure 6.61.

WORTHY OF NOTE

Since the interval $\left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$ is $P = 2\pi$ units wide, this section of the graph can generate the entire graph of $y = \sin t$.

Figure 6.61

Now try Exercises 13 and 14 ▶

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A. You've just seen how we can graph $f(t) = \sin t$ using special values and symmetry

We can verify the results of Example 3 by changing the WINDOW settings from the predefined ZOOM **7:ZTrig** to Xmin $= -\frac{\pi}{2}$, Xmax $= \frac{3\pi}{2}$, Ymin $= -2$, and Ymax $= 2$. The GRAPH of $Y_1 = \sin X$ with these window settings is shown in Figure 6.62.

Figure 6.62

B. Graphing $f(t) = \cos t$

With the graph of $f(t) = \sin t$ established, sketching the graph of $f(t) = \cos t$ is a very natural next step. First, note that when $t = 0$, $\cos t = 1$ so the graph of $y = \cos t$ will begin at $(0, 1)$ in the interval $[0, 2\pi]$. Second, we've seen $\left(\pm\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$, $\left(\pm\frac{\sqrt{3}}{2}, \pm\frac{1}{2}\right)$ and $\left(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2}\right)$ are all points on the unit circle since they satisfy $x^2 + y^2 = 1$. Since $\cos t = x$ and $\sin t = y$, the Pythagorean identity $\cos^2 t + \sin^2 t = 1$ can be obtained by direct substitution. This means if $\sin t = \pm\frac{1}{2}$, then $\cos t = \pm\frac{\sqrt{3}}{2}$ and vice versa, with the signs taken from the appropriate quadrant. Similarly, if $\sin t = 0$, then $\cos t = \pm 1$ and vice versa. The table of values for cosine then becomes a simple variation of the table for sine, as shown in Table 6.5 for $t \in [0, \pi]$.

Table 6.5

t	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\sin t$	0	$\frac{1}{2} = 0.5$	$\frac{\sqrt{2}}{2} \approx 0.71$	$\frac{\sqrt{3}}{2} \approx 0.87$	1	$\frac{\sqrt{3}}{2} \approx 0.87$	$\frac{\sqrt{2}}{2} \approx 0.71$	$\frac{1}{2} = 0.5$	0
$\cos t$	$\mathbf{1}$	$\frac{\sqrt{3}}{2} \approx \mathbf{0.87}$	$\frac{\sqrt{2}}{2} \approx \mathbf{0.71}$	$\frac{1}{2} = \mathbf{0.5}$	$\mathbf{0}$	$-\frac{1}{2} = \mathbf{-0.5}$	$-\frac{\sqrt{2}}{2} \approx \mathbf{-0.71}$	$-\frac{\sqrt{3}}{2} \approx \mathbf{-0.87}$	$\mathbf{-1}$

WORTHY OF NOTE

As with the sine function, a graphing calculator can quickly produce a table of points on the graph of the cosine function. With the independent variable in the table set to **Ask** mode, we generate the following special values, which also appear in Table 6.5 and Figure 6.63.

X	Y1
0	1
.5236	.86603
.7854	.70711
1.0472	.5
1.5708	0
2.0944	-.5
2.3562	-.7071

Y1=cos(X)

The same values can be taken from the unit circle, but this view requires much less effort and easily extends to values of t in $[\pi, 2\pi]$. Using the points from Table 6.5 and its extension through $[\pi, 2\pi]$, we can draw the graph of $y = \cos t$ in $[0, 2\pi]$ and identify where the function is increasing and decreasing in this interval. See Figure 6.63.

Figure 6.63

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The function is decreasing for t in $(0, \pi)$, and increasing for t in $(\pi, 2\pi)$. The end result appears to be the graph of $y = \sin t$ shifted to the left $\frac{\pi}{2}$ units, a fact more easily seen if we extend the graph to $-\frac{\pi}{2}$ as shown. This is in fact the case, and is a relationship we will later prove in Chapter 7. Like $y = \sin t$, the function $y = \cos t$ is periodic with period $P = 2\pi$, with the graph extending infinitely in both directions.

Finally, we note that cosine is an **even function,** meaning $\cos(-t) = \cos t$ for all t in the domain. For instance, $\cos\left(-\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) = 0$ (see Figure 6.63). Here is a summary of important characteristics of the cosine function.

Characteristics of $f(t) = \cos t$

For all real numbers t and integers k,

Domain	Range	Period
$(-\infty, \infty)$	$[-1, 1]$	2π
Symmetry	**Maximum value**	**Minimum value**
even $\cos(-t) = \cos t$	$\cos t = 1$ at $t = 2\pi k$	$\cos t = -1$ at $t = \pi + 2\pi k$
Increasing	**Decreasing**	**Zeroes**
$(\pi, 2\pi)$	$(0, \pi)$	$t = \frac{\pi}{2} + \pi k$

EXAMPLE 4 ▶ **Graphing $y = \cos t$ Using a Reference Rectangle**

Draw a sketch of $y = \cos t$ for t in $\left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$ and use a graphing calculator to check your graph.

Solution ▶ As with the graph of $y = \sin t$, begin by completing steps I and II, then extend the t-axis to include -2π. Beginning at $t = 0$, draw a reference rectangle 2π units wide and 2 units tall, centered on the t-axis. After applying the rule of fourths, we note the zeroes occur at $t = \frac{\pi}{2}$ and $t = \frac{3\pi}{2}$ with the max/min values at $t = 0$, $t = \pi$, and $t = 2\pi$. Plot these points and connect them with a smooth, dashed curve (see Figure 6.64). This is the primary period of the cosine curve.

Figure 6.64

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WORTHY OF NOTE

We also could have graphed the cosine curve in this interval using the table

t	$\cos t$
$-\frac{\pi}{2}$	0
0	1
$\frac{\pi}{2}$	0
π	-1
$\frac{3\pi}{2}$	0

and connecting these points with a smooth curve.

Using the periodic nature of the cosine function, we can also graph the cosine curve on the interval $[-2\pi, 0]$, as shown in Figure 6.65. The rule of fourths again helps to locate the zeroes and max/min values (note the bold tick-marks) over this interval.

Figure 6.65

For the graph of $y = \cos t$ in $\left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$, we simply highlight the graph in this interval using a solid curve, as shown in Figure 6.66.

Figure 6.66

Check ▶ With the WINDOW settings shown, the graphs are identical. ✓

B. You've just seen how we can graph $f(t) = \cos t$ using special values and symmetry

Now try Exercises 15 and 16 ▶

C. Graphing $y = A\sin(Bt)$ and $y = A\cos(Bt)$

In many applications, trig functions have maximum and minimum values other than 1 and -1, and periods other than 2π. For instance, in tropical regions the daily maximum and minimum temperatures may vary by no more than 20°, while for desert regions this difference may be 40° or more. This variation is modeled by the *amplitude* of the sine and cosine functions.

WORTHY OF NOTE

Because the functions $\sin t$ and $\cos t$ take on both positive and negative values, we can use the graphs of $y = |A \sin t|$ and $y = |A \cos t|$ to emphasize the role of amplitude $|A|$ as the *maximum* displacement. **See Exercises 17 through 20.**

Amplitude and the Coefficient A (assume $B = 1$)

For functions of the form $y = A \sin t$ and $y = A \cos t$, let M represent the *M*aximum value and m the *m*inimum value of the functions. Then the quantity $\frac{M + m}{2}$ gives the

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average value of the function, while $\frac{M-m}{2}$ gives the **amplitude** of the function. Amplitude is the maximum displacement from the average value in the positive or negative direction. It is represented by $|A|$, with A playing a role similar to that seen for algebraic graphs [$Af(t)$ vertically stretches or compresses the graph of f, and reflects it across the t-axis if $A < 0$]. Graphs of the form $y = \sin t$ (and $y = \cos t$) can quickly be sketched with any amplitude by noting (1) the *zeroes of the function remain fixed* since $\sin t = 0$ implies $A \sin t = 0$, and (2) the *maximum and minimum values are A and $-A$*, respectively, since $\sin t = 1$ or -1 implies $A \sin t = A$ or $-A$. Note this implies the reference rectangle will be $2A$ units tall and P units wide. Connecting the points that result with a smooth curve will complete the graph.

EXAMPLE 5 ▶ **Graphing $y = A \sin t$ Where $A \neq 1$**

Draw a sketch of $y = 4 \sin t$ in the interval $[0, 2\pi]$.

Solution ▶ With an amplitude of $|A| = 4$, the reference rectangle will be $2(4) = 8$ units tall, by 2π units wide. Using the rule of fourths, the zeroes are still $t = 0$, $t = \pi$, and $t = 2\pi$, with the max/min values spaced equally between. The maximum value is $4 \sin\left(\frac{\pi}{2}\right) = 4(1) = 4$, with a minimum value of $4 \sin\left(\frac{3\pi}{2}\right) = 4(-1) = -4$. Connecting these points with a "sine curve" gives the graph shown ($y = \sin t$ is also shown for comparison).

Now try Exercises 21 through 26 ▶

In the graph for Example 5, we note the zeroes of $y = A \sin t$ remained fixed for $A = 1$ and $A = 4$. Additionally, it appears the max/min values occur at the same t-values. We can use a graphing calculator to further investigate these observations. On the Y= screen, enter $Y_1 = \frac{1}{2} \sin X$, $Y_2 = \sin X$, $Y_3 = 2 \sin X$, and $Y_4 = 4 \sin X$, then use ZOOM **7:ZTrig** to graph the functions (note Y_2 and Y_4 were graphed in Example 5). As you see in Figure 6.67, each graph "holds on" to the zeroes, while rising to the expected amplitude $|A|$ halfway between these zeroes.

Figure 6.67

Period and the Coefficient *B*

While basic sine and cosine functions have a period of 2π, in many applications the period may be very long (tsunamis) or very short (electromagnetic waves). For the equations $y = A \sin(Bt)$ and $y = A \cos(Bt)$, the period depends on the value of B. To see why, consider the function $y = \cos(2t)$ and Table 6.6. Multiplying input values by 2 means each cycle will be completed twice as fast. The table shows that $y = \cos(2t)$

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completes a full cycle in $[0, \pi]$, giving a period of $P = \pi$. Figure 6.68 verifies these observations, with the graphs of $Y_1 = \cos(X)$ and $Y_2 = \cos(2X)$ displayed over the interval $[0, 2\pi]$.

Figure 6.68

Table 6.6

t	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
$2t$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\cos(2t)$	1	0	-1	0	1

Dividing input values by 2 (or multiplying by $\frac{1}{2}$) will cause the function to complete a cycle only half as fast, doubling the time required to complete a full cycle. Table 6.7 shows $y = \cos(\frac{1}{2}t)$ completes only one-half cycle in 2π. Figure 6.69 shows the graphs of $Y_1 = \cos(X)$ and $Y_2 = \cos((1/2)X)$.

Figure 6.69

Table 6.7
(values in blue are approximate)

t	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
$\frac{1}{2}t$	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{3\pi}{8}$	$\frac{\pi}{2}$	$\frac{5\pi}{8}$	$\frac{3\pi}{4}$	$\frac{7\pi}{8}$	π
$\cos\left(\frac{1}{2}t\right)$	1	0.92	$\frac{\sqrt{2}}{2}$	0.38	0	-0.38	$-\frac{\sqrt{2}}{2}$	-0.92	-1

The graphs of $y = \cos t$, $y = \cos(2t)$, and $y = \cos(\frac{1}{2}t)$ shown in Figure 6.70 clearly illustrate this relationship and how the value of B affects the period of a graph.

Figure 6.70

To find the period for arbitrary values of B, the formula $P = \dfrac{2\pi}{|B|}$ is used. Note for $y = \cos(2t)$, $B = 2$ and $P = \dfrac{2\pi}{2} = \pi$, as shown. For $y = \cos\left(\dfrac{1}{2}t\right)$, $|B| = \dfrac{1}{2}$, and $P = \dfrac{2\pi}{1/2} = 4\pi$.

Period Formula for Sine and Cosine

For B a real number and functions $y = A\sin(Bt)$ and $y = A\cos(Bt)$,

$$P = \frac{2\pi}{|B|}.$$

To sketch these functions for periods other than 2π, we still use a reference rectangle of height $2A$ and length P, then break the enclosed t-axis into four equal parts to help draw the graph. In general, if the period is "very large" one full cycle is appropriate for the graph. If the period is "very small," graph at least two cycles.

Note the value of B in Example 6 includes a factor of π. This actually happens quite frequently in applications of the trig functions.

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EXAMPLE 6 ▶ **Graphing $y = A\cos(Bt)$, Where $A, B \neq 1$**

Draw a sketch of $y = -2\cos(0.4\pi t)$ for t in $[-\pi, 2\pi]$.

Solution ▶ The amplitude is $|A| = 2$, so the reference rectangle will be $2(2) = 4$ units high. Since $A < 0$, the *graph will be vertically reflected across the t-axis*. The period is $P = \dfrac{2\pi}{0.4\pi} = 5$ (note the factors of π reduce to 1), so the reference rectangle will be 5 units in length. Breaking the t-axis into four parts within the frame (rule of fourths) gives $(\frac{1}{4})5 = \frac{5}{4}$ units, indicating that we should scale the t-axis in multiples of $\frac{1}{4}$. Note the zeroes occur at $\frac{5}{4}$ and $\frac{15}{4}$, with a maximum value at $\frac{10}{4}$. In cases where the π factor reduces, we scale the t-axis as a "standard" number line, and *estimate the location of multiples of* π. For practical reasons, we first draw the unreflected graph (shown in blue) for guidance in drawing the reflected graph, which is then extended to fit the given interval.

Now try Exercises 27 through 38 ▶

While graphing calculators can quickly and easily graph functions over a given interval, understanding the analytical solution presented remains essential. A true test of effective calculator use comes when the amplitude or period is a very large or a very small number. For instance, the tone you hear while pressing "5" on your telephone is actually a combination of the tones modeled by $Y_1 = \sin[2\pi(770)X]$ and $Y_2 = \sin[2\pi(1336)X]$. Graphing these functions requires a careful analysis of the period, otherwise the graph can appear garbled, misleading, or difficult to read (try graphing Y_1 on the ZOOM **7:ZTrig** screen). Firstnote for Y_1, $A = 1$ and $P = \dfrac{2\pi}{2\pi(770)}$ or $\dfrac{1}{770}$. With a period this short, even graphing the function from Xmin $= -1$ to Xmax $= 1$ gives a distorted graph (see Figure 6.71). Because of the calculated period of this function, we set Xmin to $-1/770$, Xmax to $1/770$ and Xscl to $(1/770)/10$. This gives the graph in Figure 6.72, which can then be used to investigate characteristics of the function. **See Exercises 39 and 40.**

Figure 6.71

Figure 6.72

✓ **C.** You've just seen how we can graph sine and cosine functions with various amplitudes and periods

D. Writing Equations from Graphs

Mathematical concepts are best reinforced by working with them in both "forward and reverse." Where graphs are concerned, this means we should attempt to find the equation of a given graph, rather than only using an equation to sketch the graph. Exercises

of this type require that you become very familiar with the graph's basic characteristics and how each is expressed as part of the equation.

EXAMPLE 7 ▶ **Determining the Equation of a Given Graph**

The graph shown here is of the form $y = A\sin(Bt)$. Find the values of A and B.

Solution ▶ By inspection, the graph has an amplitude of $A = 2$ and a period of $P = \frac{3\pi}{2}$. To find B we used the period formula $P = \frac{2\pi}{|B|}$, substituting $\frac{3\pi}{2}$ for P and solving.

$$P = \frac{2\pi}{|B|} \quad \text{period formula}$$

$$\frac{3\pi}{2} = \frac{2\pi}{B} \quad \text{substitute } \frac{3\pi}{2} \text{ for } P;\ B > 0$$

$$3\pi B = 4\pi \quad \text{multiply by } 2B$$

$$B = \frac{4}{3} \quad \text{solve for } B$$

The result is $B = \frac{4}{3}$, which gives us the equation $y = 2\sin(\frac{4}{3}t)$.

D. You've just seen how we can write the equation for a given graph

Now try Exercises 41 through 60 ▶

There are a number of interesting applications of this "graph to equation" process in the exercise set. **See Exercises 63 to 74.**

6.3 EXERCISES

▶ CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section if needed.

1. For the sine function, output values are _____ in the interval $\left[0, \frac{\pi}{2}\right]$.

2. For the cosine function, output values are _____ in the interval $\left[0, \frac{\pi}{2}\right]$.

3. For the sine and cosine functions, the domain is _____ and the range is _____.

4. The amplitude of sine and cosine is defined to be the maximum _____ from the _____ value in the positive and negative directions.

5. Discuss/Explain how the values generated by the unit circle can be used to graph the function $f(t) = \sin t$. Be sure to include the domain and range of this function in your discussion.

6. Discuss/Describe the four-step process outlined in this section for the graphing of basic trig functions. Include a worked-out example and a detailed explanation.

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▶ DEVELOPING YOUR SKILLS

7. Use the symmetry of the unit circle and reference arcs of standard values to complete a table of values for $y = \cos t$ in the interval $t \in [\pi, 2\pi]$.

8. Use the standard values for $y = \cos t$ for $t \in [\pi, 2\pi]$ to create a table of values for $y = \sin t$ on the same interval.

For the following, (a) simplify the given values, (b) use a calculator to evaluate the sine of any nonstandard value, and (c) compare your results to Figure 6.50 on page 544.

9. $\frac{1\pi}{12}, \frac{2\pi}{12}, \frac{3\pi}{12}$

10. $\frac{4\pi}{12}, \frac{5\pi}{12}, \frac{6\pi}{12}$

Use the characteristics of $f(t) = \sin t$ to match the given value of t to the correct value of $\sin t$.

11. a. $t = \frac{\pi}{6} + 10\pi$ **b.** $t = -\frac{\pi}{4}$ **c.** $t = \frac{-15\pi}{4}$ **d.** $t = 13\pi$ **e.** $t = \frac{21\pi}{2}$

I. $\sin t = 0$ **II.** $\sin t = \frac{1}{2}$ **III.** $\sin t = 1$ **IV.** $\sin t = \frac{\sqrt{2}}{2}$ **V.** $\sin t = -\frac{\sqrt{2}}{2}$

12. a. $t = \frac{\pi}{4} - 12\pi$ **b.** $t = \frac{11\pi}{6}$ **c.** $t = \frac{23\pi}{2}$ **d.** $t = -19\pi$ **e.** $t = -\frac{25\pi}{4}$

I. $\sin t = -\frac{1}{2}$ **II.** $\sin t = -\frac{\sqrt{2}}{2}$ **III.** $\sin t = 0$ **IV.** $\sin t = \frac{\sqrt{2}}{2}$ **V.** $\sin t = -1$

Use steps I through IV given in this section to draw a sketch of each graph. Verify your results with a graphing calculator.

13. $y = \sin t$ for $t \in \left[-\frac{3\pi}{2}, \frac{\pi}{2}\right]$

14. $y = \sin t$ for $t \in [-\pi, \pi]$

15. $y = \cos t$ for $t \in \left[-\frac{\pi}{2}, 2\pi\right]$

16. $y = \cos t$ for $t \in \left[-\frac{\pi}{2}, \frac{5\pi}{2}\right]$

Earlier we defined the amplitude of the sine and cosine functions as the maximum displacement from the average value. When the average value is 0 (no vertical translation), this "maximum displacement" can more clearly be seen using the concept of absolute value. The graphs shown in Exercises 17 to 20 are of the form $y = |A \sin t|$ or $y = |A \cos t|$, where A is the amplitude of the function. Use the graphs to state the value of A.

17.

18.

19.

20.

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Use a reference rectangle and the *rule of fourths* to draw an accurate sketch of the following functions through two complete cycles—one where $t > 0$, and one where $t < 0$. Clearly state the amplitude and period as you begin.

21. $y = 3 \sin t$

22. $y = 4 \sin t$

23. $y = -2 \cos t$

24. $y = -3 \cos t$

25. $y = \frac{1}{2} \sin t$

26. $y = \frac{3}{4} \sin t$

27. $y = -\sin(2t)$

28. $y = -\cos(2t)$

29. $y = 0.8 \cos(2t)$

30. $y = 1.7 \sin(4t)$

31. $f(t) = 4 \cos\left(\frac{1}{2}t\right)$

32. $y = -3 \cos\left(\frac{3}{4}t\right)$

33. $f(t) = 3 \sin(4\pi t)$

34. $g(t) = 5 \cos(8\pi t)$

35. $y = 4 \sin\left(\frac{5\pi}{3}t\right)$

36. $y = 2.5 \cos\left(\frac{2\pi}{5}t\right)$

37. $f(t) = 2 \sin(256\pi t)$

38. $g(t) = 3 \cos(184\pi t)$

39. Graph the tone $Y_2 = \sin[2\pi(1336)X]$ and find its value at $X = 0.00025$. Identify the settings of your "friendly" viewing window.

40. Graph the function $Y_3 = 950 \sin(0.005X)$ and find the value at $X = 550$. Identify the settings of your "friendly" viewing window.

Clearly state the amplitude and period of each function, then match it with the corresponding graph.

41. $y = -2 \cos(4t)$

42. $y = 2 \sin(4t)$

43. $y = 3 \sin(2t)$

44. $y = -3 \cos(2t)$

45. $f(t) = \frac{3}{4} \cos(0.4t)$

46. $g(t) = \frac{7}{4} \cos(0.8t)$

47. $y = 4 \sin(144\pi t)$

48. $y = 4 \cos(72\pi t)$

a.

b.

c.

d.

e.

f.

g.

h.

The graphs shown are of the form $y = A \cos(Bt)$. Use the characteristics illustrated for each graph to determine its equation.

49.

50.

51.

52.

The graphs shown are of the form $y = A \sin(Bt)$. Use the characteristics illustrated for each graph to determine its equation.

53.

54.

55.

56.

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Match each graph to its equation, then graphically estimate the points of intersection. Confirm or contradict your estimate(s) by substituting the values into the given equations using a calculator.

57. $y = -\cos x$; $y = \sin x$

58. $y = -\cos x$; $y = \sin(2x)$

59. $y = -2\cos x$; $y = 2\sin(3x)$

60. $y = 2\cos(2\pi x)$; $y = -2\sin(\pi x)$

▶ WORKING WITH FORMULAS

61. Area of a regular polygon inscribed in a circle: $\mathcal{A} = \dfrac{nr^2}{2}\sin\left(\dfrac{2\pi}{n}\right)$

The formula shown gives the area of a regular polygon inscribed in a circle, where n is the number of sides ($n \geq 3$) and r is the radius of the circle. Given $r = 10$ cm,

Exercise 61

a. What is the area of the circle?

b. What is the area of the polygon when $n = 4$? Find the length of the sides of the polygon using two different methods.

c. Calculate the area of the polygon for $n = 10$, 20, 30, and 100. What do you notice?

62. Hydrostatics, surface tension, and contact angles: $y = \dfrac{2\gamma\cos\theta}{kr}$

The height that a liquid will rise in a capillary tube is given by the formula shown, where r is the radius of the tube, θ is the contact angle of the liquid (the meniscus), γ is the surface tension of the liquid-vapor film, and k is a constant that depends on the weight-density of the liquid. How high will the liquid rise given that the surface tension $\gamma = 0.2706$, the tube has radius $r = 0.2$ cm, the contact angle $\theta = 22.5°$, and $k = 1.25$?

▶ APPLICATIONS

Tidal waves: Tsunamis, also known as tidal waves, are ocean waves produced by earthquakes or other upheavals in the Earth's crust and can move through the water undetected for hundreds of miles at great speed. While traveling in the open ocean, these waves can be represented by a sine graph with a very long wavelength (period) and a very small amplitude. Tsunami waves only attain a monstrous size as they approach the shore, and represent a very different phenomenon than the ocean swells created by heavy winds over an extended period of time.

63. A graph modeling a tsunami wave is given in the figure. (a) What is the height of the tsunami wave (from crest to trough)? Note that $h = 0$ is considered the level of a calm ocean. (b) What is the tsunami's wavelength (period)? (c) Find an equation for this wave.

64. A heavy wind is kicking up ocean swells approximately 10 ft high (from crest to trough), with wavelengths (periods) of 250 ft. (a) Find an equation that models these swells. (b) Graph the equation. (c) Determine the height of a wave measured 200 ft from the trough of the previous wave.

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Sinusoidal models: The sine and cosine functions are of great importance to meteorological studies, as when modeling the temperature based on the time of day, the illumination of the moon as it goes through its phases, or even the prediction of tidal motion.

65. The graph given shows the deviation from the average daily temperature for the hours of a given day, with $t = 0$ corresponding to 6 A.M. (a) Use the graph to determine the related equation. (b) Use the equation to find the deviation at $t = 11$ (5 P.M.) and confirm that this point is on the graph. (c) If the average temperature for this day was 72°, what was the temperature at midnight?

66. The equation $y = 7\sin\left(\frac{\pi}{6}t\right)$ models the height of the tide along a certain coastal area, as compared to average sea level. Assuming $t = 0$ is midnight, (a) graph this function over a 12-hr period. (b) What will the height of the tide be at 5 A.M.? (c) Is the tide rising or falling at this time?

Sinusoidal movements: Many animals exhibit a wavelike motion in their movements, as in the tail of a shark as it swims in a straight line or the wingtips of a large bird in flight. Such movements can be modeled by a sine or cosine function and will vary depending on the animal's size, speed, and other factors.

67. The graph shown models the position of a shark's tail at time t, as measured to the left (negative) and right (positive) of a straight line along its length. (a) Use the graph to determine the related equation. (b) Is the tail to the right, left, or at center when $t = 6.5$ sec? How far? (c) Would you say the shark is "swimming leisurely," or "chasing its prey"? Justify your answer.

68. The State Fish of Hawaii is the *humuhumunukunukuapua'a*, a small colorful fish found abundantly in coastal waters. Suppose the tail motion of an adult fish is modeled by the equation $d(t) = \sin(15\pi t)$ with $d(t)$ representing the position of the fish's tail at time t, as measured in inches to the left (negative) or right (positive) of a straight line along its length. (a) Graph the equation over two periods. (b) Is the tail to the left or right of center at $t = 2.7$ sec? How far? (c) Would you say this fish is "swimming leisurely," or "running for cover"? Justify your answer.

Kinetic energy: The kinetic energy a planet possesses as it orbits the Sun can be modeled by a cosine function. When the planet is at its apogee (greatest distance from the Sun), its kinetic energy is at its lowest point as it slows down and "turns around" to head back toward the Sun. The kinetic energy is at its highest when the planet "whips around the Sun" to begin a new orbit.

69. Two graphs are given here. (a) Which of the graphs could represent the kinetic energy of a planet orbiting the Sun if the planet is at its perigee (closest distance to the Sun) when $t = 0$? (b) For what value(s) of t does this planet possess 62.5% of its maximum kinetic energy with the kinetic energy increasing? (c) What is the orbital period of this planet?

a.

b.

70. The *potential energy* of the planet is the antipode of its kinetic energy, meaning when kinetic energy is at 100%, the potential energy is 0%, and when kinetic energy is at 0% the potential energy is at 100%. (a) How is the graph of the kinetic energy related to the graph of the potential energy? In other words, what transformation could be applied to the kinetic energy graph to obtain the potential energy graph? (b) If the kinetic energy is at 62.5% and increasing, what can be said about the potential energy in the planet's orbit at this time?

Visible light: One of the narrowest bands in the electromagnetic spectrum is the region involving visible light. The wavelengths (periods) of visible light vary from 400 nanometers (purple/violet colors) to 700 nanometers (bright red). The approximate wavelengths of the other colors are shown in the diagram.

71. The equations for the colors in this spectrum have the form $y = \sin(\gamma t)$, where $\frac{2\pi}{\gamma}$ gives the length of the sine wave. (a) What color is represented by the equation $y = \sin\left(\frac{\pi}{240}t\right)$? (b) What color is represented by the equation $y = \sin\left(\frac{\pi}{310}t\right)$?

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72. Name the color represented by each of the graphs (a) and (b) and write the related equation.

a.

b.

Alternating current: Surprisingly, even characteristics of the electric current supplied to your home can be modeled by sine or cosine functions. For alternating current (AC), the amount of current I (in amps) at time t can be modeled by $I = A\sin(\omega t)$, where A represents the maximum current that is produced, and ω is related to the frequency at which the generators turn to produce the current.

73. Find an equation of the household current modeled by the graph, then use the equation to determine I when $t = 0.045$ sec. Verify that the resulting ordered pair is on the graph.

Exercise 73

74. If the *voltage* produced by an AC circuit is modeled by the equation $E = 155\sin(120\pi t)$, (a) what is the period and amplitude of the related graph? (b) What voltage is produced when $t = 0.2$?

▶ EXTENDING THE CONCEPT

75. For $y = A\sin(Bx)$ and $y = A\cos(Bx)$, the expression $\dfrac{M + m}{2}$ gives the average value of the function, where M and m represent the maximum and minimum values, respectively. What was the average value of every function graphed in this section? Compute a table of values for $y = 2\sin t + 3$, and note its maximum and minimum values. What is the average value of this function? What transformation has been applied to change the average value of the function? Can you name the average value of $y = -2\cos t + 1$ by inspection?

76. To understand where the period formula $P = \dfrac{2\pi}{B}$ came from, consider that if $B = 1$, the graph of $y = \sin(Bt) = \sin(1t)$ completes one cycle from $1t = 0$ to $1t = 2\pi$. If $B \neq 1$, $y = \sin(Bt)$ completes one cycle from $Bt = 0$ to $Bt = 2\pi$. Discuss how this observation validates the period formula.

77. The tone you hear when pressing the digit "9" on your telephone is actually a combination of two separate tones, which can be modeled by the functions $f(t) = \sin[2\pi(852)t]$ and $g(t) = \sin[2\pi(1477)t]$. Which of the two functions has the shorter period? By carefully scaling the axes, graph the function having the shorter period using the steps I through IV discussed in this section.

▶ MAINTAINING YOUR SKILLS

78. (6.2) Given $\sin 1.12 \approx 0.9$, find an additional value of t in $[0, 2\pi)$ that makes the equation $\sin t \approx 0.9$ true.

79. (6.1) Use a special triangle to calculate the distance from the ball to the pin on the seventh hole, given the ball is in a straight line with the 100-yd plate, as shown in the figure.

Exercise 79

80. (6.1) Invercargill, New Zealand, is at 46° 14′ 24″ south latitude. If the Earth has a radius of 3960 mi, how far is Invercargill from the equator?

81. (3.1) Given $z_1 = 1 + i$ and $z_2 = 2 - 5i$, compute the following:

a. $z_1 + z_2$

b. $z_1 - z_2$

c. $z_1 z_2$

d. $\dfrac{z_2}{z_1}$

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6.4 Graphs of the Cosecant, Secant, Tangent, and Cotangent Functions

LEARNING OBJECTIVES

In Section 6.4 you will see how we can:

- ❑ **A.** Graph the functions $y = A\csc(Bt)$ and $y = A\sec(Bt)$
- ❑ **B.** Graph $y = \tan t$ using asymptotes, zeroes, and the ratio $\frac{\sin t}{\cos t}$
- ❑ **C.** Graph $y = \cot t$ using asymptotes, zeroes, and the ratio $\frac{\cos t}{\sin t}$
- ❑ **D.** Identify and discuss important characteristics of $y = \tan t$ and $y = \cot t$
- ❑ **E.** Graph $y = A\tan(Bt)$ and $y = A\cot(Bt)$ with various values of A and B

Unlike sine and cosine, the cosecant, secant, tangent, and cotangent functions have no maximum or minimum values over their domains. However, it is precisely this unique feature that adds to their value as mathematical models. Collectively, these six trig functions give scientists the tools they need to study, explore, and investigate a wide range of phenomena, extending our understanding of the world around us.

A. Graphs of $y = A\csc(Bt)$ and $y = A\sec(Bt)$

From our earlier work, we know that $y = \sin t$ and $y = \csc t$ are reciprocal functions: $\csc t = \frac{1}{\sin t}$. Likewise, we have $\sec t = \frac{1}{\cos t}$. The graphs of these reciprocal functions follow quite naturally from the graphs of $y = A\sin(Bt)$ and $y = A\cos(Bt)$ by using these observations:

1. You cannot divide by zero.
2. The reciprocal of a very small number ($|x| < 1$) is a very large number (and vice versa).
3. The reciprocal of ± 1 is ± 1.

Just as with rational functions, division by zero creates a vertical asymptote, so the graph of $y = \csc t = \frac{1}{\sin t}$ will have a vertical asymptote at every point where $\sin t = 0$. This occurs at $t = \pi k$, where k is an integer $(\ldots, -2\pi, -\pi, 0, \pi, 2\pi, \ldots)$. The table shown in Figure 6.73 shows that as $x \to \pi$, $Y_1 = \sin x$ approaches zero, while $Y_2 = \frac{1}{Y_1} = \csc X$ becomes infinitely large. Further, when $\sin(Bt) = \pm 1$, $\csc(Bt) = \pm 1$ since the reciprocals of 1 and -1 are still 1 and -1, respectively. Finally, due to observation 2, the graph of the cosecant function will be increasing when the sine function is decreasing, and decreasing when the sine function is increasing.

Figure 6.73

X	Y1	Y2
3	.14112	7.0862
3.1	.04158	24.05
3.14	.00159	627.88
3.141	5.9E-4	1687.3
3.1415	9.3E-5	10793
3.1416	2.7E-6	376848
3.1417	0	ERROR

X=π

Considering the case where $A = 1$, we can graph $y = \csc(Bt)$ by first drawing a sketch of $y = \sin(Bt)$. Then using the previous observations and a few well-known values, we can accurately complete the graph. From this graph, we discover that the period of the cosecant function is also $\frac{2\pi}{|B|}$ and that $y = \csc(Bt)$ is an *odd* function.

EXAMPLE 1 ▶ **Graphing $y = A\csc(Bt)$ for $A = 1$**

Graph the function $y = \csc t$ over the interval $[0, 4\pi]$.

Solution ▶ Begin by sketching the function $y = \sin t$, using a standard reference rectangle $2A = 2(1) = 2$ units high by $\frac{2\pi}{B} = \frac{2\pi}{1} = 2\pi$ units in length. Since $\csc t = \frac{1}{\sin t}$, the graph of $y = \csc t$ will be asymptotic at the zeroes of $y = \sin t$ ($t = 0$, π, and 2π). As in Section 6.3, we can then extend the graph

t	$\sin t$	$\csc t$
0	0	$\frac{1}{0} \to$ undefined
$\frac{\pi}{6}$	$\frac{1}{2} = 0.5$	$\frac{2}{1} = 2$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2} \approx 0.71$	$\frac{2}{\sqrt{2}} \approx 1.41$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2} \approx 0.87$	$\frac{2}{\sqrt{3}} \approx 1.15$
$\frac{\pi}{2}$	1	1

into the interval $[2\pi, 4\pi]$ by reproducing the graph from $[0, 2\pi]$. A partial table and the resulting graph are shown.

- Vertical asymptotes where $\sin(Bt)$ is zero
- When $\sin(Bt) = \pm 1$, $\csc(Bt) = \pm 1$
- Output values are reciprocated

Now try Exercises 7 through 10 ▶

WORTHY OF NOTE

Due to technological limitations, some older graphing calculators may draw what appears to be the asymptotes of $y = \csc t$. It is important to realize these are *not* part of the graph.

With Y_1 and Y_2 entered as shown in Figure 6.74, a graphing calculator set in radian MODE will confirm the results of Example 1 (Figure 6.75).

Figure 6.74

Figure 6.75

Similar to Example 1, the graph of $y = \sec t = \dfrac{1}{\cos t}$ will have vertical asymptotes where $\cos t = 0 \left(t = \dfrac{\pi}{2} \text{ and } \dfrac{3\pi}{2}\right)$. Note that if $A \neq 1$, we would then use the graph of $y = A\cos(Bt)$ to graph $y = A\sec(Bt)$, as in Example 2.

EXAMPLE 2 ▶ **Graphing $y = A\sec(Bt)$ for $A, B \neq 1$**

Graph the function $y = 3\sec\left(\dfrac{\pi}{2}t\right)$ over the interval $[-2, 6]$ and use a graphing calculator to check your graph.

Solution ▶ Begin by sketching the function $y = 3\cos\left(\dfrac{\pi}{2}t\right)$ using a reference rectangle 6 units high by $\dfrac{2\pi}{B} = \dfrac{2\pi}{\pi/2} = 4$ units long as a guide. Within the rectangle, each special feature will be 1 unit apart (rule of fourths), with the asymptotes occurring at the zeroes of $y = 3\cos\left(\dfrac{\pi}{2}t\right)$ ($t = 1$ and $t = 3$). We then extend the graph to cover the interval $[-2, 6]$ by reproducing the

t	$3\cos\left(\frac{\pi}{2}t\right)$	$3\sec\left(\frac{\pi}{2}t\right)$
0	3	3
$\frac{1}{3}$	$3 \cdot \frac{\sqrt{3}}{2} \approx 2.60$	$3 \cdot \frac{2}{\sqrt{3}} \approx 3.46$
$\frac{1}{2}$	$3 \cdot \frac{\sqrt{2}}{2} \approx 2.12$	$3 \cdot \frac{2}{\sqrt{2}} \approx 4.24$
$\frac{2}{3}$	$3 \cdot \frac{1}{2} = 1.5$	$3 \cdot \frac{2}{1} = 6$
1	0	$\frac{1}{0} \rightarrow$ undefined

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appropriate sections of the graph. A partial table and the resulting graph are shown.

- Vertical asymptotes where $\cos(Bt)$ is zero
- When $A\cos(Bt) = \pm A$, $A\sec(Bt) = \pm A$

Check ▶ With Y_1 and Y_2 entered as shown, the graphs match. ✓

Now try Exercises 11 through 14 ▶

From the graph, we discover that the period of the secant function is also $\frac{2\pi}{B}$, and that $y = A\sec(Bt)$ is an *even* function. The most important characteristics of the cosecant and secant functions are summarized in the following box. Note that for these functions, there is no discussion of amplitude, and no mention is made of their zeroes since neither graph intersects the t-axis.

A. You've just seen how we can graph the functions $y = A\csc(Bt)$ and $y = A\sec(Bt)$

Characteristics of $f(t) = \csc t$ and $f(t) = \sec t$

For all real numbers t and integers k,

$y = \csc t$

Domain	Range	Asymptotes
$t \neq k\pi$	$(-\infty, -1] \cup [1, \infty)$	$t = k\pi$

Period	Symmetry
2π	odd $\csc(-t) = -\csc t$

$y = \sec t$

Domain	Range	Asymptotes
$t \neq \frac{\pi}{2} + \pi k$	$(-\infty, -1] \cup [1, \infty)$	$t = \frac{\pi}{2} + \pi k$

Period	Symmetry
2π	even $\sec(-t) = \sec t$

B. The Graph of $y = \tan t$

Like the secant and cosecant functions, tangent is defined in terms of a ratio, creating asymptotic behavior at the zeroes of the denominator. In terms of the unit circle, $\tan t = \frac{y}{x}$, which means in $[-\pi, 2\pi]$, vertical asymptotes occur at $t = -\frac{\pi}{2}$, $t = \frac{\pi}{2}$,

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Figure 6.76

and $\frac{3\pi}{2}$, since the x-coordinate on the unit circle is zero (see Figure 6.76). We further note $\tan t = 0$ when the y-coordinate is zero, so the function will have t-intercepts at $t = -\pi$, 0, π, and 2π in the same interval. This produces the following framework for graphing the tangent function shown in Figure 6.77.

Figure 6.77

Figure 6.78

X	Y1
-1.571	ERROR
-1.57	-1256
-.7854	-1
0	0
1.5	14.101
1.57	1255.8
1.5708	ERROR

Y1=tan(X)

Knowing the graph must go through these zeroes and approach the asymptotes, we are left with determining the *direction of the approach.* This can be discovered by noting that in QI, the y-coordinates of points on the unit circle start at 0 and increase, while the x-values start at 1 and decrease. This means the ratio $\frac{y}{x}$ defining $\tan t$ is increasing, and in fact becomes infinitely large as t gets very close to $\frac{\pi}{2}$. (Figure 6.78). Using the additional points provided by $\tan\left(-\frac{\pi}{4}\right) = -1$ and $\tan\left(\frac{\pi}{4}\right) = 1$, we find the graph of $\tan t$ is increasing throughout the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and that the function has a period of π. We also note $y = \tan t$ is an odd function (symmetric about the origin), since $\tan(-t) = -\tan t$ as evidenced by the two points just computed. The completed graph is shown in Figure 6.79 with the primary interval in red.

Figure 6.79

The graph can also be developed by noting $\sin t = y$, $\cos t = x$, and $\tan t = \frac{y}{x}$. This gives $\tan t = \frac{\sin t}{\cos t}$ by direct substitution and we can quickly complete a table of values for $\tan t$, as shown in Example 3.

EXAMPLE 3 ▶ **Constructing a Table of Values for $f(t) = \tan t$**

Complete Table 6.8 shown for $\tan t = \dfrac{y}{x}$ using the values given for $\sin t$ and $\cos t$, then graph the function by plotting points.

Solution ▶ The completed table is shown here.

Table 6.8

t	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\sin t = y$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\cos t = x$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1
$\tan t = \frac{y}{x}$	0	$\frac{1}{\sqrt{3}} \approx 0.58$	1	$\sqrt{3} \approx 1.7$	undefined	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0

For the noninteger values of x and y, the "twos will cancel" each time we compute $\dfrac{y}{x}$. This means we can simply list the ratio of numerators. The resulting values are shown in red in Table 6.8, along with the corresponding plotted points. The graph shown in Figure 6.80 was completed using symmetry and the previous observations.

Figure 6.80

Now try Exercises 15 and 16 ▶

Additional values can be found using a calculator as needed. For future use and reference, it will help to recognize the approximate decimal equivalent of all special values and radian angles. In particular, note that $\sqrt{3} \approx 1.73$ and $\dfrac{1}{\sqrt{3}} \approx 0.58$. **See Exercises 17 through 22.** While we could easily use $Y_1 = \tan X$ on a graphing calculator to generate the graph shown in Figure 6.80, we would miss an opportunity to reinforce the previous observations using the ratio definition $\tan x = \dfrac{\sin x}{\cos x}$. To begin, enter $Y_1 = \sin X$, $Y_2 = \cos X$, and $Y_3 = \dfrac{Y_1}{Y_2}$, as shown in Figure 6.81 [recall that function variables are accessed using VARS ▸ **(Y-VARS)** ENTER **(1:Function)**].

Figure 6.81

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Note that Y_2 has been disabled by overlaying the cursor on the equal sign and pressing ENTER. Pressing ZOOM **7:ZTrig** at this point produces the screen shown in Figure 6.82, where we note that tan x is zero everywhere that sin x is zero. Similarly, disabling Y_1 while Y_2 is enabled emphasizes that the asymptotes of $y = \tan x$ occur everywhere cos x is zero (see Figure 6.83).

Figure 6.82

Figure 6.83

B. You've just seen how we can graph $y = \tan t$ using asymptotes, zeroes, and the ratio $\dfrac{\sin t}{\cos t}$

C. The Graph of $y = \cot t$

Since the cotangent function is also defined in terms of a ratio, it too displays asymptotic behavior at the zeroes of the denominator, with t-intercepts at the zeroes of the numerator. Like the tangent function, $\cot t = \dfrac{x}{y}$ can be written in terms of $\cos t = x$ and $\sin t = y$: $\cot t = \dfrac{\cos t}{\sin t}$, and the graph obtained by plotting points.

EXAMPLE 4 ▶ **Constructing a Table of Values for $f(t) = \cot t$**

Complete Table 6.9 for $\cot t = \dfrac{x}{y}$ for t in $[0, \pi]$ using its ratio relationship with cos t and sin t. Use the results to graph the function for t in $(-\pi, 2\pi)$.

Solution ▶ The completed table is shown here, with the computed values displayed in red.

Table 6.9

t	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\sin t = y$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\cos t = x$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1
$\cot t = \frac{x}{y}$	undefined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	$-\frac{1}{\sqrt{3}}$	-1	$-\sqrt{3}$	undefined

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In this interval, the cotangent function has asymptotes at 0 and π since $y = 0$ at these points, and has a t-intercept at $\frac{\pi}{2}$ since $x = 0$. The graph shown in Figure 6.84 was completed using the period of $y = \cot t$, $P = \pi$. Note that due to how it is defined, the function $y = \cot t$ is a decreasing function.

Figure 6.84

Now try Exercises 23 and 24 ▶

Because the cotangent (and tangent) of integer multiples of $\frac{\pi}{4}$ will always be undefined, -1, 0, or 1, these values are often used to generate the graph of the function. By entering $Y_1 = \frac{\cos X}{\sin X}$ and setting TblStart $= -\pi$ and ΔTbl $= \frac{\pi}{4}$, we can quickly generate these standard values (see Figure 6.85). Note the five points calculated in Figure 6.85 have been colored blue in Figure 6.84.

Figure 6.85

✓ **C.** You've just seen how we can graph $y = \cot t$ using asymptotes, zeroes, and the ratio $\frac{\cos t}{\sin t}$

D. Characteristics of $y = \tan t$ and $y = \cot t$

The most important characteristics of the tangent and cotangent functions are summarized in the following box. There is no discussion of amplitude, maximum, or minimum values, since maximum or minimum values do not exist. For future use and reference, perhaps the most significant characteristic distinguishing tan t from cot t is that *tan t increases*, while *cot t decreases* over their respective domains. Also note that due to symmetry, the zeroes of each function are always located halfway between the asymptotes.

Characteristics of $f(t) = \tan t$ and $f(t) = \cot t$

For all real numbers t and integers k,

$y = \tan t$			$y = \cot t$		
Domain	**Range**	**Asymptotes**	**Domain**	**Range**	**Asymptotes**
$t \neq \frac{\pi}{2} + \pi k$	$(-\infty, \infty)$	$t = \frac{\pi}{2} + \pi k$	$t \neq k\pi$	$(-\infty, \infty)$	$t = k\pi$
Period	**Behavior**	**Symmetry**	**Period**	**Behavior**	**Symmetry**
π	increasing	odd $\tan(-t) = -\tan t$	π	decreasing	odd $\cot(-t) = -\cot t$

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EXAMPLE 5 ▶ **Using the Period of $f(t) = \tan t$ to Find Additional Points**

Given $\tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$, what can you say about $\tan\left(\frac{7\pi}{6}\right)$, $\tan\left(\frac{13\pi}{6}\right)$, and $\tan\left(-\frac{5\pi}{6}\right)$?

Solution ▶ Each value of t differs from $\frac{\pi}{6}$ by a multiple of π: $\tan\left(\frac{7\pi}{6}\right) = \tan\left(\frac{\pi}{6} + \pi\right)$, $\tan\left(\frac{13\pi}{6}\right) = \tan\left(\frac{\pi}{6} + 2\pi\right)$, and $\tan\left(-\frac{5\pi}{6}\right) = \tan\left(\frac{\pi}{6} - \pi\right)$. Since the period of the tangent function is $P = \pi$, all of these expressions have a value of $\frac{1}{\sqrt{3}}$.

Now try Exercises 25 through 30 ▶

Since the tangent function is more common than the cotangent, many needed calculations will first be done using the tangent function and its properties, then reciprocated. For instance, to evaluate $\cot\left(-\frac{\pi}{6}\right)$ we reason that $\cot t$ is an odd function, so $\cot\left(-\frac{\pi}{6}\right) = -\cot\left(\frac{\pi}{6}\right)$. Since cotangent is the reciprocal of tangent and $\tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$, $-\cot\left(\frac{\pi}{6}\right) = -\sqrt{3}$. **See Exercises 31 and 32.**

✓ **D.** You've just seen how we can identify and discuss important characteristics of $y = \tan t$ and $y = \cot t$

E. Graphing $y = A\tan(Bt)$ and $y = A\cot(Bt)$

The Coefficient A: Vertical Stretches and Compressions

For the tangent and cotangent functions, the role of coefficient A is best seen through an analogy from basic algebra (the concept of amplitude is foreign to these functions). Consider the graph of $y = x^3$ (Figure 6.86). Comparing the parent function $y = x^3$ with functions $y = Ax^3$, the graph is stretched vertically if $|A| > 1$ (see Figure 6.87) and compressed if $0 < |A| < 1$. In the latter case the graph becomes very "flat" near the zeroes, as shown in Figure 6.88.

Figure 6.86
$y = x^3$

Figure 6.87
$y = 4x^3$; $A = 4$

Figure 6.88
$y = \frac{1}{4}x^3$; $A = \frac{1}{4}$

While ***cubic functions are not asymptotic,*** they are a good illustration of A's effect on the tangent and cotangent functions. Fractional values of A ($|A| < 1$) compress the graph, flattening it out near its zeroes. Numerically, this is because a fractional part of a small quantity is an even smaller quantity. For instance, compare $\tan\left(\frac{\pi}{6}\right)$ with $\frac{1}{4}\tan\left(\frac{\pi}{6}\right)$. Rounded to two decimal places, $\tan\left(\frac{\pi}{6}\right) \approx 0.57$, while $\frac{1}{4}\tan\left(\frac{\pi}{6}\right) \approx 0.14$, so the graph must be "nearer the t-axis" at this value.

EXAMPLE 6 ▶ **Comparing the Graph of $f(t) = \tan t$ and $g(t) = A \tan t$**

Draw a "comparative sketch" of $y = \tan t$ and $y = \frac{1}{4}\tan t$ on the same axis and discuss similarities and differences. Use the interval $[-\pi, 2\pi]$.

Solution ▶ Both graphs will maintain their essential features (zeroes, asymptotes, period, increasing, and so on). However, the graph of $y = \frac{1}{4}\tan t$ is vertically compressed, causing it to flatten out near its zeroes and changing how the graph approaches its asymptotes in each interval. See Figure 6.89. The table feature of a graphing calculator with $Y_1 = \tan X$ and $Y_2 = \left(\frac{1}{4}\right)\tan(X)$ reinforces these observations (Figure 6.90).

Figure 6.89

Figure 6.90

X	Y1	Y2
-1.571	ERROR	ERROR
-.7854	-1	-.25
0	0	0
.7854	1	.25
1.5708	ERROR	ERROR
2.3562	-1	-.25
3.1416	0	0

Y2=(1/4)tan(X)

Now try Exercises 33 through 36 ▶

The Coefficient B: The Period of Tangent and Cotangent

Like the other trig functions, the value of B has a material impact on the period of the function, and with the same effect. The graph of $y = \cot(2t)$ completes a cycle twice as fast as $y = \cot t \left(P = \frac{\pi}{2} \text{ versus } P = \pi\right)$, while $y = \cot\left(\frac{1}{2}t\right)$ completes a cycle one-half as fast ($P = 2\pi$ versus $P = \pi$).

This reasoning leads us to a **period formula** for tangent and cotangent, namely, $P = \frac{\pi}{|B|}$, where B is the coefficient of the input variable.

Similar to the four-step process used to graph sine and cosine functions, we can graph tangent and cotangent functions using a rectangle $P = \frac{\pi}{B}$ units in length and $2A$ units high, centered on the primary interval. After dividing the length of the rectangle into fourths, the t-intercept will always be the halfway point, with y-values of $|A|$ occurring at the $\frac{1}{4}$ and $\frac{3}{4}$ marks. See Example 7.

WORTHY OF NOTE

It may be easier to interpret the phrase "twice as fast" as $2P = \pi$ and "one-half as fast" as $\frac{1}{2}P = \pi$. In each case, solving for P gives the correct interval for the period of the new function.

EXAMPLE 7 ▶ **Graphing $y = A\cot(Bt)$ for $A, B \neq 1$**

Sketch the graph of $y = 3\cot(2t)$ over the interval $[-\pi, \pi]$.

Solution ▶ For $y = 3\cot(2t)$, $|A| = 3$, which results in a vertical stretch, and $|B| = 2$, which gives a period of $\frac{\pi}{2}$. The function is still undefined (asymptotic) at $t = 0$ and then at all integer multiples of $P = \frac{\pi}{2}$. We also know the graph is decreasing, with zeroes of the function halfway between the asymptotes. The inputs $t = \frac{\pi}{8}$ and $t = \frac{3\pi}{8}$

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$\left(\text{the } \frac{1}{4} \text{ and } \frac{3}{4} \text{ marks between 0 and } \frac{\pi}{2}\right)$ yield the points $\left(\frac{\pi}{8}, 3\right)$ and $\left(\frac{3\pi}{8}, -3\right)$, which we'll use along with the period and symmetry of the function to complete the graph in Figure 6.91.

A graphing calculator check is shown in Figure 6.92 (note $\frac{\pi}{8} \approx 0.39$).

Figure 6.91

Figure 6.92

Now try Exercises 37 through 48 ▶

As with the trig functions from Section 6.3, it is possible to determine the equation of a tangent or cotangent function from a given graph. Where previously we used the amplitude, period, and max/min values to obtain our equation, here we first determine the period of the function by calculating the "distance" between asymptotes, then choose any convenient point on the graph (other than a t-intercept) and substitute the x- and y- values into the general equation to solve for A.

EXAMPLE 8 ▶ **Constructing the Equation for a Given Graph**

Find the equation of the graph shown in Figure 6.93, given it's of the form $y = A\tan(Bt)$. Check your answer with a graphing calculator.

Figure 6.93

Solution ▶ Using the primary interval and the asymptotes at $t = -\frac{\pi}{3}$ and $t = \frac{\pi}{3}$, we find the period is $P = \frac{\pi}{3} - \left(-\frac{\pi}{3}\right) = \frac{2\pi}{3}$. To find the value of B we substitute $\frac{2\pi}{3}$ for P in $P = \frac{\pi}{B}$ and find $B = \frac{3}{2}$ (verify). This gives the equation $y = A\tan\left(\frac{3}{2}t\right)$.

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To find A, we take the point $\left(\frac{\pi}{2}, -2\right)$ shown, and use $t = \frac{\pi}{2}$ with $y = -2$ to solve for A:

$$y = A\tan\left(\frac{3}{2}t\right) \quad \text{substitute } \tfrac{3}{2} \text{ for } B$$

$$-2 = A\tan\left[\left(\frac{3}{2}\right)\left(\frac{\pi}{2}\right)\right] \quad \text{substitute } -2 \text{ for } y \text{ and } \tfrac{\pi}{2} \text{ for } t$$

$$-2 = A\tan\left(\frac{3\pi}{4}\right) \quad \text{multiply}$$

$$A = \frac{-2}{\tan\left(\frac{3\pi}{4}\right)} \quad \text{solve for } A$$

$$= 2 \quad \text{result; } \tan\left(\tfrac{3\pi}{4}\right) = -1$$

The equation of the graph is $y = 2\tan(\frac{3}{2}t)$.

Check ▶ With the WINDOW set to match the given graph $\left(\text{Xmin} = -\pi, \text{Xmax} = \pi, \text{Xscl} = \frac{\pi}{6}, \text{Ymin} = -3, \text{Ymax} = 3, \text{ and Yscl} = 1\right)$, the calculator produces the GRAPH shown in Figure 6.94. The graphs match. ✓

Figure 6.94

Now try Exercises 49 through 58 ▶

☑ **E.** You've just seen how we can graph $y = A\tan(Bt)$ and $y = A\cot(Bt)$ with various values of A and B

6.4 EXERCISES

▶ CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section if needed.

1. The period of $y = \tan t$ and $y = \cot t$ is _____. To find the period of $y = \tan(Bt)$ and $y = \cot(Bt)$, the formula _____ is used.

2. The function $y = \tan t$ is _____ everywhere it is defined. The function $y = \cot t$ is _____ everywhere it is defined.

3. Tan t and cot t are _____ functions, so $f(-t) =$ _____. If $\tan\left(-\frac{11\pi}{12}\right) \approx 0.268$, then $\tan\left(\frac{11\pi}{12}\right) \approx$ _____.

4. The asymptotes of $y =$ _____ are located at odd multiples of $\frac{\pi}{2}$. The asymptotes of $y =$ _____ are located at integer multiples of π.

5. Discuss/Explain (a) how you could obtain a table of values for $y = \sec t$ in $[0, 2\pi]$ from the values for $y = \cos t$, and (b) how you could obtain a table of values for $y = \csc\left(\frac{3}{2}t\right)t$ in $[0, 2\pi]$, given the values for $y = \sin\left(\frac{3}{2}t\right)$.

6. Explain/Discuss how the zeroes of $y = \sin t$ and $y = \cos t$ are related to the graphs of $y = \tan t$ and $y = \cot t$. How can these relationships help graph functions of the form $y = A\tan(Bt)$ and $y = A\cot(Bt)$?

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▶ DEVELOPING YOUR SKILLS

Draw the graph of each function by first sketching the related sine and cosine graphs, and applying the observations from this section. Confirm your graph using a graphing calculator.

7. $y = \csc(2t)$ **8.** $y = \csc(\pi t)$

9. $y = \sec\left(\frac{1}{2}t\right)$ **10.** $y = \sec(2\pi t)$

11. $y = 3\csc t$ **12.** $g(t) = 2\csc(4t)$

13. $y = 2\sec t$ **14.** $f(t) = 3\sec(2t)$

Use the values given for sin *t* and cos *t* to complete the tables.

15.

t	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$
$\sin t = y$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1
$\cos t = x$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan t = \frac{y}{x}$					

16.

t	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
$\sin t = y$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0
$\cos t = x$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\tan t = \frac{y}{x}$					

17. Without reference to a text or calculator, attempt to name the decimal equivalent of the following values to one decimal place.

$$\frac{\pi}{2} \quad \frac{\pi}{4} \quad \frac{\pi}{6} \quad \sqrt{2} \quad \frac{\sqrt{2}}{2} \quad \frac{2}{\sqrt{3}}$$

18. Without reference to a text or calculator, attempt to name the decimal equivalent of the following values to one decimal place.

$$\frac{\pi}{3} \quad \pi \quad \frac{3\pi}{2} \quad \sqrt{3} \quad \frac{\sqrt{3}}{2} \quad \frac{1}{\sqrt{3}}$$

19. State the value of each expression without the use of a calculator.

a. $\tan\left(-\frac{\pi}{4}\right)$ **b.** $\cot\left(\frac{\pi}{6}\right)$

c. $\cot\left(\frac{3\pi}{4}\right)$ **d.** $\tan\left(\frac{\pi}{3}\right)$

20. State the value of each expression without the use of a calculator.

a. $\cot\left(\frac{\pi}{2}\right)$ **b.** $\tan \pi$

c. $\tan\left(-\frac{5\pi}{4}\right)$ **d.** $\cot\left(-\frac{5\pi}{6}\right)$

21. State the value of t without the use of a calculator, given $t \in [0, 2\pi)$ terminates in the quadrant indicated.

a. $\tan t = -1$, t in QIV

b. $\cot t = \sqrt{3}$, t in QIII

c. $\cot t = -\frac{1}{\sqrt{3}}$, t in QIV

d. $\tan t = -1$, t in QII

22. State the value of t without the use of a calculator, given $t \in [0, 2\pi)$ terminates in the quadrant indicated.

a. $\cot t = 1$, t in QI

b. $\tan t = -\sqrt{3}$, t in QII

c. $\tan t = \frac{1}{\sqrt{3}}$, t in QI

d. $\cot t = 1$, t in QIII

Use the values given for sin *t* and cos *t* to complete the tables.

23.

t	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$
$\sin t = y$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1
$\cos t = x$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0
$\cot t = \frac{x}{y}$					

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24.

t	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
$\sin t = y$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0
$\cos t = x$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cot t = \frac{x}{y}$					

25. Given $t = \frac{11\pi}{24}$ is a solution to $\tan t \approx 7.6$, use the period of the function to name three additional solutions. Check your answer using a calculator.

26. Given $t = \frac{7\pi}{24}$ is a solution to $\cot t \approx 0.77$, use the period of the function to name three additional solutions. Check your answer using a calculator.

27. Given $t \approx 1.5$ is a solution to $\cot t = 0.07$, use the period of the function to name three additional solutions. Check your answers using a calculator.

28. Given $t \approx 1.25$ is a solution to $\tan t = 3$, use the period of the function to name three additional solutions. Check your answers using a calculator.

Verify the value shown for t is a solution to the equation given, then use the period of the function to name all real roots. Check two of these roots on a calculator.

29. $t = \frac{\pi}{10}$; $\tan t \approx 0.3249$

30. $t = -\frac{\pi}{16}$; $\tan t \approx -0.1989$

31. $t = \frac{\pi}{12}$; $\cot t = 2 + \sqrt{3}$

32. $t = \frac{5\pi}{12}$; $\cot t = 2 - \sqrt{3}$

Graph each function over the interval indicated, noting the period, asymptotes, zeroes, and value of A. Include a comparative sketch of $y = \tan t$ or $y = \cot t$ as indicated.

33. $f(t) = 2 \tan t$; $[-2\pi, 2\pi]$

34. $g(t) = \frac{1}{2} \tan t$; $[-2\pi, 2\pi]$

35. $h(t) = 3 \cot t$; $[-2\pi, 2\pi]$

36. $r(t) = \frac{1}{4} \cot t$; $[-2\pi, 2\pi]$

Graph each function over the interval indicated, noting the period, asymptotes, zeroes, and values of A and B.

37. $y = \tan(2t)$; $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

38. $y = \tan\left(\frac{1}{4}t\right)$; $[-4\pi, 4\pi]$

39. $y = \cot(4t)$; $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

40. $y = \cot\left(\frac{1}{2}t\right)$; $[-2\pi, 2\pi]$

41. $y = 2 \tan(4t)$; $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

42. $y = 4 \tan\left(\frac{1}{2}t\right)$; $[-2\pi, 2\pi]$

43. $y = 5 \cot\left(\frac{1}{3}t\right)$; $[-3\pi, 3\pi]$

44. $y = \frac{1}{2} \cot(2t)$; $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

45. $y = 3 \tan(2\pi t)$; $\left[-\frac{1}{2}, \frac{1}{2}\right]$

46. $y = 4 \tan\left(\frac{\pi}{2}t\right)$; $[-2, 2]$

47. $f(t) = 2 \cot(\pi t)$; $[-1, 1]$

48. $p(t) = \frac{1}{2} \cot\left(\frac{\pi}{4}t\right)$; $[-4, 4]$

Clearly state the period of each function, then match it with the corresponding graph.

49. $y = 2 \csc\left(\frac{1}{2}t\right)$

50. $y = 2 \sec\left(\frac{1}{4}t\right)$

51. $y = \sec(8\pi t)$

52. $y = \csc(12\pi t)$

a.

b.

c.

d.

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Find the equation of each graph, given it is of the form $y = A\csc(Bt)$. Check with a graphing calculator.

53.

54.

Find the equation of each graph, given it is of the form $y = A\tan(Bt)$. Check with a graphing calculator.

55.

56.

Find the equation of each graph, given it is of the form $y = A\cot(Bt)$. Check with a graphing calculator.

57.

58.

59. Given that $t = -\dfrac{\pi}{8}$ and $t = -\dfrac{3\pi}{8}$ are solutions to $\cot(3t) = \tan t$, use a graphing calculator to find two additional solutions in $[0, 2\pi]$.

60. Given $t = \frac{1}{6}$ is a solution to $\tan(2\pi t) = \cot(\pi t)$, use a graphing calculator to find two additional solutions in $[-1, 1]$.

▶ WORKING WITH FORMULAS

61. The height of an object calculated from a distance: $h = \dfrac{d}{\cot u - \cot v}$

The height h of a tall structure can be computed using two angles of elevation measured some distance apart along a straight line with the object. This height is given by the formula shown, where d is the distance between the two points from which angles u and v were measured. Find the height h of a building if $u = 40°$, $v = 65°$, and $d = 100$ ft.

62. Position of an image reflected from a spherical lens: $\tan\theta = \dfrac{h}{s - k}$

The equation shown is used to help locate the position of an image reflected by a spherical lens, where s is the distance of the object from the lens along a horizontal axis, θ is the angle of elevation from this axis, h is the altitude of the right triangle indicated, and k is distance from the lens to the foot of altitude h. Find the distance k given $h = 3$ mm, $\theta = \dfrac{\pi}{24}$, and that the object is 24 mm from the lens.

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▶ APPLICATIONS

Exercises 63 and 64

63. Circumscribed polygons: The *perimeter* of a regular polygon circumscribed about a circle of radius r is given by $P = 2nr\tan\left(\frac{\pi}{n}\right)$, where n is the number of sides ($n \geq 3$) and r is the radius of the circle. Given $r = 10$ cm,

a. What is the circumference of the circle?

b. What is the perimeter of the polygon when $n = 4$? Why?

c. Calculate the perimeter of the polygon for $n = 10$, 20, 30, and 100. What do you notice? See Exercise 61 from Section 6.3.

64. Circumscribed polygons: The *area* of a regular polygon circumscribed about a circle of radius r is given by $A = nr^2\tan\left(\frac{\pi}{n}\right)$, where n is the number of sides ($n \geq 3$) and r is the radius of the circle. Given $r = 10$ cm,

a. What is the area of the circle?

b. What is the area of the polygon when $n = 4$? Why?

c. Calculate the area of the polygon for $n = 10$, 20, 30, and 100. What do you notice?

Coefficients of friction: Pulling someone on a sled is much easier during the winter than in the summer, due to a phenomenon known as the *coefficient of friction*. The friction between the sled's skids and the snow is much lower than the friction between the skids and the dry ground or pavement. Basically, the coefficient of friction is defined by the relationship $\mu = \tan\theta$, where θ is the angle at which a block composed of one material will slide down an inclined plane made of another material, with a constant velocity. Coefficients of friction have been established experimentally for many materials and a short list is shown here.

Material	Coefficient
steel on steel	0.74
copper on glass	0.53
glass on glass	0.94
copper on steel	0.68
wood on wood	0.5

65. Graph the function $\mu = \tan\theta$, with θ in degrees over the interval $[0°, 60°]$ and use the intersection-of-graphs method to estimate solutions to the following. Confirm or contradict your estimates using a calculator.

a. A block of copper is placed on a sheet of steel, which is slowly inclined. Is the block of copper moving when the angle of inclination is 30°? At what angle of inclination will the copper block be moving with a constant velocity down the incline?

b. A block of copper is placed on a sheet of cast-iron. As the cast-iron sheet is slowly inclined, the copper block begins sliding at a constant velocity when the angle of inclination is approximately 46.5°. What is the coefficient of friction for copper on cast-iron?

c. Why do you suppose coefficients of friction greater than $\mu = 2.5$ are extremely rare? Give an example of two materials that likely have a high μ-value.

66. Graph the function $\mu = \tan\theta$ with θ in radians over the interval $\left[0, \frac{5\pi}{12}\right]$ and use the intersection-of-graphs method to estimate solutions to the following. Confirm or contradict your estimates using a calculator.

a. A block of glass is placed on a sheet of glass, which is slowly inclined. Is the block of glass moving when the angle of inclination is $\frac{\pi}{4}$? What is the smallest angle of inclination for which the glass block will be moving with a constant velocity down the incline (rounded to four decimal places)?

b. A block of Teflon is placed on a sheet of steel. As the steel sheet is slowly inclined, the Teflon block begins sliding at a constant velocity when the angle of inclination is approximately 0.04. What is the coefficient of friction for Teflon on steel?

c. Why do you suppose coefficients of friction less than $\mu = 0.04$ are extremely rare for two solid materials? Give an example of two materials that likely have a very low μ value.

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67. Tangent lines: The actual definition of the word *tangent* comes from the Latin *tangere*, meaning "to touch." In mathematics, a tangent line touches the graph of a circle at only one point and function values for $\tan\theta$ are obtained from the length of the line segment tangent to a unit circle.

a. What is the length of the line segment when $\theta = 80°$?

b. If the line segment is 16.35 units long, use the intersection-of-graphs method to find the value of θ.

c. Can the line segment ever be greater than 100 units long? Why or why not?

d. How does your answer to (c) relate to the asymptotic behavior of the graph?

68. Illumination: The angle α made by a light source and a point on a horizontal surface can be found using the formula $\csc\alpha = \dfrac{I}{r^2E}$, where E is the illuminance (in lumens/m^2) at the point, I is the intensity of the light source in lumens, and r is the distance in meters from the light source to the point. Use the graph from Example 1 to help determine the angle α given $I = 700$ lumens, $E = 55$ lumens/m^2, and the flashlight is held so that the distance r is 3 m.

▶ EXTENDING THE CONCEPT

69. The golden ratio $\dfrac{-1+\sqrt{5}}{2}$ has long been thought to be the most pleasing ratio in art and architecture. It is commonly believed that many forms of ancient architecture were constructed using this ratio as a guide. The ratio actually turns up in some surprising places, far removed from its original inception as a line segment cut in "mean and extreme" ratio. Given $x = 0.6662394325$, try to find a connection between $y = \cos x$, $y = \tan x$, $y = \sin x$, and the golden ratio.

70. Determine the slope of the line drawn *through* the parabola (called a **secant** line) in Figure I. Use the same method (any two points on the line) to calculate the slope of the line drawn **tangent** to the parabola in Figure II. Compare your calculations to the tangent of the angles α and β that each line makes with the x-axis. What can you conclude? Write a formula for the point/slope equation of a line using $\tan\theta$ instead of m.

Figure I

Figure II

▶ MAINTAINING YOUR SKILLS

71. **(6.1)** A lune is a section of surface area on a sphere, which is subtended by an angle θ at the circumference. For θ *in radians*, the surface area of a lune is $A = 2r^2\theta$, where r is the radius of the sphere. Find the area of a lune on the surface of the Earth which is subtended by an angle of 15°. Assume the radius of the Earth is 6373 km.

72. **(4.4/4.5)** Find the y-intercept, x-intercept(s), and all asymptotes of each function, but do not graph.

a. $h(x) = \dfrac{3x^2 - 9x}{2x^2 - 8}$ **b.** $t(x) = \dfrac{x+1}{x^2 - 4x}$ **c.** $p(x) = \dfrac{x^2 - 1}{x + 2}$

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73. **(6.2)** State the points on the unit circle that correspond to $t = 0, \frac{\pi}{4}, \frac{\pi}{2}, \pi, \frac{3\pi}{4}, \frac{3\pi}{2}$, and 2π. What is the value of $\tan\left(\frac{\pi}{2}\right)$? Why?

74. **(5.5)** The radioactive element potassium-42 is sometimes used as a tracer in certain biological experiments, and its decay can be modeled by the formula $Q(t) = Q_0 e^{-0.055t}$, where $Q(t)$ is the amount that remains after t hours. If 15 grams (g) of potassium-42 are initially present, how many hours until only 10 g remain?

MID-CHAPTER CHECK

1. The city of Las Vegas, Nevada, is located at 36° 06′ 36″ north latitude, 115° 04′ 48″ west longitude. (a) Convert both measures to decimal degrees. (b) If the radius of the Earth is 3960 mi, how far north of the equator is Las Vegas?

Exercise 2

2. Find the angle subtended by the arc shown in the figure, then determine the area of the sector.

3. Evaluate without using a calculator: (a) cot 60° and (b) $\sin\left(\frac{7\pi}{4}\right)$.

4. Evaluate using a calculator: (a) $\sec\left(\frac{\pi}{12}\right)$ and (b) tan 83.6°.

Exercise 5

5. Complete the ordered pair indicated on the unit circle in the figure and find the value of all six trigonometric functions at this point.

6. For the point on the unit circle in Exercise 5, use the intersection-of-graphs method to find the related angle t in both degrees (to tenths) and radians (to ten-thousandths).

7. Name the location of the asymptotes and graph $y = 3\tan\left(\frac{\pi}{2}t\right)$ for $t \in [-2\pi, 2\pi]$.

8. Clearly state the amplitude and period, then sketch the graph: $y = -3\cos\left(\frac{\pi}{2}t\right)$.

9. On a unit circle, if arc t has length 5.94, (a) in what quadrant does it terminate? (b) What is its reference arc? (c) Of sin t, cos t, and tan t, which are negative for this value of t?

10. For the graph given here, (a) clearly state the amplitude and period; (b) find an equation of the graph; (c) graphically find $f(\pi)$ and then confirm/contradict your estimation using a calculator.

Exercise 10

REINFORCING BASIC CONCEPTS

Trigonometry of the Real Numbers and the W rapping Function

The trigonometric functions are sometimes discussed in terms of what is called a *wrapping function*, in which the real number line is literally wrapped around the unit circle. This approach can help illustrate how the trig functions can be seen as functions of the real numbers, and apart from any reference to a right triangle. Figure 6.95 shows (1) a unit circle with the location of certain points on the circumference clearly marked and (2) a blue number line that has been marked in multiples of $\frac{\pi}{12}$ to coincide with the length of the special arcs (integers are shown in red). Figure 6.96 shows this same blue number line wrapped counterclockwise around the unit circle in the positive direction. Note how the resulting diagram confirms that an arc of length $t = \frac{\pi}{4}$ is associated with the point $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ on the unit circle: $\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$ and

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$\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$; while an arc of length of $t = \frac{5\pi}{6}$ is associated with the point $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$: $\cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$ and $\sin\frac{5\pi}{6} = \frac{1}{2}$. Use this information to complete the exercises given.

Figure 6.95

Figure 6.96

1. What is the ordered pair associated with an arc length of $t = \frac{2\pi}{3}$? What is the value of $\cos t$? $\sin t$?

2. What arc length t is associated with the ordered pair $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$? Is $\cos t$ positive or negative? Why?

3. If we continued to wrap this number line all the way around the circle, in what quadrant would an arc length of $t = \frac{11\pi}{6}$ terminate? Would $\sin t$ be positive or negative?

4. Suppose we wrapped a number line with negative values clockwise around the unit circle. In what quadrant would an arc length of $t = -\frac{5\pi}{3}$ terminate? What is $\cos t$? $\sin t$? What positive rotation terminates at the same point?

6.5 Transformations and Applications of Trigonometric Graphs

LEARNING OBJECTIVES

In Section 6.5 you will see how we can:

- ❑ **A.** Apply vertical translations in context
- ❑ **B.** Apply horizontal translations in context
- ❑ **C.** Solve applications involving harmonic motion
- ❑ **D.** Apply vertical and horizontal translations to cosecant, secant, tangent, and cotangent
- ❑ **E.** Solve applications involving the tangent, cotangent, secant, and cosecant functions

From your algebra experience, you may remember beginning with a study of linear graphs, then moving on to quadratic graphs and their characteristics. By combining and extending the knowledge you gained, you were able to investigate and understand a variety of polynomial graphs—along with some powerful applications. A study of trigonometry follows a similar pattern, and by "combining and extending" our understanding of the basic trig graphs, we'll look at some powerful applications in this section.

A. Vertical Translations: $y = A\sin(Bt) + D$ and $y = A\cos(Bt) + D$

On any given day, outdoor temperatures tend to follow a **sinusoidal pattern,** or a pattern that can be modeled by a sine function. As the sun rises, the morning temperature begins to warm and rise until reaching its high in the late afternoon, then begins to cool during the early evening and nighttime hours until falling to its nighttime low just prior to sunrise. Next morning, the cycle begins again. In the northern latitudes where the winters are very cold, it's not unreasonable to assume an average daily temperature of 0°C (32°F), and a temperature graph in degrees Celsius that looks like the one in Figure 6.97. For the moment, we'll assume that $t = 0$ corresponds to 12:00 noon. Note that $|A| = 15$ and $P = 24$, yielding $24 = \frac{2\pi}{B}$ or $B = \frac{\pi}{12}$.

Figure 6.97

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If you live in a more temperate area, the daily temperatures still follow a sinusoidal pattern, but the average temperature could be much higher. This is an example of a **vertical shift,** and is the role D plays in the equation $y = A\sin(Bt) + D$. All other aspects of a graph remain the same; it is simply shifted D units up if $D > 0$ and D units down if $D < 0$. As in Section 6.3, for maximum value M and minimum value m, $\frac{M - m}{2}$ gives the amplitude A of a sine curve, while $\frac{M + m}{2}$ gives the **average value** D.

EXAMPLE 1 ▶ **Modeling Temperature Using a Sine Function**

On a fine day in Galveston, Texas, the high temperature might be about 85°F with an overnight low of 61°F.

a. Find a sinusoidal equation model for the daily temperature.

b. Sketch the graph.

c. Approximate what time(s) of day the temperature is 65°F. Assume $t = 0$ corresponds to 12:00 noon.

Solution ▶ **a.** We first note the period is still $P = 24$, so $B = \frac{\pi}{12}$, and the equation model will have the form $y = A\sin\left(\frac{\pi}{12}t\right) + D$. Using $\frac{M + m}{2} = \frac{85 + 61}{2}$, we find the *average value* $D = 73$, with amplitude $A = \frac{85 - 61}{2} = 12$. The resulting equation is $y = 12\sin\left(\frac{\pi}{12}t\right) + 73$.

b. To sketch the graph, use a reference rectangle $2A = 24$ units tall and $P = 24$ units wide, along with the *rule of fourths* to locate zeroes and max/min values (see Figure 6.98). Then lightly sketch a sine curve through these points and within the rectangle as shown. This is the graph of $y = 12\sin\left(\frac{\pi}{12}t\right) + 0$. Using an appropriate scale, shift the rectangle and plotted points vertically upward 73 units and carefully draw the finished graph through the points and within the rectangle (see Figure 6.99).

Figure 6.98

Figure 6.99

This gives the graph of $y = 12\sin\left(\frac{\pi}{12}t\right) + 73$. Note the broken-line notation "≋" in Figure 6.99 indicates that certain values along an axis are unused (in this case, we skipped 0° to 60°), and we began scaling the axis with the values needed.

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WORTHY OF NOTE

Recall from Section 2.2 that transformations of any function $y = f(x)$ remain consistent regardless of the function f used. For the sine function, the transformation $y = af(x \pm h) \pm k$ is more commonly written $y = A\sin(t \pm C) \pm D$, and $|A|$ gives a vertical stretch or compression, C is a horizontal shift opposite the sign, and D is a vertical shift, as seen in Example 1.

c. As indicated in Figure 6.99, the temperature hits 65° twice, at about 15 and 21 hr after 12:00 noon, or at 3:00 A.M. and 9:00 A.M. This can be verified by setting

$$Y_1 = 12\sin\left(\frac{\pi}{12}X\right) + 73,$$

then going to the home screen and computing $Y_1(15)$ and $Y_1(21)$ as shown in Figure 6.100.

Figure 6.100

Now try Exercises 7 through 18 ▶

Sinusoidal graphs actually include both sine and cosine graphs, the difference being that sine graphs begin at the average value, while cosine graphs begin at the maximum value. Sometimes it's more advantageous to use one over the other, but equivalent forms can easily be found. In Example 2, a cosine function is used to model an animal population that fluctuates sinusoidally due to changes in food supplies.

EXAMPLE 2 ▶ Modeling Population Fluctuations Using a Cosine Function

The population of a certain animal species can be modeled by the function $P(t) = 1200\cos\left(\frac{\pi}{5}t\right) + 9000$, where $P(t)$ represents the population in year t.

Use the model to

a. Find the period of the function.
b. Graph the function over one period.
c. Find the maximum and minimum values.
d. Use a graphing calculator to determine the number of years the population is less than 8000.

Solution ▶

a. Since $B = \frac{\pi}{5}$, the period is $P = \frac{2\pi}{\pi/5} = 10$, meaning the population of this species fluctuates over a 10-yr cycle.

b. Use a reference rectangle ($2A = 2400$ by $P = 10$ units) and the *rule of fourths* to locate zeroes and max/min values, then sketch the unshifted graph $y = 1200\cos\left(\frac{\pi}{5}t\right)$. With $P = 10$, these important points occur at $t = 0$, 2.5, 5, 7.5, and 10 (see Figure 6.101). Shift this graph upward 9000 units (using an appropriate scale) to obtain the graph of $P(t)$ shown in Figure 6.102.

Figure 6.101

Figure 6.102

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c. The maximum value is $9000 + 1200 = 10{,}200$ and the minimum value is $9000 - 1200 = 7800$.

d. Begin by entering $Y_1 = 1200 \cos\left(\pi \frac{X}{5}\right) + 9000$ and $Y_2 = 8000$ in a graphing calculator. With the window settings identified in Figure 6.103, pressing GRAPH confirms the graph we produced in Figure 6.102. The intersection of Y_1 and Y_2 can be located using the 2nd TRACE **(CALC) 5:intersect** feature. After identifying Y_1 and Y_2 as the intersecting curves by pressing ENTER twice, the calculator asks us to make a guess. A casual observation of the graph indicates the leftmost point of intersection occurs near 4, so pressing 4 ENTER will produce the screen shown in Figure 6.104. With a minor adjustment, this process determines the rightmost point of intersection occurs when $x \approx 5.93$. We can now determine that the population drops below 8000 animals for approximately $5.93 - 4.07 = 1.86$ yr.

Figure 6.103

Figure 6.104

A. You've just seen how we can apply vertical translations in context

Now try Exercises 19 and 20 ▶

B. Horizontal Translations: $y = A \sin(Bt + C) + D$ and $y = A \cos(Bt + C) + D$

In some cases, scientists would rather "benchmark" their study of sinusoidal phenomena by placing the average value at $t = 0$ instead of a maximum value (as in Example 2), or by placing the maximum or minimum value at $t = 0$ instead of the average value (as in Example 1). Rather than make additional studies or recompute using available data, *we can simply shift these graphs using a horizontal translation*. To help understand how, consider the graph of $Y_1 = X^2$. The graph is a parabola that opens upward with a vertex at the origin. Comparing this function with $Y_2 = (X - 3)^2$ and $Y_3 = (X + 3)^2$, we note Y_2 is simply the parent graph shifted 3 units right (Figure 6.105), and Y_3 is the parent graph shifted 3 units left or "opposite the sign" (Figure 6.106).

While ***quadratic functions have no maximum value if $A > 0$,*** these graphs are a good reminder of how a basic graph can be horizontally shifted. We simply *replace the independent variable x with* $(x \pm h)$ *or t with* $(t \pm h)$, where h is the desired shift and the sign is chosen depending on the direction of the shift.

Figure 6.105

Figure 6.106

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EXAMPLE 3 ▶ **Investigating Horizontal Shifts of Trigonometric Graphs**

Use a horizontal translation to shift the graph from Example 2 so that the average population begins at $t = 0$. Verify the result on a graphing calculator, then find a sine function that gives the same graph as the shifted cosine function.

Solution ▶ For $P(t) = 1200 \cos\left(\frac{\pi}{5}t\right) + 9000$ from Example 2, the average value occurs at $t = 2.5$, and by symmetry at $t = -2.5$. For the average value to occur at $t = 0$, we shift the graph to the right 2.5 units. Replacing t with $(t - 2.5)$ gives $P(t) = 1200 \cos\left[\frac{\pi}{5}(t - 2.5)\right] + 9000$.

A graphing calculator shows the desired result is obtained (see figure). The new graph appears to be a sine function with the same amplitude and period, and in fact the equation is $y = 1200 \sin\left(\frac{\pi}{5}t\right) + 9000$.

Now try Exercises 21 and 22 ▶

Equations like $P(t) = 1200 \cos\left[\frac{\pi}{5}(t - 2.5)\right] + 9000$ from Example 3 are said to be written in **shifted form,** since changes to the input variable are seen directly and we can easily tell the magnitude and direction of the shift. To obtain the **standard form** we *distribute the value of B*: $P(t) = 1200 \cos\left(\frac{\pi}{5}t - \frac{\pi}{2}\right) + 9000$. In general, the *standard form* of a sinusoidal equation (using *either* a cosine or sine function) is written $y = A \sin(Bt \pm C) + D$, with the *shifted form* found by factoring out B from $Bt \pm C$:

$$y = A \sin(Bt \pm C) + D \rightarrow y = A \sin\left[B\left(t \pm \frac{C}{B}\right)\right] + D$$

In either case, C gives what is known as the **phase angle** of the function, and is used in a study of AC circuits and other areas, to discuss how far a given function is "out of phase" with a reference function. In the latter case, $\frac{C}{B}$ is simply the horizontal shift (or phase shift) of the function and gives the magnitude and direction of this shift (opposite the sign).

Characteristics of Sinusoidal Equations

Transformations of the graph of $y = \sin t$ can be written as $y = A \sin(Bt)$, where

1. $|A|$ gives the *amplitude* of the graph, or the maximum displacement from the average value.
2. B is related to the *period* P of the graph according to the ratio $P = \frac{2\pi}{|B|}$.

Translations of $y = A \sin(Bt)$ can be written as follows:

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WORTHY OF NOTE

It's important that you don't confuse the standard form with the shifted form. Each has a place and purpose, but the horizontal shift can be identified only by focusing on the change in an independent variable. Even though the equations $y = 4(x + 3)^2$ and $y = (2x + 6)^2$ are equivalent, only the first explicitly shows that $y = 4x^2$ has been shifted three units left. Likewise $y = \sin[2(t + 3)]$ and $y = \sin(2t + 6)$ are equivalent, but only the first explicitly gives the horizontal shift (three units left). Applications involving a horizontal shift come in an infinite variety, and the shifts are generally not uniform or standard.

Standard form	**Shifted form**
$y = A\sin(Bt \pm C) + D$	$y = A\sin\left[B\left(t \pm \frac{C}{B}\right)\right] + D$

3. In either case, C is called the *phase angle* of the graph, while $\pm\frac{C}{B}$ gives the magnitude and direction of the *horizontal shift* (opposite the given sign).
4. D gives the *vertical shift* of the graph, and the location of the average value. The shift will be in the same direction as the given sign.

Knowing where each cycle begins and ends is a helpful part of sketching a graph of the equation model. The **primary interval** for a sinusoidal graph can be found by solving the inequality $0 \le Bt \pm C < 2\pi$, with the reference rectangle and *rule of fourths* giving the zeroes, max/min values, and a sketch of the graph in this interval. The graph can then be extended in either direction, and shifted vertically as needed.

EXAMPLE 4 ▶ **Analyzing the Transformation of a Trig Function**

Identify the amplitude, period, horizontal shift, vertical shift (average value), and endpoints of the primary interval.

$$y = 2.5\sin\left(\frac{\pi}{4}t + \frac{3\pi}{4}\right) + 6$$

Solution ▶ The equation gives an amplitude of $|A| = 2.5$, with an average value of $D = 6$. The maximum value will be $y = 2.5(1) + 6 = 8.5$, with a minimum of $y = 2.5(-1) + 6 = 3.5$. With $B = \frac{\pi}{4}$, the period is $P = \frac{2\pi}{\pi/4} = 8$. To find the horizontal shift, we factor out $\frac{\pi}{4}$ to write the equation in shifted form: $\left(\frac{\pi}{4}t + \frac{3\pi}{4}\right) = \frac{\pi}{4}(t + 3)$. The horizontal shift is 3 units left. For the endpoints of the primary interval we solve $0 \le \frac{\pi}{4}(t + 3) < 2\pi$, which gives $-3 \le t < 5$.

Now try Exercises 23 through 34 ▶

The analysis of $y = 2.5\sin\left[\frac{\pi}{4}(t + 3)\right] + 6$ from Example 4 can be verified on a graphing calculator. Enter the function as Y_1 on the Y= screen and set an appropriate window size using the information gathered. Pressing the TRACE key and -3 ENTER gives the average value $y = 6$ as output. Repeating this for $x = 5$ shows one complete cycle has been completed (Figure 6.107).

Figure 6.107

To help gain a better understanding of sinusoidal functions, their graphs, and the role the coefficients A, B, C, and D play, it's often helpful to reconstruct the equation of a given graph.

EXAMPLE 5 ▶ **Determining an Equation of a Trig Function from Its Graph**

Determine an equation of the given graph using a sine function.

Solution ▶ From the graph it is apparent the maximum value is 300, with a minimum of 50. This gives a value of $\frac{300 + 50}{2} = 175$ for D and $\frac{300 - 50}{2} = 125$ for A. The graph completes one cycle from $t = 2$ to $t = 18$, showing $P = 18 - 2 = 16$ and $B = \frac{\pi}{8}$. The average value first occurs at $t = 2$, so the basic graph has been shifted to the right 2 units. The equation is $y = 125 \sin\left[\frac{\pi}{8}(t - 2)\right] + 175$.

Now try Exercises 35 through 44 ▶

In addition to the GRAPH of $Y_1 = 125 \sin\left[\frac{\pi}{8}(X - 2)\right] + 175$ from Example 5, the table feature of a calculator can provide a convincing check of the solution. After recognizing the average, maximum, and minimum values of the given graph occur at regular intervals (as is true for any sinusoidal function), setting up a table to start at 2 with step size of 4 units will produce the 2nd GRAPH **(TABLE)** shown in Figure 6.108. Note the screen shown contains the coordinates of all seven points indicated in the graph for this example.

Figure 6.108

B. You've just seen how we can apply horizontal translations in context

C. Simple Harmonic Motion: $y = A \sin(Bt)$ or $y = A \cos(Bt)$

The periodic motion of springs, tides, sound, and other phenomena all exhibit what is known as **harmonic motion,** which can be modeled using sinusoidal functions.

Harmonic Models—Springs

Consider a spring hanging from a beam with a weight attached to one end. When the weight is at rest, we say it is in **equilibrium,** or has zero displacement from center. See Figure 6.109. Stretching the spring and then releasing it causes the weight to "bounce up and down," with its displacement from center neatly modeled over time by a sine wave (see Figure 6.110).

Figure 6.109

For objects in harmonic *motion* (there are other harmonic models), the input variable t is always a time unit (seconds, minutes, days, etc.), so in addition to the period of the sinusoid, we are very interested in its **frequency**—the number of cycles it completes per unit time. Since the period gives the time required to complete one cycle, the frequency f is given by $f = \frac{1}{P} = \frac{B}{2\pi}$.

Figure 6.110

Harmonic motion

EXAMPLE 6 ▶ **Applications of Sine and Cosine: Harmonic Motion**

For the harmonic motion of the weight modeled by the sinusoid in Figure 6.110,

a. Find an equation of the form $y = A\cos(Bt)$.
b. Determine the frequency.
c. Use the equation to find the position of the weight at $t = 1.8$ sec.

Solution ▶

a. By inspection the graph has an amplitude $|A| = 3$ and a period $P = 2$. After substitution into $P = \frac{2\pi}{B}$, we obtain $B = \pi$ and the equation $y = -3\cos(\pi t)$.

b. Frequency is the reciprocal of the period so $f = \frac{1}{2}$, showing one-half a cycle is completed each second (as the graph indicates).

c. Evaluating the model at $t = 1.8$ gives $y = -3\cos[\pi(1.8)] \approx -2.43$, meaning the weight is 2.43 cm below the equilibrium point at this time.

Now try Exercises 55 through 58 ▶

Harmonic Models—Sound Waves

A second example of harmonic motion is the production of sound. For the purposes of this study, we'll look at musical notes. The vibration of matter produces a **pressure wave** or **sound energy,** which in turn vibrates the eardrum. Through the intricate structure of the middle ear, this sound energy is converted into mechanical energy and sent to the inner ear where it is converted to nerve impulses and transmitted to the brain. If the sound wave has a high frequency, the eardrum vibrates with greater frequency, which the brain interprets as a "high-pitched" sound. The *intensity* of the sound wave can also be transmitted to the brain via these mechanisms, and if the arriving sound wave has a high amplitude, the eardrum vibrates more forcefully and the sound is interpreted as "loud" by the brain. These characteristics are neatly modeled using $y = A\sin(Bt)$. For the moment we will focus on the frequency, keeping the amplitude constant at $A = 1$.

The musical note known as A_4 or "the A above middle C" is produced with a frequency of 440 vibrations per second, or 440 hertz (Hz) (this is the note most often used in the tuning of pianos and other musical instruments). For any given note, the same note one octave higher will have double the frequency, and the same note one octave lower will have one-half the frequency. In addition, with $f = \frac{1}{P}$ the value of $B = 2\pi\left(\frac{1}{P}\right)$ can always be expressed as $B = 2\pi f$, so A_4 has the equation $y = \sin[440(2\pi t)]$ (after rearranging the factors). The same note one octave lower is A_3 and has the equation $y = \sin[220(2\pi t)]$, with one-half the frequency. To draw the representative graphs, we must scale the t-axis in very small increments (seconds $\times 10^{-3}$) since $P = \frac{1}{440} \approx 0.0023$ for A_4, and $P = \frac{1}{220} \approx 0.0045$ for A_3. Both are graphed in Figure 6.111, where we see that the higher note completes two cycles in the same interval that the lower note completes one.

Figure 6.111

EXAMPLE 7 ▶ **Applications of Sine and Cosine: Sound Frequencies**

The table here gives the frequencies for three octaves of the 12 "chromatic" notes with frequencies between 110 Hz and 840 Hz. Two of the 36 notes are graphed in the figure. Which two?

Frequency by Octave			
Note	**Octave 2**	**Octave 3**	**Octave 4**
A	110.00	220.00	440.00
A#	116.54	233.08	466.16
B	123.48	**246.96**	493.92
C	130.82	261.64	523.28
C#	138.60	277.20	554.40
D	146.84	293.68	587.36
D#	155.56	311.12	622.24
E	**164.82**	329.24	659.28
F	174.62	349.24	698.48
F#	185.00	370.00	740.00
G	196.00	392.00	784.00
G#	207.66	415.32	830.64

Solution ▶ Since amplitudes are equal, the only differences are the frequency and period of the notes. It appears that y_1 has a period of about 0.004 sec, giving a frequency of $\frac{1}{0.004} = 250$ Hz—very likely a B_3 (in bold). The graph of y_2 has a period of about 0.006, for a frequency of $\frac{1}{0.006} \approx 167$ Hz—probably an E_2 (also in bold).

☑ **C.** You've just seen how we can solve applications involving harmonic motion

Now try Exercises 59 through 62 ▶

D. Vertical and Horizontal Translations of Other Trig Functions

In Section 6.4, we used the graphs of $y = A\sin(Bt)$ and $y = A\cos(Bt)$ to help graph $y = A\csc(Bt)$ and $y = A\sec(Bt)$, respectively. If a vertical or horizontal translation is being applied to a cosecant/secant function, we simply apply the same translations to the underlying sine/cosine function as an aid to graphing the function at hand.

EXAMPLE 8 ▶ **Graphing $y = A\sec(Bt + C) + D$**

Draw a sketch of $y = 2\sec\left(\frac{\pi}{3}t - \frac{\pi}{2}\right) - 3$ for t in $[0, 9]$.

Solution ▶ To graph this secant function, consider the function $y = 2\cos\left(\frac{\pi}{3}t - \frac{\pi}{2}\right) - 3$. In shifted form, we have $y = 2\cos\left[\frac{\pi}{3}\left(t - \frac{3}{2}\right)\right] - 3$. With an amplitude of $|A| = 2$ and period of $P = \frac{2\pi}{\pi/3} = 6$, the reference rectangle will be 4 units high and 6 units wide. The *rule of fourths* shows the zeroes and max/min values of the unshifted graph will occur at 0, $\frac{3}{2}$, 3, $\frac{9}{2}$, and 6. Applying a horizontal shift of $+\frac{3}{2}$ units gives

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the values $\frac{3}{2}$, 3, $\frac{9}{2}$, 6, and $\frac{15}{2}$, which we combine with the reference rectangle to produce the cosine and secant graphs shown next. Note the vertical asymptotes of the secant function occur at the zeroes of the cosine function (Figure 6.112). By shifting these results 3 units downward (centered on the line $y = -3$), we obtain the graph of $y = 2\sec\left(\frac{\pi}{3}t - \frac{\pi}{2}\right) - 3$ (Figure 6.113).

Figure 6.112

Figure 6.113

Now try Exercises 45 through 48 ▶

To horizontally and/or vertically translate the graphs of $y = A\tan(Bt)$ and $y = A\cot(Bt)$, we once again apply the translations to the basic graph. Recall for these two functions:

1. The primary interval of $y = \tan t$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, while that of $y = \cot t$ is $[0, \pi]$.
2. $y = \tan t$ increases while $y = \cot t$ decreases on their respective domains.

EXAMPLE 9 ▶ **Graphing $y = A\tan(Bt + C)$**

Draw a sketch of $y = 2.5\tan(0.5t + 1.5)$ over two periods.

Solution ▶ In shifted form, this equation is $y = 2.5\tan[0.5(t + 3)]$. With $A = 2.5$ and $B = 0.5$, the graph is vertically stretched with a period of 2π (see Figure 6.114). Note that $t = -\pi$ and π gives the location of the asymptotes for the unshifted graph, while $t = -\frac{\pi}{2}$ and $\frac{\pi}{2}$ gives -2.5 and 2.5, respectively.

Shifting the graph 3 units left produces the one shown in Figure 6.115 ($-\pi - 3 \approx -6.14$, $-\frac{\pi}{2} - 3 \approx -4.57$, $0 - 3 = -3$, $\frac{\pi}{2} - 3 \approx -1.43$, and $\pi - 3 \approx 0.14$).

WORTHY OF NOTE
The reference rectangle shown is 5 units high by 2π units wide.

Figure 6.114

Figure 6.115

Now try Exercises 49 through 52 ▶

D. You've just seen how we can apply vertical and horizontal translations to cosecant, secant, tangent, and cotangent functions

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E. Applications of the Remaining Trig Functions

We end this section with two examples of how tangent, cotangent, secant, and cosecant functions can be applied. Numerous others can be found in the exercise set.

EXAMPLE 10 ▶ **Applications of $y = A\tan(Bt)$: Modeling the Movement of a Light Beam**

One evening, in port during a *Semester at Sea*, Richard is debating a project choice for his Precalculus class. Looking out his porthole, he notices a revolving light turning at a constant speed near the corner of a long warehouse. The light throws its beam along the length of the warehouse, then disappears into the air, and then returns time and time again. Suddenly—Richard has his project. He notes the time it takes the beam to traverse the warehouse wall is very close to 4 sec, and in the morning he measures the wall's length at 127.26 m. His project? Modeling the distance of the beam from the corner of the warehouse as a function of time using a tangent function. Can you help?

Solution ▶ The equation model will have the form $D(t) = A\tan(Bt)$, where $D(t)$ is the distance (in meters) of the beam from the corner after t sec. The distance along the wall is measured in positive values so we're using only $\frac{1}{2}$ the period of the function, giving $\frac{1}{2}P = 4$ (the beam "disappears" at $t = 4$) so $P = 8$. Substitution in the period formula gives $B = \dfrac{\pi}{8}$ and the equation $D = A\tan\left(\dfrac{\pi}{8}t\right)$.

Knowing the beam travels 127.26 m in about 4 sec (when it disappears into infinity), we'll use $t = 3.9$ and $D = 127.26$ in order to solve for A and complete our equation model (see note following this example).

$$A\tan\left(\frac{\pi}{8}t\right) = D \qquad \text{equation model}$$

$$A\tan\left[\frac{\pi}{8}(3.9)\right] = 127.26 \qquad \text{substitute 127.26 for } D \text{ and 3.9 for } t$$

$$A = \frac{127.26}{\tan\left[\frac{\pi}{8}(3.9)\right]} \qquad \text{solve for } A$$

$$\approx 5 \qquad \text{result}$$

One equation approximating the distance of the beam from the corner of the warehouse is $D(t) = 5\tan\left(\dfrac{\pi}{8}t\right)$.

Now try Exercises 63 through 66 ▶

For Example 10, we should note the choice of 3.9 for t was arbitrary, and while we obtained an "acceptable" model, different values of A would be generated for other choices. For instance, $t = 3.95$ gives $A \approx 2.5$, while $t = 3.99$ gives $A \approx 0.5$. The true value of A depends on the distance of the light from the corner of the warehouse wall. In any case, it's interesting to note that at $t = 2$ sec (one-half the time it takes the beam to disappear), the beam has traveled only 5 m from the corner of the building: $D(2) = 5\tan\left(\dfrac{\pi}{4}\right) = 5$ m. Although the light is rotating at a constant angular speed, the speed of the beam along the wall increases *dramatically* as t gets close to 4 sec.

EXAMPLE 11 ▶ **Applications of $y = A\csc(Bt)$: Modeling the Length of a Shadow**

During the long winter months in southern Michigan, Daniel begins planning for his new solar-powered water heater. He needs to choose a panel location on his roof, and is primarily concerned with the moving shadows of the trees throughout the day. To this end, he begins studying the changing shadow length of a tree in front of his house. At sunrise, the shadow is too long to measure. As the morning progresses, the shadow decreases in length, until at midday it measures its shortest length of 7 ft. As the day moves on, the shadow increases in length, until at sunset it is again too long to measure. If on one particular day sunrise occurs at 6:00 A.M. and sunset at 6:00 P.M.:

a. Use the cosecant function to model the tree's shadow length.

b. Approximate the shadow length at 10:00 A.M.

Solution ▶ **a.** It appears the function $L(t) = A\csc(Bt)$ will model the shadow length, since the shadow decreases from an "infinitely large" length to a minimum length, then returns to an infinite length. Let $t = 0$ correspond to 6:00 A.M. and note Daniel's observations over the 12 hr of daylight. Since the length of the shadow is measured in positive values, we need only $\frac{1}{2}$ the period of the function, giving $\frac{1}{2}P = 12$ (so $P = 24$). The period formula gives $B = \dfrac{\pi}{12}$ and the equation model is $L(t) = A\csc\left(\dfrac{\pi}{12}t\right)$. We solve for A to complete the model, using $L = 7$ when $t = 6$ (noon).

$$L(t) = A\csc\left(\frac{\pi}{12}t\right) \quad \text{equation model}$$

$$7 = A\csc\left[\frac{\pi}{12}(6)\right] \quad \text{substitute 7 for } L \text{ and 6 for } t$$

$$A = \frac{7}{\csc\left(\dfrac{\pi}{2}\right)} \quad \text{solve for } A;\ \frac{6\pi}{12} = \frac{\pi}{2}$$

$$= \frac{7}{1} \quad \csc\left(\frac{\pi}{2}\right) = 1$$

$$= 7 \quad \text{result}$$

The equation modeling the shadow length of the tree is $L(t) = 7\csc\left(\dfrac{\pi}{12}t\right)$.

b. Evaluating the model at $t = 4$ (10 A.M.) gives $L = 7\csc\left(\dfrac{\pi}{3}\right) = 7\left(\dfrac{2\sqrt{3}}{3}\right) \approx 8.08$, meaning the shadow is approximately 8 ft, 1 in. long at 10 A.M.

E. You've just seen how we can solve applications involving the tangent, cotangent, secant, and cosecant functions

Now try Exercises 67 and 68 ▶

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6.5 EXERCISES

▶ CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section if needed.

1. A sinusoidal wave is one that can be modeled by functions of the form _____ or _____.

2. Given $h, k > 0$, the graph of $y = \sin x + k$ is the graph of $y = \sin x$ shifted _____ k units. The graph of $y = \sin(x - h)$ is the graph of $y = \sin x$ shifted _____ h units.

3. To find the primary interval of a sinusoidal graph, solve the inequality _____.

4. Given the period P, the frequency is _____, and given the frequency f, the value of B is _____.

5. Explain/Discuss the difference between the *standard form* of a sinusoidal equation, and the *shifted form*. How do you obtain one from the other? For what benefit?

6. Write out a step-by-step procedure for sketching the graph of $y = 30 \sin\left(\frac{\pi}{2}t - \frac{1}{2}\right) + 10$. Include use of the reference rectangle, primary interval, zeroes, max/mins, and so on. Be complete and thorough.

▶ DEVELOPING YOUR SKILLS

Use the graphs given to (a) state the amplitude A and period P of the function; (b) estimate the value at $x = 14$; and (c) estimate the interval in $[0, P]$ where $f(x) \geq 20$.

7.

8.

Use the graphs given to (a) state the amplitude A and period P of the function; (b) estimate the value at $x = 2$; and (c) estimate the interval in $[0, P]$, where $f(x) \leq -100$.

9.

10.

Use the information given to write a sinusoidal equation and sketch its graph. Use $B = \frac{2\pi}{P}$.

11. Max: 100, min: 20, $P = 30$

12. Max: 95, min: 40, $P = 24$

13. Max: 20, min: 4, $P = 360$

14. Max: 12,000, min: 6500, $P = 10$

Use the information given to write a sinusoidal equation, sketch its graph, and answer the question posed.

15. In Geneva, Switzerland, the daily temperature in January ranges from an average high of 39°F to an average low of 29°F. (a) Find a sinusoidal equation model for the daily temperature; (b) sketch the graph; and (c) approximate the time(s) each January day the temperature reaches the freezing point (32°F). Assume $t = 0$ corresponds to noon.

16. In Nairobi, Kenya, the daily temperature in January ranges from an average high of 77°F to an average low of 58°F. (a) Find a sinusoidal equation model for the daily temperature; (b) sketch the graph; and (c) approximate the time(s) each January day the temperature reaches a comfortable 72°F. Assume $t = 0$ corresponds to noon.

17. In Oslo, Norway, the number of hours of daylight reaches a low of 6 hr in January, and a high of nearly 18.8 hr in July. (a) Find a sinusoidal equation model for the number of daylight hours each month; (b) sketch the graph; and (c) approximate the number of *days* each year there are more than 15 hr of daylight. Use 1 month $\approx$ 30.5 days. Assume $t = 0$ corresponds to January 1.

Source: www.visitnorway.com/templates.

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18. In Vancouver, British Columbia, the number of hours of daylight reaches a low of 8.3 hr in January, and a high of nearly 16.2 hr in July. (a) Find a sinusoidal equation model for the number of daylight hours each month; (b) sketch the graph; and (c) approximate the number of *days* each year there are more than 15 hr of daylight. Use 1 month $\approx$ 30.5 days. Assume $t = 0$ corresponds to January 1.

Source: www.bcpassport.com/vital/temp.

19. Recent studies seem to indicate the population of North American porcupine (*Erethizon dorsatum*) varies sinusoidally with the solar (sunspot) cycle due to its effects on Earth's ecosystems. Suppose the population of this species in a certain locality is modeled by the function $P(t) = 250\cos\left(\frac{2\pi}{11}t\right) + 950$, where $P(t)$ represents the population of porcupines in year t. Use the model to (a) find the period of the function; (b) graph the function over one period; (c) find the maximum and minimum values; and (d) estimate the number of years the population is less than 740 animals.

Source: Ilya Klvana, McGill University (Montreal), Master of Science thesis paper, November 2002.

20. The population of mosquitoes in a given area is primarily influenced by precipitation, humidity, and temperature. In tropical regions, these tend to fluctuate sinusoidally in the course of a year. Using trap counts and statistical projections, fairly accurate estimates of a mosquito population can be obtained. Suppose the population in a certain region was modeled by the function $P(t) = 50\cos\left(\frac{\pi}{26}t\right) + 950$, where $P(t)$ was the mosquito population (in thousands) in week t of the year. Use the model to (a) find the period of the function; (b) graph the function over one period; (c) find the maximum and minimum population values; and (d) estimate the number of weeks the population is less than 915,000.

21. Use a horizontal translation to shift the graph from Exercise 19 so that the average population of the North American porcupine begins at $t = 0$. Verify results on a graphing calculator, then find a sine function that gives the same graph as the shifted cosine function.

22. Use a horizontal translation to shift the graph from Exercise 20 so that the average population of mosquitoes begins at $t = 0$. Verify results on a graphing calculator, then find a sine function that gives the same graph as the shifted cosine function.

Identify the amplitude (A), period (P), horizontal shift (HS), vertical shift (VS), and endpoints of the primary interval (PI) for each function given.

23. $y = 120\sin\left[\frac{\pi}{12}(t - 6)\right]$

24. $y = 560\sin\left[\frac{\pi}{4}(t - 4)\right]$

25. $h(t) = \sin\left(\frac{\pi}{6}t - \frac{\pi}{3}\right)$

26. $r(t) = \sin\left(\frac{\pi}{10}t - \frac{2\pi}{5}\right)$

27. $y = \sin\left(\frac{\pi}{4}t + \frac{\pi}{6}\right)$

28. $y = \sin\left(\frac{\pi}{3}t + \frac{5\pi}{12}\right)$

29. $f(t) = 24.5\sin\left[\frac{\pi}{10}(t - 2.5)\right] + 15.5$

30. $g(t) = 40.6\sin\left[\frac{\pi}{6}(t - 4)\right] + 13.4$

31. $g(t) = 28\sin\left(\frac{\pi}{6}t - \frac{5\pi}{12}\right) + 92$

32. $f(t) = 90\sin\left(\frac{\pi}{10}t - \frac{\pi}{5}\right) + 120$

33. $y = 2500\sin\left(\frac{\pi}{4}t + \frac{\pi}{12}\right) + 3150$

34. $y = 1450\sin\left(\frac{3\pi}{4}t + \frac{\pi}{8}\right) + 2050$

Find an equation of the graph given. Write answers in the form $y = A\sin(Bt + C) + D$ and check with a graphing calculator.

35.

36.

37.

38.

39.

40.

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Sketch one complete period of each function.

41. $f(t) = 25 \sin\left[\frac{\pi}{4}(t - 2)\right] + 55$

42. $g(t) = 24.5 \sin\left[\frac{\pi}{10}(t - 2.5)\right] + 15.5$

43. $h(t) = 3 \sin(4t - \pi)$

44. $p(t) = 2 \sin\left(3t - \frac{\pi}{2}\right)$

Sketch the following functions over the indicated intervals.

45. $y = 5 \sec\left[\frac{1}{3}\left(t + \frac{3\pi}{2}\right)\right] + 2; [-3\pi, 6\pi]$

46. $y = 3 \csc\left[\frac{1}{2}(t + \pi)\right] + 1; [-2\pi, 4\pi]$

47. $y = 0.7 \csc\left(\pi t - \frac{\pi}{4}\right) - 1.2; [-1.25, 1.75]$

48. $y = 1.3 \sec\left(\frac{\pi}{3}t - \frac{\pi}{6}\right) - 1.6; [-2, 6]$

Sketch two complete periods of each function.

49. $y = 0.5 \tan\left[\frac{\pi}{4}(t + 2)\right]$

50. $y = 1.5 \cot\left[\frac{\pi}{2}(t + 1)\right]$

51. $y = 10 \cot(2t - 1)$

52. $y = 8 \tan(3t - 2)$

▶ WORKING WITH FORMULAS

53. The relationship between the coefficient *B*, the frequency *f*, and the period *P*

In many applications of trigonometric functions, the equation $y = A \sin(Bt)$ is written as $y = A \sin[(2\pi f)t]$, where $B = 2\pi f$. Justify the new equation using $f = \frac{1}{P}$ and $P = \frac{2\pi}{B}$. In other words, explain how $A \sin(Bt)$ becomes $A \sin[(2\pi f)t]$, as though you were trying to help another student with the ideas involved.

54. Number of daylight hours:

$$D(t) = \frac{K}{2} \sin\left[\frac{2\pi}{365}(t - 79)\right] + 12$$

The number of daylight hours for a particular day of the year is modeled by the formula given, where $D(t)$ is the number of daylight hours on day t of the year and K is a constant related to the total variation of daylight hours, latitude of the location, and other factors. For the city of Reykjavik, Iceland, $K \approx 17$, while for Detroit, Michigan, $K \approx 6$. How many hours of daylight will each city receive on June 30 (the 182nd day of the year)?

▶ APPLICATIONS

55. Harmonic motion: A weight on the end of a spring is oscillating in harmonic motion. The equation model for the oscillations is $d(t) = 6 \sin\left(\frac{\pi}{2}t\right)$, where d is the distance (in centimeters) from the equilibrium point in t sec.

a. What is the period of the motion? What is the frequency of the motion?

b. What is the displacement from equilibrium at $t = 2.5$? Is the weight moving toward the equilibrium point or away from equilibrium at this time?

c. What is the displacement from equilibrium at $t = 3.5$? Is the weight moving toward the equilibrium point or away from equilibrium at this time?

d. How far does the weight move between $t = 1$ and $t = 1.5$ sec? What is the average velocity for this interval? Do you expect a greater or lesser velocity for $t = 1.75$ to $t = 2$? Explain why.

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56. Harmonic motion: The bob on the end of a 24-in. pendulum is oscillating in harmonic motion. The equation model for the oscillations is $d(t) = 20\cos(4t)$, where d is the distance (in inches) from the equilibrium point, t sec after being released from one side.

a. What is the period of the motion? What is the frequency of the motion?

b. What is the displacement from equilibrium at $t = 0.25$ sec? Is the weight moving toward the equilibrium point or away from equilibrium at this time?

c. What is the displacement from equilibrium at $t = 1.3$ sec? Is the weight moving toward the equilibrium point or away from equilibrium at this time?

d. How far does the bob move between $t = 0.25$ and $t = 0.35$ sec? What is its average velocity for this interval? Do you expect a greater velocity for the interval $t = 0.55$ to $t = 0.6$? Explain why.

57. Harmonic motion: A simple pendulum 36 in. in length is oscillating in harmonic motion. The bob at the end of the pendulum swings through an arc of 30 in. (from the far left to the far right, or one-half cycle) in about 0.8 sec. What is the equation model for this harmonic motion?

58. Harmonic motion: As part of a study of wave motion, the motion of a float is observed as a series of uniform ripples of water move beneath it. By careful observation, it is noted that the float bobs up and down through a distance of 2.5 cm every $\frac{1}{3}$ sec. What is the equation model for this harmonic motion?

59. Sound waves: Two of the musical notes from the chart on page 586 are graphed in the figure. Use the graphs given to determine which two.

60. Sound waves: Two chromatic notes *not on the chart from page* 586 are graphed in the figure. Use the graphs and the discussion regarding octaves to determine which two. Note the scale of the t-axis *has been changed* to hundredths of a second.

Sound waves: Use the chart on page 586 to write the equation for each note in the form $y = \sin[f(2\pi t)]$ and clearly state the period of each note.

61. notes D_2 and G_3

62. the notes A_4 and C_2

Tangent function data models: Model the data in Exercises 63 and 64 using the function $y = A\tan(Bx)$. State the period of the function, the location of the asymptotes, the value of A, and name the point (x, y) used to calculate A (answers may vary). Use your equation model to evaluate the function at $x = -2$ and $x = 2$. What observations can you make? Also see Exercise 73.

63.

Input	Output	Input	Output
−6	−∞	1	1.4
−5	−20	2	3
−4	−9.7	3	5.2
−3	−5.2	4	9.7
−2	−3	5	20
−1	−1.4	6	∞
0	0		

64.

Input	Output	Input	Output
−3	−∞	0.5	6.4
−2.5	−91.3	1	13.7
−2	−44.3	1.5	23.7
−1.5	−23.7	2	44.3
−1	−13.7	2.5	91.3
−0.5	−6.4	3	∞
0	0		

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65. As part of a lab setup, a laser pen is made to swivel on a large protractor as illustrated in the figure. For their lab project, students are asked to take the instrument to one end of a long hallway and measure the distance of the projected beam relative to the angle the pen is being held, and collect the data in a table. Use the data to find a function of the form $y = A\tan(B\theta)$. State the period of the function, the location of the asymptotes, the value of A, and name the point (θ, y) you used to calculate A (answers may vary). Based on the result, can you approximate the length of the laser pen? Note that in degrees, the period formula for tangent is $P = \dfrac{180°}{B}$.

Exercise 65

θ (degrees)	Distance (cm)
0	0
10	2.1
20	4.4
30	6.9
40	10.1
50	14.3
60	20.8
70	33.0
80	68.1
89	687.5

66. Use the equation model obtained in Exercise 65 to compare the values given by the equation with the actual data. As a percentage, what was the largest deviation between the two?

67. Shadow length: At high noon, a flagpole in Oslo, Norway, casts a 10-m-long shadow during the month of January. Using information from Exercise 17, (a) find a cosecant function that models the shadow length, and (b) use the model to find the length of the shadow at 2:00 P.M.

68. Shadow length: At high noon, the "Living Shangri-La" skyscraper in Vancouver, British Columbia, casts a 15-m-long shadow during the month of June. Given there are 16 hr of daylight that month, (a) find a cosecant function that models the shadow length, and (b) use the model to find the length of the shadow at 7:30 A.M.

Daylight hours model: Solve using a graphing calculator and the formula given in Exercise 54.

69. For the city of Caracas, Venezuela, $K \approx 1.3$, while for Tokyo, Japan, $K \approx 4.8$.

a. How many hours of daylight will each city receive on January 15th (the 15th day of the year)?

b. Graph the equations modeling the hours of daylight on the same screen. Then determine (i) what days of the year these two cities will have the same number of hours of daylight, and (ii) the number of days each year that each city receives 11.5 hr or less of daylight.

70. For the city of Houston, Texas, $K \approx 3.8$, while for Pocatello, Idaho, $K \approx 6.2$.

a. How many hours of daylight will each city receive on December 15 (the 349th day of the year)?

b. Graph the equations modeling the hours of daylight on the same screen. Then determine (i) how many days each year Pocatello receives more daylight than Houston, and (ii) the number of days each year that each city receives 13.5 hr or more of daylight.

▶ EXTENDING THE CONCEPT

71. The formulas we use in mathematics can sometimes seem very mysterious. We know they "work," and we can graph and evaluate them—but where did they come from? Consider the formula for the number of daylight hours from Exercise 54:

$$D(t) = \frac{K}{2}\sin\left[\frac{2\pi}{365}(t - 79)\right] + 12.$$

a. We know that the addition of 12 represents a vertical shift, but what does a vertical shift of 12 mean *in this context*?

b. We also know the factor $(t - 79)$ represents a phase shift of 79 to the right. But what does a horizontal (phase) shift of 79 mean *in this context*?

c. Finally, the coefficient $\dfrac{K}{2}$ represents a change in amplitude, but what does a change of amplitude mean *in this context*? Why is the coefficient bigger for the northern latitudes?

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72. Use a graphing calculator to graph the equation

$$f(x) = \frac{3x}{2} - 2\sin(2x) - 1.5.$$

a. Determine the interval between each peak of the graph. What do you notice?

b. Graph $g(x) = \frac{3x}{2} - 1.5$ on the same screen and comment on what you observe.

c. What would the graph of

$$f(x) = -\frac{3x}{2} + 2\sin(2x) + 1.5 \text{ look like?}$$

What is the x-intercept?

73. Rework Exercises 63 and 64, obtaining a new equation for the data using a different ordered pair to compute the value of A. What do you notice? Try yet another ordered pair and calculate A once again for another equation Y_2. Complete a table of values using the given inputs, with the outputs of the three equations generated (original, Y_1, and Y_2). Does any one equation seem to model the data better than the others? Are all of the equation models "acceptable"? Please comment.

74. Regarding Example 10, we can use the standard distance/rate/time formula $D = RT$ to compute the average velocity of the beam of light along the wall in any interval of time: $R = \frac{D}{T}$. For example, using $D(t) = 5\tan\left(\frac{\pi}{8}t\right)$, the average velocity in the interval [0, 2] is $\frac{D(2) - D(0)}{2 - 0} = 2.5$ m/sec. Calculate the average velocity of the beam in the time intervals [2, 3], [3, 3.5], and [3.5, 3.8] sec. What do you notice? How would the average velocity of the beam in the interval [3.9, 3.99] sec compare?

▶ MAINTAINING YOUR SKILLS

75. **(6.1)** In what quadrant does the arc $t = 3.7$ terminate? What is the reference arc?

76. **(3.2)** Given $f(x) = -3(x + 1)^2 - 4$, name the vertex and solve the inequality $f(x) > 0$.

77. **(3.1)** Compute the sum, difference, product, and quotient of $-1 + i\sqrt{5}$ and $-1 - i\sqrt{5}$.

78. **(6.3/6.4)** Sketch the graph of (a) $y = \cos t$ in the interval $[0, 2\pi)$ and (b) $y = \tan t$ in the interval $\left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$.

6.6 The Trigonometry of Right Triangles

LEARNING OBJECTIVES

In Section 6.6 you will see how we can:

- ❑ **A.** Find values of the six trigonometric functions from their ratio definitions
- ❑ **B.** Solve a right triangle given one angle and one side
- ❑ **C.** Solve a right triangle given two sides
- ❑ **D.** Use cofunctions and complements to write equivalent expressions
- ❑ **E.** Solve applications of right triangles

Over a long period of time, what began as a study of chord lengths by Hipparchus, Ptolemy, Aryabhata, and others became a systematic application of the ratios of the sides of a right triangle. In this section, we develop the sine, cosine, and tangent functions from a right triangle perspective, and explore certain relationships that exist between them. This view of the trig functions also leads to a number of significant applications.

A. Trigonometric Ratios and Their Values

In Section 6.1, we looked at applications involving 45-45-90 and 30-60-90 triangles, using the fixed ratios that exist between their sides. To apply this concept more generally using other right triangles, each side is given a specific name using its location relative to a specified angle. For the 30-60-90 triangle in Figure 6.116(a), the side **opposite (opp)** and the side **adjacent (adj)** are named with respect to the 30° angle, with the **hypotenuse (hyp)** always across from the right angle. Likewise for the 45-45-90 triangle in Figure 6.116(b).

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Figure 6.116

(a)

for 30°

$\frac{\text{opp}}{\text{hyp}} = \frac{1}{2}$

$\frac{\text{adj}}{\text{hyp}} = \frac{\sqrt{3}}{2}$

$\frac{\text{opp}}{\text{adj}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

(b)

for 45°

$\frac{\text{opp}}{\text{hyp}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

$\frac{\text{adj}}{\text{hyp}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

$\frac{\text{opp}}{\text{adj}} = \frac{1}{1}$

Using these designations to define the various trig ratios, we can now develop a systematic method for applying them. Note that the x's "cancel" in each ratio, reminding us the ratios are independent of the triangle's size (if two triangles are similar, the ratio of corresponding sides is constant).

Ancient mathematicians were able to find values for the ratios corresponding to *any acute angle* in a right triangle, and realized that *naming* each ratio would be helpful. These names are $\frac{\text{opp}}{\text{hyp}} \rightarrow$ **sine,** $\frac{\text{adj}}{\text{hyp}} \rightarrow$ **cosine,** and $\frac{\text{opp}}{\text{adj}} \rightarrow$ **tangent.** Since each ratio depends on the measure of an acute angle θ, they are often referred to as **functions of an acute angle** and written in function form.

$$\text{sine } \theta = \frac{\text{opp}}{\text{hyp}} \qquad \text{cosine } \theta = \frac{\text{adj}}{\text{hyp}} \qquad \text{tangent } \theta = \frac{\text{opp}}{\text{adj}}$$

The reciprocal of these ratios, for example, $\frac{\text{hyp}}{\text{opp}}$ instead of $\frac{\text{opp}}{\text{hyp}}$, also play a significant role in this view of trigonometry, and are likewise given names:

$$\text{cosecant } \theta = \frac{\text{hyp}}{\text{opp}} \qquad \text{secant } \theta = \frac{\text{hyp}}{\text{adj}} \qquad \text{cotangent } \theta = \frac{\text{adj}}{\text{opp}}$$

The definitions hold regardless of the triangle's orientation or which of the acute angles is used.

As before, each function name is written in abbreviated form as $\sin\theta$, $\cos\theta$, $\tan\theta$, $\csc\theta$, $\sec\theta$, and $\cot\theta$ respectively. Note that based on these designations, we have the following reciprocal relationships:

$$\sin\theta = \frac{1}{\csc\theta} \qquad \cos\theta = \frac{1}{\sec\theta} \qquad \tan\theta = \frac{1}{\cot\theta}$$

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

WORTHY OF NOTE

Over the years, a number of memory tools have been invented to help students recall these ratios correctly. One such tool is the acronym SOH CAH TOA, from the first letter of the function and the corresponding ratio. It is often recited as, "Sit On a Horse, Canter Away Hurriedly, To Other Adventures." Try making up a memory tool of your own.

In general:

Trigonometric Functions of an Acute Angle

$$\sin\alpha = \frac{a}{c} \qquad \cos\alpha = \frac{b}{c} \qquad \tan\alpha = \frac{a}{b}$$

$$\sin\beta = \frac{b}{c} \qquad \cos\beta = \frac{a}{c} \qquad \tan\beta = \frac{b}{a}$$

Now that these ratios have been formally named, we can state values of all six functions given sufficient information about a right triangle.

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EXAMPLE 1 ▶ **Finding Function Values Using a Right Triangle**

Given $\sin\theta = \frac{4}{7}$, find the values of the remaining trig functions.

Solution ▶ For $\sin\theta = \dfrac{4}{7} = \dfrac{\text{opp}}{\text{hyp}}$, we draw a triangle with a side of 4 units opposite a designated angle θ, and label a hypotenuse of 7 (see the figure). Using the Pythagorean theorem we find the length of the adjacent side: $\text{adj} = \sqrt{7^2 - 4^2} = \sqrt{33}$. The ratios are

$$\sin\theta = \frac{4}{7} \qquad \cos\theta = \frac{\sqrt{33}}{7} \qquad \tan\theta = \frac{4}{\sqrt{33}}$$

$$\csc\theta = \frac{7}{4} \qquad \sec\theta = \frac{7}{\sqrt{33}} \qquad \cot\theta = \frac{\sqrt{33}}{4}$$

Now try Exercises 7 through 12 ▶

A. You've just seen how we can find values of the six trigonometric functions from their ratio definitions

Note that due to the properties of similar triangles, identical results would be obtained using any ratio of sides that is equal to $\frac{4}{7}$. In other words, $\frac{2}{3.5} = \frac{4}{7} = \frac{8}{14} = \frac{16}{28}$ and so on, will all give the same value for $\sin\theta$.

B. Solving Right Triangles Given One Angle and One Side

Example 1 gave values of the trig functions for an *unknown angle* θ. Using the special triangles, we can state the value of each trig function for 30°, 45°, and 60° based on the related ratio (see Table 6.10). These values are used extensively in a study of trigonometry and must be committed to memory.

Table 6.10

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$	2	$\frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$	$\sqrt{3}$
45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$	2	$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

To **solve a right triangle** means to find the measure of all three angles and all three sides. This is accomplished using a combination of the Pythagorean theorem, the properties of triangles, and the trigonometric ratios. We will adopt the convention of naming each angle with a capital letter at the vertex or using a Greek letter on the interior. Each side is labeled using the related lowercase letter from the angle opposite. The complete solution should be organized in table form as in Example 2. Note the quantities shown in **bold** were given, and the remaining values were found using the relationships mentioned.

EXAMPLE 2 ▶ **Solving a Right Triangle**

Solve the triangle shown in the figure.

Solution ▶ Applying the sine ratio (since the side opposite 30° is given), we have: $\sin 30° = \dfrac{\text{opp}}{\text{hyp}}$.

$$\sin 30° = \frac{17.9}{c} \qquad \sin 30° = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$c \sin 30° = 17.9 \qquad \text{multiply by } c$$

$$c = \frac{17.9}{\sin 30°} \qquad \text{divide by } \sin 30° = \frac{1}{2}$$

$$= 35.8 \qquad \text{result}$$

Using the Pythagorean theorem shows $b \approx 31$, and since $\angle A$ and $\angle B$ are complements, $B = 60°$. Note the results would have been identical if the special ratios from the 30-60-90 triangle were applied. The hypotenuse is twice the shorter side: $c = 2(17.9) = 35.8$, and the longer side is $\sqrt{3}$ times the shorter: $b = 17.9(\sqrt{3}) \approx 31$.

Angles	Sides
$A = 30°$	$a = 17.9$
$B = 60°$	$b \approx 31$
$C = 90°$	$c = 35.8$

Now try Exercises 13 through 16 ▶

Prior to the widespread availability of handheld calculators, tables of values were used to find $\sin\theta$, $\cos\theta$, and $\tan\theta$ for nonstandard angles. Table 6.11 shows the sine of 49° 30′ is approximately 0.7604.

Table 6.11
sin θ

θ	0′	10′	20′	30′
45°	0.7071	0.7092	0.7112	0.7133
46	0.7193	0.7214	0.7234	0.7254
47	0.7314	0.7333	0.7353	0.7373
48	0.7431	0.7451	0.7470	0.7490
49	0.7547	0.7566	0.7585	0.7604

Today these trig values are programmed into your calculator and we can retrieve them with the push of a button (or two). To find the sine of 49° 30′, recall the degree and minute symbols are often found in the 2nd APPS **(ANGLE)** menu. A calculator in degree MODE can produce the screen shown in Figure 6.117 and provide an even more accurate approximation of sin 49° 30′ than the table.

Figure 6.117

EXAMPLE 3 ▶ **Solving a Right Triangle**

Solve the triangle shown in the figure.

Solution ▶ We know $\angle B = 58°$ since A and B are complements (Figure 6.118). We can find length b using the tangent function:

Figure 6.118

$$\tan 32° = \frac{24}{b} \qquad \tan 32° = \frac{\text{opp}}{\text{adj}}$$

$$b \tan 32° = 24 \qquad \text{multiply by } b$$

$$b = \frac{24}{\tan 32°} \qquad \text{divide by } \tan 32°$$

$$\approx 38.41 \text{ mm} \qquad \text{result}$$

We can find the length c by simply applying the Pythagorean theorem, as shown in the third line of Figure 6.119. Alternatively, we could find c by using another trig ratio and a known angle.

Figure 6.119

```
24/tan(32)
        38.4080287
√(Ans²+24²)
        45.28991796
```

For side c: $\sin 32° = \dfrac{24}{c} \qquad \sin 32° = \dfrac{\text{opp}}{\text{hyp}}$

$$c \sin 32° = 24 \qquad \text{multiply by } c$$

$$c = \frac{24}{\sin 32°} \qquad \text{divide by } \sin 32°$$

$$\approx 45.29 \text{ mm} \qquad \text{result}$$

The complete solution is shown in the table.

Angles	Sides
$\boldsymbol{A = 32°}$	$\boldsymbol{a = 24}$
$B = 58°$	$b \approx 38.41$
$\boldsymbol{C = 90°}$	$c \approx 45.29$

Now try Exercises 17 through 22 ▶

When solving a right triangle, any combination of the known triangle relationships can be employed:

1. The angles in a triangle must sum to 180°: $A + B + C = 180°$.
2. The sides of a right triangle are related by the Pythagorean theorem: $a^2 + b^2 = c^2$.
3. The 30-60-90 and 45-45-90 special triangles.
4. The six trigonometric functions of an acute angle.

However, the resulting equation must have only one unknown or it cannot be used. For the triangle shown in Figure 6.120, we cannot begin with the Pythagorean theorem since sides a and b are unknown, and $\tan 51°$ is unusable for the same reason. Since the hypotenuse is given, we could begin with $\cos 51° = \dfrac{b}{152}$ and solve for b, or with $\sin 51° = \dfrac{a}{152}$ and solve for a, then work out a complete solution. Verify that $a \approx 118.13$ ft and $b \approx 95.66$ ft.

Figure 6.120

B. You've just seen how we can solve a right triangle given one angle and one side

C. Solving Right Triangles Given Two Sides

In Section 6.2, we used our experience with the special values to find an angle given only the value of a trigonometric function. Unfortunately, if we did not have (or recognize) a special value, we could not determine the angle. In times past, the partial table for $\sin \theta$ given earlier was also used to find an angle whose sine was known, meaning if $\sin \theta \approx 0.7604$, then θ must be 49.5° (see the last line of Table 6.11). The modern

notation for "an angle whose sine is known" is $\theta = \mathbf{sin}^{-1}\boldsymbol{x}$ or $\theta = \mathbf{arcsin}\ \boldsymbol{x}$, where x is the known value for $\sin\theta$. The values for the acute angles $\theta = \sin^{-1}x$, $\theta = \cos^{-1}x$, and $\theta = \tan^{-1}x$ are also programmed into your calculator and are generally accessed using the 2nd key with the related SIN, COS, or TAN key. With these we are completely equipped to find all six measures of a right triangle, given at least one side and any two other measures.

CAUTION ▶ When working with the inverse trig functions, be sure to use the inverse trig keys, and not the reciprocal key x^{-1}. For instance, using special triangles, $\cos^{-1}(0.5) = 60°$. Compare this with $[\cos(0.5)]^{-1} = \dfrac{1}{\cos 0.5} = \sec 0.5$. Both calculations are done in degree MODE and shown in the figure.

EXAMPLE 4 ▶ **Solving a Right Triangle**

Solve the triangle given in the figure.

Solution ▶ Since the hypotenuse is unknown, we cannot begin with the sine or cosine ratios. The opposite and adjacent sides for α *are* known, so we use $\tan\alpha$. For $\tan\alpha = \dfrac{17}{25}$ we find $\alpha = \tan^{-1}\left(\dfrac{17}{25}\right) \approx 34.2°$ [verify that $\tan(34.2°) \approx 0.6795992982 \approx \dfrac{17}{25}$]. Since α and β are complements, $\beta \approx 90° - 34.2° = 55.8°$. The Pythagorean theorem shows the hypotenuse is about 30.23 m (verify).

Angles	Sides
$\alpha \approx 34.2°$	$\boldsymbol{a} = \mathbf{17}$
$\beta \approx 55.8°$	$\boldsymbol{b} = \mathbf{25}$
$\boldsymbol{\gamma} = \mathbf{90°}$	$c \approx 30.23$

Now try Exercises 23 through 54 ▶

☑ **C.** You've just seen how we can solve a right triangle given two sides

D. Using Cofunctions and Complements to Write Equivalent Expressions

WORTHY OF NOTE

The word *cosine* is actually a shortened form of the words "*complement of sine*," a designation suggested by Edmund Gunter around 1620 since the sine of an angle is equal to the cosine of its complement [sine(θ) = cosine($90° - \theta$)].

Figure 6.121

For the right triangle in Figure 6.121, $\angle\alpha$ and $\angle\beta$ are complements since the sum of the three angles must be 180°. The complementary angles in a right triangle have a unique relationship that is often used. Specifically, since $\alpha + \beta = 90°$, $\beta = 90° - \alpha$. Note that $\sin\alpha = \dfrac{a}{c}$ and $\cos\beta = \dfrac{a}{c}$. This means $\sin\alpha = \cos\beta$ or $\sin\alpha = \cos(90° - \alpha)$ by substitution. In words, "The sine of an angle is equal to the cosine of its complement." For this reason sine and cosine are called **cofunctions** (hence the name **co**sine), as are secant/cosecant, and tangent/cotangent. As a test, we use a calculator to check the statement $\sin 52.3° = \cos(90° - 52.3°)$

$$\sin 52.3° \stackrel{?}{=} \cos 37.7°$$
$$0.791223533 = 0.791223533 \checkmark$$

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Figure 6.122

To verify the cofunction relationship for sec θ and csc θ, recall their reciprocal relationship to cosine and sine, respectively.

$$\sec 52.3° \stackrel{?}{=} \csc 37.7°$$
$$\frac{1}{\cos 52.3°} \stackrel{?}{=} \frac{1}{\sin 37.7°}$$
$$1.635250666 = 1.635250666 \checkmark$$

The cofunction relationship for tan θ and cot θ is similarly verified in Figure 6.122.

Summary of Cofunctions

sine and cosine	tangent and cotangent	secant and cosecant
$\sin\theta = \cos(90° - \theta)$	$\tan\theta = \cot(90° - \theta)$	$\sec\theta = \csc(90° - \theta)$
$\cos\theta = \sin(90° - \theta)$	$\cot\theta = \tan(90° - \theta)$	$\csc\theta = \sec(90° - \theta)$

For use in Example 5 and elsewhere in the text, note the expression $\tan^2 15°$ is simply a more convenient way of writing $(\tan 15°)^2$.

EXAMPLE 5 ▶ **Applying the Cofunction Relationship**

Given $\cot 75° = 2 - \sqrt{3}$ in exact form, find the exact value of $\tan^2 15°$ using a cofunction. Check the result using a calculator.

Solution ▶ Using $\cot 75° = \tan(90° - 75°) = \tan 15°$ gives

$$\begin{aligned} \tan^2 15° &= \cot^2 75° && \text{cofunction; square both sides} \\ &= (2 - \sqrt{3})^2 && \text{substitute known value} \\ &= 4 - 4\sqrt{3} + 3 && \text{square as indicated} \\ &= 7 - 4\sqrt{3} && \text{result} \end{aligned}$$

Check ▶ To clearly calculate the square of the tangent, note we first evaluate tan 15° and then square the **Ans**wer. As the screen shows, the approximate forms of $\tan^2 15°$ and $7 - 4\sqrt{3}$ are equal. ✓

Now try Exercises 55 through 68 ▶

☑ **D.** You've just seen how we can use cofunctions and complements to write equivalent expressions

E. Applications of Right Triangles

While the name seems self-descriptive, in more formal terms an **angle of elevation** is defined to be the acute angle formed by a **horizontal line of orientation** (parallel to level ground) and the line of sight (see Figure 6.123). An **angle of depression** is likewise defined but involves a line of sight that is below the horizontal line of orientation (Figure 6.124).

Figure 6.123

Figure 6.124

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Angles of elevation and depression make distance and length computations of all sizes a relatively easy matter and are extensively used by surveyors, engineers, astronomers, and even the casual observer who is familiar with the basics of trigonometry.

EXAMPLE 6 ▶ **Applying Angles of Depression**

From the edge of the Perito Moreno glaciar in Patagonia, Argentina, Gary notices a kayak on the lake 30 m below. If the angle of depression at that point is 35°, what is the distance d from the kayak to the glacier?

Solution ▶ From the diagram, we note that d is the side opposite a 55° angle (the complement of the 35° angle). Since the adjacent side is known, we use the tangent function to solve for d.

$$\tan 55° = \frac{d}{30} \qquad \tan 55° = \frac{\text{opp}}{\text{adj}}$$

$$30 \tan 55° = d \qquad \text{multiply by 30 (exact form)}$$

$$42.8 \approx d \qquad \text{result (approximate form)}$$

The kayak is approximately 42.8 m (141 ft) from the glacier.

Now try Exercises 71 through 74 ▶

Closely related to angles of elevation/depression are acute angles of *rotation* from a fixed line of orientation to a fixed line of sight. In this case, the movement is simply horizontal rather than vertical. Land surveyors use a special type of measure called **bearings,** where they indicate the acute angle the line of sight makes with a due north or due south line of reference. For instance, Figure 6.125(a) shows a bearing of N 60° E and Figure 6.125(b) shows a bearing of S 40° W.

Figure 6.125

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EXAMPLE 7 ▶ **Applying Angles of Rotation**

A city building code requires a new shopping complex to be built at least 150 ft from an avenue that runs due east and west. Using a modern theodolite set up on the avenue, a surveyor finds the tip of the shopping complex closest to the avenue lies 200 ft away on a bearing of N 41°17′15″ W. How far from the avenue is this tip of the building?

Solution ▶ To find the distance d, we first convert the given angle to decimal degrees, then find its complement. For 41° 17′ 15″ we have $[41 + 17(1/60) + (15/3600)]° = 41.2875°$. The complement of this angle is $90° - 41.2875° = 48.7125°$. Since we need the side opposite this angle and the hypotenuse is known, we use a sine function to solve for d.

$$\sin 48.7125° = \frac{d}{200} \qquad \sin 48.7125° = \frac{\text{opp}}{\text{hyp}}$$

$$200 \sin 48.7125° = d \qquad \text{multiply by 200 (exact form)}$$

$$150.28 \approx d \qquad \text{result (approximate form)}$$

The shopping complex is approximately 150.28 ft from the avenue, just barely in compliance with the city code.

Now try Exercises 75 and 76 ▶

In their widest and most beneficial use, the trig functions of acute angles are used with other problem-solving skills, such as drawing a diagram, labeling unknowns, working the solution out in stages, and so on. Example 8 serves to illustrate some of these combinations.

EXAMPLE 8 ▶ **Applying Angles of Elevation and Depression**

From his hotel room window on the sixth floor, Singh notices some window washers high above him on the hotel across the street. Curious as to their height above ground, he quickly estimates the buildings are 50 ft apart, the angle of elevation to the workers is about 80°, and the angle of depression to the base of the hotel is about 50°.

a. How high above ground is the window of Singh's hotel room?

b. How high above ground are the workers?

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Solution ▶

a. Begin by drawing a diagram of the situation (see figure). To find the height of the window we'll use the tangent ratio, since the adjacent side of the angle is known, and the opposite side is the height we desire.

For the height h_1:

$$\tan 50° = \frac{h_1}{50} \qquad \tan 50° = \frac{\text{opp}}{\text{adj}}$$

$$50 \tan 50° = h_1 \qquad \text{solve for } h_1$$

$$59.6 \approx h_1 \qquad \text{result } (\tan 50° \approx 1.1918)$$

The window is approximately 59.6 ft above ground.

b. For the height h_2:

$$\tan 80° = \frac{h_2}{50} \qquad \tan 80° = \frac{\text{opp}}{\text{adj}}$$

$$50 \tan 80° = h_2 \qquad \text{solve for } h_2$$

$$283.6 \approx h_2 \qquad \text{result } (\tan 80° \approx 5.6713)$$

The workers are approximately $283.6 + 59.6 = 343.2$ ft above ground.

Now try Exercises 77 through 80 ▶

☑ **E.** You've just seen how we can solve applications of right triangles

There are a number of additional interesting applications in the Exercise Set.

6.6 EXERCISES

▶ CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section if needed.

1. The phrase, "an angle whose tangent is known," is written notationally as _____.

2. Given $\sin\theta = \frac{7}{24}$, $\csc\theta =$ _____ because they are _____.

3. The sine of an angle is the ratio of the _____ side to the _____.

4. The cosine of an angle is the ratio of the _____ side to the _____.

5. Discuss/Explain exactly what is meant when you are asked to "solve a triangle." Include an illustrative example.

6. Given an acute angle and the length of the adjacent leg, which four (of the six) trig functions could be used to begin solving the triangle?

▶ DEVELOPING YOUR SKILLS

Use the function value given to determine the values of the other five trig functions of the acute angle θ. Answer in exact form (a diagram will help).

7. $\cos\theta = \dfrac{5}{13}$

8. $\sin\theta = \dfrac{20}{29}$

9. $\tan\theta = \dfrac{84}{13}$

10. $\sec\theta = \dfrac{53}{45}$

11. $\cot\theta = \dfrac{2}{11}$

12. $\cos\theta = \dfrac{2}{3}$

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Solve each triangle using trig functions of an acute angle θ. Give a complete answer (in table form) using exact values.

13.

14.

15.

16.

Solve the triangles shown and write answers in table form. Round sides to the nearest 100th of a unit. Verify that angles sum to 180° and that the three sides satisfy (approximately) the Pythagorean theorem.

17.

18.

19.

20.

21.

22.

Use a calculator to find the value of each expression, rounded to four decimal places.

23. sin 27°
24. cos 72°
25. tan 40°
26. cot 57.3°
27. sec 40.9°
28. csc 39°
29. sin 65°
30. tan 84.1°

Use a calculator to find the acute angle whose corresponding ratio is given. Round to the nearest 10th of a degree. For Exercises 31 through 38, use Exercises 23 through 30 to answer.

31. $\sin A = 0.4540$
32. $\cos B = 0.3090$
33. $\tan \theta = 0.8391$
34. $\cot A = 0.6420$
35. $\sec B = 1.3230$
36. $\csc \beta = 1.5890$
37. $\sin A = 0.9063$
38. $\tan B = 9.6768$
39. $\tan \alpha = 0.9896$
40. $\cos \alpha = 0.7408$
41. $\sin \alpha = 0.3453$
42. $\tan \alpha = 3.1336$

Select an appropriate function to find the angle indicated (round to 10ths of a degree).

43.

44.

45.

46.

47.

48.

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Draw a right triangle *ABC* as shown, using the information given. Then select an appropriate ratio to find the side indicated. Round to the nearest 100th.

Exercises 49 to 54

49. $\angle A = 25°$
$c = 52$ mm
find side a

50. $\angle B = 55°$
$b = 31$ ft
find side c

51. $\angle A = 32°$
$a = 1.9$ mi
find side b

52. $\angle B = 29.6°$
$c = 9.5$ yd
find side a

53. $\angle A = 62.3°$
$b = 82.5$ furlongs
find side c

54. $\angle B = 12.5°$
$a = 32.8$ km
find side b

Use a calculator to evaluate each pair of functions and comment on what you notice.

55. sin 25°, cos 65°

56. sin 57°, cos 33°

57. tan 5°, cot 85°

58. sec 40°, csc 50°

Based on your observations in Exercises 55 to 58, fill in the blank so that the functions given are equal.

59. sin 47°, cos ___

60. cos ___, sin 12°

61. cot 69°, tan ___

62. csc 17°, sec ___

Complete the following tables without referring to the text or using a calculator.

63.

$\theta = 30°$	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\sin(90° - \theta)$
$\cos(90° - \theta)$	$\tan(90° - \theta)$	$\csc\theta$	$\sec\theta$	$\cot\theta$

64.

$\theta = 45°$	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\sin(90° - \theta)$
$\cos(90° - \theta)$	$\tan(90° - \theta)$	$\csc\theta$	$\sec\theta$	$\cot\theta$

Evaluate the following expressions without a calculator, using the cofunction relationship and the following exact forms: $\sec 75° = \sqrt{6} + \sqrt{2}$; $\tan 75° = 2 + \sqrt{3}$.

65. $\sqrt{6}\csc 15°$

66. $\csc^2 15°$

67. $\cot^2 15°$

68. $\sqrt{3}\cot 15°$

▶ WORKING WITH FORMULAS

69. The sine of an angle between two sides of a triangle: $\sin\theta = \dfrac{2\mathcal{A}}{ab}$

If the area $\mathcal{A}$ and two sides a and b of a triangle are known, the sine of the angle between the two sides is given by the formula shown. Find the angle θ for the triangle shown given $\mathcal{A} \approx 38.9$, and use it to solve the triangle. (*Hint:* Apply the same concept to angle β or γ.)

70. Illumination of a surface: $E = \dfrac{I\cos\theta}{d^2}$

The illumination E of a surface by a light source is a measure of the luminous flux per unit area that reaches the surface. The value of E [in lumens (lm) per square foot] is given by the formula shown, where d is the distance from the light source (in feet), I is the intensity of the light [in candelas (cd)], and θ is the angle the light source makes with the vertical. For reading a book, an illumination E of at least 18 lm/ft^2 is recommended.

Assuming the open book is lying on a horizontal surface, how far away should a light source be placed if it has an intensity of 90 cd (about 75 W) and the light flux makes an angle of 65° with the book's surface (i.e., $\theta = 25°$)?

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▶ APPLICATIONS

71. Angle of elevation: For a person standing 100 m from the center of the base of the Eiffel Tower, the angle of elevation to the top of the tower is 71.6°. How tall is the Eiffel Tower?

72. Angle of elevation: In 2001, the tallest building in the world was the Petronas Tower I in Kuala Lumpur, Malaysia. For a person standing 25.9 ft from the base of the tower, the angle of elevation to the top of the tower is 89°. How tall is the Petronas tower?

73. Angle of depression: A person standing near the top of the Eiffel Tower notices a car wreck some distance from the tower. If the angle of depression from the person's eyes to the wreck is 32°, how far away is the accident from the base of the tower? See Exercise 71.

74. Angle of depression: A person standing on the top of the Petronas Tower I looks out across the city and pinpoints her residence. If the angle of depression from the person's eyes to her home is 5°, how far away (in feet and in miles) is the residence from the base of the tower? See Exercise 72.

75. Angles of rotation: From a point 110 ft due south of the eastern end of a planned east/west bridge spanning the Illinois river, a surveyor notes the western end lies on a bearing of N 38°35′15″ W. To the nearest inch, how long will the bridge be?

76. Angles of rotation: A large sign spans an east/west highway. From a point 35 m due west of the southern base of the sign, a surveyor finds the northern base lies on a bearing of N 67°11′42″ E. To the nearest centimeter, how wide is the sign?

77. Height of a climber: A local Outdoors Club has just hiked to the south rim of a large canyon, when they spot a climber attempting to scale the taller northern face. Knowing the distance between the sheer walls of the northern and southern faces of the canyon is approximately 175 yd, they attempt to compute the distance remaining for the climbers to reach the top of the northern rim. Using a homemade transit, they sight an angle of depression of 55° to the bottom of the north face, and angles of elevation of 24° and 30° to the climbers and top of the northern rim respectively. (a) How high is the southern rim of the canyon? (b) How high is the northern rim? (c) How much farther until the climber reaches the top?

78. Observing wildlife: From her elevated observation post 300 ft away, a naturalist spots a troop of baboons high up in a tree. Using the small transit attached to her telescope, she finds the angle of depression to the bottom of this tree is 14°, while the angle of elevation to the top of the tree is 25°. The angle of elevation to the troop of baboons is 21°. Use this information to find (a) the height of the observation post, (b) the height of the baboons' tree, and (c) the height of the baboons above ground.

79. Angle of elevation: The tallest free-standing tower in the world is the CNN Tower in Toronto, Canada. The tower includes a rotating restaurant high above the ground. From a distance of 500 ft the angle of elevation to the pinnacle of the tower is 74.6°. The angle of elevation to the restaurant from the same vantage point is 66.5°. How tall is the CNN Tower? How far below the pinnacle of the tower is the restaurant located?

80. Angle of elevation: In January 2009, Burj Dubai unofficially captured the record as the world's tallest building, according to the Council on Tall Buildings and Urban Habitat (*Source:* www.ctbuh.org). Measured at a point 159 m from its base, the angle of elevation to the top of the spire is 79°. From a

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distance of about 134 m, the angle of elevation to the top of the roof is also 79°. How tall is Burj Dubai from street level to the top of the spire? How tall is the spire itself?

81. Crop duster's speed: While standing near the edge of a farmer's field, Johnny watches a crop duster dust the farmer's field for insect control. Curious as to the plane's speed during each drop, Johnny attempts an estimate using the angle of rotation from one end of the field to the other, while standing 50 ft from one corner. Using a stopwatch he finds the plane makes each pass in 2.35 sec. If the angle of rotation was 83°, how fast (in miles per hour) is the plane flying as it applies the insecticide?

82. Train speed: While driving to their next gig, Josh and the boys get stuck in a line of cars at a railroad crossing as the gates go down. As the sleek, speedy express train approaches, Josh decides to pass the time estimating its speed. He spots a large oak tree beside the track some distance away, and figures the angle of rotation from the crossing to the tree is about 80°. If their car is 60 ft from the crossing and it takes the train 3 sec to reach the tree, how fast is the train moving in miles per hour?

Alternating current: In AC (alternating current) applications, the relationship between measures known as the impedance (Z), resistance (R), and the phase angle (θ) can be demonstrated using a right triangle. Both the resistance and the impedance are measured in ohms (Ω).

83. Find the impedance Z if the phase angle θ is 34°, and the resistance R is 320 Ω.

84. Find the phase angle θ if the impedance Z is 420 Ω, and the resistance R is 290 Ω.

85. Contour maps: In the figure shown, the *contour interval* is 175 m (each concentric line represents an increase of 175 m in elevation), and the scale of horizontal distances is 1 cm = 500 m. (a) Find the vertical change from A to B (the increase in elevation); (b) use a proportion to find the horizontal change between points A and B if the measured distance on the map is 2.4 cm; and (c) draw the corresponding right triangle and use it to estimate the length of the trail up the mountain side that connects A and B, then use trig to compute the approximate angle of incline as the hiker climbs from point A to point B.

86. Contour maps: In the figure shown, the *contour interval* is 150 m (each concentric line represents an increase of 150 m in elevation), and the scale of horizontal distances is 1 cm = 250 m. (a) Find the vertical change from A to B (the increase in elevation); (b) use a proportion to find the horizontal change between points A and B if the measured distance on the map is 4.5 cm; and (c) draw the corresponding right triangle and use it to estimate the length of the trail up the mountain side that connects A and B, then use trig to compute the approximate angle of incline as the hiker climbs from point A to point B.

87. Height of a rainbow: While visiting the Lapahoehoe Memorial on the island of Hawaii, Bruce and Carma see a spectacularly vivid rainbow arching over the bay. Bruce speculates the rainbow is 500 ft away, while Carma estimates the angle of elevation to the highest point of the rainbow is about 42°. What was the approximate height of the rainbow?

88. High-wire walking: As part of a circus act, a high-wire walker not only "walks the wire," she walks a wire that is *set at an incline of* 10° to the horizontal! If the length of the (inclined) wire is 25.39 m, (a) how much higher is the wire set at the destination pole than at the departure pole? (b) How far apart are the poles?

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89. Diagonal of a cube: A cubical box has a diagonal measure of 35 cm. (a) Find the dimensions of the box and (b) the angle θ that the diagonal makes at the lower corner of the box.

90. Diagonal of a rectangular parallelepiped: A rectangular box has a width of 50 cm and a length of 70 cm. (a) Find the height h that ensures the diagonal across the middle of the box will be 90 cm and (b) the angle θ that the diagonal makes at the lower corner of the box.

▶ EXTENDING THE CONCEPT

91. One of the challenges facing any terrestrial exploration of Mars is its weak and erratic magnetic field. An unmanned rover begins a critical mission at landing site **A** where magnetic north lies on a bearing of N 30° W of true (polar) north (see figure). The rover departs in the magnetic direction of S 20° E and travels straight for 1.4 km to site **B**. At this point, magnetic north lies on a bearing of N 45° E of true north. The rover turns to magnetic direction of N 5° W and travels 2 km straight to site **C**. At **C**, magnetic north lies on a bearing of N 10° W of true north. In what magnetic direction should the rover head to return to site **A**?

Exercise 91

92. As of the year 2009, the Bailong elevator outside of Zhangjiajie, China, was the highest exterior glass elevator in the world. While Christine was descending in the red car, she noticed Simon ascending in the yellow car, below her at an angle of depression of 50°. Seven seconds later, Simon was above her at an angle of elevation of 50°. If the cars have the same velocity ascending and descending, and the horizontal distance between the two cars is 47 ft, how fast do the cars travel?

Exercise 92

93. The radius of the Earth at the equator (0° N latitude) is approximately 3960 mi. Beijing, China, is located at 39.5° N latitude, 116° E longitude. Philadelphia, Pennsylvania, is located at the same latitude, but at 75° W longitude. (a) Use the diagram given and a cofunction relationship to find the radius r of the Earth (parallel to the equator) at this latitude; (b) use the arc length formula to compute the *shortest distance* between these two cities along this latitude; and (c) if the supersonic Concorde flew a direct flight between Beijing and Philadelphia along this latitude, approximate the flight time assuming a cruising speed of 1250 mph. Note: The shortest distance is actually traversed by heading northward, using the arc of a "great circle" that goes through these two cities.

Exercise 93

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▶ MAINTAINING YOUR SKILLS

94. **(3.2)** Solve by factoring:

a. $g^2 - 9g = 0$

b. $g^2 - 9 = 0$

c. $g^2 - 9g - 10 = 0$

d. $g^2 + 9g - 10 = 0$

e. $g^3 - 9g^2 - 10g + 90 = 0$

95. **(2.1)** For the graph of $T(x)$ given, (a) name the local maximums and minimums, (b) the zeroes of T, (c) intervals where $T(x)\downarrow$ and $T(x)\uparrow$, and (d) intervals where $T(x) > 0$ and $T(x) < 0$.

96. **(6.1)** The armature for the rear windshield wiper has a length of 24 in., with a rubber wiper blade that is 20 in. long. What area of my rear windshield is cleaned as the armature swings back-and-forth through an angle of 110°?

97. **(6.1)** The boxes used to ship some washing machines are perfect cubes with edges measuring 38 in. Use a special triangle to find the length of the diagonal d of one side, and the length of the interior diagonal D (through the middle of the box).

6.7 Trigonometry and the Coordinate Plane

LEARNING OBJECTIVES

In Section 6.7 you will see how we can:

- ❑ **A.** Define the trigonometric functions using the coordinates of a point in QI
- ❑ **B.** Use reference angles to evaluate the trig functions for any angle
- ❑ **C.** Solve applications using the trig functions of any angle

This section tends to bridge the study of *static trigonometry* and the angles of a right triangle, with the study of *dynamic trigonometry* and the unit circle. This is accomplished by noting that the domain of the trig functions (unlike a triangle point of view) *need not be restricted to acute angles*. We'll soon see that the domain can be extended to include trig functions of *any* angle, a view that greatly facilitates our work in Chapter 8, where many applications involve angles greater than 90°.

A. Trigonometric Ratios and the Point $P(x, y)$

Figure 6.126

Regardless of where a right triangle is situated or how it is oriented, each trig function can be defined as a given ratio of sides with respect to a given angle. In this light, consider a 30-60-90 triangle placed in the first quadrant with the 30° angle at the origin and the longer side along the x-axis. From our previous review of similar triangles, the trig ratios will have the same value regardless of the triangle's size so for convenience, we'll use a hypotenuse of 10 giving sides of 5, $5\sqrt{3}$, and 10. From the diagram in Figure 6.126 we note the point (x, y) marking the vertex of the 60° angle has coordinates $(5\sqrt{3}, 5)$. Further, the diagram shows that sin 30°, cos 30°, and tan 30° can all be expressed in terms of triangle side lengths *or these coordinates* since $\frac{\text{opp}}{\text{hyp}} = \frac{5}{10} = \frac{y}{r}$ (sine), $\frac{\text{adj}}{\text{hyp}} = \frac{5\sqrt{3}}{10} = \frac{x}{r}$ (cosine), and $\frac{\text{opp}}{\text{adj}} = \frac{5}{5\sqrt{3}} = \frac{y}{x}$ (tangent), where r is the distance from (x, y) to the origin. Each result reduces to the more familiar values seen earlier: $\sin 30° = \frac{1}{2}$, $\cos 30° = \frac{\sqrt{3}}{2}$,

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and $\tan 30° = \dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$. This suggests we can define the six trig functions in terms of x, y, and r, where $r = \sqrt{x^2 + y^2}$.

Consider that the slope of the line coincident with the hypotenuse is $\dfrac{\text{rise}}{\text{run}} = \dfrac{5}{5\sqrt{3}} = \dfrac{\sqrt{3}}{3}$, and since the line goes through the origin its equation must be $y = \dfrac{\sqrt{3}}{3}x$. Any point (x, y) on this line will be at the 60° vertex of a right triangle formed by drawing a perpendicular line from the point (x, y) to the x-axis. As Example 1 shows, we obtain the special values for sin 30°, cos 30°, and tan 30° regardless of the point chosen.

EXAMPLE 1 ▶ **Evaluating Trig Functions Using x, y, and r**

Pick an arbitrary point in QI that satisfies $y = \dfrac{\sqrt{3}}{3}x$, then draw the corresponding right triangle and evaluate sin 30°, cos 30°, and tan 30°.

Solution ▶ The coefficient of x has a denominator of 3, so we choose a multiple of 3 for convenience. For $x = 6$ we have $y = \dfrac{\sqrt{3}}{3}(6) = 2\sqrt{3}$. As seen in the figure, the point $(6, 2\sqrt{3})$ is on the line and at the vertex of the 60° angle. Using the triangle and evaluating the trig functions at 30°, we obtain:

$$\sin 30° = \frac{y}{r} = \frac{2\sqrt{3}}{4\sqrt{3}} = \frac{1}{2} \qquad \cos 30° = \frac{x}{r} = \frac{6}{4\sqrt{3}} = \frac{6\sqrt{3}}{4\sqrt{3}\sqrt{3}} = \frac{\sqrt{3}}{2} \qquad \tan 30° = \frac{y}{x} = \frac{2\sqrt{3}}{6} = \frac{\sqrt{3}}{3}$$

Now try Exercises 7 and 8 ▶

In general, consider *any* two points (x_1, y_1) and (x_2, y_2) on an arbitrary line $y = kx$, at corresponding distances r_1 and r_2 from the origin (Figure 6.127). Because the triangles formed are similar, we have $\dfrac{y_1}{x_1} = \dfrac{y_2}{x_2}$, $\dfrac{x_1}{r_1} = \dfrac{x_2}{r_2}$, and so on, and we conclude that the values of the trig functions are indeed independent of the point chosen.

Viewing the trig functions in terms of x, y, and r produces significant results. In Figure 6.128, we note the line $y = \dfrac{\sqrt{3}}{3}x$ from Example 1 also extends into QIII, and

Figure 6.127

Figure 6.128

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creates another 30° *angle whose vertex is at the origin* (since vertical angles are equal). The sine, cosine, and tangent functions can still be evaluated for this angle, but in QIII both x and y are negative. If we consider the angle in QIII to be a positive rotation of 210° (180° + 30°), we can evaluate the trig functions using the values of x, y, and r from any point on the terminal side, since these are fixed by the 30° angle created and are the same as those in QI except for their sign:

$$\sin 210° = \frac{y}{r} = \frac{-2\sqrt{3}}{4\sqrt{3}} = -\frac{1}{2} \qquad \cos 210° = \frac{x}{r} = \frac{-6}{4\sqrt{3}} = -\frac{\sqrt{3}}{2} \qquad \tan 210° = \frac{y}{x} = \frac{-2\sqrt{3}}{-6} = \frac{\sqrt{3}}{3}$$

For *any* rotation θ and a point (x, y) on the terminal side, the distance r can be found using $r = \sqrt{x^2 + y^2}$ and the six trig functions likewise evaluated. Note that evaluating them correctly depends on the quadrant of the terminal side, since this will dictate the signs for x and y. Students are strongly encouraged to make these quadrant and sign observations the *first step* in any solution process. In summary, we have

Trigonometric Functions of Any Angle

Given $P(x, y)$ is any point on the terminal side of angle θ in standard position, with $r = \sqrt{x^2 + y^2}$ $(r > 0)$ the distance from the origin to (x, y). The six trigonometric functions of θ are

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x}, \quad x \neq 0$$

$$\csc\theta = \frac{r}{y}, \quad y \neq 0 \qquad \sec\theta = \frac{r}{x}, \quad x \neq 0 \qquad \cot\theta = \frac{x}{y}, \quad y \neq 0$$

EXAMPLE 2 ▶ **Evaluating Trig Functions Given the Terminal Side is on $y = mx$**

Given that $P(x, y)$ is a point on the terminal side of angle θ in standard position, find the values of $\sin\theta$ and $\cos\theta$, if

a. The terminal side is in QII and coincident with the line $y = -\frac{12}{5}x$,

b. The terminal side is in QIV and coincident with the line $y = -\frac{12}{5}x$.

Solution ▶

a. Select any convenient point in QII that satisfies this equation. We select $x = -5$ since x is negative in QII, which gives $y = 12$ and the point $(-5, 12)$. Solving for r gives $r = \sqrt{(-5)^2 + (12)^2} = 13$. The ratios are

$$\sin\theta = \frac{y}{r} = \frac{12}{13} \qquad \cos\theta = \frac{x}{r} = \frac{-5}{13}$$

b. In QIV we select $x = 10$ since x is positive in QIV, giving $y = -24$ and the point $(10, -24)$. Solving for r gives $r = \sqrt{(10)^2 + (-24)^2} = 26$. The ratios are

$$\sin\theta = \frac{y}{r} = \frac{-24}{26} = -\frac{12}{13} \qquad \cos\theta = \frac{x}{r} = \frac{10}{26} = \frac{5}{13}$$

Now try Exercises 9 through 12 ▶

In Example 2, note the ratios are the same in QII and QIV *except for their sign*. We will soon use this observation to great advantage.

EXAMPLE 3 ▶ **Evaluating Trig Functions Given a Point *P***

Find the values of the six trigonometric functions given $P(-5, 5)$ is on the terminal side of angle θ in standard position.

Solution ▶ For $P(-5, 5)$ we have $x < 0$ and $y > 0$ so the terminal side is in QII. Solving for r yields $r = \sqrt{(-5)^2 + (5)^2} = \sqrt{50} = 5\sqrt{2}$. For $x = -5$, $y = 5$, and $r = 5\sqrt{2}$, we obtain

$$\sin\theta = \frac{y}{r} = \frac{5}{5\sqrt{2}} = \frac{\sqrt{2}}{2} \qquad \cos\theta = \frac{x}{r} = \frac{-5}{5\sqrt{2}} = -\frac{\sqrt{2}}{2} \qquad \tan\theta = \frac{y}{x} = \frac{5}{-5} = -1$$

The remaining functions can be evaluated using reciprocals.

$$\csc\theta = \frac{2}{\sqrt{2}} = \sqrt{2} \qquad \sec\theta = -\frac{2}{\sqrt{2}} = -\sqrt{2} \qquad \cot\theta = -1$$

Note the connection between these results and the special values for $\theta = 45°$.

Now try Exercises 13 through 28 ▶

Graphing calculators offer a number of features that can assist this approach to a study of the trig functions. On many calculators, the keystrokes 2nd APPS **(ANGLE)** will bring up a menu with **options 1 through 4** which are basically used for angle conversions. Of interest to us here are options 5 and 6, which can be used to determine the radius r (option 5) or the angle (option 6) related to a given point (x, y). For $(5, -5)$ from Example 3, CLEAR the home screen and press 2nd APPS **(ANGLE) 5:R▶Pr(,** which will place the option on the home screen. This feature supplies the left parenthesis of the ordered pair, and you simply complete it: **5:R▶Pr(–5, 5).** As shown in Figure 6.129, the calculator returns 7.07…, the decimal approximation of $5\sqrt{2}$. To find the related angle, it is assumed that θ is in standard position and (x, y) is on the terminal side. Pressing 2nd APPS **(ANGLE) 6:R▶Pθ(** and completing the ordered pair as before shows the corresponding angle is 135° (Figure 6.129). Note this is a QII angle as expected, since $x < 0$ and $y > 0$, and we can check the results of Example 3 by evaluating the trig functions of 135° (see Figure 6.130). **See Exercises 29 and 30.**

Figure 6.129

```
R▸Pr(-5,5)
        7.071067812
5√(2)
        7.071067812
R▸Pθ(-5,5)
                135
```

Figure 6.130

```
sin(135)
       .7071067812
cos(135)
      -.7071067812
tan(135)
                 -1
```

Figure 6.131

Now that we've defined the trig functions in terms of ratios involving x, y, and r, the question arises as to their value at the quadrantal angles. For 90° and 270°, any point on the terminal side of the angle has an *x-value* of zero, meaning tan 90°, sec 90°, tan 270°, and sec 270° are all undefined since $x = 0$ is in the denominator. Similarly, at 180° and 360°, the y-value of any point on the terminal side is zero, so cot 180°, csc 180°, cot 360°, and csc 360° are likewise undefined (see Figure 6.131).

EXAMPLE 4 ▶ **Evaluating the Trig Functions for $\theta = 90°k$, k an Integer**

Evaluate the six trig functions for $\theta = 270°$.

Solution ▶ Here, θ is the quadrantal angle whose terminal side separates QIII and QIV. Since the evaluation is independent of the point chosen on this side, we choose $(0, -1)$ for convenience, giving $r = 1$. For $x = 0$, $y = -1$, and $r = 1$ we obtain

$$\sin 270° = \frac{-1}{1} = -1 \qquad \cos 270° = \frac{0}{-1} = 0 \qquad \tan 270° = \frac{-1}{0}\ (\textit{undefined})$$

The remaining ratios can be evaluated using reciprocals.

$$\csc 270° = -1 \qquad \sec 270° = \frac{-1}{0}\ (\textit{undefined}) \qquad \cot 270° = \frac{0}{-1} = 0$$

Now try Exercises 31 and 32 ▶

Results for the quadrantal angles are summarized in Table 6.12.

Table 6.12

θ	$\sin\theta = \frac{y}{r}$	$\cos\theta = \frac{x}{r}$	$\tan\theta = \frac{y}{x}$	$\csc\theta = \frac{r}{y}$	$\sec\theta = \frac{r}{x}$	$\cot\theta = \frac{x}{y}$
$0°/360° \to (1, 0)$	0	1	0	undefined	1	undefined
$90° \to (0, 1)$	1	0	undefined	1	undefined	0
$180° \to (-1, 0)$	0	-1	0	undefined	-1	undefined
$270° \to (0, -1)$	-1	0	undefined	-1	undefined	0

A. You've just seen how we can define the trigonometric functions using the coordinates of a point in QI

B. Reference Angles and the Trig Functions of Any Angle

Recall that for any angle θ in standard position, the acute angle θ_r formed by the terminal side and the nearest x-axis is called the **reference angle.** Several examples of this definition are illustrated in Figures 6.132 through 6.135 for $\theta > 0°$ (note that if $0° < \theta < 90°$, then $\theta_r = \theta$).

Figure 6.132

$90° < \theta < 180°$
$\theta_r = 180° - \theta$

Figure 6.133

$180° < \theta < 270°$
$\theta_r = \theta - 180°$

Figure 6.134

$270° < \theta < 360°$
$\theta_r = 360° - \theta$

Figure 6.135

$360° < \theta < 450°$
$\theta_r = \theta - 360°$

EXAMPLE 5 ▶ **Finding Reference Angles**

Determine the reference angle for

a. 315° **b.** 150° **c.** 239° **d.** 425°

Solution ▶ Begin by mentally visualizing each angle and the quadrant where it terminates.

a. 315° is a QIV angle:
$\theta_r = 360° - 315° = 45°$

b. 150° is a QII angle:
$\theta_r = 180° - 150° = 30°$

c. 239° is a QIII angle:
$\theta_r = 239° - 180° = 59°$

d. 425° is a QI angle:
$\theta_r = 425° - 360° = 65°$

Now try Exercises 33 through 44 ▶

The reference angles from Examples 5(a) and 5(b) were special angles, which means we automatically know the absolute values of the trig ratios using θ_r. The best way to remember the signs of the trig functions is to keep in mind that sine is associated with y, cosine with x, and tangent with both x and y (r is always positive). In addition, there are several mnemonic devices (memory tools) to assist you. One is to use the first letter of the function that is positive in each quadrant and create a catchy acronym. For instance **ASTC** → **A**ll **S**tudents **T**ake **C**lasses (see Figure 6.136). Note that a trig function and its reciprocal function will always have the same sign. **See Exercises 45 through 48.**

Figure 6.136

EXAMPLE 6 ▶ **Evaluating Trig Functions Using θ_r**

Use a reference angle to evaluate $\sin\theta$, $\cos\theta$, and $\tan\theta$ for $\theta = 315°$.

Solution ▶ The terminal side is in QIV where cosine is positive while sine and tangent are negative. With $\theta_r = 45°$, we have:

$$\begin{aligned} \sin 315° &= -\sin\theta_r && \text{QIV: } \sin\theta < 0 \\ &= -\sin 45° && \text{replace } \theta_r \text{ with } 45° \\ &= -\frac{\sqrt{2}}{2} && \sin 45° = \frac{\sqrt{2}}{2} \end{aligned}$$

Similarly, the values for cosine and tangent are

$$\begin{aligned} \cos 315° &= +\cos\theta_r \\ &= \cos 45° \\ &= \frac{\sqrt{2}}{2} \end{aligned} \qquad \begin{aligned} \tan 315° &= -\tan\theta_r \\ &= -\tan 45° \\ &= -1 \end{aligned}$$

Now try Exercises 49 through 56 ▶

EXAMPLE 7 ▶ **Finding Function Values Using a Quadrant and Sign Analysis**

Given $\sin\theta = \dfrac{5}{13}$ and $\cos\theta < 0$, find the value of the other ratios.

Solution ▶ Always begin with a quadrant and sign analysis: $\sin\theta$ is positive in QI and QII, while $\cos\theta$ is negative in QII and QIII. Both conditions are satisfied in QII only. For $r = 13$ and $y = 5$, the Pythagorean theorem shows $x = \pm\sqrt{13^2 - 5^2} = \pm\sqrt{144} = \pm 12$. With θ in QII, $x = -12$ and this gives $\cos\theta = \dfrac{-12}{13}$ and $\tan\theta = \dfrac{5}{-12}$. The reciprocal values are $\csc\theta = \dfrac{13}{5}$, $\sec\theta = \dfrac{13}{-12}$, and $\cot\theta = \dfrac{-12}{5}$.

Now try Exercises 57 through 64 ▶

In our everyday experience, there are many actions and activities where angles greater than or equal to 360° are applied. Some common instances are a professional basketball player who "does a three-sixty" (360°) while going to the hoop, a diver who completes a "two-and-a-half" (900°) off the high board, and a skater who executes a perfect triple axel ($3\frac{1}{2}$ turns or 1260°). As these examples suggest, angles greater than 360° must still terminate on a quadrantal axis, or in one of the four quadrants, allowing a reference angle to be found and the functions to be evaluated for any angle *regardless of size*. Figure 6.137 illustrates that $\alpha = 150°$, $\beta = -210°$, and $\theta = 510°$ are all coterminal, with *each having a reference angle of* 30°.

Figure 6.137

EXAMPLE 8 ▶ **Evaluating Trig Functions of Any Angle**

Evaluate $\sin 150°$, $\cos(-210°)$, and $\tan 510°$.

Solution ▶ The three angles are coterminal in QII where sine is positive while cosine and tangent are negative. With $\theta_r = 30°$, we have:

$$\begin{aligned}\sin 150° &= +\sin\theta_r & \cos(-210°) &= -\cos\theta_r & \tan 510° &= -\tan\theta_r\\ &= \sin 30° & &= -\cos 30° & &= -\tan 30°\\ &= \frac{1}{2} & &= -\frac{\sqrt{3}}{2} & &= -\frac{\sqrt{3}}{3}\end{aligned}$$

Now try Exercises 65 through 76 ▶

Since 360° is one full rotation, all angles $\theta + 360°k$ will be coterminal for any integer k. For angles with a very large magnitude, we can find the quadrant of the terminal side by adding or subtracting as many integer multiples of 360° as needed until $\theta < 360°$. For $\alpha = 1908°$, $\dfrac{1908°}{360°} = 5.3$ and $1908° - 360°(5) = 108°$. This angle is in QII with $\theta_r = 72°$. **See Exercises 77 through 92.**

☑ **B.** You've just seen how we can use reference angles to evaluate the trig functions for any angle

C. Applications of the Trig Functions of Any Angle

One of the most basic uses of coterminal angles is determining all values of θ that satisfy a stated relationship. For example, by now you are aware that if $\sin\theta = \frac{1}{2}$ (positive one-half), then $\theta = 30°$ or $\theta = 150°$ (see Figure 6.138). But this is also true for all angles coterminal with these two, and we would write the solutions as $\theta = 30° + 360°k$ and $\theta = 150° + 360°k$ for all integers k.

Figure 6.138

EXAMPLE 9 ▶ **Finding All Angles that Satisfy a Given Equation**

Find all angles satisfying the relationship given. Answer in degrees.

a. $\cos\theta = -\frac{\sqrt{2}}{2}$ **b.** $\tan\theta = -1.3764$

Solution ▶ **a.** Cosine is negative in QII and QIII. Recognizing $\cos 45° = \frac{\sqrt{2}}{2}$, we reason $\theta_r = 45°$ and two solutions are $\theta = 135°$ from QII (see Example 3) and $\theta = 225°$ from QIII. For all values of θ satisfying the relationship, we have $\theta = 135° + 360°k$ and $\theta = 225° + 360°k$. See Figure 6.139.

b. Tangent is negative in QII and QIV. For -1.3764 we find θ_r using a calculator: 2nd TAN **(tan^{-1})** 1.3764 ENTER shows $\tan^{-1}(1.3764) \approx 54$, so $\theta_r = 54°$. Two solutions are $\theta = 180° - 54° = 126°$ from QII, and in QIV $\theta = 360° - 54° = 306°$. The result is $\theta = 126° + 360°k$ and $\theta = 306° + 360°k$. Note these can be combined into the single statement $\theta = 126° + 180°k$. See Figure 6.140.

WORTHY OF NOTE

Since reference angles are acute, we find them by using the $\sin^{-1}$, $\cos^{-1}$, or $\tan^{-1}$ keys with positive values.

Figure 6.139

Figure 6.140

Now try Exercises 95 through 102 ▶

We close this section with an additional application of the concepts related to trigonometric functions of any angle.

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EXAMPLE 10 ▶ **Applications of Coterminal Angles: Location on Radar**

A radar operator calls the captain over to her screen saying, "Sir, we have an unidentified aircraft at bearing N 20° E (a standard 70° rotation). I think it's a UFO." The captain asks, "What makes you think so?" To which the sailor replies, "Because it's at 5000 ft and not moving!" Name all angles for which the UFO causes a "blip" to occur on the radar screen.

Solution ▶ Since radar typically sweeps out a 360° angle, a blip will occur on the screen for all angles $\theta = 70° + 360°k$, where k is an integer.

☑ **C.** You've just seen how we can solve applications using the trig functions of any angle

Now try Exercises 103 through 108 ▶

6.7 EXERCISES

▶ CONCEPTS SQAND VOCABULAR Y

Fill in each blank with the appropriate word or phrase. Carefully reread the section if needed.

1. An angle is in standard position if its vertex is at the _____ and the initial side is along the _____.

2. A(n) _____ angle is one where the _____ side is coincident with one of the coordinate axes.

3. Angles formed by a counterclockwise rotation are _____ angles. Angles formed by a _____ rotation are negative angles.

4. For any angle θ, its reference angle θ_r is the positive _____ angle formed by the _____ side and the nearest x-axis.

5. Discuss the similarities and differences between the trigonometry of right triangles and the trigonometry of *any* angle.

6. Let $T(x)$ represent any one of the six basic trig functions. Explain why the equation $T(x) = k$ will always have exactly two solutions in $[0, 2\pi)$ if x is not a quadrantal angle.

▶ DEVELOPING YOUR SKILLS

7. Draw a 30-60-90 triangle with the 60° angle at the origin and the short side along the positive x-axis. Determine the slope and equation of the line coincident with the hypotenuse, then pick any point on this line and evaluate sin 60°, cos 60°, and tan 60°. Comment on what you notice.

8. Draw a 45-45-90 triangle with a 45° angle at the origin and one side along the positive x-axis. Determine the slope and equation of the line coincident with the hypotenuse, then pick any point on this line and evaluate sin 45°, cos 45°, and tan 45°. Comment on what you notice.

Graph each linear equation and state the quadrants it traverses. Then pick one point on the line from each quadrant and evaluate the functions sin θ, cos θ and tan θ using these points.

9. $y = \frac{3}{4}x$

10. $y = \frac{5}{12}x$

11. $y = -\frac{\sqrt{3}}{3}x$

12. $y = -\frac{\sqrt{3}}{2}x$

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Find the values of the six trigonometric functions given $P(x, y)$ is on the terminal side of angle θ, with θ in standard position.

13. (8, 15) **14.** (7, 24)

15. (−20, 21) **16.** (−3, −1)

17. (7.5, −7.5) **18.** (9, −9)

19. $(4\sqrt{3}, 4)$ **20.** $(-6, 6\sqrt{3})$

21. (2, 8) **22.** (6, −15)

23. (−3.75, −2.5) **24.** (6.75, 9)

25. $\left(-\frac{5}{9}, \frac{2}{3}\right)$ **26.** $\left(\frac{3}{4}, -\frac{7}{16}\right)$

27. $\left(\frac{1}{4}, -\frac{\sqrt{5}}{2}\right)$ **28.** $\left(-\frac{\sqrt{3}}{5}, \frac{22}{25}\right)$

29. Use the **R▶Pr(** feature of a graphing calculator to find the radius corresponding to the point $(-5, 5\sqrt{3})$.

30. Use the **R▶Pθ(** feature of a graphing calculator to find the angle corresponding to (−28, −45), then evaluate $\sin\theta$, $\cos\theta$, and $\tan\theta$ for this angle. Compare each result to the values given by $\sin\theta = \frac{y}{r}$, $\cos\theta = \frac{x}{r}$, and $\tan\theta = \frac{y}{x}$.

31. Evaluate the six trig functions in terms of x, y, and r for $\theta = 90°$.

32. Evaluate the six trig functions in terms of x, y, and r for $\theta = 180°$.

Name the reference angle θ_r for the angle θ given.

33. $\theta = 120°$ **34.** $\theta = 210°$

35. $\theta = 135°$ **36.** $\theta = 315°$

37. $\theta = -45°$ **38.** $\theta = -240°$

39. $\theta = 112°$ **40.** $\theta = 179°$

41. $\theta = 500°$ **42.** $\theta = 750°$

43. $\theta = -168.4°$ **44.** $\theta = -328.2°$

State the quadrant of the terminal side of θ, using the information given.

45. $\sin\theta > 0$, $\cos\theta < 0$

46. $\cos\theta < 0$, $\tan\theta < 0$

47. $\tan\theta < 0$, $\sin\theta > 0$

48. $\sec\theta > 0$, $\tan\theta > 0$

Find the exact values of $\sin\theta$, $\cos\theta$, and $\tan\theta$ using reference angles.

49. $\theta = 330°$ **50.** $\theta = 390°$

51. $\theta = -45°$ **52.** $\theta = -120°$

53. $\theta = 240°$ **54.** $\theta = 315°$

55. $\theta = -150°$ **56.** $\theta = -210°$

For the information given, find the values of x, y, and r. Clearly indicate the quadrant of the terminal side of θ, then state the values of the six trig functions of θ.

57. $\cos\theta = \frac{4}{5}$ and $\sin\theta < 0$

58. $\tan\theta = -\frac{12}{5}$ and $\cos\theta > 0$

59. $\csc\theta = -\frac{37}{35}$ and $\tan\theta > 0$

60. $\sin\theta = -\frac{20}{29}$ and $\cot\theta < 0$

61. $\csc\theta = 3$ and $\cos\theta > 0$

62. $\csc\theta = -2$ and $\cos\theta > 0$

63. $\sin\theta = -\frac{7}{8}$ and $\sec\theta < 0$

64. $\cos\theta = \frac{5}{12}$ and $\sin\theta < 0$

Find two positive and two negative angles that are coterminal with the angle given. Answers will vary.

65. 52° **66.** 12°

67. 87.5° **68.** 22.8°

69. 225° **70.** 175°

71. −107° **72.** −215°

Evaluate in exact form as indicated.

73. $\sin 120°$, $\cos -240°$, $\tan 480°$

74. $\sin 225°$, $\cos 585°$, $\tan -495°$

75. $\sin -30°$, $\cos -390°$, $\tan -690°$

76. $\sin 210°$, $\cos 570°$, $\tan -150°$

Find the exact values of $\sin\theta$, $\cos\theta$, and $\tan\theta$ using reference angles.

77. $\theta = 600°$ **78.** $\theta = 480°$

79. $\theta = -840°$ **80.** $\theta = -930°$

81. $\theta = 570°$ **82.** $\theta = 495°$

83. $\theta = -1230°$ **84.** $\theta = 3270°$

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For each exercise, state the quadrant of the terminal side and the sign of the function in that quadrant. Then evaluate the expression using a calculator. Round to four decimal places.

85. sin 719°
86. cos 528°
87. tan −419°
88. sec −621°
89. csc 681°
90. tan 995°
91. cos 805°
92. sin 772°

▶ WORKING WITH FORMULAS

93. The area of a parallelogram: $\mathcal{A} = ab \sin \theta$

The area of a parallelogram is given by the formula shown, where a and b are the lengths of the sides and θ is the angle between them. Use the formula to complete the following: (a) find the area of a parallelogram with sides $a = 9$ and $b = 21$ given $\theta = 50°$. (b) What is the smallest integer value of θ where the area is greater than 150 units^2? (c) State what happens when $\theta = 90°$. (d) How can you find the area of a triangle using this formula?

94. The angle between two intersecting lines:
$$\tan \theta = \frac{m_2 - m_1}{1 + m_2 m_1}$$

Given line 1 and line 2 with slopes m_1 and m_2, respectively, the angle between the two lines is given by the formula shown. Find the angle θ if the equation of line 1 is $y_1 = \frac{3}{4}x + 2$ and line 2 has equation $y_2 = -\frac{2}{3}x + 5$.

▶ APPLICATIONS

Find all angles satisfying the stated relationship. For standard angles, express your answer in exact form. For nonstandard values, use a calculator and round function values to tenths.

95. $\cos \theta = \frac{1}{2}$
96. $\sin \theta = \frac{\sqrt{2}}{2}$
97. $\sin \theta = -\frac{\sqrt{3}}{2}$
98. $\tan \theta = -\frac{\sqrt{3}}{1}$
99. $\sin \theta = 0.8754$
100. $\cos \theta = 0.2378$
101. $\tan \theta = -2.3512$
102. $\cos \theta = -0.0562$

103. Nonacute angles: At a recent carnival, one of the games on the midway was played using a large spinner that turns clockwise. On Jorge's spin the number 25 began at the 12 o'clock (top/center) position, returned to this position five times during the spin and stopped at the 3 o'clock position. What angle θ did the spinner spin through? Name all angles that are coterminal with θ.

Exercise 103

104. Nonacute angles: One of the four blades on a ceiling fan has a decal on it and begins at a designated "12 o'clock" position. Turning the switch on and then immediately off, causes the blade to make over three complete, counterclockwise rotations, with the blade stopping at the 8 o'clock position. What angle θ did the blade turn through? Name all angles that are coterminal with θ.

105. High dives: As part of a diving competition, David executes a perfect reverse two-and-a-half flip. Does he enter the water feet first or head first? Through what angle did he turn from takeoff until the moment he entered the water?

Exercise 105

106. Gymnastics: While working out on a trampoline, Charlene does three complete, forward flips and then belly-flops on the trampoline before returning to the upright position. What angle did she turn through from the start of this maneuver to the moment she belly-flops?

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107. Spiral of Archimedes: The graph shown is called the spiral of Archimedes. Through what angle θ has the spiral turned, given the spiral terminates at $(6, -2)$ as indicated?

Exercise 107

108. Involute of a circle: The graph shown is called the involute of a circle. Through what angle θ has the involute turned, given the graph terminates at $(-4, -3.5)$ as indicated?

Exercise 108

▶ EXTENDING THE CONCEPT

109. In an elementary study of trigonometry, the hands of a clock are often studied because of the angle relationship that exists between the hands. For example, at 3 o'clock, the angle between the two hands is a right angle and measures 90°.

a. What is the angle between the two hands at 1 o'clock? 2 o'clock? Explain why.

b. What is the angle between the two hands at 6:30? 7:00? 7:30? Explain why.

c. Name four times at which the hands will form a 45° angle.

110. In the diagram shown, the indicated ray is of arbitrary length. (a) Through what additional angle α would the ray have to be rotated to create triangle ABC? (b) What will be the length of side AC once the triangle is complete?

Exercise 110

111. Referring to Exercise 104, suppose the fan blade has a radius of 20 in. and is turning at a rate of 12 revolutions per second. (a) Find the angle the blade turns through in 3 sec. (b) Find the circumference of the circle traced out by the tip of the blade. (c) Find the total distance traveled by the blade tip in 10 sec. (d) Find the speed, in miles per hour, that the tip of the blade is traveling.

▶ MAINTAINING YOUR SKILLS

112. **(6.1)** For emissions testing, automobiles are held stationary while a heavy roller installed in the floor allows the wheels to turn freely. If the large wheels of a customized pickup have a radius of 18 in. and are turning at 300 revolutions per minute, what speed is the odometer of the truck reading in miles per hour?

113. **(5.4)** Solve for t. Answer in both exact and approximate form:

$$-250 = -150e^{-0.05t} - 202.$$

114. **(6.6)** Jazon is standing 117 ft from the base of the Washington Monument in Washington, D.C. If his eyes are 5 ft above level ground and he must hold his head at a 78° angle from horizontal to see the top of the monument (the angle of elevation is 78°), estimate the height of the monument. Answer to the nearest tenth of a foot.

115. **(1.4)** Find an equation of the line perpendicular to $4x - 5y = 15$ that contains the point $(4, -3)$.

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6.8 Trigonometric Equation Models

LEARNING OBJECTIVES

In Section 6.8 you will see how we can:

- ❑ **A.** Create a trigonometric model from critical points or data
- ❑ **B.** Create a sinusoidal model from data using regression

In the most common use of the word, a cycle is any series of events or operations that occur in a predictable pattern and return to a starting point. This includes things as diverse as the wash cycle on a washing machine and the powers of i. There are a number of common events that occur in *sinusoidal* cycles, or events that can be modeled by a sine wave. As in Section 6.5, these include monthly average temperatures, monthly average daylight hours, and harmonic motion, among many others. Less well-known applications include alternating current, biorhythm theory, and animal populations that fluctuate over a known period of years. In this section, we develop two methods for creating a sinusoidal model. The first uses information about the critical points (where the cycle reaches its maximum or minimum values); the second involves computing the equation of best fit (a regression equation) for a set of data.

A. Critical Points and Sinusoidal Models

Although future courses will define them more precisely, we will consider **critical points** to be *inputs* where a function attains a minimum or maximum value. If an event or phenomenon is known to behave sinusoidally (regularly fluctuating between a maximum and minimum), we can create an acceptable model of the form $y = A\sin(Bx + C) + D$ given these **critical points** (x, y) and the period. For instance, many weather patterns have a period of 12 months. Using the formula $P = \frac{2\pi}{B}$, we find $B = \frac{2\pi}{P}$ and substituting 12 for P gives $B = \frac{\pi}{6}$ (always the case for phenomena with a 12-month cycle). The maximum value of $A\sin(Bx + C) + D$ will always occur when $\sin(Bx + C) = 1$, and the minimum at $\sin(Bx + C) = -1$, giving this system of equations: max value $M = A(1) + D$ and min value $m = A(-1) + D$. Solving the system for A and D gives $A = \frac{M - m}{2}$ and $D = \frac{M + m}{2}$ as before. To find C, assume the maximum and minimum values occur at (x_2, M) and (x_1, m), respectively. We can substitute the values computed for A, B, and D in $y = A\sin(Bx + C) + D$, along with either (x_2, M) or (x_1, m), and solve for C. Using the minimum value (x_1, m), where $x = x_1$ and $y = m$, we have

$$y = A\sin(Bx + C) + D \quad \text{sinusoidal equation model}$$

$$m = A\sin(Bx_1 + C) + D \quad \text{substitute } m \text{ for } y \text{ and } x_1 \text{ for } x$$

$$\frac{m - D}{A} = \sin(Bx_1 + C) \quad \text{isolate sine function}$$

Fortunately, for sine models constructed from critical points we have $\frac{y - D}{A} \rightarrow \frac{m - D}{A}$, which is always equal to -1 **(see Exercise 33).** This gives a simple result for C, since $-1 = \sin(Bx_1 + C)$ leads to $\frac{3\pi}{2} = Bx_1 + C$ or $C = \frac{3\pi}{2} - Bx_1$. **See Exercises 7 through 12** for practice with these ideas.

EXAMPLE 1 ▶ **Developing a Model for Polar Ice Cap Extent from Critical Points**

When the Spirit and Odyssey Rovers landed on Mars (January 2004), there was a renewed public interest in studying the planet. Of particular interest were the polar ice caps, which are now thought to hold frozen water, especially the northern cap. The Martian ice caps expand and contract with the seasons, just as they do here on Earth but there are about 687 days in a Martian year, making each Martian "month" just over 57 days long (1 Martian day ≈ 1 Earth day). At its smallest size, the northern ice cap covers an area of roughly 0.17 million square miles. At the height of winter, the cap covers about 3.7 million square miles (an area about the size of the 50 United States). Suppose these occur at the beginning of month 4 ($x = 4$) and month 10 ($x = 10$) respectively.

a. Use this information to create a sinusoidal model of the form $f(x) = A\sin(Bx + C) + D$.

b. Use the model to predict the area of the ice cap in the eighth Martian month.

c. Use a graphing calculator to determine the number of months the cap covers less than 1 million mi^2.

Solution ▶ **a.** Assuming a "12-month" weather pattern, $P = 12$ and $B = \frac{\pi}{6}$. The maximum and minimum points are (10, 3.7) and (4, 0.17). Using this information, $D = \frac{3.7 + 0.17}{2} = 1.935$ and $A = \frac{3.7 - 0.17}{2} = 1.765$. Using $C = \frac{3\pi}{2} - Bx_1$ gives $C = \frac{3\pi}{2} - \frac{\pi}{6}(4) = \frac{5\pi}{6}$. The equation model is $f(x) = 1.765\sin\left(\frac{\pi}{6}x + \frac{5\pi}{6}\right) + 1.935$, where $f(x)$ represents millions of square miles in month x.

b. For the size of the cap in month 8 we evaluate the function at $x = 8$.

$$f(8) = 1.765\sin\left[\frac{\pi}{6}(8) + \frac{5\pi}{6}\right] + 1.935 \quad \text{substitute 8 for } x$$

$$= 2.8175 \quad \text{result}$$

In month 8, the polar ice cap will cover about 2,817,500 mi^2.

c. Of the many options available, we opt to solve by locating the points where $Y_1 = 1.765\sin\left(\frac{\pi}{6}X + \frac{5\pi}{6}\right) + 1.935$ and $Y_2 = 1$ intersect. After entering the functions on the Y= screen, we set $x \in [0, 12]$ and $y \in [-2, 5]$ for a window with a frame around the output values.

Press 2nd TRACE **(CALC) 5:intersect** to find the intersection points. To four decimal places they occur at $x = 2.0663$ and $x = 5.9337$. The ice cap at the northern pole of Mars has an area of less than 1 million mi^2 from early in the second month to late in the fifth month. The second intersection is shown in the figure.

Now try Exercises 19 and 20 ▶

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While this form of "equation building" can't match the accuracy of a regression model (computed from a larger set of data), it does lend insight as to how sinusoidal functions work. The equation will always contain the maximum and minimum values, and using the period of the phenomena, we can create a smooth sine wave that "fills in the blanks" between these critical points.

EXAMPLE 2 ▶ **Developing a Model of Wildlife Population from Critical Points**

Naturalists have found that many animal populations, such as the arctic lynx, some species of fox, and certain rabbit breeds, tend to fluctuate sinusoidally over 10-year periods. Suppose that an extended study of a lynx population began in 2000, and in the third year of the study, the population had fallen to a minimum of 2500. In the eighth year the population hit a maximum of 9500.

a. Use this information to create a sinusoidal model of the form $P(x) = A\sin(Bx + C) + D$.

b. Use the model to predict the lynx population in the year 2006.

c. Use a graphing calculator to determine the number of years the lynx population is above 8000 in a 10-year period.

Solution ▶

a. Since $P = 10$, we have $B = \frac{2\pi}{10} = \frac{\pi}{5}$. Using 2000 as year zero, the minimum and maximum populations occur at (3, 2500) and (8, 9500). From the information given, $D = \frac{9500 + 2500}{2} = 6000$, and $A = \frac{9500 - 2500}{2} = 3500$. Using the minimum value we have $C = \frac{3\pi}{2} - \frac{\pi}{5}(3) = \frac{9\pi}{10}$, giving an equation model of $P(x) = 3500\sin\left(\frac{\pi}{5}x + \frac{9\pi}{10}\right) + 6000$, where $P(x)$ represents the lynx population in year x.

b. For the population in 2006 we evaluate the function at $x = 6$.

$$P(x) = 3500\sin\left(\frac{\pi}{5}x + \frac{9\pi}{10}\right) + 6000 \quad \text{sinusoidal function model}$$

$$P(6) = 3500\sin\left[\frac{\pi}{5}(6) + \frac{9\pi}{10}\right] + 6000 \quad \text{substitute 6 for } x$$

$$\approx 7082 \quad \text{result}$$

In 2006, the lynx population was about 7082.

c. Using a graphing calculator and the functions $Y_1 = 3500\sin\left(\frac{\pi}{5}X + \frac{9\pi}{10}\right) + 6000$ and $Y_2 = 8000$, we attempt to find points of intersection. Enter the functions (press Y=) and set a viewing window (we used $x \in [0, 12]$ and $y \in [0, 10{,}000]$). Press 2nd TRACE (**CALC**) **5:intersect** to find where Y_1 and Y_2 intersect. To four decimal places this occurs at $x = 6.4681$ and $x = 9.5319$. The lynx population exceeded 8000 for roughly 3 years. The first intersection is shown.

Now try Exercises 21 and 22 ▶

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This type of equation building isn't limited to the sine function. In fact, there are many situations where a sine model cannot be applied. Consider the length of the shadows cast by a flagpole or radio tower as the Sun makes its way across the sky. The shadow's length follows a regular pattern (shortening then lengthening) and "always returns to a starting point," yet when the Sun is low in the sky the shadow becomes (theoretically) infinitely long, unlike the output values from a sine function. If we consider *signed* shadow lengths at a latitude where the Sun passes directly overhead (in contrast with Example 11 in Section 6.5), an equation involving $\tan x$ might provide a good model. We'll attempt to model the data using $y = A\tan(Bx \pm C)$, with the D-term absent since a vertical shift in this context has no meaning. Recall that the period of the tangent function is $P = \dfrac{\pi}{|B|}$ and that $\pm\dfrac{C}{B}$ gives the magnitude and direction of the horizontal shift, in a direction opposite the sign.

EXAMPLE 3 ▶ **Using Data to Develop a Function Model for Shadow Length**

WORTHY OF NOTE
A gnomon is the protruding feature of a sundial, casting the shadow used to estimate the time of day (see photo).

The data given tracks the length of a gnomon's shadow for the 12 daylight hours at a certain location near the equator (positive and negative values indicate lengths before noon and after noon respectively). Assume $t = 0$ represents 6:00 A.M.

Hour of the Day	Length (cm)	Hour of the Day	Length (cm)
0	∞	7	-2.1
1	29.9	8	-4.6
2	13.9	9	-8.0
3	8.0	10	-13.9
4	4.6	11	-29.9
5	2.1	12	$-\infty$
6	0		

a. Use the data to find an equation model of the form $L(t) = A\tan(Bt \pm C)$.
b. Graph the function and scatterplot.
c. Find the shadow's length at 4:30 P.M.
d. If the shadow is 6.1 cm long, what time in the morning is it?

Solution ▶ **a.** We begin by noting this phenomenon has a period of $P = 12$. Using the period formula for tangent we solve for B: $P = \dfrac{\pi}{B}$ gives $12 = \dfrac{\pi}{B}$, so $B = \dfrac{\pi}{12}$. Since we want (6, 0) to be the "center" of the function [instead of (0, 0)], we desire a horizontal shift 6 units to the right. Using the ratio $\dfrac{C}{B}\left(\text{with } B = \dfrac{\pi}{12}\right)$ gives $-6 = \dfrac{12C}{\pi}$ so $C = -\dfrac{\pi}{2}$. To find A we use the equation built so far: $L(t) = A\tan\left(\dfrac{\pi}{12}t - \dfrac{\pi}{2}\right)$, and *any data point* to solve for A. Using (3, 8) we obtain $8 = A\tan\left(\dfrac{\pi}{12}(3) - \dfrac{\pi}{2}\right)$:

$$8 = A\tan\left(-\frac{\pi}{4}\right) \quad \text{simplify}$$

$$-8 = A \quad \text{solve for } A\text{: } \tan\left(-\frac{\pi}{4}\right) = -1$$

The equation model is

$$L(t) = -8\tan\left(\frac{\pi}{12}t - \frac{\pi}{2}\right).$$

b. The scatterplot and graph are shown in Figure 6.141.

Figure 6.141

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c. 4:30 P.M. indicates $t = 10.5$. Evaluating $L(10.5)$ gives

$$L(t) = -8\tan\left(\frac{\pi}{12}t - \frac{\pi}{2}\right) \quad \text{function model}$$

$$L(10.5) = -8\tan\left[\frac{\pi}{12}(10.5) - \frac{\pi}{2}\right] \quad \text{substitute 10.5 for } t$$

$$= -8\tan\left(\frac{3\pi}{8}\right) \quad \text{simplify}$$

$$\approx -19.31 \quad \text{result}$$

At 4:30 P.M., the shadow has a length of $|-19.31| = 19.31$ cm.

d. Substituting 6.1 for $L(t)$ and solving for t graphically gives the graph shown in Figure 6.142, where we note the day is about 3.5 hr old—it is about 9:30 A.M.

Figure 6.142

A. You've just seen how we can create a trigonometric model from critical points or data

Now try Exercises 23 through 26 ▶

B. Data and Sinusoidal Regression

Most graphing calculators are programmed to handle numerous forms of polynomial and nonpolynomial regression, including **sinusoidal regression.** The sequence of steps used is the same regardless of the form chosen. **Exercises 13 through 16** offer further practice with regression fundamentals. Example 4 illustrates their use in context.

EXAMPLE 4 ▶ **Calculating a Regression Equation for Seasonal Temperatures**

The data shown give the record high temperature for selected months in Bismarck, North Dakota.

Month (Jan → 1)	Temp. (°F)	Month (Jan → 1)	Temp. (°F)
1	63	9	105
3	81	11	79
5	98	12	65
7	109		

a. Use the data to draw a scatterplot, then find a sinusoidal regression model and graph both on the same screen.

b. Use the equation model to estimate the record high temperatures for months 2, 6, and 8.

c. Determine what month gives the largest difference between the actual data and the computed results.

Source: NOAA Comparative Climate Data 2004.

Solution ▶ **a.** Entering the data and running the regression (in radian mode) results in the coefficients shown in Figure 6.143. After entering the equation in Y_1 and pressing ZOOM **9:Zoom Stat** we obtain the graph shown in Figure 6.144 (indicated window settings have been rounded).

Figure 6.143

Figure 6.144

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b. Using $x = 2$, $x = 6$, and $x = 8$ as inputs, the equation model projects record high temperatures of 68.5°, 108.0°, and 108.1°, respectively.

c. In the header of L3, use Y_1(L1) ENTER to evaluate the regression model using the inputs from L1, and place the results in L3. Entering L2 − L3 in the header of L4 gives the results shown in Figure 6.145 and we note the largest difference occurs in September—about 4°.

Figure 6.145

Now try Exercises 27 through 30 ▶

Weather patterns differ a great deal depending on the locality. For example, the annual precipitation received by Seattle, Washington, far exceeds that received by Cheyenne, Wyoming. Our final example compares the two amounts and notes an interesting fact about the relationship.

EXAMPLE 5 ▶ Calculating a Regression Model for Seasonal Precipitation

The average monthly precipitation (in inches) for Cheyenne, Wyoming, and Seattle, Washington, are shown in the table.

Month (Jan. → 1)	WY Precip.	WA Precip.
1	0.45	5.13
2	0.44	4.18
3	1.05	3.75
4	1.55	2.59
5	2.48	1.77
6	2.12	1.49
7	2.26	0.79
8	1.82	1.02
9	1.43	1.63
10	0.75	3.19
11	0.64	5.90
12	0.46	5.62

a. Use the data to find a sinusoidal regression model for the average monthly precipitation in each city. Enter or paste the equation for Cheyenne in Y_1 and the equation for Seattle in Y_2.

b. Graph both equations on the same screen (without the scatterplots) and use TRACE or 2nd TRACE **(CALC) 5:intersect** to help estimate the number of months Cheyenne receives more precipitation than Seattle.

Source: NOAA Comparative Climate Data 2004.

Solution ▶

a. Setting the calculator in **Float 0 1 2 3 4 5 6 7 8 9** MODE and running sinusoidal regressions gives the equations shown in Figure 6.146.

b. Both graphs are shown in Figure 6.147. Using the TRACE feature, we find the graphs intersect at approximately (4.7, 2.0) and (8.4, 1.7). Cheyenne receives more precipitation than Seattle for about 8.4 − 4.7 = 3.7 months of the year.

Figure 6.146

Figure 6.147

B. You've just seen how we can create a sinusoidal model from data using regression

Now try Exercises 31 and 32 ▶

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6.8 EXERCISES

▶ CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section if needed.

1. For $y = A\sin(Bx + C) + D$, the maximum value occurs when _____ $= 1$, leaving $y =$ _____.

2. For $y = A\sin(Bx + C) + D$, the minimum value occurs when _____ $= -1$, leaving $y =$ _____.

3. For $y = A\sin(Bx + C) + D$, the value of C can be found using $C = -Bx$, where x is a _____ point.

4. Any phenomenon with sinusoidal behavior regularly fluctuates between a _____ and a _____ value.

5. Discuss/Expand on Example 4. Why does the equation calculated from the maximum and minimum values differ from the regression equation, even though both use the same set of data?

6. Discuss/Explain: (1) How is data concerning average rainfall related to the average discharge rate of rivers? (2) How is data concerning average daily temperature related to water demand? Are both sinusoidal with the same period? Are there other associations you can think of?

▶ DEVELOPING YOUR SKILLS

Find a sinusoidal equation for the information as given.

7. minimum value at (9, 25); maximum value at (3, 75); period: 12 min

8. minimum value at (4.5, 35); maximum value at (1.5, 121); period: 6 yr

9. minimum value at (15, 3); maximum value at (3, 7.5); period: 24 hr

10. minimum value at (3, 3.6); maximum value at (7, 12); period: 8 hr

11. minimum value at (5, 279); maximum value at (11, 1285); period: 12 yr

12. minimum value at (6, 8280); maximum value at (22, 23,126); period: 32 yr

Data and sinusoidal regression models: For the following sets of data (a) find a sinusoidal regression equation using your calculator; (b) construct an equation manually using the period and maximum/minimum values; and (c) graph both on the same screen, then use a **TABLE** to find the largest difference between output values.

13.

Day of Month	Output
1	15
4	41
7	69
10	91
13	100
16	90
19	63
22	29
25	5
28	2
31	18

14.

Day of Month	Output
1	179
4	201
7	195
10	172
13	145
16	120
19	100
22	103
25	124
28	160
31	188

15.

Month (Jan. = 1)	Output
1	16
2	19
3	21
4	22
5	21
6	19
7	16
8	13
9	11
10	10
11	11
12	13

16.

Month (Jan. = 1)	Output
1	86
2	96
3	99
4	95
5	83
6	72
7	56
8	48
9	43
10	49
11	58
12	73

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▶ WORKING WITH FORMULAS

17. Orbiting Distance North or South of the Equator: $D(t) = A\cos(Bt)$

Unless a satellite is placed in a strict equatorial orbit, its distance north or south of the equator will vary according to the sinusoidal model shown, where $D(t)$ is the distance t *minutes* after entering orbit. Negative values indicate it is south of the equator, and the distance D is actually a two-dimensional distance, as seen from a vantage point in outer space. The value of B depends on the speed of the satellite and the time it takes to complete one orbit, while $|A|$ represents the maximum distance from the equator. (a) Find the equation model for a satellite whose maximum distance north of the equator is 2000 miles and that completes one orbit every *2 hours* ($P = 120$). (b) Is the satellite north or south of the equator 257 min after entering orbit? How far north or south?

18. Biorhythm Theory: $P(d) = 50\sin(Bd) + 50$

Advocates of biorhythm theory believe that human beings are influenced by certain biological cycles that begin at birth, have different periods, and continue throughout life. The classical cycles and their periods are: physical potential (23 days), emotional potential (28 days), and intellectual potential (33 days). On any given day of life, the percent of potential in these three areas is purported to be modeled by the function shown, where $P(d)$ is the percent of available potential on day d of life. Find the value of B for each of the physical, emotional, and intellectual potentials and use it to see what the theory has to say about your potential today. Use day $d = 365.25(\text{age}) +$ days since last birthday.

▶ APPLICATIONS

19. Record monthly temperatures: The U.S. National Oceanic and Atmospheric Administration (NOAA) keeps temperature records for most major U.S. cities. For Phoenix, Arizona, they list an average high temperature of 65.0°F for the month of January (month 1) and an average high temperature of 104.2°F for July (month 7). Assuming January and July are the coolest and warmest months of the year, (a) build a sinusoidal function model for temperatures in Phoenix, and (b) use the model to find the average high temperature in September. (c) If a person has a tremendous aversion to temperatures over 95°, during what months should they plan to vacation elsewhere?

20. Seasonal size of polar ice caps: Much like the polar ice cap on Mars, the sea ice that surrounds the continent of Antarctica (the Earth's southern polar cap) varies seasonally, from about 8 million mi^2 in September to about 1 million mi^2 in March. Use this information to (a) build a sinusoidal equation that models the advance and retreat of the sea ice, and (b) determine the size of the ice sheet in May. (c) Find the months of the year that the sea ice covers more than 6.75 million mi^2.

Exercise 20

21. Body temperature cycles: A phenomenon is said to be *circadian* if it occurs in 24-hr cycles. A person's body temperature is circadian, since there are normally small, sinusoidal variations in body temperature from a low of 98.2°F to a high of 99°F throughout a 24-hr day. Use this information to (a) build the circadian equation for a person's body temperature, given $t = 0$ corresponds to midnight and that a person usually reaches their minimum temperature at 5 A.M.; (b) find the time(s) during a day when a person reaches "normal" body temperature (98.6°); and (c) find the number of hours each day that body temperature is 98.4°F or less.

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22. Position of engine piston: For an internal combustion engine, the position of a piston in the cylinder can be modeled by a sinusoidal function. For a particular engine size and idle speed, the piston head is 0 in. from the top of the cylinder (the minimum value) when $t = 0$ at the beginning of the intake stroke, and reaches a maximum distance of 4 in. from the top of the cylinder (the maximum value) when $t = \frac{1}{48}$ sec at the beginning of the compression stroke. Following the compression stroke is the power stroke ($t = \frac{2}{48}$), the exhaust stroke ($t = \frac{3}{48}$), and the intake stroke ($t = \frac{4}{48}$), after which it all begins again. Given the period of a four-stroke engine under these conditions is $P = \frac{1}{24}$ second, (a) find the sinusoidal equation modeling the position of the piston, and (b) find the distance of the piston from the top of the cylinder at $t = \frac{1}{9}$ sec. Which stroke is the engine in at this moment?

Data and tangent functions: Use the data given to find an equation model of the form $f(x) = A\tan(Bx + C)$. Then graph the function and scatterplot to help find (a) the output for $x = 2.5$, and (b) the value of x where $f(x) = 16$.

23.

x	y	x	y
0	∞	7	-1.4
1	20	8	-3
2	9.7	9	-5.2
3	5.2	10	-9.7
4	3	11	-20
5	1.4	12	$-\infty$
6	0		

24.

x	y	x	y
0	∞	7	6.4
1	-91.3	8	13.7
2	-44.3	9	23.7
3	-23.7	10	44.3
4	-13.7	11	91.3
5	-6.4	12	∞
6	0		

25. Distance and apparent height: While driving toward a Midwestern town on a long, flat stretch of highway, I decide to pass the time by measuring the apparent height of the tallest building in the downtown area as I approach. At the time the idea occurred to me, the buildings were barely visible. Three miles later I hold a 30-cm ruler up to my eyes at arm's length, and the apparent height of the tallest building is 1 cm. After three more miles the apparent height is 1.8 cm. Measurements are taken every 3 mi until I reach town and are shown in the table (assume I was 24 mi from the parking garage when I began this activity). (a) Use the data to come up with a tangent function model of the building's apparent height after traveling a distance of x mi closer. (b) What was the apparent height of the building after I had driven 19 mi? (c) How many miles had I driven when the apparent height of the building took up all 30 cm of my ruler?

Distance Traveled (mi)	Height (cm)
0	0
3	1
6	1.8
9	2.8
12	4.2
15	6.3
18	10
21	21
24	∞

26. Earthquakes and elastic rebound: The **theory of elastic rebound** has been used by seismologists to study the cause of earthquakes. As seen in the figure, the Earth's crust is stretched to a breaking point by the slow movement of one tectonic plate in a direction opposite the other along a fault line, and when the rock snaps—each half violently rebounds to its original alignment causing the Earth to quake.

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x	y	x	y
−4.5	−61	1	2.1
−4	−26	2	6.8
−3	−14.8	3	15.3
−2	−7.2	4	25.4
−1	−1.9	4.5	59
0	0		

Suppose the *misalignment* of these plates through the stress and twist of crustal movement can be modeled by a tangent graph, where x represents the horizontal distance from the original stress line, and y represents the vertical distance from the fault line. Assume a "period" of 10.2 m. (a) Use the data from the table to come up with a trigonometric model of the deformed stress line. (b) At a point 4.8 m along the fault line, what is the distance to the deformed stress line (moving parallel to the original stress line)? (c) At what point along the fault line is the vertical distance to the deformed stress line 50 m?

27. Record monthly temperatures: The highest temperature of record for the even months of the year are given in the table for the city of Pittsburgh, Pennsylvania. (a) Use the data to draw a scatterplot, then find a sinusoidal regression model and graph both on the same screen. (b) Use the equation to estimate the record high temperatures for the odd-numbered months. (c) What month shows the largest difference between the actual data and the computed results?

Month (Jan. → 1)	High Temp. (°F)
2	76
4	89
6	98
8	100
10	87
12	74

Source: 2004 Statistical Abstract of the United States, Table 378.

28. River discharge rate: The average discharge rate of the Alabama River is given in the table for the odd-numbered months of the year. (a) Use the data to draw a scatterplot, then find a sinusoidal regression model and graph both on the same screen. (b) Use the equation to estimate the flow rate for the even-numbered months. (c) Use the graph and equation to estimate the number of days per year the flow rate is below 500 m^3/sec.

Month (Jan. → 1)	Rate (m^3/sec)
1	1569
3	1781
5	1333
7	401
9	261
11	678

Source: Global River Discharge Database Project; www.rivdis.sr.unh.edu.

29. Illumination of the moon's surface: The table given indicates the percent of the Moon that is illuminated for the days of a particular month, at a given latitude. (a) Use a graphing calculator to find a sinusoidal regression model. (b) Use the model to determine what percent of the Moon is illuminated on day 20. (c) Use the maximum and minimum values with the period and an appropriate horizontal shift to create your own model of the data. How do the values for A, B, C, and D compare?

Day	% Illum.	Day	% Illum.
1	28	19	34
4	55	22	9
7	82	25	0
10	99	28	9
13	94	31	30
16	68		

30. Connections between weather and mood: The mood of persons with SAD syndrome (seasonal affective disorder) often depends on the weather. Victims of SAD are typically more despondent in rainy weather than when the sun is out, and more comfortable in the daylight hours than at night. The table shows the average number of daylight hours for Vancouver, British Columbia, for 12 months of a year. (a) Use a calculator to find a sinusoidal regression model. (b) Use the model to estimate the *number of days* per year (use 1 month ≈ 30.5 days) with more than 14 hr of daylight. (c) Use the maximum and minimum values with the period and an appropriate horizontal shift to create a model of the data. How do the values for A, B, C, and D compare?

Source: Vancouver Climate at www.bcpassport.com/vital.

Month	Hours	Month	Hours
1	8.3	7	16.2
2	9.4	8	15.1
3	11.0	9	13.5
4	12.9	10	11.7
5	14.6	11	9.9
6	15.9	12	8.5

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31. Average monthly rainfall: The average monthly rainfall (in inches) for Reno, Nevada, is shown in the table. (a) Use the data to find a sinusoidal regression model for the monthly rainfall. (b) Graph this equation model and the rainfall equation model for Cheyenne, Wyoming (from Example 5), on the same screen, and estimate the number of months that Reno gets more rainfall than Cheyenne.

Source: NOAA Comparative Climate Data 2004.

Month (Jan → 1)	Reno Rainfall	Month (Jan → 1)	Reno Rainfall
1	1.06	7	0.24
2	1.06	8	0.27
3	0.86	9	0.45
4	0.35	10	0.42
5	0.62	11	0.80
6	0.47	12	0.88

32. Hours of daylight by month: The number of daylight hours per month (as measured on the 15th of each month) is shown in the table for the cities of Beaumont, Texas, and Minneapolis, Minnesota. (a) Use the data to find a sinusoidal regression model of the daylight hours for each city. (b) Graph both equations on the same screen and use the graphs to estimate the number of days each year that Beaumont receives more daylight than Minneapolis (use 1 month = 30.5 days).

Source: www.encarta.msn.com/media_701500905/Hours_of_Daylight_by_Latitude.html.

Month (Jan → 1)	TX Sunlight	MN Sunlight
1	10.4	9.1
2	11.2	10.4
3	12.0	11.8
4	12.9	13.5
5	14.4	16.2
6	14.1	15.7
7	13.9	15.2
8	13.3	14.2
9	12.4	12.6
10	11.5	11.0
11	10.7	9.6
12	10.2	8.7

▶ EXTENDING THE CONCEPT

33. For the equations from Examples 1 and 2, use the minimum value (x, m) to show that $\frac{y - D}{A} \to \frac{m - D}{A}$ is equal to -1. Then verify this relationship in general by substituting $\frac{M - m}{2}$ for A and $\frac{M + m}{2}$ for D.

34. A *dampening factor* is any function whose product with a sinusoidal function causes a systematic reduction in amplitude. In the graph shown, the function $y = \sin(3x)$ has been dampened by the linear function $y = -\frac{1}{4}x + 2$ over the interval $[-2\pi, 2\pi]$. Notice that the peaks of the sine graph are points on the graph of this line. The table gives approximate points of intersection for $y = \sin(3x)$ with another dampening factor. Use the regression capabilities of a graphing calculator to find the approximate equation of the dampening factor. (*Hint:* It is not linear.)

x	y	x	y
$\frac{\pi}{6}$	2.12	$\frac{13\pi}{6}$	1.03
$\frac{5\pi}{6}$	1.58	$\frac{17\pi}{6}$	1.02
$\frac{3\pi}{2}$	1.22	$\frac{7\pi}{2}$	1.18

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▶ MAINTAINING YOUR SKILLS

35. **(2.4)** State the domains of the following functions:

a. $f(x) = x^{1.7}$ **b.** $g(x) = x^{\frac{4}{5}}$

c. $h(x) = \sqrt[9]{x^3}$

36. **(5.4)** Use properties of logarithms to write the following as a single term.

$$4 \log 2 - \frac{1}{2} \log 25$$

37. **(6.1)** The barrel on a winch has a radius of 3 in. and is turning at 30 rpm. As the barrel turns it winds in a cable that is pulling a heavy pallet across the warehouse floor. How fast is the pallet moving in feet per minute?

38. **(6.6)** Clarke is standing between two tall buildings. The angle of elevation to the top of the building to her north is 60°, while the angle of elevation to the top of the building to her south is 70°. If she is 400 m from the base of the northern building and 200 m from the southern, which one is taller?

MAKING CONNECTIONS

Making Connections: Graphically, Symbolically, Numerically, and Verbally

Eight graphs A through H are given. Match the characteristics given in 1 through 16 to one of the eight graphs.

1. ____ period $= \pi$

2. ____ amplitude $= 2$

3. ____ $f(t) = \sec\left(t - \frac{\pi}{2}\right) - 1$

4. ____ $f(t) = 2 \sin t$

5. ____ minimum of -1 occurs at $t = -\pi$

6. ____ $x^2 + y^2 = 1$

7. ____ $f(t) = \sin\left(t + \frac{\pi}{2}\right)$

8. ____ period $= 2$

9. ____ period $= 4$

10. ____ $f(t) = \cos\left(\frac{\pi}{2}t\right)$

11. ____ $f(t) = 2 \tan\left(\frac{\pi}{2}t\right)$

12. ____ The point $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ is on the graph

13. ____ $t \neq k\pi$, shifted up 1 unit

14. ____ $f(t) = \csc t + 1$

15. ____ $f(t) = 2 \cot\left(t - \frac{\pi}{2}\right)$

16. ____ $f(t) \downarrow$ for $t \in \mathbb{R}$

SUMMARY AND CONCEPT REVIEW

SECTION 6.1 Angle Measure, Special Triangles, and Special Angles

KEY CONCEPTS

- An angle is defined as the joining of two rays at a common endpoint called the vertex.
- An angle in standard position has its vertex at the origin and its initial side on the positive x-axis.
- Two angles in standard position are coterminal if they have the same terminal side.
- A counterclockwise rotation gives a positive angle, a clockwise rotation gives a negative angle.
- One (1°) degree is defined to be $\frac{1}{360}$ of a full revolution. One (1) radian is the measure of a central angle subtended by an arc equal in length to the radius.
- Degrees can be divided into a smaller unit called minutes: $1° = 60'$; minutes can be divided into a smaller unit called seconds: $1' = 60''$. This implies $1° = 3600''$.
- Two angles are complementary if they sum to 90° and supplementary if they sum to 180°.
- Properties of triangles: (I) the sum of the angles is 180°; (II) the combined length of any two sides must exceed that of the third side and; (III) larger angles are opposite longer sides.
- Given two triangles, if all three corresponding angles are equal, the triangles are said to be similar. If two triangles are similar, then corresponding sides are in proportion.
- In a 45-45-90 triangle, the sides are in the proportion $1x$: $1x$: $\sqrt{2}x$.
- In a 30-60-90 triangle, the sides are in the proportion $1x$: $\sqrt{3}x$: $2x$.
- The formula for arc length: $s = r\theta$, θ in radians.
- The formula for the area of a circular sector: $\mathcal{A} = \frac{1}{2}r^2\theta$, θ in radians.
- To convert degree measure to radians, multiply by $\frac{\pi}{180°}$; for radians to degrees, multiply by $\frac{180°}{\pi}$.
- Special angle conversions: $30° = \frac{\pi}{6}$, $45° = \frac{\pi}{4}$, $60° = \frac{\pi}{3}$, $90° = \frac{\pi}{2}$.
- A location north or south of the equator is given in degrees latitude; a location east or west of the Greenwich Meridian is given in degrees longitude.
- Angular velocity is a rate of rotation per unit time: $\omega = \frac{\theta}{t}$.
- Linear velocity is a change in position per unit time: $V = \frac{\theta r}{t}$ or $V = r\omega$.

EXERCISES

1. Convert 147° 36′ 48″ to decimal degrees.

2. Convert 32.87° to degrees, minutes, and seconds.

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3. All of the right triangles given are similar. Find the dimensions of the largest triangle.

Exercise 3

Exercise 4

4. Use special angles/special triangles to find the length of the bridge needed to cross the lake shown in the figure.

5. Convert to degrees: $\frac{2\pi}{3}$.

6. Convert to radians: 210°.

7. Find the arc length if $r = 5$ and $\theta = 57°$.

8. Evaluate without using a calculator: $\sin\left(\frac{7\pi}{6}\right)$.

Find the angle, radius, arc length, and/or area as needed, until all values are known.

9.

10.

11.

12. With great effort, 5-year-old Mackenzie has just rolled her bowling ball down the lane, and it is traveling painfully slow. So slow, in fact, that you can count the number of revolutions the ball makes using the finger holes as a reference. (a) If the ball is rolling at 1.5 revolutions per second, what is the angular velocity? (b) If the ball's radius is 5 in., what is its linear velocity in feet per second? (c) If the distance to the first pin is 60 feet and the ball is true, how many seconds until it hits?

SECTION 6.2 Unit Circles and the Trigonometry of Real Numbers

KEY CONCEPTS

- A central unit circle is a circle with radius 1 unit having its center at the origin.
- A central circle is symmetric to both axes and the origin. This means that if (a, b) is a point on the circle, then $(-a, b)$, $(-a, -b)$, and $(a, -b)$ are also on the circle and satisfy the equation of the circle.
- On the unit circle with θ in radians, the length of a subtended arc is numerically the same as the subtended angle, making the arc a "circular number line" and associating any given rotation with a unique real number.
- A reference angle is defined to be the acute angle formed by the terminal side of a given angle and the x-axis. For functions of a real number we refer to a reference arc rather than a reference angle.

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- For any real number t and a point on the unit circle associated with t, we have:

$$\cos t = x \qquad \sin t = y \qquad \tan t = \frac{y}{x} \quad x \neq 0 \qquad \sec t = \frac{1}{x} \quad x \neq 0 \qquad \csc t = \frac{1}{y} \quad y \neq 0 \qquad \cot t = \frac{x}{y} \quad y \neq 0$$

- Given the specific value of any function, the related real number t or angle θ can be found using a special reference arc/angle, or the $\sin^{-1}$, $\cos^{-1}$, or $\tan^{-1}$ features of a calculator.

EXERCISES

13. Given $\left(\frac{\sqrt{13}}{7}, y\right)$ is on the unit circle, find y if the point is in QIV, then use the symmetry of the circle to locate three other points.

14. Given $\left(\frac{3}{4}, -\frac{\sqrt{7}}{4}\right)$ is on the unit circle, find the value of all six trig functions of t without the use of a calculator.

15. Without using a calculator, find two values in $[0, 2\pi)$ that make the equation true: $\csc t = \frac{2}{\sqrt{3}}$.

16. Use a calculator to find the value of t that corresponds to the situation described: $\cos t = -0.7641$ with t in QII.

17. A crane used for lifting heavy equipment has a winch-drum with a 1-yd radius. (a) If 59 ft of cable has been wound in while lifting some equipment to the roof-top of a building, what radian angle has the drum turned through? (b) What angle must the drum turn through to wind in 75 ft of cable?

SECTION 6.3 Graphs of the Sine and Cosine Functions

KEY CONCEPTS

- Graphing sine and cosine functions using the special values from the unit circle results in a periodic, wavelike graph with domain $(-\infty, \infty)$.

- The characteristics of each graph play a vital role in their contextual application, and these are summarized on pages 546 and 550.
- The amplitude of a sine or cosine graph is the maximum displacement from the average value. For $y = A\sin(Bt)$ and $y = A\cos(Bt)$, the amplitude is $|A|$.
- The period of a periodic function is the smallest interval required to complete one cycle.

  For $y = A\sin(Bt)$ and $y = A\cos(Bt)$, $P = \frac{2\pi}{B}$ gives the period.

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- If $|A| > 1$, the graph is vertically stretched, if $0 < |A| < 1$ the graph is vertically compressed, and if $A < 0$ the graph is reflected across the x-axis.
- If $B > 1$, the graph is horizontally compressed (the period is smaller/shorter); if $B < 1$ the graph is horizontally stretched (the period is larger/longer).
- To graph $y = A\sin(Bt)$ or $A\cos(Bt)$, draw a reference rectangle $2A$ units high and $P = \dfrac{2\pi}{B}$ units wide, centered on the x-axis, then use the *rule of fourths* to locate zeroes and max/min values. Connect these points with a smooth curve.

EXERCISES

Use a reference rectangle and the *rule of fourths* to draw an accurate sketch of the following functions through at least one full period. Clearly state the amplitude and period as you begin.

18. $y = 3\sin t$

19. $y = -\cos(2t)$

20. $y = 1.7\sin(4t)$

21. $f(t) = 2\cos(4\pi t)$

22. $g(t) = 3\sin(398\pi t)$

23. The given graph is of the form $y = A\sin(Bt)$. Determine the equation of the graph.

24. Referring to the chart of colors visible in the electromagnetic spectrum (page 559), what color is represented by the equation $y = \sin\left(\dfrac{\pi}{270}t\right)$? By $y = \sin\left(\dfrac{\pi}{320}t\right)$?

SECTION 6.4 Graphs of the Cosecant, Secant, Tangent, and Cotangent Functions

KEY CONCEPTS

- The graph of $y = \sec t = \dfrac{1}{\cos t}$ will be asymptotic everywhere $\cos t = 0$, increasing where $\cos t$ is decreasing, and decreasing where $\cos t$ is increasing.
- The graph of $y = \csc t = \dfrac{1}{\sin t}$ will be asymptotic everywhere $\sin t = 0$, increasing where $\sin t$ is decreasing, and decreasing where $\sin t$ is increasing.
- Since $\tan t$ is defined in terms of the ratio $\dfrac{y}{x}$, the graph will be asymptotic everywhere $x = 0$ on the unit circle, meaning all odd multiples of $\dfrac{\pi}{2}$.
- Since $\cot t$ is defined in terms of the ratio $\dfrac{x}{y}$, the graph will be asymptotic everywhere $y = 0$ on the unit circle, meaning all integer multiples of π.
- The graph of $y = \tan t$ is increasing everywhere it is defined; the graph of $y = \cot t$ is decreasing everywhere it is defined.
- The characteristics of each graph play vital roles in their contextual application, and these are summarized on pages 563 and 567.

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- For the more general tangent and cotangent graphs $y = A\tan(Bt)$ and $y = A\cot(Bt)$, if $|A| > 1$, the graph is vertically stretched, if $0 < |A| < 1$ the graph is vertically compressed, and if $A < 0$ the graph is reflected across the x-axis.
- If $B > 1$, the graph is horizontally compressed (the period is smaller/shorter); if $B < 1$ the graph is horizontally stretched (the period is larger/longer).
- To graph $y = A\tan(Bt)$, note $A\tan(Bt)$ is zero at $t = 0$. Compute the period $P = \dfrac{\pi}{B}$ and draw asymptotes a distance of $\dfrac{P}{2}$ on either side of the y-axis. Plot zeroes halfway between the asymptotes and use symmetry to complete the graph.
- To graph $y = A\cot(Bt)$, note it is asymptotic at $t = 0$. Compute the period $P = \dfrac{\pi}{B}$ and draw asymptotes a distance P on either side of the y-axis. Plot zeroes halfway between the asymptotes and use symmetry to complete the graph.

EXERCISES

25. Use a reference rectangle and the *rule of fourths* to draw an accurate sketch of $y = 3\sec t$ through at least one full period. Clearly state the period as you begin.

26. The given graph is of the form $y = A\csc(Bt)$. Determine the equation of the graph.

Exercise 26

27. State the value of each expression without the aid of a calculator:

a. $\tan\left(\dfrac{7\pi}{4}\right)$ **b.** $\cot\left(\dfrac{\pi}{3}\right)$

28. State the value of each expression without the aid of a calculator, given that t terminates in QII.

a. $\tan^{-1}(-\sqrt{3})$ **b.** $\cot^{-1}\left(-\dfrac{1}{\sqrt{3}}\right)$

29. Graph $y = 6\tan\left(\dfrac{1}{2}t\right)$ in the interval $[-2\pi, 2\pi]$.

30. Graph $y = \dfrac{1}{2}\cot(2\pi t)$ in the interval $[-1, 1]$.

31. Use the period of $y = \cot t$ to name three additional solutions to $\cot t = 0.0208$, given $t = 1.55$ is a solution. Many solutions are possible.

32. Given $t = 0.4444$ is a solution to $\cot^{-1}t = 2.1$, use an analysis of signs and quadrants to name an additional solution in $[0, 2\pi)$.

33. Find the approximate height of Mount Rushmore, using $h = \dfrac{d}{\cot u - \cot v}$ and the values shown.

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SECTION 6.5 Transformations and Applications of T rigonometric Graphs

KEY CONCEPTS

- Many everyday phenomena follow a sinusoidal pattern, or a pattern that can be modeled by a sine or cosine function (e.g., daily temperatures, hours of daylight, and more).
- To obtain accurate equation models of sinusoidal phenomena, vertical and horizontal shifts of a basic function are used.
- The equation $y = A\sin(Bt \pm C) + D$ is called the *standard form* of a general sinusoid. The equation $y = A\sin\left[B\left(t \pm \frac{C}{B}\right)\right] + D$ is called the *shifted form* of a general sinusoid.
- In either form, D represents the average value of the function and a vertical shift D units upward if $D > 0$, D units downward if $D < 0$. For a maximum value M and minimum value m, $\frac{M+m}{2} = D$, $\frac{M-m}{2} = A$.
- The shifted form $y = A\sin\left[B\left(t \pm \frac{C}{B}\right)\right] + D$ enables us to quickly identify the horizontal shift of the function: $\frac{C}{B}$ units in a direction opposite the given sign.
- To graph a shifted sinusoid, locate the primary interval by solving $0 \le Bt + C < 2\pi$, then use a reference rectangle along with the *rule of fourths* to sketch the graph in this interval. The graph can then be extended as needed, then shifted vertically D units.
- One basic application of sinusoidal graphs involves phenomena in harmonic motion, or motion that can be modeled by functions of the form $y = A\sin(Bt)$ or $y = A\cos(Bt)$ (with no horizontal or vertical shift).
- If the period P and critical points (X, M) and (x, m) of a sinusoidal function are known, a model of the form $y = A\sin(Bt + C) + D$ can be obtained:

$$B = \frac{2\pi}{P} \qquad A = \frac{M-m}{2} \qquad D = \frac{M+m}{2} \qquad C = \frac{3\pi}{2} - Bx$$

EXERCISES

For each equation given, (a) identify/clearly state the amplitude, period, horizontal shift, and vertical shift; then (b) graph the equation using the primary interval, a reference rectangle, and *rule of fourths*.

34. $y = 240\sin\left[\frac{\pi}{6}(t-3)\right] + 520$

35. $y = 3.2\cos\left(\frac{\pi}{4}t + \frac{3\pi}{2}\right) + 6.4$

For each graph given, identify the amplitude, period, horizontal shift, and vertical shift, and give the equation of the graph.

36.

37.

38. Monthly precipitation in Cheyenne, Wyoming, can be modeled by a sine function, by using the average precipitation for June (2.26 in.) as a maximum (actually slightly higher in May), and the average precipitation for December (0.44 in.) as a minimum. Assume $t = 0$ corresponds to March. (a) Use the information to construct a sinusoidal model, and (b) use the model to estimate the inches of precipitation Cheyenne receives in August ($t = 5$) and September ($t = 6$).

Source: 2004 Statistical Abstract of the United States, Table 380.

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SECTION 6.6 The Trigonometry of Right Triangles

KEY CONCEPTS

- The sides of a right triangle can be named relative to their location with respect to a given angle.

- The ratios of two sides with respect to a given angle are named as follows:

$$\sin\alpha = \frac{\text{opp}}{\text{hyp}} \qquad \cos\alpha = \frac{\text{adj}}{\text{hyp}} \qquad \tan\alpha = \frac{\text{opp}}{\text{adj}}$$

- The reciprocal of the ratios above play a vital role and are likewise given special names:

$$\csc\alpha = \frac{\text{hyp}}{\text{opp}} \qquad \sec\alpha = \frac{\text{hyp}}{\text{adj}} \qquad \cot\alpha = \frac{\text{adj}}{\text{opp}}$$

$$\csc\alpha = \frac{1}{\sin\alpha} \qquad \sec\alpha = \frac{1}{\cos\alpha} \qquad \cot\alpha = \frac{1}{\tan\alpha}$$

- Each function of α is equal to the cofunction of its complement. For instance, the complement of sine is *co*sine and $\sin\alpha = \cos(90° - \alpha)$.
- To solve a right triangle means to apply any combination of the trig functions, along with the triangle properties, until all sides and all angles are known.
- An angle of elevation is the angle formed by a horizontal line of sight (parallel to level ground) and the true line of sight. An angle of depression is likewise formed, but with the line of sight below the line of orientation.

EXERCISES

39. Use a calculator to solve for A:

a. $\cos 37° = A$

b. $\cos A = 0.4340$

40. Rewrite each expression in terms of a cofunction.

a. $\tan 57.4°$

b. $\sin(19° \, 30' \, 15'')$

Solve each triangle. Round angles to the nearest tenth and sides to the nearest hundredth.

41.

42.

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43. Josephine is to weld a vertical support to a 20-m ramp so that the incline is exactly 15°. What is the height h of the support that must be used?

44. From the observation deck of a seaside building 480 m high, Armando sees two fishing boats in the distance. The angle of depression to the nearer boat is 63.5°, while for the boat farther away the angle is 45°. (a) How far out to sea is the nearer boat? (b) How far apart are the two boats?

45. A slice of bread is roughly 14 cm by 10 cm. If the slice is cut diagonally in half, what acute angles are formed?

SECTION 6.7 Trigonometry and the Coor dinate Plane

KEY CONCEPTS

- In standard position, the terminal sides of 0°, 90°, 180°, 270°, and 360° angles coincide with one of the axes and are called quadrantal angles.
- By placing a right triangle in the coordinate plane with one acute angle at the origin and one side along the x-axis, we note the trig functions can be defined in terms of a point $P(x, y)$ on the hypotenuse.
- Given $P(x, y)$ is any point on the terminal side of an angle θ in standard position. Then $r = \sqrt{x^2 + y^2}$ is the distance from the origin to this point. The six trigonometric functions of θ are defined as

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x} \qquad \csc\theta = \frac{r}{y} \qquad \sec\theta = \frac{r}{x} \qquad \cot\theta = \frac{x}{y}$$

$$x \neq 0 \qquad y \neq 0 \qquad x \neq 0 \qquad y \neq 0$$

- A reference angle θ_r is defined to be the acute angle formed by the terminal side of a given angle θ and the x-axis.
- Reference angles can be used to evaluate the trig functions of any nonquadrantal angle, since the values are fixed by the ratio of sides and the signs are dictated by the quadrant of the terminal side.
- If the value of a trig function and the quadrant of the terminal side are known, the related angle θ can be found using a reference arc/angle, or the $\sin^{-1}$, $\cos^{-1}$, or $\tan^{-1}$ features of a calculator.
- If θ is a solution to $\sin\theta = k$, then $\theta + 360°k$ is also a solution for any integer k.

EXERCISES

46. Find two positive angles and two negative angles that are coterminal with $\theta = 207°$.

47. Name the reference angle for the angles given: $\theta = -152°$, $\theta = 521°$, $\theta = 210°$

48. Find the value of the six trigonometric functions, given $P(x, y)$ is on the terminal side of angle θ in standard position.

a. $P(-12, 35)$ **b.** $(12, -18)$

49. Find the values of x, y, and r using the information given, and state the quadrant of the terminal side of θ. Then state the values of the six trig functions of θ.

a. $\cos\theta = \frac{4}{5}$; $\sin\theta < 0$ **b.** $\tan\theta = -\frac{12}{5}$; $\cos\theta > 0$

50. Find all angles satisfying the stated relationship. For standard angles, express your answer in exact form. For nonstandard angles, use a calculator and round to the nearest tenth.

a. $\tan\theta = -1$ **b.** $\cos\theta = \frac{\sqrt{3}}{2}$ **c.** $\tan\theta = 4.0108$ **d.** $\sin\theta = -0.4540$

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SECTION 6.8 Trigonometric Equation Models

KEY CONCEPTS

- If the period P and critical points (X, M) and (x, m) of a sinusoidal function are known, a model of the form $y = A\sin(Bt + C) + D$ can be obtained:

$$B = \frac{2\pi}{P} \qquad A = \frac{M - m}{2} \qquad D = \frac{M + m}{2} \qquad C = \frac{3\pi}{2} - Bx$$

- If an event or phenomenon is known to be sinusoidal, a graphing calculator can be used to find an equation model if at least four data points are given.

EXERCISES

For the following sets of data, (a) find a sinusoidal regression equation using your calculator; (b) construct an equation manually using the period and maximum/minimum values; and (c) graph both on the same screen, then use a TABLE to find the largest difference between output values.

51.

Day of Year	Output	Day of Year	Output
1	4430	184	90
31	3480	214	320
62	3050	245	930
92	1890	275	2490
123	1070	306	4200
153	790	336	4450

52.

Day of Month	Output	Day of Month	Output
1	69	19	98
4	78	22	92
7	84	25	85
10	91	28	76
13	96	31	67
16	100		

53. The highest temperature on record for the even months of the year are given in the table for the city of Juneau, Alaska. (a) Use a graphing calculator to find a sinusoidal regression model. (b) Use the equation to estimate the record high temperature for the month of July. (c) Compare the actual data to the results produced by the regression model. What comments can you make about the accuracy of the model?

Source: 2004 Statistical Abstract of the United States, Table 378.

Month (Jan → 1)	High Temp(°F)
2	57
4	72
6	86
8	83
10	61
12	54

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PRACTICE TEST

1. Name the reference angle of each angle given.

a. 225° **b.** −510°

c. $\frac{7\pi}{6}$ **d.** $\frac{25\pi}{3}$

2. Find two negative angles and two positive angles that are coterminal with $\theta = 30°$. Many solutions are possible.

3. Convert from DMS to decimal degrees or decimal degrees to DMS as indicated.

a. 100° 45′ 18″ to decimal degrees

b. 48.2125° to DMS

4. Four Corners USA is the point at which Utah, Colorado, Arizona, and New Mexico meet. The southern border of Colorado, the western border of Kansas, and the point *P* where Colorado, Nebraska, and Kansas meet, very nearly approximates a 30-60-90 triangle. If the western border of Kansas is 215 mi long, (a) what is the distance from Four Corners USA to point *P*? (b) How long is Colorado's southern border?

5. Complete the table from memory using exact values. If a function is undefined, so state.

t	sin t	cos t	tan t	csc t	sec t	cot t
0						
$\frac{2\pi}{3}$						
$\frac{7\pi}{6}$						
$\frac{5\pi}{4}$						
$\frac{5\pi}{3}$						
$\frac{13\pi}{6}$						

6. Given $\cos\theta = \frac{2}{5}$ and $\tan\theta < 0$, find the value of the other five trig functions of θ.

7. Verify that $\left(\frac{1}{3}, -\frac{2\sqrt{2}}{3}\right)$ is a point on the unit circle, then find the value of all six trig functions associated with this point.

8. In order to take pictures of a dance troupe as it performs, a camera crew rides in a cart on tracks that trace a circular arc. The radius of the arc is 75 ft, and from end to end the cart sweeps out an angle of 172.5° in 20 seconds. Use this information to find (a) the length of the track in feet and inches, (b) the angular velocity of the cart, and (c) the linear velocity of the cart in both ft/sec and mph.

9. Solve the triangle shown. Answer in table form.

Exercise 9

10. The "plow" is a yoga position in which a person lying on their back brings their feet up, over, and behind their head and touches them to the floor. If distance from hip to shoulder (at the right angle) is 57 cm and from hip to toes is 88 cm, find the

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distance from shoulders to toes and the angle formed at the hips.

Exercise 10

11. While doing some night fishing, you round a peninsula and a tall lighthouse comes into view. Taking a sighting, you find the angle of elevation to the top of the lighthouse is 25°. If the lighthouse is known to be 27 m tall, how far from the lighthouse are you?

Exercise 11

12. Find the value of $t \in [0, 2\pi]$ satisfying the conditions given.

a. $\sin t = -\frac{1}{2}$, t in QIII

b. $\sec t = \frac{2\sqrt{3}}{3}$, t in QIV

c. $\tan t = -1$, t in QII

13. In arid communities, daily water usage can often be approximated using a sinusoidal model. Suppose water consumption in the city of Caliente del Sol reaches a maximum of 525,000 gallons in the heat of the day, with a minimum usage of 157,000 gallons in the cool of the night. Assume $t = 0$ corresponds to 6:00 A.M. (a) Use the information to construct a sinusoidal model, and (b) use the model to approximate water usage at 4:00 P.M. and 4:00 A.M.

14. State the domain, range, period, and amplitude (if it exists), then graph the function over 1 period.

a. $y = 2\sin\left(\frac{\pi}{5}t\right)$ **b.** $y = \sec t$

c. $y = 2\tan(3t)$

15. The lowest temperature on record for the even months of the year are given in the table for the city of Denver, Colorado. (a) Use a graphing calculator to find a sinusoidal regression model and (b) use the equation to estimate the record low temperature for the odd numbered months.

Source: 2004 Statistical Abstract of the United States, Table 379.

Month (Jan → 1)	Low Temp (°F)
2	−30
4	−2
6	30
8	41
10	3
12	−25

16. State the amplitude, period, horizontal shift, vertical shift, and endpoints of the primary interval. Then sketch the graph using a reference rectangle and the *rule of fourths:*

$$y = 12\sin\left(3t - \frac{\pi}{4}\right) + 19.$$

17. An athlete throwing the shot-put begins his first attempt facing due east, completes three and one-half turns and launches the shot facing due west. What angle did his body turn through?

18. State the domain, range, and period, then sketch the graph in $[0, 2\pi)$.

a. $y = \tan(2t)$ **b.** $y = \cot\left(\frac{1}{2}t\right)$

19. Due to tidal motions, the depth of water in Brentwood Bay varies sinusoidally as shown in the diagram, where time is in hours and depth is in feet. Find an equation that models the depth of water at time t.

Exercise 19

20. Find the value of t satisfying the given conditions.

a. $\sin t = -0.7568$; t in QIII

b. $\sec t = -1.5$; t in QII

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CALCULATOR EXPLORATION AND DISCOVERY

Variable Amplitudes and Modeling the Tides

Tidal motion is often too complex to be modeled by a single sine function. In this *Exploration and Discovery*, we'll look at a method that combines two sine functions to help model a tidal motion with variable amplitude. In the process, we'll use much of what we know about the amplitude, horizontal shifts, and vertical shifts of a sine function, helping to reinforce these important concepts and broaden our understanding about how they can be applied. The graph in Figure 6.148 shows three days of tidal motion for Davis Inlet, Canada.

As you can see, the amplitude of the graph varies, and there is no *single* sine function that can serve as a model. However, notice that the amplitude *varies predictably*, and that the high tides and low tides can independently be modeled by sine functions. To simplify our exploration, we will use the assumption that tides have an exact 24-hr period (close, but no), that variations between high and low tides takes place every 12 hr (again close but not exactly true), and the variation between the "low-high" (1.9 m) and the "high-high" (2.4 m) is uniform. A similar assumption is made for the low tides. The result is the graph in Figure 6.149.

Figure 6.148

Figure 6.149

First consider the high tides, which vary from a maximum of 2.4 to a minimum of 1.9. Using the ideas from Section 6.7 to construct an equation model gives $A = \frac{2.4 - 1.9}{2} = 0.25$ and $D = \frac{2.4 + 1.9}{2} = 2.15$. With a period of $P = 24$ hr we obtain the equation $Y_1 = 0.25 \sin\left(\frac{\pi}{12}x\right) + 2.15$. Using 0.9 and 0.7 as the maximum and minimum low tides, similar calculations yield the equation $Y_2 = 0.1 \sin\left(\frac{\pi}{12}x\right) + 0.8$ (verify this). Graphing these two functions over a 24-hr period yields the graph in Figure 6.150, where we note the high and low values are correct, but the two functions are in phase with each other. As can be determined from Figure 6.149, we want the high tide model to start at the average value and decrease, and the low tide equation model to start at high-low and decrease. Replacing x with $x - 12$ in Y_1 and x with $x + 6$ in Y_2 accomplishes this result (see Figure 6.151). Now comes the fun part! Since Y_1 represents the low/high maximum values for high tide, and Y_2 represents the low/high minimum values for low tide,

Figure 6.150

Figure 6.151

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the amplitude and average value for the tidal motion at Davis Inlet are $A = \dfrac{Y_1 - Y_2}{2}$ and $D = \dfrac{Y_1 + Y_2}{2}$! By entering $Y_3 = \dfrac{Y_1 - Y_2}{2}$ and $Y_4 = \dfrac{Y_1 + Y_2}{2}$, the equation for the tidal motion (with its variable amplitude) will have the form $Y_5 = Y_3 \sin(Bx \pm C) + Y_4$, where the value of B and C must be determined. The key here is to note there is only a 12-hr difference between the changes in amplitude, so $P = 12$ (instead of 24) and $B = \dfrac{\pi}{6}$ for this function. Also, from the graph (Figure 6.149) we note the tidal motion begins at a minimum and increases, indicating a shift of 3 units to the right is required. Replacing x with $x - 3$ gives the equation modeling these tides, and the final equation is $Y_5 = Y_3 \sin\left[\dfrac{\pi}{6}(X - 3)\right] + Y_4$. Figure 6.152 gives a screen shot of Y_1, Y_2, and Y_5 in the interval [0, 24]. The tidal graph from Figure 6.149 is shown in Figure 6.153 with Y_1 and Y_2 superimposed on it.

Figure 6.152

Figure 6.153

Exercise 1: The website www.tides.com/*tcpred*.htm offers both *t*ide and *c*urrent *pred*ictions for various locations around the world, in both numeric and graphical form. In addition, data for the "two" high tides and "two" low tides are clearly highlighted. Select a coastal area where tidal motion is similar to that of Davis Inlet, and repeat this exercise. Compare your model to the actual data given on the website. How good was the fit?

STRENGTHENING CORE SKILLS

Standard Angles, Refer ence Angles, and the T rig Functions

A review of the main ideas discussed in this chapter indicates there are four of what might be called "core skills." These are skills that (a) play a fundamental part in the acquisition of concepts, (b) hold the overall structure together as we move from concept to concept, and (c) are ones we return to again and again throughout our study. The first of these is *(1) knowing the standard angles and standard values.* These values are "standard" because no estimation, interpolation, or special methods are required to name their values, and each can be expressed as a single factor. This gives them a great advantage in that further conceptual development can take place without the main points being obscured by large expressions or decimal approximations. Knowing the values of the trig functions for each standard angle will serve you very well throughout this study. *Know* the chart on page 537 and the ideas that led to it.

The standard angles/values brought us to the trigonometry of any angle, forming a strong bridge to the second core skill: *(2) using reference angles to determine the values of the trig functions in each quadrant.* For review, a 30-60-90 triangle will always have sides that are in the proportion $1x$: $\sqrt{3}x$: $2x$, regardless of its size. This means for any angle θ, where $\theta_r = 30°$, $\sin\theta = \frac{1}{2}$ or $\sin\theta = -\frac{1}{2}$ since the *ratio is fixed* but the sign *depends on the quadrant of* θ: $\sin 30° = \frac{1}{2}$ [QI], $\sin 150° = \frac{1}{2}$ [QII], $\sin 210° = -\frac{1}{2}$ [QIII], $\sin 330° = -\frac{1}{2}$ [QII], and so on (see Figure 6.154).

Figure 6.154

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In turn, the reference angles led us to a third core skill, helping us realize that if θ was not a quadrantal angle, *(3) equations like* $\cos(\theta) = -\frac{\sqrt{3}}{2}$ *must have two solutions in* $[0, 360°)$. From the standard angles and standard values we learn to recognize that for $\cos\theta = -\frac{\sqrt{3}}{2}$, $\theta_r = 30°$, which will occur as a reference angle in the two quadrants where cosine is negative, QII and QIII. The solutions in $[0, 360°)$ are $\theta = 150°$ and $\theta = 210°$ (see Figure 6.155).

Figure 6.155

Of necessity, this brings us to the fourth core skill, *(4) effective use of a calculator.* The standard angles are a wonderful vehicle for introducing the basic ideas of trigonometry, and actually occur quite frequently in real-world applications. But by far, most of the values we encounter will be nonstandard values where θ_r must be found using a calculator. However, once θ_r is found, the reason and reckoning inherent in these ideas can be directly applied.

The *Summary and Concept Review Exercises*, as well as the *Practice Test* offer ample opportunities to refine these skills, so that they will serve you well in future chapters as we continue our attempts to explain and understand the world around us in mathematical terms.

Exercise 1: Fill in the table from memory.

t	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$
$\sin t = y$											
$\cos t = x$											
$\tan t = \frac{y}{x}$											

Exercise 2: Solve each equation in $[0, 2\pi)$ without the use of a calculator.

a. $2\sin t + \sqrt{3} = 0$

b. $-3\sqrt{2}\cos t + 4 = 1$

c. $-\sqrt{3}\tan t + 2 = 1$

d. $\sqrt{2}\sec t + 1 = 3$

Exercise 3: Solve each equation in $[0, 2\pi)$ using a calculator and rounding answers to four decimal places.

a. $\sqrt{6}\sin t - 2 = 1$

b. $-3\sqrt{2}\cos t + \sqrt{2} = 0$

c. $3\tan t + \frac{1}{2} = -\frac{1}{4}$

d. $2\sec t = -5$

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CUMULATIVE REVIEW CHAPTERS 1–6

1. Solve the inequality: $2|x + 1| - 3 < 5$

2. Find the domain of the function: $y = \sqrt{x^2 - 2x - 15}$

3. Given that $\tan\theta = \frac{80}{39}$, draw a right triangle that corresponds to this ratio, then use the Pythagorean theorem to find the length of the missing side. Finally, find the two acute angles.

4. Without a calculator, what values in $[0, 2\pi)$ make the equation true: $\sin t = -\frac{\sqrt{3}}{2}$?

5. Given $\left(\frac{3}{4}, -\frac{\sqrt{7}}{4}\right)$ is a point on the unit circle corresponding to t, find all six trig functions of t.

State the domain and range of each function:

6. $y = f(x)$

7. a. $f(x) = \sqrt{2x - 3}$

b. $g(x) = \frac{2x}{x^2 - 49}$

8. $y = T(x)$

x	$T(x)$
0	−7
1	−5
2	−3
3	−1
4	1
5	3
6	5

9. Analyze the graph of the function in Exercise 6, including: (a) local maximum and minimum values; (b) intervals where $f(x) \geq 0$ and $f(x) < 0$; (c) intervals where f is increasing or decreasing; and (d) any symmetry noted. Assume the features you are to describe have integer values.

10. The attractive force that exists between two magnets varies inversely as the square of the distance between them. If the attractive force is 1.5 newtons (N) at a distance of 10 cm, how close are the magnets when the attractive force reaches 5 N?

11. The world's tallest indoor waterfall is in Detroit, Michigan, in the lobby of the International Center Building. Standing 66 ft from the base of the falls, the angle of elevation to the top of the waterfall is 60°. How tall is the waterfall?

12. It's a warm, lazy Saturday and Hank is watching a county maintenance crew mow the park across the street. He notices the mower takes 29 sec to pass through 77° of rotation from one end of the park to the other. If the corner of the park is 60 ft directly across the street from his house, (a) how wide is the park? (b) How fast (in mph) does the mower travel as it cuts the grass?

13. Graph using transformations of a parent function: $f(x) = \frac{1}{x + 1} - 2$.

14. Graph using transformations of a parent function: $g(x) = e^{x-1} - 2$.

15. Find $f(\theta)$ for all six trig functions, given the point $P(-9, 40)$ is a point on the terminal side of the angle. Then find the angle θ in degrees, rounded to tenths.

16. Given $t = 5.37$, (a) in what quadrant does the arc terminate? (b) What is the reference arc? (c) Find the value of $\sin t$ rounded to four decimal places.

17. A jet-stream water sprinkler shoots water a distance of 15 m and turns back-and-forth through an angle of $t = 1.2$ rad. (a) What is the length of the arc that the sprinkler reaches? (b) What is the area in m^2 of the yard that is watered?

18. Determine the equation of graph shown given it is of the form $y = A\tan(Bt)$.

19. Determine the equation of the graph shown given it is of the form $y = A\sin(Bt \pm C) + D$.

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20. In London, the average temperatures on a summer day range from a high of 72°F to a low of 56°F. Use this information to write a sinusoidal equation model, assuming the low temperature occurs at 6:00 A.M. Clearly state the amplitude, average value, period, and horizontal shift.

21. The graph of a function $f(x)$ is given. Sketch the graph of $f^{-1}(x)$.

22. The volume of a spherical cap is given by $V = \frac{\pi h^2}{3}(3r - h)$. Solve for r in terms of V and h.

23. Find the slope and y-intercept: $3x - 4y = 8$.

24. Solve by factoring: $4x^3 - 8x^2 - 9x + 18 = 0$.

25. At what interest rate will \$1000 grow to \$2275 in 12 yr if compounded continuously?

Exercises 26 through 30 require the use of a graphing calculator

26. Determine the minimum value of the function $f(x) = 0.5x^4 - 3x^3 + e$. Round your answers to one decimal place.

27. Use the **TABLE** feature of a calculator to find the unknown coordinates of the following QII points on the unit circle. Round your answers to two decimal places when necessary.

x	y
−0.2	
−0.4	
−0.6	
−0.8	

28. In 2006, the Podcast Alley directory had cataloged about 26,000 different podcasts with over a million episodes. In 2010, there were over 77,000 different podcasts in this directory. This exponential growth can be modeled by the equation $p = 19.82(1.31)^t$, where p is the number of podcasts (in thousands) registered in a given year t ($t = 0$ corresponds to 2005). According to this model, when did the number of registered podcasts exceed (a) 50,000 and (b) 100,000?

29. Solve the equation and round your answer to two decimal places.

$-2x^2 + 27x + 62 = 11 \ln x$

30. Use the **R → Pθ(** feature of a graphing calculator to find the angle corresponding to the point (12, −32), then evaluate tan θ. Compare your result to $\tan\theta = \frac{y}{x}$.

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CONNECTIONS TO CALCULUS

While right triangles have a number of meaningful applications as a problem-solving tool, they can also help to rewrite certain expressions in preparation for the tools of calculus, and introduce us to an alternative method for graphing relations and functions using polar coordinates.

Right Triangle Relationships

Drawing a diagram to visualize relationships and develop information is an important element of good problem solving. This is no less true in calculus, where it is often a fundamental part of understanding the question being asked. As a precursor to applications involving *trig substitutions*, we'll illustrate how right triangle diagrams are used to rewrite trigonometric functions of θ as algebraic functions of x.

EXAMPLE 1 ▶ Using Right Triangle Diagrams to Rewrite Trig Expressions

Use the equation $x = 5 \sin \theta$ and a right triangle diagram to write $\cos \theta$, $\tan \theta$, $\sec \theta$, and $\csc \theta$ as functions of x.

Solution ▶ Using $x = 5 \sin \theta$, we obtain $\sin \theta = \dfrac{x}{5}$. From our work in Chapter 6, we know the right triangle definition of $\sin \theta$ is $\dfrac{\text{opp}}{\text{hyp}}$, and we draw a triangle with side x oriented opposite an angle θ, and label a hypotenuse of 5 (see figure). To find an expression for the adjacent side, we use the Pythagorean theorem:

$$(\text{adj})^2 + x^2 = 5^2 \qquad \text{Pythagorean theorem}$$
$$(\text{adj})^2 = 25 - x^2 \qquad \text{isolate term}$$
$$\text{adj} = \sqrt{25 - x^2} \qquad \text{result (length must be positive); } -5 < x < 5$$

Using this triangle and the standard definition of the remaining trig functions, we find $\cos \theta = \dfrac{\sqrt{25 - x^2}}{5}$, $\tan \theta = \dfrac{x}{\sqrt{25 - x^2}}$, $\sec \theta = \dfrac{5}{\sqrt{25 - x^2}}$, and $\csc \theta = \dfrac{5}{x}$.

Now try Exercises 1 through 4 ▶

EXAMPLE 2 ▶ Using Right Triangle Diagrams to Rewrite Trig Expressions

Find expressions for $\tan \theta$ and $\csc \theta$, given $\sec \theta = \dfrac{\sqrt{u^2 + 144}}{u}$.

Solution ▶ With $\sec \theta = \dfrac{\text{hyp}}{\text{adj}}$, we draw a right triangle diagram as in Example 1, with a hypotenuse of $\sqrt{u^2 + 144}$ and a side u adjacent to angle θ. For $\tan \theta = \dfrac{\text{opp}}{\text{adj}}$ and $\csc \theta = \dfrac{\text{hyp}}{\text{opp}}$, we use the Pythagorean theorem to find an expression for the opposite side:

$$(\text{opp})^2 + u^2 = (\sqrt{u^2 + 144})^2 \qquad \text{Pythagorean theorem}$$
$$(\text{opp})^2 + u^2 = u^2 + 144 \qquad \text{square radical}$$
$$(\text{opp})^2 = 144 \qquad \text{subtract } u^2$$
$$\text{opp} = 12 \qquad \text{result (length must be positive)}$$

With an opposite side of 12 units, the figure shows $\tan \theta = \dfrac{12}{u}$ and $\csc \theta = \dfrac{\sqrt{u^2 + 144}}{12}$.

Now try Exercises 5 and 6 ▶

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Converting from Rectangular Coordinates to Trigonometric (Polar) Form

Using the equations $\cos\theta = \frac{x}{r}$, $\sin\theta = \frac{y}{r}$, and $r = \sqrt{x^2 + y^2}$ from Section 6.7, we can derive $x = r\cos\theta$, $y = r\sin\theta$, and $x^2 + y^2 = r^2$. With these equations, we can rewrite functions of x given in rectangular form as equations of θ in trigonometric (*polar*) form. In Chapter 10 we'll see how this offers us certain advantages. Note that while algebraic equations are often written with y in terms of x, polar equations are written with r in terms of θ.

EXAMPLE 3 ▶ **Converting from Rectangular to Trigonometric Form**

Rewrite the equation $2x + 3y = 6$ in trigonometric form using the substitutions indicated and solving for r. Note the given equation is that of a line with x-intercept (3, 0) and y-intercept (0, 2).

Solution ▶ Using $x = r\cos\theta$ and $y = r\sin\theta$ we proceed as follows:

$$2x + 3y = 6 \quad \text{given}$$
$$2(r\cos\theta) + 3(r\sin\theta) = 6 \quad \text{substitute } r\cos\theta \text{ for } x,\ r\sin\theta \text{ for } y$$
$$r[2\cos\theta + 3\sin\theta] = 6 \quad \text{factor out } r$$
$$r = \frac{6}{2\cos\theta + 3\sin\theta} \quad \text{solve for } r$$

This is the equation of the same line, but in trigonometric form.

Now try Exercises 7 through 10 ▶

While it is somewhat simplistic (there are other subtleties involved), we can verify the equation obtained in Example 3 produces the same line as $2x + 3y = 6$ by evaluating the equation at $\theta = 0°$ to find a point on the positive x-axis, $\theta = 90°$ to find a point of the positive y-axis, and $\theta = 135°$ to find a point in QII.

For $\theta = 0°$:
$$r = \frac{6}{2\cos 0° + 3\sin 0°} = \frac{6}{2(1) + 3(0)} = \frac{6}{2} = 3$$

For $\theta = 90°$:
$$r = \frac{6}{2\cos 90° + 3\sin 90°} = \frac{6}{2(0) + 3(1)} = \frac{6}{3} = 2$$

For $\theta = 135°$:
$$r = \frac{6}{2\cos 135° + 3\sin 135°} = \frac{6}{2\left(-\frac{\sqrt{2}}{2}\right) + 3\left(\frac{\sqrt{2}}{2}\right)} = \frac{6}{\frac{\sqrt{2}}{2}} = \frac{12}{\sqrt{2}} = 6\sqrt{2} \approx 8.5$$

Using a distance r from the origin at each θ given, we note that all three points are on the line $2x + 3y = 6$, as shown in the figure.

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EXAMPLE 4 ▶ **Converting from Polar to Rectangular Form**

Rewrite the equation $r = 4\cos\theta$ in rectangular form using the relationships $x = r\cos\theta$, $y = r\sin\theta$, and/or $x^2 + y^2 = r^2$. Identify the resulting equation.

Solution ▶ From $x = r\cos\theta$ we have $\cos\theta = \frac{x}{r}$. Substituting into the given equation we have

$r = 4\cos\theta$	given
$r = 4\left(\frac{x}{r}\right)$	substitute $\frac{x}{r}$ for $\cos\theta$
$r^2 = 4x$	multiply by r
$x^2 + y^2 = 4x$	substitute $x^2 + y^2$ for r^2
$x^2 - 4x + y^2 = 0$	set equal to 0
$(x^2 - 4x + 4) + y^2 = 4$	complete the square in x
$(x - 2)^2 + y^2 = 4$	standard form

The result shows $r = 4\cos\theta$ is a trigonometric form for the equation of a circle with center at (2, 0) and radius $r = 2$.

Now try Exercises 11 through 14 ▶

Connections to Calculus Exercises

Use the diagram given to find the remaining side of the triangle. Then write the six trig functions as functions of x.

1.

2.

Use the equation given and a sketch of the corresponding right triangle diagram to write the remaining five trig functions as functions of x.

3. $x = 4\tan\theta$

4. $x = 5\sec\theta$

5. Find expressions for $\cot\theta$ and $\sec\theta$, given $\csc\theta = \frac{\sqrt{u^2 + 169}}{u}$.

6. Find expressions for $\sin\theta$ and $\cos\theta$, given $\cot\theta = \frac{2\sqrt{5}}{x}$.

Use $x = r\cos\theta$, $y = r\sin\theta$, and $x^2 + y^2 = r^2$ to rewrite the expressions in trigonometric form.

7. $y = 2$

8. $y = \frac{1}{4}x^2$

9. $y = -2x + 3$

10. $(x - 2)^2 + y^2 = 4$

Use $x = r\cos\theta$, $y = r\sin\theta$ and $x^2 + y^2 = r^2$ to rewrite the expressions in rectangular form, then identify the equation as that of a line, circle, vertical parabola, or horizontal parabola.

11. $r = 5\sin\theta$

12. $r = \frac{4}{1 + \sin\theta}$ $\left(\textit{Hint}: \sin\theta = \frac{y}{r}.\right)$

13. $r(3\cos\theta - 2\sin\theta) = 6$

14. $r = \frac{4}{2 - 2\cos\theta}$ $\left(\textit{Hint}: \cos\theta = \frac{x}{r}.\right)$