STAT 5140 Hmwk #2 Allen Baumgarten

#1 Which of the processes is stationary (weak)? If stationary, compute the μ and autocovariance (ACF) functions.

Given $\{Z_t\}$ be iid variable such that $Z_t \sim N(0,\sigma^2)$

(a.) $X_t = a + bZ_t + cZ_{t-2}$ [we have 3 constants a, b, +c w/ current period Z_t and a 2-period lag of Z_{t-2}]

$\mu[X_t] = E[X_t] = E[a + bZ_t + cZ_{t-2}]$

$= E[a] + bE[Z_t] + cE[Z_{t-2}]$ Both have mean of zero

$= a + b[0] + c[0]$

$= a$ our expected mean is just the slope of our constant 'a'

$s \le t$

$\text{cov}[X_s, X_T] = E[X_s, X_t] - E[X_s]E[X_t]$

$= \underbrace{E[X_s, X_t]} - a^2$

$E[X_s X_t] = E[(a + bZ_t + cZ_{t-2})(a + bZ_s + cZ_{s-2})]$

$= E[(a^2 + \underbrace{ab Z_s}_{=0} + \underbrace{ac Z_{s-2}}_{=0} + \underbrace{abZ_t}_{=0} + b^2 Z_t Z_s + bc Z_t Z_{s-2} + \underbrace{ac Z_{t-2}}_{=0} + bcZ_s Z_{t-2} + c^2 Z_{t-2} Z_{s-2})$

$= \cancel{a^2} + b^2 E[Z_t Z_s] + bc E[Z_t Z_{s-2}] - bcE[Z_s Z_{t-2}] + c^2 E[Z_{t-2} Z_{s-2}] - \cancel{a^2}$

If $t = s \rightarrow (b^2 + c^2)\sigma^2$

If $t = s+1 \rightarrow 0$

If $t = s+2 \rightarrow bc$

If $t \ge s+3 \rightarrow 0$

weak stationary

(b.) $X_t = Z_t \cos(ct) + Z_{t-1} \sin(ct)$

$E[X_t] = \cos(ct)\underbrace{E[Z_t]}_{=0} + \sin(ct)\underbrace{E[Z_{t-1}]}_{=0} = 0$

$\text{Cov}(X_t X_s) = E[X_t X_s]$

$= E[\underbrace{(Z_t \cos(ct) + Z_{t-1}\sin(ct))}_{X_t}\underbrace{(Z_s \cos(cs) + Z_{s-1}\sin(cs))}_{X_s}]$

$= E[(Z_t Z_s \cos(ct)\cos(cs)) + (Z_{t-1}Z_s \sin(ct)\cos(cs)) + (Z_t Z_{s-1}\cos(ct)\sin(cs))$
$(Z_{t-1}Z_{s-1}\sin(ct)\sin(cs))]$

$= \sigma^2(\cos^2(cs) + \sin^2(cs)) = \sigma^2$ stationary for $\gamma(s,s)$

(c.) $X_t = a + bZ_0$ $\qquad E[X] = E[a + bZ_0] = E[a] + bE[Z_0] = a + b(0) = a$

(a constant)

$\text{cov}(X_t, X_s) = E[X_t X_s] - E[X_t]E[X_s] = E[(a+bZ_0)(a+bZ_0)] - a^2$

$= E[a^2 + abZ_0 + abZ_0 + b^2 Z_0^2] - a^2$

$= \cancel{a^2} + b^2 E[Z_0^2] - \cancel{a^2} = b^2\sigma^2$

I will argue this is <u>not</u> stationary, given a constant slope for the $\mu = a$

(d.) $X_t = Z_t Z_{t-1}$ $\qquad E[X_t] = E[Z_t Z_{t-1}] = E[Z_t]E[Z_{t-1}] = 0$

$\text{cov}[X_t X_s] = E[X_t X_s] - E[X_t]E[X_s] = E[X_t X_s] - 0$

$= E[(Z_t Z_{t-1})(Z_t Z_{t-1})]$ $\qquad$ <u>stationary</u>

#2 The time series process $\{X_t\}$ is given by: $X_t = e_t + \theta e_{t-1}$

This is a moving average (order 1 or "lag 1") we're told that θ is $\mathbb{R}$ + that $e_t = WN(0, \sigma^2)$ + $\theta = 0.8$

(a.) Comput the ACF for the process when $\theta = 0.8$

$\text{cov}(X_t X_s) = E[X_t X_s] = E[(e_t + \theta e_{t-1})(e_s + \theta e_{s-1})]$

$= E[e_t e_s + \theta e_s e_{t-1} + \theta e_{s-1} e_t + \theta^2 e_{s-1} e_{t-1}]$

Now test:

$t = s \Rightarrow E[e_s^2] + \theta^2 E[e_{s-1}^2] = (1+\theta^2)\sigma^2$

$t = s+1 \Rightarrow \theta E[e_s^2] = \theta\sigma^2$

$t \geq s+2 \Rightarrow 0$

(b.) Compute the variance of the sample mean $(X_1 + X_2 + X_3 + X_4)/4$ when $\theta = 0.8$

3] Let $\{Z_t\}$ be iid $N(0,1)$ + define

$$X_t = \begin{cases} Z_t & \text{if } t \text{ is even} \\ \frac{1}{\sqrt{2}}(Z_{t-1}^2 - 1) & \text{if } t \text{ is odd} \end{cases}$$

(a.) show that X_t is a White Noise Process w/ $\mu = 0$ + $\sigma^2 = 1$

$$E[X_t] = \begin{cases} E[Z_t] = 0 & \text{if } t \text{ is even} \\ \frac{1}{\sqrt{2}}(E[Z_t^2]-1) = \frac{1}{\sqrt{2}}[1-1] = 0 & \text{if } t \text{ is odd} \end{cases}$$

$$\operatorname{var}[X_t] = \begin{matrix} \operatorname{Var}[Z_t] = 1 & \text{if } t \text{ is even} \\ (\frac{1}{\sqrt{2}})^2 \operatorname{Var}[Z_t^2] = \frac{1}{2}\cdot 2 = 1 & \text{if } t \text{ is odd} \end{matrix}$$

(b.) Also prove that they are not iid noise:

4] Consider a time series process as $X_t = U_1 \sin(2\pi\omega t) + U_2 \cos(2\pi\omega t)$ where U_1 + U_2 are independent random variables w/ $\mu = 0$ + variance σ^2

Prove that this process is weak stationary:

$$E[X_t] = \underbrace{E[U_1]}_{=0} \sin(2\pi\omega t) + \underbrace{E[U_2]}_{=0} \cos(2\pi\omega t) = 0$$

$$\operatorname{Var}[X_t] = \underbrace{\operatorname{var}[U_1]}_{=\sigma^2} \sin^2(\ldots) + \underbrace{\operatorname{Var}[U_2]}_{=\sigma^2} \cos^2(\ldots) = \sigma^2$$

weak stationary

5] Predict a stationary time series $\{X_t\}$ w/ $\mu = 0$ + autocov function $\gamma(h)$ at some future time, say $t+l$ for $l > 0$. Use predictor as $\hat{X}_{t+l} = AX_t$ for $A \in \mathbb{R}$.

Prove the MSE is minimized by choosing $A = \frac{\gamma(l)}{\gamma(0)}$

$$\text{MSE}: E[(X_{t+e} - AX_t)^2]$$

$$= E[X_{t+e}^2 + A^2X_t^2 - 2A X_t X_{t+e}]$$

$$= E[X_{t+e}^2] + A^2 E[X_t^2] - 2A E[X_t X_{t+l}]$$

#6 Two uncorrelated stationary sequences $\{X_t\}$ + $\{Y_t\}$, show that $\{X_t + Y_t\}$ is a stationary sequence + the ACF $\{X_t + Y_t\}$ is the sum of the ACFs of $\{X_t\}$ + $\{Y_t\}$

$$\mu = E[Z_t] = E[X_t] + E[Y_t] = \mu_x + \mu_y \quad \text{(constants)}$$

$$\sigma^2 = Var[Z_t] = Var[X_t] + Var[Y_t]$$

$$= \sigma_x^2 + \sigma_y^2 + 2(0) = \underset{\text{constant}}{\sigma_x^2} + \underset{\text{constant}}{\sigma_y^2}$$

#7 Consider a Random walk w/ Drift defined as $X_t = \delta + X_{t-1} + e_t$ w/ $X_0 = 0$ + $e_t \sim WN(0, \sigma^2)$

(a). Prove that the process can also be written as $X_t = \delta t + \sum_{k=1}^{t} e_k$

Use Proof by Induction: $X_0 = 0$

$$X_1 = \delta + X_0 + e_1 = \delta + 0 + e_1$$

$$X_2 = \delta + X_1 + e_2 = \delta + (\delta + e_1) + e_2 = 2\delta + (e_1 + e_2)$$

assume formula is true for t: $X_t = \delta t + \sum_{k=1}^{t} e_k$

now $X_{t+1} = \delta + X_t + e_{t+1} = \delta_t \left(\delta t + \sum_{k=1}^{t} e_k\right) + e_{t+1} = \delta(t+1) + \sum_{k=1}^{t} e_k$

(b). Compute its mean, variance, + ACF:

$$E[X_t] = E[\delta t + \sum_{k=1}^{t} e_k] = E[\delta t] + E[\sum_{k=1}^{t} e_k]$$

$$= \delta t + \sum_{k=1}^{t} \underbrace{E[e_k]}_{0} = \delta t$$

$$var[X_t] = Var[\delta t + \sum e_k] = \underbrace{Var[\delta t]}_{0} + Var[\sum e_k] + \underbrace{2\,cov(\delta t, \sum e_k)}_{0}$$

$$= t\sigma^2$$

(c). Prove that the process is not stationary: ?

(d.) Suggest a tranformation to make the process stationary: ?

(e). see other pages (#5 + #6)

#8 Consider the process $X_t = \beta_1 + \beta_2 t e_t$ (assume $e \sim WN(0, \sigma^2)$)

we see that X_t is not stationary in its variance:

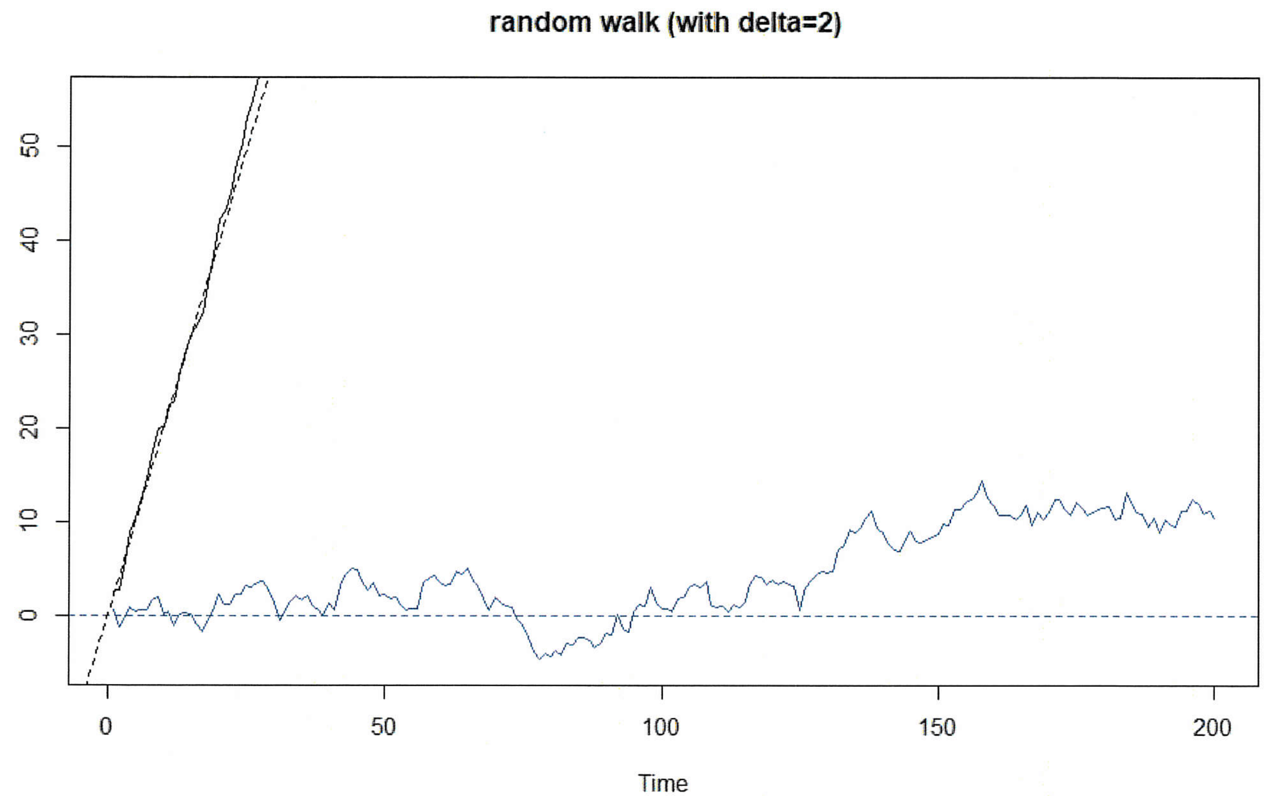
$$\mu = E[X_t] = \beta_1 + \beta_2 t \underbrace{E[e]}_{0} = \beta_1 \leftarrow \text{constant}$$

$$var[X_t] = var[\beta_2 t e]$$

L #5

Problem 7 (e) simulate and plot the process for positive and negative values of δ (maybe δ = 2 and δ = -2)
Option 1: δ = 2

```
set.seed(154)
w = rnorm(200); x = cumsum(w)
wd = w + 2; xd = cumsum(wd)
plot.ts(xd, ylim=c(-5,55), main = "random walk (with delta=2)", ylab='')
lines(x, col= 4); abline(h=0, col=4, lty=2); abline(a=0, b=2, lty=2)
```



#6

Problem 7 (e) simulate and plot the process for positive and negative values of δ (maybe δ = 2 and δ = -2)
Option 2: δ = -2

```
set.seed(154)
w = rnorm(200); x = cumsum(w)
wd = w + -2; xd = cumsum(wd)
plot.ts(xd, ylim=c(-5,55), main = "random walk (with delta=-2)", ylab='')
lines(x, col= 4); abline(h=0, col=4, lty=2); abline(a=0, b=-2, lty=2)
```