Lecture Notes #5

Time series analysis (STAT 5140/4140)

1 Auto regressive process

In general a linear random process in invertiable if all the roots of $H(B) = 0$, where $H(B)$ is the transfer function of the process, are outside of a unit circle. An Auto Regressive process of order p, $AR(P)$ is defined as:

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} \ldots \phi_p X_{t-p} + e_t$$

Recall linear random process can be written as:

$$X_t = \sum_{j=0}^{\infty} \psi_j e_{t-j}$$

Where $\sum_{j=0}^{\infty} |\psi_j| < \infty$. Now assume a parametric structure $\psi_j = \alpha\lambda^j$ for $\alpha \neq 0$ and $|\lambda| < 1$.Under this assumptions the above equation reduces to:

$$X_t = \sum_{j=0}^{\infty} \alpha\lambda^j e_{t-j} \tag{1}$$

Also X_{t-1} cab be written as:

$$X_{t-1} = \sum_{j=0}^{\infty} \alpha\lambda^j e_{t-1-j} \tag{2}$$

Using equation 1 and 2

$$X_t - \lambda X_{t-1} = \alpha e_t$$

Assume $\psi_0 = 1$, which means $\alpha = 1$. The equation of the process is :

$$X_t = \lambda X_{t-1} + e_t$$

Which is an AR(1) process.Using the backward shift operator B an AR(p) model can also be written as :

$$X_t = \phi_1 B X_t + \phi_2 B^2 X_t + \cdots + \phi_p B^p X_t + e_t$$
$$\Rightarrow \quad (1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p) X_t = e_t$$

Let $\Phi(B) = 1 - \sum_{i=1}^{p} \phi_i B^i$ then the equation of a AR(P) model is :

$$\Phi(B) X_t = e_t \tag{3}$$

The equation $\Phi(B) = 0$ is called characteristic equation for an AR(p) process. Now consider an AR(1) process $X_t = \phi_1 Xt - 1 = e_t$. Is it stationary ?? To answer this question we need to write the model as a linear random process:

$$
\begin{aligned}
& X_t = \phi_1 Xt - 1 = e_t \\
\Rightarrow\ & (1 - \phi_1 B)X_t = e_t \\
\Rightarrow\ & X_t = (1 - \phi_1 B)^{-1} e_t \\
\Rightarrow\ & X_t = (1 + \phi_1 B + \phi_1^2 B^2 + \phi_1^3 B^3 + \dots) \\
\Rightarrow\ & X_t = \sum_{j=0}^{\infty} \phi_1^j e_{t-j}
\end{aligned} \tag{4}
$$

So, this linear random process is stationary if $\sum_{j=0}^{\infty} |\phi^j| < \infty$ or $|\phi_1| < 1$. NOTE: If $\phi_1 = 1$ the process is a random walk process. Also AR(P) processes is always invertiable

Theorem *A AR(p) process, $\Phi(B)X_t = e_t$, where $\Phi(B) = 1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p$, stationary if all the roots of $\Phi(B) = 0$ are outside of an unit circle.*

Example Consider the process $Y_t = 0.8Y_{t-1} + 0.09Y_{t-2} + e_t$. In this case $\Phi(B) = 1 - 0.8B - 0.09B^2$ and roots of $\Phi(B) = 0$ are outside of an unit circle. So the process is stationary.

2 Moments of AR(p) models

Consider an AR(p) model as:

$$
\begin{aligned}
& \Phi(B)X_t = e_t \\
& \Rightarrow X_t = \Phi(B)^{-1} e_t \\
& \Rightarrow E[X_t] = \Phi(B)^{-1} E[e_t] \\
& \Rightarrow E[X_t] = 0
\end{aligned} \tag{5}
$$

Variance of the process is $V(X_t) = \Phi(B)^{-2}\sigma^2$. An AR(p) process can also be written as:

$$
\begin{aligned}
& X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \cdots + \phi_p X_{t-p} + e_t \\
\Rightarrow\ & X_t X_{t+h} = \phi_1 X_{t+h} X_{t-1} + \phi_2 X_{t+h} X_{t-2} + \cdots + \phi_p X_{t+h} X_{t-p} + e_t \\
\Rightarrow\ & E[X_t X_{t+h}] = \phi_1 E[X_{t+h} X_{t-1}] + \phi_2 E[X_{t+h} X_{t-2}] + \cdots + \phi_p E[X_{t+h} X_{t-p}] + E[X_{t+h} e_t] \\
\Rightarrow\ & \gamma(h) = \phi_1 \gamma_{h-1} + \phi_2 \gamma(h-2) + \cdots + \phi_p \gamma(h-p); if h \neq 0, as E[X_{t+h} e_t] = 0 && (6) \\
\Rightarrow\ & \gamma(0) = \phi_1 \gamma_1 + \phi_2 \gamma(2) + \cdots + \phi_p \gamma(p) + \sigma^2; if h = 0, as E[X_t e_t] = \sigma^2. && (7)
\end{aligned}
$$

Equations (6) and (7) are called Yule-Walker. They can be used to estimate parameters.

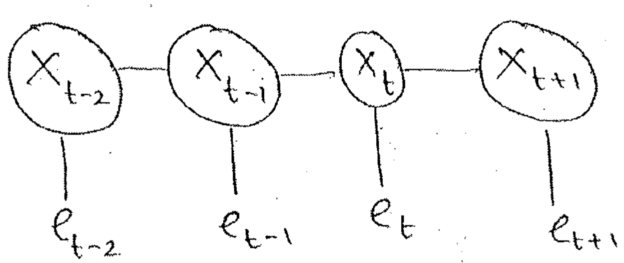
Theorem *AR(p) models, $\Phi(B)X_t = e_t$, are causal if all roots of $\Phi(B) = 0$ are outside of a unit circle.*

Example Consider the model $X_t = 2X_{t-1} + e_t$. This model is not stationary as 1/2 is a root of $\Phi(B) = 1 - 2B = 0$. But if we rewrite the model as $X_{t-1} = 0.5X_t + \tilde{e}_t$, where $\tilde{e}_t = -0.5e_t$, this model is stationary. But we can't use it for prediction as this is non-causal model and requires future values. This model can be used to estimate missing values. Note: For prediction of a AR(p) model we only need p past information.

2.1 Model Identification

For an AR(1) process the ACF function is given by $\rho(h) = (\phi_1)^{|h|}$, where $|\phi_1| < 1$ and $h = 0, 1, 2, \ldots$. ACF exhibits exponentially decay. In general the ACF function of an AR(p) process exhibits exponentially decay. Unlike MA(q) the function is never equal to zero after finite lag. So by looking ACF plot of a AR(p) process one can identify the process but can't determine the order of the process.

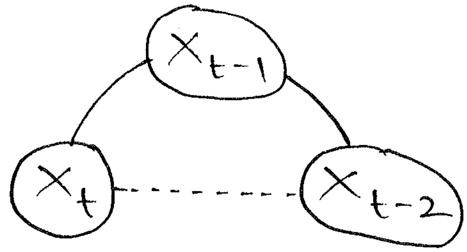
For an AR(1) process $X_t = \phi_1 X_{t-1} + e_t$ we can see that the X_{t-1} is effecting X_t directly. Order of a AR process can be thought of number of observations in the past that have DIRECT influence on present. For example in AR(1) X_{t-1} is directly effecting X_t.



To identify the order of a AR process we need to look at the partial auto-correlation of the process.

2.1.1 Partial auto correlation:

Let ρ_{hh} be the auto-correlation between X_t and X_{t-1} removing the linear effect of $X_{t-1}, \ldots, X_{t-h+1}$. By definition $\rho_{11} = 1$.



Example For a Ar(1) process $\rho_{22} = 0$, that is the correlation between X_t and X_{t-2} removing the linear effect of X_{t-1} is zero.

First fit $\hat{X}_{t-2} = \beta X_{t-1}$ and choose β such that

$$L(\beta) = E(X_{t-2} - \hat{X}_{t-2})^2 \text{ is minimized.}$$

$$= E(X_{t-2} - \beta X_{t-1})^2$$

$$= E(X_{t-2}^2 - 2\beta X_{t-2} X_{t-1} + X_{t-1}^2)$$

$$= E(X_{t-2}^2) - 2\beta E(X_{t-2} X_{t-1}) + E(X_{t-1})^2$$

$$= \gamma(0) - 2\beta\gamma(1) + \gamma(0)$$

$$\boxed{\begin{array}{l} \text{As } E(X_t) = 0 \\ \Rightarrow V(X_t) = E(X_t^2) \\ \gamma(h) = Cov(X_t\, X_{t-h}) \end{array}}$$

$$\frac{\partial L(\beta)}{\partial \beta} = 0$$

$$\Rightarrow \quad \beta = \frac{\gamma(1)}{\gamma(0)} = \rho(1) = \phi$$

As For AR(1) : $\rho(h) = (\phi)^{|h|}$

Similarly fitting $\hat{X}_t = \eta X_{t-1}$ such that $\min_{\eta} E(X_t - \hat{X}_{t-1})^2$

$\Rightarrow \eta = \phi$

then $Cov(X_{t-2} - \hat{X}_{t-2}, X_t - \hat{X}_t)$

$$= Cov(X_{t-2} - \phi X_{t-1}, X_t - \phi X_{t-1})$$

$$= Cov(X_{t-2} - \phi X_{t-1}, e_t) \quad \text{as } X_t = \phi X_{t-1} + e_t$$

$$= 0 \quad \longleftarrow *$$

As e_t is uncorrelated with it's past and $\{X_t\}$ is a linear Combinaton of $\{e_t\}$ [equation 4]

So $\rho_{22} = 0$ [Using $*$]

Another way of finding PACF:

1. Fit AR(p) model successively increasing orders $p = 1, 2, \ldots, P_{max}$. You need to guess P_{max}. For each value of p fit:

$$X_t = \sum_{j=1}^{p} \phi_{pj} X_{t-j} + e_t$$

for $p = 1, 2, \ldots, P_{max}$. Collect all the coefficient ϕ_{pp} for each model. The value of p for which ϕpp persistently remains zero is the true order of the process.

2.1.2 Durbin Levinson Algorithm

Durbin Levinson algorithm can be used to get ϕ_{pp} values.

Fit AR(1) model: $\phi_{11} = \rho(1)$

the Coefficient of AR(P) model can be obtained

$$\phi_{p+1,p+1} = \frac{\rho(p+1) - \sum_{j=1}^{p} \phi_{pj}\, \rho(p+1-j)}{1 - \sum_{j=1}^{p} \phi_{pj}\, \rho(p+1-j)}$$

$$p = 1, 2 \ldots P_{max}$$

$$\phi_{p+1,j} = \phi_{pj} - \phi_{p+1,p+1}\, \phi_{p,p-j+1} \quad ; j = 1, 2 \ldots P_{max}$$

Note:- For $P=1$: $X_t = \phi_{11} X_{t-1} + e_t$

$p = 2$: $X_t = \phi_{21} X_{t-1} + \phi_{22} X_{t-2} + e_t$

ϕ_{22} is measuring the "direct effect" of X_{t-2} on X_t in presense of X_{t-1}, which is PACF at lag 2.

So

$$\rho_{hh} = \phi_{hh} \quad ; h \geq 1$$

In general for a AR(p) model $\rho_{hh} = 0$ for all $h > p$. PACF function behaves exactly same as ACF function of MA(q) process. So to get the order of a AR process we need to look at the partial auto correlation function of the process.

Example Let the true model be AR(1). If we fit AR(2) model the value of $\phi_2 = 0$.

For AR(1), $\rho(h) = \phi^{|h|}$

Fit an AR(2) model

$$X_t = \phi_1 X_{t-1} - \phi_2 X_{t-2} + e_t$$

Using Yule-Walker equation

$$\begin{pmatrix} \rho(0) & \rho(1) \\ \rho(1) & \rho(0) \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} \rho(1) \\ \rho(2) \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} 1 & \rho(1) \\ \rho(1) & 1 \end{pmatrix}^{-1} \begin{pmatrix} \rho(1) \\ \rho(2) \end{pmatrix}$$

$$\Rightarrow \phi_1 = \frac{\rho(1) - \rho(1)\rho(2)}{1 - \rho^2(1)} \quad \text{———} \; (a)$$

$$\phi_2 = \frac{\rho(2) - \rho^2(1)}{1 - \rho^2(1)} \quad \text{———} \; (b)$$

As the actual model is AR(1) $\rho(1) = \phi$ and $\rho(2) = \phi^2$ — (c)

Using (b) and (c)

$$\phi_2 = \frac{\phi^2 - \phi^2}{1 - \phi^2} = 0$$

So, If the true model is AR(1) and you fit AR(2) model, the value of ϕ_2 will be zero.

3 Auto Regressive Moving Average model ARMA

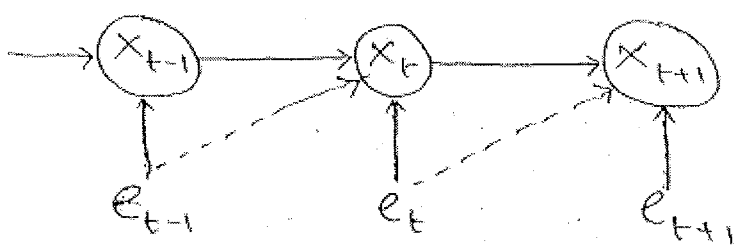
Combining AR(p) and MA(q) we can construct mode general model as:

$$X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p} = e_t - \theta_1 e_{t-1} - \cdots - \theta_q x_{t-q}$$

Using Transfer functions we can also rewrite the model as:

$$\Phi(B)X_t = \Theta(B)e_t$$

Where $\Phi(B) = 1 - \phi_1 B - \cdots - \phi_p B^p$ is calles AR polynomial and $\Theta(B) == 1 - \theta_1 B - \cdots - \theta_q B^q$ is called MA polynomial. We will write the model as $ARMA(p,q)$.



Theorem *An ARMA(p,q) model is stationary/causal iff all the roots of $\Phi(B) = 0$ are outside of an unit circle.*

Theorem *An ARMA(p,q) model is invertiable iff all the roots of $\Theta(B) = 0$ are outside of an unit circle.*

Example Using the above theorems we can easily verify that ARMA(1,1) process is stationary and invertiable iff $|\phi_1| < 1$ and $|\theta_1| < 1$. As an ARMA(1,1) process can be written as:

$$\begin{aligned} & X_t - \phi_1 X_{t-1} = e_t + \theta_1 et - 1 \\ \Rightarrow \ & (1 - \phi_1 B)X_t = (1 - \theta_1 B)e_t \\ \Rightarrow \ & \Phi(B)X_t = \Theta(B)e_t \end{aligned} \tag{8}$$

Where $\Phi(B) = (1 - \phi_1 B)$ and $\Theta(B) = (1 - \theta_1 B)$

3.0.1 MA representation of ARMA(1,1)

Consider an AR(1,1) process as:

$$X_t - \phi_1 X_{t-1} = e_t + \theta_1 et - 1$$

If the process is stationary and causal, the process can also be written as:

$$X_t = \sum_{i=0}^{\infty} \psi_j e_{t-j}$$

Where $\psi_0 = 1$ and $\psi_i = (\phi_1 + \theta_1)\phi_1^{i-1}$. ARMAtoMA function in R can be used to compute the coefficients.

Proof

$$X_t - \phi_1 X_{t-1} = e_t + \theta_1 e_{t-1}$$

$$\Rightarrow (1 - \phi_1 B) X_t = (1 + \theta_1 B) e_t$$

$$\Rightarrow X_t = (1 - \phi_1 B)^{-1} (1 + \theta_1 B) e_t$$

$$= (1 + \phi_1 B + \phi_1^2 B^2 + \phi_1^3 B^3 + \cdots)(1 + \theta_1 B) e_t$$

$$= \left[1 + (\phi_1 + \theta_1) B + (\phi_1 + \theta_1) \phi B^2 + \cdots \right] e_t$$

So $X_t = \sum_{j=0}^{\infty} \psi_j e_{t-j}$ where $\psi_0 = 1$

$\psi_i = (\phi_1 + \theta_1)\phi_1^{i-1}$

$i = 1, 2, \ldots$

3.0.2 Invertiable representation of ARMA(1,1)

Consider an AR(1,1) process as:

$$X_t - \phi_1 X_{t-1} = e_t + \theta_1 et - 1$$

If the process is stationary and causal, the process can also be written as:

$$e_t = \sum_{j=0}^{\infty} \pi_j X_{t-j}$$

Where $\pi_0 = 1$ and $\pi_i = (-1)^i(\phi_1 + \theta_1)\theta_1^{i-1}$. ARMAtoAR function in R can be used to compute the coefficients.

Proof

$$X_t - \phi_1 X_{t-1} = e_t + \theta_1 e_{t-1}$$

$$\Rightarrow (1 + \theta B) e_t = (1 - \phi_1 B) X_t$$

$$\Rightarrow e_t = (1 + \theta B)^{-1} (1 - \phi_1 B) X_t$$

$$= (1 - \theta_1 B + \theta_1^2 B^2 - \theta_1^3 B^3 + \cdots) (1 - \phi_1 B) X_t$$

$$= [1 - (\theta_1 + \phi_1) B + \theta_1 (\theta_1 + \phi_1) B^2 - \theta_1^2 (\theta_1 + \phi_1) B^3 + \cdots]$$

So $$e_t = \sum_{j=0}^{\infty} \pi_j X_{t-j}$$

where $\pi_0 = 1$

$$\pi_j = (-1)^j \theta_1^{j-1} (\theta_1 + \phi_1)$$

3.1 ACF of ARMA(1,1)

Auto-covariance function of an ARMA(1,1) is given by:

$$\gamma(0) = \sigma^2 \frac{1 + 2\phi_1\theta_1 + \theta_1^2}{1 - \phi_1^2}$$

for $|h| > 1$

$$\gamma(h) = \sigma^2 \frac{(1 + \theta_1\phi_1)(\theta_1 + \phi_1)}{1 - \phi_1^2}$$

The Auto correlation function is:

$$\rho(h) = \frac{(1 + \theta_1\phi_1)(\theta_1 + \phi_1)}{(1 + 2\phi_1\theta_1 + \theta_1^2)} \phi_1^{h-1}$$

Note: ACF of AR(1) and ARMA(1,1) has same pattern.
Note: When $\theta_1 = -\phi_1$ then $\rho(h) = 0$ for all h. This means the process is White noise process.

3.2 Parameter redundancy, Invertiability Causality

Consider the ARMA model.

$$X_t = 0.4X_{t-1} + 0.45X_{t-2} + e_t + e_{t-1} + 0.25\,e_{t-2}$$

This appears to be AR (2, 2), but:—

$$(1 - 0.4B - 0.45B^2)X_t = (1 + B + 0.25B^2)\,e_t$$

$$\Rightarrow (1 + 0.5B)(1 - 0.9B)X_t = (1 + 0.5B)^2\,e_t$$

$$\Rightarrow \cancel{(1+0.5B)}\,(1 - 0.9B)X_t = (1 + 0.5B)\,e_t$$

This is actually an ARMA(1,1) model, all the extra parameters are not required.

So the actual model is

$$\phi(B) X_t = \theta(B) e_t$$

where
$$\phi(B) = 1 - 0.9B$$
$$\theta(B) = 1 + 0.5B$$

Root of $\phi(B) = 0 \Rightarrow B = \frac{10}{9} \Rightarrow |B| > 1$

" " $\theta(B) = 0 \Rightarrow B = -2 \Rightarrow |B| > 1$.

So the process is stationary/causal and invertiable.