Lecture Notes #6

Time series analysis (STAT 5140/4140)

1 Non Stationary process

Till now we have been discussing stationary processes. But most of the real life processes are not stationary. There several type of non-stationary processes. We will mainly discuss two type:

1. Deterministic trend

2. Stochastic type trend

1.1 Deterministic trend

Deterministic trend in a time series can be removed by fitting a polynomial trend or one cal also estimate the trend using non parametric methods.

Example Consider a time series model as:

$$X_t = a + bt + Y_t$$

Where $\{Y_t\}$ is a stationary process, with $E[Y_t] = 0$. The linear trend can removed by fitting a a linear regression model. Note that $E[X_t] = a + bt$ the trend is nothing but the mean of the process.

In general a deterministic trend model can be written as:

$$X_t = m(t, \beta) + Y_t$$

where $m(t, \beta)$ is a polynomial of degree k with parameters $\beta = (\beta_1, \ldots, \beta_k)$. The set of parameter can e estimate by minimizing mean squre error:

$$S(\beta) = \sum_{t=1}^{n} [X_t - m(t, \beta)]^2$$

For example please see the R code in D2L.

1.1.1 Removing deterministic trend using non-parametric method.

Several non parametric methods can be used to

Moving average

For a time series process $X_t = m(t, \beta) + Y_t$ we define a moving average (of size q) smoother as:

$$W_t = \frac{1}{(2q+1)} \sum_{j=-q}^{q} X_{t-j}$$

for a sample of size n, $p + 1 \leq t \leq n - q$.

Example Consider a sample from a time series $\{X_1 = 5, X_2 = 3, X_3 = 4, X_4 = 5, X_5 = 6, X_6 = 8, X_7 = 9, X_8 = 5, X_9 = 2\}$. Consider $q = 1$, then $W_1 = \frac{1}{3}(X_0 + X_1 + X_2)$. But X_0 is not available. Similarly we need the value of X_10, which we do not have.

So this moving average method has issue at the boundary points. In other words at the boundary it requires future/past values. This is the reason moving average method is also called lagging indicator. One can assume $X_t = X_1$ if $t < 1$ and $X_t = X_n$ for all t¿n. In the example once can use $X_0 = X_1 = 5$ and $X_{10} = X_9 = 2$. In particular if the trend is linear over $[t - q, t + q]$ moving average process can be used to extract the trend. That is the moving average has no effect on linear trend. If the time series process is $X_t = a + bt + Y_t$. Then the moving average of the process is:

$$\begin{aligned} W_t &= \frac{1}{(2q+1)} \sum_{j=-q}^{q} X_{t-j} \\ &= \frac{1}{(2q+1)} \sum_{j=-q}^{q} a + b(t-j) + Y_{t-j} \\ &= a + bt + \frac{1}{(2q+1)} \sum_{j=-q}^{q} Y_{t-j} \end{aligned} \tag{1}$$

If you choose q such that $\sum_{j=-q}^{q} Y_{t-j} = 0$. Under such conditions we can estimate the trend as

$$\hat{m}_t = \frac{1}{2q+1} \sum_{j=-q}^{q} X_{t-j}$$

. But we can not choose large value of q. As it is subject to the condition that the trend is linear over $(t - q, t + q)$. In general moving average "filter" can also be used with different weights as:

$$W_t = \sum_{j=-q}^{q} a_j X_{t-j}$$

One can choose weights carefully to remove quadratic trend.

Exponential Smoothing

Another way of smoothing a time series is using Exponential smoother.

$$m(t) = \alpha X_t + (1-\alpha)m(t-1)$$

Where m(t) is the estimated value of the process at time t, with $m(1) = X_1$, and $\alpha \in (0,1)$ is smoothing parameter. For $t \geq 2$ this equation can also be written as:

$$m(t) = (1-\alpha)^{t-1}X_1 + \sum_{j=0}^{t-2}(1-\alpha)^j X_{t-j}$$

This is also a moving average process with exponentially decaying weights. What will happen if you choose $\alpha = 0$ or $\alpha = 1$???

Kernel Smoothing

This is also a moving average process with weights:

$$a_t = \sum_{i=1}^{n} \omega_i(t) X_i$$

Where

$$\omega_t = \frac{K(\frac{t-i}{b})}{\sum_{j=1}^{n} K(\frac{t-i}{b})}$$

K is a Kernel function. One example of such function is Gaussian Kernel.

$$K(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

Differencing

A special type of moving average method when weights are +1 and -1. Lag 1 differencing of $\{X_t\}$ is

$$\nabla X_t = X_t - X_{t-1}$$

Using backward shift operator we ca also write.

$$\nabla X_t = (1-B)X_t$$

In general $\nabla^d = (1-B)^d$. Consider a time series with polynomial trend m(t) as:

$$X_t = c_0 + c_1 t + c_2 t^2 + \cdots + c_n t^n + Y_t$$

Where $\{Y_t\}$ is a stationary process, then:

$$\nabla^k X_t = k! c_k + \nabla^k Y_t$$

This means if the noise has zero mean the k^{th} differencing of the time series with a polynomial trend of degree k should give a stationary process.

Eliminating trend and seasonality

Assume a time series with deterministic trend and seasonality component as:

$$X_t = m(t) + S_t + Y_t$$

Where $\{Y_t\}$ is the stationary process with $E[Y_t] = 0$ $m(t)$ is the trend and S_t is the seasonal component with period d. That is $S_t = S_{t-d}$ and $\sum_{i=1}^{d} S_i = 0$. Idea of estimating the trend using the differencing is to consider a moving average of size $q = d$. For a monthly data (d=12) for b years, estimated trend and seasonality is:

$$\hat{m}_j = \frac{1}{d} \sum_{k=1}^{d} X_{jk} \tag{2}$$

$$\hat{S}_k = \frac{1}{b} \sum_{j=1}^{b} (X_{kj} - \hat{m}_j) \tag{3}$$

To estimate stationary component use:

$$\hat{Y}_{jk} = X_{jk} - \hat{m}_j - \hat{S}_k$$

Other algorithms are also available to extract seasonality and trend from a data set. We will use R to extract trend and seasonality. See R code in D2L.

Differencing can also be use to remove trend and seasonality. For example:

Example

Consider $X_t = m_t + S_t + Y_t$

with $S_t = S_{t-d}$

Seasonality can be removed by $\nabla_d = (1-B^d)$

$$
\begin{aligned}
\nabla_d X_t &= X_t - X_{t-d} \\
&= (1-B^d)X_t \\
&= X_t - X_{t-d} \\
&= m_t + S_t + Y_t - (\cancel{*}\, m_{t-d} + S_{t-d} + Y_{t-d}) \\
&= m_t + \cancel{S_t} + Y_t - m_{t-d} - \cancel{S_{t-d}} - Y_{t-d}
\end{aligned}
$$

$$\nabla_d X_t = m_t - m_{t-d} + Y_t - Y_{t-d}$$

In this process we used ∇_d operator to remove seasonality. The trend can be removed by further differencing.

Although this process of differencing is great one should be careful as this may lead in to a non-stationary process.

Example

Consider $X_t = \alpha + \beta t + Y_t$, assume $\{Y_t\}$ is a MA(1) process $Y_t = e_t + \theta e_{t-1}$; $|\theta| < 1$

then

$$
\begin{aligned}
\nabla X_t &= X_t - X_{t-1} \\
&= \alpha + \beta t + Y_t - \alpha - \beta(t-1) - Y_{t-1} \\
&\cancel{= \beta + \alpha(1-B)X_t} \\
&= \beta + (Y_t - Y_{t-1}) \\
&= \beta + (1-B)Y_t
\end{aligned}
$$

Consider

$$
\begin{aligned}
W_t &= (1-B)Y_t \\
&= Y_t - Y_{t-1} \\
&= e_t + \theta e_{t-1} - \{e_{t-1} + \theta e_{t-2}\} \\
&= (1 + (\theta - 1)B - \theta B^2)Y_t \\
&= \Theta(B)Y_t
\end{aligned}
$$

So W_t is ~~not~~ now a MA(2) process, But $\theta = 1$ is a root of $\Theta(B) = 0$. So the process $\{W_t\}$ is not invertiable.