Lecture Notes #4

Time series analysis (STAT 5140/4140)

1 Moving Average process

We have derived the ACVF function for MA(1) process, using that we can get the auto correlation function for this process. ACF of MA(1) process is:

$$\rho(h) = \begin{cases} 1 & \text{for } h = 0 \\ \frac{\theta_1}{1+\theta_1{}^2} & \text{for } |h| = 1 \\ 0 & \text{for } |h| > 1 \end{cases}$$

Note: In R ARMAacf functuion can be used to plot this.

Example Consider a function below:

$$f(h) = \begin{cases} 1 & \text{for } h = 0 \\ c & \text{for } |h| = 1 \\ 0 & \text{for } |h| > 1 \end{cases}$$

Can this function be a ACF of MA(1) process for any value of c ?

Equating the functions $\rho(h)$ and $f(h)$ at $h = 1$

$$\begin{aligned} \frac{\theta_1}{1+\theta_1{}^2} &= c \\ \Rightarrow \theta_1{}^2 c + c - \theta_1 &= 0 \\ \Rightarrow \theta_1{}^2 - \frac{1}{c}\theta_1 + 1 &= 0 \end{aligned} \tag{1}$$

We must have real roots for the above equation. So we must have:

$$\begin{aligned} \frac{1}{c^2} - 4 \geq 0 \\ |c| \leq \frac{1}{2} \end{aligned} \tag{2}$$

So, a MA(1) process can never produce an ACF $\rho(1) > \frac{1}{2}$. The function $f(h)$ can be ACF of a stationary MA(1) process iff $|c| \leq \frac{1}{2}$.

Definition *Auto covariance generating function of a random process is defined as:*

$$g(z) = \sum_{l=-\infty}^{\infty} \gamma(l) z^{-l} \tag{3}$$

The covariance generating function can be used to find ACVF of a random process. Note that the coefficient of z^{-l} of the function is $\gamma(l)$. Any causal (no future white noise term) random process can be written as:

$$\begin{aligned} X_t &= \sum_{i=0}^{i=\infty} \theta_t e_{t-i} \\ &= \sum_{i=0}^{i=\infty} \theta_t B^n e_t \\ &= H(B) e_t \end{aligned} \tag{4}$$

Where B is called backward shift operator defined as $B^i X_t = X_{t-i}$, and H(B) is called transfer function, with $H(B) = \sum_{i=0}^{i=\infty} \theta_t B^n$ Using the backward shift operator, B^i, we can write auto covariance generating function as:

$$g_\sigma(z) = \sigma^2 H(z) H(z^{-1})$$

We will use this to compute ACVF of a function.

Example Consider a MA(1) process

$$\begin{aligned} X_t &= e_t + \theta_1 e_{t-1} \\ &= e_t + \theta_1 B e_t \\ &= (1 + \theta_1 B) e_t \end{aligned} \tag{5}$$

For this process the transfer function $H(B) = 1 + \theta_1 B$

Auto covariance generating function for this process is:

$$\begin{aligned} g_\sigma(z) &= \sigma^2 H(z) H(z^{-1}) \\ &= \sigma^2 (1 + \theta_1 z)(1 + \theta_1 z_{-1}) \\ &= \sigma^2 (1 + {\theta_1}^2) + \theta_1 \sigma^2 z + {\theta_1}^2 \sigma^2 z^{-1} \end{aligned} \tag{6}$$

Comparing the Equation (3) and (6): $\gamma(0) =$ coefficient of z^0, $\gamma(1) =$ coefficient of z^1 etc. So the ACVF function is given by:

$$\gamma(h) = \begin{cases} \sigma^2(1+{\theta_1}^2), & \text{for } h = 0 \\ \theta_1\sigma^2 & \text{for } |h| = 1 \\ 0 & \text{for } |h| > 1 \end{cases}$$

It is easy to get the auto covariance function using the generating function.

Example Consider the ptocess $X_t = -d_1X_{t-1} - e_t$. Use the auto-covariance generating function find ACVF. Hint: Transfer function for this process is $H(B) = (1+d_1B)^{-1} = 1 - d_1B + {d_1}^2B - \dots$
Ans: ACF of this process $\rho(h) = (-d_1)^{|h|}$

2 Non-uniqueness of MA models

In real life we will have data from a MA process. Using the data we need to estimate the parameter. Consider a MA(1) process $X_t = e_t + \theta_1 e_{t-1}$, where $\{e_t\} \sim WN(0, \sigma^2)$.To estimate the model we need to estimate two parameters σ^2 and θ_1. We know the $ACVF$ function of a $MA(1)$ process is:

$$\gamma(h) = \begin{cases} \sigma^2(1+{\theta_1}^2), & \text{for } h = 0 \\ \theta_1\sigma^2 & \text{for } |h| = 1 \\ 0 & \text{for } |h| > 1 \end{cases}$$

Note that there are two MA(1) models can produce this ACVF:

$$X_t = e_t + \theta_1 e_{t-1}$$

and

$$X_t = e_t + \frac{1}{\theta_1} e_{t-1}$$

Now the question is which model to choose. We need to keep that in mind that the purpose of model fitting is prediction. That our model should be useful to predict X_{t+1} given all history upto time t, that is goal is to find $X\widehat{[t+1}|t] = E[X_{t+1}|t]$.

$$X\widehat{[t+1}|t] = E[X_{t+1}|t] = E[e_{t+1} + \theta_1 e_t|t] = \theta_1 E[e_t|t] \tag{7}$$

Note : $E[e_t|t]$ no not equal to zero. As X_t and X_{t-1} contains information about e_t So, for forecasting X_{t+1}

we need to estimate the shock e_t. From the model:

$$\begin{aligned} e_t &= X_t - \theta_1 e_{t-1} \\ &= X_t - \theta_1\{X_{t-1} - \theta_1 e_{t-2}\} \\ &= X_t - \theta_1\ X_{t-1} + \theta_1^2 e_{t-1} \\ &= \ldots\ldots \\ &= X_t + \sum_{k=1}^{\infty} (-\theta_1)^k X_{t-k} \end{aligned} \tag{8}$$

In the above equation the infinite sum converges if $|\theta_1| < 1$. In other words the $\{e_t\}$ is stationary if $|\theta_1| < 1$. So to resolve the non uniqueness of MA(1) process we should choose the value of such that $|theta_1| < 1$. Also note that Equation (8) implies that to recover $\{e_t\}$ we need to have complete history of the process.

If the actual series is generated using $\theta_1 = 2$, and let this information is known. For prediction purpose we can't use the model with $\theta_1 = 2$. We must use the model with $\theta = \frac{1}{2}$. As $\theta_1 = 2$ model will not generate stable $\{e_t\}$.

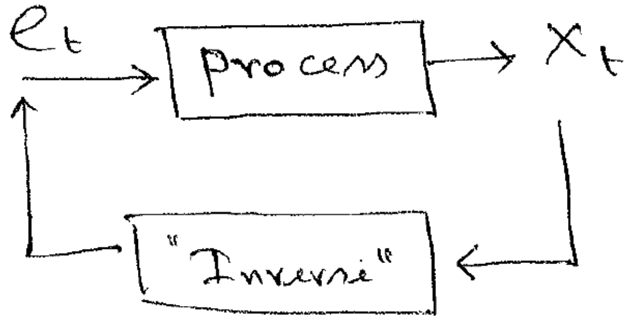

Figure 1: Invertiable Random Process

In general any linear random process can be written as

$$X_t = H(B)e_t$$

, where $H(B)$ is the transfer function discussed before. We are seeking a inverse operator $H(\tilde{B})$ such that

$$e_t = H(\tilde{B})X_t$$

with $H(\tilde{B})H(B) = 1$ and $\{e_t\}$ is stationary. It can be shown that $H(\tilde{B}) = H^{-1}(B)$.

Example Consider MA(1) process $X_t = e_t + \theta e_{t-1}$. Using the transfer function notation we can rewrite the model as $X_t = H(B)e_t$ where $H(B) = 1 + \theta B$.

$$\begin{aligned} H^{-1}(B) &= (1+\theta B)^{-1} \\ &= 1 - \theta B + \theta^2 B^2 - \theta^3 B^3 + \dots \end{aligned} \tag{9}$$

provided $|\theta| < 1$. So we can write the model in Equation (9) as

$$e_t = H^{-1}(B)X_t$$

. A time series us called invertiable if $H^{-1}(B)$ exist and it must produce a stationary $\{e_t\}$.

Definition *A linear random process $\{X_t\}$ is called invertiable if there is function $\Phi(B) = \phi_0 + \phi_1 B + \phi_2 B^2 + \dots$ with $\sum_{i=0}^{\infty} |\phi_i| < infty$ and $e_t = \Phi(B)X_t$, where $B^i X_t = X_{t-i}$*

In short this definition means a stationary $\{e_t\}$ can be recovered using X_t.

Definition *A linear random process $\{X_t\}$ is called causal if there ts a function $\Psi(B) = \psi_0 + \psi_1 B^1 + \psi_2 B^2 + \dots$ with $\sum_{i=0}^{\infty} |\psi_i| < \infty$ and $X_t = \Psi(B)e_t$.*

This defination simply means in your model you do not have future values. That is X_t is only dependent on past shocks.

Example Consider MA(q) process

$$X_t = e_t + \theta_1 e_{t-1} + \cdots + \theta_q e_{t-q}$$

. Another way writing the process is

$$X_t = \Theta(B)e_t$$

, where $\Theta(B) = \sum_{i=0}^{q} \theta_i B^i$, $\theta_0 = 1$. So the process is always casual. But the process is not always invertiable. This process is invertiable if $\Theta^{-1}(B)$ exist.

Theorem *A MA(q) process, $X_t = \Theta(B)e_t$, is invertiable if all the roots of $\Theta(B) = 0$ are outside of a unit circle.*

Example Consider the process

$$X_t = e_t + 0.5e_{t-1} - 0.2e_{t-2} + 0.6e_{t-3}$$

. Is this process invertiable ? Lets rewrite the process as:

$$\begin{aligned} X_t &= (1 + 0.5B - 0.2B^2 + 0.6B^3)e_t \\ &= \Theta(B)e_t \end{aligned} \tag{10}$$

where $\Theta(B) = 1 + 0.5B - 0.2B^2 + 0.6B^3$ is the transfer function. Solving $\Theta(B) = 0$ we get:

$$z_1 = 0.606 - 1.237i$$

$$z_2 = 0.606 + 1.237i$$

$$z_3 = -0.878 + 0i$$

As $|z_3| < 1$, the process is not invertiable. But the process is causal.