Mathematics Learning Centre

The University of Sydney

Introduction to Trigonometric Functions

Peggy Adamson and Jackie Nicholas

Acknowledgements

A significant part of this manuscript has previously appeared in a version of this booklet published in 1986 by Peggy Adamson. In rewriting this booklet, I have relied a great deal on Peggy's ideas and approach for Chapters 1, 2, 3, 4, 5 and 7. Chapter 6 appears in a similar form in the booklet, Introduction to Differential Calculus, which was written by Christopher Thomas.

In her original booklet, Peggy acknowledged the contributions made by Mary Barnes and Sue Gordon. I would like to extend this list and thank Collin Phillips for his hours of discussion and suggestions.

Jackie Nicholas
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Contents

1 Introduction

You have probably met the trigonometric ratios cosine, sine, and tangent in a right angled triangle, and have used them to calculate the sides and angles of those triangles.

In this booklet we review the definition of these trigonometric ratios and extend the concept of cosine, sine and tangent. We define the cosine, sine and tangent as functions of all real numbers. These trigonometric functions are extremely important in science, engineering and mathematics, and some familiarity with them will be assumed in most first year university mathematics courses.

In Chapter 2 we represent an angle as radian measure and convert degrees to radians and radians to degrees. In Chapter 3 we review the definition of the trigonometric ratios in a right angled triangle. In Chapter 4, we extend these ideas and define cosine, sine and tangent as functions of real numbers. In Chapter 5, we discuss the properties of their graphs. Chapter 6 looks at derivatives of these functions and assumes that you have studied calculus before. If you haven't done so, then skip Chapter 6 for now. You may find the Mathematics Learning Centre booklet: *Introduction to Differential Calculus* useful if you need to study calculus. Chapter 7 gives a brief look at inverse trigonometric functions.

1.1 How to use this booklet

You will not gain much by just reading this booklet. Mathematics is not a spectator sport! Rather, have pen and paper ready and try to work through the examples before reading their solutions. Do **all** the exercises. It is important that you try hard to complete the exercises, rather than refer to the solutions as soon as you are stuck.

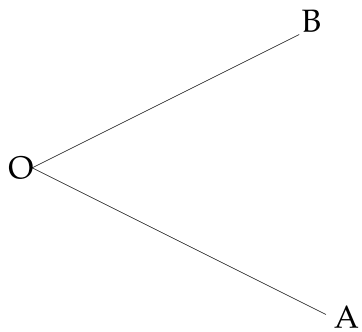
1.2 Objectives

By the time you have completed this booklet you should:

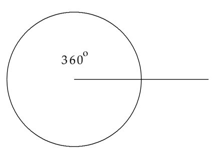
- know what a radian is and know how to convert degrees to radians and radians to degrees;
- know how cos, sin and tan can be defined as ratios of the sides of a right angled triangle;
- know how to find the cos, sin and tan of $\frac{\pi}{6}$, $\frac{\pi}{4}$ and $\frac{\pi}{2}$;
- know how cos, sin and tan functions are defined for all real numbers;
- be able to sketch the graph of certain trigonometric functions;
- know how to differentiate the cos, sin and tan functions;
- understand the definition of the inverse function $f^{-1}(x) = \cos^{-1}(x)$.

2 Angles and Angular Measure

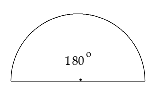
An angle can be thought of as the amount of rotation required to take one straight line to another line with a common point. Angles are often labelled with Greek letters, for example θ. Sometimes an arrow is used to indicate the direction of the rotation. If the arrow points in an anticlockwise direction, the angle is positive. If it points clockwise, the angle is negative.



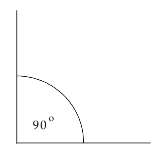
Angles can be measured in degrees or radians. Measurement in degrees is based on dividing the circumference of the circle into 360 equal parts. You are probably familiar with this method of measurement.



A complete revolution is 360°.



A straight angle is 180°.

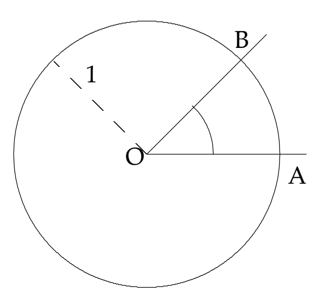


A right angle is 90°.

Fractions of a degree are expressed in minutes ($'$) and seconds ($''$). There are sixty seconds in one minute, and sixty minutes in one degree. So an angle of $31°17'$ can be expressed as $31 + \frac{17}{60} = 31.28°$.

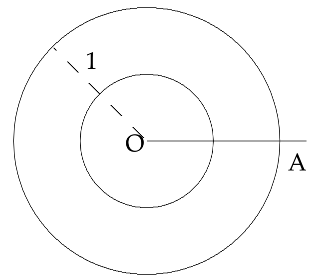
The radian is a natural unit for measuring angles. We use radian measure in calculus because it makes the derivatives of trigonometric functions simple. You should try to get used to thinking in radians rather than degrees.

To measure an angle in radians, construct a unit circle (radius 1) with centre at the vertex of the angle. The radian measure of an angle AOB is defined to be the length of the circular arc AB around the circumference.

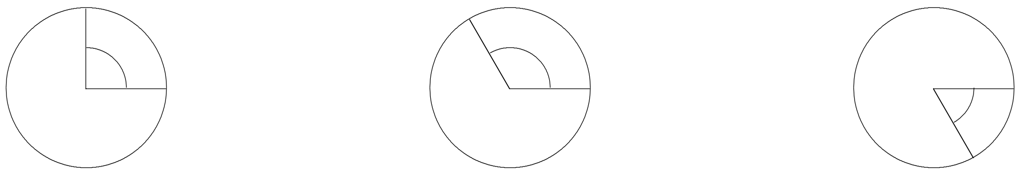


This definition can be used to find the number of radians corresponding to one complete revolution.

In a complete revolution, A moves anticlockwise around the whole circumference of the unit circle, a distance of 2π. So a complete revolution is measured as 2π radians. That is, 2π radians corresponds to $360°$.



Fractions of a revolution correspond to angles which are fractions of 2π.



$\frac{1}{4}$ revolution $90°$ or $\frac{\pi}{2}$ radians

$\frac{1}{3}$ revolution $120°$ or $\frac{2\pi}{3}$ radians

$-\frac{1}{6}$ revolution $-60°$ or $-\frac{\pi}{3}$ radians

2.1 Converting from radians to degrees and degrees to radians

Since 2π radians is equal to $360°$

$$\begin{aligned} \pi \text{ radians} &= 180°, \\ 1 \text{ radian} &= \frac{180°}{\pi} \\ &= 57.3°, \\ y \text{ radians} &= y \times \frac{180°}{\pi}, \end{aligned}$$

and similarly

$$\begin{aligned} 1° &= \frac{\pi}{180} \text{ radians}, \\ &\approx 0.017, \\ y° &= y \times \frac{\pi}{180} \text{ radians}. \end{aligned}$$

Your calculator has a key that enters the approximate value of π.

If you are going to do calculus, it is important to get used to thinking in terms of radian measure. In particular, think of:

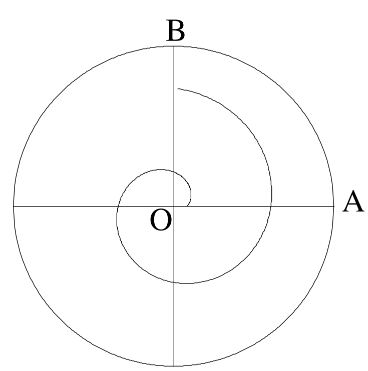
$$\begin{array}{rcll} 180^\circ & \text{as} & \pi & \text{radians,} \\ 90^\circ & \text{as} & \dfrac{\pi}{2} & \text{radians,} \\ 60^\circ & \text{as} & \dfrac{\pi}{3} & \text{radians,} \\ 45^\circ & \text{as} & \dfrac{\pi}{4} & \text{radians,} \\ 30^\circ & \text{as} & \dfrac{\pi}{6} & \text{radians.} \end{array}$$

You should make sure you are really familiar with these.

2.2 Real numbers as radians

Any real number can be thought of as a radian measure if we express the number as a multiple of 2π.

For example, $\dfrac{5\pi}{2} = 2\pi \times (1 + \dfrac{1}{4}) = 2\pi + \dfrac{\pi}{2}$ corresponds to the arc length of $1\frac{1}{4}$ revolutions of the unit circle going anticlockwise from A to B.



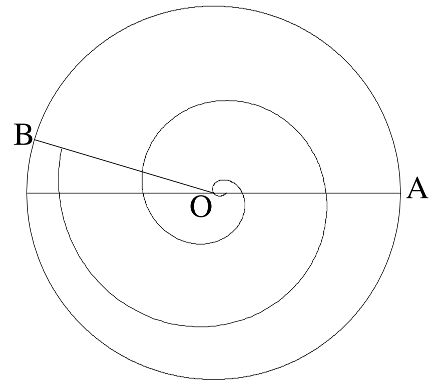
Similarly,

$$\begin{aligned} 27 &\approx 4.297 \times 2\pi \\ &= 4 \times 2\pi + 0.297 \times 2\pi \end{aligned}$$

corresponds to an arc length of 4.297 revolutions of the unit circle going anticlockwise.

We can also think of negative numbers in terms of radians. Remember for negative radians we measure arc length **clockwise** around the unit circle.

For example,
$-16 \approx -2.546 \times 2\pi = -2 \times 2\pi + -0.546 \times 2\pi$
corresponds to the arc length of approximately 2.546 revolutions of the unit circle going clockwise from A to B.



We are, in effect, wrapping the positive real number line anticlockwise around the unit circle and the negative real number line clockwise around the unit circle, starting in each case with 0 at A, $(1, 0)$.

By doing so we are associating each and every real number with exactly one point on the unit circle. Real numbers that have a difference of 2π (or a multiple of 2π) correspond to the same point on the unit circle. Using one of our previous examples, $\frac{5\pi}{2}$ corresponds to $\frac{\pi}{2}$ as they differ by a multiple of 2π.

2.2.1 Exercise

Write the following in both degrees and radians and represent them on a diagram.

a. $30°$ **b.** 1 **c.** $120°$

d. $\frac{3\pi}{4}$ **e.** 2 **f.** $\frac{4\pi}{3}$

g. $270°$ **h.** -1 **i.** $-\frac{\pi}{2}$

Note that we do not indicate the units when we are talking about radians.

In the rest of this booklet, we will be using radian measure only. You'll need to make sure that your calculator is in radian mode.

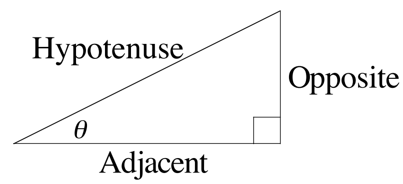
3 Trigonometric Ratios in a Right Angled Triangle

If you have met trigonometry before, you probably learned definitions of $\sin\theta$, $\cos\theta$ and $\tan\theta$ which were expressed as ratios of the sides of a right angled triangle.

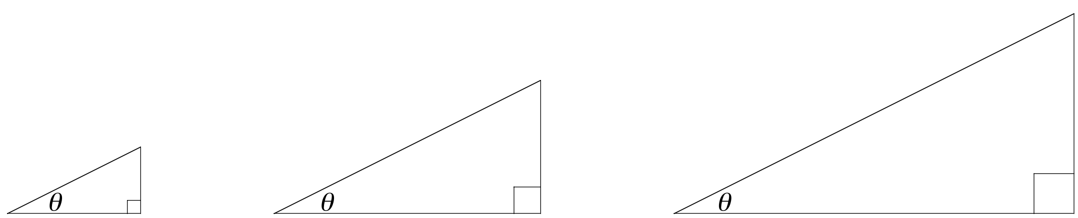
These definitions are repeated here, just to remind you, but we shall go on, in the next section, to give a much more useful definition.

3.1 Definition of sine, cosine and tangent

In a right angled triangle, the side opposite to the right angle is called the *hypotenuse*. If we choose one of the other angles and label it θ, the other sides are often called *opposite* (the side opposite to θ) and *adjacent* (the side next to θ).



For a given θ, there is a whole family of right angled triangles, that are triangles of different sizes but are the same shape.



For each of the triangles above, the ratios of corresponding sides have the same values.

The ratio $\dfrac{\text{adjacent}}{\text{hypotenuse}}$ has the same value for each triangle. This ratio is given a special name, the *cosine* of θ or $\cos\theta$.

The ratio $\dfrac{\text{opposite}}{\text{hypotenuse}}$ has the same value for each triangle. This ratio is the *sine* of θ or $\sin\theta$.

The ratio $\dfrac{\text{opposite}}{\text{adjacent}}$ takes the same value for each triangle. This ratio is called the *tangent* of θ or $\tan\theta$.

Summarising,

$$\begin{aligned}\cos\theta &= \frac{\text{adjacent}}{\text{hypotenuse}},\\ \sin\theta &= \frac{\text{opposite}}{\text{hypotenuse}},\\ \tan\theta &= \frac{\text{opposite}}{\text{adjacent}}.\end{aligned}$$

The values of these ratios can be found using a calculator. Remember, we are working in radians so your calculator must be in radian mode.

3.1.1 Exercise

Use your calculator to evaluate the following. Where appropriate, compare your answers with the exact values for the special trigonometric ratios given in the next section.

a. $\sin\frac{\pi}{6}$ **b.** $\tan 1$ **c.** $\cos\frac{\pi}{3}$ **d.** $\tan\frac{\pi}{4}$

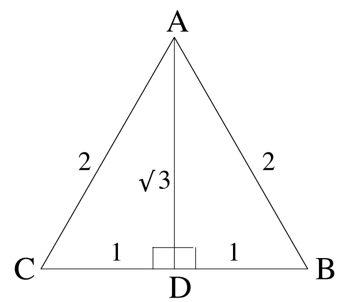
e. $\sin 1.5$ **f.** $\tan\frac{\pi}{3}$ **g.** $\cos\frac{\pi}{6}$ **h.** $\sin\frac{\pi}{3}$

3.2 Some special trigonometric ratios

You will need to be familiar with the trigonometric ratios of $\frac{\pi}{6}$, $\frac{\pi}{3}$ and $\frac{\pi}{4}$.
The ratios of $\frac{\pi}{6}$ and $\frac{\pi}{3}$ are found with the aid of an equilateral triangle ABC with sides of length 2.

$\angle$BAC is bisected by AD, and $\angle$ADC is a right angle. Pythagoras' theorem tells us that the length of AD = $\sqrt{3}$.

$\angle$ACD $= \frac{\pi}{3}$.
$\angle$DAC $= \frac{\pi}{6}$.

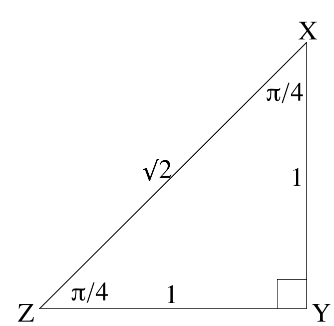


$$\cos\frac{\pi}{3} = \frac{1}{2}, \quad \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}, \quad \tan\frac{\pi}{3} = \sqrt{3}.$$

$$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}, \quad \sin\frac{\pi}{6} = \frac{1}{2}, \quad \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}.$$

The ratios of $\frac{\pi}{4}$ are found with the aid of an isosceles right angled triangle XYZ with the two equal sides of length 1.

Pythagoras' theorem tells us that the hypotenuse of the triangle has length $\sqrt{2}$.



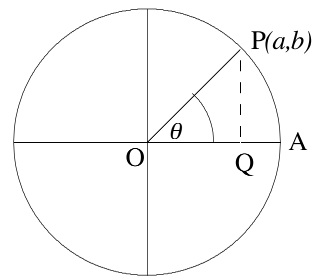
$$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}, \quad \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}, \quad \tan\frac{\pi}{4} = 1.$$

4 The Trigonometric Functions

The definitions in the previous section apply to θ between 0 and $\frac{\pi}{2}$, since the angles in a right angle triangle can never be greater than $\frac{\pi}{2}$. The definitions given below are useful in calculus, as they extend $\sin\theta$, $\cos\theta$ and $\tan\theta$ without restrictions on the value of θ.

4.1 The cosine function

Let's begin with a definition of $\cos\theta$.
Consider a circle of radius 1, with centre O at the origin of the (x, y) plane. Let A be the point on the circumference of the circle with coordinates $(1, 0)$. OA is a radius of the circle with length 1. Let P be a point on the circumference of the circle with coordinates (a, b). We can represent the angle between OA and OP, θ, by the arc length along the unit circle from A to P. This is the radian representation of θ.



The cosine of θ is defined to be the x coordinate of P.

Let's, for the moment, consider values of θ between 0 and $\frac{\pi}{2}$. The cosine of θ is written $\cos\theta$, so in the diagram above, $\cos\theta = a$. Notice that as θ increases from 0 to $\frac{\pi}{2}$, $\cos\theta$ decreases from 1 to 0.

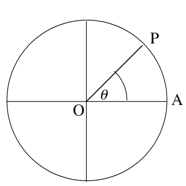
For values of θ between 0 and $\frac{\pi}{2}$, this definition agrees with the definition of $\cos\theta$ as the ratio $\dfrac{\text{adjacent}}{\text{hypotenuse}}$ of the sides of a right angled triangle.

Draw PQ perpendicular OA. In $\triangle$ OPQ, the hypotenuse OP has length 1, while OQ has length a.

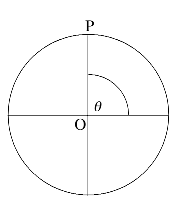
$$\text{The ratio } \frac{\text{adjacent}}{\text{hypotenuse}} = a = \cos\theta.$$

The definition of $\cos\theta$ using the unit circle makes sense for all values of θ. For now, we will consider values of θ between 0 and 2π.

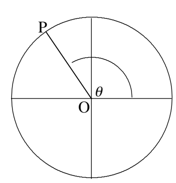
The x coordinate of P gives the value of $\cos\theta$. When $\theta = \frac{\pi}{2}$, P is on the y axis, and it's x coordinate is zero. As θ increases beyond $\frac{\pi}{2}$, P moves around the circle into the second quadrant and therefore it's x coordinate will be negative. When $\theta = \pi$, the x coordinate is -1.



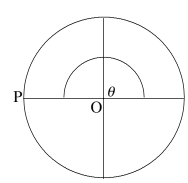
$\cos\theta$ is positive



$\cos\frac{\pi}{2} = 0$

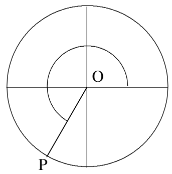


$\cos\theta$ negative

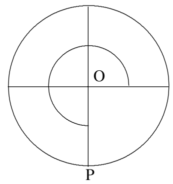


$\cos\pi = -1$

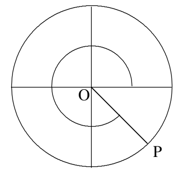
As θ increases further, P moves around into the third quadrant and its x coordinate increases from -1 to 0. Finally as θ increases from $\frac{3\pi}{2}$ to 2π the x coordinate of P increases from 0 to 1.



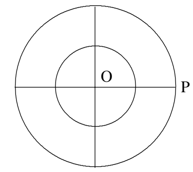
$\cos\theta$ is negative



$\cos\frac{3\pi}{2} = 0$



$\cos\theta$ positive



$\cos 2\pi = 1$

4.1.1 Exercise

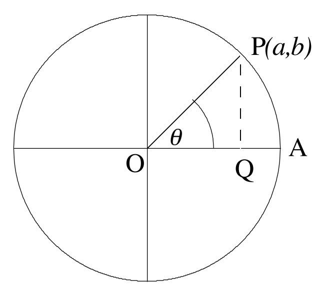
1. Use the cosine (cos) key on your calculator to complete this table. (Make sure your calculator is in radian mode.)

θ	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$
$\cos\theta$									
θ	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	2π
$\cos\theta$									

2. Using this table plot the graph of $y = \cos\theta$ for values of θ ranging from 0 to 2π.

4.2 The sine function

The sine of θ is defined using the same unit circle diagram that we used to define the cosine.



The sine of θ is defined to be the y coordinate of P.

The sine of θ is written as $\sin\theta$, so in the diagram above, $\sin\theta = b$.

For values of θ between 0 and $\frac{\pi}{2}$, this definition agrees with the definition of $\sin\theta$ as the ratio $\dfrac{\text{opposite}}{\text{hypotenuse}}$ of sides of a right angled triangle.

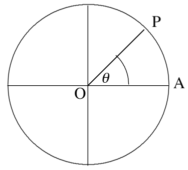
In the right angled triangle OQP, the hypotenuse OP has length 1 while PQ has length b.

$$\text{The ratio } \frac{\text{opposite}}{\text{hypotenuse}} = \frac{b}{1} = \sin\theta.$$

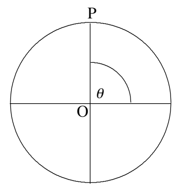
This definition of $\sin\theta$ using the unit circle extends to all values of θ. Here, we will consider values of θ between 0 and 2π.

As P moves anticlockwise around the circle from A to B, θ increases from 0 to $\frac{\pi}{2}$. When P is at A, $\sin\theta = 0$, and when P is at B, $\sin\theta = 1$. So as θ increases from 0 to $\frac{\pi}{2}$, $\sin\theta$ increases from 0 to 1. The largest value of $\sin\theta$ is 1.

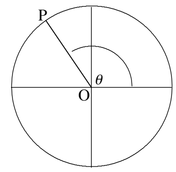
As θ increases beyond $\frac{\pi}{2}$, $\sin\theta$ decreases and equals zero when $\theta = \pi$. As θ increases beyond π, $\sin\theta$ becomes negative.



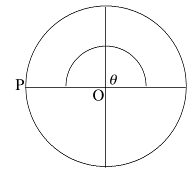
$\sin\theta$ is positive



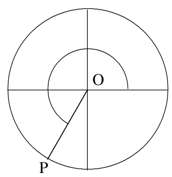
$\sin\frac{\pi}{2} = 1$



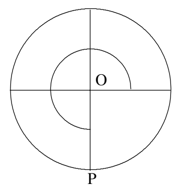
$\sin\theta$ positive



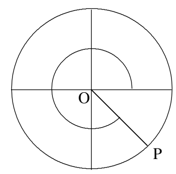
$\sin\pi = 0$



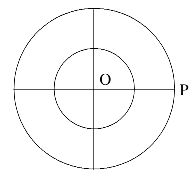
$\sin\theta$ is negative



$\sin\frac{3\pi}{2} = -1$



$\sin\theta$ negative



$\sin 2\pi = 0$

4.2.1 Exercise

1. Use the sin key on your calculator to complete this table. Make sure your calculator is in radian mode.

θ	0	0.2	0.4	0.6	0.8	1	1.2	1.4	1.6
$\sin\theta$									
θ	2	2.4	2.8	3.2	3.6	4.0	4.6	5.4	6.2
$\sin\theta$									

2. Plot the graph of the $y = \sin\theta$ using the table in the previous exercise.

4.3 The tangent function

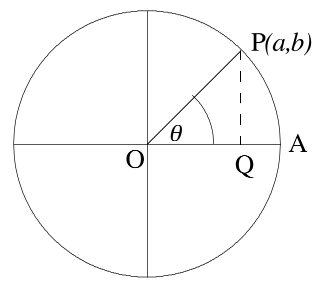
We can define the tangent of θ, written $\tan\theta$, in terms of $\sin\theta$ and $\cos\theta$.

$$\tan\theta = \frac{\sin\theta}{\cos\theta}.$$

Using this definition we can work out $\tan\theta$ for values of θ between 0 and 2π. You will be asked to do this in Exercise 3.3. In particular, we know from this definition that $\tan\theta$ is *not* defined when $\cos\theta = 0$. This occurs when $\theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}$.

When $0 < \theta < \frac{\pi}{2}$ this definition agrees with the definition of $\tan\theta$ as the ratio $\dfrac{\text{opposite}}{\text{adjacent}}$ of the sides of a right angled triangle.

As before, consider the unit circle with points O, A and P as shown. Drop a perpendicular from the point P to OA which intersects OA at Q. As before P has coordinates (a, b) and Q coordinates $(a, 0)$.



$$\begin{aligned}\frac{\text{opposite}}{\text{adjacent}} &= \frac{\text{PQ}}{\text{OQ}} \quad \text{(in triangle OPQ)}\\ &= \frac{b}{a}\\ &= \frac{\sin\theta}{\cos\theta}\\ &= \tan\theta.\end{aligned}$$

If you try to find $\tan\frac{\pi}{2}$ using your calculator, you will get an error message. Look at the definition. The tangent of $\frac{\pi}{2}$ is *not defined* as $\cos\frac{\pi}{2} = 0$. For values of θ near $\frac{\pi}{2}$, $\tan\theta$ is very large. Try putting some values in your calculator. (eg $\frac{\pi}{2} \approx 1.570796$. Try $\tan(1.57)$, $\tan(1.5707)$, $\tan(1.57079)$.)

4.3.1 Exercise

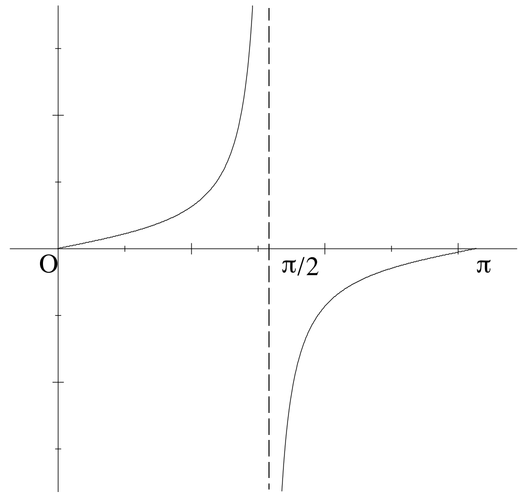
1. Use the tan key on your calculator to complete this table. Make sure your calculator is in radian mode.

θ	0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.50	1.65	2	2.4
$\tan\theta$												
θ	2.8	3.2	3.6	4.0	4.4	4.6	4.65	4.78	5.0	5.6	6.0	6.28
$\tan\theta$												

2. Use the table above to graph $\tan\theta$.

Your graph should look like this for values of θ between 0 and π.

Notice that there is a vertical asymptote at $\theta = \frac{\pi}{2}$. This is because $\tan\theta$ is not defined at $\theta = \frac{\pi}{2}$. You will find another vertical asymptote at $\theta = \frac{3\pi}{2}$. When $\theta = 0$ or π, $\tan\theta = 0$. For θ greater than 0 and less than $\frac{\pi}{2}$, $\tan\theta$ is positive. For values of θ greater than $\frac{\pi}{2}$ and less than π, $\tan\theta$ is negative.



4.4 Extending the domain

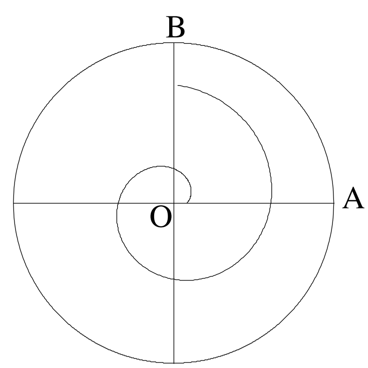
The definitions of sine, cosine and tangent can be extended to all real values of θ in the following way.

$\dfrac{5\pi}{2} = 2\pi + \dfrac{\pi}{2}$ corresponds to the arc length of $1\frac{1}{4}$ revolutions around the unit circle going anticlockwise from A to B.
Since B has coordinates $(0, 1)$ we can use the previous definitions to get:

$$\sin\tfrac{5\pi}{2} = 1,$$

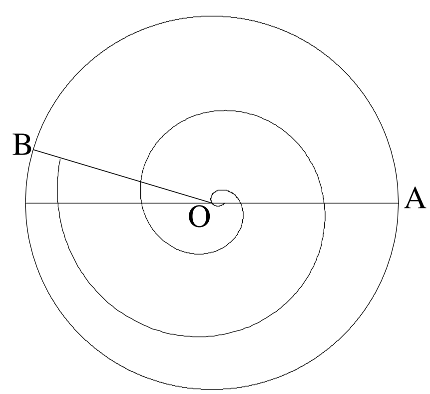
$$\cos\tfrac{5\pi}{2} = 0,$$

$$\tan\tfrac{5\pi}{2} \text{ is undefined.}$$



Similarly,

$$\begin{aligned}
-16 &\approx -2.546 \times 2\pi \\
&= -2 \times 2\pi + -0.546 \times 2\pi, \\
\sin(-16) &\approx \sin(-0.546 \times 2\pi) \\
&\approx 0.29, \\
\cos(-16) &\approx -0.96, \\
\tan(-16) &\approx -0.30.
\end{aligned}$$



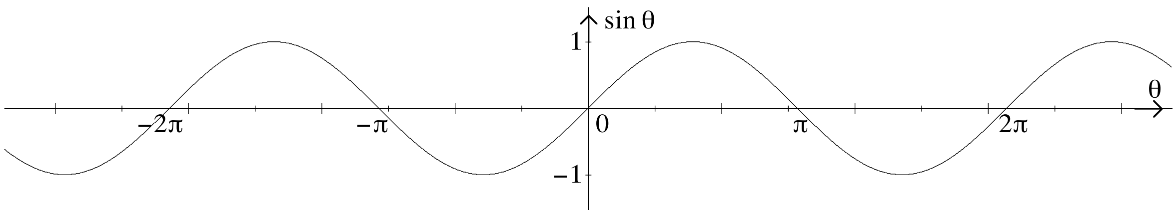
4.4.1 Exercise

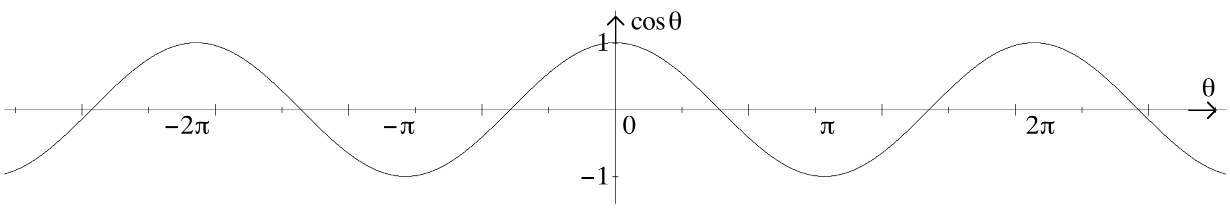
Evaluate the following trig functions giving exact answers where you are able.

1. $\sin \frac{15\pi}{2}$
2. $\tan \frac{13\pi}{6}$
3. $\cos 15$
4. $\tan \frac{-14\pi}{3}$
5. $\sin \frac{23\pi}{6}$

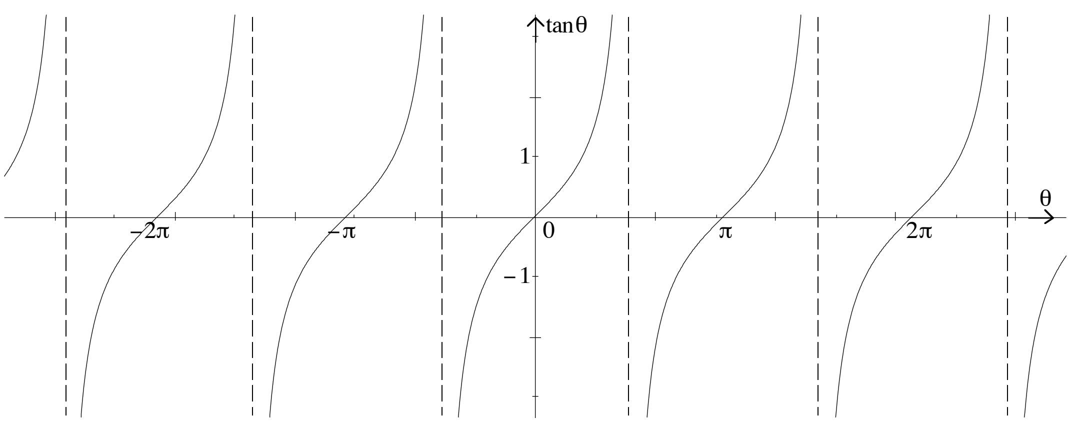
Notice

The values of sine and cosine functions repeat after every interval of length 2π. Since the real numbers x, $x + 2\pi$, $x - 2\pi$, $x + 4\pi$, $x - 4\pi$ etc differ by a multiple of 2π, they correspond to the same point on the unit circle. So, $\sin x = \sin(x + 2\pi) = \sin(x - 2\pi) = \sin(x + 4\pi) = \sin(x - 4\pi)$ etc. We can see the effect of this in the functions below and will discuss it further in the next chapter.



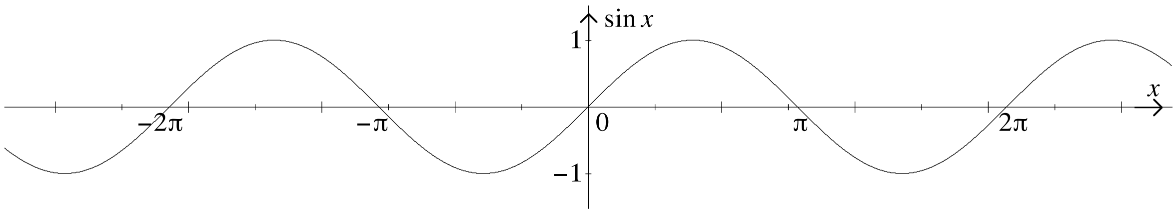


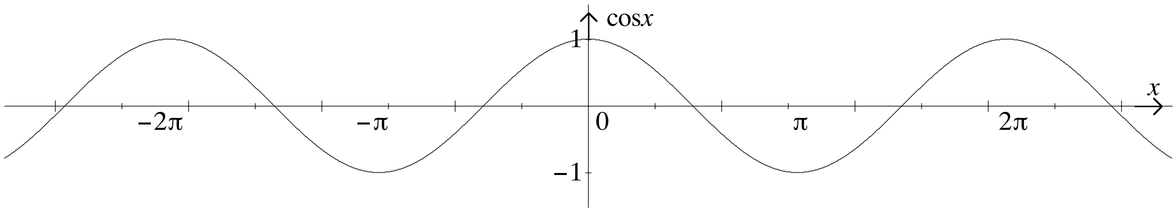
The tangent function repeats after every interval of length π.



5 Graphs of Trigonometric Functions

In this section we use our knowledge of the graphs $y = \sin x$ and $y = \cos x$ to sketch the graphs of more complex trigonometric functions.





Let's look first at some important features of these two graphs.

The shape of each graph is repeated after every interval of length 2π.

This makes sense when we think of the way we have defined sin and cos using the unit circle.

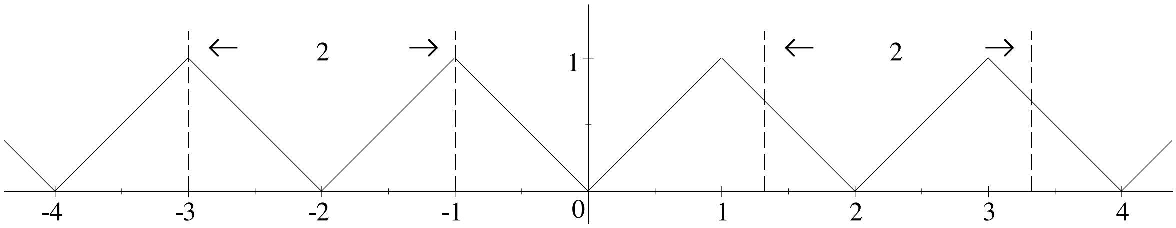
We say that these functions are *periodic* with *period* 2π.

The sin and cos functions are the most famous examples of a class of functions called periodic functions.

> Functions with the property that $f(x) = f(x + a)$ for all x are called periodic functions. Such a function is said to have period a.

This means that the function repeats itself after every interval of length a.

Note that you can have periodic functions that are *not* trigonometric functions. For example, the function below is periodic with period 2.



The values of the functions $y = \sin x$ and $y = \cos x$ oscillate between -1 and 1. We say that $y = \sin x$ and $y = \cos x$ have *amplitude* 1. A general definition for the amplitude of any periodic function is:

> The amplitude of a periodic function is half the distance between its minimum and maximum values.

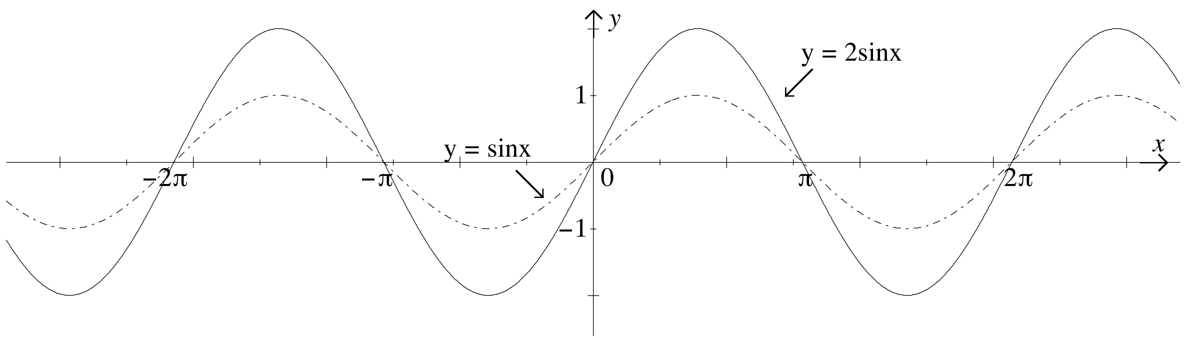
Also, the functions $y = \sin x$ and $y = \cos x$ oscillate about the x-axis. We refer to the x-axis as the *mean level* of these functions, or say that they have a mean level of 0.

We notice that the graphs of $\sin x$ and $\cos x$ have the same shape. The graph of $\sin x$ looks like the graph of $\cos x$ shifted to the right by $\frac{\pi}{2}$ units. We say that the *phase difference* between the two functions is $\frac{\pi}{2}$.

Other trigonometric functions can be obtained by modifying the graphs of $\sin x$ and $\cos x$.

5.1 Changing the amplitude

Consider the graph of the function $y = 2\sin x$.



The graph of $y = 2\sin x$ has the same period as $y = \sin x$ but has been stretched in the y direction by a factor of 2. That is, for every value of x the y value for $y = 2\sin x$ is twice the y value for $y = \sin x$.

So, the amplitude of the function $y = 2\sin x$ is 2. Its period is 2π.

In general we can say that the amplitude of the function $y = a\sin x$ is a, since in this case $y = a\sin x$ oscillates between $-a$ and a.

What happens if a is negative? See the solution to number 3 of the following exercise.

5.1.1 Exercise

Sketch the graphs of the following functions.

1. $y = 3\cos x$

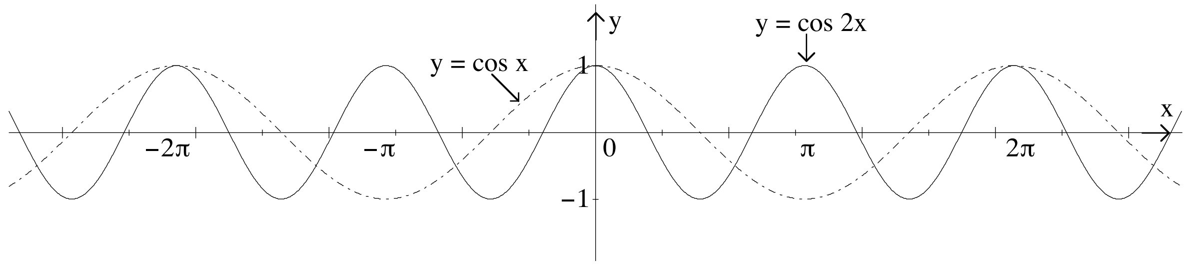
2. $y = \frac{1}{2}\sin x$

3. $y = -3\cos x$

5.2 Changing the period

Let's consider the graph of $y = \cos 2x$.

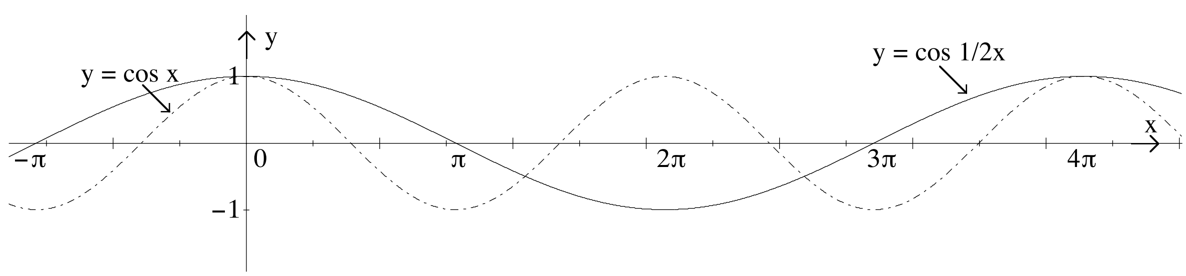
To sketch the graph of $y = \cos 2x$, first think about some specific points. We will look at the points where the function $y = \cos x$ equals 0 or ± 1.



$\cos x = 1$	when $x = 0$,	so $\cos 2x = 1$	when $2x = 0$,	ie $x = 0$.
$\cos x = 0$	when $x = \frac{\pi}{2}$,	so $\cos 2x = 0$	when $2x = \frac{\pi}{2}$,	ie $x = \frac{\pi}{4}$.
$\cos x = -1$	when $x = \pi$,	so $\cos 2x = -1$	when $2x = \pi$,	ie $x = \frac{\pi}{2}$.
$\cos x = 0$	when $x = \frac{3\pi}{2}$,	so $\cos 2x = 0$	when $2x = \frac{3\pi}{2}$,	ie $x = \frac{3\pi}{4}$.
$\cos x = 1$	when $x = 2\pi$,	so $\cos 2x = 1$	when $2x = 2\pi$,	ie $x = \pi$.

As we see from the graph, the function $y = \cos 2x$ has a period of π. The function still oscillates between the values -1 and 1, so its amplitude is 1.

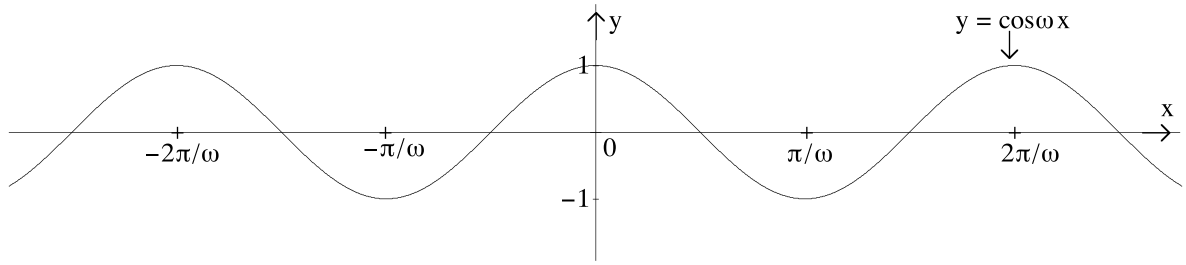
Now, let's consider the function $y = \cos \frac{1}{2}x$. Again we will sketch the graph by looking at the points where $y = \cos x$ equals 0 or ± 1.



$\cos x = 1$	when $x = 0$,	so $\cos \frac{1}{2}x = 1$	when $\frac{1}{2}x = 0$,	ie $x = 0$.
$\cos x = 0$	when $x = \frac{\pi}{2}$,	so $\cos \frac{1}{2}x = 0$	when $\frac{1}{2}x = \frac{\pi}{2}$,	ie $x = \pi$.
$\cos x = -1$	when $x = \pi$,	so $\cos \frac{1}{2}x = -1$	when $\frac{1}{2}x = \pi$,	ie $x = 2\pi$.
$\cos x = 0$	when $x = \frac{3\pi}{2}$,	so $\cos \frac{1}{2}x = 0$	when $\frac{1}{2}x = \frac{3\pi}{2}$,	ie $x = 3\pi$.
$\cos x = 1$	when $x = 2\pi$,	so $\cos \frac{1}{2}x = 1$	when $\frac{1}{2}x = 2\pi$,	ie $x = 4\pi$.

In this case our modified function $y = \cos \frac{1}{2}x$ has period 4π. It's amplitude is 1.

What happens if we take the function $y = \cos \omega x$ where $\omega > 0$?



$\cos x = 1$	when $x = 0$,	so $\cos \omega x = 1$	when $\omega x = 0$,	ie $x = 0$.
$\cos x = 0$	when $x = \frac{\pi}{2}$,	so $\cos \omega x = 0$	when $\omega x = \frac{\pi}{2}$,	ie $x = \frac{\pi}{2\omega}$.
$\cos x = -1$	when $x = \pi$,	so $\cos \omega x = -1$	when $\omega x = \pi$,	ie $x = \frac{\pi}{\omega}$.
$\cos x = 0$	when $x = \frac{3\pi}{2}$,	so $\cos \omega x = 0$	when $\omega x = \frac{3\pi}{2}$,	ie $x = \frac{3\pi}{2\omega}$.
$\cos x = 1$	when $x = 2\pi$,	so $\cos \omega x = 1$	when $\omega x = 2\pi$,	ie $x = \frac{2\pi}{\omega}$.

In general, if we take the function $y = \cos \omega x$ where $\omega > 0$, the period of the function is $\frac{2\pi}{\omega}$.

What happens if we have a function like $y = \cos(-2x)$? See the solution to number 3 of the following exercise.

5.2.1 Exercise

Sketch the graphs of the following functions. Give the amplitude and period of each function.

1. $y = \cos \frac{1}{2}x$

2. $y = 2 \sin \frac{x}{4}$

3. $y = \cos(-2x)$

4. $y = \sin(-2x)$

5. $y = 3 \sin \pi x$

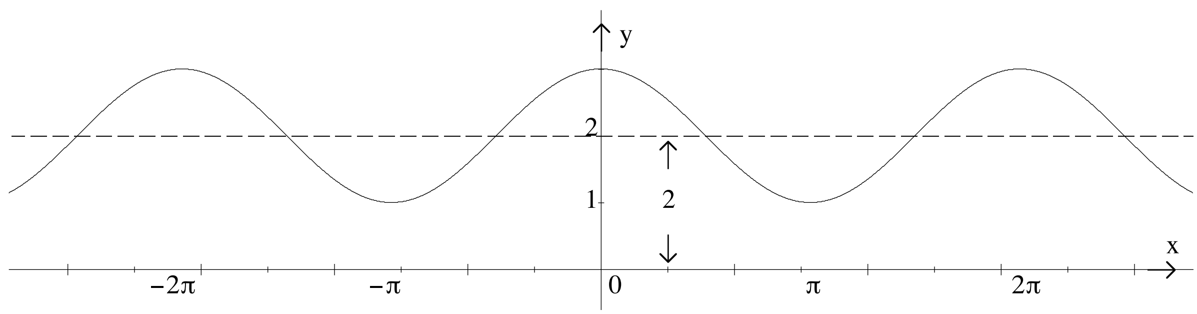
6. $y = -\frac{1}{2} \sin 2\pi x$

7. Find the equation of a sin or cos function which has amplitude 4 and period 2.

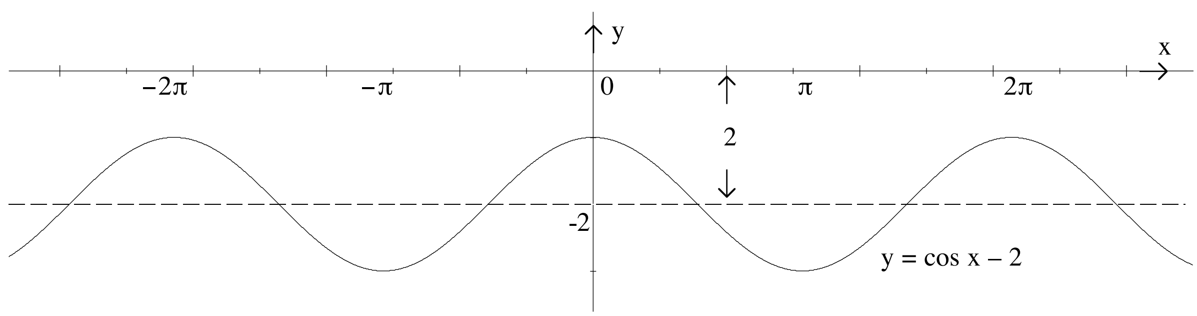
5.3 Changing the mean level

We saw above that the functions $y = \sin x$ and $y = \cos x$ both oscillate about the x-axis which is sometimes refered to as the *mean level* for $y = \sin x$ and $y = \cos x$.

We can change the mean level of the function by adding or subtracting a constant. For example, adding the constant 2 to $y = \cos x$ gives us $y = \cos x + 2$ and has the effect of shifting the whole graph up by 2 units. So, the mean level of $y = \cos x + 2$ is 2.



Similarly, we can shift the graph of $y = \cos x$ down by two units. In this case, we have $y = \cos x - 2$, and this function has mean level -2.



In general, if $d > 0$, the function $y = \cos x + d$ looks like the function $y = \cos x$ shifted *up* by d units. If $d > 0$, then the function $y = \cos x - d$ looks like the function $y = \cos x$ shifted *down* by d units. If $d < 0$, say $d = -2$, the function $y = \cos x + d = \cos x + (-2)$ can be writen as $y = \cos x - 2$ so again looks like the function $y = \cos x$ shifted down by 2 units.

5.3.1 Exercise

Sketch the graphs of the following functions.

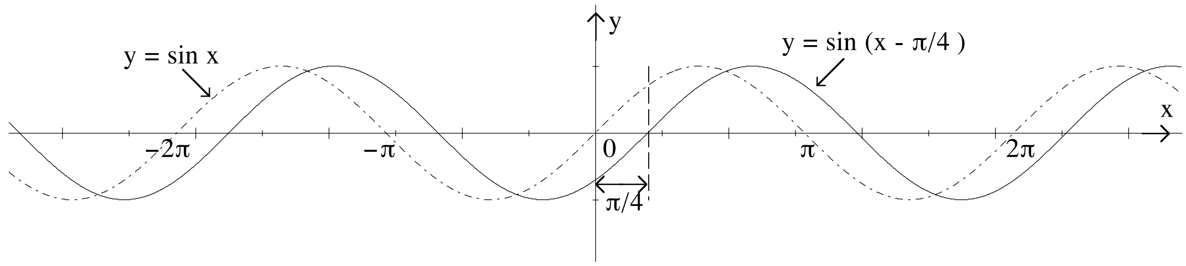
1. $y = \sin 2x + 3$

2. $y = 2\cos \pi x - 1$

3. Find a cos or sin function which has amplitude 2, period 1, and mean level -1.

5.4 Changing the phase

Consider the function $y = \sin(x - \frac{\pi}{4})$.

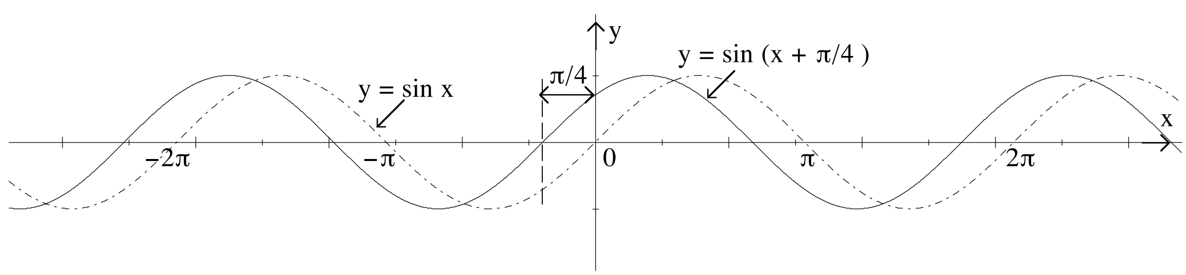


To sketch the graph of $y = \sin(x - \frac{\pi}{4})$ we again use the points at which $y = \sin x$ is 0 or ± 1.

$\sin x = 0$	when $x = 0$,	so $\sin(x - \frac{\pi}{4}) = 0$	when $x - \frac{\pi}{4} = 0$,	ie $x = 0 + \frac{\pi}{4} = \frac{\pi}{4}$.
$\sin x = 1$	when $x = \frac{\pi}{2}$,	so $\sin(x - \frac{\pi}{4}) = 1$	when $x - \frac{\pi}{4} = \frac{\pi}{2}$,	ie $x = \frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4}$.
$\sin x = 0$	when $x = \pi$,	so $\sin(x - \frac{\pi}{4}) = 0$	when $x - \frac{\pi}{4} = \pi$,	ie $x = \pi + \frac{\pi}{4} = \frac{5\pi}{4}$.
$\sin x = -1$	when $x = \frac{3\pi}{2}$,	so $\sin(x - \frac{\pi}{4}) = -1$	when $x - \frac{\pi}{4} = \frac{3\pi}{2}$,	ie $x = \frac{3\pi}{2} + \frac{\pi}{4} = \frac{7\pi}{4}$.
$\sin x = 0$	when $x = 2\pi$,	so $\sin(x - \frac{\pi}{4}) = 0$	when $x - \frac{\pi}{4} = 2\pi$,	ie $x = 2\pi + \frac{\pi}{4} = \frac{9\pi}{4}$.

The graph of $y = \sin(x - \frac{\pi}{4})$ looks like the graph of $y = \sin x$ shifted $\frac{\pi}{4}$ units to the right. We say that there has been a phase shift to the right by $\frac{\pi}{4}$.

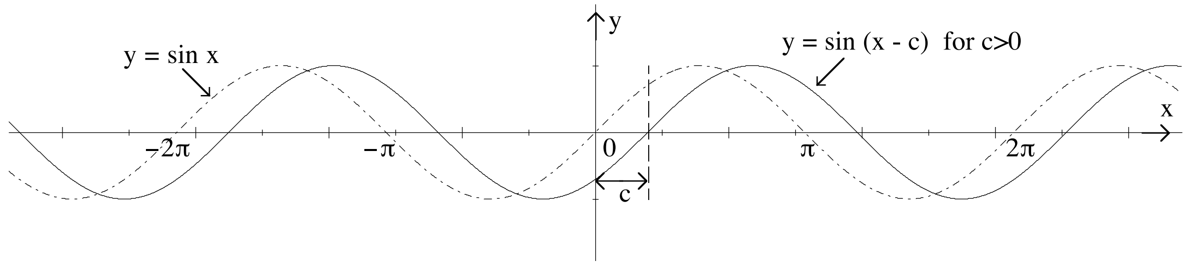
Now consider the function $y = \sin(x + \frac{\pi}{4})$.



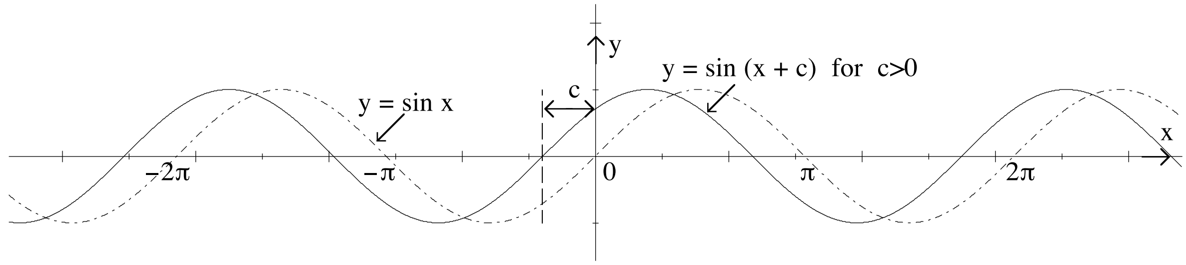
$\sin x = 0$	when $x = 0$,	so $\sin(x + \frac{\pi}{4}) = 0$	when $x + \frac{\pi}{4} = 0$,	ie $x = 0 - \frac{\pi}{4} = -\frac{\pi}{4}$.
$\sin x = 1$	when $x = \frac{\pi}{2}$,	so $\sin(x + \frac{\pi}{4}) = 1$	when $x + \frac{\pi}{4} = \frac{\pi}{2}$,	ie $x = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$.
$\sin x = 0$	when $x = \pi$,	so $\sin(x + \frac{\pi}{4}) = 0$	when $x + \frac{\pi}{4} = \pi$,	ie $x = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$.
$\sin x = -1$	when $x = \frac{3\pi}{2}$,	so $\sin(x + \frac{\pi}{4}) = -1$	when $x + \frac{\pi}{4} = \frac{3\pi}{2}$,	ie $x = \frac{3\pi}{2} - \frac{\pi}{4} = \frac{5\pi}{4}$.
$\sin x = 0$	when $x = 2\pi$,	so $\sin(x + \frac{\pi}{4}) = 0$	when $x + \frac{\pi}{4} = 2\pi$,	ie $x = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$.

The graph of $y = \sin(x + \frac{\pi}{4})$ looks like the graph of $y = \sin x$ shifted $\frac{\pi}{4}$ units to the left.

More generally, the graph of $y = \sin(x - c)$ where $c > 0$ can be drawn by shifting the graph of $y = \sin x$ to the right by c units.



Similarly, the graph of $y = \sin(x + c)$ where $c > 0$ can be drawn by shifting the graph of $y = \sin x$ to the left by c units.



What happens to $y = \sin(x - c)$ if $c < 0$? Consider, for example, what happens when $c = \frac{-\pi}{4}$. In this case we can write $x - \frac{-\pi}{4} = x + \frac{\pi}{4}$. So, $y = \sin(x - \frac{-\pi}{4}) = \sin(x + \frac{\pi}{4})$ and we have a shift to the left by $\frac{\pi}{4}$ units as before.

5.4.1 Exercise

Sketch the graphs of the following functions.

1. $y = \cos(x - \frac{\pi}{2})$

2. $y = \sin 2(x + \frac{\pi}{2})$ (Hint: For this one you'll need to think about the period as well.)

3. $y = 3\cos(x + \pi)$

4. $y = -3\cos(x + \pi) + 2$ (Hint: Use the previous exercise.)

5. Find a sin or cos function that has amplitude 2, period π, and for which $f(0) = 2$.

6 Derivatives of Trigonometric Functions

This Chapter assumes you have a knowledge of differential calculus. If you have not studied differential calculus before, go on to the next chapter.

6.1 The calculus of trigonometric functions

When differentiating all trigonometric functions there are two things that we need to remember.

$$\begin{aligned} \frac{d}{dx}\sin x &= \cos x \\ \frac{d}{dx}\cos x &= -\sin x. \end{aligned}$$

Of course all the rules of differentiation apply to the trigonometric functions. Thus we can use the product, quotient and chain rules to differentiate combinations of trigonometric functions.

For example, $\tan x = \dfrac{\sin x}{\cos x}$, so we can use the quotient rule to calculate the derivative.

$$\begin{aligned} f(x) &= \tan x = \frac{\sin x}{\cos x} \\ f'(x) &= \frac{\cos x.(\cos x) - \sin x.(-\sin x)}{(\cos x)^2} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos x} = \frac{1}{\cos^2 x} \qquad (\text{since } \cos^2 x + \sin^2 x = 1) \\ &= \sec^2 x. \end{aligned}$$

Note also that

$$\frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{\cos^2 x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} = 1 + \tan^2 x$$

so it is also true that

$$\frac{d}{dx}\tan x = \sec^2 x = 1 + \tan^2 x.$$

Example

Differentiate $f(x) = \sin^2 x$.

Solution

$f(x) = \sin^2 x$ is just another way of writing $f(x) = (\sin x)^2$. This is a composite function, with the outside function being $(\cdot)^2$ and the inside function being $\sin x$. By the chain rule,

$f'(x) = 2(\sin x)^1 \times \cos x = 2\sin x \cos x$. Alternatively, setting $u = \sin x$ we get $f(u) = u^2$ and

$$\frac{df(x)}{dx} = \frac{df(u)}{du} \times \frac{du}{dx} = 2u \times \frac{du}{dx} = 2\sin x \cos x.$$

Example

Differentiate $g(z) = \cos(3z^2 + 2z + 1)$.

Solution

Again we should recognise this as a composite function, with the outside function being $\cos(\cdot)$ and the inside function being $3z^2 + 2z + 1$. By the chain rule

$$g'(z) = -\sin(3z^2 + 2z + 1) \times (6z + 2) = -(6z + 2)\sin(3z^2 + 2z + 1).$$

Example

Differentiate $f(t) = \dfrac{e^t}{\sin t}$.

Solution

By the quotient rule

$$f'(t) = \frac{e^t \sin t - e^t \cos t}{\sin^2 t} = \frac{e^t(\sin t - \cos t)}{\sin^2 t}.$$

Example

Use the quotient rule or the composite function rule to find the derivatives of $\cot x$, $\sec x$, and cosec x.

Solution

These functions are defined as follows:

$$\begin{aligned} \cot x &= \frac{\cos x}{\sin x} \\ \sec x &= \frac{1}{\cos x} \\ \csc x &= \frac{1}{\sin x}. \end{aligned}$$

By the quotient rule

$$\frac{d\cot x}{dx} = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = \frac{-1}{\sin^2 x}.$$

Using the composite function rule

$$\frac{d\sec x}{dx} = \frac{d(\cos x)^{-1}}{dx} = -(\cos x)^{-2} \times (-\sin x) = \frac{\sin x}{\cos^2 x}.$$

$$\frac{d\csc x}{dx} = \frac{d(\sin x)^{-1}}{dx} = -(\sin x)^{-2} \times \cos x = -\frac{\cos x}{\sin^2 x}.$$

6.1.1 Exercise

Differentiate the following:

1. $\cos 3x$ **2.** $\sin(4x + 5)$ **3.** $\sin^3 x$ **4.** $\sin x \cos x$ **5.** $x^2 \sin x$

6. $\cos(x^2 + 1)$ **7.** $\dfrac{\sin x}{x}$ **8.** $\sin \dfrac{1}{x}$ **9.** $\tan(\sqrt{x})$ **10.** $\dfrac{1}{x} \sin \dfrac{1}{x}$

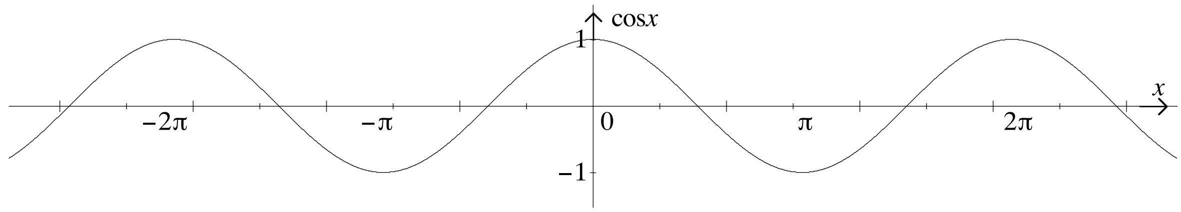
7 A Brief Look at Inverse Trigonometric Functions

Before we define the inverse trigonometric functions we need to think about exactly what we mean by a *function*.

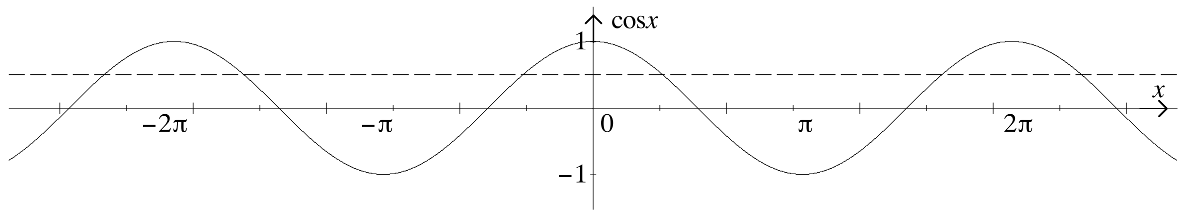
> A function f from a set of elements A to a set of elements B is a rule that assigns to each element x in A exactly one element $f(x)$ in B.

$y = \sin x$, $y = \cos x$ and $y = \tan x$ are functions in the sense of this definition with A and B being sets of real numbers.

Let's look at the function $y = \cos x$. As you can see, whatever value we choose for x, there is only ever one accompanying value for y. For example, when $x = \frac{-\pi}{2}$, $y = 0$.

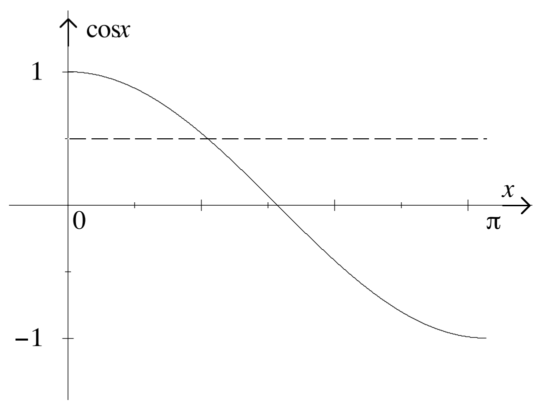


Now lets consider the following question. Suppose we have $\cos x = 0.5$ and we want to find the value of x.



As you can see from the diagram above, there are (infinitely) many values of x for which $y = \cos x = 0.5$. Indeed, there are infinitely many solutions to the equation $\cos x = a$ where $-1 \leq a \leq 1$. (There are no solutions if a is outside this interval.)

If we want an interval for x where there is only one solution to $\cos x = a$ for $-1 \leq a \leq 1$, then we can choose the interval from 0 to π. We could also choose the interval $\pi \leq x \leq 2\pi$ or many others. It is a mathematical convention to choose $0 \leq x \leq \pi$.



In the interval from 0 to π, we can find a unique solution to the equation $\cos x = a$ where a is in the interval $-1 \le a \le 1$. We write this solution as $x = \cos^{-1} a$. Another way of saying this is that x is the number in the interval $0 \le x \le \pi$ whose cosine is a.

Now that we have found an interval of x for which there is only one solution of the equation $\cos x = a$ where $-1 \le a \le 1$, we can define an inverse *function* for $\cos x$.

7.1 Definition of the inverse cosine function

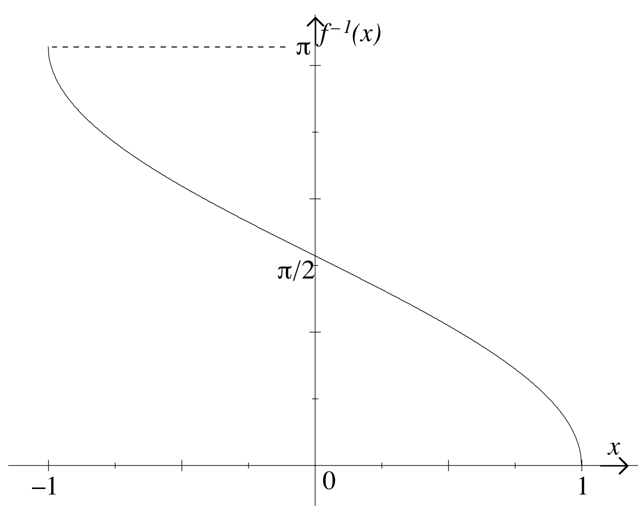
We will describe an *inverse function* for $\cos x$ where $0 \le x \le \pi$.

> For $-1 \le x \le 1$, $f^{-1}(x) = \cos^{-1}(x)$ is the number in the interval 0 to π whose cosine is x.

So, we have:

$\cos^{-1}(-1) = \pi$	since	$\cos \pi = -1,$
$\cos^{-1}(\frac{1}{\sqrt{2}}) = \frac{\pi}{4}$	since	$\cos \frac{\pi}{4} = \frac{1}{\sqrt{2}},$
$\cos^{-1}(0) = \frac{\pi}{2}$	since	$\cos \frac{\pi}{2} = 0,$
$\cos^{-1}(1) = 0$	since	$\cos 0 = 1.$

Provided we take the function $\cos x$ where $0 \le x \le 1$, we can define an inverse function $f^{-1}(x) = \cos^{-1}(x)$. This function is defined for x in the interval $-1 \le x \le 1$, and is sketched below.



Inverse functions for $\sin x$ where $\frac{-\pi}{2} \le x \le \frac{\pi}{2}$, and $\tan x$ where $\frac{-\pi}{2} < x < \frac{\pi}{2}$ can be defined in a similar way.

For a more detailed discussion of inverse functions see the Mathematics Learning Centre booklet: *Functions*.

7.1.1 Exercise

1. The inverse function, $f^{-1}(x) = \sin^{-1}(x)$, is defined for the function $f(x) = \sin x$ where $\frac{-\pi}{2} \leq x \leq \frac{\pi}{2}$. Complete the following table of values.

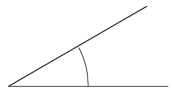
$\sin^{-1}(\quad) =$	since	$\sin\frac{-\pi}{2} = -1,$
$\sin^{-1}(\quad) =$	since	$\sin\frac{-\pi}{4} = -\frac{1}{\sqrt{2}},$
$\sin^{-1}(\quad) =$	since	$\sin 0 = 0,$
$\sin^{-1}(\quad) =$	since	$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2},$
$\sin^{-1}(\quad) =$	since	$\sin\frac{\pi}{2} = 1.$

2. Sketch the function $f^{-1}(x) = \sin^{-1}(x)$ using the values in the previous exercise.

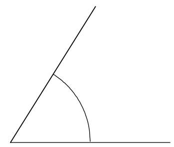
8 Solutions to Exercises

Exercise 2.2.1

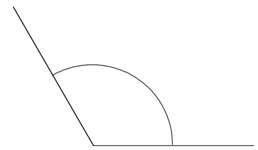
a. $30°$ or $\frac{\pi}{6}$ radians



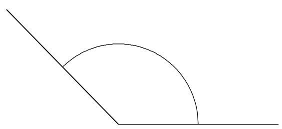
b. 1 radian or $57.3°$



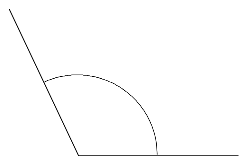
c. $120°$ or $\frac{2\pi}{3}$ radians



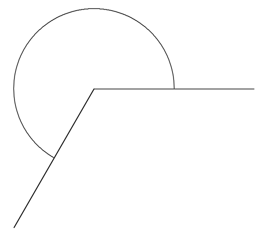
d. $\frac{3\pi}{4}$ radians or $135°$



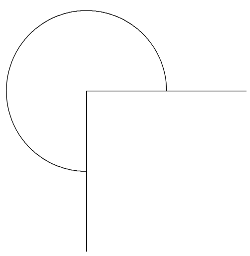
e. 2 radians or $114.6°$



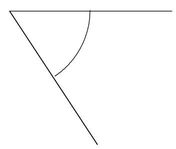
f. $\frac{4\pi}{3}$ radians or $240°$



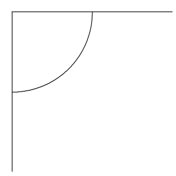
g. $270°$ or $\frac{3\pi}{2}$ radians



h. -1 radians or $-57.3°$



i. $-\frac{\pi}{2}$ radians or $-90°$



Exercise 3.1.1

a. $\sin\frac{\pi}{6} = 0.5$ **b.** $\tan 1 = 1.557$ **c.** $\cos\frac{\pi}{3} = 0.5$ **d.** $\tan\frac{\pi}{4} = 1$

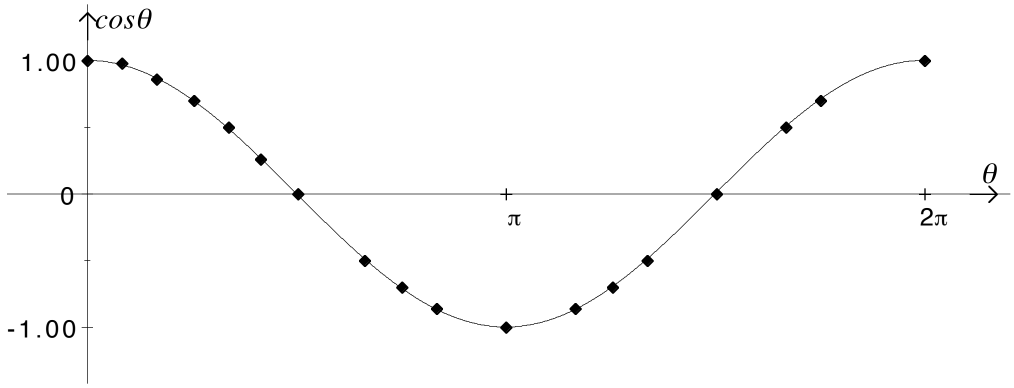
e. $\sin 1.5 = 0.997$ **f.** $\tan\frac{\pi}{3} = 1.732$ **g.** $\cos\frac{\pi}{6} = 0.866$ **h.** $\sin\frac{\pi}{3} = 0.866$

Exercise 4.1.1

1.

θ	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$
$\cos\theta$	1	0.97	0.87	0.71	0.5	0.26	0	-0.5	-0.71
θ	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	2π
$\cos\theta$	-0.87	-1	-0.87	-0.71	-0.5	0	0.5	0.71	1

2.

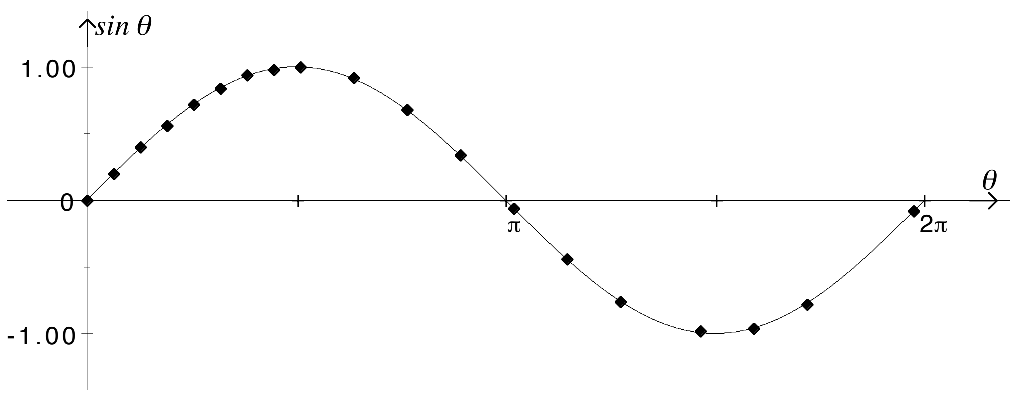


Exercise 4.2.1

1.

θ	0	0.2	0.4	0.6	0.8	1	1.2	1.4	1.6
$\sin\theta$	0	0.20	0.39	0.56	0.72	0.84	0.93	0.99	1.00
θ	2	2.4	2.8	3.2	3.6	4.0	4.6	5.4	6.5
$\sin\theta$	0.91	0.68	0.33	-0.06	-0.44	-0.76	-0.99	-0.77	-0.083

2.

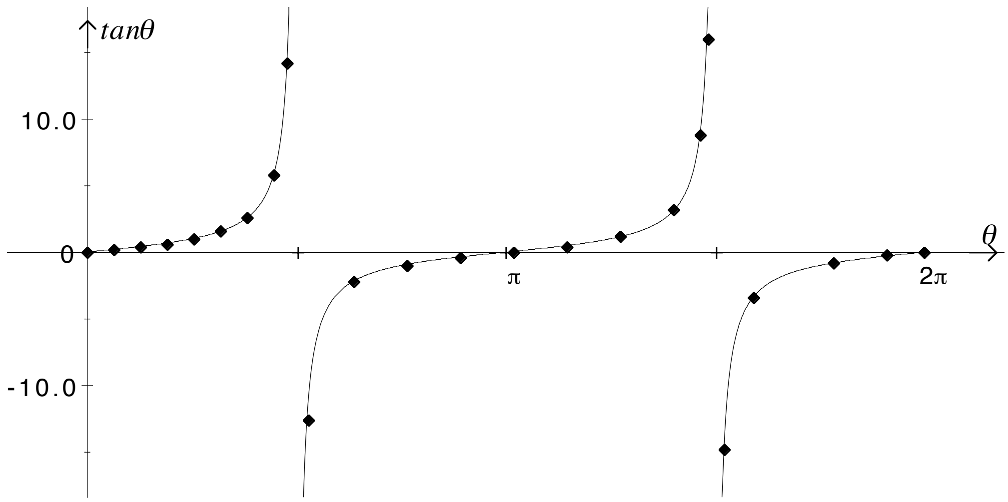


Exercise 4.3.1

1.

θ	0	0.2	0.4	0.6	0.8	1.0	1.2	1.4
$\tan\theta$	0	0.20	0.42	0.68	1.03	1.56	2.57	5.80
θ	1.50	1.65	2	2.4	2.8	3.2	3.6	4.0
$\tan\theta$	14.10	-12.60	-2.19	-0.92	-0.36	0.06	0.49	1.16
θ	4.4	4.6	4.65	4.78	5.0	5.6	6.0	6.28
$\tan\theta$	3.10	8.86	16.01	-14.77	-3.38	-0.81	-0.29	-0.00

2.



Exercise 4.4.1

1. $\sin\frac{15\pi}{2} = \sin\frac{3\pi}{2} = -1$ since $\frac{15\pi}{2}$ and $\frac{3\pi}{2}$ differ by $6\pi = 3 \times 2\pi$.

2. $\tan\frac{13\pi}{6} = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$.

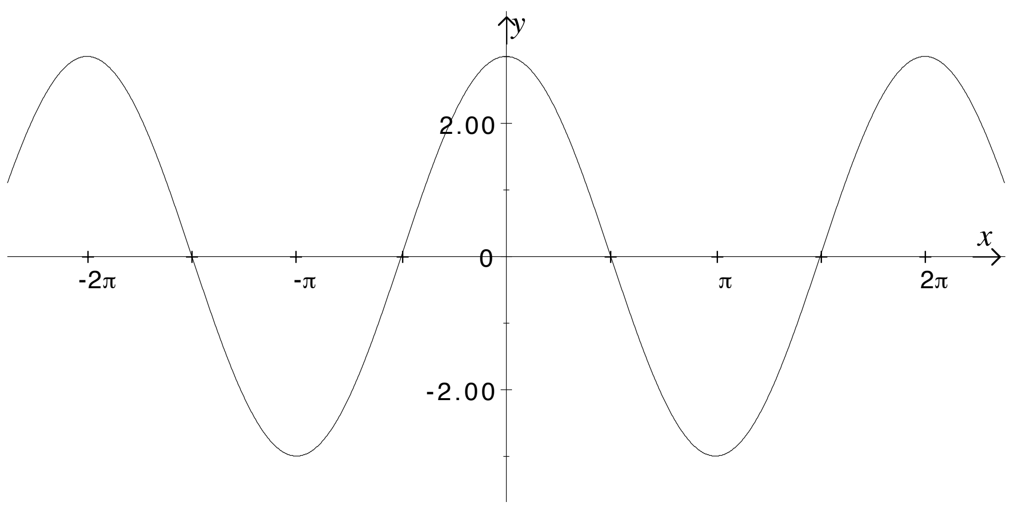
3. $\cos 15 = -0.76$.

4. $\tan -\frac{14\pi}{3} = \tan -\frac{2\pi}{3} = \tan\frac{4\pi}{3} = \tan\frac{\pi}{3} = \sqrt{3}$.

5. $\sin\frac{23\pi}{3} = \sin\frac{5\pi}{3} = \sin -\frac{\pi}{3} = -\frac{\sqrt{3}}{2}$.

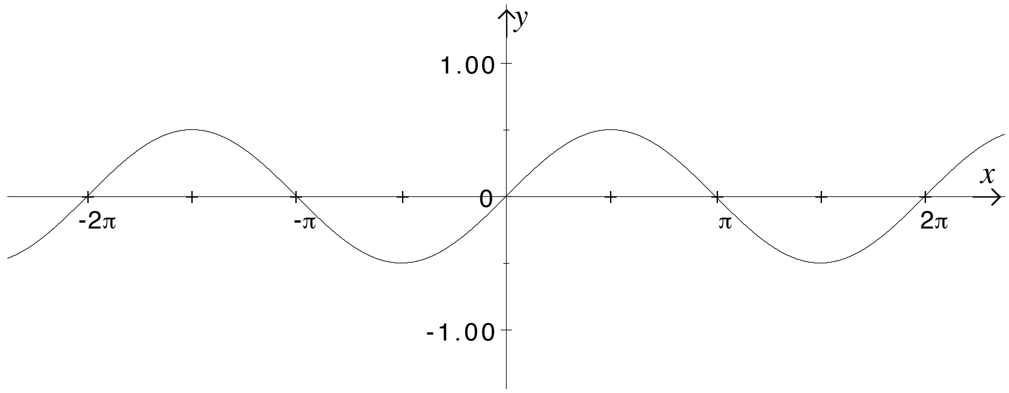
Exercise 5.1.1

1. $y = 3cosx$



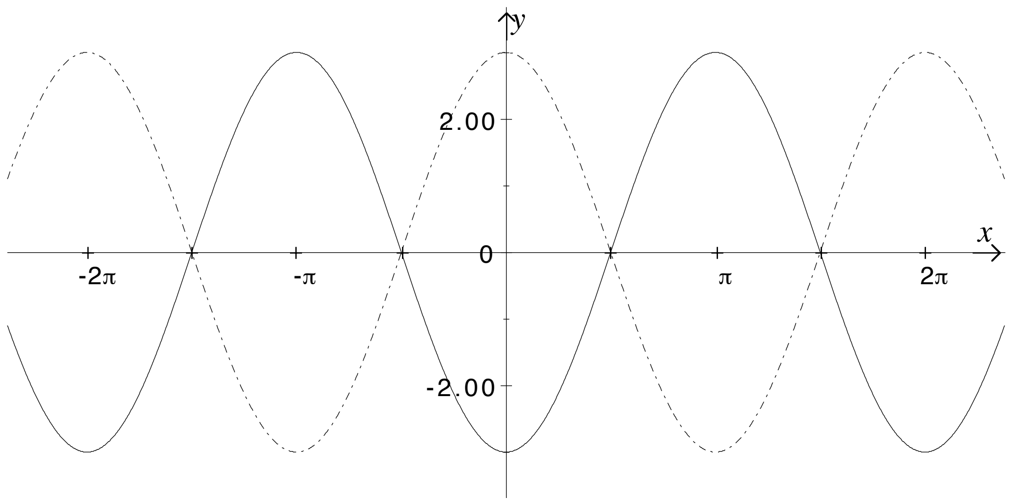
The amplitude of this function is 3.

2. $y = \frac{1}{2}\sin x$



The amplitude of this function is $\frac{1}{2}$.

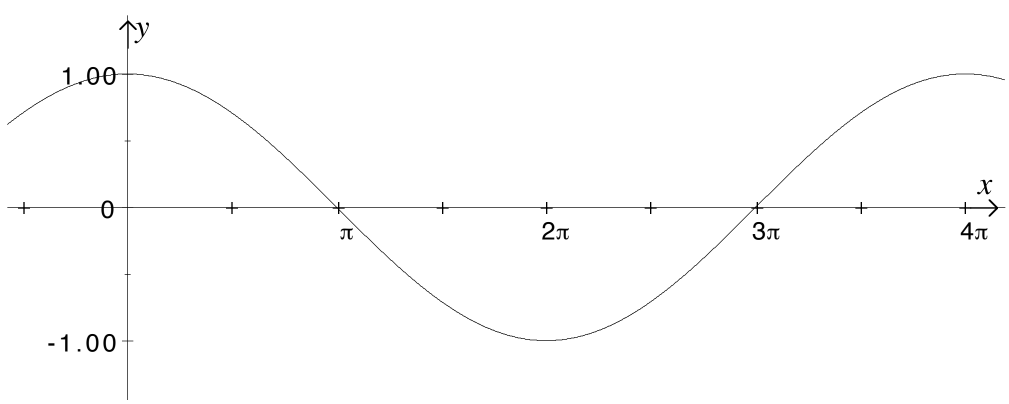
3. $y = -3\cos x$



The amplitude of this function is 3. Notice that it is just the graph of $y = 3\cos x$ (in dots) reflected in the x-axis.

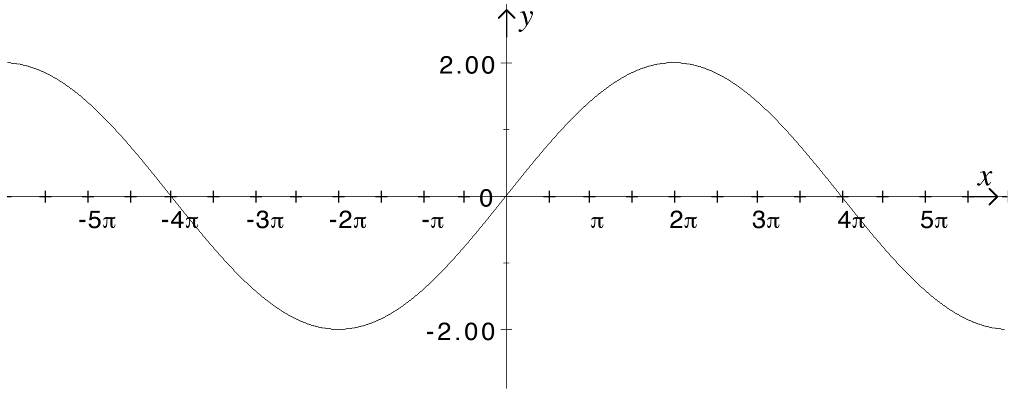
Exercise 5.2.1

1. $y = \cos\frac{1}{2}x$



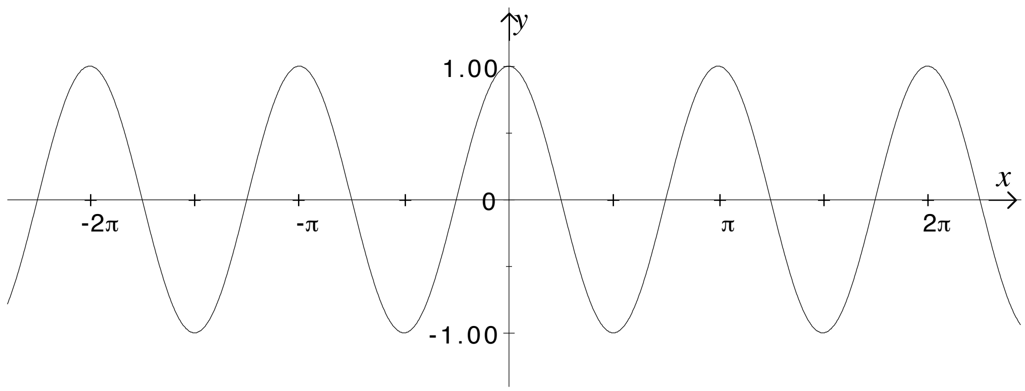
This function has period 4π and amplitude 1.

2. $y = 2\sin\frac{x}{4}$



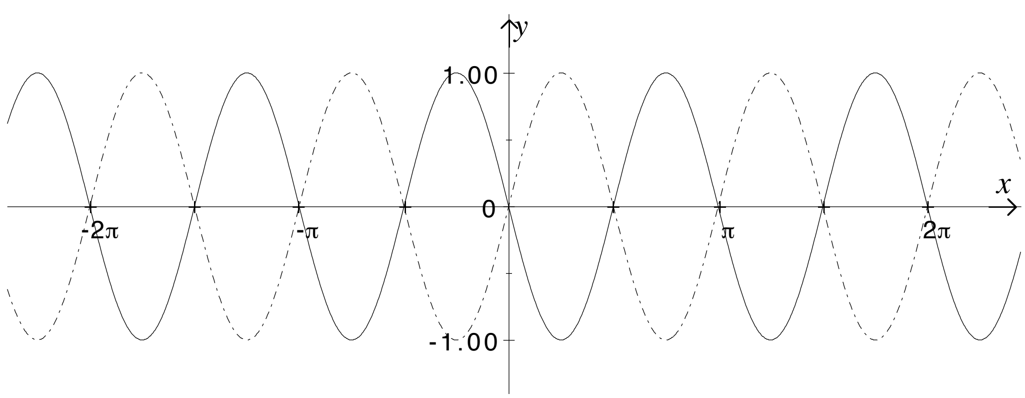
This function has period 8π and amplitude 2.

3. $y = \cos(-2x)$



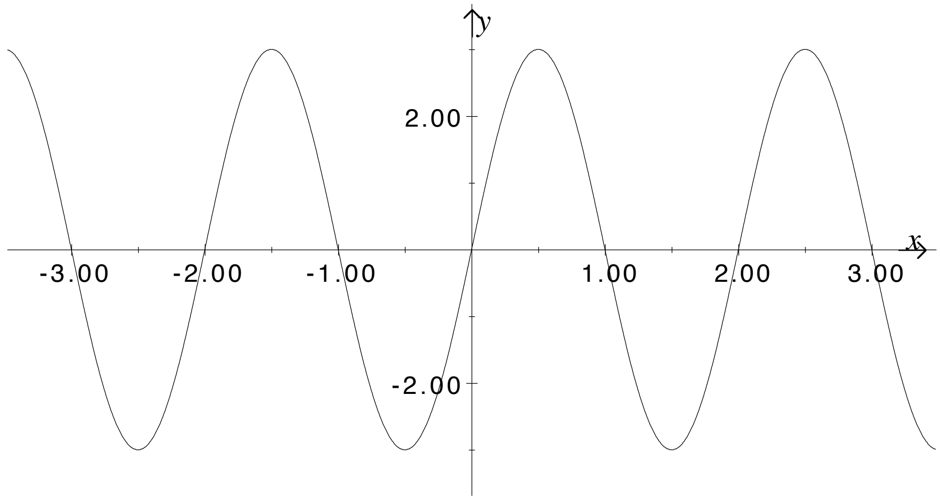
This function has period π and amplitude 1. Notice that this function looks like $y = \cos 2x$. The cosine function is an even function, so, $\cos(-2x) = \cos 2x$ for all values of x.

4. $y = \sin(-2x)$



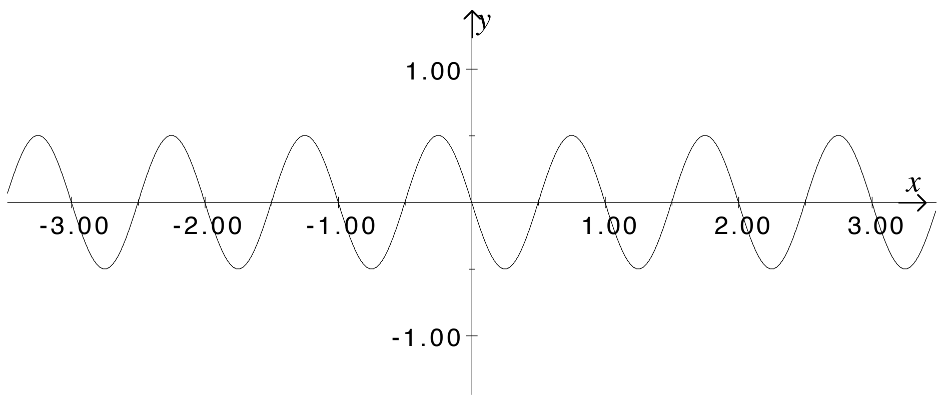
This function has period π and amplitude 1. Notice that this function is a reflection of $y = \sin 2x$ in the x-axis. The sine function is an odd function, so, $\sin(-2x) = -\sin 2x$ for all values of x.

5. $y = 3\sin \pi x$



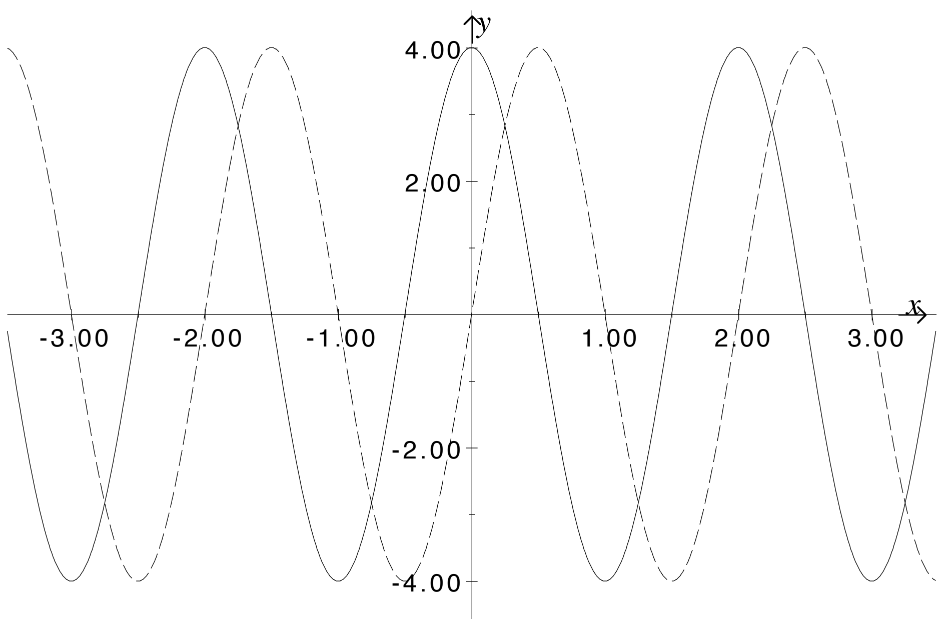
This function has period 2 and amplitude 3.

6. $y = -\frac{1}{2}\sin 2\pi x$



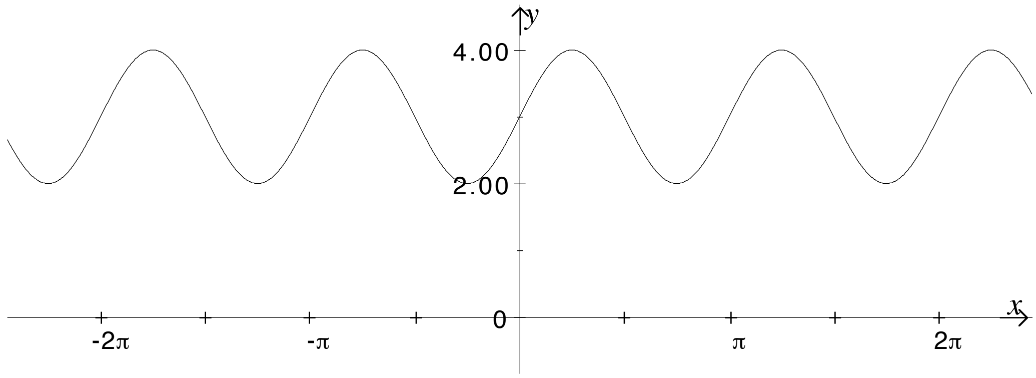
This function has period 1 and amplitude $\frac{1}{2}$.

7. $y = 4\cos \pi x$ (solid line) or $y = 4\sin \pi x$ (in dashes). There are many other solutions.



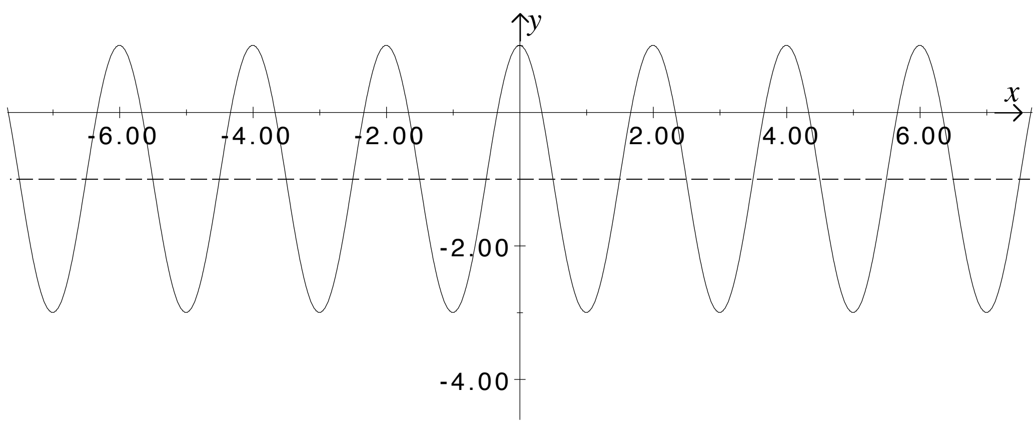
Exercise 5.3.1

1. $y = \sin 2x + 3$



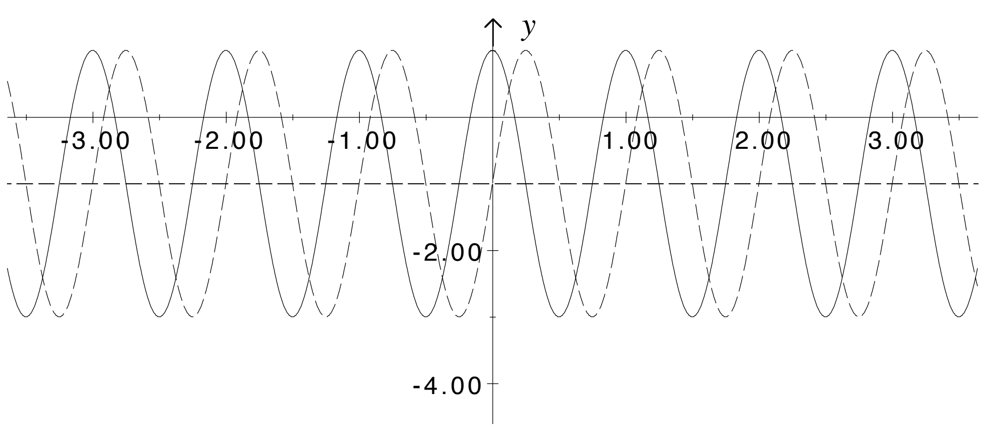
This function has amplitude 1, period π and mean level 3.

2. $y = 2\cos \pi x - 1$



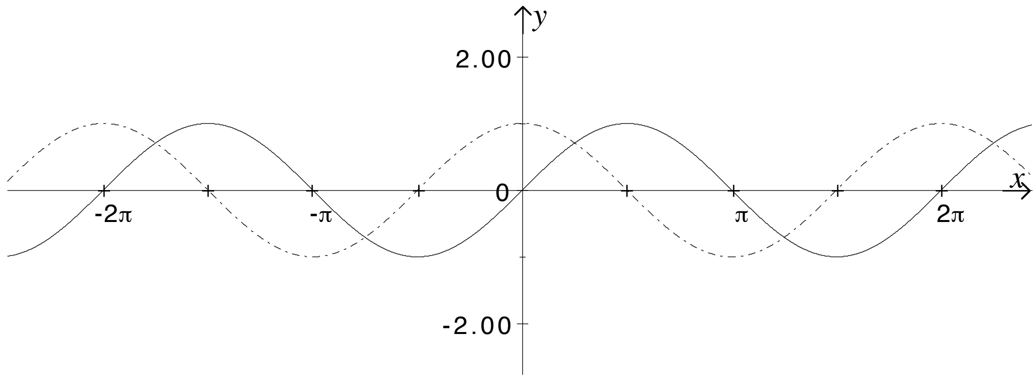
This function has amplitude 2, period 2 and mean level -1.

3. $y = 2\cos 2\pi x - 1$ (solid line) or $y = 2\sin 2\pi x - 1$ (in dashes) or many others.



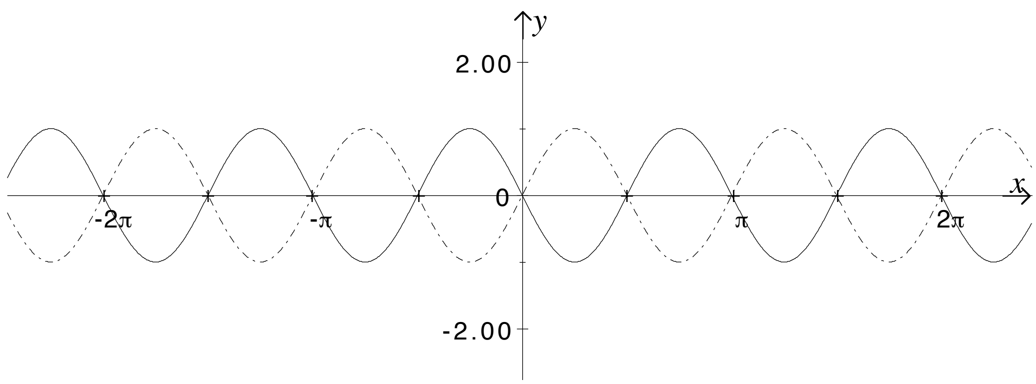
Exercise 5.4.1

1. $y = \cos(x - \frac{\pi}{2})$



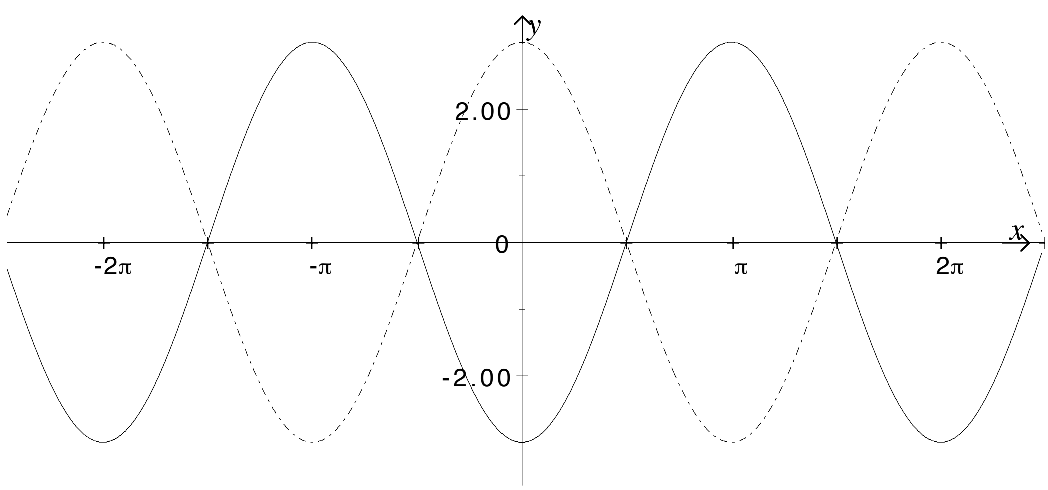
The function $y = \cos x$ (in dots) has been shifted to the right by $\frac{\pi}{2}$ units. Notice that this function looks like $y = \sin x$.

2. $y = \sin 2(x + \frac{\pi}{2})$



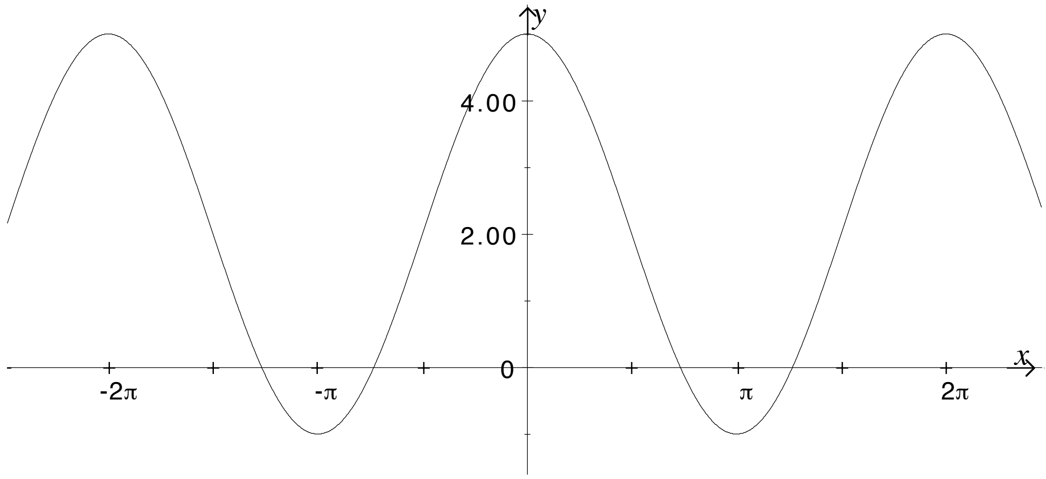
The period of this function is π. The function $y = \sin 2x$ (in dots) has been shifted to the left by $\frac{\pi}{2}$ units.

3. $y = 3\cos(x + \pi)$



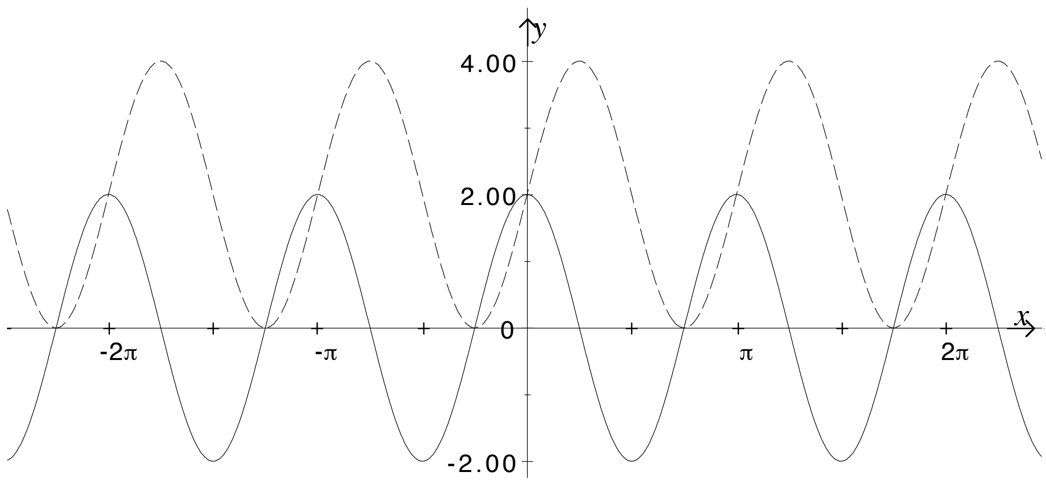
This function has period 2π and amplitude 3. The function $y = 3\cos x$ (in dots) has been shifted to the left by π units.

4. $y = -3\cos(x+\pi) + 2$



This function has period 2π and amplitude 3. The function $3\cos(x+\pi)$ (see prevoius exercise) has been reflected in the x-axis and shifted up by 2 units.

5. $y = 2\sin 2x + 2$ (in dashes) or $y = 2\cos 2x$ (solid line) or many others.



Exercise 6.1.1

1. $\dfrac{d}{dx}(\cos 3x) = -3\sin 3x.$

2. $\dfrac{d}{dx}(\sin(4x+5)) = 4\cos(4x+5).$

3. $\dfrac{d}{dx}(\sin^3 x) = 3\sin^2 x\cos x.$

4. $\dfrac{d}{dx}(\sin x\cos x) = \sin x(-\sin x) + \cos x(\cos x) = \cos^2 x - \sin^2 x.$

5. $\dfrac{d}{dx}(x^2\sin x) = x^2\cos x + 2x\sin x.$

6. $\dfrac{d}{dx}(\cos(x^2+1)) = -\sin(x^2+1)(2x) = -2x\sin(x^2+1).$

7. $\dfrac{d}{dx}(\dfrac{\sin x}{x}) = \dfrac{x\cos x - \sin x}{x^2}.$

8. $\dfrac{d}{dx}(\sin\dfrac{1}{x}) = (\cos\dfrac{1}{x})(-x^{-2}) = \dfrac{\cos\frac{1}{x}}{x^2}.$

9. $\dfrac{d}{dx}(\tan(\sqrt{x})) = (\sec^2(\sqrt{x}))(\dfrac{1}{2}x^{-\frac{1}{2}}) = \dfrac{\sec^2\sqrt{x}}{2\sqrt{x}}.$

10. $\dfrac{d}{dx}(\dfrac{1}{x}\sin\dfrac{1}{x}) = \dfrac{1}{x}(\cos\dfrac{1}{x})(-x^{-2}) + (\sin\dfrac{1}{x})(-x^{-2}) = -\dfrac{1}{x^3}(\cos\dfrac{1}{x} + x\sin\dfrac{1}{x}).$

Exercise 7.1.1

1.

$\sin^{-1}(-1) = -\frac{\pi}{2}$	since	$\sin\frac{-\pi}{2} = -1$
$\sin^{-1}(-\frac{1}{\sqrt{2}}) = -\frac{\pi}{4}$	since	$\sin\frac{-\pi}{4} = -\frac{1}{\sqrt{2}}$
$\sin^{-1}(0) = 0$	since	$\sin 0 = 0$
$\sin^{-1}(\frac{\sqrt{3}}{2}) = \frac{\pi}{3}$	since	$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$
$\sin^{-1}(1) = \frac{\pi}{2}$	since	$\sin\frac{\pi}{2} = 1$

2. $f^{-1}(x) = \sin^{-1}(x)$