Homework #2 Solution :-

1 (a) Given process $X_t = a + bZ_t + cZ_{t-2}$

$E(X_t) = a$ as $E(Z_t) = 0$

$$\gamma(h) = Cov(X_t, X_{t+h})$$
$$= Cov(a + bZ_t + cZ_{t-2}, a + bZ_{t+h} + cZ_{t+h-2})$$
$$= Cov(bZ_t, bZ_{t+h}) + Cov(bZ_t, cZ_{t+h-2})$$
$$+ Cov(cZ_{t-2}, bZ_{t+h}) + Cov(cZ_{t-2}, cZ_{t+h-2})$$

If $h=0$

$$\gamma(0) = Cov(bZ_t, bZ_t) + Cov(cZ_{t-2}, cZ_{t+h-2})$$
$$= b^2 Cov(Z_t, Z_t) + c^2 Cov(Z_{t-2}, Z_{t-2})$$
$$= b^2 V(Z_t) + c^2 V(Z_{t-2})$$
$$= b^2\sigma^2 + c^2\sigma^2.$$

for $h = 2$

$$\gamma(2) = Cov(bZ_t, cZ_t) = bc\sigma^2$$

the autocovariance function is

$$\gamma(h) = \begin{cases} b^2\sigma^2 + c^2\sigma^2 & ; h = 0 \\ bc\sigma^2 & ; h = \pm 2 \\ 0 & ; h = \pm 1, |h| > 2 \end{cases}$$

As $E(X_t) = a$, not a function of t, $V(X_t) < \infty$ and $Cov(X_t, X_{t+h})$ is a function of h only. The process is stationary.

(b) The process

$$X_t = Z_t \cos(ct) + Z_{t-1} \sin(ct)$$

$$E(X_t) = 0$$

$$\begin{aligned}
\gamma(h) &= \operatorname{Cov}(X_t, X_{t+h}) \\
&= \operatorname{Cov}\big(Z_t \cos(ct) + Z_{t-1} \sin(ct),\ Z_{t+h} \cos(ct+ch) \\
&\qquad + Z_{t+h-1} \sin(ct+ch)\big)
\end{aligned}$$

if $h = 1$

$$\begin{aligned}
\gamma(1) &= \operatorname{Cov}\big(Z_t \cos(ct) + Z_{t-1} \sin(ct),\ Z_{t+1} \cos(ct+K) \\
&\qquad + Z_t \sin(ct+c)\big) \\
&= \operatorname{Cov}\big(Z_t \cos(ct),\ Z_t \sin(ct+c)\big) \\
&= \sin(ct+c) \cos(ct)\, \sigma^2
\end{aligned}$$

$\gamma(1)$ is a function of t, so the process is not stationary.

(c) Given process

$$X_t = a + b Z_0$$

$$E(X_t) = 0 \quad , \quad \begin{aligned}[t]
\gamma(h) &= \operatorname{Cov}(X_t, X_{t+h}) \\
&= \operatorname{Cov}(a + b Z_0, a + b Z_0) \\
&= b^2 \operatorname{Cov}(Z_0, Z_0) \\
&= b^2 \sigma^2
\end{aligned}$$

The process is stationary.

(d) The process is $X_t = Z_t \cdot Z_{t-1}$

$$E(X_t) = E(Z_t Z_{t-1})$$

$= E(Z_t) E(Z_{t-1})$ as $\{Z_t\}$ are independent.

$= 0$

$$\gamma(h) = Cov(X_t, X_{t+h})$$

$= \cancel{Cov(Z_t Z_{t+})} = Cov(Z_t Z_{t-1}, Z_{t+h} Z_{t+h-1})$

$= E(Z_t Z_{t-1} Z_{t+h} Z_{t+h-1})$

$= E(Z_t) E(Z_{t-1}) E(Z_{t+h}) E(Z_{t+h-1})$

as $\{Z_t\}$ are independent.

$\gamma(0) = E(Z_t^2 Z_{t-1}^2) = E(Z_t^2) E(Z_{t-1}^2)$

$= \sigma^4$

$$\gamma(h) = \begin{cases} \sigma^4 & ; h = 0 \\ 0 & ; h \geq 1. \end{cases}$$

The process is stationary.

(2) This is a MA(1) process

$$\gamma(h) = \begin{cases} \sigma^2 (1+\theta^2) & ; h = 0 \\ \theta \sigma^2 & ; h = 1 \\ 0 & ; |h| \geq 2 \end{cases}$$

when $\theta = 0.8$

$$\gamma(h) = \begin{cases} 1.64\sigma^2 & ; \quad h = 0 \\ 0.8\sigma^2 & ; \quad h = 1 \\ 0 & ; \quad |h| \geq 2 \end{cases} \quad \text{——— (1)}$$

(b) $V(\bar{X})$

$= V\left(\frac{1}{4}(X_1 + X_2 + X_3 + X_4)\right)$

$= \frac{1}{16} V(X_1 + X_2 + X_3 + X_4)$

$= \frac{1}{16}\Big[V(X_1) + V(X_2) + V(X_3) + V(X_4) + 2\,Cov(X_1, X_2)$
$+ 2\,Cov(X_1, X_3) + 2\,Cov(X_1, X_4)$
$+ 2\,Cov(X_2, X_3) + 2\,Cov(X_2, X_4)$
$+ 2\,Cov(X_3, X_4)\Big]$

$= 0.71\sigma^2$ Using (1)

(3)

$E(X_t) = 0$; for all t.

$$V(X_t) = \begin{cases} V(Z_t) = 1 & ; \quad \text{if } t \text{ is even} \\ V\left(\frac{1}{\sqrt{2}}(Z_{t-1}^2 - 1)\right) & \quad \text{if } t \text{ is even} \\ = \frac{1}{2} V(Z_{t-1}^2) & \\ = \frac{1}{2} 2 = 1 & \end{cases}$$

$Z_{t-1} \sim N(0,1)$

$Z_{t-1}^2 \sim \chi^2_{(1)}$

So $V(Z_t) = 0$ for all t.

Consider t even and $h \neq 0$ such that $t+h$ is even

$$\text{Cov}(X_t, X_{t+h})$$
$$= \text{Cov}(Z_t, Z_{t+h}) = 0 \quad \text{as } \{Z_t\} \text{ is an iid process}$$

for t odd and $h \neq 0$, $t+h$ odd

$$\text{Cov}(X_t, X_{t+h})$$
$$= \text{Cov}\left(\frac{1}{\sqrt{2}}(Z_{t-1}^2 - 1), \frac{1}{\sqrt{2}}(Z_{t+h-1}^2 - 1)\right)$$
$$= \frac{1}{2}\text{Cov}(Z_{t-1}^2, Z_{t+h-1})$$
$$= 0 \quad \text{as they are independent}$$

Similarly you can prove for all other cases the covariance is zero.

The process is uncorrelated $\Rightarrow$ white noise.

$P(X_t \le x) = P(Z_t \le x)$ for t even

$P(X_t \le x) = P\left(\frac{1}{\sqrt{2}}(Z_{t-1}^2 - 1) \le x\right)$ for t odd.

$$= P\left(-\sqrt{\sqrt{2}x+1} < Z_t < \sqrt{\sqrt{2}x+1}\right)$$

In particular if $x = 0$

$P(X_t \le 0) = 0.5$ if t even

$P(X_t \le 0) = P(-1 < Z_t < 1) = 0.68$. if t odd.

CDF of $\{X_t\}$ is dependent on t. $\Rightarrow$ Not an iid process

(4) As $X_t = U_1 \sin(2\pi\omega t) + U_2 \cos(2\pi\omega t)$

$E(X_t) = 0$ as $E(U_1) = E(U_2) = 0$

$V(X_t) = V(U_1 \sin(2\pi\omega t) + U_2 \cos(2\pi\omega t))$

$= V(U_1) \sin^2(2\pi\omega t) + V(U_2) \cos^2(2\pi\omega t)$

$= \sigma^2 (\sin^2(2\pi\omega t) + \cos^2(2\pi\omega t))$

$= 1$

for $h \neq 0$

$\gamma(h) = Cov(X_t, X_{t+h})$

$= Cov((U_1 \sin(2\pi\omega t) + U_2 \cos(2\pi\omega t)), U_1 \sin(2\pi\omega(t+h)) + U_2 \cos(2\pi\omega(t+h)))$

$= Cov(U_1 \sin(2\pi\omega t), U_1 \sin(2\pi\omega(t+h))) + Cov(U_2 \cos(2\pi\omega t) + U_2 \cos(2\pi\omega(t+h)))$

$= \sigma^2 [\sin(2\pi\omega t) \sin(2\pi\omega(t+h)) + \cos(2\pi\omega t) \cos(2\pi\omega(t+h))]$

$= \sigma^2 \cos(2\pi\omega h)$ $\quad [\cos(A-B) = \sin A \sin B + \cos A \cos B]$

$$\gamma(h) = \begin{cases} \sigma^2; & h = 0 \\ \sigma^2 \cos(2\pi\omega h); & h \neq 0 \end{cases}$$

5 $MSE_A(\hat{X}_{t+l})$

$= E\left[\left(X_{t+l} - \hat{X}_{t+l}\right)^2\right]$

$= E\left[\left(X_{t+l} - AX_t\right)^2\right]$

$= E\left(X_{t+l}^2 - 2AX_tX_{t+l} + A^2X_t^2\right)$

To minimize $MSE_A(\hat{X}_{t+l})$

$$\frac{d}{dA} MSE_A(\hat{X}_{t+l}) = 0$$

$\Rightarrow 2AE(X_t^2) - 2E(X_tX_{t+l}) = 0$

$\Rightarrow A = \dfrac{E(X_t, X_{t+l})}{E(X_t^2)} = \dfrac{\gamma(l)}{\gamma(0)}$, $\boxed{E(X_t) = 0}$

6 Let $E(X_t) = \mu_x$, $E(Y_t) = \mu_y$

$Cov(X_t, X_{t+h}) = \gamma_X(h)$ as $\{X_t\}$ is stationary

$Cov(Y_t, Y_{t+h}) = \gamma_Y(h)$ as $\{Y_t\}$ " "

~~Consider~~ $Cov(X_t + Y_t, X_{t+h} + Y_{t+h})$. $E(X_t + Y_t) = \mu_X + \mu_Y$

$= Cov(X_t, X_{t+h}) + Cov(Y_t, Y_{t+h})$

$= \gamma_X(h) + \gamma_Y(h)$ not a function of t

So $\{X_t + Y_t\}$ is a stationary process

(7) (a) The process $X_t = \delta + X_{t-1} + e_t$, ; $t = 1, 2, \ldots$

$X_0 = 0$

$$X_t = \delta + X_{t-1} + e_t$$

$$= \delta + (\delta + X_{t-2} + e_{t-1}) + e_t$$

$$= 2\delta + X_{t-2} + e_{t-1} + e_t$$

$$= 2\delta + (\delta + X_{t-3} + e_{t-2}) + e_{t-1} + e_t$$

$$\vdots$$

$$= t\delta + \sum_{k=1}^{t} e_{t-k} \quad \text{as} \quad X_0 = 0$$

(b) ~~$E(X_t) = \delta t$~~

As $X_t = \delta t + \sum_{k=1}^{t} e_{t-k}$

$$E(X_t) = E(\delta t) + E\left(\sum_{k=1}^{t} e_{t-k}\right)$$

$$= \delta t + 0$$

$$= \delta t$$

$$V(X_t) = V\left(\delta t + \sum_{k=1}^{t} e_{t-k}\right)$$

$$= V\left(\sum_{k=1}^{t} e_{t-k}\right)$$

$$= \sum_{k=1}^{t} V(e_{t-k}) = \sigma^2 t.$$

as $\{e_t\}$ is uncorrelated: $Cov(e_t, e_{t+h}) = 0$, $h \neq 0$

Now, $Y_t = X_t - X_{t-1}$

$$= \beta_1 + \beta_2 t e_t - (\beta_1 + \beta_2 (t-1) e_{t-1})$$

$$= \cancel{\beta_1} + \beta_2 t e_t - \cancel{\beta_1} - \beta_2 (t-1) e_{t-1}$$

$$= \beta_2 t e_t - \beta_2 t e_{t-1} + \beta_2 e_{t-1}$$

As $\{e_t\}$ is WN and stationary process.

Y_t is sum of stationary processes.

So $\{Y_t\}$ is stationary.

(c) As $E(X_t) = \delta t$ depends on time, so $\{X_t\}$ is not stationary.

(d) $$X_t - X_{t-1} = \delta + e_t$$

So $Y_t = X_t - X_{t-1}$ is stationary

(8)

$$E(X_t) = E(\beta_1 + \beta_2 t\, e_t)$$

$$= E(\beta_1) + E(\beta_2 t\, e_t)$$

$$= \beta_1 + \beta_2 t\, E(e_t)$$

$$= \beta_1 \quad \text{as } \{e_t\} \sim WN(0, \sigma^2)$$
$$E(e_t) = 0$$

$$V(X_t) = V(\beta_1 + \beta_2 t\, e_t)$$

$$= V(\beta_2 t\, e_t) \qquad [\text{As } V(a + X) = V(X)]$$

$$= \beta_2^2 t^2 V(e_t) \qquad [V(aX) = a^2 V(X)]$$

$$= \sigma^2 \beta_2^2 t^2$$

$V(X_t)$ is a function of time. The process $\{X_t\}$ is not stationary