A course in Time Series Analysis

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December 5, 2018

Contents

Preface

- The material for these notes come from several different places, in particular:
    - Brockwell and Davis (1998) (yellow book)
    - Shumway and Stoffer (2006) (a shortened version is Shumway and Stoffer EZ).
    - Fuller (1995)
    - Pourahmadi (2001)
    - Priestley (1983)
    - Box and Jenkins (1970)
    - Brockwell and Davis (2002) (the red book), is a very nice introduction to Time Series, which may be useful for students who don't have a rigourous background in mathematics.
    - A whole bunch of articles.
- Tata Subba Rao and Piotr Fryzlewicz were very generous in giving advice and sharing homework problems.
- When doing the homework, you are encouraged to use all materials available, including Wikipedia, Mathematica/Maple (software which allows you to easily derive analytic expressions, a web-based version which is not sensitive to syntax is Wolfram-alpha).
- You are encouraged to use `R` (see David Stoffer's tutorial). I have tried to include Rcode in the notes so that you can replicate some of the results.
- Exercise questions will be in the notes and will be set at regular intervals.

- Finally, these notes are dedicated to my wonderful Father, whose inquisitive questions, and unconditional support inspired my quest in time series.

Chapter 1

Time series

A time series is a series of observations x_t, observed over a period of time. Typically the observations can be over an entire interval, randomly sampled on an interval or at fixed time points. Different types of time sampling require different approaches to the data analysis.

In this course we will focus on the case that observations are observed at fixed equidistant time points, hence we will suppose we observe $\{x_t : t \in \mathbb{Z}\}$ ($\mathbb{Z} = \{\ldots, 0, 1, 2 \ldots\}$).

Let us start with a simple example, independent, uncorrelated random variables (the simplest example of a time series). A plot is given in Figure 1.1. We observe that there aren't any clear patterns in the data. Our best forecast (predictor) of the next observation is zero (which appears to be the mean). The feature that distinguishes a time series from classical statistics is that there is dependence in the observations. This allows us to obtain better forecasts of future observations. Keep Figure 1.1 in mind, and compare this to the following real examples of time series (observe in all these examples you see patterns).

1.1 Time Series data

Below we discuss four different data sets.

The Southern Oscillation Index from 1876-present

The Southern Oscillation Index (SOI) is an indicator of intensity of the El Nino effect (see wiki). The SOI measures the fluctuations in air surface pressures between Tahiti and Darwin.

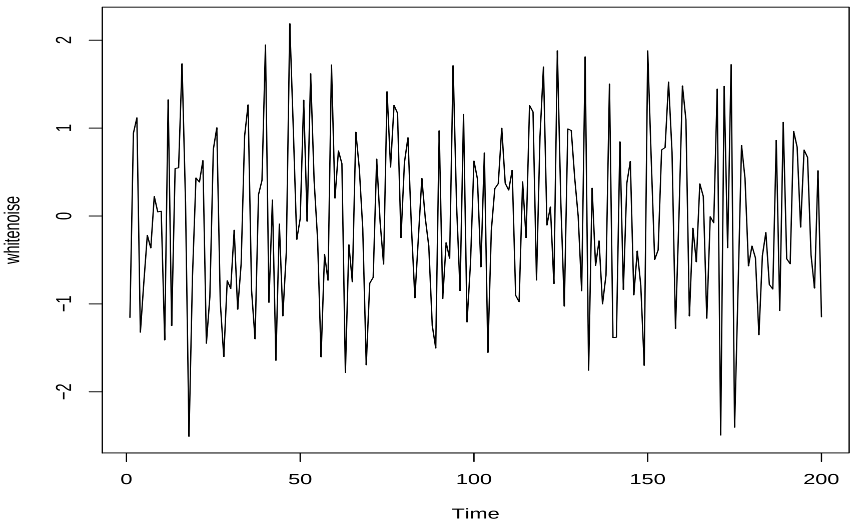


Figure 1.1: Plot of independent uncorrelated random variables

In Figure 1.2 we give a plot of monthly SOI from January 1876 - July 2014 (note that there is some doubt on the reliability of the data before 1930). The data was obtained from `http://www.bom.gov.au/climate/current/soihtm1.shtml`. Using this data set one major goal is to look for patterns, in particular periodicities in the data.

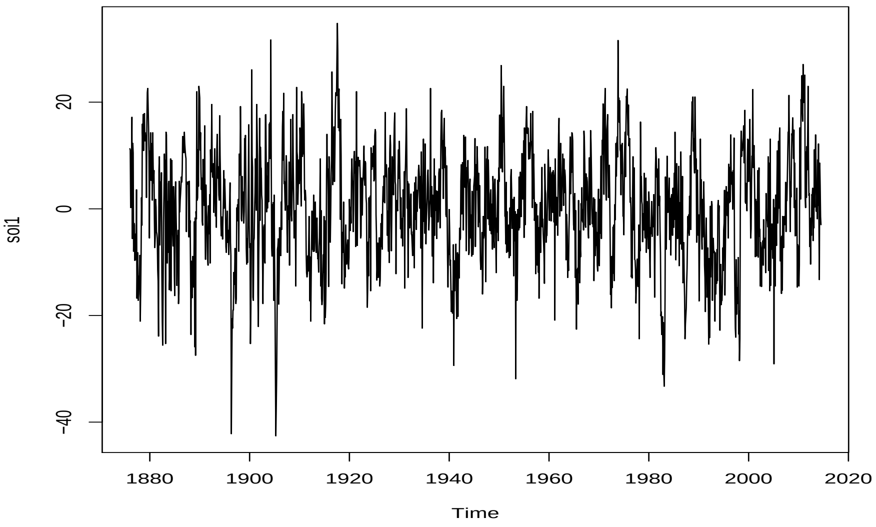


Figure 1.2: Plot of monthly Southern Oscillation Index, 1876-2014

Nasdaq Data from 1985-present

The daily closing Nasdaq price from 1st October, 1985- 8th August, 2014 is given in Figure 1.3. The (historical) data was obtained from `https://uk.finance.yahoo.com`. See also `http://www.federalreserve.gov/releases/h10/Hist/`. Of course with this type of data the goal is to make money! Therefore the main object is to forecast (predict future volatility).

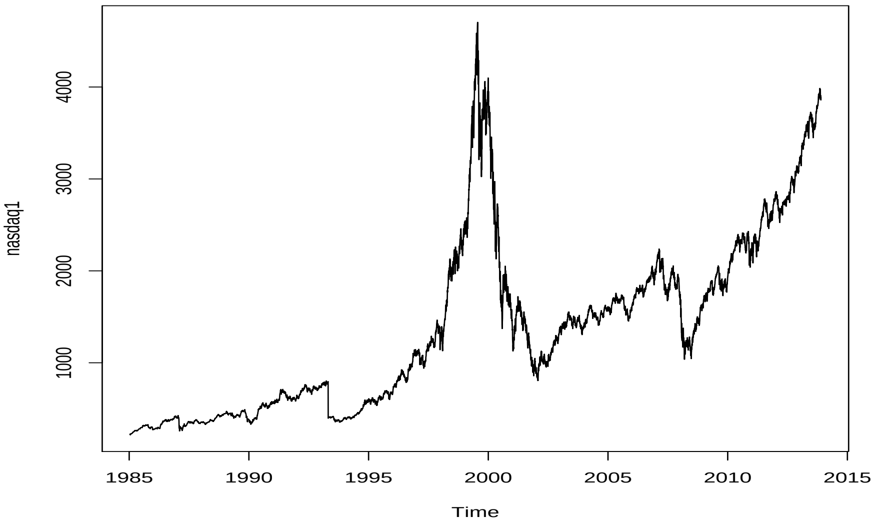


Figure 1.3: Plot of daily closing price of Nasdaq 1985-2014

Yearly sunspot data from 1700-2013

Sunspot activity is measured by the number of sunspots seen on the sun. In recent years it has had renewed interest because times in which there are high activity causes huge disruptions to communication networks (see wiki and NASA).

In Figure 1.4 we give a plot of yearly sunspot numbers from 1700-2013. The data was obtained from `http://www.sidc.be/silso/datafiles`. For this type of data the main aim is to both look for patterns in the data and also to forecast (predict future sunspot activity).

Yearly and monthly temperature data

Given that climate change is a very topical subject we consider global temperature data. Figure 1.5 gives the yearly temperature anomalies from 1880-2013 and in Figure 1.6 we plot

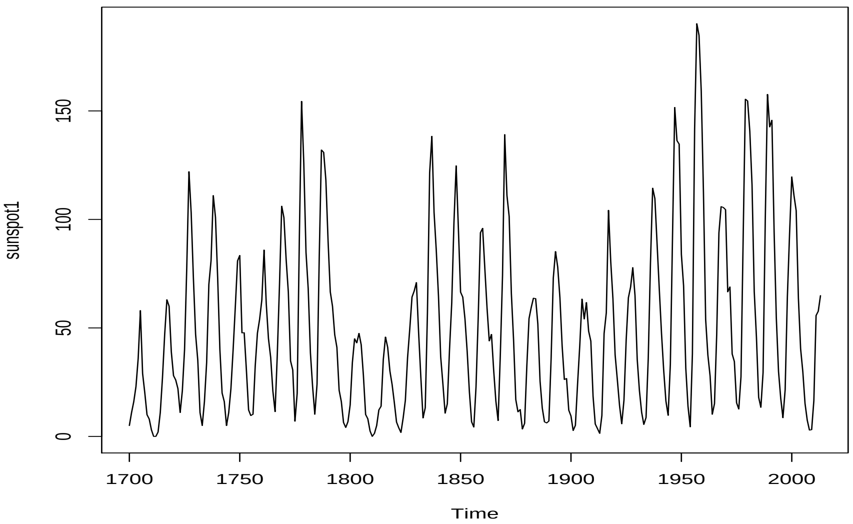


Figure 1.4: Plot of Sunspot numbers 1700-2013

the monthly temperatures from January 1996 - July 2014. The data was obtained from `http://data.giss.nasa.gov/gistemp/graphs_v3/Fig.A2.txt` and `http://data.giss.nasa.gov/gistemp/graphs_v3/Fig.C.txt` respectively. For this type of data one may be trying to detect for global warming (a long term change/increase in the average temperatures). This would be done by fitting trend functions through the data. However, sophisticated time series analysis is required to determine whether these estimators are statistically significant.

1.2 `R` code

A large number of the methods and concepts will be illustrated in `R`. If you are not familar with this language please learn the basics.

Here we give the `R` code for making the plots above.

```
# assuming the data is stored in your main directory we scan the data into R
soi <-  scan("~/soi.txt")
soi1 <-  ts(monthlytemp,start=c(1876,1),frequency=12)
# the function ts creates a timeseries object, start = starting year,
```

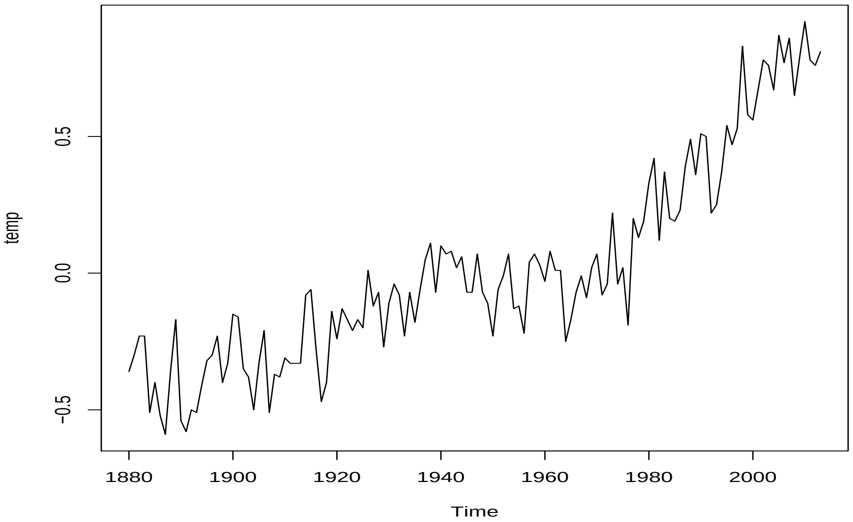


Figure 1.5: Plot of global, yearly average, temperature anomalies, 1880 - 2013

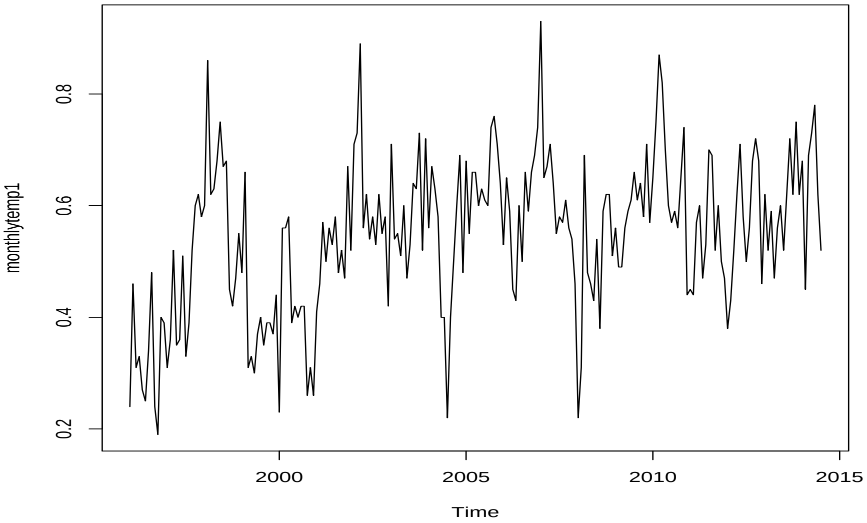


Figure 1.6: Plot of global, monthly average, temperatures January, 1996 - July, 2014.

```
# where 1 denotes January. Frequency = number of observations in a
# unit of time (year). As the data is monthly it is 12.
plot.ts(soi1)
```

Dating plots properly is very useful. This can be done using the package `zoo` and the function `as.Date`.

1.3 Detrending a time series

In time series, the main focus is on understanding and modelling the relationship between observations. Time series analysis is often performed after the data has been detrended. In other words, if $Y_t = \mu_t + \varepsilon_t$, where $\{\varepsilon_t\}$ is zero mean time series, typically we first estimate μ_t and then conduct the time series analysis on the resulting estimated residuals. Once the analysis has been performed, we return to the trend estimators and use the results from the time series analysis to construct confidence intervals etc. In this course the focus will be on the time series after detrending. However, we start by reviewing some well known detrending methods.

A very good primer on detrending is given in Shumway and Stoffer, Chapter 2, and Brockwell and Davis (2002), Chapter 1.

1.3.1 Parametric trend

Often a parametric trend is assumed. Common examples include a linear trend

$$Y_t = \beta_0 + \beta_1 t + \varepsilon_t \tag{1.1}$$

and the quadratic trend

$$Y_t = \beta_0 + \beta_1 t + \beta_2 t^2 + \varepsilon_t. \tag{1.2}$$

For example we may fit such models to the yearly average temperature data. Alternatively, it the time series appears to have include seasonal behaviour we may want to include seasonal terms

$$Y_t = \beta_0 + \beta_1 \sin\left(\frac{2\pi t}{\Omega}\right) + \beta_3 \cos\left(\frac{2\pi t}{\Omega}\right) + \varepsilon_t.$$

For example, we may believe that the Southern Oscillation Index has a period 12 (since the observations are taken monthly). A simple method for modelling the seasonality is to include sine and cosine functions with $\Omega = 12$. For these type of models, least squares can be used to estimate the parameters.

1.3.2 Taking differences to avoid fitting linear and higher order polynomial trends

Let us return to the Nasdaq data. Here we see a different type of "trend" behaviour. This is often refered to as stochastic trend. For most financial data the stochastic trend is removed by taking first difference (after taking logarithms). First differencing also avoids the need of fitting a linear trend to a model. For example if $Y_t = \beta_0 + \beta_1 t + \varepsilon_t$, then

$$Z_t = Y_{t+1} - Y_t = \beta_1 + \varepsilon_{t+1} - \varepsilon_t.$$

Taking higher order differences (ie. taking first differences of $\{Z_t\}$ removes quadratic terms) removes higher order polynomials. The number of differences corresponding to the order of the polynomial.

Beware, taking too many differences can induce "ugly" dependences in the data. Eg. If X_t are iid random variables then $Z_t = X_t - X_{t-1}$ is a dependent random variable. So one should avoid over differencing the data.

Exercise 1.1 *(i) Import the yearly temperature data (file* `global_mean_temp.txt`*) into* `R` *and fit the linear model in (1.1) to the data (use the* `R` *command* `lsfit`*).*

(ii) Suppose the errors in (1.1) are correlated. Under the correlated assumption, explain why the standard errors reported in the `R` *output are unreliable. Actually they are not reported in the output! But they are usually calculated as*

$$\left(\sum_{t=1}^{n}(1,t)'(1,t)\right)^{-1}\frac{1}{n-2}\sum_{t=1}^{n}\widehat{\varepsilon}_t^2.$$

(iii) Make a plot of the residuals after fitting the linear model in (i).

Make a plot of the first differences of the temperature data.

What do you notice about the two plots, are they similar?

The AIC (Akaike Information Criterion) is usually used to select the parameters in the model (see wiki).

1.3.3 Estimation using nonparametric methods

In Section 1.3.1 we assumed that the mean had a certain known parametric form. This may not always be the case. If we have no apriori knowledge of the features in the mean, we can estimate the mean using a nonparametric approach. Of course some assumptions on the mean are still required. And the most common is to assume that the mean μ_t is a sample from a 'smooth' function, i.e. $\mu_t = \mu(\frac{t}{n})$. Under this assumption the following approaches are valid.

Rolling window Possibly one of the simplest methods is to use a 'rolling window'. There are several windows that one can use. We describe, below, the exponential window, since it can be 'evaluated' in an online way. For $t = 1$ let $\hat{\mu}_1 = Y_1$, then for $t > 1$ define

$$\widehat{\mu}_t = (1-\lambda)\widehat{\mu}_{t-1} + \lambda Y_t,$$

where $0 < \lambda < 1$. The choice of λ depends on how much weight one wants to give the present observation. The rolling window is related to the regular window often used in nonparametric regression. To see this, we note that it is straightforward to show that

$$\widehat{\mu}_t \quad = \quad \sum_{j=1}^{t-1}(1-\lambda)^{t-j}\lambda Y_j = \sum_{j=1}^{t}[1-\exp(-\gamma)]\exp\left[-\gamma(t-j)\right]Y_j$$

where $1-\lambda = \exp(-\gamma)$. Set $\gamma = (nb)^{-1}$ and $K(u) = \exp(-u)I(u \geq 0)$. Note that we treat n as a "sample size" (it is of the same order as n and for convenience one can let $n = t$), whereas b is a bandwidth, the smaller b the larger the weight on the current observations. Then, $\widehat{\mu}_t$ can be written as

$$\widehat{\mu}_t \quad = \quad \underbrace{(1-e^{-1/(nb)})}_{\approx(nb)^{-1}}\sum_{j=1}^{n}K\left(\frac{t-j}{nb}\right)Y_j,$$

where the above approximation is due to a Taylor expansion of $e^{-1/(nb)}$. This we observe that the exponential rolling window estimator is very close to a nonparametric kernel smoothing, which typically takes the form

$$\widetilde{\mu}_t = \sum_{j=1}^{n} \frac{1}{nb} K\left(\frac{t-j}{nb}\right) Y_j.$$

it is likely you came across such estimators in your nonparametric classes (a classical example is the local average where $K(u) = 1$ for $u \in [-1/2, 1/2]$ but zero elsewhere). The main difference between the rolling window estimator and the nonparametric kernel estimator is that the kernel/window for the rolling window is not symmetric. This is because we are trying to estimate the mean at time t, given only the observations up to time t. Whereas for general nonparametric kernel estimators one can use observations on both sides of t.

Sieve-estimators Suppose that $\{\phi_k(\cdot)\}_k$ is an orthonormal basis of $L_2[0,1]$ ($L_2[0,1] = \{f; \int_0^1 f(x)^2 dx < \infty\}$, so it includes all bounded and continuous functions)[1]. Then every function in L_2 can be represented as a linear sum of the basis. Suppose $\mu(\cdot) \in L_2[0,1]$ (for example the function is simply bounded). Then

$$\mu(u) = \sum_{k=1}^{\infty} a_k \phi_k(u).$$

Examples of basis functions are the Fourier $\phi_k(u) = \exp(iku)$, Haar/other wavelet functions etc. We observe that the unknown coefficients a_k are a linear in the 'regressors' ϕ_k. Since $\sum_k |a_k|^2 < \infty$, $a_k \to 0$. Therefore, for a sufficiently large M the finite truncation of the above is such that

$$Y_t \approx \sum_{k=1}^{M} a_k \phi_k \left(\frac{t}{n}\right) + \varepsilon_t.$$

Based on the above we observe that we can use least squares to estimate the coefficients, $\{a_k\}$. To estimate these coefficients, we truncate the above expansion to order M, and use

[1] Orthonormal basis means that for all k $\int_0^1 \phi_k(u)^2 du = 1$ and for any $k_1 \neq k_2$ we have $\int_0^1 \phi_{k_1}(u)\phi_{k_2}(u) du = 0$

least squares to estimate the coefficients

$$\sum_{t=1}^{n}\left[Y_t - \sum_{k=1}^{M} a_k\phi_k\left(\frac{t}{n}\right)\right]^2. \tag{1.3}$$

The orthogonality of the basis means that the corresponding design matrix $(X'X)$ is close to identity, since

$$n^{-1}(X'X)_{k_1,k_2} = \frac{1}{n}\sum_t \phi_{k_1}\left(\frac{t}{n}\right)\phi_{k_2}\left(\frac{t}{n}\right) \approx \int \phi_{k_1}(u)\phi_{k_2}(u)du = \begin{cases} 0 & k_1 \neq k_2 \\ 1 & k_1 = k_2 \end{cases}.$$

This means that the least squares estimator of a_k is $\widehat{a}_k$ where

$$\widehat{a}_k \approx \frac{1}{n}\sum_{t=1}^{n} Y_t\phi_k\left(\frac{t}{n}\right).$$

What is trend and what is noise?

Once we allow the "noise" ε_t to be dependent it becomes extremely difficult to discriminate between mean trend and noise. In Figure 1.7 two plots are given. The top plot is a realisation from independent normal noise the bottom plot is a realisation from dependent noise (the AR(1) process $X_t = 0.95X_{t-1} + \varepsilon_t$). Both realisations have zero mean (no trend), but the lower plot does give the appearance of an underlying mean trend.

This effect because more problematic when analysing data where there is mean term plus dependent noise. The smoothness in the dependent noise may give the appearance of additional features mean function. This makes estimating the mean function more difficult, especially the choice of bandwidth b. To understand why, suppose the mean function is $\mu_t = \mu(\frac{t}{200})$ (the sample size $n = 200$), where $\mu(u) = 5 \times (2u - 2.5u^2) + 20$. We corrupt this quadratic function with both iid and dependent noise (the dependent noise is the AR(2) process defined in equation (1.7)). The plots are given in Figure 1.8. We observe that the dependent noise looks 'smooth' (dependence can induce smoothness in a realisation). This means that in the case that the mean has been corrupted by dependent noise it difficult to see that the underlying trend is a simple quadratic function. In a very interesting paper Hart

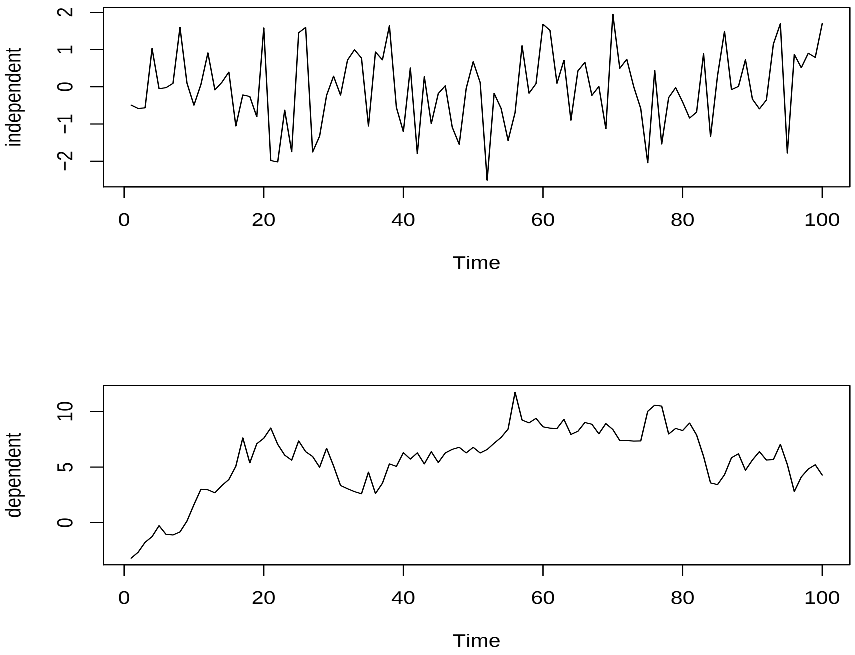


Figure 1.7: Top: realisations from iid random noise. Bottom: Realisation from dependent noise

(1991), shows that cross-validation (which is the classical method for choosing the bandwidth parameter b) is terrible when the errors are correlated.

Exercise 1.2 *The purpose of this exercise is to understand how correlated errors in a non-parametric model influence local smoothing estimators. We will use a simple local average.*

Define the smooth signal $f(u) = 5 * (2u - 2.5u^2) + 20$ *and suppose we observe* $Y_i = f(i/200) + \varepsilon_i$ *(*$n = 200$*). To simular* $f(u)$ *with* $n = 200$ *define* `temp <- c(1:200)/200` *and* `quadratic <- 5*(2*temp - 2.5*(temp**2)) + 20`.

(i) Simulate from the above model using iid noise. You can use the code `iid=rnom(200)` *and* `quadraticiid = (quadratic + iid)`.

Our aim is to estimate f*. To do this take a local average (the average can have different lengths* m*) (you can use* `mean(quadraticiid[c(k:(k+m-1))])` *for* $k = 1, \ldots, 200-m$*).*

(ii) Simulate from the above model using correlated noise (we simulate from an $AR(2)$*)* `ar2`

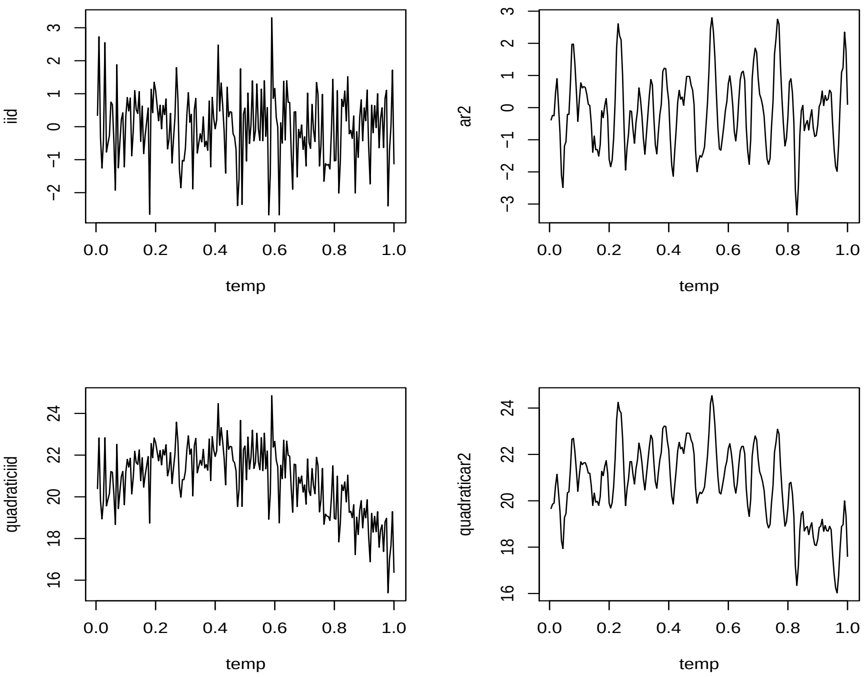


Figure 1.8: Top: realisations from iid random noise and dependent noise (left = iid and right = dependent). Bottom: Quadratic trend plus corresponding noise.

`= 0.5*arima.sim(list(order=c(2,0,0), ar = c(1.5, -0.75)), n=200)` *and define* `quadraticar2 = (quadratic +ar2)`*.*

Again estimate f using local averages.

By making plots of you estimators against `temp` *compare them.*

1.3.4 Periodic functions

Periodic mean functions arise in several applications, from ECG (which measure heart rhythms), monthly rainfall to astrostatistics. Often the aim is to estimate the period or of a periodic function. To model a periodic mean functions let

$$Y_t = d_P(t) + \varepsilon_t \qquad t = 1, \ldots, n,$$

where for all t, $d_P(t) = d_P(t + P)$. One classical method for detecting periodicities is to evaluate the inner product of Y_t with sin and cosine functions, since these functions are periodic. This is called the discrete Fourier transform;

$$J_n(\omega_k) = \sum_{t=1}^{n} Y_t \left(\cos(t\omega_k) + i\sin(t\omega_k)\right) = \sum_{t=1}^{n} Y_t \exp(it\omega_k)$$

where $\{\omega_k = \frac{2\pi k}{n}\}$ and $i = \sqrt{-1}$. Note it is often far easier to use $e^{it\omega_k}$ than $\cos(t\omega_k) + i\sin(t\omega_k)$. When the periodicity in the cosine and sin function matches the periodicity of the mean function $J_n(\omega)$ will be large. To see why, we rewrite $J_n(\omega)$ as

$$\begin{aligned} J_n(\omega_k) &= \sum_{t=0}^{n/P-1}\sum_{s=1}^{P} d_P(Pt+s)\exp(iPt\omega_k + is\omega_k) + \sum_{t=1}^{n}\varepsilon_t e^{it\omega_k} \\ &= \sum_{t=0}^{n/P-1}\exp(iPt\omega_k)\sum_{s=1}^{P} d_P(s)\exp(is\omega_k) + \sum_{t=1}^{n}\varepsilon_t e^{it\omega_k} \\ &= \sum_{s=1}^{P} d_P(s)\exp(is\omega_k)\sum_{t=0}^{n/P-1}\exp(iPt\omega_k) + \sum_{t=1}^{n}\varepsilon_t e^{it\omega_k}. \end{aligned}$$

Basic algebra gives

$$\sum_{t=0}^{n/P-1}\exp(iPt\omega_k) = \begin{cases} \frac{\exp(i2\pi k)}{1-\exp(iPt\omega_k)} = 0 & k \neq \frac{n}{P}\mathbb{Z} \\ n/P & k \in \frac{n}{P}\mathbb{Z} \end{cases}$$

Thus

$$J_n(\omega_k) = \begin{cases} nD_p(r) + \sum_{t=1}^{n}\varepsilon_t e^{it\omega_k} & k = \frac{n}{P}r, \quad r = 0,\ldots,P-1. \\ \sum_{t=1}^{n}\varepsilon_t e^{it\omega_k} & k \neq \frac{n}{P}\mathbb{Z} \end{cases} \tag{1.4}$$

where $D_P(r) = P^{-1}\sum_{s=1}^{P} d_P(s)\exp(2\pi isr/P)$. Given that $\sum_{t=1}^{n}\varepsilon_t e^{it\omega_k}$ should be low lying noise (we discuss this in detail later), what we should see is P large spikes, each corresponding to $D_P(r)$. Though the above is simply an algebraic calculation. The reason for the term n in (1.4) (recall n is the sample size) is because there are n/P repetitions of the period. As n grows do the repetitions (this growth in n will be important later on when we estimate the period of a function). An example of a periodic function and its Fourier transform (without added noise) is given in Figure 1.9. The period $P = 5$ and length is $n = 250$. We observe that for both the real and imaginary parts there are "blips" every $250/5 = 50$ points, which matches with the above calculations. Moreover, there is a symmetry in the real part (cosine term) and an asymmetry in the imaginary part. Due to the nature of the real and imaginary parts

$$\begin{aligned}\Re J_{250}(\omega_k) &= \sum_{t=1}^{250} Y_t \cos\left(t\frac{2\pi k}{250}\right) = \sum_{t=1}^{250} Y_t \cos\left(t\frac{2\pi(250-k)}{250}\right) = \Re J_{250}(\omega_{250-n}) \\ \Im J_{250}(\omega_k) &= \sum_{t=1}^{250} Y_t \sin\left(t\frac{2\pi k}{250}\right) = -\sum_{t=1}^{250} Y_t \sin\left(t\frac{2\pi(250-k)}{250}\right) = -\Im J_{250}(\omega_{250-n}).\end{aligned}$$

Due to this symmetry all information about the data is contained within the first $n/2$ frequencies i.e. $\{J_n(\omega_k); k = 0, \ldots, (n-1)/2\}$.

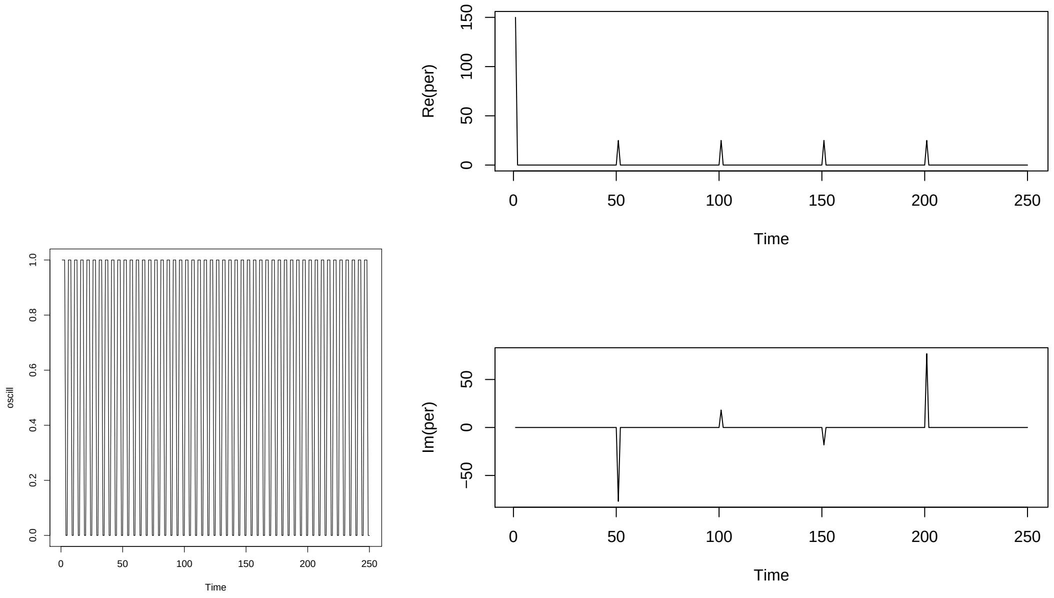


Figure 1.9: Left: Periodic function $d_5(s) = 1$ for $s = 1, 2$, $d_5(s) = 0$ for $s = 3, 4, 5$ (length 250, period 5), Right: The real and imaginary parts of its Fourier transform

Remark 1.3.1 *In the case that* $d_P(t)$ *is a pure sine or cosine function* $\sin(2\pi t/P)$ *or* $\cos(2\pi t/P)$, *then* $D_P(r)$ *will only be non-zero at* $r = 1$ *and* $r = P - 1$. *To see why we note that*

$$\frac{1}{P}\sum_{s=0}^{P-1} \cos\left(\frac{2\pi s}{P}\right) \exp\left(i\frac{2\pi sr}{P}\right) = \frac{1}{2P}\sum_{s=0}^{P-1}\left(e^{i2\pi s/P} + e^{-i2\pi s/P}\right) e^{i\frac{2\pi sr}{P}} = \begin{cases} 1/2 & r = 1 \textit{ or } P-1 \\ 0 & \textit{otherwise} \end{cases}$$

$$\frac{1}{P}\sum_{s=0}^{P-1}\sin\left(\frac{2\pi s}{P}\right)\exp\left(i\frac{2\pi sr}{P}\right) = \frac{-i}{2P}\sum_{s=0}^{P-1}\left(e^{i2\pi s/P} - e^{-i2\pi s/P}\right)e^{i\frac{2\pi sr}{P}} = \begin{cases} i/2 & r = 1 \\ -i/2 & r = P-1 \\ 0 & otherwise \end{cases}$$

Therefore, for pure sinusoids on the periodogram of $\{Y_t\}$ *there will be a large blip when* $\omega_k = 2\pi(n/P)$ *and* $\omega_k = 2\pi(n - n/P)$*. Below, we explore this idea further.*

For general time series, the DFT, $\{J_n(\frac{2\pi k}{n}); 1 \leq k \leq n\}$ is simply an (orthogonal) decomposition of the time series $\{X_t; t = 1, \ldots, n\}$ into sins and cosines of different frequencies. The magnitude of $J_n(\omega_k)$ informs on how much of the functions $\sin(t\omega)$ and $\cos(t\omega_k)$ are in the $\{X_t; t = 1, \ldots, n\}$. Below we define the periodogram. The periodogram effectively removes the complex part in $J_n(\omega_k)$ and only measures the absolute magnitude.

Definition 1.3.1 (The periodogram) $J_n(\omega)$ *is complex random variables. Often the absolute square of* $J_n(\omega)$ *is analyzed, this is called the periodogram*

$$I_n(\omega) = n^{-1}|J_n(\omega)|^2 = \frac{1}{n}\left|\sum_{t=1}^{n} X_t\cos(t\omega)\right|^2 + \frac{1}{n}\left|\sum_{t=1}^{n} X_t\sin(t\omega)\right|^2.$$

$I_n(\omega)$ *combines the information in the real and imaginary parts of* $J_n(\omega)$ *and has the advantage that it is real. However, it is positive this means that individual information on the amplitude of* $\Re J_n(\omega_k)$ *and* $\Im J_n(\omega)$ *(this is sometimes referred to as the phase information) is lost. For example, consider the pure sine and cosine functions discussed in Remark 1.3.1, the periodogram for both these functions is identical. But the only difference between a sine function and cosine function is the "phase shift" of* $\pi/2$*.*

Note that besides being symmetric about π*,* $I_n(\omega)$ *is periodic every* $[0, 2\pi]$*, thus* $I_n(\omega + 2\pi) = I_n(\omega)$*. Hence one only needs to consider* $I_n(\omega)$ *in the range* $[0, \pi]$*.*

For the remainder of this section we focus on the case that $d_P(t)$ is a pure sine or cosine function. Our aim is to estimate the period P. We show that the Fourier transform method above is effectively equivalent to period estimation using least squares. Suppose that the observations $\{Y_t; t = 1, \ldots, n\}$ satisfy the following regression model

$$Y_t = A\cos(\Omega t) + B\sin(\Omega t) + \varepsilon_t$$

where $\{\varepsilon_t\}$ are iid standard normal random variables and $0 < \Omega < \pi$ (using the periodic notation we can set $\Omega = \frac{2\pi}{P}$). The parameters A, B, and Ω are real and unknown. Unlike the regression models given in (1.3.1) the model here is nonlinear, since the unknown parameter, Ω, is inside a trignometric function. Standard least squares methods cannot be used to estimate the parameters. Assuming Gaussianity of $\{\varepsilon_t\}$ (though this assumption is not necessary), the maximum likelihood corresponding to the model is

$$\mathcal{L}_n(A, B, \Omega) = -\frac{1}{2}\sum_{t=1}^{n}(Y_t - A\cos(\Omega t) - B\sin(\Omega t))^2$$

(alternatively one can think of it in terms use least squares which is negative of the above). The above criterion is a negative nonlinear least squares criterion in A, B and Ω. It does not yield an analytic solution and would require the use of a numerical maximisation scheme. However, using some algebraic manipulations, explicit expressions for the estimators can be obtained (see Walker (1971) and Exercise 1.4). The result of these manipulations give the frequency estimator

$$\widehat{\Omega}_n = \arg\max_{\omega} I_n(\omega)$$

where

$$I_n(\omega) = \frac{1}{n}\left|\sum_{t=1}^{n} Y_t \exp(it\omega)\right|^2 = \frac{1}{n}\left(\sum_{t=1}^{n} Y_t\cos(t\Omega)\right)^2 + \frac{1}{n}\left(\sum_{t=1}^{n} Y_t\sin(t\omega)\right)^2. \qquad (1.5)$$

Typically we search for the maximum over the grid $\omega_k = \frac{2\pi k}{n}$ for $1 \leq k \leq n$. Using $\widehat{\Omega}_k$ we estimate A and B with

$$\widehat{A}_n = \frac{2}{n}\sum_{t=1}^{n} Y_t\cos(\widehat{\Omega}_n t) \text{ and } \widehat{B}_n = \frac{2}{n}\sum_{t=1}^{n} Y_t\sin(\widehat{\Omega}_n t).$$

The rather remarkable aspect of this result is that the rate of convergence of $|\widehat{\Omega}_n - \Omega| = O(n^{-3/2})$, which is faster than the standard $O(n^{-1/2})$ that we usually encounter (we will see this in Example 1.3.1)[2]. The heuristic behind this result can be see in Remark 1.3.1. Roughly speaking

$$I_n(\omega) = \underbrace{\frac{1}{n}\left|\sum_{t=1}^{n}\left[A\cos(t\Omega) + B\sin(t\Omega)\right]e^{it\omega}\right|}_{\text{signal}} + \underbrace{\frac{1}{n}\left|\sum_{t=1}^{n}\varepsilon_t e^{it\omega}\right|^2}_{\text{noise}} + \text{additional term.}$$

The "signal" in $I_n(\omega_k)$ is the periodogram corresponding to the cos and/or sine function. For example setting $\Omega = 2\pi/P$, $A = 1$ and $B = 0$. The signal is

$$\frac{1}{n}\left|\sum_{t=1}^{n}\cos\left(\frac{2\pi t}{P}\right)e^{it\omega_k}\right|^2 = \begin{cases} \frac{n}{4} & k = \frac{n}{P} \text{ or } k = \frac{n-P}{P} \\ 0 & \text{other wise} \end{cases}.$$

Observe there is a peak at $\frac{2\pi P}{n}$ and $\frac{2\pi(n-P)}{n}$, which is of size n, elsewhere it is zero. On the other hand the noise is

$$\frac{1}{n}\left|\sum_{t=1}^{n}\varepsilon_t e^{it\omega_k}\right|^2 = \left|\underbrace{\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\varepsilon_t e^{it\omega_k}}_{\text{treat as a rescaled mean}}\right|^2 = O_p(1),$$

where $O_p(1)$ means that it is bounded in probability (it does not grow as $n \to \infty$). Putting these two facts together, we observe that the contribution of the signal is extremely obvious in the periodogram $I_n(\omega)$, and is the heuristic behind this extremely fast rate of convergence.

Remark 1.3.2 *In practice, we only evaluate $J_n(\omega)$ and $I_n(\omega)$ at the so called fundamental frequencies $\omega_k = \frac{2\pi k}{n}$.*

$$\{Y_t\}_{t=1}^{n} \to \left\{J_n(\frac{2\pi k}{n}) = \frac{1}{\sqrt{n}}\sum_{t=1}^{n}Y_t\cos\left(t\frac{2\pi k}{n}\right) + i\frac{1}{\sqrt{n}}\sum_{t=1}^{n}Y_t\sin\left(t\frac{2\pi k}{n}\right)\right\}_{k=1}^{n}.$$

[2] As a contrast consider the iid random variables $\{X_t\}_{t=1}^{n}$, where $\mathrm{E}[X_t] = \mu$ and $\mathrm{var}(X_t) = \sigma^2$. The variance of the sample mean $\bar{X} = n^{-1}\sum_{t=1}^{n}$ is $\mathrm{var}[\bar{X}] = \sigma^2/n$ (where $\mathrm{var}(X_t) = \sigma^2$). This means $|\bar{X} - \mu| = O_p(n^{-1/2})$. This means there exists a random variable U such that $|\bar{X} - \mu| \leq n^{-1/2}U$. Roughly, this means as $n \to \infty$ the distance between $\bar{X}$ and μ declines at the rate $n^{-1/2}$.

$J_n(\omega_k)$ is simply a linear *one to one transformation of the data (nothing is lost in this transformation). Statistical analysis can be applied on any transformation of the data, it so happens that for stationary time series this so called Fourier transform has some advantages.*

We consider an example below.

Example 1.3.1 *Consider the following model*

$$Y_t = 2\sin\left(\frac{2\pi t}{8}\right) + \varepsilon_t \qquad t = 1, \ldots, n. \tag{1.6}$$

where ε_t are iid standard normal random variables. Note by using Remark 1.3.1 and equation (1.4) we have

$$\frac{1}{n}\left|2\sum_{t=1}^{n}\sin\left(\frac{2\pi t}{8}\right)\exp(it\omega_k)\right|^2 = \begin{cases} n & k = \frac{n}{P} \text{ or } n - \frac{n}{P} \\ 0 & \text{otherwise} \end{cases}$$

It is clear that $\{Y_t\}$ is made up of a periodic signal with period eight. We make a plot of one realisation (using sample size $n = 128$) together with the periodogram $I(\omega)$ (defined in (1.5)). In Figure 1.10 we give a plot of one realisation together with a plot of the periodogram. From the realisation, it is not clear what the period is (the noise has made it difficult to see the period). On the other hand, the periodogram clearly shows a peak at frequenct $2\pi/8 \approx 0.78$ (where we recall that 8 is the period) and $2\pi - 2\pi/8$ (since the periodogram is symmetric about π).

Searching for peaks in the periodogram is a long established method for detecting periodicities. The method outlined above can easily be generalized to the case that there are multiple periods. However, distinguishing between two periods which are very close in frequency (such data arises in astronomy) is a difficult problem and requires more subtle methods (see Quinn and Hannan (2001)).

The Fisher's g-statistic The discussion above motivates Fisher's test for hidden period, where the objective is to detect a period in the signal. The null hypothesis is H_0 : The signal is just white noise with no periodicities the alternative is H_1 : The signal contains a periodicity. The original test statistic was constructed under the assumption that the noise

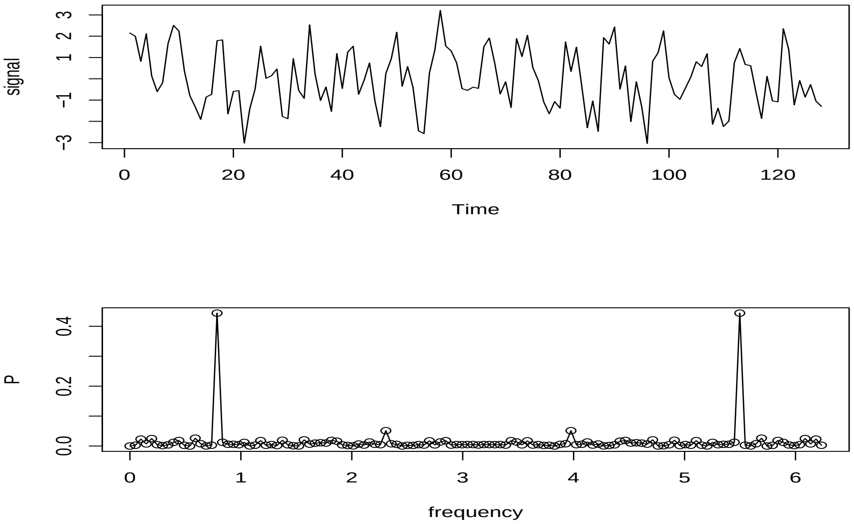


Figure 1.10: Left: Realisation of (1.6) plus iid noise, Right: Periodogram of signal plus iid noise.

was iid Gaussian. As we have discussed above, if a period exists, $I_n(\omega_k)$ will contain a few "large" values, which correspond to the periodicities. The majority of $I_n(\omega_k)$ will be "small". Based on this notion, the Fisher's g-statistic is defined as

$$\eta_n = \frac{\max_{1 \leq k \leq (n-1)/2} I_n(\omega_k)}{\frac{2}{n-1}\sum_{k=1}^{(n-1)/2} I_n(\omega_k)},$$

where we note that the denominator can be treated as the average noise. Under the null (and iid normality of the noise), this ratio is pivotal (it does not depend on any unknown nuisance parameters).

Period detection and correlated noise

The methods described in the previous section are extremely effective if the error process $\{\varepsilon_t\}$ is uncorrelated. However, problems arise when the errors are correlated. To illustrate this issue, consider again model (1.6)

$$Y_t = 2\sin\left(\frac{2\pi t}{8}\right) + \varepsilon_t \qquad t = 1, \ldots, n.$$

but this time the errors are correlated. More precisely, the are generated by the AR(2) model,

$$\varepsilon_t = 1.5\varepsilon_{t-1} - 0.75\varepsilon_{t-2} + \epsilon_t, \tag{1.7}$$

where $\{\epsilon_t\}$ are iid random variables (do not worry if this does not make sense to you we define this class of models precisely in Chapter 3). As in the iid case we use a sample size $n = 128$. In Figure 1.11 we give a plot of one realisation and the corresponding periodogram. We observe that the peak at $2\pi/8$ is not the highest. The correlated errors (often called coloured noise) is masking the peak by introducing new peaks. To see what happens for larger sample sizes, we consider exactly the same model (1.6) with the noise generated as in (1.7). But this time we use $n = 1024$ (8 time the previous sample size). A plot of one realisation, together with the periodogram is given in Figure 1.12. In contrast to the smaller sample size, a large peak is visible at $2\pi/8$. These examples illustrates two important points:

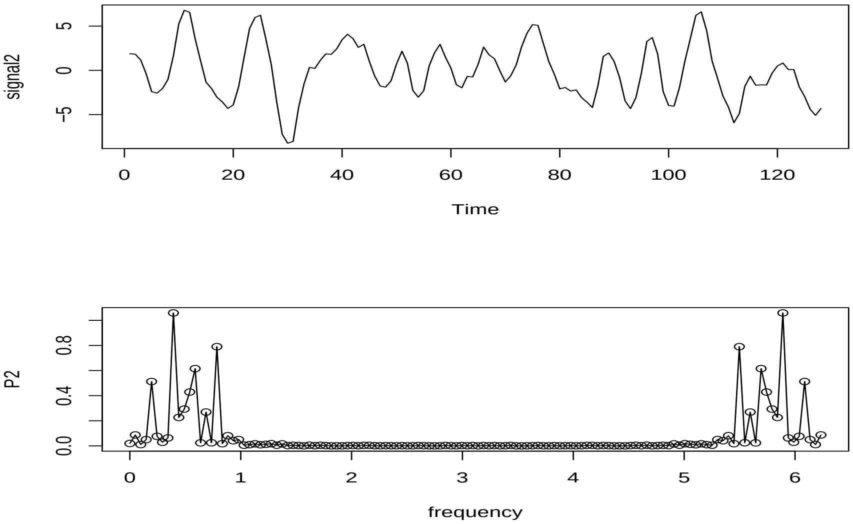


Figure 1.11: Top: Realisation of (1.6) plus correlated noise and $n = 128$, Bottom: Periodogram of signal plus correlated noise.

(i) When the noise is correlated and the sample size is relatively small it is difficult to disentangle the deterministic period from the noise. Indeed we will show in Chapters 3 and 4 that linear time series (such as the AR(2) model described in (1.7)) can exhibit

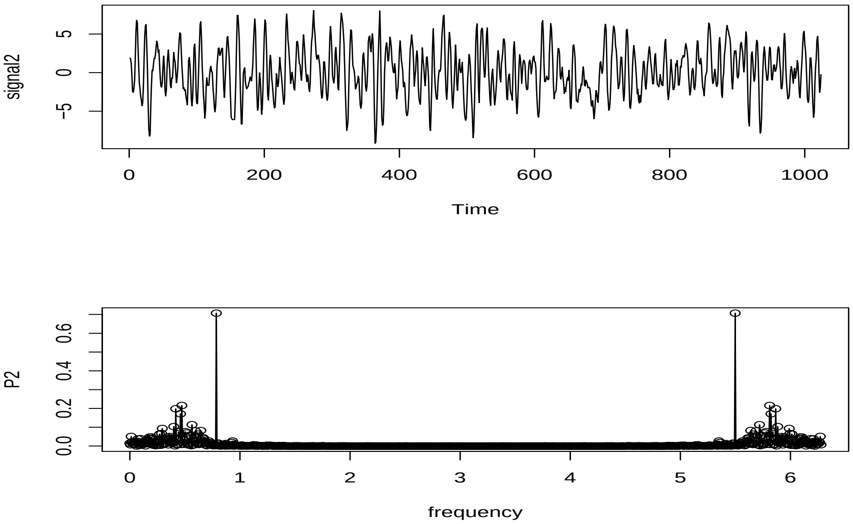


Figure 1.12: Top: Realisation of (1.6) plus correlated noise and $n = 1024$, Bottom: Periodogram of signal plus correlated noise.

similar types of behaviour to a periodic deterministic signal. This is a subject of on going research that dates back at least 60 years (see Quinn and Hannan (2001) and the P-statistic proposed by Priestley).

However, the similarity is only to a point. Given a large enough sample size (which may in practice not be realistic), the deterministic frequency dominates again (as we have seen when we increase n to 1024).

(ii) The periodogram holds important information about oscillations in the both the signal and also the noise $\{\varepsilon_t\}$. If the noise is iid then the corresponding periodogram tends to be flatish (see Figure 1.10). This informs us that no frequency dominates others. And is the reason that iid time series (or more precisely uncorrelated time series) is called "white noise".

Comparing Figure 1.10 with 1.11 and 1.12) we observe that the periodogram does not appear completely flat. Some frequencies tend to be far larger than others. This is because when data is dependent, certain patterns are seen, which are registered by the periodogram (see Section 1.4).

Understanding the DFT and the periodogram is called spectral analysis and is explored in Chapters 9 and 10. Indeed a lot of time series analysis can be done within the so called frequency <u>or</u> time domain.

1.3.5 Historic Background

The use of the periodogram, $I_n(\omega)$ to detect for periodocities in the data dates back to Schuster in the 1890's. One of Schuster's interest was sunspot data. He analyzed the number of sunspot through the lense of the periodogram. A plot is given in Figure 1.13. Assuming that the the sunspot data roughly follows the period trend plus noise model

$$Y_t = A\cos(\Omega t) + B\sin(\Omega t) + \varepsilon_t,$$

$\Omega = 2\pi/P$. The periodogram below shows a peak at $\Omega = 2 \times 30\pi/314$, which corresponds to a period of $P = 314/30 \approx 10.5$ years. This suggests that the number of sunspots follow a periodic cycle with a peak every $P = 10.5$ years. The general view until the 1920s is that most time series were a mix of periodic function with additive noise

$$Y_t = \sum_{j=1}^{P}[A_j\cos(t\Omega_j) + B_j\sin(t\Omega_j)] + \varepsilon_t.$$

However, in the 1920's, Udny Yule, a statistician, and Gilbert Walker, a Meterologist (working in Pune, India) believed an alternative model could be used to explain the features seen in the periodogram. Yule fitted an Autoregressive model of order two to the Sunspot data and obtained the AR(2) model

$$X_t = 1.381X_{t-1} - 0.6807X_{t-2} + \varepsilon_t,$$

this corresponds to the characteristic function $1-1.381x+0.68x^2$, whose roots are $0.77^{-1}\exp(\pm i0.57)$. Yule showed that if the roots of the characteristic polynomial of an AR(2) process were complex (not on the unit circle), then the solution would have so called 'pseudo periodicities'.

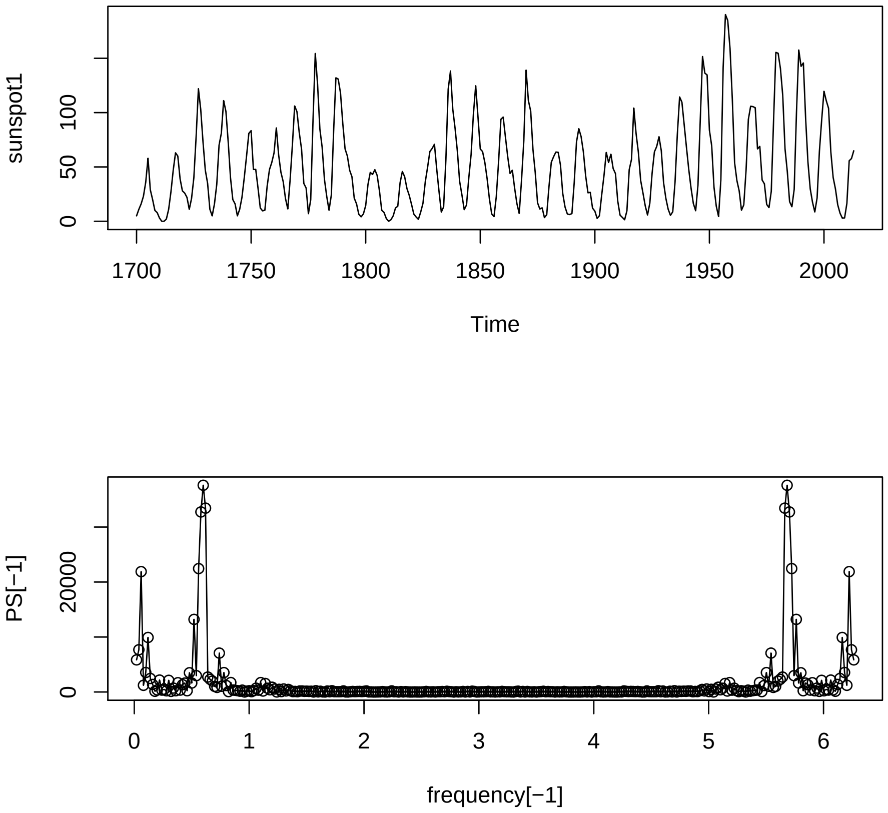


Figure 1.13: Sunspot data from 1700 to 2014. There is a peak at about 30 along the line which corresponds to $2 \times 30\pi/314$ and $314/30 \approx 10.5$ years.

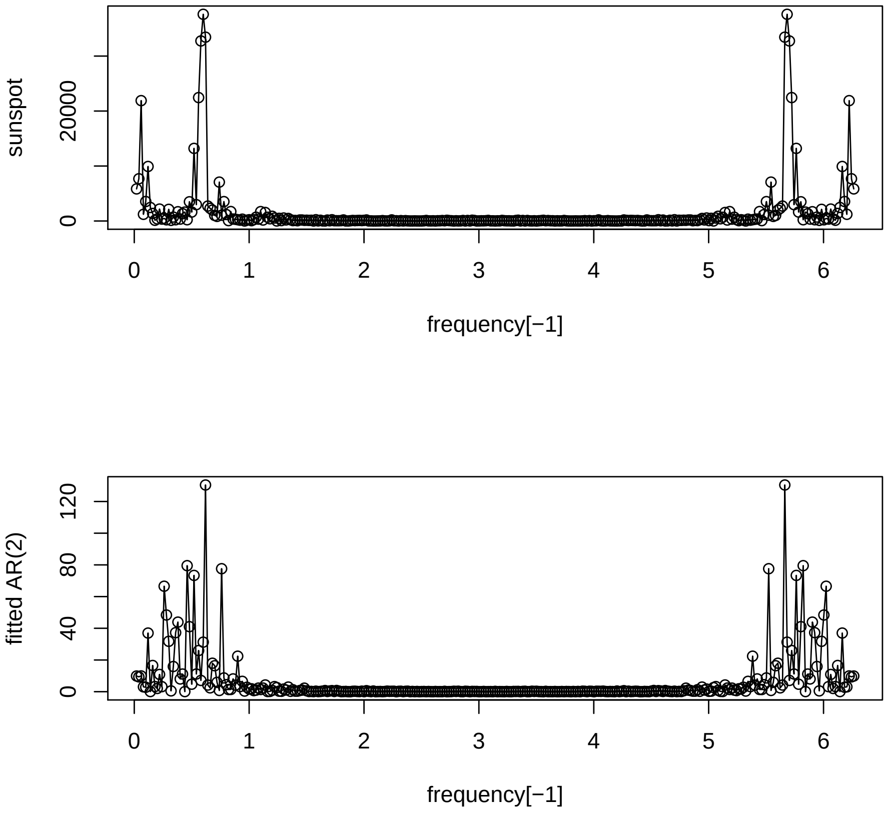


Figure 1.14: The periodogram of the Sunspot data is the top plot and the periodogram of the fitted AR(2) model is the lower plot. They do not look exactly the same, but the AR(2) model is able to capture some of the periodicities.

The above model has the solution

$$X_t = \sum_{j=0}^{\infty} 0.77^j \sin\left[0.57(j+1)\right] \varepsilon_{t-j},$$

(we learn why later on in this the course). We see that the solution is completely stochastic (no deterministic mean), but the sin/cos functions make a typical realisation 'look' periodic (though there is no real periodic). Thus giving the peaks in the corresponding periodogram. In Figure 1.14 we compare a periodogram of the sunspot data and a realisation from the fitted AR(2) process. In Figure 1.15 we make a plot of the sunspot data and a realisation of the AR(2) process.

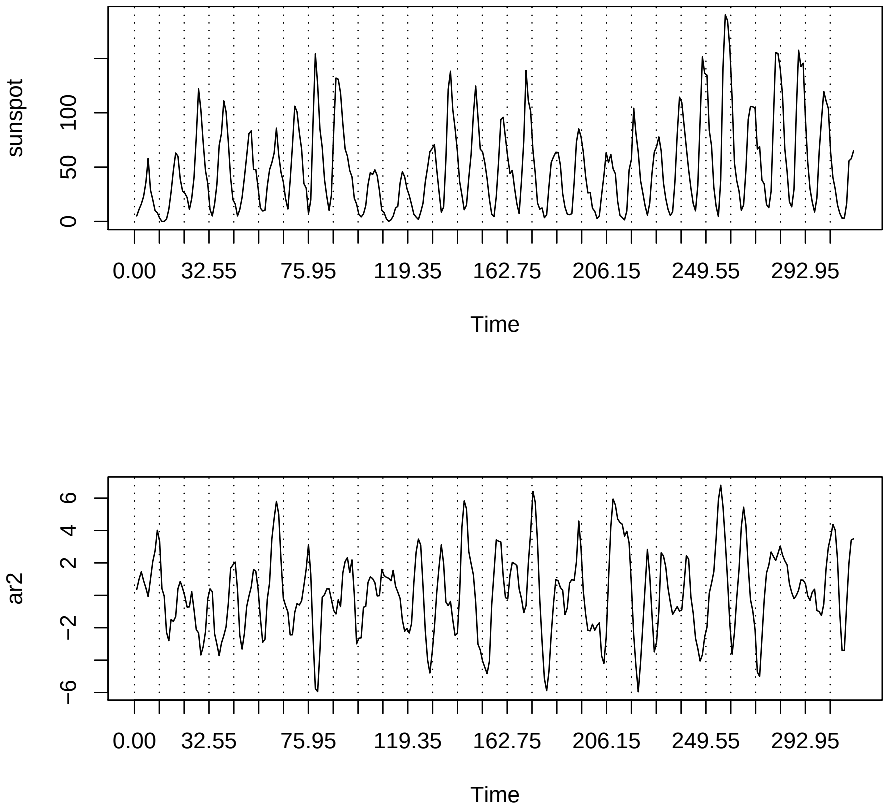


Figure 1.15: Top: Sunspot, Lower: a realisation from the AR(2) process. Lines correspond to period of $P = 2\pi/0.57 = 10.85$ years.

1.4 Pseudo Periodic functions

The entirely different modelling approaches of Schuster and Yule-Walker illustrate the difference between periodic functions and "pseudo-periodic" functions. The periodogram starkly illustrates these differences.

Summarizing the the discussion in the section above, in Figure 1.16 a plot of $\sin(2\pi t/5)$ for $t = 1, \ldots, 200$ is given together with the corresponding periodogram. As expected the periodogram is zero everywhere except at frequency $2\pi/5$.

In contrast in Figures 1.17 and 1.18 two different realisations from the stochastic process

$$X_t = 1.9 \cos\left(\frac{2\pi}{5}\right) X_{t-1} - 0.95^2 X_{t-2} + \varepsilon_t \tag{1.8}$$

are given together with the corresponding periodogram. Both realisations are very different, as are their periodograms. The periodograms tell us what frequencies are contained in the time series $\{X_t\}$. We observe that for both realisations the frequencies are concentrated around $2\pi/5$. However, (a) there's no pure single frequency (b) the exact frequencies are different for the two realisations. What we have seen is typical of a random process/time series. The time series will often appear to have a periodicity. But these periodicities are not pure periods, instead they tend to concentrated about a certain period, in what can be thought of as a pseudo-period. Why (1.8) exhibits this type of behaviour will be explained in future chapters.

1.4.1 Preliminary analysis of EEG data

In this section we conduct a preliminary analysis of an EEG data set. A plot of one EEG of one participant at one channel (probe on skull) over 2 seconds (about 512 observations, so 256 observations per second) is given in Figure 1.19. The neuroscientists who analysis such data use the periodogram to associate the EEG to different types of brain activity. The periodogram of the EEG is given in Figure 1.19 (lower plot). However, the x-axis is from $[0, \pi]$, which is not typically used in the neurosciences. The neurologists think in terms of neural oscillations per second (Hz). Therefore, in order to understand the periodogram from

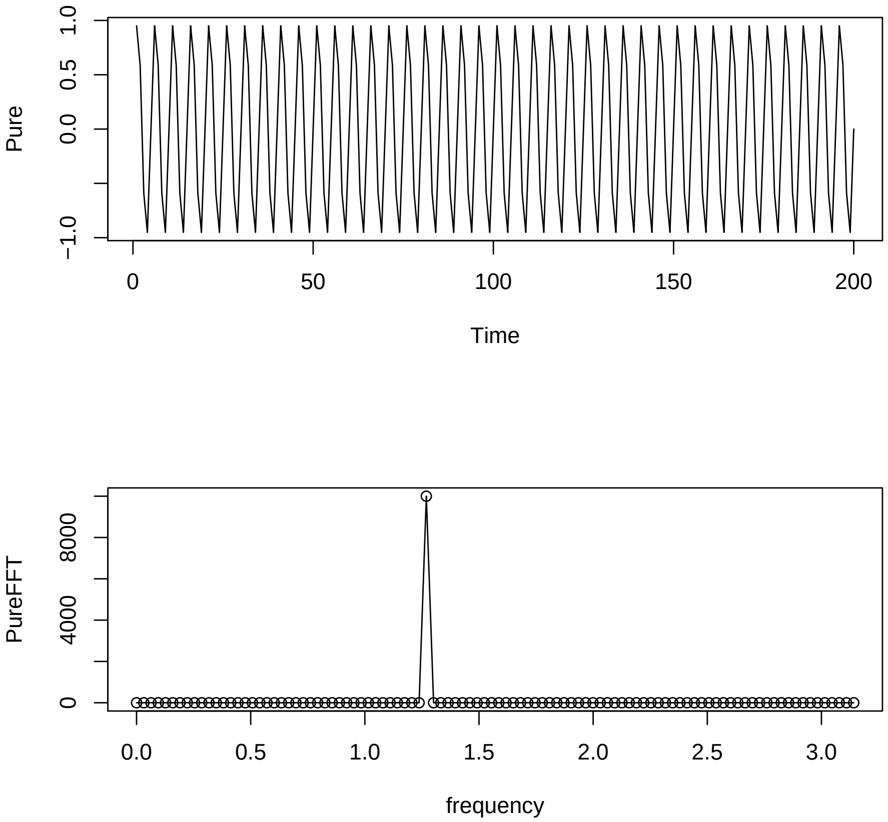


Figure 1.16: Top: Plot of $\sin(2\pi t/5)$, Lower: Corresponding periodogram.

the perspective of the neurologists the same periodogram is given in terms of Hz (number of cycles per second), which is essentially the periodogam $I(\omega_k)$ plotted against

$$\frac{\text{k}}{\text{no. of seconds observed}} \qquad k = 1, \ldots, n/2.$$

This plot is given in Figure 1.20. Observe that the EEG contains a large amount of low frequency information, this is probably due to the slowly changing trend in the original EEG. The neurologists have banded the cycles into bands and associated to each band different types of brain activity (see `https://en.wikipedia.org/wiki/Alpha_wave#Brain_waves`). Very low frequency waves, such as delta, theta and to some extent alpha waves are often associated with low level brain activity (such as breathing). Higher frequencies (alpha and gamma waves) in the EEG are often associated with conscious thought (though none of this is

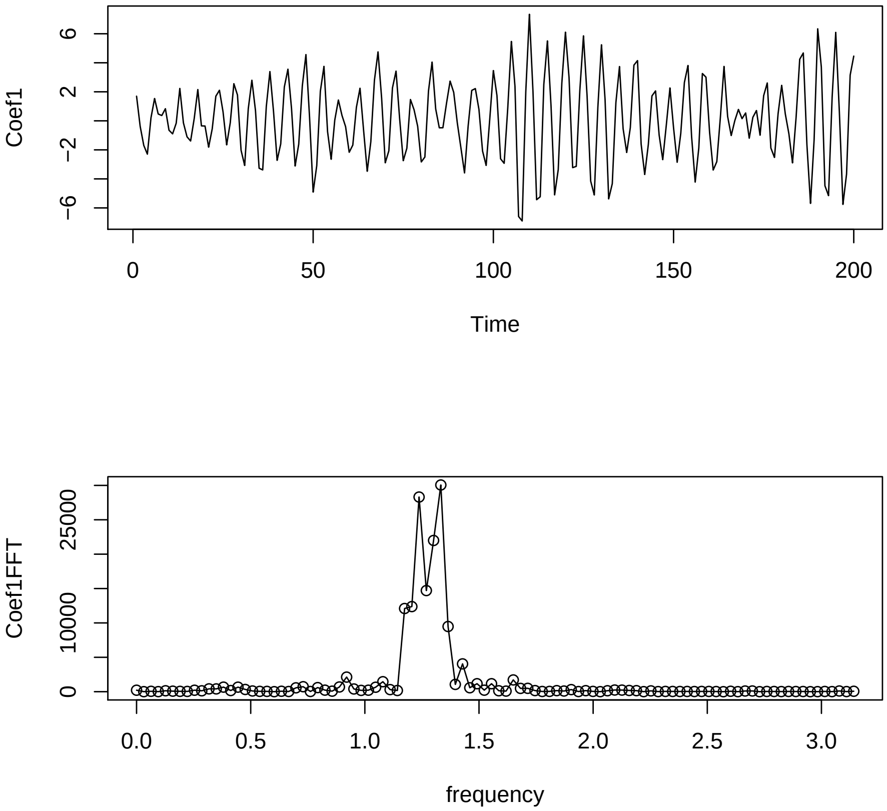


Figure 1.17: Top: Realisation of time series in (1.8), Lower: Corresponding periodogram.

completely understood and there are many debates on this). Studying the periodogram of the EEG in Figures 1.19 and 1.20, we observe that the low frequency information dominates the signal. Therefore, the neuroscientists prefer to decompose the signal into different frequency bands to isolate different parts of the signal. This is usually done by means of a band filter.

As mentioned above, higher frequencies in the EEG are believed to be associated with conscious thought. However, the lower frequencies dominate the EEG. Therefore to put a "microscope" on the higher frequencies in the EEG we isolate them by removing the lower delta and theta band information. This allows us to examine the higher frequencies without being being "drowned out" by the more prominent lower frequencies (which have a much larger amplitude). In this data example, we use a Butterworth filter which removes most of the low frequency and very high information (by convolving the original signal with a filter, see Remark 1.4.1). A plot of the periodogam of the orignal EEG together with the EEG

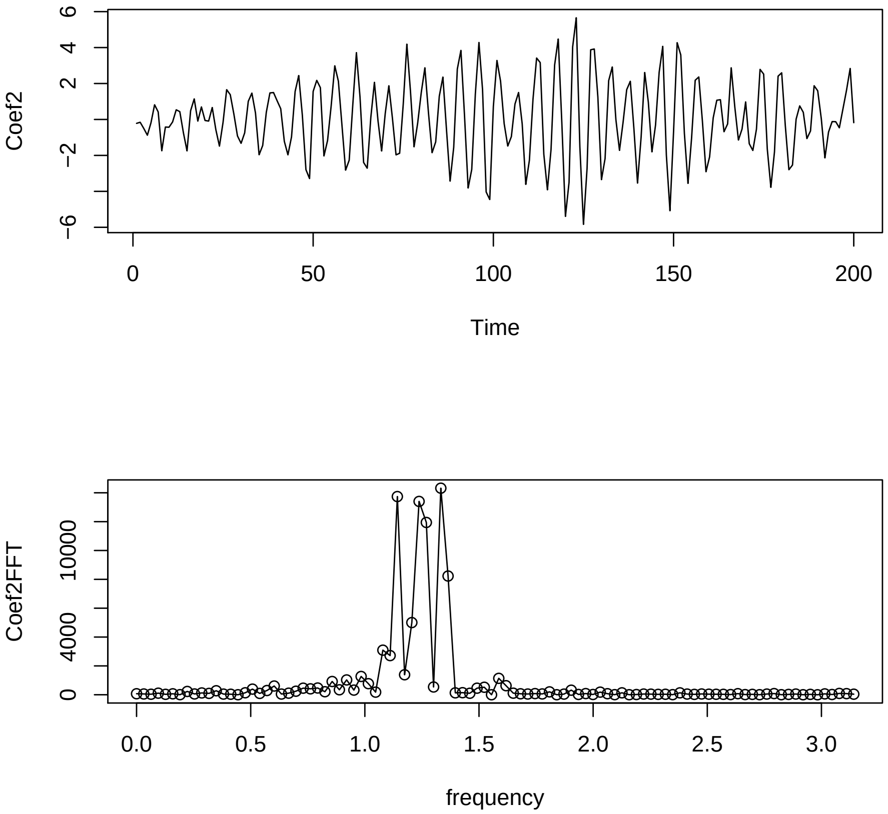


Figure 1.18: Top: Realisation of time series in (1.8), Lower: Corresponding periodogram.

after processing with a filter is given in Figure 1.21. Except for a few artifacts (since the Butterworth filter is a finite impulse response filter, and thus only has a finite number of non-zero coefficients), the filter has completely removed the very low frequency information, from $0 - 0.2$ and for the higher frequencies beyond 0.75; we see from the lower plot in Figure 1.21 this means the focus is on 8-32Hz (Hz = number of cycles per second). We observe that most of the frequencies in the interval $[0.2, 0.75]$ have been captured with only a slight amount of distortion. The processed EEG after passing it through the filter is given in Figure 1.22, this data set corresponds to the red periodogram plot seen in Figure 1.21. The corresponding processed EEG clearly shows the evidence of pseudo frequencies described in the section above, and often the aim is to model this processed EEG.

The plot of the original, filtered and the differences in the EEG is given in Figure 1.23. We see the difference (bottom plot) contains the trend in the original EEG and also the small very high frequency fluctuations (probably corresponding to the small spike in the original periodogram in the higher frequencies).

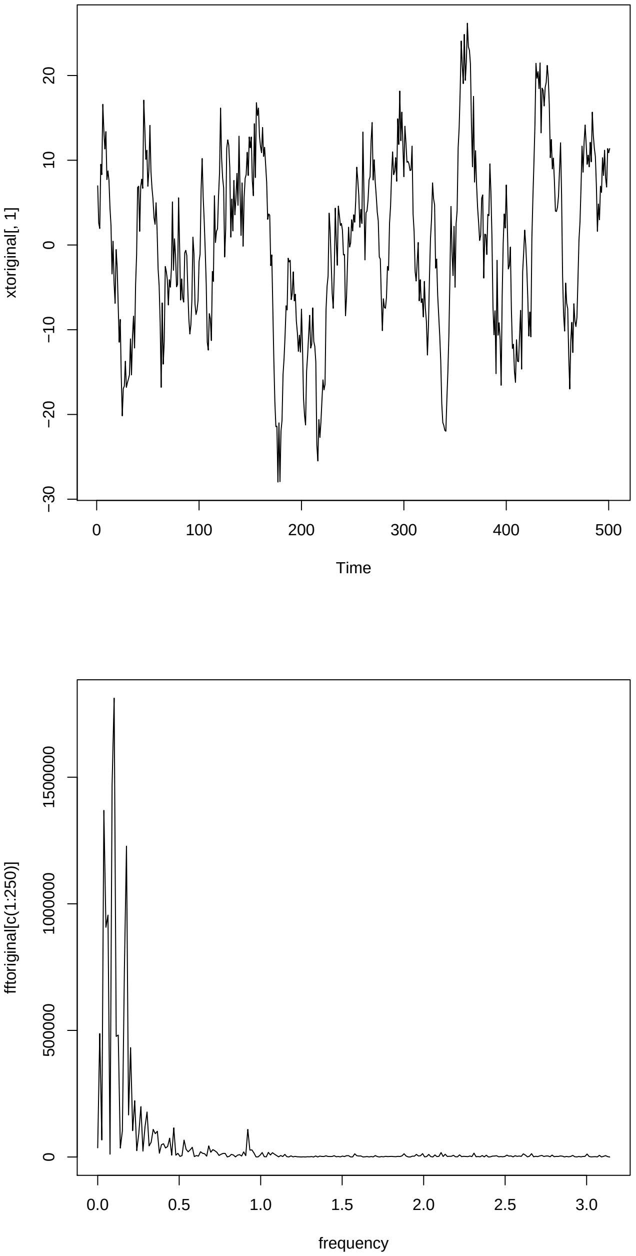


Figure 1.19: Top: Original EEG. Bottom: Corresponding periodogram.

Remark 1.4.1 (How filtering works) *A linear filter is essentially a linear combination of the time series with some weights. The weights are moved along the time series. For*

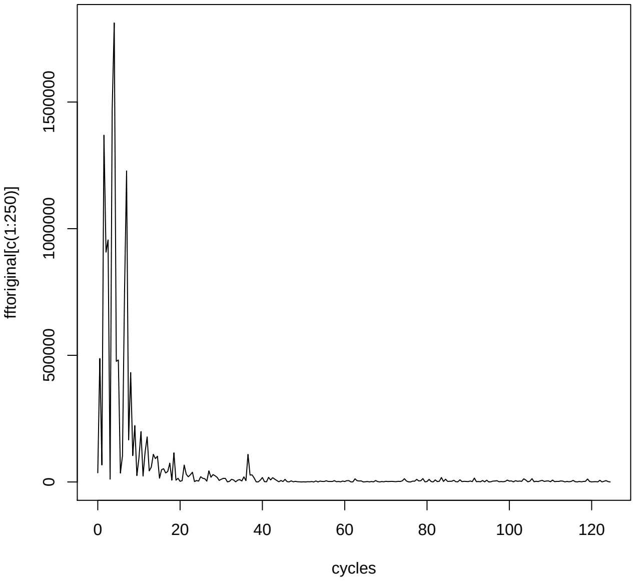


Figure 1.20: Periodogram of original EEG in cycles per second (How the neurologists interprete the EEG).

example, if $\{h_k\}$ is the filter. Then the filtered time series $\{X_t\}$ is the convolution

$$Y_t = \sum_{s=0}^{\infty} h_s X_{t-s},$$

note that h_s can be viewed as a moving window. However, the moving window (filter) considered in Section 1.3 "smooth" and is used to isolate low frequency trend (mean) behaviour. Whereas the general filtering scheme described above can isolate any type of frequency behaviour. To isolate high frequencies the weights $\{h_s\}$ should not be smooth (should not change slowly over k). To understand the impact $\{h_s\}$ has on $\{X_t\}$ we evaluate the Fourier transform of $\{Y_t\}$.

The periodogram of $\{Y_t\}$ is

$$\begin{aligned} |J_Y(\omega)|^2 = \left|\frac{1}{\sqrt{n}}\sum_{t=1}^{n} Y_t e^{it\omega}\right| &= \left|\sum_{s=1}^{n} h_s e^{is\omega}\right|^2 \left|\frac{1}{\sqrt{n}}\sum_{t=1}^{n} X_t e^{it\omega}\right|^2 \\ &= |H(\omega)|^2 \, |J_X(\omega)|^2 . \end{aligned}$$

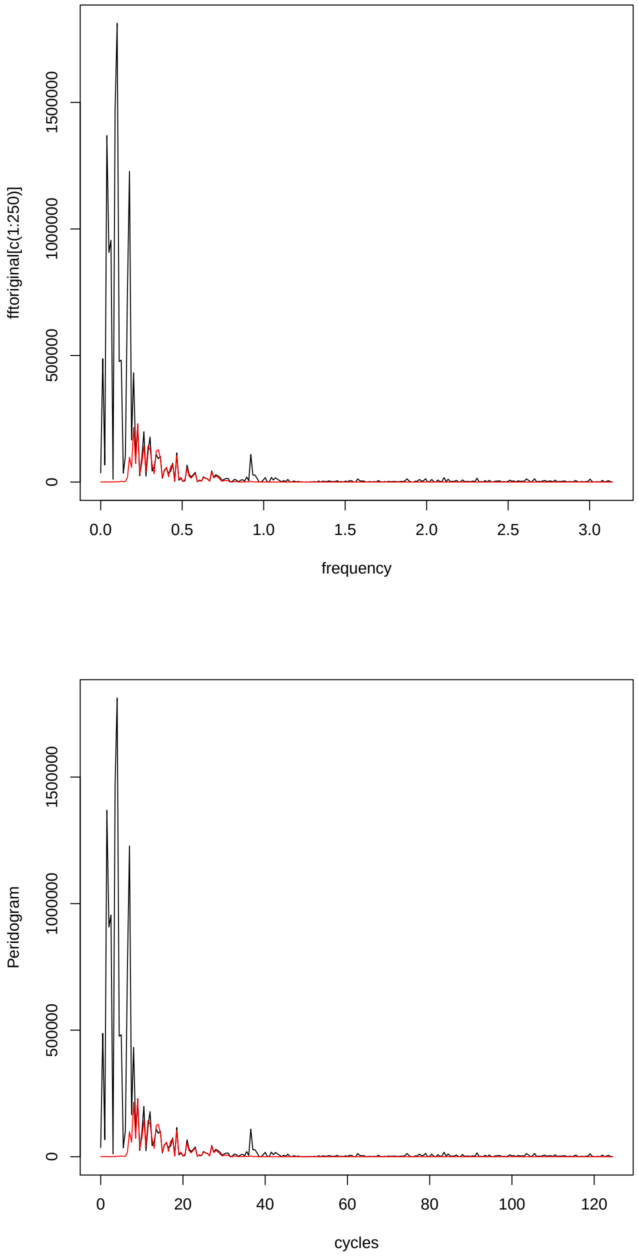


Figure 1.21: The periodogram of original EEG overlayed with processed EEG (in red). The same plot is given below, but the x-axis corresponds to cycles per second (measured in Hz)

If $H(\omega)$ is close to zero at certain frequencies it is removing those frequencies in $\{Y_t\}$. Hence using the correct choice of h_s we can isolate certain frequency bands.

Note, if a filter is finite (only a finite number of coefficients), then it is impossible to make the function drop from zero to one. But one can approximately the step by a smooth

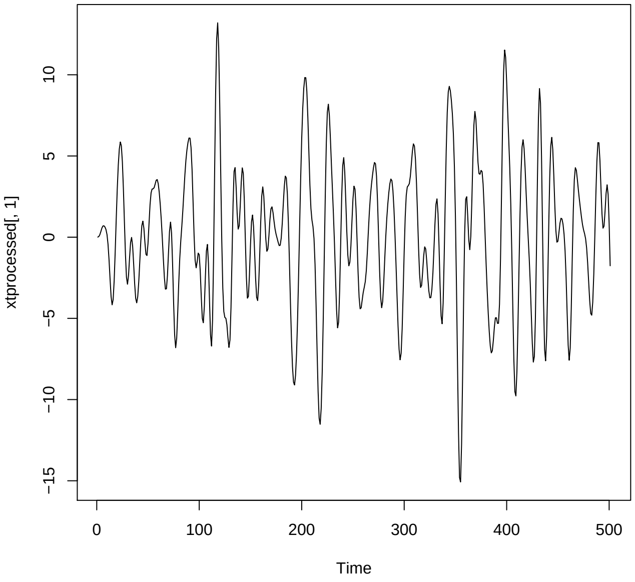


Figure 1.22: Time series after processing with a Buttersworth filter.

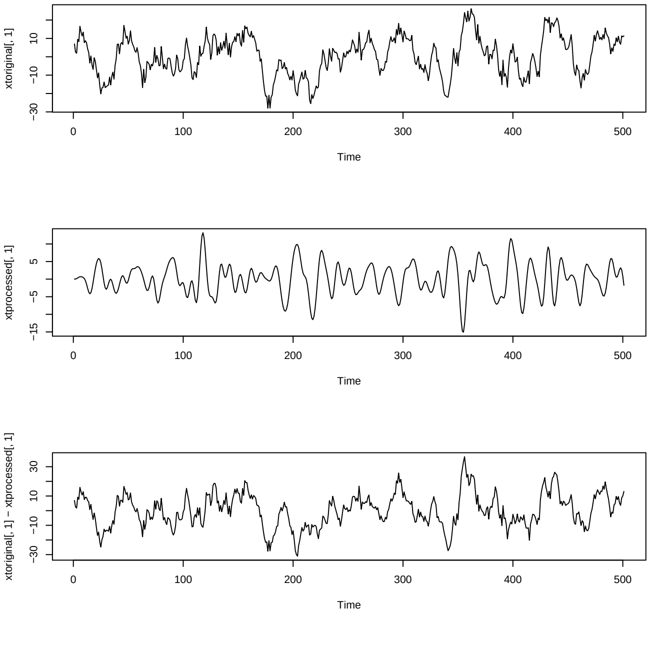


Figure 1.23: Top: Original EEG. Middle: Filtered EEG and Bottom: Diffence between Original and Filtered EEG

function (see `https://en.wikipedia.org/wiki/Butterworth_filter`).

1.5 Exercises

Exercise 1.3 (Understanding Fourier transforms) *(i) Let $Y_t = 1$. Plot the Periodogram of $\{Y_t; t = 1, \ldots, 128\}$.*

(ii) Let $Y_t = 1 + \varepsilon_t$, where $\{\varepsilon_t\}$ are iid standard normal random variables. Plot the Periodogram of $\{Y_t; t = 1, \ldots, 128\}$.

(iii) Let $Y_t = \mu(\frac{t}{128})$ where $\mu(u) = 5 \times (2u - 2.5u^2) + 20$. Plot the Periodogram of $\{Y_t; t = 1, \ldots, 128\}$.

(iv) Let $Y_t = 2 \times \sin(\frac{2\pi t}{8})$. Plot the Periodogram of $\{Y_t; t = 1, \ldots, 128\}$.

(v) Let $Y_t = 2 \times \sin(\frac{2\pi t}{8}) + 4 \times \cos(\frac{2\pi t}{12})$. Plot the Periodogram of $\{Y_t; t = 1, \ldots, 128\}$.

You can locate the maximum by using the function `which.max`

Exercise 1.4 *This exercise is designed only for statistics graduate students.*

(i) Let

$$\mathcal{S}_n(A, B, \Omega) = \left(\sum_{t=1}^{n} Y_t^2 - 2\sum_{t=1}^{n} Y_t\big(A\cos(\Omega t) + B\sin(\Omega t)\big) + \frac{1}{2}n(A^2 + B^2)\right).$$

Show that

$$2\mathcal{L}_n(A, B, \Omega) + \mathcal{S}_n(A, B, \Omega) = -\frac{(A^2 - B^2)}{2}\sum_{t=1}^{n}\cos(2t\Omega) - AB\sum_{t=1}^{n}\sin(2t\Omega).$$

and thus $|\mathcal{L}_n(A, B, \Omega) + \frac{1}{2}\mathcal{S}_n(A, B, \Omega)| = O(1)$ (ie. the difference does not grow with n).

Since $\mathcal{L}_n(A, B, \Omega)$ and $-\frac{1}{2}\mathcal{S}_n(A, B, \Omega)$ are asymptotically equivalent (i) shows that we can maximise $\frac{-1}{2}\mathcal{S}_n(A, B, \Omega)$ instead of the likelihood $\mathcal{L}_n(A, B, \Omega)$.

(ii) By profiling out the parameters A and B, use the the profile likelihood to show that $\widehat{\Omega}_n = \arg\max_{\omega} |\sum_{t=1}^{n} Y_t \exp(it\omega)|^2$.

(iii) By using the identity (which is the one-sided Dirichlet kernel)

$$\sum_{t=1}^{n}\exp(i\Omega t)=\begin{cases}\frac{\exp(\frac{1}{2}i(n+1)\Omega)\sin(\frac{1}{2}n\Omega)}{\sin(\frac{1}{2}\Omega)} & 0<\Omega<2\pi\\ n & \Omega=0 \textit{ or } 2\pi.\end{cases}\qquad(1.9)$$

we can show that for $0<\Omega<2\pi$ we have

$$\begin{aligned}\sum_{t=1}^{n}t\cos(\Omega t)&=O(n) & \sum_{t=1}^{n}t\sin(\Omega t)&=O(n)\\ \sum_{t=1}^{n}t^2\cos(\Omega t)&=O(n^2) & \sum_{t=1}^{n}t^2\sin(\Omega t)&=O(n^2).\end{aligned}$$

Using the above identities, show that the Fisher Information of $\mathcal{L}_n(A,B,\omega)$ (denoted as $I(A,B,\omega)$) is asymptotically equivalent to

$$2I(A,B,\Omega)=E\big(\frac{\partial^2\mathcal{S}_n}{\partial\omega^2}\big)=\left(\begin{array}{ccc} n & 0 & \frac{n^2}{2}B+O(n)\\ 0 & n & -\frac{n^2}{2}A+O(n)\\ \frac{n^2}{2}B+O(n) & -\frac{n^2}{2}A+O(n) & \frac{n^3}{3}(A^2+B^2)+O(n^2)\end{array}\right).$$

(iv) Use the Fisher information to show that $|\widehat{\Omega}_n-\Omega|=O(n^{-3/2})$.

Exercise 1.5 *(i) Simulate three hundred times from model (1.6) using $n=128$. Estimate ω, A and B for each simulation and obtain the empirical mean squared error $\frac{1}{300}\sum_{i=1}^{300}(\hat{\theta}_i-\theta)^2$ (where θ denotes the parameter and $\hat{\theta}_i$ the estimate).*

In your simulations, is the estimate of the period, ω superior to the estimator of coefficients, A and B?

(ii) Do the same as above but now use coloured noise given in (1.7) as the errors. How do your estimates compare to (i)?

`R` Code

Simulation and periodogram for model (1.6) with iid errors:

```
temp <- rnorm(128)
```

```
signal <- 1.5*sin(2*pi*c(1:128)/8) + temp # this simulates the series

# Use the command fft to make the periodogram

P <- abs(fft(signal)/128)**2

frequency <- 2*pi*c(0:127)/128

# To plot the series and periodogram

par(mfrow=c(2,1))

plot.ts(signal)

plot(frequency, P,type="o")
```

Simulation and periodogram for model (1.6) with correlated errors:

```
set.seed(10)

ar2 <- arima.sim(list(order=c(2,0,0), ar = c(1.5, -0.75)), n=128)

signal2 <- 1.5*sin(2*pi*c(1:128)/8) + ar2

P2 <- abs(fft(signal2)/128)**2

frequency <- 2*pi*c(0:127)/128

par(mfrow=c(2,1))

plot.ts(signal2)

plot(frequency, P2,type="o")
```

Chapter 2

Stationary Time Series

2.1 Preliminaries

Different types of convergence

(i) Almost sure convergence: $X_n \overset{\text{a.s.}}{\to} a$ as $n \to \infty$ (in this course a will always be a constant). This means for every $\omega \in \Omega$ $X_n(\omega) \to a$, where $P(\Omega) = 1$ as $n \to \infty$ (this is classical limit of a sequence, see Wiki for a definition).

(ii) Convergence in probability: $X_n \overset{\mathcal{P}}{\to} a$. This means that for every $\varepsilon > 0$, $P(|X_n - a| > \varepsilon) \to 0$ as $n \to \infty$ (see Wiki)

(iii) Convergence in mean square $X_n \overset{2}{\to} a$. This means $\mathrm{E}|X_n - a|^2 \to 0$ as $n \to \infty$ (see Wiki).

(iv) Convergence in distribution. This means the distribution of X_n converges to the distribution of X, ie. for all x where F_X is continuous, we have $F_n(x) \to F_X(x)$ as $n \to \infty$ (where F_n and F_X are the distribution functions of X_n and X respectively). This is the simplest definition (see Wiki).

- Implies:
  - (i), (ii) and (iii) imply (iv).
  - (i) implies (ii).
  - (iii) implies (ii).
- Comments:

- Central limit theorems require (iv).
- It is often easy to show (iii) (since this only requires mean and variance calculations).

The "$O_p(\cdot)$" notation.

- We use the notation $|\widehat{\theta}_n - \theta| = O_p(n^{-1/2})$ if there exists a random variable A (which does not depend on n) such that $|\widehat{\theta}_n - \theta| \leq An^{-1/2}$.

  Example of when you can use $O_p(n^{-1/2})$. If $\mathrm{E}[\widehat{\theta}_n] = 0$ but $\mathrm{var}[\widehat{\theta}_n] \leq Cn^{-1}$. Then we can say that $\mathrm{E}|\widehat{\theta} - \theta| \leq Cn^{-1/2}$ and thus $|\widehat{\theta} - \theta| = O_p(n^{-1/2})$.

Definition of expectation

- Suppose X is a random variable with density f_X, then

$$\mathrm{E}(X) = \int_{-\infty}^{\infty} x f_X(x) dx.$$

  The sample mean $\bar{X} = n^{-1}\sum_{i=1}^{n} X_i$ is an estimator of $\mathrm{E}(X)$.

- Suppose (X, Y) is a bivariate random variable with joint density $f_{X,Y}$, then

$$\mathrm{E}(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{X,Y}(x, y) dx dy.$$

- The covariance is defined as

$$\mathrm{cov}(X, Y) = \mathrm{E}\left((X - \mathrm{E}(X))(Y - \mathrm{E}(Y))\right) = \mathrm{E}(XY) - \mathrm{E}(X)\mathrm{E}(Y).$$

- The variance is $\mathrm{var}(X) = \mathrm{E}(X - \mathrm{E}(X))^2 = \mathrm{E}(X^2) = \mathrm{E}(X)^2$.
- Observe $\mathrm{var}(X) = \mathrm{cov}(X, X)$.
- Rules of covariances. If a, b, c are finite constants and X, Y, Z are random variables with $\mathrm{E}(X^2) < \infty$, $\mathrm{E}(Y^2) < \infty$ and $\mathrm{E}(Z^2) < \infty$ (which immediately implies their means are finite). Then the covariance satisfies the linearity property

$$\mathrm{cov}\left(aX + bY + c, Z\right) = a\mathrm{cov}(X, Z) + b\mathrm{cov}(Y, Z).$$

  Observe the shift c plays no role in the covariance (since it simply shifts the data).

- The variance of vectors. Suppose that A is a matrix and $\underline{X}$ a random vector with variance/-covariance matrix Σ. Then

$$\mathrm{var}(A\underline{X}) = A\mathrm{var}(\underline{X})A' = A\Sigma A', \tag{2.1}$$

which can be proved using the linearity property of covariances.

- The correlation between X and Y is

$$\mathrm{cor}(X,Y) = \frac{\mathrm{cov}(X,Y)}{\sqrt{\mathrm{var}(X)\mathrm{var}(Y)}}$$

and lies between $[-1,1]$. If $\mathrm{var}(X) = \mathrm{var}(Y)$ then $\mathrm{cor}(X,Y)$ is the coefficient of the best linear predictor of X given Y and visa versa.

Remark 2.1.1 (What is a covariance?) *A covariance measures the linear dependence between two random variables. If you plot realisations of the bivariate random variable (X,Y) (X on x-axis and Y on y-axis), then the best line of best fit*

$$\widehat{Y} = \beta_0 + \beta_1 X$$

gives the best linear predictor of Y given X. β_1 is closely related to the covariance. To see how, consider the following example. Given the observation $\{(X_i,Y_i); i = 1,\ldots,n\}$ the gradient of the linear of the line of best fit is

$$\widehat{\beta}_1 = \frac{\sum_{i=1}^n (X_i - \bar{X})(Y_i - \bar{Y})}{\sum_{i=1}^n (X_i - \bar{X})^2}.$$

As the sample size $n \to \infty$ we recall that

$$\widehat{\beta}_1 \overset{\mathcal{P}}{\to} \frac{\mathrm{cov}(X,Y)}{\mathrm{var}(Y)} = \beta_1.$$

Therefore, the covariance between two random variables measures the amount of predictive information (in terms of linear prediction) one variable contains about the other.

Exercise 2.1 (Covariance calculations practice) *Suppose $\{\varepsilon_t\}$ are uncorrelated random variables with $\mathrm{E}[\varepsilon_t] = 0$ and $\mathrm{E}[\varepsilon_t^2] = \sigma^2$*

- *Let* $X_t = \varepsilon_t + 0.5\varepsilon_{t-1}$*. Evaluate* $\mathrm{cov}(X_t, X_{t+r})$ *for* $r = 0, \pm1, \pm2, \pm3, \pm4, \pm5$.

- *Let* $X_t = \sum_{j=0}^{\infty} \rho^j \varepsilon_{t-j}$ *where* $|\rho| < 1$*. Evaluate* $\mathrm{cov}(X_t, X_{t+r})$ *for* $r \in \mathbb{Z}$ *(*$0, \pm1, \pm2, \pm3, \pm4, \ldots$*).*

2.1.1 Formal definition of a time series

When we observe the time series $\{x_t\}$, usually we assume that $\{x_t\}$ is a realisation from a random process $\{X_t\}$. We formalise this notion below. The random process $\{X_t; t \in \mathbb{Z}\}$ (where $\mathbb{Z}$ denotes the integers) is defined on the probability space $\{\Omega, \mathcal{F}, P\}$. We explain what these mean below:

(i) Ω is the set of all possible outcomes. Suppose that $\omega \in \Omega$, then $\{X_t(\omega)\}$ is one realisation from the random process. For any given ω, $\{X_t(\omega)\}$ is not random. In time series we will usually assume that what we observe $x_t = X_t(\omega)$ (for some ω) is a typical realisation. That is, for any other $\omega^* \in \Omega$, $X_t(\omega^*)$ will be different, but its general or overall characteristics will be similar.

(ii) $\mathcal{F}$ is known as a sigma algebra. It is a set of subsets of Ω (though not necessarily the set of all subsets, as this can be too large). But it consists of all sets for which a probability can be assigned. That is if $A \in \mathcal{F}$, then a probability is assigned to the set A.

(iii) P is the probability.

This is very general definition. But it is too general for modelling. Below we define the notion of stationarity and weak dependence, that allows for estimators to have a meaningful interpretation.

2.2 The sample mean and its standard error

We start with the simplest case, estimating the mean when the data is dependent. This is usually estimated with the sample mean. However, for the sample mean to be estimating something reasonable we require a very weak form of stationarity. That is the time series has the same mean for all t i.e.

$$X_t = \underbrace{\mu}_{=\mathrm{E}(X_t)} + \underbrace{(X_t - \mu)}_{=\varepsilon_t},$$

where $\mu = \mathrm{E}(X_t)$ for all t. This is analogous to say that the independent random variables $\{X_t\}$ all have a common mean. Under this assumption $\bar{X}$ is an unbiased estimator of μ. Next, our aim is to obtain conditions under which $\bar{X}$ is a "reasonable" estimator of the mean.

Based on just one realisation of a time series we want to make inference about the parameters associated with the process $\{X_t\}$, such as the mean. We recall that in classical statistics we usually assume we observe several independent realisations, $\{X_t\}$ all with the same distribution, and use $\bar{X} = \frac{1}{n}\sum_{t=1}^{n} X_t$ to estimate the mean. Roughly speaking, with several independent realisations we are able to sample over the entire probability space and thus obtain a "good" (meaning consistent or close to true mean) estimator of the mean. On the other hand, if the samples were highly dependent, then it is likely that $\{X_t\}$ is concentrated over a small part of the probability space. In this case, the sample mean will not converge to the mean (be close to the true mean) as the sample size grows.

A typical time series is a half way house between "fully" dependent data and independent data. Unlike classical statistics, in time series, parameter estimation is based on only one realisation $x_t = X_t(\omega)$ (not multiple, independent, replications). Therefore, it would appear impossible to obtain a good estimator of the mean. However good estimators of the mean are still possible, based on just one realisation of the time series so long as certain assumptions are satisfied (i) the process has a constant mean (a type of stationarity) and (ii) despite the fact that each time series is generated from one realisation there is 'short' memory in the observations. That is, what is observed today, x_t has little influence on observations in the future, x_{t+k} (when k is relatively large). Hence, even though we observe one tragectory, that trajectory traverses much of the probability space. The amount of dependency in the time series determines the 'quality' of the estimator. There are several ways to measure the dependency. We know that the most common is the measure of linear dependency, known as the covariance. Formally, the covariance in the stochastic process $\{X_t\}$ is defined as

$$\mathrm{cov}(X_t, X_{t+k}) = \mathrm{E}\left[\left(X_t - \mathrm{E}(X_t)\right)\left(X_{t+k} - \mathrm{E}\left(X_{t+k}\right)\right)\right] = \mathrm{E}(X_t X_{t+k}) - \mathrm{E}(X_t)\mathrm{E}(X_{t+k}).$$

Noting that if $\{X_t\}$ has zero mean, then the above reduces to $\mathrm{cov}(X_t, X_{t+k}) = \mathrm{E}(X_t X_{t+k})$.

Remark 2.2.1 (Covariance in a time series) *To illustrate the covariance within a time series setting, we generate the time series*

$$X_t = 1.8\cos\left(\frac{2\pi}{5}\right)X_{t-1} - 0.9^2 X_{t-2} + \varepsilon_t \tag{2.2}$$

for $t = 1, \ldots, n$. A scatter plot of X_t against X_{t+r} for $r = 1, \ldots, 4$ and $n = 200$ is given in Figure 2.1. The corresponding sample autocorrelation (ACF) plot (as defined in equation (2.7) is given in Figure 2.2). Focus on the lags $r = 1, \ldots, 4$ in the ACF plot. Observe that they match what is seen in the scatter plots.

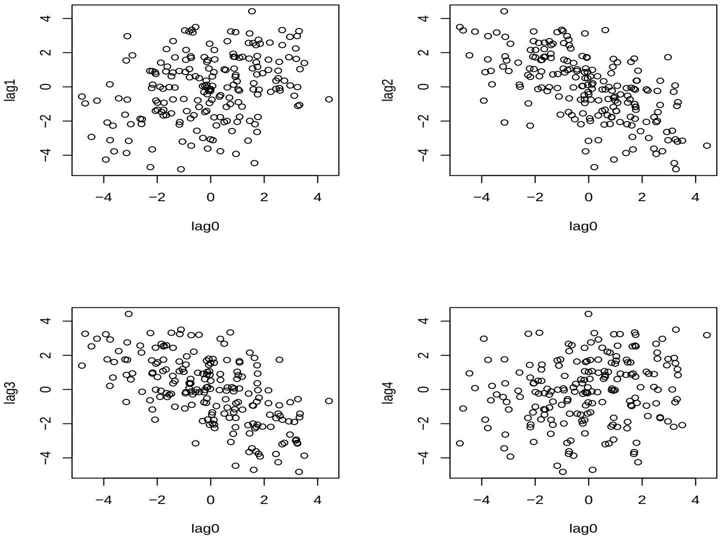


Figure 2.1: From model (2.2). Plot of X_t against X_{t+r} for $r = 1, \ldots, 4$. Top left: $r = 1$. Top right: $r = 2$, Bottom left: $r = 3$ and Bottom right: $r = 4$.

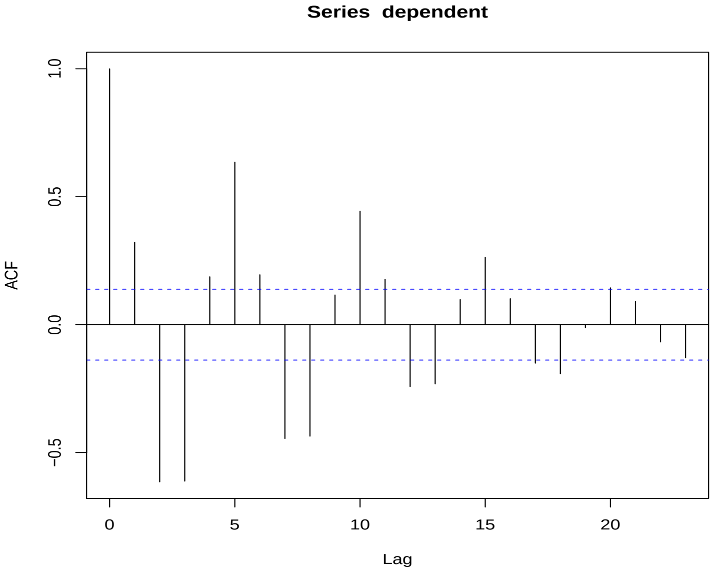


Figure 2.2: ACF plot of realisation from model (2.2).

Remark 2.2.2 *It is worth bearing in mind that the covariance only measures linear dependence. For some statistical analysis, such as deriving an expression for the variance of an estimator, the covariance is often sufficient as a measure. However, given* $\mathrm{cov}(X_t, X_{t+k})$ *we cannot say anything about* $\mathrm{cov}(g(X_t), g(X_{t+k}))$*, where* g *is a nonlinear function. There are occassions where we require a more general measure of dependence (for example, to show asymptotic normality). Examples of more general measures include mixing (and other related notions, such as Mixingales, Near-Epoch dependence, approximate m-dependence, physical dependence, weak dependence), first introduced by Rosenblatt in the 50s (Rosenblatt and Grenander (1997)). In this course we will not cover mixing.*

Returning to the sample mean example suppose that $\{X_t\}$ is a time series. In order to estimate the mean we need to be sure that the mean is constant over time (else the estimator will be meaningless). Therefore we will assume that $\{X_t\}$ is a time series with constant mean μ, that is $\mathrm{E}[X_t] = \mu$ for all t. We observe $\{X_t\}_{t=1}^{n}$ and estimate the mean μ with the sample mean $\bar{X} = \frac{1}{n}\sum_{t=1}^{n} X_t$. It is clear that this is an unbiased estimator of μ, since $\mathrm{E}(\bar{X}) = \mu$ (it is unbiased). Thus to see whether it converges in mean square to μ we consider its variance

$$\begin{aligned}
\mathrm{var}(\bar{X}) &= \frac{1}{n^2}\sum_{t,\tau=1}^{n}\mathrm{cov}(X_t, X_\tau) = \frac{1}{n^2}\sum_{t=1}^{n}\mathrm{var}(X_t) + \frac{2}{n^2}\sum_{t=1}^{n}\sum_{\tau=t+1}^{n}\mathrm{cov}(X_t, X_\tau) \\
&= \frac{1}{n^2}\sum_{t=1}^{n}\mathrm{var}(X_t) + \frac{2}{n^2}\sum_{r=1}^{n-1}\sum_{t=1}^{n-|r|}\mathrm{cov}(X_t, X_{t+r}). \qquad (2.3)
\end{aligned}$$

Remark 2.2.3 *The last line of (2.4) is clear from viewing the sample mean as the inner product between ones and the vector* $\underline{X}_n' = (X_1, \ldots, X_n)$*, that is*

$$\bar{X} = n^{-1}(1, \ldots, 1)\underline{X}_n.$$

Using (2.1) we have

$$\mathrm{var}(\bar{X}) = n^{-2}(1, \ldots, 1)\underbrace{\mathrm{var}(\underline{X}_n)}_{\textit{matrix, } \Sigma}\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

We recall that

$$\mathrm{var}(\underline{X}_n) = \begin{pmatrix} \mathrm{cov}(X_1, X_1) & \mathrm{cov}(X_1, X_2) & \mathrm{cov}(X_1, X_3) & \ldots & \mathrm{cov}(X_1, X_n) \\ \mathrm{cov}(X_2, X_1) & \mathrm{cov}(X_2, X_2) & \mathrm{cov}(X_2, X_3) & \ldots & \mathrm{cov}(X_2, X_n) \\ \mathrm{cov}(X_3, X_1) & \mathrm{cov}(X_3, X_2) & \mathrm{cov}(X_3, X_3) & \ldots & \mathrm{cov}(X_3, X_n) \\ \vdots & \vdots & \vdots & \ddots & \ldots \\ \mathrm{cov}(X_n, X_1) & \mathrm{cov}(X_n, X_2) & \ldots & \ldots & \mathrm{cov}(X_n, X_n) \end{pmatrix}.$$

Now pre-multiplying and post multipling a matrix by a vector of ones, simple means adding all the elements in the matrix. From the symmetry of variance matrix and by summing each off-diagonal (the rth off-diagonal is r shifts from the diagonal) we obtain (2.4).

Using the expression in (2.4) we can deduce under what conditions on the time series we can obtain a reasonable estimator of the mean. If the covariance structure decays at such a rate that the sum of all lags is finite, that is

$$\sup_t \sum_{r=-\infty}^{\infty} |\mathrm{cov}(X_t, X_{t+r})| < \infty,$$

often called short memory), then the variance is

$$\begin{aligned} \mathrm{var}(\bar{X}) &\leq \frac{1}{n^2}\sum_{t=1}^{n}\mathrm{var}(X_t) + \frac{2}{n^2}\sum_{r=1}^{n-1}\sum_{t=1}^{n-|r|}|\mathrm{cov}(X_t, X_{t+r})| \\ &\leq \frac{1}{n^2}\sum_{t=1}^{n}\mathrm{var}(X_t) + \frac{2}{n^2}\sum_{t=1}^{n-1}\underbrace{\sum_{r=1}^{\infty}|\mathrm{cov}(X_t, X_{t+r})|}_{\text{finite for all } t \text{ and } n} \leq Cn^{-1} = O(n^{-1}). \qquad (2.4) \end{aligned}$$

This rate of convergence is the same as if $\{X_t\}$ were iid/uncorrelated data. However, if the correlations are positive it will be larger than the case that $\{X_t\}$ are uncorrelated.

However, even with this assumption we need to be able to estimate $\mathrm{var}(\bar{X})$ in order to test/construct CI for μ. Usually this requires the stronger assumption of stationarity, which we define in Section 2.3.

2.2.1 The variance of the estimated regressors in a linear regression model with correlated errors

Let us return to the parametric models discussed in Section 1.3.1. The general model is

$$Y_t = \beta_0 + \sum_{j=1}^{p} \beta_j u_{t,j} + \varepsilon_t = \boldsymbol{\beta}'\mathbf{u}_t + \varepsilon_t,$$

where $\mathrm{E}[\varepsilon_t] = 0$ and we will assume that $\{u_{t,j}\}$ are nonrandom regressors. Note this includes the parametric trend models discussed in Section 1.3.1. We use least squares to estimate $\boldsymbol{\beta}$

$$\mathcal{L}_n(\boldsymbol{\beta}) = \sum_{t=1}^{n} (Y_t - \boldsymbol{\beta}'\mathbf{u}_t)^2,$$

with

$$\hat{\beta}_n = \arg\min \mathcal{L}_n(\boldsymbol{\beta}) = (\sum_{t=1}^{n} \mathbf{u}_t\mathbf{u}_t')^{-1} \sum_{t=1}^{n} Y_t\mathbf{u}_t.$$

Thus $\frac{\partial \mathcal{L}_n(\hat{\boldsymbol{\beta}}_n)}{\partial \boldsymbol{\beta}} = 0$. To evaluate the variance of $\hat{\boldsymbol{\beta}}_n$ we will derive an expression for $\hat{\boldsymbol{\beta}}_n - \boldsymbol{\beta}$ (this expression also applies to many nonlinear estimators too). We note that by using straightforward algebra we can show that

$$\frac{\partial \mathcal{L}_n(\hat{\boldsymbol{\beta}}_n)}{\partial \boldsymbol{\beta}} - \frac{\partial \mathcal{L}_n(\boldsymbol{\beta})}{\partial \boldsymbol{\beta}} = \left[\hat{\boldsymbol{\beta}}_n - \boldsymbol{\beta}\right]' \sum_{t=1}^{n} \mathbf{u}_t\mathbf{u}_t'. \tag{2.5}$$

Moreoover because $\frac{\partial \mathcal{L}_n(\hat{\boldsymbol{\beta}}_n)}{\partial \boldsymbol{\beta}} = 0$ we have

$$\begin{aligned} \frac{\partial \mathcal{L}_n(\hat{\boldsymbol{\beta}}_n)}{\partial \boldsymbol{\beta}} - \frac{\partial \mathcal{L}_n(\boldsymbol{\beta})}{\partial \boldsymbol{\beta}} &= -\frac{\partial \mathcal{L}_n(\boldsymbol{\beta})}{\partial \boldsymbol{\beta}} \\ &= \sum_{t=1}^{n} \underbrace{\left[Y_t - \boldsymbol{\beta}'\mathbf{u}_t\right]}_{\varepsilon_t} \mathbf{u}_t = \sum_{t=1}^{n} \mathbf{u}_t\varepsilon_t. \end{aligned} \tag{2.6}$$

Altogether (2.5) and (2.6) give

$$\left[\hat{\boldsymbol{\beta}}_n - \boldsymbol{\beta}\right]' \sum_{t=1}^{n} \mathbf{u}_t\mathbf{u}_t' = \sum_{t=1}^{n} \mathbf{u}_t'\varepsilon_t.$$

and

$$\left[\hat{\boldsymbol{\beta}}_n - \boldsymbol{\beta}\right] = \left(\sum_{t=1}^{n} \mathbf{u}_t \mathbf{u}_t'\right)^{-1} \sum_{t=1}^{n} \mathbf{u}_t \varepsilon_t.$$

Using this expression we can see that

$$\text{var}\left[\hat{\boldsymbol{\beta}}_n - \boldsymbol{\beta}\right] = \left(\frac{1}{n}\sum_{t=1}^{n} \mathbf{u}_t \mathbf{u}_t'\right)^{-1} \text{var}\left(\frac{1}{n}\sum_{t=1}^{n} \mathbf{u}_t \varepsilon_t\right) \left(\frac{1}{n}\sum_{t=1}^{n} \mathbf{u}_t \mathbf{u}_t'\right)^{-1}.$$

Finally we need only evaluate $\text{var}\left(\frac{1}{n}\sum_{t=1}^{n} \mathbf{u}_t \varepsilon_t\right)$ which is

$$\begin{aligned}\text{var}\left(\frac{1}{n}\sum_{t=1}^{n} \mathbf{u}_t \varepsilon_t\right) &= \frac{1}{n^2}\sum_{t,\tau=1}^{n} \text{cov}[\varepsilon_t, \varepsilon_\tau] \mathbf{u}_t \mathbf{u}_\tau' \\ &= \underbrace{\frac{1}{n^2}\sum_{t=1}^{n} \text{var}[\varepsilon_t] \mathbf{u}_t \mathbf{u}_t'}_{\text{expression if independent}} + \underbrace{\frac{2}{n^2}\sum_{t=1}^{n}\sum_{\tau=t+1}^{n} \text{cov}[\varepsilon_t, \varepsilon_\tau] \mathbf{u}_t \mathbf{u}_\tau'}_{\text{additional term due to correlation in the errors}}.\end{aligned}$$

Under the assumption that $\left(\frac{1}{n}\sum_{t=1}^{n} \mathbf{u}_t \mathbf{u}_t'\right)$ is non-singular, $\sup_t \|\mathbf{u}_t\|_1 < \infty$ and $\sup_t \sum_{\tau=-\infty}^{\infty} |\text{cov}(\varepsilon_t, \varepsilon_\tau)| < \infty$, we can see that $\text{var}\left[\hat{\boldsymbol{\beta}}_n - \boldsymbol{\beta}\right] = O(n^{-1})$, but just as in the case of the sample mean we need to impose some additional conditions on $\{\varepsilon_t\}$ if we want to construct confidence intervals/test $\boldsymbol{\beta}$.

2.3 Stationary processes

We have established that one of the main features that distinguish time series analysis from classical methods is that observations taken over time (a time series) can be dependent and this dependency tends to decline the further apart in time these two observations. However, to do any sort of analysis of this time series we have to assume some sort of invariance in the time series, for example the mean or variance of the time series does not change over time. If the marginal distributions of the time series were totally different no sort of inference would be possible (suppose in classical statistics you were given independent random variables all with different distributions, what parameter would you be estimating, it is not possible to estimate anything!).

The typical assumption that is made is that a time series is stationary. Stationarity is a rather intuitive concept, it is an invariant property which means that statistical characteristics of the time series do not change over time. For example, the yearly rainfall may vary year by year, but the average rainfall in two equal length time intervals will be roughly the same as would the number of times the rainfall exceeds a certain threshold. Of course, over long periods of time this assumption may not be so plausible. For example, the climate change that we are currently experiencing is causing changes in the overall weather patterns (we will consider nonstationary time series towards the end of this course). However in many situations, including short time intervals, the assumption of stationarity is quite a plausible. Indeed often the statistical analysis of a time series is done under the assumption that a time series is stationary.

2.3.1 Types of stationarity and Ergodicity

There are two definitions of stationarity, weak stationarity which only concerns the covariance of a process and strict stationarity which is a much stronger condition and supposes the distributions are invariant over time.

Definition 2.3.1 (Strict stationarity) *The time series $\{X_t\}$ is said to be strictly stationary if for any finite sequence of integers $t_1, \ldots, t_k$ and shift h the distribution of $(X_{t_1}, \ldots, X_{t_k})$ and $(X_{t_1+h}, \ldots, X_{t_k+h})$ are the same.*

The above assumption is often considered to be rather strong (and given a data it is very hard to check). Often it is possible to work under a weaker assumption called weak/second order stationarity.

Definition 2.3.2 (Second order stationarity/weak stationarity) *The time series $\{X_t\}$ is said to be second order stationary if the mean is constant for all t and if for any t and k the covariance between X_t and X_{t+k} only depends on the lag difference k. In other words there exists a function $c : \mathbb{Z} \to \mathbb{R}$ such that for all t and k we have*

$$c(k) = \text{cov}(X_t, X_{t+k}).$$

Remark 2.3.1 (Strict and second order stationarity) *(i) If a process is strictly stationarity $\underline{\text{and}}$ $\text{E}|X_t^2| < \infty$, then it is also second order stationary. But the converse is not necessarily true. To show that strict stationarity (with $\text{E}|X_t^2| < \infty$) implies second order stationarity,* *suppose that $\{X_t\}$ is a strictly stationary process, then*

$$\begin{aligned}\mathrm{cov}(X_t, X_{t+k}) &= \mathrm{E}(X_t X_{t+k}) - \mathrm{E}(X_t)\mathrm{E}(X_{t+k}) \\ &= \int xy \left[P_{X_t,X_{t+k}}(dx,dy) - P_{X_t}(dx)P_{X_{t+k}}(dy)\right] \\ &= \int xy \left[P_{X_0,X_k}(dx,dy) - P_{X_0}(dx)P_{X_k}(dy)\right] = \mathrm{cov}(X_0, X_k),\end{aligned}$$

where $P_{X_t,X_{t+k}}$ and P_{X_t} is the joint distribution and marginal distribution of X_t, X_{t+k} respectively. The above shows that $\mathrm{cov}(X_t, X_{t+k})$ does not depend on t and $\{X_t\}$ is second order stationary.

(ii) If a process is strictly stationary but the second moment is <u>not</u> finite, then it is not second order stationary.

(iii) It should be noted that a weakly stationary Gaussian time series is also strictly stationary too (this is the only case where weakly stationary implies strictly stationary).

Example 2.3.1 (The sample mean and its variance under second order stationarity) *Returning the variance of the sample mean discussed (2.4), if a time series is second order stationary, then the sample mean $\bar{X}$ is estimating the mean μ and the variance of $\bar{X}$ is*

$$\begin{aligned}\mathrm{var}(\bar{X}) &= \frac{1}{n^2}\sum_{t=1}^{n}\underbrace{\mathrm{var}(X_t)}_{c(0)} + \frac{2}{n^2}\sum_{r=1}^{n-1}\sum_{t=1}^{n-r}\underbrace{\mathrm{cov}(X_t, X_{t+r})}_{=c(r)} \\ &= \frac{1}{n}c(0) + \frac{2}{n}\sum_{r=1}^{n}\underbrace{\left(\frac{n-r}{n}\right)}_{=1-r/n} c(r),\end{aligned}$$

where we note that above is based on the expansion in (2.4). We approximate the above, by using that the covariances $\sum_r |c(r)| < \infty$. Therefore for all r, $(1-r/n)c(r) \to c(r)$ and $|\sum_{r=1}^{n}(1-|r|/n)c(r)| \leq \sum_r |c(r)|$, thus by dominated convergence (see Appendix A) $\sum_{r=1}^{n}(1-r/n)c(r) \to \sum_{r=1}^{\infty} c(r)$. This implies that

$$\mathrm{var}(\bar{X}) \approx \frac{1}{n}c(0) + \frac{2}{n}\sum_{r=1}^{\infty} c(r) = O(\frac{1}{n}).$$

The above is often called the long term variance. The above implies that

$$\mathrm{E}(\bar{X}-\mu)^2 = \mathrm{var}(\bar{X}) \to 0, \qquad n \to \infty,$$

which we recall is convergence in mean square. This immediately implies convergence in probability $\bar{X} \xrightarrow{\mathcal{P}} \mu$*.*

The example above illustrates how second order stationarity gives an elegant expression for the variance and can be used to estimate the standard error associated with $\bar{X}$.

Example 2.3.2 *In Chapter 7 we consider estimation of the autocovariance function. However for now rely on the* `R` *command* `acf`*. For the curious, it evaluates* $\widehat{\rho}(r) = \widehat{c}(r)/\widehat{c}(0)$*, where*

$$\widehat{c}(r) = \frac{1}{n}\sum_{t=1}^{n-r}(X_t - \bar{X})(X_{t+r} - \bar{X}) \qquad (2.7)$$

for $r = 1, \ldots, m$ *(*m *is some value that* `R` *defines), you can change the maximum number of lags by using* `acf(data, lag = 30)`*, say). Observe that even if* $X_t = \mu_t$ *(nonconstant mean), from the way* $\widehat{c}(r)$ *(sum of* $(n-r)$ *terms) is defined,* $\widehat{\rho}(r)$ *will decay to zero as* $r \to n$*.*

In Figure 2.3 we give the sample acf plots of the Southern Oscillation Index and the Sunspot data. We observe that are very different. The acf of the SOI decays rapidly, but there does appear to be some sort of 'pattern' in the correlations. On the other hand, there is more "persistence" in the acf of the Sunspot data. The correlations of the acf appear to decay but over a longer period of time and there is a clear periodicity.

We now motivate the concept of ergodicity. Sometimes, it is difficult to evaluate the mean and variance of an estimator. Therefore, we may want an alternative form of convergence (instead of the mean squared error). To see whether this is possible we recall that for iid random variables we have the very useful law of large numbers

$$\frac{1}{n}\sum_{t=1}^{n} X_t \overset{\text{a.s.}}{\to} \mu$$

and in general $\frac{1}{n}\sum_{t=1}^{n} g(X_t) \overset{\text{a.s.}}{\to} \mathrm{E}[g(X_0)]$ (if $\mathrm{E}[g(X_0)] < \infty$). Does such a result exists in time series? It does, but we require the slightly stronger condition that a time series is ergodic (which is a slightly stronger condition than the strictly stationary).

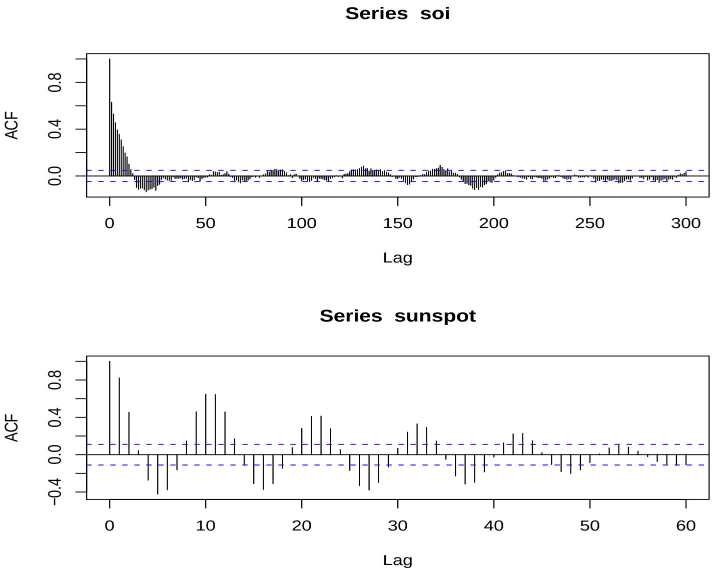


Figure 2.3: Top: ACF of Southern Oscillation data. Bottom ACF plot of Sunspot data.

Definition 2.3.3 (Ergodicity: Formal definition) *Let $(\Omega, \mathcal{F}, P)$ be a probability space. A transformation $T : \Omega \to \Omega$ is said to be measure preserving if for every set $A \in \mathcal{F}$, $P(T^{-1}A) = P(A)$. Moreover, it is said to be an ergodic transformation if $T^{-1}A = A$ implies that $P(A) = 0$ or 1.*

It is not obvious what this has to do with stochastic processes, but we attempt to make a link. Let us suppose that $X = \{X_t\}$ is a strictly stationary process defined on the probability space $(\Omega, \mathcal{F}, P)$. By strict stationarity the transformation (shifting a sequence by one)

$$T(x_1, x_2, \ldots) = (x_2, x_3, \ldots),$$

is a measure preserving transformation. To understand ergodicity we define the set A, where

$$A = \{\omega : (X_1(\omega), X_0(\omega), \ldots) \in H\}. = \{\omega : X_{-1}(\omega), \ldots, X_{-2}(\omega), \ldots) \in H\}.$$

The stochastic process is said to be ergodic, if the only sets which satisfies the above are such that $P(A) = 0$ or 1. Roughly, this means there cannot be too many outcomes ω which generate sequences which 'repeat' itself (are periodic in some sense). An equivalent definition is given in (2.8). From this definition is can be seen why "repeats" are a bad idea. If a sequence repeats the time average is unlikey to converge to the mean.

See Billingsley (1994), page 312-314, for examples and a better explanation.

The definition of ergodicity, given above, is quite complex and is rarely used in time series analysis. However, one consequence of ergodicity is the ergodic theorem, which is extremely useful in time series. It states that if $\{X_t\}$ is an ergodic stochastic process then

$$\frac{1}{n}\sum_{t=1}^{n} g(X_t) \overset{\text{a.s.}}{\to} \mathrm{E}[g(X_0)]$$

for any function $g(\cdot)$. And in general for any shift $\tau_1, \ldots, \tau_k$ and function $g : \mathbb{R}^{k+1} \to \mathbb{R}$ we have

$$\frac{1}{n}\sum_{t=1}^{n} g(X_t, X_{t+\tau_1}, \ldots, X_{t+\tau_k}) \overset{\text{a.s.}}{\to} \mathrm{E}[g(X_0, \ldots, X_{t+\tau_k})] \tag{2.8}$$

(often (2.8) is used as the definition of ergodicity, as it is an iff with the ergodic definition). This result generalises the strong law of large numbers (which shows almost sure convergence for iid random variables) to dependent random variables. It is an extremely useful result, as it shows us that "mean-type" estimators consistently estimate their mean (without any real effort). The only drawback is that we do not know the speed of convergence.

(2.8) gives us an idea of what constitutes an ergodic process. Suppose that $\{\varepsilon_t\}$ is an ergodic process (a classical example are iid random variables) then any reasonable (meaning measurable) function of X_t is also ergodic. More precisely, if X_t is defined as

$$X_t = h(\ldots, \varepsilon_t, \varepsilon_{t-1}, \ldots), \tag{2.9}$$

where $\{\varepsilon_t\}$ are iid random variables and $h(\cdot)$ is a measureable function, then $\{X_t\}$ is an Ergodic process. For full details see Stout (1974), Theorem 3.4.5.

Remark 2.3.2 *As mentioned above all Ergodic processes are stationary, but a stationary process is not necessarily ergodic. Here is one simple example. Suppose that $\{\varepsilon_t\}$ are iid random variables and Z is a Bernoulli random variable with outcomes $\{1, 2\}$ (where the chance of either outcome is half). Suppose that Z stays the same for all t. Define*

$$X_t = \begin{cases} \mu_1 + \varepsilon_t & Z = 1 \\ \mu_2 + \varepsilon_t & Z = 2. \end{cases}$$

It is clear that $\mathrm{E}(X_t|Z = i) = \mu_i$ and $\mathrm{E}(X_t) = \frac{1}{2}(\mu_1 + \mu_2)$. This sequence is stationary. However, we observe that $\frac{1}{T}\sum_{t=1}^{T} X_t$ will only converge to one of the means, hence we do not have almost *sure convergence (or convergence in probability) to* $\frac{1}{2}(\mu_1 + \mu_2)$.

Exercise 2.2 *State, with explanation, which of the following time series is second order stationary, which are strictly stationary and which are both.*

(i) $\{\varepsilon_t\}$ *are iid random variables with mean zero and variance one.*

(ii) $\{\varepsilon_t\}$ *are iid random variables from a Cauchy distributon.*

(iii) $X_{t+1} = X_t + \varepsilon_t$, *where* $\{\varepsilon_t\}$ *are iid random variables with mean zero and variance one.*

(iv) $X_t = Y$ *where* Y *is a random variable with mean zero and variance one.*

(iv) $X_t = U_t + U_{t-1} + V_t$, *where* $\{(U_t, V_t)\}$ *is a strictly stationary vector time series with* $\mathrm{E}[U_t^2] < \infty$ *and* $\mathrm{E}[V_t^2] < \infty$.

Exercise 2.3 *(i) Make an ACF plot of the monthly temperature data from 1996-2014.*

(ii) Make and ACF plot of the yearly temperature data from 1880-2013.

(iii) Make and ACF plot of the residuals (after fitting a line through the data (using the command `lsfit(..)$res`*)) of the yearly temperature data from 1880-2013.*

Briefly describe what you see.

`R` code

To make the above plots we use the commands

```
par(mfrow=c(2,1))
acf(soi,lag.max=300)
acf(sunspot,lag.max=60)
```

2.3.2 Towards statistical inference for time series

Returning to the sample mean Example 2.3.1. Suppose we want to construct CIs or apply statistical tests on the mean. This requires us to estimate the long run variance (assuming stationarity)

$$\mathrm{var}(\bar{X}) \approx \frac{1}{n}c(0) + \frac{2}{n}\sum_{r=1}^{\infty} c(r).$$

There are several ways this can be done, either by fitting a model to the data and from the model estimate the covariance or doing it nonparametrically. This example motivates the contents of the course:

(i) Modelling, finding suitable time series models to fit to the data.

(ii) Forecasting, this is essentially predicting the future given current and past observations.

(iii) Estimation of the parameters in the time series model.

(iv) The spectral density function and frequency domain approaches, sometimes within the frequency domain time series methods become extremely elegant.

(v) Analysis of nonstationary time series.

(vi) Analysis of nonlinear time series.

(vii) How to derive sampling properties.

2.4 What makes a covariance a covariance?

The covariance of a stationary process has several very interesting properties. The most important is that it is positive semi-definite, which we define below.

Definition 2.4.1 (Positive semi-definite sequence) *(i) A sequence $\{c(k); k \in \mathbb{Z}\}$ ($\mathbb{Z}$ is the set of all integers) is said to be positive semi-definite if for any $n \in \mathbb{Z}$ and sequence $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ the following is satisfied*

$$\sum_{i,j=1}^{n} c(i-j)x_i x_j \geq 0.$$

(ii) A function is said to be an <u>even</u> positive semi-definite sequence if (i) is satisfied and $c(k) = c(-k)$ for all $k \in \mathbb{Z}$.

An extension of this notion is the positive semi-definite function.

Definition 2.4.2 (Positive semi-definite function) *(i) A function $\{c(u); u \in \mathbb{R}\}$ is said to be positive semi-definite if for any $n \in \mathbb{Z}$ and sequence $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ the following* *is satisfied*

$$\sum_{i,j=1}^{n} c(u_i - u_j) x_i x_j \geq 0.$$

(ii) A function is said to be an <u>*even*</u> *positive semi-definite function if (i) is satisfied and $c(u) = c(-u)$ for all $u \in \mathbb{R}$.*

Remark 2.4.1 *You have probably encountered this positive definite notion before, when dealing with positive definite matrices. Recall the $n \times n$ matrix Σ_n is positive semi-definite if for all $\underline{x} \in \mathbb{R}^n$ $\underline{x}'\Sigma_n\underline{x} \geq 0$. To see how this is related to positive semi-definite matrices, suppose that the matrix Σ_n has a special form, that is the elements of Σ_n are $(\Sigma_n)_{i,j} = c(i-j)$. Then $\underline{x}'\Sigma_n\underline{x} = \sum_{i,j}^{n} c(i-j)x_i x_j$. We observe that in the case that $\{X_t\}$ is a stationary process with covariance $c(k)$, the variance covariance matrix of $\underline{X}_n = (X_1, \ldots, X_n)$ is Σ_n, where $(\Sigma_n)_{i,j} = c(i-j)$.*

We now take the above remark further and show that the covariance of a stationary process is positive semi-definite.

Theorem 2.4.1 *Suppose that $\{X_t\}$ is a discrete time/continuous stationary time series with covariance function $\{c(k)\}$, then $\{c(k)\}$ is an even positive semi-definite sequence/function. Conversely for any* <u>*even*</u> *positive semi-definite sequence/function there exists a stationary time series with this positive semi-definite sequence/function as its covariance function.*

PROOF. We prove the result in the case that $\{X_t\}$ is a discrete time time series, ie. $\{X_t; t \in \mathbb{Z}\}$.

We first show that $\{c(k)\}$ is a positive semi-definite sequence. Consider any sequence $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, and the double sum $\sum_{i,j}^{n} x_i c(i-j) x_j$. Define the random variable $Y = \sum_{i=1}^{n} x_i X_i$. It is straightforward to see that $\text{var}(Y) = \underline{x}'\text{var}(\underline{X}_n)\underline{x} = \sum_{i,j=1}^{n} c(i-j)x_i x_j$ where $\underline{X}_n = (X_1, \ldots, X_n)$. Since for any random variable Y, $\text{var}(Y) \geq 0$, this means that $\sum_{i,j=1}^{n} x_i c(i-j) x_j \geq 0$, hence $\{c(k)\}$ is a positive definite sequence.

To show the converse, that is for any positive semi-definite sequence $\{c(k)\}$ we can find a corresponding stationary time series with the covariance $\{c(k)\}$ is relatively straightfoward, but depends on defining the characteristic function of a process and using Komologorov's extension theorem. We omit the details but refer an interested reader to Brockwell and Davis (1998), Section 1.5. □

In time series analysis usually the data is analysed by fitting a *model* to the data. The model (so long as it is correctly specified, we will see what this means in later chapters) guarantees the covariance function corresponding to the model (again we cover this in later chapters) is positive definite. This means, in general we do not have to worry about positive definiteness of the covariance function, as it is implicitly implied.

On the other hand, in spatial statistics, often the object of interest is the covariance function and specific classes of covariance functions are fitted to the data. In which case it is necessary to ensure that the covariance function is semi-positive definite (noting that once a covariance function has been found by Theorem 2.4.1 there must exist a spatial process which has this covariance function). It is impossible to check for positive definiteness using Definitions 2.4.1 or 2.4.1. Instead an alternative but equivalent criterion is used. The general result, which does not impose any conditions on $\{c(k)\}$ is stated in terms of positive measures (this result is often called Bochner's theorem). Instead, we place some conditions on $\{c(k)\}$, and state a simpler version of the theorem.

Theorem 2.4.2 *Suppose the coefficients $\{c(k); k \in \mathbb{Z}\}$ are absolutely summable (that is $\sum_k |c(k)| < \infty$). Then the sequence $\{c(k)\}$ is positive semi-definite if an only if the function $f(\omega)$, where*

$$f(\omega) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} c(k)\exp(ik\omega),$$

is nonnegative for all $\omega \in [0, 2\pi]$.

We also state a variant of this result for positive semi-definite functions. Suppose the function $\{c(u); k \in \mathbb{R}\}$ is absolutely summable (that is $\int_{\mathbb{R}} |c(u)|du < \infty$). Then the function $\{c(u)\}$ is positive semi-definite if and only if the function $f(\omega)$, where

$$f(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} c(u)\exp(iu\omega)du \geq 0$$

for all $\omega \in \mathbb{R}$.

The generalisation of the above result to dimension d is that $\{c(\boldsymbol{u}); \boldsymbol{u} \in \mathbb{R}^d\}$ is a positive semi-definite sequence if and if

$$f(\boldsymbol{\omega}) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} c(\boldsymbol{u})\exp(i\boldsymbol{u}'\boldsymbol{\omega})d\boldsymbol{u} \geq 0$$

for all $\boldsymbol{\omega}^d \in \mathbb{R}^d$.

PROOF. See Section 9.3.1.

Example 2.4.1 *We will show that sequence $c(0) = 1$, $c(1) = 0.5$, $c(-1) = 0.5$ and $c(k) = 0$ for $|k| > 1$ a positive definite sequence.*

From the definition of spectral density given above we see that the 'spectral density' corresponding to the above sequence is

$$f(\omega) = 1 + 2 \times 0.5 \times cos(\omega).$$

Since $|\cos(\omega)| \leq 1$, $f(\omega) \geq 0$, thus the sequence is positive definite. An alternative method is to find a model which has this as the covariance structure. Let $X_t = \varepsilon_t + \varepsilon_{t-1}$, where ε_t are iid random variables with $\mathrm{E}[\varepsilon_t] = 0$ and $\mathrm{var}(\varepsilon_t) = 0.5$. This model has this covariance structure.

2.5 Spatial covariances

Theorem 2.4.2 is extremely useful in finding valid spatial covariances. We recall that $c_d : \mathbb{R}^d \to \mathbb{R}$ is a positive semi-definite covariance (on the spatial plane $\mathbb{R}^d$) if there exists a positive function f_d where

$$c_d(\boldsymbol{u}) = \int_{\mathbb{R}^d} f_d(\boldsymbol{\omega}) \exp(-i\boldsymbol{u}'\boldsymbol{\omega}) d\boldsymbol{\omega} \tag{2.10}$$

for all $\boldsymbol{u} \in \mathbb{R}^d$ (the inverse Fourier transform of what was written). This result allows one to find parametric covariance spatial processes.

However, beyond dimension $d = 1$ (which can be considered a "time series"), there exists conditions stronger than spatial (second order) stationarity. Probably the the most popular is spatial isotropy, which is even stronger than stationarity. A covariance c_d is called spatially isotropic if it is stationary and there exist a function $c : \mathbb{R} \to \mathbb{R}$ such that $c_d(\boldsymbol{u}) = c(\|\boldsymbol{u}\|_2)$. It is clear that in the case $d = 1$, a stationary covariance is isotropic since $\mathrm{cov}(X_t, X_{t+1}) = c(1) = c(-1) == \mathrm{cov}(X_t, X_{t-1}) = \mathrm{cov}(X_{t-1}, X_t)$. For $d > 1$, isotropy is a stronger condition than stationarity. The appeal of an isotropic covariance is that the actual directional difference between two observations *does not* impact the covariance, it is simply the Euclidean distance between the two locations (see picture on board). To show that the covariance $c(\cdot)$ is a valid isotropic covariance in dimension d (that is there exists a positive semi-definite function $c_d : \mathbb{R}^d \to \mathbb{R}$ such that $c(\|\boldsymbol{u}\|) = c_d(\boldsymbol{u})$), conditions analogous but not the same as (2.10) are required. We state them now.

Theorem 2.5.1 *If a covariance $c_d(\cdot)$ is isotropic, its corresponding spectral density function f_d is also isotropic. That is, there exists a positive function $f : \mathbb{R} \rightarrow \mathbb{R}^+$ such that $f_d(\boldsymbol{\omega}) = f(\|\boldsymbol{\omega}\|_2)$.*

A covariance $c(\cdot)$ is a valid isotropic covariance in $\mathbb{R}^d$ iff there exists a positive function $f(\cdot; d)$ defined in $\mathbb{R}^+$ such that

$$c(r) = (2\pi)^{d/2}\int_0^\infty \rho^{d/2}J_{(d/2)-1}(\rho)f(\rho;d)d\rho \tag{2.11}$$

where J_n is the order n Bessel function of the first kind.

PROOF. To give us some idea of where this result came from, we assume the first statement is true and prove the second statement for the case the dimension $d = 2$.

By the spectral representation theorem we know that if $c(u_1, u_r)$ is a valid covariance then there exists a positive function f_2 such that

$$c(u_1,u_2) = \int_{\mathbb{R}^2} f_2(\omega_1,\omega_2)\exp(i\omega_1u_1 + i\omega_2u_2)d\omega_1 d\omega_2.$$

Next we change variables moving from Euclidean coordinates to polar coordinates (see `https://en.wikipedia.org/wiki/Polar_coordinate_system`), where $s = \sqrt{\omega_1^2+\omega_2^2}$ and $\theta = tan^{-1}\omega_1/\omega_2$. In this way the spectral density can be written in terms of $f_2(\omega_1,\omega_2) = f_{P,2}(r,\theta)$ and we have

$$c(u_1,u_2) = \int_0^\infty\int_0^{2\pi} rf_{P,2}(s,\theta)\exp(isu_1\cos\theta + isu_2\sin\theta)dsd\theta.$$

We convert the covariance in terms of polar coordinates $c(u_1,u_2) = c_{P,2}(r,\Omega)$ (where $u_1 = r\cos\Omega$ and $u_2 = r\sin\Omega$) to give

$$\begin{aligned} c_{P,2}(r,\Omega) &= \int_0^\infty\int_0^{2\pi} sf_{P,2}(s,\theta)\exp\left[isr\left(\cos\Omega\cos\theta + \sin\Omega\sin\theta\right)\right]dsd\theta \\ &= \int_0^\infty\int_0^{2\pi} sf_{P,2}(s,\theta)\exp\left[isr\cos\left(\Omega-\theta\Omega\right)\right]dsd\theta. \end{aligned} \tag{2.12}$$

So far we have not used isotropy of the covariance, we have simply rewritten the spectral representation in terms of polar coordinates.

Now, we consider the special case that the covariance is isotropic, this means that there exists a function c such that $c_{P,2}(r,\Omega) = c(r)$ for all r and Ω. Furthermore, by the first statement of the theorem, if the covariance is isotropic, then there exists a positive function $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $f_{P,2}(s,\theta) = f(s)$ for all s and θ. Using these two facts and substituting them into (2.12) gives

$$\begin{aligned} c(r) &= \int_0^\infty \int_0^{2\pi} s f(s) \exp\left[isr\cos\left(\Omega - \theta\Omega\right)\right] ds d\theta \\ &= \int_0^\infty s f(s) \underbrace{\int_0^{2\pi} \exp\left[isr\cos\left(\Omega - \theta\Omega\right)\right] d\theta\, ds.}_{=2\pi J_0(s)} \end{aligned}$$

For the case, $d = 2$ we have obtained the desired result. Note that the Bessel function $J_0(\cdot)$ is effectively playing the same role as the exponential function in the general spectral representation theorem. □

The above result is extremely useful. It allows one to construct a valid isotropic covariance function in dimension d with a positive function f. Furthermore, it shows that an isotropic covariance $c(r)$ may be valid in dimension in $d = 1, \ldots, 3$, but for $d > 3$ it may not be valid. That is for $d > 3$, there does not exist a positive function $f(\cdot; d)$ which satisfies (2.11). Schoenberg showed that an isotropic covariance $c(r)$ was valid in all dimensions d iff there exists a representation

$$c(r) = \int_0^\infty \exp(-r^2t^2) dF(t),$$

where F is a probability measure. In most situations the above can be written as

$$c(r) = \int_0^\infty \exp(-r^2t^2) f(t) dt,$$

where $f : \mathbb{R}^+ \to \mathbb{R}^+$. This representation turns out to be a very fruitful method for generating parametric families of isotropic covariances which are valid on all dimensions d. These include the Matern class, Cauchy class, Powered exponential family. The feature in common to all these isotropic covariance functions is that all the covariances are strictly positive and strictly decreasing. In other words, the cost for an isotropic covariance to be valid in all dimensions is that it can only model positive, monotonic correlations. The use of such covariances have become very popular in modelling Gaussian processes for problems in machine learning (see `http://www.gaussianprocess.org/gpml/chapters/RW1.pdf`).

For an excellent review see **?**, Section 2.5.

2.6 Exercises

Exercise 2.4 *Which of these sequences can used as the autocovariance function of a second order stationary time series?*

(i) $c(-1) = 1/2$, $c(0) = 1$, $c(1) = 1/2$ *and for all* $|k| > 1$, $c(k) = 0$.

(ii) $c(-1) = -1/2$, $c(0) = 1$, $c(1) = 1/2$ *and for all* $|k| > 1$, $c(k) = 0$.

(iii) $c(-2) = -0.8$, $c(-1) = 0.5$, $c(0) = 1$, $c(1) = 0.5$ *and* $c(2) = -0.8$ *and for all* $|k| > 2$, $c(k) = 0$.

Exercise 2.5 *(i) Show that the function* $c(u) = \exp(-a|u|)$ *where* $a > 0$ *is a positive semi-definite function.*

(ii) Show that the commonly used exponential spatial covariance defined on $\mathbb{R}^2$, $c(u_1, u_2) = \exp(-a\sqrt{u_1^2 + u_2^2})$, *where* $a > 0$, *is a positive semi-definite function.*

Hint: One method is to make a change of variables using Polar coordinates. You may also want to harness the power of Mathematica or other such tools.

Chapter 3

Linear time series

Prerequisites

- Familarity with linear models.
- Solve polynomial equations.
- Be familiar with complex numbers.
- Understand under what conditions the partial sum $S_n = \sum_{j=1}^{n} a_j$ has a well defined limits (ie. if $\sum_{j=1}^{\infty} |a_j| < \infty$, then $S_n \to S$, where $S = \sum_{j=1}^{\infty} a_j$.

Objectives

- Understand what causal and invertible is.
- Know what an AR, MA and ARMA time series model is.
- Know how to find a solution of an ARMA time series, and understand why this is important (how the roots determine causality and why this is important to know - in terms of characteristics in the process and also simulations).
- Understand how the roots of the AR can determine 'features' in the time series and covariance structure (such as pseudo periodicities).

3.1 Motivation

The objective of this chapter is to introduce the linear time series model. Linear time series models are designed to model the covariance structure in the time series. There are two popular subgroups of linear time models (a) the autoregressive and (a) the moving average models, which can be combined to make the autoregressive moving average models.

We motivate the autoregressive from the perspective of classical linear regression. We recall one objective in linear regression is to predict the response variable given variables that are observed. To do this, typically linear dependence between response and variable is assumed and we model Y_i as

$$Y_i = \sum_{j=1}^{p} a_j X_{ij} + \varepsilon_i,$$

where ε_i is such that $\mathrm{E}[\varepsilon_i | X_{ij}] = 0$ and more commonly ε_i and X_{ij} are independent. In linear regression once the model has been defined, we can immediately find estimators of the parameters, do model selection etc.

Returning to time series, one major objective is to predict/forecast the future given current and past observations (just as in linear regression our aim is to predict the response given the observed variables). At least formally, it seems reasonable to represent this as

$$X_t = \sum_{j=1}^{p} \phi_j X_{t-j} + \varepsilon_t, \qquad t \in \mathbb{Z} \tag{3.1}$$

where we assume that $\{\varepsilon_t\}$ are independent, identically distributed, zero mean random variables. Model (3.1) is called an autoregressive model of order p (AR(p) for short). Further, it would appear that

$$\mathrm{E}(X_t | X_{t-1}, \ldots, X_{t-p}) = \sum_{j=1}^{p} \phi_j X_{t-j}$$

(the expected value of X_t given that $X_{t-1}, \ldots, X_{t-p}$ have already been observed), thus the past values of X_t have a linear influence on the conditional mean of X_t. However this is not necessarily the case; the autoregressive model appears to be a straightforward extension of the linear regression model but don't be fooled by this, it is a more complex object. It may seem suprising, but for certain parameters $\{\phi_j\}$ a *stationary* time series which satisfies model (3.1) may be such that

$\mathrm{E}[\varepsilon_t|X_{t-1}] \neq 0$ and $\mathrm{E}[X_t|X_{t-1}, X_{t-2}, \ldots] \neq \sum_{j=1}^{p} \phi_j X_{t-j}$.

Unlike the linear regression model, (3.1) is an infinite set of linear difference equations. This means, for this systems of equations to be well defined, it needs to have a solution which is meaningful. To understand why, recall that (3.1) is defined for all $t \in \mathbb{Z}$, so let us start the equation at the beginning of time ($t = -\infty$) and run it on. Without any constraint on the parameters $\{\phi_j\}$, there is no reason to believe the solution is finite (contrast this with linear regression where these issues are not relevant). Therefore, the first thing to understand is under what conditions will the AR model (3.1) have a well defined stationary solution and what features in a time series is the solution able to capture.

Of course, one could ask why go through to the effort. One could simply use least squares to estimate the parameters. This is possible, but there are two related problems (a) without a proper analysis it is not clear whether model has a meaningful solution (for example in Section 4.4 we show that the least squares estimator can lead to misspecified models), it's not even possible to make simulations of the process (b) it is possible that $\mathrm{E}(\varepsilon_t|X_{t-p}) \neq 0$, this means that least squares is not estimating ϕ_j and is instead estimating an entirely different set of parameters! Therefore, there is a practical motivation behind our theoretical treatment.

In this chapter we will be deriving conditions for a strictly stationary solution of (3.1). Under these moment conditions we obtain a strictly stationary solution of (3.1). In Chapter 4 we obtain conditions for (3.1) to have be both a strictly stationary and second order stationary solution. It is worth mentioning that it is possible Ito obtain strictly stationary solution to (3.1) under weaker conditions (see Theorem 5.0.1).

How would you simulate from the following model? One simple method for understanding a model is to understand how you would simulate from it:

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-1} + \varepsilon_t \qquad t = \ldots, -1, 0, 1, \ldots.$$

Remark 3.1.1 *The main objective in this Chapter is to look for stationary solutions to (3.1). If, however, we define the equation*

$$X_t = \sum_{j=1}^{p} \phi_j X_{t-j} + \varepsilon_t, \qquad t = 1, 2, \ldots \tag{3.2}$$

(note $t \geq 0$), then (3.2) will always yield a nonstationary solution. However, depending on the

nature of the coefficients $\{\phi_j\}$ as $t \to \infty$ the solution may or may not converge to a stationary process (effectively, it will do so if the variance does not grow). If it does not converge to a stationary process, it will either be a so called unit root process or an explosive process (where the variance grows linearly or exponentially). The sampling properties of estimators for explosive processes are very different to those in the stationary case and were considered in Anderson (1959).

3.2 Linear time series and moving average models

3.2.1 Infinite sums of random variables

Before defining a linear time series, we define the MA(q) model which is a subclass of linear time series. Let us supppose that $\{\varepsilon_t\}$ are iid random variables with mean zero and finite variance. The time series $\{X_t\}$ is said to have a MA(q) representation if it satisfies

$$X_t = \sum_{j=0}^{q} \psi_j \varepsilon_{t-j},$$

where $\mathrm{E}(\varepsilon_t) = 0$ and $\mathrm{var}(\varepsilon_t) = 1$. It is clear that X_t is a rolling finite weighted sum of $\{\varepsilon_t\}$, therefore $\{X_t\}$ must be well defined. We extend this notion and consider infinite sums of random variables. Now, things become more complicated, since care must be always be taken with anything involving *infinite sums*. More precisely, for the sum

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j \varepsilon_{t-j},$$

to be well defined (has a finite limit), the partial sums $S_n = \sum_{j=-n}^{n} \psi_j \varepsilon_{t-j}$ should be (almost surely) finite and the sequence S_n should converge (ie. $|S_{n_1} - S_{n_2}| \to 0$ as $n_1, n_2 \to \infty$). A random variable makes no sense if it is infinite. Therefore we must be sure that X_t is finite (this is what we mean by being well defined).

Below, we give conditions under which this is true.

Lemma 3.2.1 *Suppose $\sum_{j=-\infty}^{\infty} |\phi_j| < \infty$ and $\{X_t\}$ is a strictly stationary time series with $\mathrm{E}|X_t| < \infty$. Then $\{Y_t\}$, defined by*

$$Y_t = \sum_{j=-\infty}^{\infty} \psi_j X_{t-j},$$

is a strictly stationary time series. Furthermore, the partial sum converges almost surely, $Y_{n,t} = \sum_{j=-n}^{n} \psi_j X_{t-j} \to Y_t$. *If* $\mathrm{var}(X_t) < \infty$, *then* $\{Y_t\}$ *is second order stationary and converges in mean square (that is* $\mathrm{E}(Y_{n,t} - Y_t)^2 \to 0$*).*

PROOF. See Brockwell and Davis (1998), Proposition 3.1.1 or Fuller (1995), Theorem 2.1.1 (page 31) (also Shumway and Stoffer (2006), page 86). □

Example 3.2.1 *Suppose* $\{X_t\}$ *is a strictly stationary time series with* $\mathrm{var}(X_t) < \infty$. *Define* $\{Y_t\}$ *as the following infinite sum*

$$Y_t = \sum_{j=0}^{\infty} j^k \rho^j |X_{t-j}|$$

where $|\rho| < 1$. *Then* $\{Y_t\}$ *is also a strictly stationary time series with a finite variance.*

We will use this example later in the course.

Having derived conditions under which infinite sums are well defined, we can now define the general class of linear and MA(∞) processes.

Definition 3.2.1 (The linear process and moving average (MA)(∞)) *Suppose that* $\{\varepsilon_t\}$ *are iid random variables,* $\sum_{j=0}^{\infty} |\psi_j| < \infty$ *and* $\mathrm{E}(|\varepsilon_t|) < \infty$.

(i) A time series is said to be a linear time series if it can be represented as

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j \varepsilon_{t-j},$$

where $\{\varepsilon_t\}$ *are iid random variables with finite variance. Note that since that as these sums are well defined by equation (2.9)* $\{X_t\}$ *is a strictly stationary (ergodic) time series.*

This is a rather strong definition of a linear process. A more general definition is $\{X_t\}$ *has the representation*

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j \varepsilon_{t-j},$$

where $\{\varepsilon_t\}$ *are* uncorrelated *random variables with mean zero and variance one (thus the independence assumption has been dropped).*

(ii) The time series $\{X_t\}$ has a MA(∞) representation if it satisfies

$$X_t = \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}. \tag{3.3}$$

[1]

The difference between an MA(∞) process and a linear process is quite subtle. A linear process involves both past, present and future innovations $\{\varepsilon_t\}$, whereas the MA(∞) uses only past and present innovations.

Definition 3.2.2 (Causal and invertible) *Consider the ARMA(p, q) model defined by*

$$X_t + \sum_{j=1}^{p} \psi_j X_{t-j} = \sum_{i=1}^{q} \theta_i \varepsilon_t,$$

where $\{\varepsilon_t\}$ are iid random variables with mean zero and constant variance.

(i) An ARMA process is said to be causal if it has the representation

$$X_t = \sum_{j=0}^{\infty} a_j \varepsilon_{t-j}.$$

(ii) An ARMA(p, q) process $X_t + \sum_{j=1}^{p} \psi_j X_{t-j} = \sum_{i=1}^{q} \theta_i \varepsilon_t$ (where $\{\varepsilon_t\}$ are uncorrelated random variables with mean zero and constant variance) is said to be invertible if it has the representation

$$X_t = \sum_{j=1}^{\infty} b_j X_{t-j} + \varepsilon_t.$$

Causal and invertible solutions are useful in both estimation and forecasting (predicting the future based on the current and past).

[1] Note that late on we show that all second order stationary time series $\{X_t\}$ have the representation

$$X_t = \sum_{j=1}^{\infty} \psi_j Z_{t-j}, \tag{3.4}$$

where $\{Z_t = X_t - P_{X_{t-1},X_{t-2},\ldots}(X_t)\}$ (where $P_{X_{t-1},X_{t-2},\ldots}(X_t)$ is the best linear predictor of X_t given the past, $X_{t-1}, X_{t-2}, \ldots$). In this case $\{Z_t\}$ are uncorrelated random variables. It is called Wold's representation theorem (see Section 6.9). The representation in (3.4) has many practical advantages. For example Krampe et al. (2016) recently used it to define the so called "MA bootstrap".

A very interesting class of models which have MA(∞) representations are autoregressive and autoregressive moving average models. In the following sections we prove this.

3.3 The autoregressive model and the solution

In this section we will examine under what conditions the AR(p) model has a stationary solution.

3.3.1 Difference equations and back-shift operators

The autoregressive model is defined in terms of inhomogenuous difference equations. Difference equations can often be represented in terms of backshift operators, so we start by defining them and see why this representation may be useful (and why it should work).

The time series $\{X_t\}$ is said to be an autoregressive (AR(p)) if it satisfies the equation

$$X_t - \phi_1 X_{t-1} - \ldots - \phi_p X_{t-p} = \varepsilon_t, \quad t \in \mathbb{Z},$$

where $\{\varepsilon_t\}$ are zero mean, finite variance random variables. As we mentioned previously, the autoregressive model is a difference equation (which can be treated as a infinite number of simultaneous equations). Therefore for it to make any sense it must have a solution. To obtain a general solution we write the autoregressive model in terms of backshift operators:

$$X_t - \phi_1 B X_t - \ldots - \phi_p B^p X_t = \varepsilon_t, \quad \Rightarrow \quad \phi(B) X_t = \varepsilon_t$$

where $\phi(B) = 1 - \sum_{j=1}^{p} \phi_j B^j$, B is the backshift operator and is defined such that $B^k X_t = X_{t-k}$. Simply rearranging $\phi(B)X_t = \varepsilon_t$, gives the 'solution' of the autoregressive difference equation to be $X_t = \phi(B)^{-1}\varepsilon_t$, however this is just an algebraic manipulation, below we investigate whether it really has any meaning. To do this, we start with an example.

3.3.2 Solution of two particular AR(1) models

Below we consider two different AR(1) models and obtain their solutions.

Remark 3.3.1 (What is meant by a solution?) *Let us suppose the model is*

$$X_t = \phi X_{t-1} + \varepsilon_t \textit{ for } t \in \mathbb{Z},$$

where ε_t are iid random variables and ϕ is a known parameter. Suppose our objective is to generate the above, we need to first draw ε_t. In one draw we get $\varepsilon_t = 0.5$, $\varepsilon_{t+1} = 3.1$, $varepsilon_{t+2} = -1.2$ etc. This gives the system of equations

$$\begin{aligned} X_t &= \phi X_{t-1} + 0.5 \\ X_{t+1} &= \phi X_t + 3.1 \\ X_{t+2} &= \phi X_{t+1} - 1.2 \end{aligned}$$

and so forth. We see this is an equation in terms of unknown $\{X_t\}$ which we want to solve for. The solution will be in terms of ϕ and $0.5, 3.1, -1.2$ etc.

(i) Consider the AR(1) process

$$X_t = 0.5X_{t-1} + \varepsilon_t, \quad t \in \mathbb{Z}. \tag{3.5}$$

Notice this is an equation (rather like $3x^2 + 2x + 1 = 0$, or an infinite number of simultaneous equations), which may or may not have a solution. To obtain the solution we note that $X_t = 0.5X_{t-1} + \varepsilon_t$ and $X_{t-1} = 0.5X_{t-2} + \varepsilon_{t-1}$. Using this we get $X_t = \varepsilon_t + 0.5(0.5X_{t-2} + \varepsilon_{t-1}) = \varepsilon_t + 0.5\varepsilon_{t-1} + 0.5^2X_{t-2}$. Continuing this backward iteration we obtain at the kth iteration, $X_t = \sum_{j=0}^{k}(0.5)^j\varepsilon_{t-j} + (0.5)^{k+1}X_{t-k}$. Because $(0.5)^{k+1} \to 0$ as $k \to \infty$ by taking the limit we can show that $X_t = \sum_{j=0}^{\infty}(0.5)^j\varepsilon_{t-j}$ is almost surely finite and a solution of (3.5). Of course like any other equation one may wonder whether it is the unique solution (recalling that $3x^2 + 2x + 1 = 0$ has two solutions). We show in Section 3.3.3 that this is the unique stationary solution of (3.5).

Let us see whether we can obtain a solution using the difference equation representation. We recall, that by crudely taking inverses, the solution is $X_t = (1 - 0.5B)^{-1}\varepsilon_t$. The obvious question is whether this has any meaning. Note that $(1 - 0.5B)^{-1} = \sum_{j=0}^{\infty}(0.5B)^j$, for $|B| \leq 2$, hence substituting this power series expansion into X_t we have

$$X_t = (1 - 0.5B)^{-1}\varepsilon_t = (\sum_{j=0}(0.5B)^j)\varepsilon_t = (\sum_{j=0}(0.5^jB^j))\varepsilon_t = \sum_{j=0}^{\infty}(0.5)^j\varepsilon_{t-j},$$

which corresponds to the solution above. Hence the backshift operator in this example helps us to obtain a solution. Moreover, because the solution can be written in terms of past values of ε_t, it is causal.

(ii) Let us consider the AR model, which we will see has a very different solution:

$$X_t = 2X_{t-1} + \varepsilon_t. \tag{3.6}$$

Doing what we did in (i) we find that after the kth back iteration we have $X_t = \sum_{j=0}^{k} 2^j \varepsilon_{t-j} + 2^{k+1} X_{t-k}$. However, unlike example (i) 2^k does not converge as $k \to \infty$. This suggest that if we continue the iteration $X_t = \sum_{j=0}^{\infty} 2^j \varepsilon_{t-j}$ is not a quantity that is finite (when ε_t are iid). Therefore $X_t = \sum_{j=0}^{\infty} 2^j \varepsilon_{t-j}$ cannot be considered as a solution of (3.6). We need to write (3.6) in a slightly different way in order to obtain a meaningful solution.

Rewriting (3.6) we have $X_{t-1} = 0.5X_t + 0.5\varepsilon_t$. Forward iterating this we get $X_{t-1} = -(0.5)\sum_{j=0}^{k}(0.5)^j \varepsilon_{t+j} - (0.5)^{k+1} X_{t+k}$. Since $(0.5)^{k+1} \to 0$ as $k \to \infty$ we have

$$X_{t-1} = -(0.5)\sum_{j=0}^{\infty}(0.5)^j \varepsilon_{t+j}$$

as a solution of (3.6).

Let us see whether the difference equation can also offer a solution. Since $(1-2B)X_t = \varepsilon_t$, using the crude manipulation we have $X_t = (1-2B)^{-1}\varepsilon_t$. Now we see that

$$(1-2B)^{-1} = \sum_{j=0}^{\infty}(2B)^j \qquad \text{for } |B| < 1/2.$$

Using this expansion gives the solution $X_t = \sum_{j=0}^{\infty} 2^j B^j X_t$, but as pointed out above this sum is not well defined. What we find is that $\phi(B)^{-1}\varepsilon_t$ only makes sense (is well defined) if the series expansion of $\phi(B)^{-1}$ converges in a region that includes the unit circle $|B| = 1$.

What we need is another series expansion of $(1-2B)^{-1}$ which converges in a region which includes the unit circle $|B| = 1$ (as an aside, we note that a function does not necessarily have a unique series expansion, it can have difference series expansions which may converge in different regions). We now show that a convergent series expansion needs to be defined in terms of negative powers of B not positive powers. Writing $(1-2B) = -(2B)(1-(2B)^{-1})$, therefore

$$(1-2B)^{-1} = -(2B)^{-1}\sum_{j=0}^{\infty}(2B)^{-j},$$

which converges for $|B| > 1/2$. Using this expansion we have

$$X_t = -\sum_{j=0}^{\infty}(0.5)^{j+1}B^{-j-1}\varepsilon_t = -\sum_{j=0}^{\infty}(0.5)^{j+1}\varepsilon_{t+j+1},$$

which we have shown above is a well defined solution of (3.6).

In summary $(1-2B)^{-1}$ has two series expansions

$$\frac{1}{(1-2B)} = \sum_{j=0}^{\infty}(2B)^{-j}$$

which converges for $|B| < 1/2$ and

$$\frac{1}{(1-2B)} = -(2B)^{-1}\sum_{j=0}^{\infty}(2B)^{-j},$$

which converges for $|B| > 1/2$. The one that is useful for us is the series which converges when $|B| = 1$.

It is clear from the above examples how to obtain the solution of a general AR(1). We now show that this solution is the unique stationary solution.

Exercise 3.1 *(i) Find the stationary solution of the AR*(1) *model*

$$X_t = 0.8X_{t-1} + \varepsilon_t$$

where ε_t are iid random variables with mean zero and variance one.

(ii) Find the stationary solution of the AR(1) *model*

$$X_t = \frac{5}{4}X_{t-1} + \varepsilon_t$$

where ε_t are iid random variables with mean zero and variance one.

(iii) [Optional] Obtain the autocovariance function of the stationary solution for both the models in (i) and (ii).

3.3.3 Showing that the AR(1) model has a unique solution

Consider the AR(1) process $X_t = \phi X_{t-1} + \varepsilon_t$, where $|\phi| < 1$. Using the method outlined in (i), it is straightforward to show that $X_t = \sum_{j=0}^{\infty} \phi^j \varepsilon_{t-j}$ is its stationary solution, we now show that this solution is unique. This may seem obvious, but recall that many equations have multiple solutions. The techniques used here generalize to nonlinear models too.

We first show that $X_t = \sum_{j=0}^{\infty} \phi^j \varepsilon_{t-j}$ is well defined (that it is almost surely finite). We note that $|X_t| \leq \sum_{j=0}^{\infty} |\phi^j| \cdot |\varepsilon_{t-j}|$. Thus we will show that $\sum_{j=0}^{\infty} |\phi^j| \cdot |\varepsilon_{t-j}|$ is almost surely finite, which will imply that X_t is almost surely finite. By montone convergence we can exchange sum and expectation and we have $\mathrm{E}(|X_t|) \leq \mathrm{E}(\lim_{n\to\infty} \sum_{j=0}^{n} |\phi^j \varepsilon_{t-j}|) = \lim_{n\to\infty} \sum_{j=0}^{n} |\phi^j| \mathrm{E}|\varepsilon_{t-j}|) = \mathrm{E}(|\varepsilon_0|) \sum_{j=0}^{\infty} |\phi^j| < \infty$. Therefore since $\mathrm{E}|X_t| < \infty$, $\sum_{j=0}^{\infty} \phi^j \varepsilon_{t-j}$ is a well defined solution of $X_t = \phi X_{t-1} + \varepsilon_t$.

To show that it is the unique, stationary, causal solution, let us suppose there is another (causal) solution, call it Y_t. Clearly, by recursively applying the difference equation to Y_t, for every s we have

$$Y_t = \sum_{j=0}^{s} \phi^j \varepsilon_{t-j} + \phi^s Y_{t-s-1}.$$

Evaluating the difference between the two solutions gives $Y_t - X_t = A_s - B_s$ where $A_s = \phi^s Y_{t-s-1}$ and $B_s = \sum_{j=s+1}^{\infty} \phi^j \varepsilon_{t-j}$ for all s. To show that Y_t and X_t coincide almost surely we will show that for every $\epsilon > 0$, $\sum_{s=1}^{\infty} P(|A_s - B_s| > \varepsilon) < \infty$ (and then apply the Borel-Cantelli lemma). We note if $|A_s - B_s| > \varepsilon$), then either $|A_s| > \varepsilon/2$ or $|B_s| > \varepsilon/2$. Therefore $P(|A_s - B_s| > \varepsilon) \leq P(|A_s| > \varepsilon/2) + P(|B_s| > \varepsilon/2)$. To bound these two terms we use Markov's inequality. It is straightforward to show that $P(|B_s| > \varepsilon/2) \leq C\phi^s/\varepsilon$. To bound $\mathrm{E}|A_s|$, we note that $|Y_s| \leq |\phi| \cdot |Y_{s-1}| + |\varepsilon_s|$, since $\{Y_t\}$ is a stationary solution then $\mathrm{E}|Y_s|(1 - |\phi|) \leq \mathrm{E}|\varepsilon_s|$, thus $\mathrm{E}|Y_t| \leq \mathrm{E}|\varepsilon_t|/(1 - |\phi|) < \infty$. Altogether this gives $P(|A_s - B_s| > \varepsilon) \leq C\phi^s/\varepsilon$ (for some finite constant C). Hence $\sum_{s=1}^{\infty} P(|A_s - B_s| > \varepsilon) < \sum_{s=1}^{\infty} C\phi^s/\varepsilon < \infty$. Thus by the Borel-Cantelli lemma, this implies that the event $\{|A_s - B_s| > \varepsilon\}$ happens only finitely often (almost surely). Since for every ε, $\{|A_s - B_s| > \varepsilon\}$ occurs (almost surely) only finitely often for all ε, then $Y_t = X_t$ almost surely. Hence $X_t = \sum_{j=0}^{\infty} \phi^j \varepsilon_{t-j}$ is (almost surely) the unique causal solution.

3.3.4 The solution of a general AR(p)

Let us now summarise our observation for the general AR(1) process $X_t = \phi X_{t-1} + \varepsilon_t$. If $|\phi| < 1$, then the solution is in terms of past values of $\{\varepsilon_t\}$, if on the other hand $|\phi| > 1$ the solution is in terms of future values of $\{\varepsilon_t\}$.

Now we try to understand this in terms of the expansions of the characteristic polynomial $\phi(B) = 1 - \phi B$ (using the AR(1) as a starting point). From what we learnt in the previous section, we require the characteristic polynomial of the AR process to have a convergent power series expansion in the region including the ring $|B| = 1$. In terms of the AR(1) process, if the root of $\phi(B)$ is greater than one, then the power series of $\phi(B)^{-1}$ is in terms of positive powers, if it is less than one, then $\phi(B)^{-1}$ is in terms of negative powers.

Generalising this argument to a general polynomial, if the roots of $\phi(B)$ are greater than one, then the power series of $\phi(B)^{-1}$ (which converges for $|B| = 1$) is in terms of positive powers (hence the solution $\phi(B)^{-1}\varepsilon_t$ will be in past terms of $\{\varepsilon_t\}$). On the other hand, if the roots are both less than and greater than one (but do not lie on the unit circle), then the power series of $\phi(B)^{-1}$ will be in both negative and positive powers. Thus the solution $X_t = \phi(B)^{-1}\varepsilon_t$ will be in terms of both past and future values of $\{\varepsilon_t\}$. We summarize this result in a lemma below.

Lemma 3.3.1 *Suppose that the AR(p) process satisfies the representation $\phi(B)X_t = \varepsilon_t$, where none of the roots of the characteristic polynomial lie on the unit circle and $\mathrm{E}|\varepsilon_t| < \infty$. Then $\{X_t\}$ has a stationary, almost surely unique, solution.*

We see that where the roots of the characteristic polynomial $\phi(B)$ lie defines the solution of the AR process. We will show in Sections 3.3.6 and 4.1.2 that it not only defines the solution but also determines some of the characteristics of the time series.

Exercise 3.2 *Suppose $\{X_t\}$ satisfies the AR(p) representation*

$$X_t = \sum_{j=1}^{p} \phi_j X_{t-j} + \varepsilon_t,$$

where $\sum_{j=1}^{p} |\phi_j| < 1$ and $\mathrm{E}|\varepsilon_t| < \infty$. Show that $\{X_t\}$ will always have a causal stationary solution.

3.3.5 Obtaining an explicit solution of an AR(2) model

Specific example

Suppose $\{X_t\}$ satisfies

$$X_t = 0.75X_{t-1} - 0.125X_{t-2} + \varepsilon_t,$$

where $\{\varepsilon_t\}$ are iid random variables. We want to obtain a solution for the above equations.

It is not easy to use the backward (or forward) iterating techique for AR processes beyond order one. This is where using the backshift operator becomes useful. We start by writing $X_t = 0.75X_{t-1} - 0.125X_{t-2} + \varepsilon_t$ as $\phi(B)X_t = \varepsilon$, where $\phi(B) = 1 - 0.75B + 0.125B^2$, which leads to what is commonly known as the characteristic polynomial $\phi(z) = 1 - 0.75z + 0.125z^2$. If we can find a power series expansion of $\phi(B)^{-1}$, which is valid for $|B| = 1$, then the solution is $X_t = \phi(B)^{-1}\varepsilon_t$.

We first observe that $\phi(z) = 1 - 0.75z + 0.125z^2 = (1 - 0.5z)(1 - 0.25z)$. Therefore by using partial fractions we have

$$\frac{1}{\phi(z)} = \frac{1}{(1-0.5z)(1-0.25z)} = \frac{-1}{(1-0.5z)} + \frac{2}{(1-0.25z)}.$$

We recall from geometric expansions that

$$\frac{-1}{(1-0.5z)} = -\sum_{j=0}^{\infty}(0.5)^j z^j \quad |z| \leq 2, \qquad \frac{2}{(1-0.25z)} = 2\sum_{j=0}^{\infty}(0.25)^j z^j \quad |z| \leq 4.$$

Putting the above together gives

$$\frac{1}{(1-0.5z)(1-0.25z)} = \sum_{j=0}^{\infty}\{-(0.5)^j + 2(0.25)^j\}z^j \quad |z| < 2.$$

The above expansion is valid for $|z| = 1$, because $\sum_{j=0}^{\infty}|-(0.5)^j + 2(0.25)^j| < \infty$ (see Lemma 3.3.2). Hence

$$X_t = \{(1-0.5B)(1-0.25B)\}^{-1}\varepsilon_t = \big(\sum_{j=0}^{\infty}\{-(0.5)^j + 2(0.25)^j\}B^j\big)\varepsilon_t = \sum_{j=0}^{\infty}\{-(0.5)^j + 2(0.25)^j\}\varepsilon_{t-j},$$

which gives a stationary solution to the AR(2) process (see Lemma 3.2.1). Moreover since the roots lie outside the unit circle the solution is *causal*.

The discussion above shows how the backshift operator can be applied and how it can be used to obtain solutions to AR(p) processes.

The solution of a general AR(2) model

We now generalise the above to general AR(2) models

$$X_t = (a+b)X_{t-1} - abX_{t-2} + \varepsilon_t,$$

the characteristic polynomial of the above is $1-(a+b)z+abz^2 = (1-az)(1-bz)$. This means the solution of X_t is

$$X_t = (1-Ba)^{-1}(1-Bb)^{-1}\varepsilon_t,$$

thus we need an expansion of $(1-Ba)^{-1}(1-Bb)^{-1}$. Assuming that $a \neq b$, and using partial fractions we have

$$\frac{1}{(1-za)(1-zb)} = \frac{1}{b-a}\left(\frac{b}{1-bz} - \frac{a}{1-az}\right)$$

Cases:

(i) $|a|<1$ and $|b|<1$, this means the roots lie outside the unit circle. Thus the expansion is

$$\frac{1}{(1-za)(1-zb)} = \frac{1}{(b-a)}\big(b\sum_{j=0}^{\infty}b^jz^j - a\sum_{j=0}^{\infty}a^jz^j\big),$$

which leads to the causal solution

$$X_t = \frac{1}{b-a}\left(\sum_{j=0}^{\infty}\big(b^{j+1}-a^{j+1}\big)\varepsilon_{t-j}\right). \tag{3.7}$$

(ii) Case that $|a|>1$ and $|b|<1$, this means the roots lie inside and outside the unit circle and we have the expansion

$$\begin{aligned}\frac{1}{(1-za)(1-zb)} &= \frac{1}{b-a}\left(\frac{b}{1-bz} - \frac{a}{(az)((az)^{-1}-1)}\right)\\ &= \frac{1}{(b-a)}\left(b\sum_{j=0}^{\infty}b^jz^j + z^{-1}\sum_{j=0}^{\infty}a^{-j}z^{-j}\right),\end{aligned} \tag{3.8}$$

which leads to the non-causal solution

$$X_t = \frac{1}{b-a}\big(\sum_{j=0}^{\infty} b^{j+1}\varepsilon_{t-j} + \sum_{j=0}^{\infty} a^{-j}\varepsilon_{t+1+j}\big). \tag{3.9}$$

2

Returning to (3.9), we see that this solution throws up additional interesting results. Let us return to the expansion in (3.8) and apply it to X_t

$$\begin{aligned} X_t &= \frac{1}{(1-Ba)(1-Bb)}\varepsilon_t = \frac{1}{b-a}\left(\underbrace{\frac{b}{1-bB}\varepsilon_t}_{\text{causal AR(1)}} + \underbrace{\frac{1}{B(1-a^{-1}B^{-1})}\varepsilon_t}_{\text{noncausal AR(1)}}\right) \\ &= \frac{1}{b-a}\left(Y_t + Z_{t+1}\right) \end{aligned}$$

where $Y_t = bY_{t-1} + \varepsilon_t$ and $Z_{t+1} = a^{-1}Z_{t+2} + \varepsilon_{t+1}$. In other words, the noncausal AR(2) process is the sum of a causal and a'future' AR(1) process. This is true for all noncausal time series (except when there is multiplicity in the roots) and is discussed further in Section 3.7.

We mention that several authors argue that noncausal time series can model features in data which causal time series cannot.

(iii) $a = b < 1$ (both roots are the same and lie outside the unit circle). The characteristic polynomial is $(1-az)^2$. To obtain the convergent expansion when $|z| = 1$ we note that

[2] Later we show that the non-causal X_t, has the same correlation as an AR(2) model whose characteristic polynomial has the roots a^{-1} and b, since both these roots lie out side the unit this model has a causal solution. Moreover, it is possible to rewrite this non-causal AR(2) as an MA infinite type process but where the innovations are no independent but uncorrelated instead. I.e. we can write X_t as

$$(1-a^{-1}B)(1-bB)X_t = \widetilde{\varepsilon}_t,$$

where $\widehat{\varepsilon}_t$ are uncorrelated (and are a linear sum of the iid $varepsilon_t$), which as the solution

$$X_t = \frac{1}{b-a}\bigg(\sum_{j=0}^{\infty}\big(b^{j+1} - a^{j+1})\widetilde{\varepsilon}_{t-j}\bigg). \tag{3.10}$$

$(1-az)^{-2} = (-1)\frac{d(1-az)^{-1}}{d(az)}$. Thus

$$\frac{(-1)}{(1-az)^2} = (-1)\sum_{j=0}^{\infty} j(az)^{j-1}.$$

This leads to the causal solution

$$X_t = (-1)\sum_{j=1}^{\infty} j a^{j-1}\varepsilon_{t-j}.$$

In many respects this is analogous to Matern covariance defined over $\mathbb{R}^d$ (and used in spatial statistics). However, unlike autocovarianced defined over $\mathbb{R}^d$ the behaviour of the autocovariance at zero is not an issue.

Exercise 3.3 *Show for the AR(2) model $X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \varepsilon_t$ to have a causal stationary solution the parameters ϕ_1, ϕ_2 must lie in the region defined by the three conditions*

$$\phi_2 + \phi_1 < 1, \quad \phi_2 - \phi_1 < 1 \quad |\phi_2| < 1.$$

Exercise 3.4 *(a) Consider the AR(2) process*

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \varepsilon_t,$$

where $\{\varepsilon_t\}$ are iid random variables with mean zero and variance one. Suppose the absolute of the roots of the characteristic polynomial $1 - \phi_1 z - \phi_2 z^2$ are greater than one. Show that $|\phi_1| + |\phi_2| < 4$.

(b) Now consider a generalisation of this result. Consider the AR(p) process

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \ldots \phi_p X_{t-p} + \varepsilon_t.$$

Suppose the absolute of the roots of the characteristic polynomial $1 - \phi_1 z - \ldots - \phi_p z^p$ are greater than one. Show that $|\phi_1| + \ldots + |\phi_p| \leq 2^p$.

3.3.6 Features of a realisation from an AR(2)

We now explain why the AR(2) (and higher orders) can characterise some very interesting behaviour (over the rather dull AR(1)). For now we assume that X_t is a causal time series which satisfies the AR(2) representation

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \varepsilon_t$$

where $\{\varepsilon_t\}$ are iid with mean zero and finite variance. The characteristic polynomial is $\phi(B) = 1 - \phi_1 B - \phi_2 B^2$. Let us assume the roots of $\phi(B)$ are complex, since ϕ_1 and ϕ_2 are real, the roots are complex conjugates. We assume the process is causal, that is the roots lie outside the unit circle. Therefore

$$\phi(B) = (1 - \lambda B)(1 - \overline{\lambda} B)$$

where $|\lambda| < 1$. Thus by using case (i) with $a = \lambda$ and $b = \overline{\lambda}$ we have

$$\frac{1}{1 - \phi_1 B - \phi_2 B^2} = \frac{1}{\lambda - \overline{\lambda}} \left(\frac{\lambda}{1 - \lambda B} - \frac{\bar{\lambda}}{1 - \bar{\lambda} B} \right),$$

Using (i) we have

$$X_t = \frac{1}{\lambda - \overline{\lambda}} \sum_{j=0}^{\infty} \left(\lambda^{j+1} - \overline{\lambda^{j+1}} \right) \varepsilon_{t-j}.$$

We reparameterize $\lambda = re^{i\theta}$ (noting that $|r| < 1$). Then

$$X_t = \frac{1}{2r \sin\theta} \sum_{j=0}^{\infty} 2r^{j+1} \sin\left((j+1)\theta\right) \varepsilon_{t-j}. \qquad (3.11)$$

We can see that X_t is effectively the sum of cosines/sines with frequency θ that have been modulated by the iid errors and exponentially damped. This is why for realisations of autoregressive processes you will often see periodicities (depending on the roots of the characteristic). Often to include periodicities in a time series in an AR(2) model one can reparameterise the model as

$$X_t = 2\phi \cos(\Omega) X_{t-1} - \phi^2 X_{t-2} + \varepsilon_t \qquad |\phi| < 1.$$

These arguments can be generalised to higher orders p.

Exercise 3.5 *(a) Obtain the stationary solution of the $AR(2)$ process*

$$X_t = \frac{7}{3}X_{t-1} - \frac{2}{3}X_{t-2} + \varepsilon_t,$$

where $\{\varepsilon_t\}$ are iid random variables with mean zero and variance σ^2.

Does the solution have an MA(∞) representation?

(b) Obtain the stationary solution of the $AR(2)$ process

$$X_t = \frac{4 \times \sqrt{3}}{5}X_{t-1} - \frac{4^2}{5^2}X_{t-2} + \varepsilon_t,$$

where $\{\varepsilon_t\}$ are iid random variables with mean zero and variance σ^2.

Does the solution have an MA(∞) representation?

(c) Obtain the stationary solution of the $AR(2)$ process

$$X_t = X_{t-1} - 4X_{t-2} + \varepsilon_t,$$

where $\{\varepsilon_t\}$ are iid random variables with mean zero and variance σ^2.

Does the solution have an MA(∞) representation?

Exercise 3.6 *Construct a causal stationary $AR(2)$ process with pseudo-period* 17*. Using the* `R` *function* `arima.sim` *simulate a realisation from this process (of length* 200*) and make a plot of the periodogram. What do you observe about the peak in this plot?*

Below we now consider solutions to general AR(∞) processes.

3.3.7 Solution of the general AR(∞) model

The AR(∞) model generalizes the AR(p)

$$X_t = \sum_{j=1}^{\infty} \phi_j X_{t-j} + \varepsilon_t$$

where $\{\varepsilon_t\}$ are iid random variables. AR(∞) models are more general than the AR(p) model and are able to model more complex behaviour, such as slower decay of the covariance structure.

In order to obtain the stationary solution of an AR(∞), we need to define an analytic function and its inverse.

Definition 3.3.1 (Analytic functions in the region Ω) *Suppose that $z \in \mathbb{C}$. $\phi(z)$ is an analytic complex function in the region Ω, if it has a power series expansion which converges in Ω, that is $\phi(z) = \sum_{j=-\infty}^{\infty} \phi_j z^j$.*

If there exists a function $\tilde{\phi}(z) = \sum_{j=-\infty}^{\infty} \tilde{\phi}_j z^j$ such that $\tilde{\phi}(z)\phi(z) = 1$ for all $z \in \Omega$, then $\tilde{\phi}(z)$ is the inverse of $\phi(z)$ in the region Ω.

Example 3.3.1 (Analytic functions)

(i) Clearly $a(z) = 1 - 0.5z$ is analytic for all $z \in \mathbb{C}$, and has no zeros for $|z| < 2$. The inverse is $\frac{1}{a(z)} = \sum_{j=0}^{\infty}(0.5z)^j$ is well defined in the region $|z| < 2$.

(ii) Clearly $a(z) = 1 - 2z$ is analytic for all $z \in \mathbb{C}$, and has no zeros for $|z| > 1/2$. The inverse is $\frac{1}{a(z)} = (-2z)^{-1}(1 - (1/2z)) = (-2z)^{-1}(\sum_{j=0}^{\infty}(1/(2z))^j)$ well defined in the region $|z| > 1/2$.

(iii) The function $a(z) = \frac{1}{(1-0.5z)(1-2z)}$ is analytic in the region $0.5 < z < 2$.

(iv) $a(z) = 1 - z$, is analytic for all $z \in \mathbb{C}$, but is zero for $z = 1$. Hence its inverse is not well defined for regions which involve $|z| = 1$ (see Example 3.6).

(v) Finite order polynomials such as $\phi(z) = \sum_{j=0}^{p} \phi_j z^j$ for $\Omega = \mathbb{C}$.

(vi) The expansion $(1 - 0.5z)^{-1} = \sum_{j=0}^{\infty}(0.5z)^j$ for $\Omega = \{z; |z| \leq 2\}$.

We observe that for AR processes we can represent the equation as $\phi(B)X_t = \varepsilon_t$, which formally gives the solution $X_t = \phi(B)^{-1}\varepsilon_t$. This raises the question, under what conditions on $\phi(B)^{-1}$ is $\phi(B)^{-1}\varepsilon_t$ a valid solution. For $\phi(B)^{-1}\varepsilon_t$ to make sense $\phi(B)^{-1}$ should be represented as a power series expansion. Below, we state a technical lemma on $\phi(z)$ which we use to obtain a stationary solution.

Lemma 3.3.2 (Technical lemma) *Suppose that $\psi(z) = \sum_{j=-\infty}^{\infty} \psi_j z^j$ is finite on a region that includes $|z| = 1$ (we say it is analytic in the region $|z| = 1$). Then $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$.*

An immediate consequence of the lemma above is that if $\psi(z) = \sum_{j=-\infty}^{\infty} \psi_j z^j$ is analytic in the region and $\{X_t\}$ is a strictly stationary time series, where $\mathrm{E}|X_t|$ we define the time series

$Y_t = \psi(B)X_t = \sum_{j=-\infty}^{\infty} \psi_j X_{t-j}$. Then by the lemma above and Lemma 3.2.1, $\{Y_t\}$ is almost surely finite and strictly stationary time series. We use this result to obtain a solution of an AR(∞) (which includes an AR(p) as a special case).

Lemma 3.3.3 *Suppose $\phi(z) = 1+\sum_{j=1}^{\infty} \phi_j$ is analytic in the region $|z| = 1$. We define the AR(∞) process*

$$X_t = \sum_{j=1}^{\infty} \phi_j X_{t-j} + \varepsilon_t.$$

If $\phi(z)$ has the inverse $\psi(z) = \sum_{j=-\infty}^{\infty} \psi_j z^j$ which is analytic in a region including $|z| = 1$, then the unique stationary solution of X_t is

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j \varepsilon_{t-j}.$$

We can immediately apply the lemma to find conditions under which the AR(p) process will admit a stationary solution. Note that this is simply a reformulation of Lemma 3.3.1.

Corollary 3.3.1 *Let X_t be an AR(p) time series, where*

$$X_t = \sum_{j=1}^{p} \phi_j X_{t-j} + \varepsilon_t.$$

Suppose the roots of the characteristic polynomial $\phi(B) = 1 - \sum_{j=1}^{p} \phi_j B^j$ do not lie on the unit circle $|B| = 1$, then X_t admits a strictly stationary solution.

In addition suppose the roots of $\phi(B)$ all lie outside the unit circle, then X_t admits a strictly stationary, causal solution.

This summarises what we observed in Section 3.3.4.

Rules of the back shift operator:

(i) If $a(z)$ is analytic in a region Ω which includes the unit circle $|z| = 1$ in its interior and $\{Y_t\}$ is a well defined time series, then X_t defined by $Y_t = a(B)X_t$ is a well defined random variable.

(ii) The operator is commutative and associative, that is $[a(B)b(B)]X_t = a(B)[b(B)X_t] = [b(B)a(B)]X_t$ (the square brackets are used to indicate which parts to multiply first). This may seems obvious, but remember matrices are not commutative!

(iii) Suppose that $a(z)$ and its inverse $\frac{1}{a(z)}$ are both have solutions in the region Ω which includes the unit circle $|z| = 1$ in its interior. If $a(B)X_t = Z_t$, then $X_t = \frac{1}{a(B)}Z_t$.

3.3.8 The AR(∞) representation of an MA(q) process

Consider the MA(1) process

$$X_t = \varepsilon_t + \theta\varepsilon_{t-1},$$

where $\{\varepsilon_t\}$ are iid random variables. Our aim is understand when X_t can have an AR(∞) representation. We do this using the backshift notation. Recall $B\varepsilon_t = \varepsilon_{t-1}$ substituting this into the MA(1) model above gives

$$X_t = (1 + \theta B)\varepsilon_t.$$

Thus at least formally

$$\varepsilon_t = (1 + \theta B)^{-1} X_t.$$

We recall that the following equality holds

$$(1 + \theta B)^{-1} = \sum_{j=0}^{\infty}(-\theta)^j B^j,$$

when $|\theta B| < 1$. Therefore the following result holds

$$\varepsilon_t = (1 + B\theta)^{-1} X_t = \sum_{j=0}^{\infty}(-\theta)^j B^j X_t = \sum_{j=0}^{\infty}(-\theta)^j X_{t-j}$$

when $|\theta| < 1$. Rearranging the above gives the AR(∞) representation

$$X_t = \sum_{j=1}^{\infty}(-\theta)^j X_{t-j} + \varepsilon_t,$$

but observe this representation only holds if $|\theta| < 1$. In general the MA(q) process

$$X_t = \sum_{j=1}^{q} \theta_j \varepsilon_{t-j} + \varepsilon_t$$

will have an AR(∞) representation if the polynomial $\theta(B) = 1 + \sum_{j=1}^{q} \theta_j B^j$ which lie outside the unit circle (i.e. all the roots are greater than one in absolute), which we denote

$$X_t = \sum_{j=1}^{\infty} a_j X_{t-j} + \varepsilon_t.$$

Remark 3.3.2 (Why is this useful?) *The AR(∞) representation of an MA(q) process is very useful for prediction. We observe the time series $\{X_t\}$ not the innovations $\{\varepsilon_t\}$. Using AR(∞) representation of the MA(∞), the best predictor of X_{t+1} given the past $\{X_t\}$ is*

$$\sum_{j=1}^{\infty} a_j X_{t+1-j}$$

Remark 3.3.3 (AR(∞) representations and stationary time series) *If a time series is second order stationary and its spectral density function $f(\omega) = (2\pi)^{-1} \sum_{r\in\mathbb{Z}} c(r) e^{ir\omega}$ is bounded away from zero (is not zero) and is finite on $[0, \pi]$. Then it will have an type of AR(∞) representation*

$$X_t = \sum_{j=1}^{\infty} a_j X_{t-j} + \varepsilon_t,$$

the difference is that $\{\varepsilon_t\}$ are **uncorrelated random variables** *and* **may not be** *iid random variables. This result is useful when finding the best linear predictors of X_t given the past.*

3.4 Some additional discussion

3.4.1 An explanation as to why the backshift operator method works

To understand why the magic backshift operator works, we use matrix notation to rewrite the AR(p) model as an infinite set of difference equations

$$\begin{pmatrix} \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & 0 & 1 & -\phi_1 & \ldots & -\phi_p & \ldots \\ \ldots & 0 & 0 & 1 & -\phi_1 & \ldots & -\phi_p \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix} \begin{pmatrix} \vdots \\ X_t \\ X_{t-1} \\ X_{t-2} \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ \varepsilon_t \\ \varepsilon_{t-1} \\ \varepsilon_{t-2} \\ \vdots \end{pmatrix}.$$

The above is an infinite dimensional equation (and the matrix is an infinite upper triangular matrix). Formally to obtain a simulation we invert the matrix to get a solution of X_t in terms of ε_t. Of course in reality it is not straightfoward to define this inverse. Instead let us consider a finite (truncated) version of the above matrix equation. Except for the edge effects this is a circulant matrix (where the rows are repeated, but each time shifted by one, see wiki for a description). Truncating the matrix to have dimension n, we approximate the above by the finite set of n-equations

$$\begin{pmatrix} 1 & -\phi_1 & \ldots & -\phi_p & 0 & \ldots \\ 0 & 1 & -\phi_1 & \ldots & -\phi_p & \\ \ldots & \ldots & \ldots & \ldots & \ldots & \\ -\phi_1 & -\phi_2 & \ldots & \ldots & 0 & 1 \end{pmatrix} \begin{pmatrix} X_n \\ X_{n-1} \\ \vdots \\ X_0 \end{pmatrix} = \begin{pmatrix} \varepsilon_n \\ \varepsilon_{n-1} \\ \vdots \\ \varepsilon_0 \end{pmatrix}$$

$$\Rightarrow C_n \underline{X}_n \approx \varepsilon_n.$$

The approximation of the AR(p) equation only arises in the first p-equations, where

$$\begin{aligned}
X_0 - \sum_{j=1}^{p} \phi_j X_{n-j} &= \varepsilon_0 \\
X_1 - \phi_1 X_0 - \sum_{j=2}^{p} \phi_j X_{n+1-j} &= \varepsilon_1 \\
\vdots \quad &\quad \vdots \\
X_p - \sum_{j=1}^{p-1} \phi_j X_{p-j} - \phi_p X_n &= \varepsilon_p.
\end{aligned}$$

We now define the $n \times n$ matrix U_n, where

$$U_n = \begin{pmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & \ldots & 0 \end{pmatrix}.$$

We observe that U_n is a 'deformed diagonal matrix' where all the ones along the diagonal have been shifted once to the right, and the 'left over' one is placed in the bottom left hand corner. U_n is another example of a circulant matrix, moreover U_n^2 shifts once again all the ones to the right

$$U_n^2 = \begin{pmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & 0 & \ldots & 0 \end{pmatrix}.$$

U_n^3 shifts the ones to the third off-diagonal and so forth until $U_n^n = I$. Thus all circulant matrices can be written in terms of powers of U_n (the matrix U_n can be considered as the building blocks of circulant matrices). In particular

$$C_n = I_n - \sum_{j=1}^{p} \phi_j U_n^j,$$

$[I_n - \sum_{j=1}^{p}\phi_j U_n^j]\underline{X}_n = \underline{\varepsilon}_n$ and the solution to the equation is

$$\underline{X}_n = (I_n - \sum_{j=1}^{p}\phi_j U_n^j)^{-1}\underline{\varepsilon}_n.$$

Our aim is to write $(I_n - \sum_{j=1}^{p}\phi_j U_n^j)^{-1}$ as a power series in U_n, with U_n playing the role of the backshift operator.

To do this we recall the similarity between the matrix $I_n - \sum_{j=1}^{p}\phi_j U_n^j$ and the characteristic equation $\phi(B) = 1 - \sum_{j=1}^{p}\phi_j z^j$. In particular since we can factorize the characteristic equation as $\phi(B) = \prod_{j=1}^{p}[1 - \lambda_j B]$, we can factorize the matrix $I_n - \sum_{j=1}^{p}\phi_j U_n^j = \prod_{j=1}^{p}[I_n - \lambda_j U_n]$. To obtain the inverse, for simplicity, we assume that the roots of the characteristic function are greater than one (ie. $|\lambda_j| < 1$, which we recall corresponds to a causal solution) and are all different. Then there exists constants c_j where

$$[I_n - \sum_{j=1}^{p}\phi_j U_n^j]^{-1} = \sum_{j=1}^{p} c_j (I_n - \lambda_j U_n)^{-1}$$

(just as in partial fractions) - to see why multiply the above by $[I_n - \sum_{j=1}^{p}\phi_j U_n^j]$. Finally, we recall that if the eigenvalues of A are less than one, then $(1 - A)^{-1} = \sum_{j=0}^{\infty} A^j$. The eigenvalues of U_n are $\{\exp(\frac{2\pi i j}{n}); j = 1, \ldots, n\}$, thus the eigenvalues of $\lambda_j U_n$ are less than one. This gives $(I_n - \lambda_j U_n)^{-1} = \sum_{k=0}^{\infty}\lambda_j^k U_n^k$ and

$$[I_n - \sum_{j=1}^{p}\phi_j U_n^j]^{-1} = \sum_{j=1}^{p} c_j \sum_{k=0}^{\infty}\lambda_j^k U_n^k. \tag{3.12}$$

Therefore, the solution of $C_n\underline{X}_n = \underline{\varepsilon}_n$ is

$$\underline{X}_n = C_n^{-1}\underline{\varepsilon}_n = \left(\sum_{j=1}^{p} c_j \sum_{k=0}^{\infty}\lambda_j^k U_n^k\right)\underline{\varepsilon}_n.$$

Let us focus on the first element of the vector $\underline{X}_n$, which is X_n. Since $U_n^k\underline{\varepsilon}_n$ shifts the elements of $\underline{\varepsilon}_n$ up by k (note that this shift is with wrapping of the vector) we have

$$X_n = \sum_{j=1}^{p} c_j \sum_{k=0}^{n}\lambda_j^k \varepsilon_{n-k} + \underbrace{\sum_{j=1}^{p} c_j \sum_{k=n+1}^{\infty}\lambda_j^k \varepsilon_{n-k \mod (n)}}_{\to 0}. \tag{3.13}$$

Note that the second term decays geometrically fast to zero. Thus giving the stationary solution $X_n = \sum_{j=1}^{p} c_j \sum_{k=0}^{\infty} \lambda_j^k \varepsilon_{n-k}$.

To recollect, we have shown that $[I_n - \sum_{j=1}^{p} \phi_j U_n^j]^{-1}$ admits the solution in (3.12) (which is the same as the solution of the inverse of $\phi(B)^{-1}$) and that $U_n^j \underline{\varepsilon}_n$ plays the role of the backshift operator. Therefore, we can use the backshift operator in obtaining a solution of an AR process because it plays the role of the matrix U_n.

Example 3.4.1 *The AR(1) model, $X_t - \phi_1 X_{t-1} = \varepsilon_t$ is written as*

$$\begin{pmatrix} 1 & -\phi_1 & \ldots & 0 & 0 & 0 & 0 \\ 0 & 1 & -\phi_1 & \ldots & 0 & 0 & \\ \ldots & \ldots & \ldots & \ldots & \ldots & & \\ -\phi_1 & 0 & 0 & 0 & 0 & 1 & \end{pmatrix} \begin{pmatrix} X_n \\ X_{n-1} \\ \vdots \\ X_0 \end{pmatrix} = \begin{pmatrix} \varepsilon_n \\ \varepsilon_{n-1} \\ \vdots \\ \varepsilon_0 \end{pmatrix}$$

$$\Rightarrow C_n \underline{X}_n = \underline{\varepsilon}_n.$$

The approximation of the AR(1) is only for the first equation, where $X_0 - \phi_1 X_n = \varepsilon_0$. Using the matrix U_n, the above equation can be written as $(I_n - \phi_1 U_n)\underline{X}_n = \underline{\varepsilon}_n$, which gives the solution

$$\underline{X}_n = (I_n - \phi_1 U_n)^{-1} \underline{\varepsilon}_n.$$

Let us suppose that $|\phi_1| > 1$ (ie, the root lies inside the unit circle and the solution is noncausal), then to get a convergent expansion of $(1_n - \phi_1 U_n)^{-1}$ we rewrite $(I_n - \phi_1 U_n) = -\phi_1 U_n (I_n - \phi_1^{-1} U_n^{-1})$. Thus we have

$$(I_n - \phi_1 U_n)^{-1} = -\left[\sum_{k=0}^{\infty} \phi_1^{-k} U_n^{-k}\right] (\phi_1 U_n)^{-1}.$$

Therefore the solution is

$$\underline{X}_n = \left(-\sum_{k=0}^{\infty} \phi_1^{-k+1} U_n^{-k+1}\right) \underline{\varepsilon}_n,$$

which in its limit gives the same solution as Section 3.3.2(ii).

Notice that U_n^j and B^j are playing the same role.

A rigourous explanation on extending this argument to stationary time series defined on $\mathbb{Z}$ can be found in Pourahmadi (2001), Sections 5.3 and 9.5.3. The rough argument is that one defines a Hilbert space $H(X)$ which is the closure of all linear combinations of $\{X_t\}$. Note that the metric on this Hilbert space is simply the covariance i.e. if $Y = \sum_{j\in\mathbb{Z}} a_j X_j$, $Z = \sum_{j\in\mathbb{Z}} b_j X_j$ and $Y, Z \in H(X)$ then $\langle Y, Z\rangle = \text{cov}[Y, Z] = \sum_{j_1,j_2} a_{j_1} b_{j_2} c(j_1 - j_2)$ where $c(\cdot)$ is an autocovariance function of $\{X_t\}$. We define the operator U where $UX_t = X_{t+1}$ and $U(\sum_{j=1}^{m} a_j X_{t+j}) = \sum_{j=1}^{m} a_j X_{t+j+1}$. It can be shown that U extends to $H(X)$ and is a continuous, linear, surjective operator (see Pourahmadi (2001), Section 9.5.3). Moreover since U is an isometric operator (i.e. it is measure preserving; $\text{cov}[UY, UZ] = \text{cov}[Y, Z]$ if $Y, Z \in H(X)$, this is easy to show), then it is a unitary operator (this means its adjoint operator is also its inverse i.e. $U^*U = I$). It is clear that $U^*X_t = X_{t-1}$. All this implies if $Y \in H(X)$ and $Z = UY$ then $Y = U^*Z$. To jump between ARMA and its solutions we need to extend these arguments to two processes $\{X_t, \varepsilon_t\}$ (see Section Pourahmadi (2001), Section 5.3.2). Once these details are clarified we can jump between $\phi(B)X_t = \varepsilon_t$ and $X_t = \phi(B)^{-1}\varepsilon_t$ and back again.

3.4.2 Representing the AR(p) as a vector AR(1)

Let us suppose X_t is an AR(p) process, with the representation

$$X_t = \sum_{j=1}^{p} \phi_j X_{t-j} + \varepsilon_t.$$

For the rest of this section we will assume that the roots of the characteristic function, $\phi(z)$, lie outside the unit circle, thus the solution causal. We can rewrite the above as a Vector Autoregressive (VAR(1)) process

$$\underline{X}_t = A\underline{X}_{t-1} + \underline{\varepsilon}_t \tag{3.14}$$

where

$$\begin{pmatrix} \phi_1 & \phi_2 & \dots & \phi_{p-1} & \phi_p \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}, \tag{3.15}$$

$\underline{X}'_t = (X_t, \ldots, X_{t-p+1})$ and $\underline{\varepsilon}'_t = (\varepsilon_t, 0, \ldots, 0)$. It is straightforward to show that the eigenvalues of A are the inverse of the roots of $\phi(z)$ (since

$$\det(A - zI) = z^p - \sum_{i=1}^{p} \phi_i z^{p-i} = \underbrace{z^p\,(1 - \sum_{i=1}^{p} \phi_i z^{-i})}_{=z^p\phi(z^{-1})}),$$

thus the eigenvalues of A lie inside the unit circle. It can be shown that for any $|\lambda_{max}(A)| < \delta < 1$, there exists a constant C_δ such that $|\|A^j\|_{spec} \leq C_\delta \delta^j$ (see Appendix A). Note that result is extremely obvious if the eigenvalues are distinct (in which case the spectral decomposition can be used), in which case $|\|A^j\|_{spec} \leq C_\delta |\lambda_{max}(A)|^j$ (note that $\|A\|_{spec}$ is the spectral norm of A, which is the largest eigenvalue of the symmetric matrix AA').

We can apply the same back iterating that we did for the AR(1) to the vector AR(1). Iterating (5.4) backwards k times gives

$$\underline{X}_t = \sum_{j=0}^{k-1} A^j \underline{\varepsilon}_{t-j} + A^k \underline{X}_{t-k}.$$

Since $\|A^k \underline{X}_{t-k}\|_2 \leq \|A^k\|_{spec} \|\underline{X}_{t-k}\| \xrightarrow{\mathcal{P}} 0$ we have

$$\underline{X}_t = \sum_{j=0}^{\infty} A^j \underline{\varepsilon}_{t-j}.$$

3.5 The ARMA model

Up to now, we have defined the moving average and the autoregressive model. The MA(q) average has the feature that after q lags there isn't any correlation between two random variables. On the other hand, there are correlations at all lags for an AR(p) model. In addition as we shall see later on, it is much easier to estimate the parameters of an AR model than an MA. Therefore, there are several advantages in fitting an AR model to the data (note that when the roots are of the characteristic polynomial lie inside the unit circle, then the AR can also be written as an MA(∞), since it is causal). However, if we do fit an AR model to the data, what order of model should we use? Usually one uses the AIC (BIC or similar criterion) to determine the order. But for many data sets, the selected order tends to be relative large, for example order 14. The large order is usually chosen when correlations tend to decay slowly and/or the autcorrelations structure is quite complex (not just monotonically decaying). However, a model involving 10-15 unknown parameters is not particularly parsimonious and more parsimonious models which can model the same behaviour would be useful. A very useful generalisation which can be more flexible (and parsimonious) is the ARMA(p,q) model, in this case X_t satisfies

$$X_t - \sum_{i=1}^{p} \phi_i X_{t-i} = \varepsilon_t + \sum_{j=1}^{q} \theta_j \varepsilon_{t-j}.$$

Definition 3.5.1 (Summary of AR, ARMA and MA models) *(i) The autoregressive AR(p) model: $\{X_t\}$ satisfies*

$$X_t = \sum_{i=1}^{p} \phi_i X_{t-i} + \varepsilon_t. \tag{3.16}$$

Observe we can write it as $\phi(B)X_t = \varepsilon_t$

(ii) The moving average $MA(q)$ model: $\{X_t\}$ satisfies

$$X_t = \varepsilon_t + \sum_{j=1}^{q} \theta_j \varepsilon_{t-j}. \tag{3.17}$$

Observe we can write $X_t = \theta(B)\varepsilon_t$

(iii) The autoregressive moving average $ARMA(p,q)$ model: $\{X_t\}$ satisfies

$$X_t - \sum_{i=1}^{p} \phi_i X_{t-i} = \varepsilon_t + \sum_{j=1}^{q} \theta_j \varepsilon_{t-j}. \tag{3.18}$$

We observe that we can write X_t as $\phi(B)X_t = \theta(B)\varepsilon_t$.

Below we give conditions for the ARMA to have a causal solution and also be invertible. We also show that the coefficients of the MA(∞) representation of X_t will decay exponentially.

Lemma 3.5.1 *Let us suppose X_t is an $ARMA(p,q)$ process with representation given in Definition 3.5.1.*

(i) If the roots of the polynomial $\phi(z)$ lie outside the unit circle, and are greater than $(1+\delta)$ (for

some $\delta > 0$), then X_t almost surely has the solution

$$X_t = \sum_{j=0}^{\infty} a_j \varepsilon_{t-j}, \tag{3.19}$$

where for $j > q$, $a_j = [A^j]_{1,1} + \sum_{i=1}^{q} \theta_i [A^{j-i}]_{1,1}$, with

$$A = \left(\begin{array}{ccccc} \phi_1 & \phi_2 & \ldots & \phi_{p-1} & \phi_p \\ 1 & 0 & \ldots & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & \ldots & 1 & 0 \end{array} \right).$$

where $\sum_j |a_j| < \infty$ (we note that really $a_j = a_j(\phi, \theta)$ since its a function of $\{\phi_i\}$ and $\{\theta_i\}$). Moreover for all j,

$$|a_j| \leq K\rho^j \tag{3.20}$$

for some finite constant K and $1/(1+\delta) < \rho < 1$.

(ii) If the roots of $\phi(z)$ lie both inside or outside the unit circle and are larger than $(1+\delta)$ or less than $(1+\delta)^{-1}$ for some $\delta > 0$, then we have

$$X_t = \sum_{j=-\infty}^{\infty} a_j \varepsilon_{t-j}, \tag{3.21}$$

(a vector $AR(1)$ is not possible), where

$$|a_j| \leq K\rho^{|j|} \tag{3.22}$$

for some finite constant K and $1/(1+\delta) < \rho < 1$.

(iii) If the absolute value of the roots of $\theta(z) = 1 + \sum_{j=1}^{q} \theta_j z^j$ are greater than $(1+\delta)$, then (3.18) can be written as

$$X_t = \sum_{j=1}^{\infty} b_j X_{t-j} + \varepsilon_t. \tag{3.23}$$

where

$$|b_j| \leq K\rho^j \tag{3.24}$$

for some finite constant K and $1/(1+\delta) < \rho < 1$.

PROOF. We first prove (i) There are several way to prove the result. The proof we consider here, uses the VAR expansion given in Section 3.4.2; thus we avoid using the Backshift operator (however the same result can easily proved using the backshift). We write the ARMA process as a vector difference equation

$$\underline{X}_t = A\underline{X}_{t-1} + \underline{\varepsilon}_t \tag{3.25}$$

where $\underline{X}'_t = (X_t, \ldots, X_{t-p+1})$, $\underline{\varepsilon}'_t = (\varepsilon_t + \sum_{j=1}^{q}\theta_j\varepsilon_{t-j}, 0, \ldots, 0)$. Now iterating (3.25), we have

$$\underline{X}_t = \sum_{j=0}^{\infty} A^j \underline{\varepsilon}_{t-j}, \tag{3.26}$$

concentrating on the first element of the vector $\underline{X}_t$ we see that

$$X_t = \sum_{i=0}^{\infty}[A^i]_{1,1}(\varepsilon_{t-i} + \sum_{j=1}^{q}\theta_j\varepsilon_{t-i-j}).$$

Comparing (3.19) with the above it is clear that for $j > q$, $a_j = [A^j]_{1,1} + \sum_{i=1}^{q}\theta_i[A^{j-i}]_{1,1}$. Observe that the above representation is very similar to the AR(1). Indeed as we will show below the A^j behaves in much the same way as the ϕ^j in AR(1) example. As with ϕ^j, we will show that A^j converges to zero as $j \to \infty$ (because the eigenvalues of A are less than one). We now show that $|X_t| \leq K\sum_{j=1}^{\infty}\rho^j|\varepsilon_{t-j}|$ for some $0 < \rho < 1$, this will mean that $|a_j| \leq K\rho^j$. To bound $|X_t|$ we use (3.26)

$$|X_t| \leq \|\underline{X}_t\|_2 \leq \sum_{j=0}^{\infty}\|A^j\|_{spec}\|\underline{\varepsilon}_{t-j}\|_2.$$

Hence, by using Gelfand's formula (see Appendix A) we have $|\|A^j\|_{spec} \leq C_\rho\rho^j$ (for any $|\lambda_{\max}(A)| < \rho < 1$, where $\lambda_{\max}(A)$ denotes the largest maximum eigenvalue of the matrix A), which gives the corresponding bound for $|a_j|$.

To prove (ii) we use the backshift operator. This requires the power series expansion of $\frac{\theta(z)}{\phi(z)}$. If the roots of $\phi(z)$ are distinct, then it is straightforward to write $\phi(z)^{-1}$ in terms of partial fractions which uses a convergent power series for $|z| = 1$. This expansion immediately gives the the linear coefficients a_j and show that $|a_j| \leq C(1+\delta)^{-|j|}$ for some finite constant C. On the other hand, if there are multiple roots, say the roots of $\phi(z)$ are $\lambda_1, \ldots, \lambda_s$ with multiplicity $m_1, \ldots, m_s$ (where $\sum_{j=1}^{s} m_s = p$) then we need to adjust the partial fraction expansion. It can be shown that $|a_j| \leq C|j|^{\max_s |m_s|}(1+\delta)^{-|j|}$. We note that for every $(1+\delta)^{-1} < \rho < 1$, there exists a constant such that $|j|^{\max_s |m_s|}(1+\delta)^{-|j|} \leq C\rho^{|j|}$, thus we obtain the desired result.

To show (iii) we use a similar proof to (i), and omit the details. □

Corollary 3.5.1 *An ARMA process is invertible if the roots of $\theta(B)$ (the MA coefficients) lie outside the unit circle and causal if the roots of $\phi(B)$ (the AR coefficients) lie outside the unit circle.*

An AR(p) process and an MA(q) process is identifiable (meaning there is only one model associated to one solution). However, the ARMA is not necessarily identifiable. The problem arises when the characteristic polynomial of the AR and MA part of the model share common roots. A simple example is $X_t = \varepsilon_t$, this also satisfies the representation $X_t - \phi X_{t-1} = \varepsilon_t - \phi\varepsilon_{t-1}$ etc. Therefore it is not possible to identify common factors in the polynomials.

One of the main advantages of the invertibility property is in prediction and estimation. We will consider this in detail below. It is worth noting that even if an ARMA process is not invertible, one can generate a time series which has identical correlation structure but is invertible (see Section 4.4).

3.6 Unit roots, integrated, long memory and non-invertible processes

3.6.1 Unit roots

If the difference equation has a root which is one, then an (almost sure) stationary solution of the AR model does not exist. The simplest example is the 'random walk' $X_t = X_{t-1} + \varepsilon_t$ ($\phi(z) = (1-z)$). This is an example of an Autoregressive Integrated Moving Average ARIMA$(0,1,0)$ model

$(1-B)X_t = \varepsilon_t$.

To see that it does not have a stationary solution, we iterate the equation n steps backwards; $X_t = \sum_{j=0}^{n} \varepsilon_{t-j} + X_{t-n}$. $S_{t,n} = \sum_{j=0}^{n} \varepsilon_{t-j}$ is the partial sum, but it is clear that the partial sum $S_{t,n}$ does not have a limit, since it is not a Cauchy sequence, ie. $|S_{t,n} - S_{t,m}|$ does not have a limit. However, given some initial value X_0, for $t > 0$ the so called "unit process" $X_t = X_{t-1} + \varepsilon_t$ is well defined. Notice that the nonstationary solution of this sequence is $X_t = X_0 + \sum_{j=1}^{t} \varepsilon_{t-j}$ which has variance $\text{var}(X_t) = \text{var}(X_0) + t$ (assuming that $\{\varepsilon_t\}$ are iid random variables with variance one and independent of X_0).

We observe that we can 'stationarize' the process by taking first differences, i.e. defining $Y_t = X_t - X_{t-1} = \varepsilon_t$.

Unit roots for higher order differences The unit process described above can be generalised to taking d differences (often denoted as an ARIMA$(0, d, 0)$) where $(1-B)^d X_t = \varepsilon_t$ (by taking d-differences we can remove d-order polynomial trends). We elaborate on this below.

To stationarize the sequence we take d differences, i.e. let $Y_{t,0} = X_t$ and for $1 \leq i \leq d$ define the iteration

$$Y_{t,i} = Y_{t,i-1} - Y_{t-1,i-1}$$

and $Y_t = Y_{t,d}$ will be a stationary sequence. Note that this is equivalent to

$$Y_t = \sum_{j=0}^{d} \frac{d!}{j!(d-j)!} (-1)^j X_{t-j}.$$

The ARIMA(p, d, q) model The general ARIMA(p, d, q) is defined as $(1-B)^d \phi(B) X_t = \theta(B)\varepsilon_t$, where $\phi(B)$ and $\theta(B)$ are p and q order polynomials respectively and the roots of $\phi(B)$ lie outside the unit circle.

Another way of describing the above model is that after taking d differences (as detailed in (ii)) the resulting process is an ARMA(p, q) process (see Section 3.5 for the definition of an ARMA model).

To illustrate the difference between stationary ARMA and ARIMA processes, in Figure 3.1

Suppose $(1-B)\phi(B)X_t = \varepsilon_t$ and let $\widetilde{\phi}(B) = (1-B)\phi(B)$. Then we observe that $\widetilde{\phi}(1) = 0$. This property is useful when checking for unit root behaviour (see Section 3.8).

More exotic unit roots

The unit root process need not be restricted to the case that the characteristic polynomial associated the AR model is one. If the absolute of the root is equal to one, then a stationary solution cannot exist. Consider the AR(2) model

$$X_t = 2\cos\theta X_{t-1} - X_{t-2} + \varepsilon_t.$$

The associated characteristic polynomial is $\phi(B) = 1 - 2\cos(\theta)B + B^2 = (1 - e^{i\theta}B)(1 - e^{-i\theta}B)$. Thus the roots are $e^{i\theta}$ and $e^{-i\theta}$ both of which lie on the unit circle. Simulate this process.

3.6.2 Long memory

We have mentioned previously that that the coefficients of an ARMA processes which admit a stationary solution decay geometrically. This means that they are unable to model "persistant" behaviour between random variables which are separately relatively far in time. However, the ARIMA offers a solution on how this could be done. We recall that $(1 - B)X_t = \varepsilon_t$ is a process which is nonstationary. However we can no replace $(1 - B)^d$ (where d is a fraction) and see if one can obtain a compromise between persistance (long memory) and nonstatonary (in the sense of differencing). Suppose

$$(1 - B)^d X_t = \varepsilon_t.$$

If $0 \leq d \leq 1/2$ we have the expansions

$$(1 - B)^d = \sum_{j=0}^{\infty} \psi_j B^j \qquad (1 - B)^{-d} = \sum_{j=0}^{\infty} \phi_j B^j$$

where

$$\phi_j = \frac{\Gamma(j - d)}{\Gamma(j + 1)\Gamma(-d)} \qquad \psi_j = \frac{\Gamma(j + d)}{\Gamma(j + 1)\Gamma(d)}$$

and $\Gamma(1 + k) = k\Gamma(k)$ is the Gamma function. Note that $\sum_{j=0}^{\infty} \psi_j^2 < \infty$ but $\sum_{j=0}^{\infty} \psi_j = \infty$. This means that X_t has the stationary solution

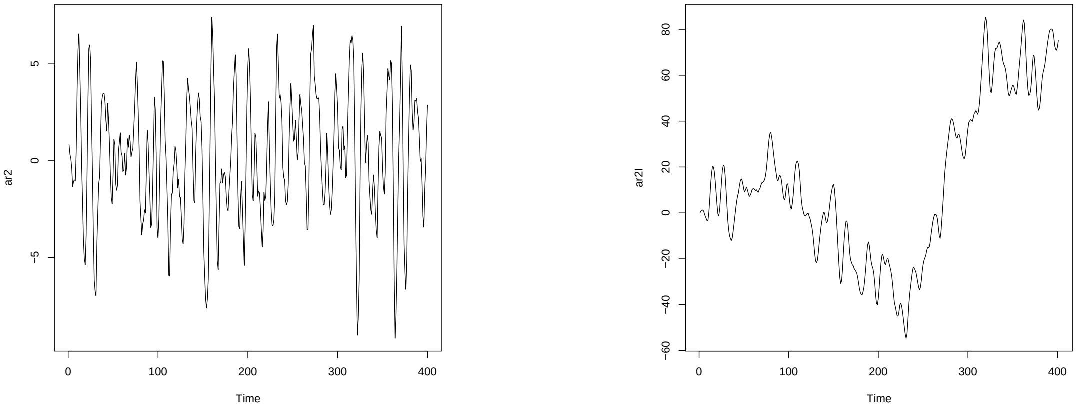
$$X_t = \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}.$$

Noting to show that the above is true requires weaker conditions than those given in Lemma 3.2.1. It above process does not decay geometrically fast, and it can be shown that the sample covariance is such that $c(r) \sim |r|^{2d-1}$ (hence is not absolutely summable).

3.6.3 Non-invertible processes

In the examples above a stationary solution does not exist. We now consider an example where the process is stationary but an autoregressive representation does not exist (this matters when we want to forecast).

Consider the MA(1) model $X_t = \varepsilon_t - \varepsilon_{t-1}$. We recall that this can be written as $X_t = \phi(B)\varepsilon_t$ where $\phi(B) = 1 - B$. From Example 3.3.1(iv) we know that $\phi(z)^{-1}$ does not exist, therefore it does not have an AR(∞) representation since $(1 - B)^{-1}X_t = \varepsilon_t$ is not well defined.



(a) $X_t = 1.5X_{t-1} - 0.75X_{t-2} + \varepsilon_t$

(b) $(1 - B)Y_t = X_t$, where X_t is defined in (a)

Figure 3.1: Realisations from an AR process and its corresponding integrated process, using $N(0, 1)$ innovations (generated using the same seed).

3.7 Simulating from an Autoregressive process

Simulating from a Gaussian AR process

It is straightforward to simulate from an AR process with Gaussian innovations, $\{\varepsilon_t\}$. Given the autoregressive structure we can deduce the correlation structure (see Chapter 4) (regardless of the distribution of the innovations). Furthermore, from Lemma 3.5.1(ii) we observe that all AR processes can be written as the infinite sum of the innovations. Thus if the innovations are Gaussian, so is the AR process. This allows us to deduce the joint distribution of $X_1, \ldots, X_p$, which in turn allows us generate the AR(p) process.

We illustrate the details with with an AR(1) process. Suppose $X_t = \phi_1 X_{t-1} + \varepsilon_t$ where $\{\varepsilon_t\}$ are iid standard normal random variables (note that for Gaussian processes it is impossible to discriminate between causal and non-causal processes - see Section 4.4, therefore we will assume $|\phi_1| < 1$). We will show in Section 4.1, equation (4.1) that the autocovariance of an AR(1) is

$$c(r) \quad = \quad \phi_1^r \sum_{j=0}^{\infty} \phi_1^{2j} = \frac{\phi_1^r}{1-\phi_1^2}.$$

Therefore, the marginal distribution of X_t is Gaussian with variance $(1-\phi_1^2)^{-1}$. Therefore, to simulate an AR(1) Gaussian time series, we draw from a Gaussian time series with mean zero and variance $(1-\phi_1^2)^{-1}$, calling this X_1. We then iterate for $2 \leq t$, $X_t = \phi_1 X_{t-1} + \varepsilon_t$. This will give us a stationary realization from an AR(1) Gaussian time series.

Note the function `arima.sim` is a routine in `R` which does the above. See below for details.

Simulating from a non-Gaussian AR model

Unlike the Gaussian AR process it is difficult to simulate a non-Gaussian model, but we can obtain a very close approximation. This is because if the innovations are non-Gaussian but known it is not clear what the distribution of X_t will be. Here we describe how to obtain a close approximation in the case that the AR process is causal.

Again we describe the method for the AR(1). Let $\{X_t\}$ be an AR(1) process, $X_t = \phi_1 X_{t-1} + \varepsilon_t$, which has stationary, causal solution

$$X_t = \sum_{j=0}^{\infty} \phi_1^j \varepsilon_{t-j}.$$

To simulate from the above model, we set $\tilde{X}_1 = 0$. Then obtain the iteration $\tilde{X}_t = \phi_1 \tilde{X}_{t-1} + \varepsilon_t$ for $t \geq 2$. We note that the solution of this equation is

$$\tilde{X}_t = \sum_{j=0}^{t} \phi_1^j \varepsilon_{t-j}.$$

We recall from Lemma 3.5.1 that $|X_t - \tilde{X}_t| \leq |\phi_1|^t \sum_{j=0}^{\infty} |\phi_1^j \varepsilon_{-j}|$, which converges geometrically fast to zero. Thus if we choose a large n to allow 'burn in' and use $\{\tilde{X}_t; t \geq n\}$ in the simulations we have a simulation which is close to a stationary solution from an AR(1) process.

Simulating from an Integrated process

To simulate from an integrated process ARIMA$(p,1,q)$ $(1-B)Y_t = X_t$, where X_t is a causal ARMA(p,q) process. We first simulate $\{X_t\}$ using the method above. Then we define the recursion $Y_1 = X_1$ and for $t > 1$

$$Y_t = Y_{t-1} + X_t.$$

Thus giving a realisation from an ARIMA$(p,1,q)$.

Simulating from a non-Gaussian non-causal model

Suppose that X_t satisfies the representation

$$X_t = \sum_{j=1}^{p} \phi_j X_{t-j} + \varepsilon_t,$$

whose characteristic function have roots both inside and outside the unit circle. Thus, the stationary solution of this equation is not causal. It is not possible to simulate from this equation. To see why, consider directly simulating from $X_t = 2X_{t-1} + \varepsilon_t$ without rearranging it as $X_{t-1} = \frac{1}{2}X_t - \frac{1}{2}\varepsilon_t$, the solution would explode. Now if the roots are both inside and outside the unit circle, there would not be a way to rearrange the equation to iterate a stationary solution. There are two methods to remedy this problem:

(i) From Lemma 3.5.1(ii) we recall that X_t has the solution

$$X_t = \sum_{j=-\infty}^{\infty} a_j \varepsilon_{t-j}, \qquad (3.27)$$

where the coefficients a_j are determined from the characteristic equation. Thus to simulate the process we use the above representation, though we do need to truncate the number of terms in (3.27) and use

$$\tilde{X}_t = \sum_{j=-M}^{M} a_j \varepsilon_{t-j}.$$

(ii) The above is a brute force method is an approximation which is also difficult to evaluate. There are simpler methods, if one studies the roots of the characteristic equation.

Let us suppose that $\{\lambda_{j_1}; j_1 = 1, \ldots, p_1\}$ are the roots of $\phi(z)$ which lie outside the unit circle and $\{\mu_{j_2}; j_2 = 1, \ldots, p_2\}$ are the roots which lie inside the unit circle. For ease of calculation we will assume the roots are distinct.

(a) We can rewrite $\phi(z)^{-1}$ as

$$\begin{aligned}\phi(z)^{-1} &= \frac{1}{\left[\prod_{j_1=1}^{p_1}(1-\lambda_{j_1}z)\right]\cdot\left[\prod_{j_2=1}^{p_2}(1-\mu_{j_2}z)\right]} \\ &= \sum_{j_1=1}^{p_1}\frac{c_{j_1}}{(1-\lambda_{j_1}z)} + \sum_{j_2=1}^{p_2}\frac{d_{j_d}}{(1-\mu_{j_d}z)} \\ &= \sum_{j_1=1}^{p_1}\frac{c_{j_1}}{(1-\lambda_{j_1}z)} - \sum_{j_2=1}^{p_2}\frac{d_{j_d}}{\mu_{j_d}z(1-\mu_{j_d}^{-1}z^{-1})}\end{aligned}$$

Thus the solution of X_t is

$$X_t = \phi(B)^{-1}\varepsilon_t \;=\; \sum_{j_1=1}^{p_1}\frac{c_{j_1}}{(1-\lambda_{j_1}B)}\varepsilon_t - \sum_{j_2=1}^{p_2}\frac{d_{j_d}}{\mu_{j_d}B(1-\mu_{j_d}^{-1}B^{-1})}\varepsilon_t$$

Let $Y_{j_1,t} = \lambda_{j_1}Y_{j_1,t-1} + \varepsilon_t$ and $Z_{j_2,t} = \mu_{j_2}Z_{j_2,t-1} + \varepsilon_t$ (thus the stationary solution is generated with $Z_{j_2,t-1} = \mu_{j_2}^{-1}Z_{j_2,t} - \mu_{j_2}^{-1}\varepsilon_t$). Generate the time series $\{Y_{j_1,t}; j_1 = 1, \ldots, p_1\}$ and $\{Y_{j_1,t}; j_1 = 1, \ldots, p_1\}$ using the method described above. Then the non-causal time series can be generated by using

$$X_t = \sum_{j_1=1}^{p_1} c_{j_1}Y_{j_1,t} - \sum_{j_2=1}^{p_2} d_{j_2}Z_{j_2,t}.$$

(b) An even easier method is represent $\phi(z)$ as the product of two polynomial, one whose roots are outside the unit circle ($\phi_1(z) = \prod_{i=1}^{p_1}(1-\lambda_{j_1}z)$) and one whose roots are inside the unit circle ($\phi_2(z) = \prod_{i=1}^{p_1}(1-\mu_{j_1}z)$). Then

$$\underbrace{\phi_1(B)\phi_2(B))}_{\text{commutative}} X_t = \varepsilon_t \Rightarrow \phi_2(B)X_t = \phi_1(B)^{-1}\varepsilon_t.$$

Thus first define a causal stationary time series defined using the equation

$$\phi_1(B)Y_t = \varepsilon_t.$$

Next, using $\{Y_t\}$ as the innovations, define a noncausal stationary time series defined using the recursion

$$\phi_2(B)X_t = Y_t.$$

Comments:

- Remember $Y_{j,t}$ is generated using the past ε_t and $Z_{j,t}$ is generated using future innovations. Therefore to ensure that the generated $\{Y_{j,t}\}$ and $\{Z_{j,t}\}$ are close to the stationary we need to ensure that the initial value of $Y_{j,t}$ is far in the past and the initial value for $Z_{j,t}$ is far in the future.
- If the roots are complex conjugates, then the corresponding $\{Y_{j,t}\}$ or $\{Z_{j,t}\}$ should be written as AR(2) models (to avoid complex processes).

`R` functions

Shumway and Stoffer (2006) and David Stoffer's website gives a comprehensive introduction to time series `R`-functions.

The function `arima.sim` simulates from a Gaussian ARIMA process. For example, `arima.sim(list(order=c(2,0,0), ar = c(1.5, -0.75)), n=150)` simulates from the AR(2) model $X_t = 1.5X_{t-1} - 0.75X_{t-2} + \varepsilon_t$, where the innovations are Gaussian.

Exercise 3.7 *In the following simulations, use <u>non-Gaussian</u> innovations.*

(i) Simulate an AR(4) process with characteristic function

$$\phi(z) = \left[1 - 0.8\exp(i\frac{2\pi}{13})z\right]\left[1 - 0.8\exp(-i\frac{2\pi}{13})z\right]\left[1 - 1.5\exp(i\frac{2\pi}{5})z\right]\left[1 - 1.5\exp(-i\frac{2\pi}{5})z\right].$$

(ii) Simulate an $AR(4)$ process with characteristic function

$$\phi(z) = \left[1 - 0.8\exp(i\frac{2\pi}{13})z\right]\left[1 - 0.8\exp(-i\frac{2\pi}{13})z\right]\left[1 - \frac{2}{3}\exp(i\frac{2\pi}{5})z\right]\left[1 - \frac{2}{3}\exp(-i\frac{2\pi}{5})z\right].$$

Do you observe any differences between these realisations?

3.8 Some diagnostics

Here we discuss some guidelines which allows us to discriminate between a pure autoregressive process and a pure moving average process; both with low orders. And also briefly discuss how to identify a “unit root” in the time series and whether the data has been over differenced.

3.8.1 ACF and PACF plots for checking for MA and AR behaviour

The ACF and PACF plots are the autocorrelations and partial autocorrelations estimated from the time series data (estimated assuming the time series is second order stationary). The ACF we came across is Chapter 1, the PACF we define in Chapter 4, however roughly it is the correlation between two time points after removing the linear dependence involving the observations inbetween. In `R` the functions are `acf` and `pacf`. Note that the PACF at lag zero is not given (as it does not make any sense).

The ACF and PACF of an AR(1), AR(2), MA(1) and MA(2) are given in Figures 3.2-3.5.

We observe from Figure 3.2 and 3.3 (which give the ACF of and AR(1) and AR(2) process) that there is correlation at all lags (though it reduces for large lags). However, we see from the PACF for the AR(1) has only one large coefficient at lag one and the PACF plot of the AR(2) has two large coefficients at lag one *and* two. This suggests that the ACF and PACF plot can be used to diagnose autoregressive behaviour and its order.

Similarly, we observe from Figures 3.4 and 3.5 (which give the ACF of and MA(1) and MA(2) process) that there is no real correlation in the ACF plots after lag one and two respectively, but the PACF plots are more ambigious (there seems to be correlations at several lags).

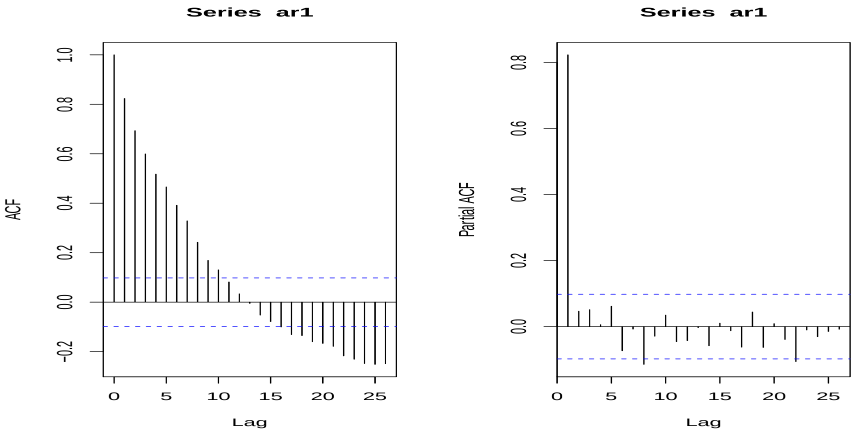


Figure 3.2: ACF and PACF plot of an AR(1), $X_t = 0.5X_{t-1} + \varepsilon_t$, $n = 400$

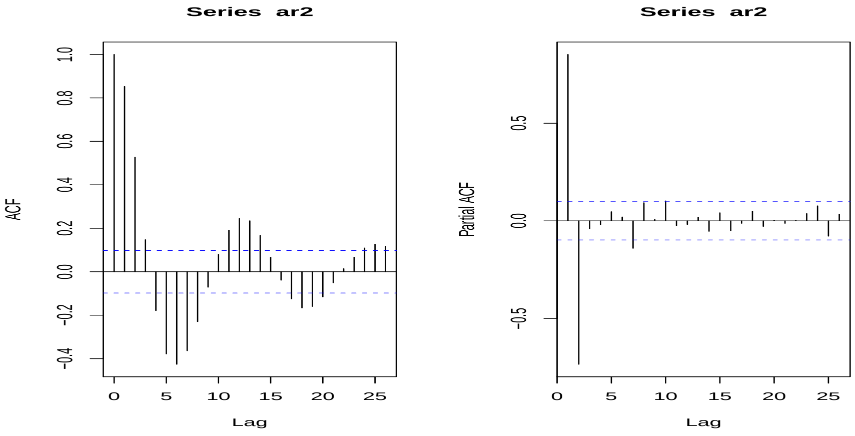


Figure 3.3: ACF and PACF plot of an AR(2), $n = 400$

3.8.2 Checking for unit roots

We recall that for an AR(1) process, the unit root corresponds to $X_t = X_{t-1} + \varepsilon_t$ i.e. $\phi = 1$. Thus to check for unit root type behaviour we estimate ϕ and see how close ϕ is to one. We can formally turn this into a statistical test $H_0 : \phi = 1$ vs. $H_A : |\phi| < 1$ and there several tests for this, the most famous is the Dickey-Fuller test. Rather intriguingly, the distribution of $\widehat{\phi}$ (using the least squares estimator) does not follow a normal distribution with a $\sqrt{n}$-rate!

Extending the the unit root to the AR(p) process, the unit root corresponds to $(1-B)\phi(B)X_t = \varepsilon_t$ where $\phi(B)$ is an order $(p-1)$-polynomial (this is the same as saying $X_t - X_{t-1}$ is a stationary AR$(p-1)$ process). Checking for unit root is the same as checking that the sum of all the AR coefficients is equal to one. This is easily seen by noting that $\widetilde{\phi}(1) = 0$ where $\widetilde{\phi}(B) = (1-B)\phi(B)$

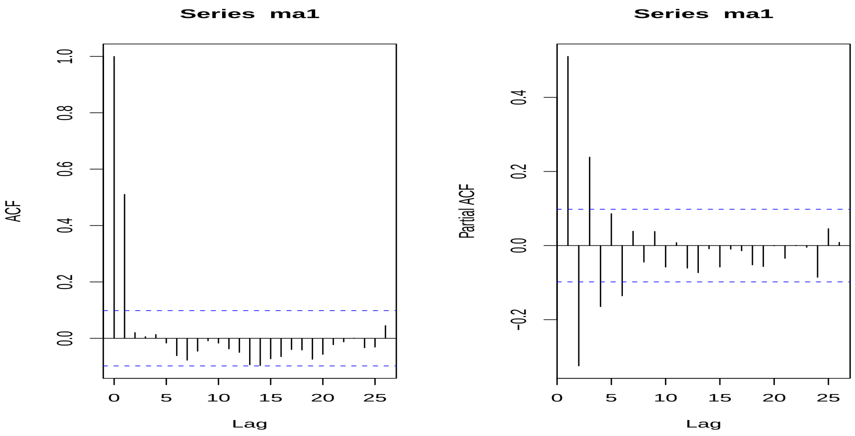


Figure 3.4: ACF and PACF plot of an MA(1), $X_t = \varepsilon_t + 0.8\varepsilon_{t-1}$, $n = 400$

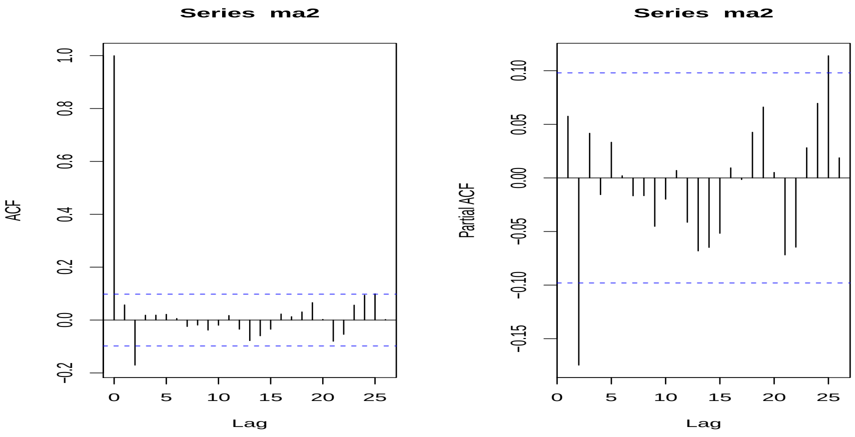


Figure 3.5: ACF and PACF plot of an MA(2), $n = 400$

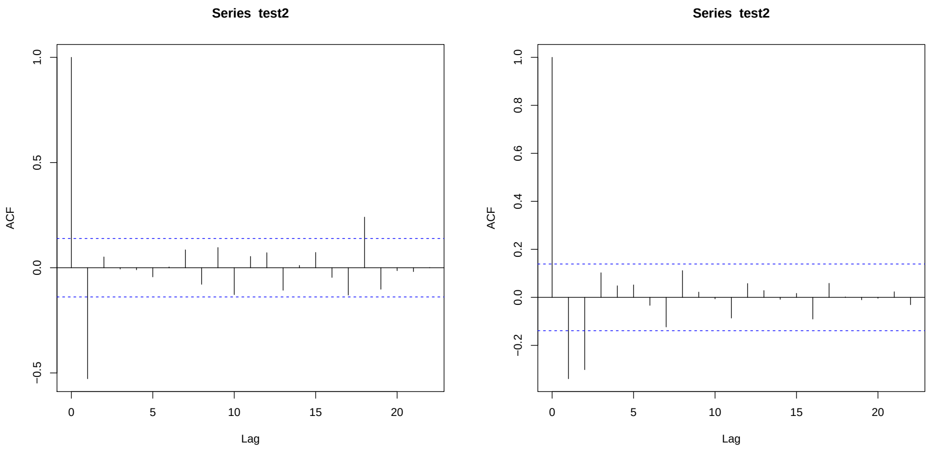


Figure 3.6: ACF of differenced data $Y_t = X_t - X_{t-1}$. Left $X_t = \varepsilon_t$, Right $X_t = 1.5X_{t-1} - 0.75X_{t-2} + \varepsilon_t$.

or

$$(1-B)\phi(B)X_t = X_t - (\phi_1 - 1)X_{t-1} - (\phi_2 - \phi_1)X_{t-2} - (\phi_{p-1} - \phi_{p-2})X_{t-p+1} + \phi_{p-1}X_{t-p} = \varepsilon_t.$$

Thus we see that the sum of the AR coefficients is equal to one. Therefore to check for unit root behaviour in AR(p) processes one can see how close the sum of the estimate AR coefficients $\sum_{j=1}^{p}\widehat{\phi}_j$ is to one. Again this can be turned into a formal test.

In order to remove stochastic or deterministic trend one may difference the data. But if the data is over differenced one can induce spurious dependence in the data which is best avoided (estimation is terrible and prediction becomes a nightmare). One indicator of over differencing is the appearance of negative correlation at lag one in the data. This is illustrated in Figure 3.6, where for both data sets (difference of iid noise and differenced of an AR(2) process) we observe a large negative correlation at lag one.

Chapter 4

The autocovariance function of a linear time series

Objectives

- Be able to determine the rate of decay of an ARMA time series.
- Be able 'solve' the autocovariance structure of an AR process.
- Understand what partial correlation is and how this may be useful in determining the order of an AR model.
- Understand why autocovariance is 'blind' to processes which are non-causal. But the higher order cumulants are not 'blind' to causality.

4.1 The autocovariance function

The autocovariance function (ACF) is defined as the sequence of covariances of a stationary process. That is suppose that $\{X_t\}$ is a stationary process with mean zero, then $\{c(k) : k \in \mathbb{Z}\}$ is the ACF of $\{X_t\}$ where $c(k) = \mathrm{E}(X_0 X_k)$. Clearly different time series give rise to different features in the ACF. We will explore some of these features below.

Before investigating the structure of ARMA processes we state a general result connecting linear time series and the summability of the autocovariance function.

Lemma 4.1.1 *Suppose the stationary time series X_t satisfies the linear representation $\sum_{j=-\infty}^{\infty}\psi_j\varepsilon_{t-j}$. The covariance is $c(r)=\sum_{j=-\infty}^{\infty}\psi_j\psi_{j+r}$.*

(i) If $\sum_{j=\infty}^{\infty}|\psi_j|<\infty$, then $\sum_k|c(k)|<\infty$.

(ii) If $\sum_{j=\infty}^{\infty}|j\psi_j|<\infty$, then $\sum_k|k\cdot c(k)|<\infty$.

(iii) If $\sum_{j=\infty}^{\infty}|\psi_j|^2<\infty$, then we cannot say anything about summability of the covariance.

PROOF. It is straightforward to show that

$$c(k)=\text{var}[\varepsilon_t]\sum_j\psi_j\psi_{j-k}.$$

Using this result, it is easy to see that $\sum_k|c(k)|\leq\sum_k\sum_j|\psi_j|\cdot|\psi_{j-k}|$, thus $\sum_k|c(k)|<\infty$, which proves (i).

The proof of (ii) is similar. To prove (iii), we observe that $\sum_j|\psi_j|^2<\infty$ is a weaker condition then $\sum_j|\psi_j|<\infty$ (for example the sequence $\psi_j=|j|^{-1}$ satisfies the former condition but not the latter). Thus based on the condition we cannot say anything about summability of the covariances. □

First we consider a general result on the covariance of a causal ARMA process (always to obtain the covariance we use the MA(∞) expansion - you will see why below).

4.1.1 The rate of decay of the autocovariance of an ARMA process

We evaluate the covariance of an ARMA process using its MA(∞) representation. Let us suppose that $\{X_t\}$ is a causal ARMA process, then it has the representation in (3.21) (where the roots of $\phi(z)$ have absolute value greater than $1+\delta$). Using (3.21) and the independence of $\{\varepsilon_t\}$ we have

$$\begin{aligned}\text{cov}(X_t,X_\tau) &= \text{cov}(\sum_{j_1=0}^{\infty}a_{j_1}\varepsilon_{t-j_1},\sum_{j_2=0}^{\infty}a_{j_2}\varepsilon_{\tau-j_2})\\ &= \sum_{j=0}^{\infty}a_{j_1}a_{j_2}\text{cov}(\varepsilon_{t-j},\varepsilon_{\tau-j})=\sum_{j=0}^{\infty}a_ja_{j+|t-\tau|}\text{var}(\varepsilon_t) \qquad (4.1)\end{aligned}$$

(here we see the beauty of the MA(∞) expansion). Using (3.22) we have

$$|\text{cov}(X_t,X_\tau)|\leq\text{var}(\varepsilon_t)C_\rho^2\sum_{j=0}^{\infty}\rho^j\rho^{j+|t-\tau|}\leq C_\rho^2\rho^{|t-\tau|}\sum_{j=0}^{\infty}\rho^{2j}=\frac{\rho^{|t-\tau|}}{1-\rho^2}, \qquad (4.2)$$

for any $1/(1+\delta) < \rho < 1$.

The above bound is useful, it tells us that the ACF of an ARMA process decays exponentially fast. In other words, there is very little memory in an ARMA process. However, it is not very enlightening about features within the process. In the following we obtain an explicit expression for the ACF of an autoregressive process. So far we have used the characteristic polynomial associated with an AR process to determine whether it was causal. Now we show that the roots of the characteristic polynomial also give information about the ACF and what a 'typical' realisation of a autoregressive process could look like.

4.1.2 The autocovariance of an autoregressive process and the Yule-Walker equations

Let us consider the zero mean AR(p) process $\{X_t\}$ where

$$X_t = \sum_{j=1}^{p} \phi_j X_{t-j} + \varepsilon_t. \tag{4.3}$$

From now onwards we will assume that $\{X_t\}$ is causal (the roots of $\phi(z)$ lie outside the unit circle). Given that $\{X_t\}$ is causal we can derive a recursion for the covariances. It can be shown that multipying both sides of the above equation by X_{t-k} ($k \leq 0$) and taking expectations, gives the equation

$$\mathrm{E}(X_t X_{t-k}) = \sum_{j=1}^{p} \phi_j \mathrm{E}(X_{t-j} X_{t-k}) + \underbrace{\mathrm{E}(\varepsilon_t X_{t-k})}_{=0} = \sum_{j=1}^{p} \phi_j \mathrm{E}(X_{t-j} X_{t-k}). \tag{4.4}$$

Note the above normal equations are the parameters which minimise the mean squared error $\mathrm{E}(X_t - \sum_{j=1}^{p} \theta_j X_{t-j})^2$. It is worth mentioning that if the process were not causal this equation would not hold, since ε_t and X_{t-k} are not necessarily independent. (4.4) are the Yule-Walker equations, we will discuss them in detail when we consider estimation. For now letting $c(k) = \mathrm{E}(X_0 X_k)$ and using the above we see that the autocovariance satisfies the homogenuous difference equation

$$c(k) - \sum_{j=1}^{p} \phi_j c(k-j) = 0, \tag{4.5}$$

for $k \geq 0$. In other words, the autocovariance function of $\{X_t\}$ is the solution of this difference equation. The study of difference equations is a entire field of research, however we will now scratch the surface to obtain a solution for (4.5). Solving (4.5) is very similar to solving homogenuous differential equations, which some of you may be familar with (do not worry if you are not).

Recall the characteristic polynomial of the AR process $\phi(z) = 1 - \sum_{j=1}^{p} \phi_j z^j = 0$, which has the roots $\lambda_1, \ldots, \lambda_p$. In Section 3.3.4 we used the roots of the characteristic equation to find the stationary solution of the AR process. In this section we use the roots characteristic to obtain the solution (4.5). It can be shown if the roots are distinct (the roots are all different) the solution of (4.5) is

$$c(k) = \sum_{j=1}^{p} C_j \lambda_j^{-k}, \tag{4.6}$$

where the constants $\{C_j\}$ are chosen depending on the initial values $\{c(k) : 1 \leq k \leq p\}$ and are such that they ensure that $c(k)$ is real (recalling that λ_j) can be complex.

The simplest way to prove (4.6) is to use a plugin method. Plugging $c(k) = \sum_{j=1}^{p} C_j \lambda_j^{-k}$ into (4.5) gives

$$\begin{aligned} c(k) - \sum_{j=1}^{p} \phi_j c(k-j) &= \sum_{j=1}^{p} C_j \left(\lambda_j^{-k} - \sum_{i=1}^{p} \phi_i \lambda_j^{-(k-i)} \right) \\ &= \sum_{j=1}^{p} C_j \lambda_j^{-k} \underbrace{\left(1 - \sum_{i=1}^{p} \phi_i \lambda_j^{i} \right)}_{\phi(\lambda_i)} = 0. \end{aligned}$$

In the case that the roots of $\phi(z)$ are not distinct, let the roots be $\lambda_1, \ldots, \lambda_s$ with multiplicity $m_1, \ldots, m_s$ ($\sum_{k=1}^{s} m_k = p$). In this case the solution is

$$c(k) = \sum_{j=1}^{s} \lambda_j^{-k} P_{m_j}(k), \tag{4.7}$$

where $P_{m_j}(k)$ is m_jth order polynomial and the coefficients $\{C_j\}$ are now 'hidden' in $P_{m_j}(k)$. We now study the covariance in greater details and see what it tells us about a realisation. As a motivation consider the following example.

Example 4.1.1 *Consider the $AR(2)$ process*

$$X_t = 1.5X_{t-1} - 0.75X_{t-2} + \varepsilon_t, \tag{4.8}$$

where $\{\varepsilon_t\}$ are iid random variables with mean zero and variance one. The corresponding characteristic polynomial is $1 - 1.5z + 0,75z^2$, which has roots $1 \pm i3^{-1/2} = \sqrt{4/3}\exp(i\pi/6)$. Using the discussion above we see that the autocovariance function of $\{X_t\}$ is

$$c(k) = (\sqrt{4/3})^{-k}(C_1\exp(-ik\pi/6) + \bar{C}_1\exp(ik\pi/6)),$$

for a particular value of C_1. Now write $C_1 = a\exp(ib)$, then the above can be written as

$$c(k) = a(\sqrt{4/3})^{-k}\cos\left(k\frac{\pi}{6} + b\right).$$

We see that the covariance decays at an exponential rate, but there is a periodicity within the decay. This means that observations separated by a lag $k = 12$ are more closely correlated than other lags, this suggests a quasi-periodicity in the time series. The ACF of the process is given in Figure 4.1. Notice that it decays to zero (relatively fast) but it also undulates. A plot of a realisation of the time series is given in Figure 4.2, notice the quasi-periodicity of about $2\pi/12$. Let is briefly return to the definition of the periodogram given in Section 1.3.4 ($I_n(\omega) = \frac{1}{n}\left|\sum_{t=1}^{n} X_t\exp(it\omega)\right|^2$). We used the periodogram to identify the periodogram of a deterministic signal. But when dependent, correlated noise was added to the periodic signal the periodogram exhibited more complex behaviour than in the iid case. In Figure 7.1 we give a plot of the periodogram corresponding to Figure 4.2. Recall that this $AR(2)$ gives a quasi-periodicity of 12, which corresponds to the frequency $2\pi/12 \approx 0.52$, which matches the main peaks in periodogram. We will learn later that the periodogram is a 'crude' (meaning inconsistent) estimator of the spectral density function. The spectral density if given in the lower plot of Figure 7.1.

We now generalise the above example. Let us consider the general AR(p) process defined in (4.3). Suppose the roots of the corresponding characteristic polynomial are *distinct* and we split them into real and complex roots. Because the characteristic polynomial is comprised of real coefficients, the complex roots come in complex conjugate pairs. Hence let us suppose the real roots are $\{\lambda_j\}_{j=1}^{r}$

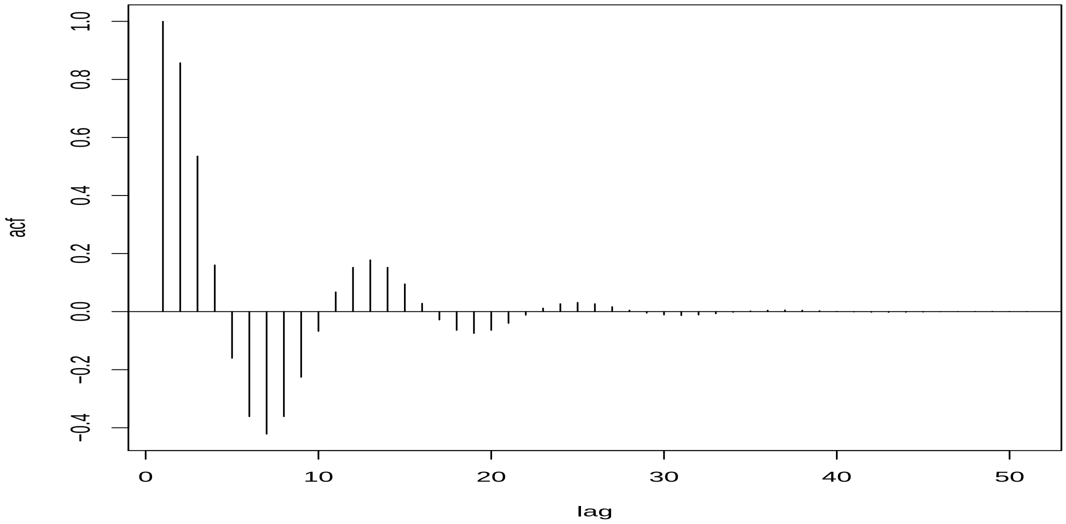


Figure 4.1: The ACF of the time series $X_t = 1.5X_{t-1} - 0.75X_{t-2} + \varepsilon_t$

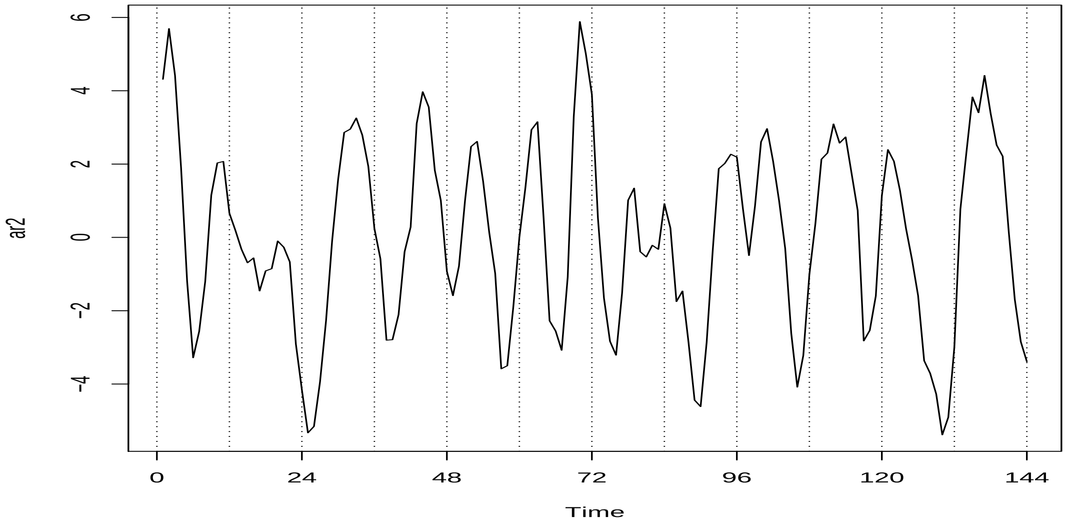


Figure 4.2: The a simulation of the time series $X_t = 1.5X_{t-1} - 0.75X_{t-2} + \varepsilon_t$

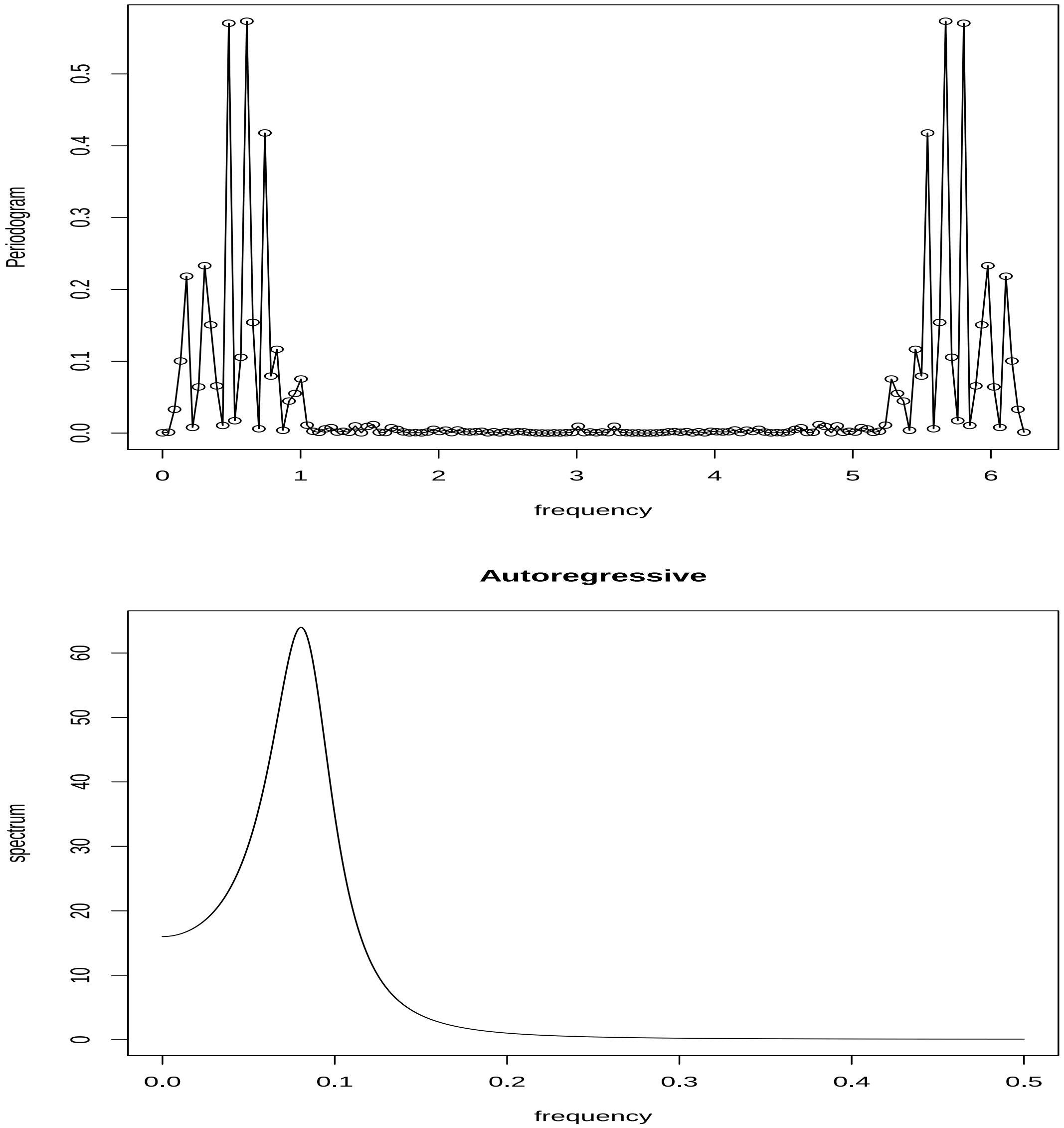


Figure 4.3: Top: Periodogram of $X_t = 1.5X_{t-1} - 0.75X_{t-2} + \varepsilon_t$ for sample size $n = 144$. Lower: The corresponding spectral density function (note that 0.5 of the x-axis on spectral density corresponds to π on the x-axis of the periodogram).

and the complex roots are $\{\lambda_j, \overline{\lambda}_j\}_{j=r+1}^{(p-r)/2}$. The covariance in (4.6) can be written as

$$c(k) = \sum_{j=1}^{r} C_j \lambda_j^{-k} + \sum_{j=r+1}^{(p-2)/2} a_j |\lambda_j|^{-k} \cos(k\theta_j + b_j) \tag{4.9}$$

where for $j > r$ we write $\lambda_j = |\lambda_j| \exp(i\theta_j)$ and a_j and b_j are real constants. Notice that as the example above the covariance decays exponentially with lag, but there is undulation. A typical realisation from such a process will be quasi-periodic with periods at $\theta_{r+1}, \ldots, \theta_{(p-r)/2}$, though the magnitude of each period will vary.

An interesting discussion on covariances of an AR process and realisation of an AR process is given in Shumway and Stoffer (2006), Chapter 3.3 (it uses the example above). A discussion of difference equations is also given in Brockwell and Davis (1998), Sections 3.3 and 3.6 and Fuller (1995), Section 2.4.

Example 4.1.2 (Autocovariance of an AR(2)**)** *Let us suppose that X_t satisfies the model $X_t = (a+b)X_{t-1} - abX_{t-2} + \varepsilon_t$. We have shown that if $|a| < 1$ and $|b| < 1$, then it has the solution*

$$X_t = \frac{1}{b-a}\big(\sum_{j=0}^{\infty} \big(b^{j+1} - a^{j+1})\varepsilon_{t-j}\big).$$

By writing a 'timeline' it is straightfoward to show that for $r > 1$

$$\mathrm{cov}(X_t, X_{t-r}) = \sum_{j=0}^{\infty} (b^{j+1} - a^{j+1})(b^{j+1+r} - a^{j+1+r}).$$

Example 4.1.3 *The autocorrelation of a causal and noncausal time series Let us consider the two AR(1) processes considered in Section 3.3.2. We recall that the model*

$$X_t = 0.5X_{t-1} + \varepsilon_t$$

has the stationary causal solution

$$X_t = \sum_{j=0}^{\infty} 0.5^j \varepsilon_{t-j}.$$

Assuming the innovations has variance one, the ACF of X_t is

$$c_X(0) = \frac{1}{1-0.5^2} \qquad c_X(k) = \frac{0.5^{|k|}}{1-0.5^2}$$

On the other hand the model

$$Y_t = 2Y_{t-1} + \varepsilon_t$$

has the noncausal stationary solution

$$Y_t = -\sum_{j=0}^{\infty}(0.5)^{j+1}\varepsilon_{t+j+1}.$$

Thus process has the ACF

$$c_Y(0) = \frac{0.5^2}{1-0.5^2} \qquad c_X(k) = \frac{0.5^{2+|k|}}{1-0.5^2}.$$

Thus we observe that except for a factor $(0.5)^2$ both models has an identical autocovariance function. Indeed their autocorrelation function would be same. Furthermore, by letting the innovation of X_t have standard deviation 0.5, both time series would have the same autocovariance function.

Therefore, we observe an interesting feature, that the non-causal time series has the same correlation structure of a causal time series. In Section 4.4 that for every non-causal time series there exists a causal time series with the same autocovariance function. Therefore autocorrelation is 'blind' to non-causality.

Exercise 4.1 *Recall the AR(2) models considered in Exercise 3.5. Now we want to derive their ACF functions.*

(i) *(a)* *Obtain the ACF corresponding to*

$$X_t = \frac{7}{3}X_{t-1} - \frac{2}{3}X_{t-2} + \varepsilon_t,$$

where $\{\varepsilon_t\}$ are iid random variables with mean zero and variance σ^2.

(b) *Obtain the ACF corresponding to*

$$X_t = \frac{4\times\sqrt{3}}{5}X_{t-1} - \frac{4^2}{5^2}X_{t-2} + \varepsilon_t,$$

where $\{\varepsilon_t\}$ are iid random variables with mean zero and variance σ^2.

(c) *Obtain the ACF corresponding to*

$$X_t = X_{t-1} - 4X_{t-2} + \varepsilon_t,$$

where $\{\varepsilon_t\}$ are iid random variables with mean zero and variance σ^2.

(ii) *For all these models plot the true ACF in* R. *You will need to use the function* ARMAacf. *BEWARE of the ACF it gives for non-causal solutions. Find a method of plotting a causal solution in the non-causal case.*

Exercise 4.2 *In Exercise 3.6 you constructed a causal $AR(2)$ process with period 17.*

Load Shumway and Stoffer's package astsa *into R (use the command* install.packages("astsa") *and then* library("astsa").

Use the command arma.spec *to make a plot of the corresponding spectral density function. How does your periodogram compare with the 'true' spectral density function?*

R code

We use the code given in Shumway and Stoffer (2006), page 101 to make Figures 4.1 and 4.2.

To make Figure 4.1:

```
acf = ARMAacf(ar=c(1.5,-0.75),ma=0,50)
plot(acf,type="h",xlab="lag")
abline(h=0)
```

To make Figures 4.2 and 7.1:

```
set.seed(5)
ar2 <- arima.sim(list(order=c(2,0,0), ar = c(1.5, -0.75)), n=144)
plot.ts(ar2, axes=F); box(); axis(2)
axis(1,seq(0,144,24))
abline(v=seq(0,144,12),lty="dotted")
Periodogram <- abs(fft(ar2)/144)**2
frequency = 2*pi*c(0:143)/144
plot(frequency, Periodogram,type="o")
```

```
library("astsa")
arma.spec( ar = c(1.5, -0.75), log = "no", main = "Autoregressive")
```

4.1.3 The autocovariance of a moving average process

Suppose that $\{X_t\}$ satisfies

$$X_t = \varepsilon_t + \sum_{j=1}^{q} \theta_j \varepsilon_{t-j}.$$

The covariance is

$$\mathrm{cov}(X_t, X_{t-k}) = \begin{cases} \sum_{i=0}^{p} \theta_i \theta_{i-k} & k = -q, \ldots, q \\ 0 & \text{otherwise} \end{cases}$$

where $\theta_0 = 1$ and $\theta_i = 0$ for $i < 0$ and $i \geq q$. Therefore we see that there is no correlation when the lag between X_t and X_{t-k} is greater than q.

4.1.4 The autocovariance of an autoregressive moving average process

We see from the above that an MA(q) model is only really suitable when we believe that there is no correlaton between two random variables separated by more than a certain distance. Often autoregressive models are fitted. However in several applications we find that autoregressive models of a very high order are needed to fit the data. If a very 'long' autoregressive model is required a more suitable model may be the autoregressive moving average process. It has several of the properties of an autoregressive process, but can be more parsimonuous than a 'long' autoregressive process. In this section we consider the ACF of an ARMA process.

Let us suppose that the causal time series $\{X_t\}$ satisfies the equations

$$X_t - \sum_{i=1}^{p} \phi_i X_{t-i} = \varepsilon_t + \sum_{j=1}^{q} \theta_j \varepsilon_{t-j}.$$

We now define a recursion for ACF, which is similar to the ACF recursion for AR processes. Let us suppose that the lag k is such that $k > q$, then it can be shown that the autocovariance function of the ARMA process satisfies

$$\mathrm{E}(X_t X_{t-k}) - \sum_{i=1}^{p} \phi_i \mathrm{E}(X_{t-i}X_{t-k}) = 0$$

On the other hand, if $k \leq q$, then we have

$$\mathrm{E}(X_t X_{t-k}) - \sum_{i=1}^{p} \phi_i \mathrm{E}(X_{t-i}X_{t-k}) \;=\; \sum_{j=1}^{q} \theta_j \mathrm{E}(\varepsilon_{t-j}X_{t-k}) = \sum_{j=k}^{q} \theta_j \mathrm{E}(\varepsilon_{t-j}X_{t-k}).$$

We recall that X_t has the MA(∞) representation $X_t = \sum_{j=0}^{\infty} a_j \varepsilon_{t-j}$ (see (3.21)), therefore for $k \leq j \leq q$ we have $\mathrm{E}(\varepsilon_{t-j}X_{t-k}) = a_{j-k}\mathrm{var}(\varepsilon_t)$ (where $a(z) = \theta(z)\phi(z)^{-1}$). Altogether the above gives the difference equations

$$\begin{aligned} c(k) - \sum_{i=1}^{p} \phi_i c(k-i) &= \mathrm{var}(\varepsilon_t) \sum_{j=k}^{q} \theta_j a_{j-k} \quad \text{for } 1 \leq k \leq q \\ c(k) - \sum_{i=1}^{p} \phi_i c(k-i) &= 0, \text{ for } k > q, \end{aligned} \tag{4.10}$$

where $c(k) = \mathrm{E}(X_0 X_k)$. (4.10) is homogenuous difference equation, then it can be shown that the solution is

$$c(k) = \sum_{j=1}^{s} \lambda_j^{-k} P_{m_j}(k),$$

where $\lambda_1, \ldots, \lambda_s$ with multiplicity $m_1, \ldots, m_s$ ($\sum_k m_s = p$) are the roots of the characteristic polynomial $1 - \sum_{j=1}^{p} \phi_j z^j$. Observe the similarity to the autocovariance function of the AR process (see (4.7)). The coefficients in the polynomials P_{m_j} are determined by the initial condition given in (4.10).

You can also look at Brockwell and Davis (1998), Chapter 3.3 and Shumway and Stoffer (2006), Chapter 3.4.

4.2 A review of multivariate analysis

We see that by using the true autocovariance function we are able to identify the order of an MA(q) process: when the covariance lag is greater than q the covariance is zero[1]. However the same is not true for AR(p) processes. The autocovariances do not enlighten us on the order p. However a variant of the autocovariance, called the partial autocovariance is quite informative about order of AR(p). We start by reviewing the partial autocovariance, and it's relationship to the inverse variance/covariance matrix (often called the precision matrix).

4.2.1 The best linear predictor

Suppose $(Y, \mathbf{X})$, where $\mathbf{X} = (X_1, \ldots, X_p)$ is a random vector. The best linear predictor of Y given $\mathbf{X}$ is given by

$$\widehat{Y} = \sum_{j=1}^{p} \beta_j X_j$$

where $\boldsymbol{\beta} = \Sigma_{XX}^{-1}\Sigma_{XY}$, with $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_p)$ and $\Sigma_{XX} = \mathrm{var}(\mathbf{X})$, $\Sigma_{XY} = \mathrm{cov}[\mathbf{X}, Y]$.

To understand why the above is true, we need to find the θ which minimises

$$\mathrm{E}\left(Y - \sum_{j=1}^{p} \theta_j X_j\right)^2,$$

we assume that X_j has zero mean. Differentiating the above wrt θ_i leads to the normal equations

$$-2\left(\mathrm{E}\left(YX_i\right) - \sum_{j=1}^{p} \theta_j \mathrm{E}\left(X_j X_i\right)\right) \qquad i = 1, \ldots, p.$$

Equating to zero (since we want to find the θ_i which minimises the above) is

$$\underbrace{\mathrm{E}\left(YX_i\right)}_{=\mathrm{cov}(Y,X_i)} - \sum_{j=1}^{p} \theta_j \underbrace{\mathrm{E}\left(X_j X_i\right)}_{=\mathrm{cov}(X_i,X_j)} = 0 \qquad i = 1, \ldots, p.$$

[1] However, the order is not straightforward to diagnose from the sample ACF plot, as there are errors in the estimators and the error bars are not reliable for non iid noise

Writing the above as a matrix equation gives the solution

$$\underline{\beta} = \mathrm{var}\left(\mathbf{X}\right)^{-1}\mathrm{cov}\left(Y, \mathbf{X}\right).$$

Substituting the above into the mean squared error gives

$$\mathrm{E}\left(Y - \sum_{j=1}^{p}\beta_j X_j\right)^2 = \mathrm{E}(Y^2) - 2\mathrm{E}(Y\widehat{Y}) + \mathrm{E}(\widehat{Y}^2).$$

Using that

$$Y = \widehat{Y} + e$$

where e is uncorrelated with $\{X_j\}$, thus it is uncorrelated with $\widehat{Y}$. This means $\mathrm{E}[Y\widehat{Y}] = \mathrm{E}[\widehat{Y}^2]$. Therefore

$$\begin{aligned}\mathrm{E}\left(Y - \sum_{j=1}^{p}\beta_j X_j\right)^2 &= \mathrm{E}(Y^2) - \mathrm{E}(\widehat{Y}^2) = \mathrm{E}(Y^2) - \underline{\beta}'\mathrm{var}(\mathbf{X})\underline{\beta} \\ &= \mathrm{E}(Y^2) - \Sigma_{YX}\Sigma_{XX}^{-1}\Sigma_{XY}.\end{aligned}$$

4.2.2 Partial correlation

Suppose $\boldsymbol{X} = (X_1, \ldots, X_d)$ is a zero mean random vector (we impose the zero mean condition to simplify notation and it's not necessary). The partial correlation is the covariance between X_i and X_j, conditioned on the other elements in the vector. In other words, the covariance between the residuals of X_i conditioned on $\boldsymbol{X}_{-(ij)}$ (the vector not containing X_i and X_j) and the residual of X_j conditioned on $\boldsymbol{X}_{-(ij)}$. Note that by using the results in Section 4.2.1

$$\widehat{X}_i = \mathrm{var}[\boldsymbol{X}_{-(ij)}]^{-1}\mathrm{E}[\boldsymbol{X}_{-(ij)}X_i]\boldsymbol{X}_{-(ij)} \text{ and } \widehat{X}_j\mathrm{var}[\boldsymbol{X}_{-(ij)}]^{-1}\mathrm{E}[\boldsymbol{X}_{-(ij)}X_j]\boldsymbol{X}_{-(ij)}$$

are the best linear predictors of X_i and X_j given $\boldsymbol{X}_{-(ij)}$ respectively. Using this the *partial covariance* between X_i and X_j given $\boldsymbol{X}_{-(ij)}$ is defined as

$$\begin{aligned}\mathrm{cov}(X_i - \widehat{X}_i, X_j - \widehat{X}_j) &= \mathrm{cov}\big(X_i - \mathrm{var}[\boldsymbol{X}_{-(ij)}]^{-1}\mathrm{E}[\boldsymbol{X}_{-(ij)}X_i]\boldsymbol{X}_{-(ij)}, X_j - \mathrm{var}[\boldsymbol{X}_{-(ij)}]^{-1}\mathrm{E}[\boldsymbol{X}_{-(ij)}X_j]\boldsymbol{X}_{-(ij)}\big) \\ &= \mathrm{cov}[X_iX_j] - \mathrm{E}[\boldsymbol{X}_{-(ij)}X_i]'\mathrm{var}[\boldsymbol{X}_{-(ij)}]^{-1}\mathrm{E}[\boldsymbol{X}_{-(ij)}X_j].\end{aligned}$$

Taking the above argument further, the variance/covariance matrix of the residual of $\boldsymbol{X}_{ij} = (X_i, X_j)'$ given $\boldsymbol{X}_{-(ij)}$ is defined as

$$\mathrm{var}\big(\boldsymbol{X}_{ij} - \mathrm{E}[\boldsymbol{X}_{ij} \otimes \boldsymbol{X}_{-(ij)}]'\mathrm{var}[\boldsymbol{X}_{-(ij)}]^{-1}\boldsymbol{X}_{-(ij)}\big) = \Sigma_{ij} - \underline{c}_{ij}'\Sigma_{-(ij)}^{-1}\underline{c}_{ij} \tag{4.11}$$

where $\Sigma_{ij} = \mathrm{var}(\boldsymbol{X}_{ij})$, $\underline{c}_{ij} = \mathrm{E}(\boldsymbol{X}_{ij} \otimes \boldsymbol{X}_{-(ij)})$ (=cov$(\boldsymbol{X}_{ij}, \boldsymbol{X}_{-(ij)})$) and $\Sigma_{-(ij)} = \mathrm{var}(\boldsymbol{X}_{-(ij)})$ ($\otimes$ denotes the tensor product). Let s_{ij} denote the (i,j)th element of the (2×2) matrix $\Sigma_{ij} - \boldsymbol{c}_{ij}'\Sigma_{-(ij)}^{-1}\boldsymbol{c}_{ij}$. The *partial correlation* between X_i and X_j given $\boldsymbol{X}_{-(ij)}$ is

$$\rho_{ij} = \frac{s_{12}}{\sqrt{s_{11}s_{22}}},$$

observing that

(i) s_{12} is the partial covariance between X_i and X_j.

(ii) $s_{11} = \mathrm{E}(X_i - \sum_{k \neq i,j} \beta_{i,k} X_k)^2$ (where $\beta_{i,k}$ are the coefficients of the best linear predictor of X_i given $\{X_k; k \neq i,j\}$).

(ii) $s_{22} = \mathrm{E}(X_j - \sum_{k \neq i,j} \beta_{j,k} X_k)^2$ (where $\beta_{j,k}$ are the coefficients of the best linear predictor of X_j given $\{X_k; k \neq i,j\}$).

In a later section we relate partial correlation to the inverse of the variance/covariance matrix (often called the precision matrix).

4.2.3 The prediction error in a regression model and the precision matrix

We start with a nice (well known) expression from linear regression which expresses the prediction errors in terms of determinants matrices.

It is well know (see the above derivation) that the prediction error is

$$\mathrm{E}[Y - \widehat{Y}]^2 = \sigma_Y - \Sigma_{YX}\Sigma_{XX}^{-1}\Sigma_{XY} \tag{4.12}$$

with $\sigma_Y = \text{var}[Y]$. Let

$$\Sigma = \begin{pmatrix} \text{var}[Y] & \Sigma_{YX} \\ \Sigma_{XY} & \Sigma_{XX} \end{pmatrix}. \tag{4.13}$$

We show below that the prediction error can be rewritten as

$$\mathrm{E}[Y - \widehat{Y}]^2 = \sigma_Y - \Sigma_{YX}\Sigma_{XX}^{-1}\Sigma_{XY} = \frac{\det(\Sigma)}{\det(\Sigma_{XX})}. \tag{4.14}$$

To prove this result we use

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(D)\det\left(A - BD^{-1}C\right). \tag{4.15}$$

Applying this to (4.15) gives

$$\begin{aligned} \det(\Sigma) &= \det(\Sigma_{XX})\left(\sigma_Y - \Sigma_{YX}\Sigma_{XX}^{-1}\Sigma_{XY}\right) \\ \Rightarrow \det(\Sigma) &= \det(\Sigma_{XX})\mathrm{E}[Y - \widehat{Y}]^2, \end{aligned} \tag{4.16}$$

thus giving (4.14).

The above result leads to two more useful relations, which we now summarize. The first uses the following result on the inverse of block matrices

$$\begin{aligned} &\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} \\ = &\begin{pmatrix} A^{-1} + A^{-1}BP^{-1}CA^{-1} & -A^{-1}BP^{-1} \\ -P^{-1}CA^{-1} & P^{-1} \end{pmatrix} = \begin{pmatrix} P_1^{-1} & -P_1^{-1}BD^{-1} \\ -D^{-1}CP_1^{-1} & D^{-1} + D^{-1}CP_1^{-1}BD^{-1} \end{pmatrix}, \end{aligned} \tag{4.17}$$

where $P = (D - CA^{-1}B)$ and $P_1 = (A - BD^{-1}C)$. This block inverse turns out to be crucial in deriving many of the interesting properties associated with the inverse of a matrix. We now show that the the inverse of the matrix Σ, Σ^{-1} (usually called the precision matrix) contains the mean squared error.

Comparing the above with (4.13) and (4.12) we see that

$$\left(\Sigma^{-1}\right)_{11} = \frac{1}{\sigma_Y - \Sigma_{YX}\Sigma_{XX}^{-1}\Sigma_{XY}} = \frac{1}{\mathrm{E}[Y - \widehat{Y}]^2}.$$

In other words, the inverse of the top left hand side of the matrix Σ gives the inverse mean squared error of Y given X. Furthermore, by using (4.14) this implies that

$$\left(\Sigma^{-1}\right)_{11} = \frac{1}{\mathrm{E}[Y - \widehat{Y}]^2}. \tag{4.18}$$

This we have two equivalent expressions for $\mathrm{E}[Y - \widehat{Y}]^2$ in terms of Σ.

$$\mathrm{E}[Y - \widehat{Y}]^2 = \left(\Sigma^{-1}\right)_{11}^{-1} \text{ and } \mathrm{E}[Y - \widehat{Y}]^2 = \frac{\det(\Sigma)}{\det(\Sigma_{XX})}.$$

4.2.4 The precision matrix and partial correlation

Let us suppose that $\boldsymbol{X} = (X_1, \ldots, X_d)$ is a zero mean random vector with variance Σ. The $(i,j)th$ element of Σ the covariance $\mathrm{cov}(X_i, X_j) = \Sigma_{ij}$. Here we consider the inverse of Σ, and what information the $(i,j)th$ of the inverse tells us about the correlation between X_i and X_j. Let Σ^{ij} denote the $(i,j)th$ element of Σ^{-1}. We will show that with appropriate standardisation, Σ^{ij} is the negative partial correlation between X_i and X_j. More precisely,

$$\frac{\Sigma^{ij}}{\sqrt{\Sigma^{ii}\Sigma^{jj}}} = -\rho_{ij}. \tag{4.19}$$

The proof uses the inverse of block matrices. To simplify the notation, we will focus on the $(1,2)th$ element of Σ and Σ^{-1} (which concerns the correlation between X_1 and X_2).

Remark 4.2.1 *Remember the reason we can always focus on the top two elements of* $\mathbf{X}$ *is because we can always use a permutation matrix to permute the* X_i *and* X_j *such that they become the top two elements. Since the inverse of the permutation matrix is simply its transpose everything still holds.*

Let $\boldsymbol{X}_{1,2} = (X_1, X_2)'$, $\boldsymbol{X}_{-(1,2)} = (X_3, \ldots, X_d)'$, $\Sigma_{-(1,2)} = \mathrm{var}(\boldsymbol{X}_{-(1,2)})$, $\underline{c}_{1,2} = \mathrm{cov}(\boldsymbol{X}_{(1,2)}, \boldsymbol{X}_{-(1,2)})$

and $\Sigma_{1,2} = \mathrm{var}(\boldsymbol{X}_{1,2})$. Using this notation it is clear that

$$\mathrm{var}(\boldsymbol{X}) = \Sigma = \begin{pmatrix} \Sigma_{1,2} & \underline{c}_{1,2} \\ \underline{c}'_{1,2} & \Sigma_{-(1,2)} \end{pmatrix}. \tag{4.20}$$

By using (4.17) we have

$$\Sigma^{-1} = \begin{pmatrix} P^{-1} & -P^{-1}\underline{c}'_{1,2}\Sigma^{-1}_{-(1,2)} \\ -\Sigma^{-1}_{-(1,2)}\underline{c}_{1,2}P^{-1} & P^{-1} + \Sigma^{-1}_{-(1,2)}\underline{c}_{1,2}P^{-1}\underline{c}'_{1,2}\Sigma^{-1}_{-(1,2)} \end{pmatrix}, \tag{4.21}$$

where $P = (\Sigma_{1,2} - \underline{c}'_{1,2}\Sigma^{-1}_{-(1,2)}\underline{c}_{1,2})$. Comparing P with (4.11), we see that P is the 2×2 variance/-covariance matrix of the residuals of $X_{(1,2)}$ conditioned on $\boldsymbol{X}_{-(1,2)}$. Thus the partial correlation between X_1 and X_2 is

$$\rho_{1,2} = \frac{P_{1,2}}{\sqrt{P_{1,1}P_{2,2}}} \tag{4.22}$$

where P_{ij} denotes the elements of the matrix P. Inverting P (since it is a two by two matrix), we see that

$$P^{-1} = \frac{1}{P_{1,1}P_{2,2} - P^2_{1,2}} \begin{pmatrix} P_{2,2} & -P_{1,2} \\ -P_{1,2} & P_{11} \end{pmatrix}. \tag{4.23}$$

Thus, by comparing (4.21) and (4.23) and by the definition of partial correlation given in (4.22) we have

$$\frac{P^{(1,2)}}{\sqrt{P^{(1,1)}P^{(2,2)}}} = -\rho_{1,2}.$$

Let Σ^{ij} denote the (i,j)th element of Σ^{-1}. Thus we have shown (4.19):

$$\rho_{ij} = -\frac{\Sigma^{ij}}{\sqrt{\Sigma^{ii}\Sigma^{jj}}}.$$

In other words, the (i,j)th element of Σ^{-1} divided by the square root of it's diagonal gives negative partial correlation. Therefore, if the partial correlation between X_i and X_j given $\mathbf{X}_{ij}$ is zero, then $\Sigma^{i,j} = 0$.

4.2.5 The precision matrix and coefficients in a regression

The precision matrix, Σ^{-1}, contains many other hidden treasures. For example, the coefficients of Σ^{-1} convey information about the best linear predictor X_i given $\boldsymbol{X}_{-i} = (X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_d)$ (all elements of $\boldsymbol{X}$ except X_i). Let

$$X_i = \sum_{j \neq i} \beta_{i,j} X_j + \varepsilon_i,$$

where $\{\beta_{i,j}\}$ are the coefficients of the best linear predictor. Then it can be shown that

$$\beta_{i,j} = -\frac{\Sigma^{ij}}{\Sigma^{ii}} \quad \text{and} \quad \Sigma^{ii} = \frac{1}{\mathrm{E}[X_i - \sum_{j \neq i} \beta_{i,j} X_j]^2}. \tag{4.24}$$

The proof uses the same arguments as those in (4.20).

Therefore, we see that

$$\beta_{ij} = \rho_{ij} \sqrt{\frac{\Sigma^{jj}}{\Sigma^{ii}}}. \tag{4.25}$$

Exercise 4.3 *By using the decomposition*

$$\mathrm{var}(\boldsymbol{X}) = \Sigma = \begin{pmatrix} \Sigma_1 & \underset{\sim}{c}_1 \\ \underset{\sim}{c}_1' & \Sigma_{-(1)} \end{pmatrix} \tag{4.26}$$

where $\Sigma_1 = \mathrm{var}(X_1)$*,* $\underset{\sim}{c}_1 = \mathrm{E}[X_1 \boldsymbol{X}'_{-1}]$ *and* $\Sigma_{-(1)} = \mathrm{var}[\boldsymbol{X}_{-1}]$ *prove (4.24).*

The Cholesky decomposition and the precision matrix

We now represent the precision matrix through its Cholesky decomposition. It should be mentioned that Mohsen Pourahmadi has done a lot of interesting research in this area and he recently wrote a review paper, which can be found here.

We define the sequence of linear equations

$$X_t = \sum_{j=1}^{t-1} \beta_{t,j} X_j + \varepsilon_t, \quad t = 2, \ldots, k, \tag{4.27}$$

where $\{\beta_{t,j}; 1 \leq j \leq t-1\}$ are the coefficeints of the best linear predictor of X_t given $X_1, \ldots, X_{t-1}$.

Let $\sigma_t^2 = \mathrm{var}[\varepsilon_t] = \mathrm{E}[X_t - \sum_{j=1}^{\boldsymbol{t-1}} \beta_{t,j} X_j]^2$ and $\sigma_1^2 = \mathrm{var}[X_1]$. We standardize (4.27) and define

$$\sum_{j=1}^{t} \gamma_{t,j} X_j = \frac{1}{\sigma_t}\left(X_t - \sum_{j=1}^{\boldsymbol{t-1}} \beta_{t,j} X_j\right), \tag{4.28}$$

where we set $\gamma_{t,t} = 1/\sigma_t$ and for $1 \leq j < t-1$, $\gamma_{t,j} = -\beta_{t,j}/\sigma_i$. By construction it is clear that $\mathrm{var}(L\underline{X}) = I_k$, where

$$L = \begin{pmatrix} \gamma_{1,1} & 0 & 0 & \ldots & 0 & 0 \\ \gamma_{2,1} & \gamma_{2,2} & 0 & \ldots & 0 & 0 \\ \gamma_{3,1} & \gamma_{3,2} & \gamma_{3,3} & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \gamma_{k,1} & \gamma_{k,2} & \gamma_{k,3} & \cdots & \gamma_{k,k-1} & \gamma_{k,k} \end{pmatrix} \tag{4.29}$$

and $LL = \Sigma^{-1}$ (see Pourahmadi, equation (18)), where $\Sigma = \mathrm{var}(\boldsymbol{X}_k)$. Let $\Sigma = \mathrm{var}[\boldsymbol{X}_k]$, then

$$\Sigma^{ij} = \sum_{s=1}^{k} \gamma_{is}\gamma_{js} \qquad \text{(note many of the elements will be zero).}$$

Remark 4.2.2 (The Cholesky decomposition of a matrix) *All positive definite matrices admit a Cholesky decomposition. That is $H'H = Sigma$, where H is a lower triangular matrix. Similarly, $Sigma^{-1} = LL'$, where L is a lower triangular matrix and $L = H^{-1}$. Therefore we observe that if $\Sigma = \mathrm{var}(\underline{X})$ (where $\underline{X}$ is a p-dimension random vector), then*

$$\mathrm{var}\left(L\underline{X}\right) = L'\Sigma L = L'H'HL = I_p.$$

Therefore, the lower triangular matrix L "finds" a linear combination of the elements $\underline{X}$ such that the resulting random vector is uncorrelated.

We use apply these results to the analysis of the partial correlations of autoregressive processes and the inverse of its variance/covariance matrix.

4.3 Partial correlation in time series

The partial covariance/correlation of a time series is defined in a similar way.

Definition 4.3.1 *The partial covariance/correlation between X_t and X_{t+k+1} is defined as the partial covariance/correlation between X_t and X_{t+k+1} after conditioning out the 'inbetween' time series $X_{t+1},\ldots,X_{t+k}$.*

We now obtain an expression for the partial correlation between X_t and X_{t+k+1} in terms of their autocovariance function (for the final result see equation (4.30)). As the underlying assumption is that the time series is stationary it is the same as the partial covariance/correlation X_{k+1} and X_0. In Chapter 6 we will introduce the idea of linear predictor of a future time point given the present and the past (usually called forecasting) this can be neatly described using the idea of projections onto subspaces. This notation is quite succinct, therefore we derive an expression for the partial correlation using projection notation. The projection of X_{k+1} onto the space spanned by $\boldsymbol{X}_k = (X_1, X_2, \ldots, X_k)$, is the best linear predictor of X_{k+1} given $\boldsymbol{X}_k$. We will denote the projection of X_k onto the space spanned by $X_1, X_2, \ldots, X_k$ as $P_{\boldsymbol{X}_k}(X_{k+1})$ (note that this is the same as the best linear predictor). Thus

$$P_{\boldsymbol{X}_k}(X_{k+1}) = \boldsymbol{X}_k'(\text{var}[\boldsymbol{X}_k]^{-1}\text{E}[X_{k+1}\boldsymbol{X}_k])^{-1} = \boldsymbol{X}_k'\Sigma_k^{-1}\boldsymbol{c}_k := \sum_{j=1}^{k}\phi_{k,j}X_j,$$

where $\Sigma_k = \text{var}(\boldsymbol{X}_k)$ and $\boldsymbol{c}_k = \text{E}(X_{k+1}\boldsymbol{X}_k)$. To derive a similar expression for $P_{\boldsymbol{X}_k}(X_0)$ we use the stationarity property

$$\begin{aligned}
P_{\boldsymbol{X}_k}(X_0) &= \boldsymbol{X}_k'(\text{var}[\boldsymbol{X}_k]^{-1}\text{E}[X_0\boldsymbol{X}_k]) \\
&= \boldsymbol{X}_k'(\text{var}[\boldsymbol{X}_k]^{-1}E_k\text{E}[X_{k+1}\boldsymbol{X}_k]) \\
&= \boldsymbol{X}_k'\Sigma_k^{-1}E_k\boldsymbol{c}_k = \boldsymbol{X}_k'E_k\Sigma_k^{-1}\boldsymbol{c}_k := \sum_{j=1}^{k}\phi_{k,k+1-j}X_j,
\end{aligned}$$

where E_k is a matrix which swops round all the elements in a vector

$$E_k = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & 0 & \ldots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ 1 & 0 & \vdots & 0 & 0 & 0 \end{pmatrix}.$$

Thus the partial correlation between X_t and X_{t+k} (where $k > 0$) is the correlation $X_0 - P_{\boldsymbol{X}_k}(X_0)$

and $X_{k+1} - P_{\mathbf{X}_k}(X_{k+1})$, some algebra gives

$$\begin{aligned}
\mathrm{cov}(X_{k+1} - P_{\mathbf{X}_k}(X_{k+1}), X_0 - P_{\mathbf{X}_k}(X_0)) &= \mathrm{cov}(X_{k+1}X_0) - \underline{c}_k'\Sigma_k^{-1}E_k\underline{c}_k \qquad (4.30)\\
\Rightarrow \mathrm{cor}(X_{k+1} - P_{\mathbf{X}_k}(X_{k+1}), X_0 - P_{\mathbf{X}_k}(X_0)) &= \frac{\mathrm{cov}(X_{k+1}X_0) - \underline{c}_k'\Sigma_k^{-1}E_k\underline{c}_k}{\mathrm{var}[X_k - P_{\mathbf{X}_k}(X_0)]}.
\end{aligned}$$

We use this expression later to show that the partial correlations is also the last coefficient for the best linear predictor of X_{k+1} given $\underline{X}_k$. Note this can almost be seen from equation (4.25) i.e. $\beta_{t+1,1} = \rho_{t+1,1}\sqrt{\frac{\Sigma^{t+1,t+1}}{\Sigma^{1,1}}}$, however the next step is to show that $\Sigma^{t+1,t+1} = \Sigma^{1,1}$ (however this can be reasoned by using (4.18)).

We consider an example.

Example 4.3.1 (The PACF of an AR(1) process) *Consider the causal AR(1) process $X_t = 0.5X_{t-1} + \varepsilon_t$ where $\mathrm{E}(\varepsilon_t) = 0$ and $\mathrm{var}(\varepsilon_t) = 1$. Using (4.1) it can be shown that $\mathrm{cov}(X_t, X_{t-2}) = 2\times 0.5^2$ (compare with the MA(1) process $X_t = \varepsilon_t + 0.5\varepsilon_{t-1}$, where the covariance $\mathrm{cov}(X_t, X_{t-2}) = 0$). We evaluate the partial covariance between X_t and X_{t-2}. Remember we have to 'condition out' the random variables inbetween, which in this case is X_{t-1}. It is clear that the projection of X_t onto X_{t-1} is $0.5X_{t-1}$ (since $X_t = 0.5X_{t-1} + \varepsilon_t$). Therefore $X_t - P_{\bar{sp}(X_{t-1})}X_t = X_t - 0.5X_{t-1} = \varepsilon_t$. The projection of X_{t-2} onto X_{t-1} is a little more complicated, it is $P_{\bar{sp}(X_{t-1})}X_{t-2} = \frac{\mathrm{E}(X_{t-1}X_{t-2})}{\mathrm{E}(X_{t-1}^2)}X_{t-1}$. Therefore the partial correlation between X_t and X_{t-2}*

$$\mathrm{cov}\left(X_t - P_{X_{t-1}}X_t, X_{t-2} - P_{X_{t-1})}X_{t-2}\right) = \mathrm{cov}\left(\varepsilon_t, X_{t-2} - \frac{\mathrm{E}(X_{t-1}X_{t-2})}{\mathrm{E}(X_{t-1}^2)}X_{t-1}\right) = 0.$$

In fact the above is true for the partial covariance between X_t and X_{t-k}, for all $k \geq 2$. Hence we see that despite the covariance not being zero for the autocovariance of an AR process greater than order two, the partial covariance is zero for all lags greater than or equal to two.

Using the same argument as above, it is easy to show that partial covariance of an AR(p) for lags greater than p is zero. Hence in may respects the partial covariance can be considered as an analogue of the autocovariance. It should be noted that though the covariance of MA(q) is zero for lag greater than q, the same is not true for the parial covariance. Whereas partial covariances removes correlation for autoregressive processes it seems to 'add' correlation for moving average processes!

Model identification:

- If the autocovariances after a certain lag are zero q, it may be appropriate to fit an MA(q) model to the time series.

  On the other hand, the autocovariances of any AR(p) process will only decay to zero as the lag increases.

- If the partial autocovariances after a certain lag are zero p, it may be appropriate to fit an AR(p) model to the time series.

  On the other hand, the partial covariances of any MA(p) process will only decay to zero as the lag increases.

Exercise 4.4 (The partial correlation of an invertible MA(1)) *Let $\phi_{t,t}$ denote the partial correlation between X_{t+1} and X_1. It is well known (this is the Levinson-Durbin algorithm, which we cover in Chapter 6) that $\phi_{t,t}$ can be deduced recursively from the autocovariance funciton using the algorithm:*

Step 1 $\phi_{1,1} = c(1)/c(0)$ *and* $r(2) = \mathrm{E}[X_2 - X_{2|1}]^2 = \mathrm{E}[X_2 - \phi_{1,1}X_1]^2 = c(0) - \phi_{1,1}c(1)$.

Step 2 *For* $j = t$

$$\begin{aligned}
\phi_{t,t} &= \frac{c(t) - \sum_{j=1}^{t-1}\phi_{t-1,j}c(t-j)}{r(t)} \\
\phi_{t,j} &= \phi_{t-1,j} - \phi_{t,t}\phi_{t-1,t-j} \qquad 1 \leq j \leq t-1, \\
\textit{and } r(t+1) &= r(t)(1-\phi_{t,t}^2).
\end{aligned}$$

(i) Using this algorithm and induction to show that the PACF of the MA(1) process $X_t = \varepsilon_t + \theta\varepsilon_{t-1}$, where $|\theta| < 1$ (so it is invertible) is

$$\phi_{t,t} = \frac{(-1)^{t+1}(\theta)^t(1-\theta^2)}{1-\theta^{2(t+1)}}.$$

(ii) Explain how this partial correlation is similar to the ACF of the AR(1) model $X_t = -\theta X_{t-1} + \varepsilon_t$.

Exercise 4.5 (Comparing the ACF and PACF of an AR process) *Compare the below plots:*

(i) Compare the ACF and PACF of the AR(2) model $X_t = 1.5X_{t-1} - 0.75X_{t-2} + \varepsilon_t$ using `ARIMAacf(ar=c(1.5,-0.75),ma=0,30)` *and* `ARIMAacf(ar=c(1.5,-0.75),ma=0,pacf=T,30)`.

(ii) Compare the ACF and PACF of the MA(1) *model* $X_t = \varepsilon_t - 0.5\varepsilon_t$ *using* `ARIMAacf(ar=0,ma=c(-1.5),30)` *and* `ARIMAacf(ar=0,ma=c(-1.5),pacf=T,30)`.

(ii) Compare the ACF and PACF of the ARMA$(2,1)$ *model* $X_t - 1.5X_{t-1} + 0.75X_{t-2} = \varepsilon_t - 0.5\varepsilon_t$ *using* `ARIMAacf(ar=c(1.5,-0.75),ma=c(-1.5),30)` *and* `ARIMAacf(ar=c(1.5,0.75),ma=c(-1.5),pacf=T,30)`.

Exercise 4.6 *Compare the ACF and PACF plots of the monthly temperature data from 1996-2014. Would you fit an AR, MA or ARMA model to this data?*

`Rcode`

The sample partial autocorrelation of a time series can be obtained using the command `pacf`. However, remember just because the sample PACF is not zero, does not mean the true PACF is non-zero. This is why we require the error bars. In Section 7.4 we show how these error bars are derived. The surprisingly result is that the error bars of a PACF can be used "quite" reliably to determine the order of an AR(p) process. We will use Remark 4.3.1 to show that if the order of the autoregressive process is p the for lag $r > p$, the partial correlation is such that $\widehat{\phi}_{rr} = N(0, n^{-1/2})$ (thus giving rise to the $[-1.96n^{-1/2}, 1.96n^{-1/2}]$ error bars). However, it should be noted that there will still be correlation between the sample partial correlations. The surprising result, is that the error bars for an ACF plot *cannot* be reliably used to determine the order of an MA(q) model.

4.3.1 The variance/covariance matrix and precision matrix of an autoregressive and moving average process

Let us suppose that $\{X_t\}$ is a stationary time series. In this section we consider the variance/covariance matrix $\text{var}(\underline{X}_k) = \Sigma_k$, where $\boldsymbol{X}_k = (X_1, \ldots, X_k)'$. We will consider two cases (i) when X_t follows an MA(p) models and (ii) when X_t follows an AR(p) model. The variance and inverse of the variance matrices for both cases yield quite interesting results. We will use classical results from multivariate analysis, stated in Section 4.2.

We recall that the variance/covariance matrix of a stationary time series has a (symmetric)

Toeplitz structure (see wiki for a definition). Let $\boldsymbol{X}_k = (X_1, \ldots, X_k)'$, then

$$\Sigma_k = \mathrm{var}(\boldsymbol{X}_k) = \begin{pmatrix} c(0) & c(1) & 0 & \ldots & c(k-2) & c(k-1) \\ c(1) & c(0) & c(1) & \ldots & c(k-3) & c(k-2) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ c(k-1) & c(k-2) & \vdots & \ldots & c(1) & c(0) \end{pmatrix}.$$

Σ_k for AR(p) and MA(p) models

(i) If $\{X_t\}$ satisfies an MA(p) model and $k > p$, then Σ_k will be bandlimited, where p off-diagonals above and below the diagonal will be non-zero and the rest of the off-diagonal will be zero.

(ii) If $\{X_t\}$ satisfies an AR(p) model, then Σ_k will not be bandlimited.

Σ_k^{-1} for an AR(p) model

We now consider the inverse of Σ_k. Warning: note that the inverse of a Toeplitz is not necessarily Toeplitz (unlike the circulant which is). We use the results in Section 4.2. Suppose that we have an AR(p) process and we consider the precision matrix of $\boldsymbol{X}_k = (X_1, \ldots, X_k)$, where $k > p$.

Recall the (i,j)th element of Σ_k^{-1} divided by the square roots of the corresponding diagonals is the negative partial correlation of between X_i and X_j conditioned on all the elements in $\boldsymbol{X}_k$. In Section 4.3 we showed that if $|i-j| > p$, then the partial correlation between X_i and X_j given $X_{i+1}, \ldots, X_{j-1}$ (assuming without loss of generality that $i < j$) is zero. We now show that the precision matrix of Σ_k^{-1} will be bandlimited (note that it is not immediate obvious since Σ_k^{ij} is the negative partial correlation between X_i and X_j given $\mathbf{X}_{-(ij)}$ not just the elements between X_i and X_j). To show this we use the Cholesky decomposition given in (4.27). Since X_t is an autoregressive process of order p and plugging this information into (4.27), for $t > p$ we have

$$X_t = \sum_{j=1}^{t-1} \beta_{t,j} X_j + \varepsilon_t = \sum_{j=1}^{p} \phi_j X_{t-j} + \varepsilon_t,$$

thus $\beta_{t,t-j} = \phi_j$ for $1 \leq j \leq p$ otherwise $\beta_{t,t-j} = 0$. Moreover, for $t > p$ we have $\sigma_t^2 = \mathrm{var}(\varepsilon_t) = 1$. For $t \leq p$ we use the same notation as that used in (4.27). This gives the lower triangular p-

bandlimited matrix

$$L_k = \begin{pmatrix} \gamma_{1,1} & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ \gamma_{2,1} & \gamma_{2,2} & \ldots & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -\phi_p & -\phi_{p-1} & \ldots & -\phi_1 & 1 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & -\phi_p & -\phi_{p-1} & \ldots & -\phi_1 & 1 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & -\phi_p & \ldots & -\phi_2 & -\phi_1 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 1 \end{pmatrix} \qquad (4.31)$$

(the above matrix has not been formated well, but after the first $p-1$ rows, there are ones along the diagonal and the p lower off-diagonals are non-zero).

We recall that $\Sigma_k^{-1} = L_k L_k'$, thus we observe that since L_k is a lower triangular bandlimited matrix, $\Sigma_k^{-1} = L_k L_k'$ is a bandlimited matrix with the p off-diagonals either side of the diagonal non-zero. Let Σ^{ij} denote the (i,j)th element of Σ_k^{-1}. Then we observe that $\Sigma^{(i,j)} = 0$ if $|i-j| > p$. Moreover, if $0 < |i-j| \leq p$ and either i or j is greater than p, then $\Sigma^{ij} = 2\sum_{k=|i-j|}^{p} \phi_k \phi_{k-|i-j|+1} - \phi_{|i-j|}$.

The coefficients $\Sigma^{(i,j)}$ gives us a fascinating insight into the prediction of X_t given the past and future observations. We recall from equation (4.24) that $-\Sigma^{ij}/\Sigma^{ii}$ are the coffficients of the best linear predictor of X_i given $\boldsymbol{X}_{-i}$. This result tells if the observations came from a stationary AR(p) process, then the best linear predictor of X_i given $X_{i-1}, \ldots, X_{i-a}$ and $X_{i+1}, \ldots, X_{i+b}$ (where a and $b > p$) is the same as the best linear predictor of X_i given $X_{i-1}, \ldots, X_{i-p}$ and $X_{i+1}, \ldots, X_{i+p}$ (knowledge of other values will not improve the prediction).

Exercise 4.7 *Suppose that the time series $\{X_t\}$ has the causal AR(2) representation*

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \varepsilon_t.$$

Let $\underline{X}_n' = (X_1, \ldots, X_n)$ and $\Sigma_n = \mathrm{var}(\underline{X}_n)$. Suppose $L_n L_n' = Sigma_n^{-1}$, where L_n is a lower triangular matrix.

What does L_n looks like?

Remark 4.3.1 *Suppose that X_t is an autoregressive process $X_t = \sum_{j=1}^{p} \phi_j X_{t-j} + \varepsilon_t$ where $\mathrm{var}[\varepsilon_t] = \sigma^2$ and $\{\varepsilon_t\}$ are uncorrelated random variables with zero mean. Let $\Sigma_m = \mathrm{var}[\boldsymbol{X}_m]$ where $\boldsymbol{X}_m = (X_1, \ldots, X_m)$. If $m > p$ then*

$$\left[\Sigma_m^{-1}\right]_{mm} = \Sigma^{mm} = \sigma^{-2}$$

and $\det(\Sigma_m) = \det(\Sigma_p)\sigma^{2(m-p)}$.

Exercise 4.8 *Prove Remark 4.3.1.*

4.4 Correlation and non-causal time series

Here we demonstrate that it is not possible to identify whether a process is noninvertible/noncausal from its covariance structure. The simplest way to show result this uses the spectral density function, which will now define and then return to and study in depth in Chapter 9.

Definition 4.4.1 (The spectral density) *Given the covariances $c(k)$ (with $\sum_k |c(k)|^2 < \infty$) the spectral density function is defined as*

$$f(\omega) = \sum_k c(k) \exp(ik\omega).$$

The covariances can be obtained from the spectral density by using the inverse fourier transform

$$c(k) = \frac{1}{2\pi} \int_0^{2\pi} f(\omega) \exp(-ik\omega).$$

Hence the covariance yields the spectral density and visa-versa.

For reference below, we point out that the spectral density function uniquely identifies the autocovariance function.

Let us suppose that $\{X_t\}$ satisfies the AR(p) representation

$$X_t = \sum_{i=1}^{p} \phi_i X_{t-i} + \varepsilon_t$$

where $\mathrm{var}(\varepsilon_t) = 1$ and the roots of $\phi(z) = 1 - \sum_{j=1}^{p} \phi_j z^j$ can lie inside and outside the unit circle, but not on the unit circle (thus it has a stationary solution). We will show in Chapter 9 that the spectral density of this AR process is

$$f(\omega) = \frac{1}{|1 - \sum_{j=1}^{p} \phi_j \exp(ij\omega)|^2}. \qquad (4.32)$$

- Factorizing $f(\omega)$.

  Let us supose the roots of the characteristic polynomial $\phi(z) = 1 + \sum_{j=1}^{q} \phi_j z^j$ are $\{\lambda_j\}_{j=1}^{p}$, thus we can factorize $\phi(x)$ $1 + \sum_{j=1}^{p} \phi_j z^j = \prod_{j=1}^{p}(1 - \lambda_j z)$. Using this factorization we have (4.32) can be written as

  $$f(\omega) \;=\; \frac{1}{\prod_{j=1}^{p} |1 - \lambda_j \exp(i\omega)|^2}. \qquad (4.33)$$

  As we have not assumed $\{X_t\}$ is causal, the roots of $\phi(z)$ can lie both inside and outside the unit circle. We separate the roots, into those outside the unit circle $\{\lambda_{O,j_1}; j_1 = 1, \ldots, p_1\}$ and inside the unit circle $\{\lambda_{I,j_2}; j_2 = 1, \ldots, p_2\}$ $(p_1 + p_2 = p)$. Thus

  $$\begin{aligned} \phi(z) &= [\prod_{j_1=1}^{p_1} (1 - \lambda_{O,j_1} z)][\prod_{j_2=1}^{p_2} (1 - \lambda_{I,j_2} z)] \\ &= (-1)^{p_2} \lambda_{I,j_2} z^{-p_2} [\prod_{j_1=1}^{p_1} (1 - \lambda_{O,j_1} z)][\prod_{j_2=1}^{p_2} (1 - \lambda_{I,j_2}^{-1} z)]. \qquad (4.34) \end{aligned}$$

  Thus we can rewrite the spectral density in (4.35)

  $$f(\omega) \;=\; \frac{1}{\prod_{j_2=1}^{p_2} |\lambda_{I,j_2}|^2} \frac{1}{\prod_{j_1=1}^{p_1} |1 - \lambda_{O,j} \exp(i\omega)|^2 \prod_{j_2=1}^{p_2} |1 - \lambda_{I,j_2}^{-1} \exp(i\omega)|^2}. \qquad (4.35)$$

  Let

  $$f_O(\omega) \;=\; \frac{1}{\prod_{j_1=1}^{p_1} |1 - \lambda_{O,j} \exp(i\omega)|^2 \prod_{j_2=1}^{p_2} |1 - \lambda_{I,j_2}^{-1} \exp(i\omega)|^2}.$$

  Then $f(\omega) = \prod_{j_2=1}^{p_2} |\lambda_{I,j_2}|^{-2} f_O(\omega)$.

- A parallel causal AR(p) process with the same covariance structure always exists.

  We now define a process which has the same autocovariance function as $\{X_t\}$ but is causal.

Using (4.34) we define the polynomial

$$\widetilde{\phi}(z) = [\prod_{j_1=1}^{p_1}(1-\lambda_{O,j_1}z)][\prod_{j_2=1}^{p_2}(1-\lambda_{I,j_2}^{-1}z)]. \tag{4.36}$$

By construction, the roots of this polynomial lie outside the unit circle. We then define the AR(p) process

$$\widetilde{\phi}(B)\widetilde{X}_t = \varepsilon_t, \tag{4.37}$$

from Lemma 3.3.1 we know that $\{\widetilde{X}_t\}$ has a stationary, almost sure unique solution. Moreover, because the roots lie outside the unit circle the solution is causal.

By using (4.32) the spectral density of $\{\widetilde{X}_t\}$ is $\widetilde{f}(\omega)$. We know that the spectral density function uniquely gives the autocovariance function. Comparing the spectral density of $\{\widetilde{X}_t\}$ with the spectral density of $\{X_t\}$ we see that they both are the same up to a multiplicative constant. Thus they both have the same autocovariance structure up to a multiplicative constant (which can be made the same, if in the definition (4.37) the innovation process has variance $\prod_{j_2=1}^{p_2}|\lambda_{I,j_2}|^{-2}$).

Therefore, for every non-causal process, there exists a causal process with the same autocovariance function.

By using the same arguments above, we can generalize to result to ARMA processes.

Definition 4.4.2 *An ARMA process is said to have minimum phase when the roots of $\phi(z)$ and $\theta(z)$ both lie outside of the unit circle.*

Remark 4.4.1 *For Gaussian random processes it is impossible to discriminate between a causal and non-causal time series, this is because the mean and autocovariance function uniquely identify the process.*

However, if the innovations are non-Gaussian, even though the autocovariance function is 'blind' to non-causal processes, by looking for other features in the time series we are able to discriminate between a causal and non-causal process.

4.4.1 The Yule-Walker equations of a non-causal process

Once again let us consider the zero mean AR(p) model

$$X_t = \sum_{j=1}^{p} \phi_j X_{t-j} + \varepsilon_t,$$

and $\text{var}(\varepsilon_t) < \infty$. Suppose the roots of the corresponding characteristic polynomial lie outside the unit circle, then $\{X_t\}$ is strictly stationary where the solution of X_t is only in terms of past and present values of $\{\varepsilon_t\}$. Moreover, it is second order stationary with covariance $\{c(k)\}$. We recall from Section 4.1.2, equation (4.4) that we derived the Yule-Walker equations for causal AR(p) processes, where

$$\text{E}(X_t X_{t-k}) \;=\; \sum_{j=1}^{p} \phi_j \text{E}(X_{t-j} X_{t-k}) \Rightarrow c(k) - \sum_{j=1}^{p} \phi_j c(k-j) = 0. \tag{4.38}$$

Let us now consider the case that the roots of the characteristic polynomial lie both outside and inside the unit circle, thus X_t does not have a causal solution but it is still strictly and second order stationary (with autocovariance, say $\{c(k)\}$). In the previous section we showed that there exists a causal AR(p) $\widetilde{\phi}(B)\widetilde{X}_t = \varepsilon_t$ (where $\phi(B)$ and $\widetilde{\phi}(B) = 1 - \sum_{j=1}^{p} \tilde{\phi}_j z^j$ are the characteristic polynomials defined in (4.34) and (4.36)). We showed that both have the same autocovariance structure. Therefore,

$$c(k) - \sum_{j=1}^{p} \tilde{\phi}_j c(k-j) = 0$$

This means the Yule-Walker equations for $\{X_t\}$ would actually give the AR(p) coefficients of $\{\tilde{X}_t\}$. Thus if the Yule-Walker equations were used to estimate the AR coefficients of $\{X_t\}$, in reality we would be estimating the AR coefficients of the corresponding causal $\{\tilde{X}_t\}$.

4.4.2 Filtering non-causal AR models

Here we discuss the surprising result that filtering a non-causal time series with the corresponding causal AR parameters leaves a sequence which is uncorrelated but not independent. Let us suppose that

$$X_t = \sum_{j=1}^{p} \phi_j X_{t-j} + \varepsilon_t,$$

where ε_t are iid, $\mathrm{E}(\varepsilon_t) = 0$ and $\mathrm{var}(\varepsilon_t) < \infty$. It is clear that given the input X_t, if we apply the filter $X_t - \sum_{j=1}^{p} \phi_j X_{t-j}$ we obtain an iid sequence (which is $\{\varepsilon_t\}$).

Suppose that we filter $\{X_t\}$ with the causal coefficients $\{\widetilde{\phi}_j\}$, the output $\widetilde{\varepsilon}_t = X_t - \sum_{j=1}^{p} \widetilde{\phi}_j X_{t-j}$ is not an independent sequence. However, it is an *uncorrelated sequence*. We illustrate this with an example.

Example 4.4.1 *Let us return to the AR(1) example, where $X_t = \phi X_{t-1} + \varepsilon_t$. Let us suppose that $\phi > 1$, which corresponds to a non-causal time series, then X_t has the solution*

$$X_t = -\sum_{j=1}^{\infty} \frac{1}{\phi^j} \varepsilon_{t+j+1}.$$

The causal time series with the same covariance structure as X_t is $\widetilde{X}_t = \frac{1}{\phi}\widetilde{X}_{t-1} + \varepsilon$ (which has backshift representation $(1 - 1/(\phi B))X_t = \varepsilon_t$). Suppose we pass X_t through the causal filter

$$\begin{aligned} \widetilde{\varepsilon}_t &= (1 - \frac{1}{\phi}B)X_t = X_t - \frac{1}{\phi}X_{t-1} = -\frac{(1 - \frac{1}{\phi}B)}{B(1 - \frac{1}{\phi B})}\varepsilon_t \\ &= -\frac{1}{\phi}\varepsilon_t + (1 - \frac{1}{\phi^2}) \sum_{j=1}^{\infty} \frac{1}{\phi^{j-1}} \varepsilon_{t+j}. \end{aligned}$$

Evaluating the covariance of the above (assuming wlog that $\mathrm{var}(\varepsilon) = 1$*) is*

$$\mathrm{cov}(\widetilde{\varepsilon}_t, \widetilde{\varepsilon}_{t+r}) = -\frac{1}{\phi}(1 - \frac{1}{\phi^2})\frac{1}{\phi^r} + (1 - \frac{1}{\phi^2})^2 \sum_{j=0}^{\infty} \frac{1}{\phi^{2j}} = 0.$$

Thus we see that $\{\widetilde{\varepsilon}_t\}$ is an uncorrelated sequence, but unless it is Gaussian it is clearly not independent. One method to study the higher order dependence of $\{\widetilde{\varepsilon}_t\}$, by considering it's higher order cumulant structure etc.

The above above result can be generalised to general AR models, and it is relatively straightforward to prove using the Crámer representation of a stationary process (see Section 9.4, Theorem **??**).

Exercise 4.9 *(i) Consider the causal AR(p) process*

$$X_t = 1.5X_{t-1} - 0.75X_{t-2} + \varepsilon_t.$$

Derive a parallel process with the same autocovariance structure but that is non-causal (it should be real).

(ii) Simulate both from the causal process above and the corresponding non-causal process with non-Gaussian innovations (see Section 3.7). Show that they have the same ACF function.

(iii) Find features which allow you to discriminate between the causal and non-causal process.

Chapter 5

Nonlinear Time Series Models

Prerequisites

- A basic understanding of expectations, conditional expectations and how one can use conditioning to obtain an expectation.

Objectives:

- Use relevant results to show that a model has a stationary, solution.
- Derive moments of these processes.
- Understand the differences between linear and nonlinear time series.

So far we have focused on linear time series, that is time series which have the representation

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j \varepsilon_{t-j}, \tag{5.1}$$

where $\{\varepsilon_t\}$ are iid random variables. Such models are extremely useful, because they are designed to model the autocovariance structure and are straightforward to use for forecasting. These are some of the reasons that they are used widely in several applications. Note that all stationary Gaussian time series have a linear form (of the type given in (5.1)), where the innovations $\{\varepsilon_t\}$ are Gaussian.

A typical realisation from a linear time series, will be quite regular with no suddent bursts or jumps. This is due to the linearity of the system. However, if one looks at financial data, for example, there are sudden bursts in volatility (variation) and extreme values, which calm down after a while. It is not possible to model such behaviour well with a linear time series. In order to capture ‘nonlinear behaviour several nonlinear models have been proposed. The models typically consists of products of random variables which make possible the sudden irratic bursts seen in the data. Over the past 30 years there has been a lot research into nonlinear time series models. Probably one of the first nonlinear models proposed for time series analysis is the bilinear model, this model is used extensively in signal processing and engineering. A popular model for modelling financial data are (G)ARCH-family of models. Other popular models are random autoregressive coefficient models and threshold models, to name but a few (see, for example, Subba Rao (1977), Granger and Andersen (1978), Nicholls and Quinn (1982), Engle (1982), Subba Rao and Gabr (1984), Bollerslev (1986), Terdik (1999), Fan and Yao (2003), Straumann (2005) and Douc et al. (2014)).

Once a model has been defined, the first difficult task is to show that it actually has a solution which is almost surely finite (recall these models have dynamics which start at the $-\infty$, so if they are not well defined they could be ‘infinite’), with a stationary solution. Typically, in the nonlinear world, we look for causal solutions. I suspect this is because the mathematics behind existence of non-causal solution makes the problem even more complex.

We state a result that gives sufficient conditions for a stationary, causal solution of a certain class of models. These models include ARCH/GARCH and Bilinear models. We note that the theorem guarantees a solution, but does not give conditions for it’s moments. The result is based on Brandt (1986), but under stronger conditions.

Theorem 5.0.1 (Brandt (1986)) *Let us suppose that $\{\boldsymbol{X}_t\}$ is a d-dimensional time series defined by the stochastic recurrence relation*

$$\boldsymbol{X}_t = A_t \boldsymbol{X}_{t-1} + \boldsymbol{B}_t, \tag{5.2}$$

where $\{A_t\}$ and $\{B_t\}$ are iid random matrices and vectors respectively. If $\mathrm{E}\log\|A_t\| < 0$ *and* $\mathrm{E}\log\|\boldsymbol{B}_t\| < \infty$ *(where $\|\cdot\|$ denotes the spectral norm of a matrix), then*

$$\boldsymbol{X}_t = \boldsymbol{B}_t + \sum_{k=1}^{\infty}\left(\prod_{i=0}^{k-1} A_{t-i}\right)\boldsymbol{B}_{t-k} \tag{5.3}$$

converges almost surely and is the unique strictly stationary causal solution.

Note: The conditions given above are very strong and Brandt (1986) states the result under *which weaker conditions, we outline the differences here. Firstly, the assumption* $\{A_t, B_t\}$ *are iid can be relaxed to their being Ergodic sequences. Secondly, the assumption* $\mathrm{E}\log\|A_t\| < 0$ *can be relaxed to* $\mathrm{E}\log\|A_t\| < \infty$[1] *and that* $\{A_t\}$ *has a negative Lyapunov exponent, where the Lyapunov exponent is defined as* $\lim_{n\to\infty}\frac{1}{n}\|\prod_{j=1}^{n} A_j\| = \gamma$*, with* $\gamma < 0$ *(see Brandt (1986)).*

The conditions given in the above theorem may appear a little cryptic. However, the condition $\mathrm{E}\log|A_t| < 0$ (in the unvariate case) becomes quite clear if you compare the SRE model with the AR(1) model $X_t = \rho X_{t-1} + \varepsilon_t$, where $|\rho| < 1$ (which is the special case of the SRE, where the coefficients is deterministic). We recall that the solution of the AR(1) is $X_t = \sum_{k=1}^{\infty}\rho^j\varepsilon_{t-j}$. The important part in this decomposition is that $|\rho^j|$ decays geometrically fast to zero. Now let us compare this to (5.3), we see that ρ^j plays a similar role to $\prod_{i=0}^{k-1} A_{t-i}$. Given that there are similarities between the AR(1) and SRE, it seems reasonable that for (5.3) to converge, $\prod_{i=0}^{k-1} A_{t-i}$ should converge geometrically too (at least almost surely). However analysis of a product is not straight forward, therefore we take logarithms to turn it into a sum

$$\frac{1}{k}\log\prod_{i=0}^{k-1} A_{t-i} = \frac{1}{k}\sum_{i=0}^{k-1}\log A_{t-i} \overset{\text{a.s.}}{\to} \mathrm{E}[\log A_t] := \gamma,$$

since it is the sum of iid random variables. Thus taking anti-logs

$$\prod_{i=0}^{k-1} A_{t-i} \approx \exp[k\gamma],$$

which only converges to zero if $\gamma < 0$, in other words $\mathrm{E}[\log A_t] < 0$. Thus we see that the condition $\mathrm{E}\log|A_t| < 0$ is quite a logical conditional afterall.

5.0.1 Examples

<u>The AR(1) model</u>

It is straightforward to see that the causal, stationary AR(1) model satisfies the conditions in Theorem 5.0.1. Observe that since

$$X_t = \phi X_{t-1} + \varepsilon_t$$

has a stationary causal solution when $|\phi| < 1$, then $\mathrm{E}[\log|\phi|] = \log|\phi| < 0$ (since $|\phi < 1|$).

[1] Usually we use the spectral norm, which is defined as the $\sqrt{\lambda_{\max}(A'A)}$

The AR(2) model

Things become a little tricker with the AR(2) case. We recall from Section 3.4.2 that the causal AR(2) model

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \varepsilon_t$$

can be written as an VAR(1) model

$$\underline{X}_t = A\underline{X}_{t-1} + \underline{\varepsilon}_t = \sum_{j=0}^{\infty} A^j \underline{\varepsilon}_{t-j} \tag{5.4}$$

where

$$\begin{pmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{pmatrix}, \tag{5.5}$$

and $\underline{\varepsilon}_t' = (\varepsilon_t, 0)$. For the process to be causal the eigenvalues of the matrix A should be less than one. Thus in the above example $\phi_1 = a + b$ and $\phi_2 = -ab$, for some $|a|, |b| < 1$. This implies that the eigenvalues of A will be less than one, however the eigenvalues of $A'A$ may not be less than one. For example consider the AR(2) model

$$X_t = 2 \times 0.2 X_{t-1} - 0.2^2 X_{t-2} + \varepsilon_t$$

which correspond to A with the eigenvalues 0.2 and 0.2.

```
A = matrix(c(2*phi,-phi**2,1,0),byrow =T, ncol = 2)
>eigen(A)
eigen() decomposition
$values
[1] 0.2+0i 0.2-0i
> eigen(A%*%t(A))
eigen() decomposition
$values
[1] 1.160220952 0.001379048
```

From the code above, we see that the spectral radius of A (largest eigenvalue of A) is 0.2, but

$\|A\|_{spec} = 1.16$. However, if we evaluate the spectral norm of A^2, it is less than one;

```
> A2 = A%*%A
> eigen(A2%*%t(A2))
eigen() decomposition
$values
[1] 1.762415e-01 1.452553e-05
```

In this example we see that $\|A^2\|_{spec} = \sqrt{0.176}$. Since we can group the product A^k into the products of A^2, this is what gives the contraction. This will happen for any matrix, A, whose eigenvalues are less than one. For a large enough k, the spectral norm of A^k will be less than one[2]. Therefore the conditions of Theorem 5.0.1 are not satisfied. But the weaker conditions (given below the main conditions of the theorem) is satisfied.

5.1 Data Motivation

5.1.1 Yahoo data from 1996-2014

We consider here the closing share price of the Yahoo daily data downloaded from `https://uk.finance.yahoo.com/q/hp?s=YHOO`. The data starts from from 10th April 1996 to 8th August 2014 (over 4000 observations). A plot is given in Figure 5.1. Typically the logarithm of such data taken, and in order to remove linear and/or stochastic trend the first difference of the logarithm is taken (ie. $X_t = \log S_t - \log S_{t-1}$). The hope is that after taking differences the data has been stationarized (see Example 3.6). However, the data set spans almost 20 years and this assumption is rather precarious and will be investigated later. A plot of the data after taking first differences together with the QQplot is given in Figure 5.2. From the QQplot in Figure 5.2, we observe that log differences $\{X_t\}$ appears to have very thick tails, which may mean that higher order moments of the log returns do not exist (not finite).

In Figure 5.3 we give the autocorrelation (ACF) plots of the log differences, absolute log differences and squares of the log differences. Note that the sample autocorrelation is defined as

$$\widehat{\rho}(k) = \frac{\widehat{c}(k)}{\widehat{c}(0)}, \quad \text{where} \quad \widehat{c}(k) = \frac{1}{T}\sum_{t=1}^{T-|k|}(X_t - \bar{X})(X_{t+k} - \bar{X}). \tag{5.6}$$

[2] This result is due to Gelfand's lemma

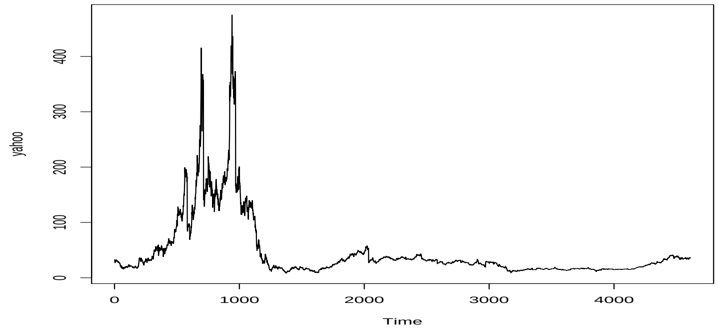


Figure 5.1: Plot of daily closing Yahoo share price 1996-2014

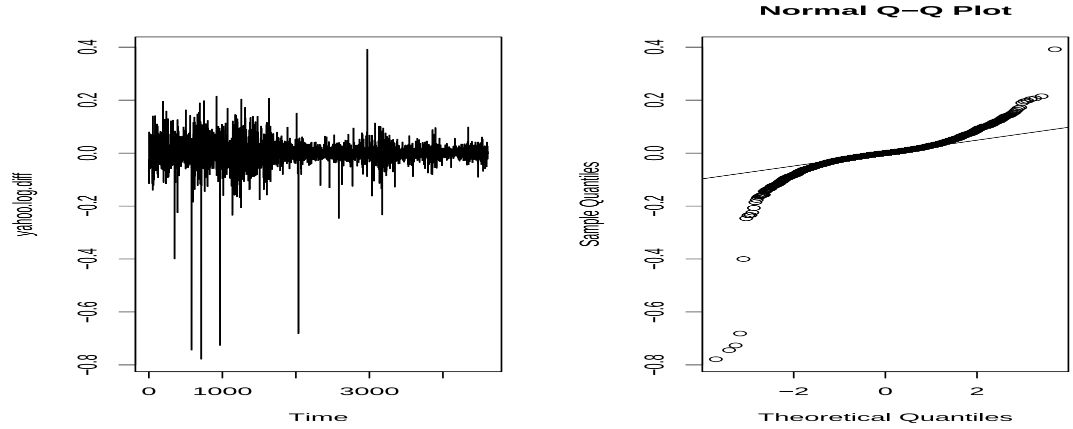
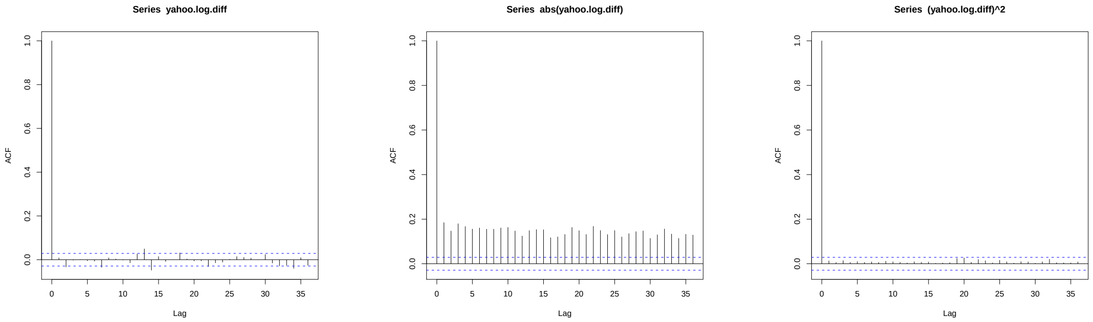


Figure 5.2: Plot of log differences of daily Yahoo share price 1996-2014 and the corresponding QQplot

The dotted lines are the errors bars (the 95% confidence of the sample correlations constructed under the assumption the observations are independent, see Section 7.2.1). From Figure 5.3a we see that there appears to be no correlation in the data. More precisely, most of the sample correlations are within the errors bars, the few that are outside it could be by chance, as the error bars are constructed pointwise. However, Figure 5.3b the ACF plot of the absolutes gives significant large correlations. In contrast, in Figure 5.3c we give the ACF plot of the squares, where there does not appear to be any significant correlations.

To summarise, $\{X_t\}$ appears to be uncorrelated (white noise). However, once absolutes have been taken there does appear to be dependence. This type of behaviour cannot be modelled with a linear model. What is quite interesting is that there does not appear to be any significant correlation in the squares. However, on explanation for this can be found in the QQplot. The data has extremely thick tails which suggest that the forth moment of the process may not exist



(a) ACF plot of the log differences

(b) ACF plot of the absolute of the log differences

(c) ACF plot of the square of the log differences

Figure 5.3: ACF plots of the transformed Yahoo data

(the empirical variance of X_t will be extremely large). Since correlation is defined as (5.6) involves division by $\widehat{c}(0)$, which could be extremely large, this would 'hide' the sample covariance.

`R` code for Yahoo data

Here we give the `R` code for making the plots above.

```
yahoo <- scan("~/yahoo304.96.8.14.txt")
yahoo <- yahoo[c(length(yahoo):1)] # switches the entries to ascending order 1996-2014
yahoo.log.diff <- log(yahoo[-1]) - log(yahoo[-length(yahoo)])
# Takelog differences
par(mfrow=c(1,1))
plot.ts(yahoo)
par(mfrow=c(1,2))
plot.ts(yahoo.log.diff)
qqnorm(yahoo.log.diff)
qqline(yahoo.log.diff)
par(mfrow=c(1,3))
acf(yahoo.log.diff) # ACF plot of log differences
acf(abs(yahoo.log.diff)) # ACF plot of absolute log differences
acf((yahoo.log.diff)**2) # ACF plot of square of log differences
```

5.1.2 FTSE 100 from January - August 2014

For completeness we discuss a much shorter data set, the daily closing price of the FTSE 100 from 20th January - 8th August, 2014 (141 observations). This data was downloaded from `http://markets.ft.com/research//Tearsheets/PriceHistoryPopup?symbol=FTSE:FSI`.

Exactly the same analysis that was applied to the Yahoo data is applied to the FTSE data and the plots are given in Figure 5.4-5.6.

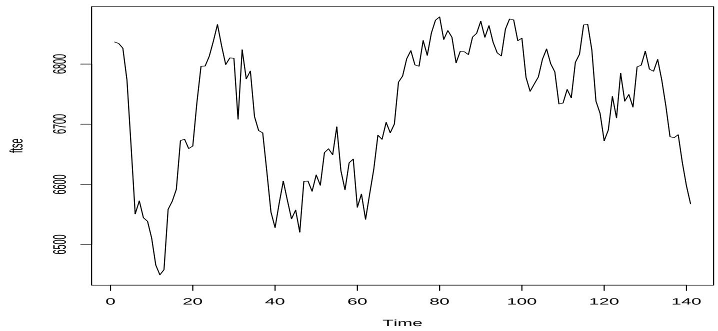


Figure 5.4: Plot of daily closing FTSE price Jan-August, 2014

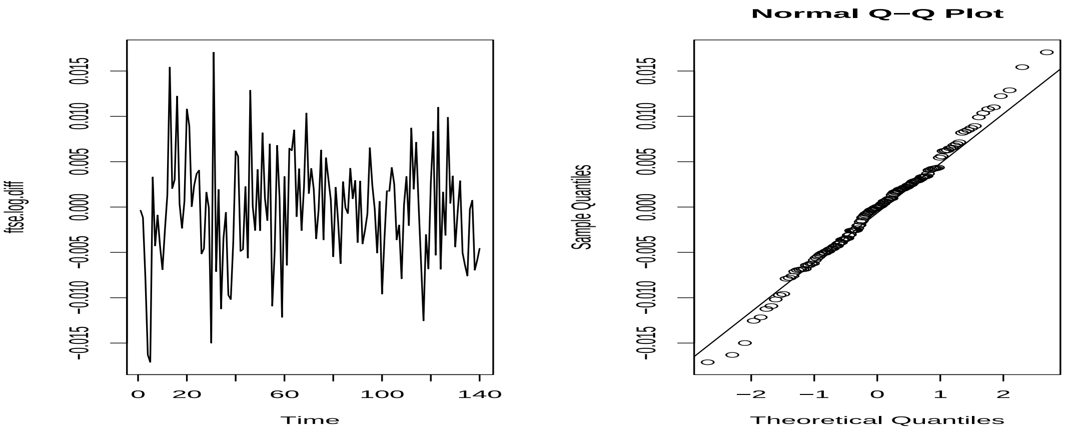
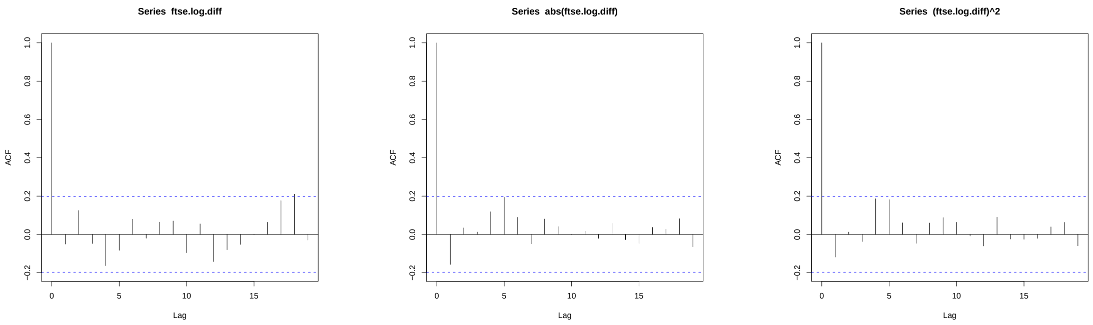


Figure 5.5: Plot of log differences of daily FTSE price Jan-August, 2014 and the corresponding QQplot

We observe that for this (much shorter) data set, the marginal observations do not appear to deviate much from normality (note just because the marginal is Gaussian does not mean the entire time series is Gaussian). Furthermore, the ACF plot of the log differences, absolutes and squares do not suggest any evidence of correlation. Could it be, that after taking log differences, there is



(a) ACF plot of the log differences

(b) ACF plot of the absolute of the log differences

(c) ACF plot of the square of the log differences

Figure 5.6: ACF plots of the transformed FTSE data

no dependence in the data (the data is a realisation from iid random variables). Or that there is dependence but it lies in a 'higher order structure' or over more sophisticated transformations.

Comparing this to the Yahoo data, may be we 'see' dependence in the Yahoo data because it is actually nonstationary. The mystery continues (we look into this later). It would be worth while conducting a similar analysis on a similar portion of the Yahoo data.

5.2 The ARCH model

During the early 80s Econometricians were trying to find a suitable model for forecasting stock prices. They were faced with data similar to the log differences of the Yahoo data in Figure 5.2. As Figure 5.3a demonstrates, there does not appear to be any linear dependence in the data, which makes the best linear predictor quite useless for forecasting. Instead, they tried to predict the variance of future prices given the past, $\mathrm{var}[X_{t+1}|X_t, X_{t-1}, \ldots]$. This called for a model that has a zero autocorrelation function, but models the conditional variance.

To address this need, Engle (1982) proposed the autoregressive conditionally heteroskadastic (ARCH) model (note that Rob Engle, together with Clive Granger, in 2004, received the Noble prize for Economics for Cointegration). He proposed the ARCH(p) which satisfies the representation

$$X_t = \sigma_t Z_t \qquad \sigma_t^2 = a_0 + \sum_{j=1}^{p} a_j X_{t-j}^2,$$

where Z_t are iid random variables where $\mathrm{E}(Z_t) = 0$ and $\mathrm{var}(Z_t) = 1$, $a_0 > 0$ and for $1 \leq j \leq p$ $a_j \geq 0$.

Before, worrying about whether a solution of such a model exists, let us consider the reasons behind why this model was first proposed.

5.2.1 Features of an ARCH

Let us suppose that a causal, stationary solution of the ARCH model exists (X_t is a function of $Z_t, Z_{t-1}, Z_{t-1}, \ldots$) and all the necessary moments exist. Then we obtain the following.

(i) The first moment:

$$\begin{aligned}
\mathrm{E}[X_t] &= \mathrm{E}[Z_t\sigma_t] = \mathrm{E}[\mathrm{E}(Z_t\sigma_t|X_{t-1}, X_{t-2}, \ldots)] = \underbrace{\mathrm{E}[\sigma_t \mathrm{E}(Z_t|X_{t-1}, X_{t-2}, \ldots)]}_{\sigma_t \text{ function of } X_{t-1},\ldots,X_{t-p}} \\
&= \mathrm{E}[\sigma_t \underbrace{\mathrm{E}(Z_t)}_{\text{by causality}}] = \mathrm{E}[0 \cdot \sigma_t] = 0.
\end{aligned}$$

Thus the ARCH process has a zero mean.

(ii) The conditional variance:

$$\begin{aligned}
\mathrm{var}(X_t|X_{t-1}, X_{t-2}, \ldots, X_{t-p}) &= \mathrm{E}(X_t^2|X_{t-1}, X_{t-2}, \ldots, X_{t-p}) \\
&= \mathrm{E}(Z_t^2\sigma_t^2|X_{t-1}, X_{t-2}, \ldots, X_{t-p}) = \sigma_t^2 \mathrm{E}[Z_t^2] = \sigma_t^2.
\end{aligned}$$

Thus the conditional variance is $\sigma_t^2 = a_0 + \sum_{j=1}^p a_j X_{t-j}^2$ (a weighted sum of the squared past).

(iii) The autocovariance function:

Without loss of generality assume $k > 0$

$$\begin{aligned}
\mathrm{cov}[X_t, X_{t+k}] &= \mathrm{E}[X_t X_{t+k}] = \mathrm{E}[X_t \mathrm{E}(X_{t+k}|X_{t+k-1}, \ldots, X_t)] \\
&= \mathrm{E}[X_t\sigma_{t+k}\mathrm{E}(Z_{t+k}|X_{t+k-1}, \ldots, X_t)] = \mathrm{E}[X_t\sigma_{t+k}\mathrm{E}(Z_{t+k})] = \mathrm{E}[X_t\sigma_{t+k} \cdot 0] = 0.
\end{aligned}$$

The autocorrelation function is zero (it is a white noise process).

(iv) We will show in Section 5.2.2 that $\mathrm{E}[X^{2d}] < \infty$ iff $[\sum_{j=1}^p a_j]\mathrm{E}[Z_t^{2d}]^{1/d} < 1$. It is well known that even for Gaussian innovations $\mathrm{E}[Z_t^{2d}]^{1/d}$ grows with d, therefore if any of the a_j are non-zero (recall all need to be positive), there will exist a d_0 such that for all $d \geq d_0$ $\mathrm{E}[X_t^d]$ will not be finite. Thus the we see that the ARCH process has thick tails.

Usually we measure the thickness of tails in data using the Kurtosis measure (see wiki).

Points (i-iv) demonstrate that the ARCH model is able to model many of the features seen in the stock price data.

In some sense the ARCH model can be considered as a generalisation of the AR model. That is the squares of ARCH model satisfy

$$X_t^2 = \sigma^2 Z_t^2 = a_0 + \sum_{j=1}^{p} a_j X_{t-j}^2 + (Z_t^2 - 1)\sigma_t^2, \tag{5.7}$$

with characteristic polynomial $\phi(z) = 1 - \sum_{j=1}^{p} a_j z^j$. It can be shown that if $\sum_{j=1}^{p} a_j < 1$, then the roots of the characteristic polynomial $\phi(z)$ lie outside the unit circle (see Exercise 3.2). Moreover, the 'innovations' $\epsilon_t = (Z_t^2 - 1)\sigma_t^2$ are *martingale differences* (see wiki). This can be shown by noting that

$$\mathrm{E}[(Z_t^2 - 1)\sigma_t^2 | X_{t-1}, X_{t-2}, \ldots] = \sigma_t^2 \mathrm{E}(Z_t^2 - 1 | X_{t-1}, X_{t-2}, \ldots) = \sigma_t^2 \underbrace{\mathrm{E}(Z_t^2 - 1)}_{=0} = 0.$$

Thus $\mathrm{cov}(\epsilon_t, \epsilon_s) = 0$ for $s \neq t$. Martingales are a useful asymptotic tool in time series, we demonstrate how they can be used in Chapter 11.

To summarise, in many respects the ARCH(p) model resembles the AR(p) <u>except</u> that the innovations $\{\epsilon_t\}$ are martingale differences and not iid random variables. This means that despite the resemblence, it is not a linear time series.

We show that a unique, stationary causal solution of the ARCH model exists and derive conditions under which the moments exist.

5.2.2 Existence of a strictly stationary solution and second order stationarity of the ARCH

To simplify notation we will consider the ARCH(1) model

$$X_t = \sigma_t Z_t \qquad \sigma_t^2 = a_0 + a_1 X_{t-1}^2. \tag{5.8}$$

It is difficult to directly obtain a solution of X_t, instead we obtain a solution for σ_t^2 (since X_t can immediately be obtained from this). Using that $X_{t-1}^2 = \sigma_{t-1}^2 Z_{t-1}^2$ and substituting this into (5.8) we obtain

$$\sigma_t^2 = a_0 + a_1 X_{t-1}^2 = (a_1 Z_{t-1}^2)\sigma_{t-1}^2 + a_0. \tag{5.9}$$

We observe that (5.9) can be written in the stochastic recurrence relation form given in (5.2) with $A_t = a_1 Z_{t-1}^2$ and $B_t = a_0$. Therefore, by using Theorem 5.0.1, if $\mathrm{E}[\log a_1 Z_{t-1}^2] = \log a_1 + \mathrm{E}[\log Z_{t-1}^2] < 0$, then σ_t^2 has the strictly stationary causal solution

$$\sigma_t^2 = a_0 + a_0 \sum_{k=1}^{\infty} a_1^k \prod_{j=1}^{k} Z_{t-j}^2.$$

The condition for *existence* using Theorem 5.0.1 and (5.9) is

$$\mathrm{E}[\log(a_1 Z_t^2)] = \log a_1 + \mathrm{E}[\log Z_t^2] < 0, \tag{5.10}$$

which is immediately implied if $a_1 < 1$ (since $\mathrm{E}[\log Z_t^2] \leq \log \mathrm{E}[Z_t^2] = 0$), but it is also satisfied under weaker conditions on a_1.

To obtain the moments of X_t^2 we use that it has the solution is

$$X_t^2 = Z_t^2 \left(a_0 + a_0 \sum_{k=1}^{\infty} a_1^k \prod_{j=1}^{k} Z_{t-j}^2 \right), \tag{5.11}$$

therefore taking expectations we have

$$\mathrm{E}[X_t^2] \;\; = \;\; \mathrm{E}[Z_t^2]\mathrm{E}\left(a_0 + a_0 \sum_{k=1}^{\infty} a_1^k \prod_{j=1}^{k} Z_{t-j}^2 \right) = a_0 + a_0 \sum_{k=1}^{\infty} a_1^k.$$

Thus $\mathrm{E}[X_t^2] < \infty$ if and only if $a_1 < 1$ (heuristically we can see this from $\mathrm{E}[X_t^2] = \mathrm{E}[Z_2^2](a_0 + a_1 \mathrm{E}[X_{t-1}^2])$).

By placing stricter conditions on a_1, namely $a_1 \mathrm{E}(Z_t^{2d})^{1/d} < 1$, we can show that $\mathrm{E}[X_t^{2d}] < \infty$ (note that this is an iff condition). To see why consider the simple case d is an integer, then by using (5.11) we have

$$\begin{aligned} X_t^{2d} &\geq Z_t^{2d} a_0^d \sum_{k=1}^{\infty} a_1^{dk} \left(\prod_{j=1}^{k} Z_{t-j}^2 \right)^{2d} \\ \Rightarrow \mathrm{E}[X_t^{2d}] &\geq \mathrm{E}[Z_t^{2d}] a_0^d \sum_{k=1}^{\infty} a_1^{dk} \prod_{j=1}^{k} \mathrm{E}[Z_{t-j}^{2d}] = \mathrm{E}[Z_t^{2d}] a_0^d \sum_{k=1}^{\infty} a_1^{dk} \mathrm{E}[Z_t^{2d}]^k \\ &= \mathrm{E}[Z_t^{2d}] a_0^d \sum_{k=1}^{\infty} \left(a_1^d \mathrm{E}[Z_t^{2d}] \right)^k. \end{aligned}$$

It is immediately clear the above is only finite if $a_1 \mathrm{E}[Z_t^{2d}]^{1/d} < 1$.

The ARCH(p) model

We can generalize the above results to ARCH(p) processes (but to show existence of a solution we need to write the ARCH(p) process as a vector process similar to the Vector AR(1) representation of an AR(p) given in Section 3.4.2). It can be shown that under sufficient conditions on the coefficients $\{a_j\}$ that the stationary, causal solution of the ARCH(p) model is

$$\begin{aligned} X_t^2 &= a_0 Z_t^2 + \sum_{k \geq 1} m_t(k) \qquad (5.12) \\ \text{where } m_t(k) &= \sum_{j_1,\ldots,j_k \geq 1} a_0 \Big(\prod_{r=1}^{k} a_{j_r} \Big) \prod_{r=0}^{k} Z^2_{t-\sum_{s=0}^{r} j_s} \quad (j_0 = 0). \end{aligned}$$

The above solution belongs to a general class of functions called a Volterra expansion. We note that $\mathrm{E}[X_t^2] < \infty$ iff $\sum_{j=1}^{p} a_j < 1$.

5.3 The GARCH model

A possible drawback of the ARCH(p) model is that the conditional variance only depends on finite number of the past squared observations/log returns (in finance, the share price is often called the return). However, when fitting the model to the data, analogous to order selection of an autoregressive model (using, say, the AIC), often a large order p is selected. This suggests that the conditional variance should involve a large (infinite?) number of past terms. This observation motivated the GARCH model (first proposed in Bollerslev (1986) and Taylor (1986)), which in many respects is analogous to the ARMA. The conditional variance of the GARCH model is a weighted average of the squared returns, the weights decline with the lag, but never go completely to zero. The GARCH class of models is a rather parsimonous class of models and is extremely popular in finance. The GARCH(p,q) model is defined as

$$X_t = \sigma_t Z_t \qquad \sigma_t^2 = a_0 + \sum_{j=1}^{p} a_j X_{t-j}^2 + \sum_{i=1}^{q} b_i \sigma_{t-i}^2 \tag{5.13}$$

where Z_t are iid random variables where $\mathrm{E}(Z_t) = 0$ and $\mathrm{var}(Z_t) = 1$, $a_0 > 0$ and for $1 \leq j \leq p$ $a_j \geq 0$ and $1 \leq i \leq q$ $b_i \geq 0$.

Under the assumption that a causal solution with sufficient moments exist, the same properties defined for the ARCH(p) in Section 5.2.1 also apply to the GARCH(p,q) model.

It can be shown that under suitable conditions on $\{b_j\}$ that X_t satisfies an ARCH(∞) representation. Formally, we can write the conditional variance σ_t^2 (assuming that a stationarity solution exists) as

$$(1 - \sum_{i=1}^{q} b_i B^i)\sigma_t^2 = (a_0 + \sum_{j=1}^{p} a_j X_{t-j}^2),$$

where B denotes the backshift notation defined in Chapter 3. Therefore if the roots of $b(z) = (1 - \sum_{i=1}^{q} b_i z^i)$ lie outside the unit circle (which is satisfied if $\sum_i b_i < 1$) then

$$\sigma_t^2 \quad = \quad \frac{1}{(1 - \sum_{j=1}^{q} b_j B^j)}(a_0 + \sum_{j=1}^{p} a_j X_{t-j}^2) = \alpha_0 + \sum_{j=1}^{\infty} \alpha_j X_{t-j}^2, \tag{5.14}$$

where a recursive equation for the derivation of α_j can be found in Berkes et al. (2003). In other words the GARCH(p,q) process can be written as a ARCH(∞) process. This is analogous to the invertibility representation given in Definition 3.2.2. This representation is useful when estimating the parameters of a GARCH process (see Berkes et al. (2003)) and also prediction. The expansion in (5.14) helps explain why the GARCH(p,q) process is so popular. As we stated at the start of this section, the conditional variance of the GARCH is a weighted average of the squared returns, the weights decline with the lag, but never go completely to zero, a property that is highly desirable.

Example 5.3.1 (Inverting the GARCH$(1,1)$) *If $b_1 < 1$, then we can write σ_t^2 as*

$$\sigma_t^2 \quad = \quad \left[\sum_{j=0}^{\infty} b^j B^j\right] \cdot \left[a_0 + a_1 X_{t-1}^2\right] = \frac{a_0}{1-b} + a_1 \sum_{j=0}^{\infty} b^j X_{t-1-j}^2.$$

This expansion offers us a clue as to why the GARCH$(1,1)$ is so popular in finance. In finance one important objective is to predict future volatility, this is the variance of say a stock tomorrow given past information. Using the GARCH model this is σ_t^2, which we see is

$$\sigma_t^2 = \frac{a_0}{1-b} + a_1 \sum_{j=0}^{\infty} b^j X_{t-1-j}^2.$$

This can be viewed as simply an exponentially weighted average of X_{t-j}^2. Some researchers argue that other models can lead to the same predictor of future volatility and there is nothing intrinsically specially about the GARCH process. We discuss this in more detail in Chapter 5.

In the following section we derive conditions for existence of the GARCH model and also it's moments.

5.3.1 Existence of a stationary solution of a GARCH$(1,1)$

We will focus on the GARCH$(1,1)$ model as this substantially simplifies the conditions. We recall the conditional variance of the GARCH$(1,1)$ can be written as

$$\sigma_t^2 = a_0 + a_1 X_{t-1}^2 + b_1 \sigma_{t-1}^2 = \left(a_1 Z_{t-1}^2 + b_1\right) \sigma_{t-1}^2 + a_0. \tag{5.15}$$

We observe that (5.15) can be written in the stochastic recurrence relation form given in (5.2) with $A_t = (a_1 Z_{t-1}^2 + b_1)$ and $B_t = a_0$. Therefore, by using Theorem 5.0.1, if $\mathrm{E}[\log(a_1 Z_{t-1}^2 + b_1)] < 0$, then σ_t^2 has the strictly stationary causal solution

$$\sigma_t^2 = a_0 + a_0 \sum_{k=1}^{\infty} \prod_{j=1}^{k} (a_1 Z_{t-j}^2 + b_1). \tag{5.16}$$

These conditions are relatively weak and depend on the distribution of Z_t. They are definitely satisfied if $a_1 + b_1 < 1$, since $\mathrm{E}[\log(a_1 Z_{t-1}^2 + b_1)] \leq \log \mathrm{E}[a_1 Z_{t-1}^2 + b_1] = \log(a_1 + b_1)$. However existence of a stationary solution does not require such a strong condition on the coefficients (and there can still exist a stationary solution if $a_1 + b_1 > 1$, so long as the distribution of Z_t^2 is such that $\mathrm{E}[\log(a_1 Z_t^2 + b_1)] < 0$).

By taking expectations of (5.16) we can see that

$$\mathrm{E}[X_t^2] = \mathrm{E}[\sigma_t^2] = a_0 + a_0 \sum_{k=1}^{\infty} \prod_{j=1}^{k} (a_1 + b_1) = a_0 + a_0 \sum_{k=1}^{\infty} (a_1 + b_1)^k.$$

Thus $\mathrm{E}[X_t^2] < \infty$ iff $a_1 + b_1 < 1$ (noting that a_1 and b_1 are both positive). Expanding on this argument, if $d > 1$ we can use Minkowski inequality to show

$$(\mathrm{E}[\sigma_t^{2d}])^{1/d} \leq a_0 + a_0 \sum_{k=1}^{\infty} (\mathrm{E}[\prod_{j=1}^{k} (a_1 Z_{t-j}^2 + b_1)]^d)^{1/d} \leq a_0 + a_0 \sum_{k=1}^{\infty} (\prod_{j=1}^{k} \mathrm{E}[(a_1 Z_{t-j}^2 + b_1)^d])^{1/d}.$$

Therefore, if $\mathrm{E}[(a_1 Z_{t-j}^2 + b_1)^d] < 1$, then $\mathrm{E}[X_t^{2d}] < \infty$. This is an iff condition, since from the definition in (5.15) we have

$$\mathrm{E}[\sigma_t^{2d}] \quad = \quad \mathrm{E}[\underbrace{a_0 + (a_1 Z_{t-1}^2 + b_1)\sigma_{t-1}^2}_{\text{every term is positive}}]^d \geq \mathrm{E}[(a_1 Z_{t-1}^2 + b_1)\sigma_{t-1}^2]^d = \mathrm{E}[(a_1 Z_{t-1}^2 + b_1)^d]\mathrm{E}[\sigma_{t-1}^{2d}],$$

since σ_{t-1}^2 has a causal solution, it is independent of Z_{t-1}^2. We observe that by stationarity and if $\mathrm{E}[\sigma_t^{2d}] < \infty$, then $\mathrm{E}[\sigma_t^{2d}] = \mathrm{E}[\sigma_{t-1}^{2d}]$. Thus the above inequality only holds if $\mathrm{E}[(a_1 Z_{t-1}^2 + b_1)^d] < 1$. Therefore, $\mathrm{E}[X_t^{2d}] < \infty$ iff $\mathrm{E}[(a_1 Z_{t-1}^2 + b_1)^d] < 1$.

Indeed in order for $\mathrm{E}[X_t^{2d}] < \infty$ a huge constraint needs to be placed on the parameter space of a_1 and b_1.

Exercise 5.1 *Suppose* $\{Z_t\}$ *are standard normal random variables. Find conditions on* a_1 *and* b_1 *such that* $\mathrm{E}[X_t^4] < \infty$*.*

The above results can be generalised to GARCH(p, q) model. Conditions for existence of a stationary solution hinge on the random matrix corresponding to the SRE representation of the GARCH model (see Bougerol and Picard (1992a) and Bougerol and Picard (1992b)), which are nearly impossible to verify. Sufficient and necessary conditions for both a stationary (causal) solution and second order stationarity ($\mathrm{E}[X_t^2] < \infty$) is $\sum_{j=1}^{p} a_j + \sum_{i=1}^{q} b_i < 1$. However, many econometricians believe this condition places an unreasonable constraint on the parameter space of $\{a_j\}$ and $\{b_j\}$. A large amount of research has been done on finding consistent parameter estimators under weaker conditions. Indeed, in the very interesting paper by Berkes et al. (2003) (see also Straumann (2005)) they derive consistent estimates of GARCH parameters on far milder set of conditions on $\{a_j\}$ and $\{b_i\}$ (which don't require $\mathrm{E}(X_t^2) < \infty$).

Definition 5.3.1 *The IGARCH model is a GARCH model where*

$$X_t = \sigma_t Z_t \qquad \sigma_t^2 = a_0 + \sum_{j=1}^{p} a_j X_{t-j}^2 + \sum_{i=1}^{q} b_i \sigma_{t-i}^2 \tag{5.17}$$

where the coefficients are such that $\sum_{j=1}^{p} a_j + \sum_{i=1}^{q} b_i = 1$. This is an example of a time series model which has a strictly stationary solution but it is not second order stationary.

Exercise 5.2 *Simulate realisations of ARCH(1) and GARCH$(1,1)$ models. Simulate with both iid Gaussian and t-distribution errors ($\{Z_t\}$ where $\mathrm{E}[Z_t^2] = 1$). Remember to 'burn-in' each realisation.*

In all cases fix $a_0 > 0$. Then

(i) Simulate an ARCH(1) with $a_1 = 0.3$ and $a_1 = 0.9$.

(ii) Simulate a GARCH$(1,1)$ with $a_1 = 0.1$ and $b_1 = 0.85$, and a GARCH$(1,1)$ with $a_1 = 0.85$ and $b_1 = 0.1$. Compare the two behaviours.

5.3.2 Extensions of the GARCH model

One criticism of the GARCH model is that it is 'blind' to negative the sign of the return X_t. In other words, the conditional variance of X_t only takes into account the magnitude of X_t and does not depend on increases or a decreases in S_t (which corresponds to X_t being positive or negative). In contrast it is largely believed that the financial markets react differently to negative or positive X_t. The general view is that there is greater volatility/uncertainity/variation in the market when the price goes down.

This observation has motivated extensions to the GARCH, such as the EGARCH which take into account the sign of X_t. Deriving conditions for such a stationary solution to exist can be difficult task, and the reader is refered to Straumann (2005) and more the details.

Other extensions to the GARCH include an Autoregressive type model with GARCH innovations.

5.3.3 `R code`

`install.packages("tseries")`, `library("tseries")` recently there have been a new package developed `library("fGARCH")`.

5.4 Bilinear models

The Bilinear model was first proposed in Subba Rao (1977) and Granger and Andersen (1978) (see also Subba Rao (1981)). The general Bilinear (BL(p,q,r,s)) model is defined as

$$X_t - \sum_{j=1}^{p} \phi_j X_{t-j} = \varepsilon_t + \sum_{i=1}^{q} \theta_i \varepsilon_{t-i} + \sum_{k=1}^{r} \sum_{k'=1}^{s} b_{k,k'} X_{t-k} \varepsilon_{t-k'},$$

where $\{\varepsilon_t\}$ are iid random variables with mean zero and variance σ^2.

To motivate the Bilinear model let us consider the simplest version of the model BL$(1,0,1,1)$

$$X_t = \phi_1 X_{t-1} + b_{1,1} X_{t-1}\varepsilon_{t-1} + \varepsilon_t = [\phi_1 + b_{1,1}\varepsilon_{t-1}]X_{t-1} + \varepsilon_t. \tag{5.18}$$

Comparing (5.20) with the conditional variane of the GARCH$(1,1)$ in (5.15) we see that they are very similar, the main differences are that (a) the bilinear model does not constrain the coefficients to be positive (whereas the conditional variance requires the coefficients to be positive) (b) the ε_{t-1} depends on X_{t-1}, whereas in the GARCH$(1,1)$ Z_{t-1}^2 and σ_{t-1}^2 are independent coefficients and (c) the innovation in the GARCH$(1,1)$ model is deterministic, whereas in the innovation in the Bilinear model is random. (b) and (c) makes the analysis of the Bilinear more complicated than the GARCH model. From model (5.20) we observe that when ε_{t-1} and X_{t-1} "couple" (both are large, mainly because ε_{t-1} is large) it leads to large burst in X_t. We observe this from the simulations below. Therefore this model has been used to model seismic activity etc.

5.4.1 Features of the Bilinear model

In this section we assume a causal, stationary solution of the bilinear model exists, the appropriate number of moments and that it is invertible in the sense that there exists a function g such that $\varepsilon_t = g(X_{t-1}, X_{t-2}, \ldots)$.

Under the assumption that the Bilinear process is invertible we can show that

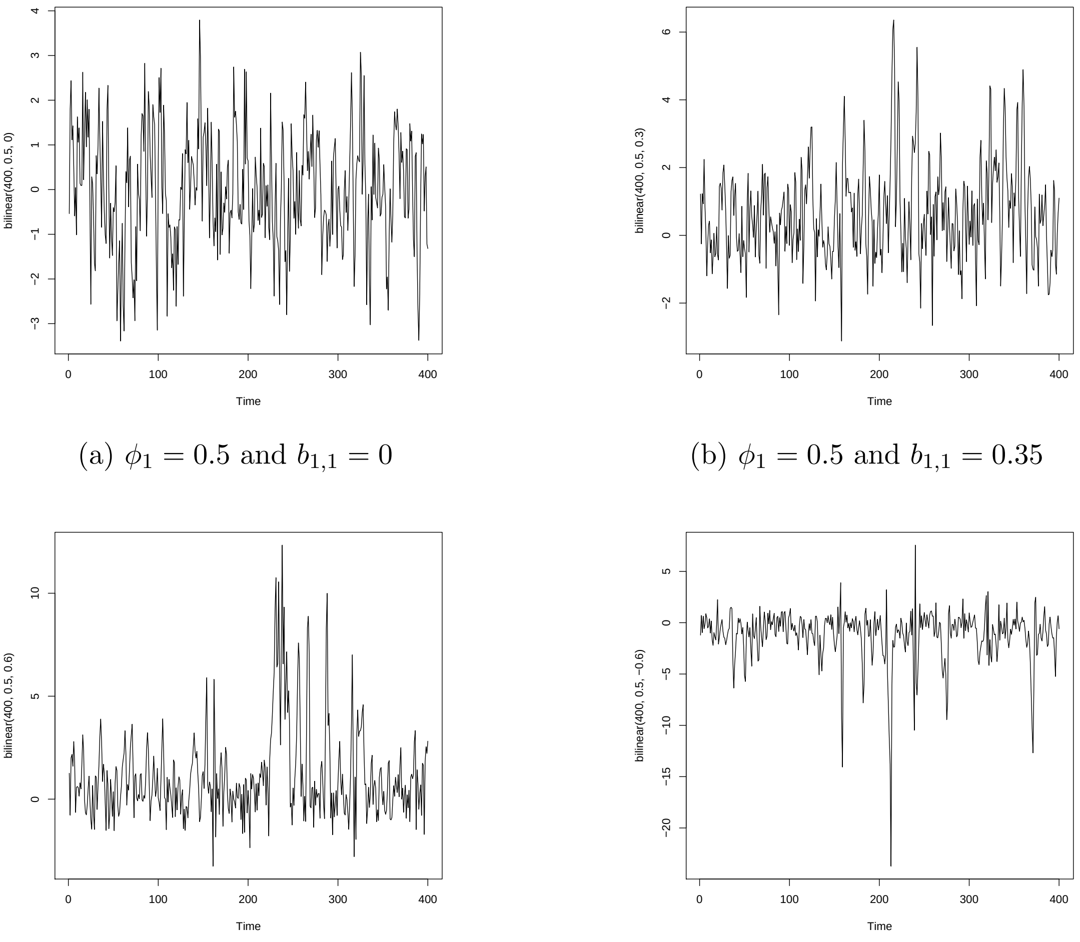
$$\begin{aligned} \mathrm{E}[X_t|X_{t-1}, X_{t-2}, \ldots] &= \mathrm{E}[(\phi_1 + b_{1,1}\varepsilon_{t-1})X_{t-1}|X_{t-1}, X_{t-2}, \ldots] + \mathrm{E}[\varepsilon_t|X_{t-1}, X_{t-2}, \ldots] \\ &= (\phi_1 + b_{1,1}\varepsilon_{t-1})X_{t-1}, \end{aligned} \tag{5.19}$$

thus unlike the autoregressive model the conditional expectation of the X_t given the past is a nonlinear function of the past. It is this nonlinearity that gives rise to the spontaneous peaks that we see a typical realisation.

To see how the bilinear model was motivated in Figure 5.7 we give a plot of

$$X_t = \phi_1 X_{t-1} + b_{1,1} X_{t-1}\varepsilon_{t-1} + \varepsilon_t, \tag{5.20}$$

where $\phi_1 = 0.5$ and $b_{1,1} = 0, 0.35, 0.65$ and -0.65. and $\{\varepsilon_t\}$ are iid standard normal random variables. We observe that Figure 5.7a is a realisation from an AR(1) process and the subsequent plots are for different values of $b_{1,1}$. Figure 5.7a is quite 'regular', whereas the sudden bursts in activity become more pronounced as $b_{1,1}$ grows (see Figures 5.7b and 5.7c). In Figure 5.7d we give a plot realisation from a model where $b_{1,1}$ is negative and we see that the fluctation has changed direction.



(a) $\phi_1 = 0.5$ and $b_{1,1} = 0$

(b) $\phi_1 = 0.5$ and $b_{1,1} = 0.35$

(c) $\phi_1 = 0.5$ and $b_{1,1} = 0.65$

(d) $\phi_1 = 0.5$ and $b_{1,1} = -0.65$

Figure 5.7: Realisations from different BL$(1, 0, 1, 1)$ models

Remark 5.4.1 (Markov Bilinear model) *Some authors define the $BL(1,0,1,1)$ as*

$$Y_t \quad = \quad \phi_1 Y_{t-1} + b_{1,1} Y_{t-1}\varepsilon_t + \varepsilon_t = [\phi_1 + b_{11}\varepsilon_t]Y_{t-1} + \varepsilon_t.$$

The fundamental difference between this model and (5.20) is that the multiplicative innovation (using ε_t rather than ε_{t-1}) does not depend on Y_{t-1}. This means that $\mathrm{E}[Y_t|Y_{t-1}, Y_{t-2}, \ldots] = \phi_1 Y_{t-1}$ and the autocovariance function is the same as the autocovariance function of an $AR(1)$ model with the same AR parameter. Therefore, it is unclear the advantage of using this version of the model if the aim is to forecast, since the forecast of this model is the same as a forecast using the corresponding $AR(1)$ process $X_t = \phi_1 X_{t-1} + \varepsilon_t$. Forecasting with this model does not take into account its nonlinear behaviour.

5.4.2 Solution of the Bilinear model

We observe that (5.20) can be written in the stochastic recurrence relation form given in (5.2) with $A_t = (\phi_1 + b_{11}\varepsilon_{t-1})$ and $B_t = a_0$. Therefore, by using Theorem 5.0.1, if $\mathrm{E}[\log(\phi_1 + b_{11}\varepsilon_{t-1})^2] < 0$ and $\mathrm{E}[\varepsilon_t] < \infty$, then X_t has the strictly stationary, causal solution

$$X_t = \sum_{k=1}^{\infty}\left[\prod_{j=1}^{k-1}(\phi_1 + b_{1,1}\varepsilon_{t-j})\right]\cdot[(\phi_1 + b_{1,1}\varepsilon_{t-k})\varepsilon_{t-k}] + \varepsilon_t. \tag{5.21}$$

To show that it is second order stationary we require that $\mathrm{E}[X_t^2] < \infty$, which imposes additional conditions on the parameters. To derive conditions for $\mathrm{E}[X_t^2]$ we use (5.22) and the Minkowski inequality to give

$$\begin{aligned}(\mathrm{E}[X_t^2])^{1/2} &\leq \sum_{k=1}^{\infty}\mathrm{E}\left(\left[\prod_{j=1}^{k-1}(\phi_1 + b_{1,1}\varepsilon_{t-j})\right]^2\right)^{1/2}\cdot\left(\mathrm{E}\left[(\phi_1 + b_{11}\varepsilon_{t-k})\varepsilon_{t-k}\right]^2\right)^{1/2} \\ &= \sum_{k=1}^{\infty}\prod_{j=1}^{k-1}\mathrm{E}\left([(\phi_1 + b_{1,1}\varepsilon_{t-j})]^2\right)^{1/2}\cdot\left(\mathrm{E}\left[(\phi_1 + b_{1,1}\varepsilon_{t-k})\varepsilon_{t-k}\right]^2\right)^{1/2}. \end{aligned} \tag{5.22}$$

Therefore if $\mathrm{E}[\varepsilon_t^4] < \infty$ and

$$\mathrm{E}\left[(\phi_1 + b_{1,1}\varepsilon_t)\right]^2 = \phi^2 + b_{11}^2\mathrm{var}(\varepsilon_t) < 1,$$

then $\mathrm{E}[X_t^2] < \infty$ (note that the above equality is due to $\mathrm{E}[\varepsilon_t] = 0$).

Remark 5.4.2 (Inverting the Bilinear model) *We note that*

$$\varepsilon_t \quad = \quad -(bX_{t-1})\varepsilon_{t-1} + [X_t - \phi X_{t-1}],$$

thus by iterating backwards with respect to ε_{t-j} we have

$$\varepsilon_t = \sum_{j=0}^{\infty} \left((-b)^{j-1} \prod_{i=0}^{j} X_{t-1-j} \right) [X_{t-j} - \phi X_{t-j-1}].$$

This invertible representation is useful both in forecasting and estimation (see Section 6.7.3).

Exercise 5.3 *Simulate the BL$(2,0,1,1)$ model (using the AR(2) parameters $\phi_1 = 1.5$ and $\phi_2 = -0.75$). Experiment with different parameters to give different types of behaviour.*

Exercise 5.4 *The random coefficient AR model is a nonlinear time series proposed by Barry Quinn (see Nicholls and Quinn (1982) and Aue et al. (2006)). The random coefficient AR(1) model is defined as*

$$X_t = (\phi + \eta_t)X_{t-1} + \varepsilon_t$$

where $\{\varepsilon_t\}$ and $\{\eta_t\}$ are iid random variables which are independent of each other.

(i) State sufficient conditions which ensure that $\{X_t\}$ has a strictly stationary solution.

(ii) State conditions which ensure that $\{X_t\}$ is second order stationary.

(iii) Simulate from this model for different ϕ and $\mathrm{var}[\eta_t]$.

5.4.3 R code

Code to simulate a BL$(1,0,1,1)$ model:

```
# Bilinear Simulation
# Bilinear(1,0,1,1) model, we use the first n0 observations are burn-in
# in order to get close to the stationary solution.
bilinear <- function(n,phi,b,n0=400) {
y <- rnorm(n+n0)
w <- rnorm(n + n0)
for (t in 2:(n+n0)) {
y[t] <- phi * y[t-1] + b * w[t-1] * y[t-1] + w[t]
}
return(y[(n0+1):(n0+n)])
}
```

5.5 Nonparametric time series models

Many researchers argue that fitting parametric models can lead to misspecification and argue that it may be more realistic to fit nonparametric or semi-parametric time series models instead. There exists several nonstationary and semi-parametric time series (see Fan and Yao (2003) and Douc et al. (2014) for a comprehensive summary), we give a few examples below. The most general nonparametric model is

$$X_t = m(X_{t-1},\ldots,X_{t-p},\varepsilon_t),$$

but this is so general it looses all meaning, especially if the need is to predict. A slight restriction is make the innovation term additive (see Jones (1978))

$$X_t = m(X_{t-1},\ldots,X_{t-p}) + \varepsilon_t,$$

it is clear that for this model $\mathrm{E}[X_t|X_{t-1},\ldots,X_{t-p}] = m(X_{t-1},\ldots,X_{t-p})$. However this model has the distinct disadvantage that without placing any structure on $m(\cdot)$, for $p > 2$ nonparametric estimators of $m(\cdot)$ are lousy (as the suffer from the curse of dimensionality).

Thus such a generalisation renders the model useless. Instead semi-parametric approaches have been developed. Examples include the functional AR(p) model defined as

$$X_t = \sum_{j=1}^{p} \phi_j(X_{t-p})X_{t-j} + \varepsilon_t$$

the semi-parametric AR(1) model

$$X_t = \phi X_{t-1} + \gamma(X_{t-1}) + \varepsilon_t,$$

the nonparametric ARCH(p)

$$X_t = \sigma_t Z_t \qquad \sigma_t^2 = a_0 + \sum_{j=1}^{p} a_j(X_{t-j}^2).$$

In the case of all these models it is not easy to establish conditions in which a stationary solution exists. More often then not, if conditions are established they are similar in spirit to those that are used in the parametric setting. For some details on the proof see Vogt (2013) (also here), who considers nonparametric and nonstationary models (note the nonstationarity he considers is when the covariance structure changes over time, not the unit root type). For example in the case of the the semi-parametric AR(1) model, a stationary causal solution exists if $|\phi + \gamma'(0)| < 1$.

Chapter 6

Prediction

Prerequisites

- The best linear predictor.
- Some idea of what a basis of a vector space is.

Objectives

- Understand that prediction using a long past can be difficult because a large matrix has to be inverted, thus alternative, recursive method are often used to avoid direct inversion.
- Understand the derivation of the Levinson-Durbin algorithm, and why the coefficient, $\phi_{t,t}$, corresponds to the partial correlation between X_1 and X_{t+1}.
- Understand how these predictive schemes can be used write space of $\overline{sp}(X_t, X_{t-1}, \ldots, X_1)$ in terms of an orthogonal basis $\overline{sp}(X_t - P_{X_{t-1},X_{t-2},\ldots,X_1}(X_t), \ldots, X_1)$.
- Understand how the above leads to the Wold decomposition of a second order stationary time series.
- To understand how to approximate the prediction for an ARMA time series into a scheme which explicitly uses the ARMA structure. And this approximation improves geometrically, when the past is large.

One motivation behind fitting models to a time series is to forecast future unobserved observations - which would not be possible without a model. In this chapter we consider forecasting, based on the assumption that the model and/or autocovariance structure is known.

6.1 Why prediction is important

There are various reasons prediction is important. The first is that forecasting has a vast number of applications from finance to climatology. The second reason is that it forms the basis of most estimation schemes. To understand why forecasting is important in the latter, we now obtain the "likelihood" of the observed time series $\{X_t\}_{t=1}^n$. We assume the joint density of $\underline{X}_n = (X_1, \ldots, X_n)$ is $f_n(\underline{x}_n;\theta)$. By using conditioning it is clear that the likelihood is

$$f_n(\underline{x}_n;\theta) = f_1(x_1)f_2(x_2|x_1;\theta)f_3(x_3|x_2,x_1;\theta)\ldots f_n(x_n|x_{n-1},\ldots,x_1;\theta)$$

Therefore the log-likelihood is

$$\log f_n(\underline{x}_n;\theta) = \log f_1(x_1) + \sum_{t=1}^{n} \log f_t(x_t|x_{t-1},\ldots,x_1;\theta).$$

The parameters may be the AR, ARMA, ARCH, GARCH etc parameters. However, usually the conditional distributions $f_t(x_t|x_{t-1},\ldots,x_1;\theta)$ which make up the joint density $f(\underline{x};\theta)$ is completely unknown. However, often we can get away with assuming that the conditional distribution is Gaussian and we can still consistently estimate the parameters so long as the model has been correctly specified. Now, if we can "pretend" that the conditional distribution is Gaussian, then all we need is the conditional mean and the conditional variance

$$E(X_t|X_{t-1},\ldots,X_1;\theta) = \mathrm{E}\left(X_t|X_{t-1},\ldots,X_1;\theta\right) \text{ and } V(X_t|X_{t-1},\ldots,X_1,\theta) = \mathrm{var}\left(X_t|X_{t-1},\ldots,X_1;\theta\right).$$

Using this above and the "Gaussianity" of the conditional distribution gives

$$\log f_t(x_t|x_{t-1},\ldots,x_1;\theta) = -\frac{1}{2}\log V(x_t|x_{t-1},\ldots,x_1,\theta) - \frac{(x_t - E(x_t|x_{t-1},\ldots,x_1,\theta))^2}{V(x_t|x_{t-1},\ldots,x_1,\theta)}.$$

Using the above the log density

$$\log f_n(\underline{x}_n;\theta) = -\frac{1}{2}\sum_{t=1}^{n}\left(\log V(x_t|x_{t-1},\ldots,x_1,\theta) + \frac{(x_t - E(x_t|x_{t-1},\ldots,x_1,\theta))^2}{V(x_t|x_{t-1},\ldots,x_1,\theta)}\right).$$

Thus the log-likelihood

$$\mathcal{L}(\underline{X}_n;\theta) = -\frac{1}{2}\sum_{t=1}^{n}\left(\log V(X_t|X_{t-1},\ldots,X_1,\theta) + \frac{(X_t - E(X_t|X_{t-1},\ldots,X_1,\theta))^2}{V(X_t|X_{t-1},\ldots,X_1,\theta)}\right).$$

Therefore we observe that in order to evaluate the log-likelihood, and estimate the parameters, we require the conditonal mean and the conditional variance

$$E(X_t|X_{t-1},\ldots,X_1;\theta) \qquad \text{and} \qquad V(X_t|X_{t-1},\ldots,X_1;\theta).$$

This means that in order to do any form of estimation we need a clear understanding of what the conditional mean (which is simply the best predictor of the observation tomorrow given the past) and the conditional variance is for various models.

Note:

- Often expressions for conditional mean and variance can be extremely unwieldy. Therefore, often we require approximations of the conditonal mean and variance which are tractable (this is reminiscent of the Box-Jenkins approach and is till used when the conditional expectation and variance are difficult to estimate).

- Suppose we "pretend" that the time series $\{X_t\}$ is Gaussian. Which we can if it is linear, even if it is not. But we *cannot* if the time series is nonlinear (since nonlinear time series are not Gaussian), then the conditional variance $\text{var}(X_t|X_{t-1},\ldots,X_1)$ will *not* be random (this is a well known result for Gaussian random variables). If X_t is nonlinear, it can be conditionally Gaussian but not Gaussian.

- If the model is linear usually the conditonal expectation $E(X_t|X_{t-1},\ldots,X_1;\theta)$ is replaced with the best linear predictor of X_t given $X_{t-1},\ldots,X_1$. This means if the model is in fact non-causal the estimator will give a causal solution instead. Though not critical it is worth bearing in mind.

6.2 Forecasting given the present and infinite past

In this section we will assume that the linear time series $\{X_t\}$ is both causal and invertible, that is

$$X_t = \sum_{j=0}^{\infty} a_j \varepsilon_{t-j} = \sum_{i=1}^{\infty} b_i X_{t-i} + \varepsilon_t, \tag{6.1}$$

where $\{\varepsilon_t\}$ are iid random variables (recall Definition 3.2.2). Both these representations play an important role in prediction. Furthermore, in order to predict X_{t+k} given $X_t, X_{t-1}, \ldots$ we will assume that the infinite past is observed. In later sections we consider the more realistic situation that only the finite past is observed. We note that since $X_t, X_{t-1}, X_{t-2}, \ldots$ is observed that we can obtain ε_τ (for $\tau \leq t$) by using the invertibility condition

$$\varepsilon_\tau = X_\tau - \sum_{i=1}^{\infty} b_i X_{\tau-i}.$$

Now we consider the prediction of X_{t+k} given $\{X_\tau; \tau \leq t\}$. Using the MA(∞) presentation (since the time series is causal) of X_{t+k} we have

$$X_{t+k} = \underbrace{\sum_{j=0}^{\infty} a_{j+k}\varepsilon_{t-j}}_{\text{innovations are 'observed'}} + \underbrace{\sum_{j=0}^{k-1} a_j \varepsilon_{t+k-j}}_{\text{future innovations impossible to predict}},$$

since $\mathrm{E}[\sum_{j=0}^{k-1} a_j\varepsilon_{t+k-j}|X_t, X_{t-1}, \ldots] = \mathrm{E}[\sum_{j=0}^{k-1} a_j\varepsilon_{t+k-j}] = 0$. Therefore, the best linear predictor of X_{t+k} given $X_t, X_{t-1}, \ldots$, which we denote as $X_t(k)$ is

$$X_t(k) \;=\; \sum_{j=0}^{\infty} a_{j+k}\varepsilon_{t-j} = \sum_{j=0}^{\infty} a_{j+k}(X_{t-j} - \sum_{i=1}^{\infty} b_i X_{t-i-j}). \tag{6.2}$$

$X_t(k)$ is called the k-step ahead predictor and it is straightforward to see that it's mean squared error is

$$\mathrm{E}\left[X_{t+k} - X_t(k)\right]^2 = \mathrm{E}\left[\sum_{j=0}^{k-1} a_j\varepsilon_{t+k-j}\right]^2 = \mathrm{var}[\varepsilon_t]\sum_{j=0}^{k} a_j^2, \tag{6.3}$$

where the last line is due to the uncorrelatedness and zero mean of the innovations.

Often we would like to obtain the k-step ahead predictor for $k = 1, \ldots, n$ where n is some time in the future. We now explain how $X_t(k)$ can be evaluated recursively using the invertibility assumption.

Step 1 Use invertibility in (6.1) to give

$$X_t(1) = \sum_{i=1}^{\infty} b_i X_{t+1-i},$$

and $\mathrm{E}\left[X_{t+1} - X_t(1)\right]^2 = \mathrm{var}[\varepsilon_t]$

Step 2 To obtain the 2-step ahead predictor we note that

$$\begin{aligned} X_{t+2} &= \sum_{i=2}^{\infty} b_i X_{t+2-i} + b_1 X_{t+1} + \varepsilon_{t+2} \\ &= \sum_{i=2}^{\infty} b_i X_{t+2-i} + b_1 [X_t(1) + \varepsilon_{t+1}] + \varepsilon_{t+2}, \end{aligned}$$

thus it is clear that

$$X_t(2) = \sum_{i=2}^{\infty} b_i X_{t+2-i} + b_1 X_t(1)$$

and $\mathrm{E}\left[X_{t+2} - X_t(2)\right]^2 = \mathrm{var}[\varepsilon_t]\left(b_1^2 + 1\right) = \mathrm{var}[\varepsilon_t]\left(a_2^2 + a_1^2\right)$.

Step 3 To obtain the 3-step ahead predictor we note that

$$\begin{aligned} X_{t+3} &= \sum_{i=3}^{\infty} b_i X_{t+2-i} + b_2 X_{t+1} + b_1 X_{t+2} + \varepsilon_{t+3} \\ &= \sum_{i=3}^{\infty} b_i X_{t+2-i} + b_2 \left(X_t(1) + \varepsilon_{t+1}\right) + b_1 \left(X_t(2) + b_1 \varepsilon_{t+1} + \varepsilon_{t+2}\right) + \varepsilon_{t+3}. \end{aligned}$$

Thus

$$X_t(3) = \sum_{i=3}^{\infty} b_i X_{t+2-i} + b_2 X_t(1) + b_1 X_t(2)$$

and $\mathrm{E}\left[X_{t+3} - X_t(3)\right]^2 = \mathrm{var}[\varepsilon_t]\left[(b_2 + b_1^2)^2 + b_1^2 + 1\right] = \mathrm{var}[\varepsilon_t]\left(a_3^2 + a_2^2 + a_1^2\right)$.

Step k Using the arguments above it is easily seen that

$$X_t(k) = \sum_{i=k}^{\infty} b_i X_{t+k-i} + \sum_{i=1}^{k-1} b_i X_t(k-i).$$

Thus the k-step ahead predictor can be recursively estimated.

We note that the predictor given above it based on the assumption that the infinite past is observed. In practice this is not a realistic assumption. However, in the special case that time series is an autoregressive process of order p (with AR parameters $\{\phi_j\}_{j=1}^{p}$) and $X_t, \ldots, X_{t-m}$ is observed where $m \geq p-1$, then the above scheme can be used for forecasting. More precisely,

$$\begin{aligned}
X_t(1) &= \sum_{j=1}^{p} \phi_j X_{t+1-j} \\
X_t(k) &= \sum_{j=k}^{p} \phi_j X_{t+k-j} + \sum_{j=1}^{k-1} \phi_j X_t(k-j) \text{ for } 2 \leq k \leq p \\
X_t(k) &= \sum_{j=1}^{p} \phi_j X_t(k-j) \text{ for } k > p. \qquad (6.4)
\end{aligned}$$

However, in the general case more sophisticated algorithms are required when only the finite past is known.

Example: Forecasting yearly temperatures

We now fit an autoregressive model to the yearly temperatures from 1880-2008 and use this model to forecast the temperatures from 2009-2013. In Figure 6.1 we give a plot of the temperature time series together with its ACF. It is clear there is some trend in the temperature data, therefore we have taken second differences, a plot of the second difference and its ACF is given in Figure 6.2. We now use the command `ar.yule(res1,order.max=10)` (we will discuss in Chapter 8 how this function estimates the AR parameters) to estimate the the AR parameters.

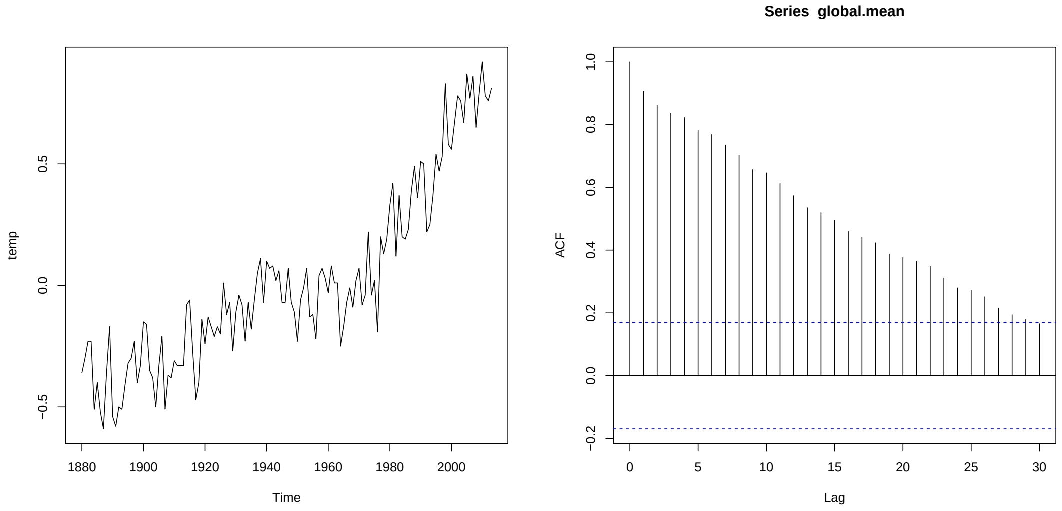


Figure 6.1: Yearly temperature from 1880-2013 and the ACF.

Remark 6.2.1 (The Yule-Walker estimator in prediction) *The least squares estimator (or*

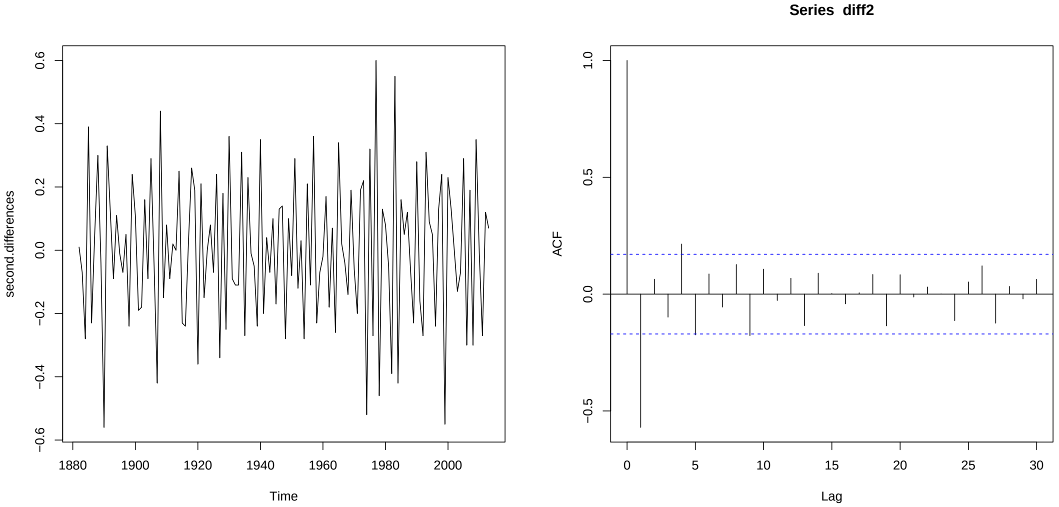


Figure 6.2: Second differences of yearly temperature from 1880-2013 and its ACF.

equivalently the conditional likelihood) is likely to give a causal estimator of the AR parameters. But it is not guaranteed. On the other hand the Yule-Walker estimator is guaranteed to give a causal solution. This will matter for prediction. We emphasize here that the least squares estimator cannot consistently estimate non-causal solutions, it is only a quirk of the estimation method that means at times the solution may be noncausal.

If the time series $\{X_t\}_t$ is linear and stationary with mean zero, then if we predict several steps into the future we would expect our predictor to be close to zero (since $\mathrm{E}(X_t) = 0$). This is guaranteed if one uses AR parameters which are causal (since the eigenvalues of the VAR matrix is less than one); such as the Yule-Walker estimators. On the other hand, if the parameter estimators do not correspond to a causal solution (as could happen for the least squares estimator), the predictors may explode for long term forecasts which makes no sense.

The function `ar.yule` uses the AIC to select the order of the AR model. When fitting the second differences from (from 1880-2008 - a data set of length of 127) the AIC chooses the AR(7) model

$$X_t = -1.1472X_{t-1} - 1.1565X_{t-2} - 1.0784X_{t-3} - 0.7745X_{t-4} - 0.6132X_{t-5} - 0.3515X_{t-6} - 0.1575X_{t-7} + \varepsilon_t,$$

with $\mathrm{var}[\varepsilon_t] = \sigma^2 = 0.02294$. An ACF plot after fitting this model and then estimating the residuals $\{\varepsilon_t\}$ is given in Figure 6.3. We observe that the ACF of the residuals 'appears' to be uncorrelated, which suggests that the AR(7) model fitted the data well. Later we define the Ljung-Box test, which is a method for checking this claim. However since the residuals are *estimated* residuals and *not* the true residual, the results of this test need to be taken with a large pinch of salt. We will show that when the residuals are estimated from the data the error bars given in the ACF plot are not correct and the Ljung-Box test is not pivotal (as is assumed when deriving the limiting distribution under the null the model is correct). By using the sequence of equations

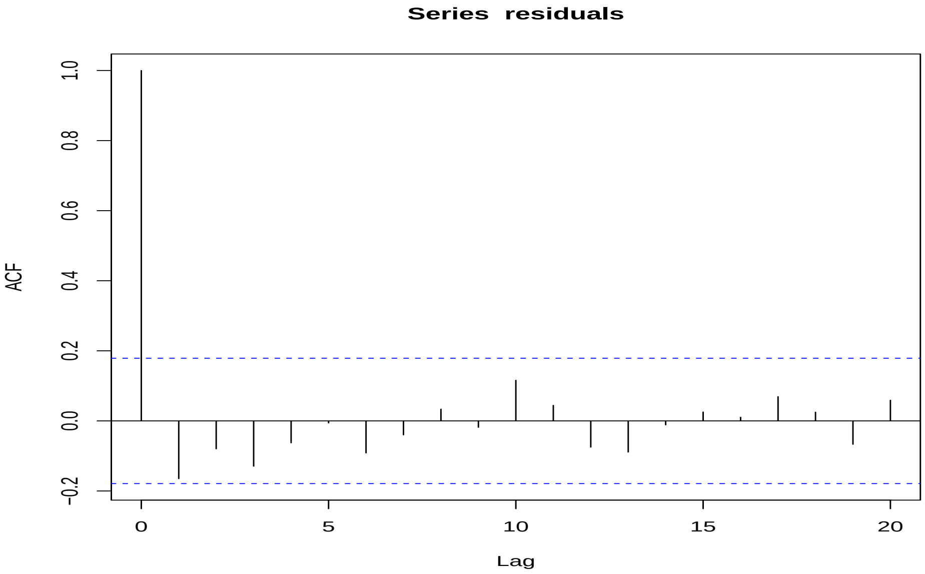


Figure 6.3: An ACF plot of the estimated residuals $\{\widehat{\varepsilon_t}\}$.

$$\begin{aligned}
\hat{X}_{127}(1) &= -1.1472X_{127} - 1.1565X_{126} - 1.0784X_{125} - 0.7745X_{124} - 0.6132X_{123} \\
&\quad -0.3515X_{122} - 0.1575X_{121} \\
\hat{X}_{127}(2) &= -1.1472\hat{X}_{127}(1) - 1.1565X_{127} - 1.0784X_{126} - 0.7745X_{125} - 0.6132X_{124} \\
&\quad -0.3515X_{123} - 0.1575X_{122} \\
\hat{X}_{127}(3) &= -1.1472\hat{X}_{127}(2) - 1.1565\hat{X}_{127}(1) - 1.0784X_{127} - 0.7745X_{126} - 0.6132X_{125} \\
&\quad -0.3515X_{124} - 0.1575X_{123} \\
\hat{X}_{127}(4) &= -1.1472\hat{X}_{127}(3) - 1.1565\hat{X}_{127}(2) - 1.0784\hat{X}_{127}(1) - 0.7745X_{127} - 0.6132X_{126} \\
&\quad -0.3515X_{125} - 0.1575X_{124} \\
\hat{X}_{127}(5) &= -1.1472\hat{X}_{127}(4) - 1.1565\hat{X}_{127}(3) - 1.0784\hat{X}_{127}(2) - 0.7745\hat{X}_{127}(1) - 0.6132X_{127} \\
&\quad -0.3515X_{126} - 0.1575X_{125}.
\end{aligned}$$

We can use $\hat{X}_{127}(1),\ldots,\hat{X}_{127}(5)$ as forecasts of $X_{128},\ldots,X_{132}$ (we recall are the second differences), which we then use to construct forecasts of the temperatures. A plot of the second difference forecasts together with the true values are given in Figure 6.4. From the forecasts of the second differences we can obtain forecasts of the original data. Let Y_t denote the temperature at time t and X_t its second difference. Then $Y_t = -Y_{t-2} + 2Y_{t-1} + X_t$. Using this we have

$$\begin{aligned}
\widehat{Y}_{127}(1) &= -Y_{126} + 2Y_{127} + X_{127}(1) \\
\widehat{Y}_{127}(2) &= -Y_{127} + 2Y_{127}(1) + X_{127}(2) \\
\widehat{Y}_{127}(3) &= -Y_{127}(1) + 2Y_{127}(2) + X_{127}(3)
\end{aligned}$$

and so forth.

We note that (6.3) can be used to give the mse error. For example

$$\begin{aligned}
\mathrm{E}[X_{128} - \hat{X}_{127}(1)]^2 &= \sigma_t^2 \\
\mathrm{E}[X_{128} - \hat{X}_{127}(1)]^2 &= (1+\phi_1^2)\sigma_t^2
\end{aligned}$$

If we believe the residuals are Gaussian we can use the mean squared error to construct confidence intervals for the predictions. Assuming for now that the parameter estimates are the true parameters (this is not the case), and $X_t = \sum_{j=0}^{\infty}\psi_j(\widehat{\phi})\varepsilon_{t-j}$ is the MA(∞) representation of the AR(7) model, the mean square error for the kth ahead predictor is

$$\sigma^2\sum_{j=0}^{k-1}\psi_j(\widehat{\phi})^2 \text{ (using (6.3))}$$

thus the 95% CI for the prediction is

$$\left[X_t(k) \pm 1.96\sigma^2\sum_{j=0}^{k-1}\psi_j(\widehat{\phi})^2\right],$$

however this confidence interval for not take into account $X_t(k)$ uses only parameter estimators and not the true values. In reality we need to take into account the approximation error here too.

If the residuals are not Gaussian, the above interval is not a 95% confidence interval for the prediction. One way to account for the non-Gaussianity is to use bootstrap. Specifically, we rewrite the AR(7) process as an MA(∞) process

$$X_t = \sum_{j=0}^{\infty} \psi_j(\widehat{\phi})\varepsilon_{t-j}.$$

Hence the best linear predictor can be rewritten as

$$X_t(k) = \sum_{j=k}^{\infty} \psi_j(\widehat{\phi})\varepsilon_{t+k-j}$$

thus giving the prediction error

$$X_{t+k} - X_t(k) = \sum_{j=0}^{k-1} \psi_j(\widehat{\phi})\varepsilon_{t+k-j}.$$

We have the prediction estimates, therefore all we need is to obtain the distribution of $\sum_{j=0}^{k-1} \psi_j(\widehat{\phi})\varepsilon_{t+k-j}$. This can be done by estimating the residuals and then using bootstrap[1] to estimate the distribution of $\sum_{j=0}^{k-1} \psi_j(\widehat{\phi})\varepsilon_{t+k-j}$, using the empirical distribution of $\sum_{j=0}^{k-1} \psi_j(\widehat{\phi})\varepsilon^*_{t+k-j}$. From this we can construct the 95% CI for the forecasts.

A small criticism of our approach is that we have fitted a rather large AR(7) model to time series of length of 127. It may be more appropriate to fit an ARMA model to this time series.

Exercise 6.1 *In this exercise we analyze the Sunspot data found on the course website. In the data analysis below only use the data from 1700 - 2003 (the remaining data we will use for prediction). In this section you will need to use the function* `ar.yw` *in* `R`.

(i) Fit the following models to the data and study the residuals (using the ACF). Using this decide which model

$$\begin{aligned} X_t &= \mu + A\cos(\omega t) + B\sin(\omega t) + \underbrace{\varepsilon_t}_{AR} \quad \textit{or} \\ X_t &= \mu + \underbrace{\varepsilon_t}_{AR} \end{aligned}$$

[1] Residual bootstrap is based on sampling from the empirical distribution of the residuals i.e. construct the "bootstrap" sequence $\{\varepsilon^*_{t+k-j}\}_j$ by sampling from the empirical distribution $\widehat{F}(x) = \frac{1}{n}\sum_{t=p+1}^{n} I(\widehat{\varepsilon}_t \leq x)$ (where $\widehat{\varepsilon}_t = X_t - \sum_{j=1}^{p} \widehat{\phi}_j X_{t-j}$). This sequence is used to construct the bootstrap estimator $\sum_{j=0}^{k-1} \psi_j(\widehat{\phi})\varepsilon^*_{t+k-j}$. By doing this several thousand times we can evaluate the empirical distribution of $\sum_{j=0}^{k-1} \psi_j(\widehat{\phi})\varepsilon^*_{t+k-j}$ using these bootstrap samples. This is an estimator of the distribution function of $\sum_{j=0}^{k-1} \psi_j(\widehat{\phi})\varepsilon_{t+k-j}$.

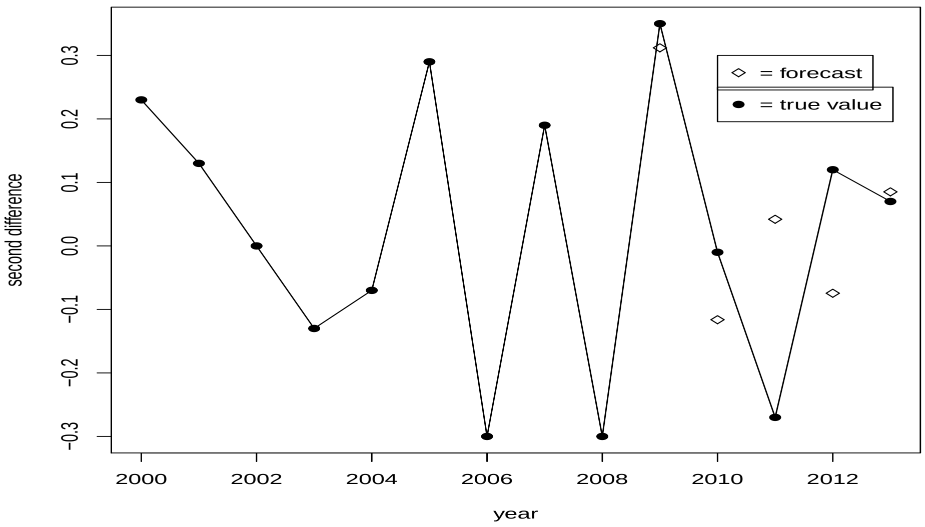


Figure 6.4: Forecasts of second differences.

is more appropriate (take into account the number of parameters estimated overall).

(ii) Use these models to forecast the sunspot numbers from 2004-2013.

```
diff1 = global.mean[c(2:134)] - global.mean[c(1:133)]

diff2 = diff1[c(2:133)] - diff1[c(1:132)]

res1 = diff2[c(1:127)]

residualsar7 <- ar.yw(res1,order.max=10)$resid

residuals <- residualsar7[-c(1:7)]

Forecast using the above model

res = c(res1,rep(0,5))

res[128] = -1.1472*res[127]  -1.1565*res[126]  -1.0784*res[125]  -0.7745*res[124]  -0.6132*res

res[129] = -1.1472*res[128]  -1.1565*res[127]  -1.0784*res[126]  -0.7745*res[125]  -0.6132*res

res[130] = -1.1472*res[129]  -1.1565*res[128]  -1.0784*res[127]  -0.7745*res[126]  -0.6132*res

res[131] = -1.1472*res[130]  -1.1565*res[129]  -1.0784*res[128]  -0.7745*res[127]  -0.6132*res

res[132] = -1.1472*res[131]  -1.1565*res[130]  -1.0784*res[129]  -0.7745*res[128]  -0.6132*res
```

6.3 Review of vector spaces

In next few sections we will consider prediction/forecasting for stationary time series. In particular to find the best linear predictor of X_{t+1} given the finite past $X_t, \ldots, X_1$. Setting up notation our aim is to find

$$X_{t+1|t} = P_{X_1,\ldots,X_t}(X_{t+1}) = X_{t+1|t,\ldots,1} = \sum_{j=1}^{t} \phi_{t,j} X_{t+1-j},$$

where $\{\phi_{t,j}\}$ are chosen to minimise the mean squared error $\min_{\underline{\phi}_t} \mathrm{E}(X_{t+1} - \sum_{j=1}^{t} \phi_{t,j} X_{t+1-j})^2$. Basic results from multiple regression show that

$$\begin{pmatrix} \phi_{t,1} \\ \vdots \\ \phi_{t,t} \end{pmatrix} = \Sigma_t^{-1} \underline{r}_t,$$

where $(\Sigma_t)_{i,j} = \mathrm{E}(X_i X_j)$ and $(\underline{r}_t)_i = \mathrm{E}(X_{t-i} X_{t+1})$. Given the covariances this can easily be done. However, if t is large a brute force method would require $O(t^3)$ computing operations to calculate (6.7). Our aim is to exploit stationarity to reduce the number of operations. To do this, we will briefly discuss the notion of projections on a space, which help in our derivation of computationally more efficient methods.

Before we continue we first discuss briefly the idea of a a vector space, inner product spaces, Hilbert spaces, spans and basis. A more complete review is given in Brockwell and Davis (1998), Chapter 2.

First a brief definition of a vector space. $\mathcal{X}$ is called an vector space if for every $x, y \in \mathcal{X}$ and $a, b \in \mathbb{R}$ (this can be generalised to $\mathbb{C}$), then $ax + by \in \mathcal{X}$. An inner product space is a vector space which comes with an inner product, in other words for every element $x, y \in \mathcal{X}$ we can defined an innerproduct $\langle x, y \rangle$, where $\langle \cdot, \cdot \rangle$ satisfies all the conditions of an inner product. Thus for every element $x \in \mathcal{X}$ we can define its norm as $\|x\| = \langle x, x \rangle$. If the inner product space is complete (meaning the limit of every sequence in the space is also in the space) then the innerproduct space is a Hilbert space (see wiki).

Example 6.3.1 *(i) The classical example of a Hilbert space is the Euclidean space $\mathbb{R}^n$ where the innerproduct between two elements is simply the scalar product, $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{n} x_i y_i$.*

(ii) The subset of the probability space $(\Omega, \mathcal{F}, P)$, where all the random variables defined on Ω have a finite second moment, ie. $\mathrm{E}(X^2) = \int_\Omega X(\omega)^2 dP(\omega) < \infty$. This space is denoted as $L^2(\Omega, \mathcal{F}, P)$. In this case, the inner product is $\langle X, Y\rangle = \mathrm{E}(XY)$.

(iii) The function space $L^2[\mathbb{R}, \mu]$, where $f \in L^2[\mathbb{R}, \mu]$ if f is mu-measureable and

$$\int_{\mathbb{R}} |f(x)|^2 d\mu(x) < \infty,$$

is a Hilbert space. For this space, the inner product is defined as

$$\langle f, g\rangle = \int_{\mathbb{R}} f(x)g(x)d\mu(x).$$

In this chapter we will not use this function space, but it will be used in Chapter **??** *(when we prove the Spectral representation theorem).*

It is straightforward to generalize the above to complex random variables and functions defined on $\mathbb{C}$. We simply need to remember to take conjugates when defining the innerproduct, ie. $\langle X, Y\rangle = \mathrm{E}(X\overline{Y})$ and $\langle f, g\rangle = \int_{\mathbb{C}} f(z)\overline{g(z)}d\mu(z)$.

In this chapter our focus will be on certain spaces of random variables which have a finite variance.

Basis

The random variables $\{X_t, X_{t-1}, \ldots, X_1\}$ span the space $\mathcal{X}_t^1$ (denoted as $\overline{\mathrm{sp}}(X_t, X_{t-1}, \ldots, X_1)$), if for every $Y \in \mathcal{X}_t^1$, there exists coefficients $\{a_j \in \mathbb{R}\}$ such that

$$Y = \sum_{j=1}^{t} a_j X_{t+1-j}. \tag{6.5}$$

Moreover, $\overline{\mathrm{sp}}(X_t, X_{t-1}, \ldots, X_1) = \mathcal{X}_t^1$ if for every $\{a_j \in \mathbb{R}\}$, $\sum_{j=1}^t a_j X_{t+1-j} \in \mathcal{X}_t^1$. We now define the basis of a vector space, which is closely related to the span. The random variables $\{X_t, \ldots, X_1\}$ form a basis of the space $\mathcal{X}_t^1$, if for every $Y \in \mathcal{X}_t^1$ we have a representation (6.5) and this representation is unique. More precisely, there does not exist another set of coefficients $\{b_j\}$ such that $Y = \sum_{j=1}^t b_j X_{t+1-j}$. For this reason, one can consider a basis as the minimal span, that is the smallest set of elements which can span a space.

Definition 6.3.1 (Projections) *The projection of the random variable Y onto the space spanned* *by* $\overline{\text{sp}}(X_t, X_{t-1}, \ldots, X_1)$ *(often denoted as* $P_{X_t,X_{t-1},\ldots,X_1}(Y)$*) is defined as* $P_{X_t,X_{t-1},\ldots,X_1}(Y) = \sum_{j=1}^{t} c_j X_{t+1-j}$*, where* $\{c_j\}$ *is chosen such that the difference* $Y - P_{(X_t,X_{t-1},\ldots,X_1)}(Y_t)$ *is uncorrelated (orthogonal/perpendicular) to any element in* $\overline{\text{sp}}(X_t, X_{t-1}, \ldots, X_1)$*. In other words,* $P_{X_t,X_{t-1},\ldots,X_1}(Y_t)$ *is the best linear predictor of* Y *given* $X_t, \ldots, X_1$*.*

Orthogonal basis

An orthogonal basis is a basis, where every element in the basis is orthogonal to every other element in the basis. It is straightforward to orthogonalize any given basis using the method of projections.

To simplify notation let $X_{t|t-1} = P_{X_{t-1},\ldots,X_1}(X_t)$. By definition, $X_t - X_{t|t-1}$ is orthogonal to the space $\overline{\text{sp}}(X_{t-1}, X_{t-1}, \ldots, X_1)$. In other words $X_t - X_{t|t-1}$ and X_s $(1 \leq s \leq t)$ are orthogonal $(\text{cov}(X_s, (X_t - X_{t|t-1}))$, and by a similar argument $X_t - X_{t|t-1}$ and $X_s - X_{s|s-1}$ are orthogonal.

Thus by using projections we have created an orthogonal basis $X_1, (X_2 - X_{2|1}), \ldots, (X_t - X_{t|t-1})$ of the space $\overline{\text{sp}}(X_1, (X_2 - X_{2|1}), \ldots, (X_t - X_{t|t-1}))$. By construction it clear that $\overline{\text{sp}}(X_1, (X_2 - X_{2|1}), \ldots, (X_t - X_{t|t-1}))$ is a subspace of $\overline{\text{sp}}(X_t, \ldots, X_1)$. We now show that
$\overline{\text{sp}}(X_1, (X_2 - X_{2|1}), \ldots, (X_t - X_{t|t-1})) = \overline{\text{sp}}(X_t, \ldots, X_1)$.

To do this we define the sum of spaces. If U and V are two orthogonal vector spaces (which share the same innerproduct), then $y \in U \oplus V$, if there exists a $u \in U$ and $v \in V$ such that $y = u + v$. By the definition of $\mathcal{X}_t^1$, it is clear that $(X_t - X_{t|t-1}) \in \mathcal{X}_t^1$, but $(X_t - X_{t|t-1}) \notin \mathcal{X}_{t-1}^1$. Hence $\mathcal{X}_t^1 = \bar{sp}(X_t - X_{t|t-1}) \oplus \mathcal{X}_{t-1}^1$. Continuing this argument we see that $\mathcal{X}_t^1 = \bar{sp}(X_t - X_{t|t-1}) \oplus \bar{sp}(X_{t-1} - X_{t-1|t-2}) \oplus, \ldots, \oplus \bar{sp}(X_1)$. Hence $\bar{sp}(X_t, \ldots, X_1) = \bar{sp}(X_t - X_{t|t-1}, \ldots, X_2 - X_{2|1}, X_1)$. Therefore for every $P_{X_t,\ldots,X_1}(Y) = \sum_{j=1}^{t} a_j X_{t+1-j}$, there exists coefficients $\{b_j\}$ such that

$$P_{X_t,\ldots,X_1}(Y) = P_{X_t - X_{t|t-1},\ldots,X_2 - X_{2|1},X_1}(Y) = \sum_{j=1}^{t} P_{X_{t+1-j} - X_{t+1-j|t-j}}(Y) = \sum_{j=1}^{t-1} b_j (X_{t+1-j} - X_{t+1-j|t-j}) + b_t X_1,$$

where $b_j = \text{E}(Y(X_j - X_{j|j-1}))/\text{E}(X_j - X_{j|j-1}))^2$. A useful application of orthogonal basis is the ease of obtaining the coefficients b_j, which avoids the inversion of a matrix. This is the underlying idea behind the innovations algorithm proposed in Brockwell and Davis (1998), Chapter 5.

6.3.1 Spaces spanned by infinite number of elements

The notions above can be generalised to spaces which have an infinite number of elements in their basis (and are useful to prove Wold's decomposition theorem). Let now construct the space spanned by infinite number random variables $\{X_t, X_{t-1}, \ldots\}$. As with anything that involves ∞ we need to define precisely what we mean by an infinite basis. To do this we construct a sequence of subspaces, each defined with a finite number of elements in the basis. We increase the number of elements in the subspace and consider the limit of this space. Let $\mathcal{X}_t^{-n} = \overline{\text{sp}}(X_t, \ldots, X_{-n})$, clearly if $m > n$, then $\mathcal{X}_t^{-n} \subset \mathcal{X}_t^{-m}$. We define $X_t^{-\infty}$, as $X_t^{-\infty} = \cup_{n=1}^{\infty} \mathcal{X}_t^{-n}$, in other words if $Y \in \mathcal{X}_t^{-\infty}$, then there exists an n such that $Y \in \mathcal{X}_t^{-n}$. However, we also need to ensure that the limits of all the sequences lie in this infinite dimensional space, therefore we close the space by defining defining a new space which includes the old space and also includes all the limits. To make this precise suppose the sequence of random variables is such that $Y_s \in \mathcal{X}_t^{-s}$, and $\mathrm{E}(Y_{s_1} - Y_{s_2})^2 \to 0$ as $s_1, s_2 \to \infty$. Since the sequence $\{Y_s\}$ is a Cauchy sequence there exists a limit. More precisely, there exists a random variable Y, such that $\mathrm{E}(Y_s - Y)^2 \to 0$ as $s \to \infty$. Since the closure of the space, $\overline{\mathcal{X}_t}^{-n}$, contains the set $\mathcal{X}_t^{-n}$ and all the limits of the Cauchy sequences in this set, then $Y \in \overline{\mathcal{X}_t^{-\infty}}$. We let

$$\overline{\mathcal{X}_t^{-\infty}} = \overline{\text{sp}}(X_t, X_{t-1}, \ldots), \tag{6.6}$$

The orthogonal basis of $\overline{\text{sp}}(X_t, X_{t-1}, \ldots)$

An orthogonal basis of $\overline{\text{sp}}(X_t, X_{t-1}, \ldots)$ can be constructed using the same method used to orthogonalize $\overline{\text{sp}}(X_t, X_{t-1}, \ldots, X_1)$. The main difference is how to deal with the initial value, which in the case of $\overline{\text{sp}}(X_t, X_{t-1}, \ldots, X_1)$ is X_1. The analogous version of the initial value in infinite dimension space $\overline{\text{sp}}(X_t, X_{t-1}, \ldots)$ is $X_{-\infty}$, but this it not a well defined quantity (again we have to be careful with these pesky infinities).

Let $X_{t-1}(1)$ denote the best linear predictor of X_t given $X_{t-1}, X_{t-2}, \ldots$. As in Section 6.3 it is clear that $(X_t - X_{t-1}(1))$ and X_s for $s \leq t-1$ are uncorrelated and $\overline{X_t^{-\infty}} = \overline{\text{sp}}(X_t - X_{t-1}(1)) \oplus \overline{X_{t-1}^{-\infty}}$, where $\overline{X_t^{-\infty}} = \overline{\text{sp}}(X_t, X_{t-1}, \ldots)$. Thus we can construct the orthogonal basis $(X_t - X_{t-1}(1)), (X_{t-1} - X_{t-2}(1)), \ldots$ and the corresponding space $\overline{\text{sp}}((X_t - X_{t-1}(1)), (X_{t-1} - X_{t-2}(1)), \ldots)$. It is clear that $\overline{\text{sp}}((X_t - X_{t-1}(1)), (X_{t-1} - X_{t-2}(1)), \ldots) \subset \overline{\text{sp}}(X_t, X_{t-1}, \ldots)$. However, unlike the finite dimensional case it is not clear that they are equal, roughly speaking this is because $\overline{\text{sp}}((X_t - X_{t-1}(1)), (X_{t-1} - X_{t-2}(1)), \ldots)$ lacks the inital value $X_{-\infty}$. Of course the time $-\infty$ in the past is not really a well defined quantity. Instead, the way we overcome this issue is that we define the initial starting random variable as the intersection of the subspaces, more precisely let $\mathcal{X}_{-\infty} = \cap_{n=-\infty}^{\infty} \mathcal{X}_t^{-\infty}$. Furthermore, we note that since $X_n - X_{n-1}(1)$ and X_s (for any $s \leq n$) are orthogonal, then $\overline{\text{sp}}((X_t - X_{t-1}(1)), (X_{t-1} - X_{t-2}(1)), \ldots)$ and $\mathcal{X}_{-\infty}$ are orthogonal spaces. Using $\mathcal{X}_{-\infty}$, we have

$\oplus_{j=0}^{t}\overline{\text{sp}}((X_{t-j} - X_{t-j-1}(1)) \oplus \mathcal{X}_{-\infty} = \overline{\text{sp}}(X_t, X_{t-1}, \ldots)$.

We will use this result when we prove the Wold decomposition theorem (in Section 6.9).

6.4 Levinson-Durbin algorithm

We recall that in prediction the aim is to predict X_{t+1} given $X_t, X_{t-1}, \ldots, X_1$. The best linear predictor is

$$X_{t+1|t} = P_{X_1,\ldots,X_t}(X_{t+1}) = X_{t+1|t,\ldots,1} = \sum_{j=1}^{t} \phi_{t,j} X_{t+1-j}, \qquad (6.7)$$

where $\{\phi_{t,j}\}$ are chosen to minimise the mean squared error, and are the solution of the equation

$$\begin{pmatrix} \phi_{t,1} \\ \vdots \\ \phi_{t,t} \end{pmatrix} = \Sigma_t^{-1} \underline{r}_t, \qquad (6.8)$$

where $(\Sigma_t)_{i,j} = \mathrm{E}(X_i X_j)$ and $(\underline{r}_t)_i = \mathrm{E}(X_{t-i} X_{t+1})$. Using standard methods, such as Gauss-Jordan elimination, to solve this system of equations requires $O(t^3)$ operations. However, we recall that $\{X_t\}$ is a stationary time series, thus Σ_t is a Toeplitz matrix, by using this information in the 1940s Norman Levinson proposed an algorithm which reduced the number of operations to $O(t^2)$. In the 1960s, Jim Durbin adapted the algorithm to time series and improved it.

We first outline the algorithm. We recall that the best linear predictor of X_{t+1} given $X_t, \ldots, X_1$ is

$$X_{t+1|t} = \sum_{j=1}^{t} \phi_{t,j} X_{t+1-j}. \qquad (6.9)$$

The mean squared error is $r(t+1) = \mathrm{E}[X_{t+1} - X_{t+1|t}]^2$. Given that the second order stationary covariance structure, the idea of the Levinson-Durbin algorithm is to recursively estimate $\{\phi_{t,j}; j = 1, \ldots, t\}$ given $\{\phi_{t-1,j}; j = 1, \ldots, t-1\}$ (which are the coefficients of the best linear predictor of X_t given $X_{t-1}, \ldots, X_1$). Let us suppose that the autocovariance function $c(k) = \mathrm{cov}[X_0, X_k]$ is known. The Levinson-Durbin algorithm is calculated using the following recursion.

Step 1 $\phi_{1,1} = c(1)/c(0)$ and $r(2) = \mathrm{E}[X_2 - X_{2|1}]^2 = \mathrm{E}[X_2 - \phi_{1,1}X_1]^2 = 2c(0) - 2\phi_{1,1}c(1)$.

Step 2 For $j = t$

$$\begin{aligned}
\phi_{t,t} &= \frac{c(t) - \sum_{j=1}^{t-1} \phi_{t-1,j} c(t-j)}{r(t)} \\
\phi_{t,j} &= \phi_{t-1,j} - \phi_{t,t}\phi_{t-1,t-j} \qquad 1 \leq j \leq t-1, \\
\text{and } r(t+1) &= r(t)(1-\phi_{t,t}^2).
\end{aligned}$$

We give two proofs of the above recursion.

Exercise 6.2 *(i) Suppose $X_t = \phi X_{t-1} + \varepsilon_t$ (where $|\phi| < 1$). Use the Levinson-Durbin algorithm, to deduce an expression for $\phi_{t,j}$ for ($1 \leq j \leq t$).*

(ii) Suppose $X_t = \phi\varepsilon_{t-1} + \varepsilon_t$ (where $|\phi| < 1$). Use the Levinson-Durbin algorithm (and possibly Maple/Matlab), deduce an expression for $\phi_{t,j}$ for ($1 \leq j \leq t$). (recall from Exercise 4.4 that you already have an analytic expression for $\phi_{t,t}$).

6.4.1 A proof based on projections

Let us suppose $\{X_t\}$ is a zero mean stationary time series and $c(k) = \mathrm{E}(X_k X_0)$. Let $P_{X_t,\ldots,X_2}(X_1)$ denote the best linear predictor of X_1 given $X_t, \ldots, X_2$ and $P_{X_t,\ldots,X_2}(X_{t+1})$ denote the best linear predictor of X_{t+1} given $X_t, \ldots, X_2$. Stationarity means that the following predictors share the same coefficients

$$\begin{aligned}
X_{t|t-1} &= \sum_{j=1}^{t-1} \phi_{t-1,j} X_{t-j} \qquad P_{X_t,\ldots,X_2}(X_{t+1}) = \sum_{j=1}^{t-1} \phi_{t-1,j} X_{t+1-j} \qquad (6.10) \\
P_{X_t,\ldots,X_2}(X_1) &= \sum_{j=1}^{t-1} \phi_{t-1,j} X_{j+1}.
\end{aligned}$$

The last line is because stationarity means that flipping a time series round has the same correlation structure. These three relations are an important component of the proof.

Recall our objective is to derive the coefficients of the best linear predictor of $P_{X_t,\ldots,X_1}(X_{t+1})$ based on the coefficients of the best linear predictor $P_{X_{t-1},\ldots,X_1}(X_t)$. To do this we partition the space $\overline{\text{sp}}(X_t, \ldots, X_2, X_1)$ into two orthogonal spaces $\overline{\text{sp}}(X_t, \ldots, X_2, X_1) = \overline{\text{sp}}(X_t, \ldots, X_2, X_1) \oplus$

$\overline{\text{sp}}(X_1 - P_{X_t,\ldots,X_2}(X_1))$. Therefore by uncorrelatedness we have the partition

$$
\begin{aligned}
X_{t+1|t} &= P_{X_t,\ldots,X_2}(X_{t+1}) + P_{X_1 - P_{X_t,\ldots,X_2}(X_1)}(X_{t+1}) \\
&= \underbrace{\sum_{j=1}^{t-1} \phi_{t-1,j} X_{t+1-j}}_{\text{by (6.10)}} + \underbrace{\phi_{tt}\left(X_1 - P_{X_t,\ldots,X_2}(X_1)\right)}_{\text{by projection onto one variable}} \\
&= \sum_{j=1}^{t-1} \phi_{t-1,j} X_{t+1-j} + \phi_{t,t}\left(X_1 - \underbrace{\sum_{j=1}^{t-1} \phi_{t-1,j} X_{j+1}}_{\text{by (6.10)}} \right). \qquad (6.11)
\end{aligned}
$$

We start by evaluating an expression for $\phi_{t,t}$ (which in turn will give the expression for the other coefficients). It is straightforward to see that

$$
\begin{aligned}
\phi_{t,t} &= \frac{\text{E}(X_{t+1}(X_1 - P_{X_t,\ldots,X_2}(X_1)))}{\text{E}(X_1 - P_{X_t,\ldots,X_2}(X_1))^2} \qquad (6.12) \\
&= \frac{\text{E}[(X_{t+1} - P_{X_t,\ldots,X_2}(X_{t+1}) + P_{X_t,\ldots,X_2}(X_{t+1}))(X_1 - P_{X_t,\ldots,X_2}(X_1))]}{\text{E}(X_1 - P_{X_t,\ldots,X_2}(X_1))^2} \\
&= \frac{\text{E}[(X_{t+1} - P_{X_t,\ldots,X_2}(X_{t+1}))(X_1 - P_{X_t,\ldots,X_2}(X_1))]}{\text{E}(X_1 - P_{X_t,\ldots,X_2}(X_1))^2}
\end{aligned}
$$

Therefore we see that the numerator of $\phi_{t,t}$ is the partial covariance between X_{t+1} and X_1 (see Section 4.3), furthermore the denominator of $\phi_{t,t}$ is the mean squared prediction error, since by stationarity

$$
\text{E}(X_1 - P_{X_t,\ldots,X_2}(X_1))^2 = \text{E}(X_t - P_{X_{t-1},\ldots,X_1}(X_t))^2 = r(t) \qquad (6.13)
$$

Returning to (6.12), expanding out the expectation in the numerator and using (6.13) we have

$$
\phi_{t,t} = \frac{\text{E}(X_{t+1}(X_1 - P_{X_t,\ldots,X_2}(X_1)))}{r(t)} = \frac{c(0) - \text{E}[X_{t+1} P_{X_t,\ldots,X_2}(X_1))]}{r(t)} = \frac{c(0) - \sum_{j=1}^{t-1} \phi_{t-1,j} c(t-j)}{r(t)}, \qquad (6.14)
$$

which immediately gives us the first equation in Step 2 of the Levinson-Durbin algorithm. To obtain the recursion for $\phi_{t,j}$ we use (6.11) to give

$$\begin{aligned} X_{t+1|t} &= \sum_{j=1}^{t} \phi_{t,j} X_{t+1-j} \\ &= \sum_{j=1}^{t-1} \phi_{t-1,j} X_{t+1-j} + \phi_{t,t} \left(X_1 - \sum_{j=1}^{t-1} \phi_{t-1,j} X_{j+1} \right). \end{aligned}$$

To obtain the recursion we simply compare coefficients to give

$$\phi_{t,j} = \phi_{t-1,j} - \phi_{t,t}\phi_{t-1,t-j} \qquad 1 \leq j \leq t-1.$$

This gives the middle equation in Step 2. To obtain the recursion for the mean squared prediction error we note that by orthogonality of $\{X_t, \ldots, X_2\}$ and $X_1 - P_{X_t,\ldots,X_2}(X_1)$ we use (6.11) to give

$$\begin{aligned} r(t+1) &= \mathrm{E}(X_{t+1} - X_{t+1|t})^2 = \mathrm{E}[X_{t+1} - P_{X_t,\ldots,X_2}(X_{t+1}) - \phi_{t,t}(X_1 - P_{X_t,\ldots,X_2}(X_1)]^2 \\ &= \mathrm{E}[X_{t+1} - P_{X_2,\ldots,X_t}(X_{t+1})]^2 + \phi_{t,t}^2 \mathrm{E}[X_1 - P_{X_t,\ldots,X_2}(X_1)]^2 \\ &\quad -2\phi_{t,t}\mathrm{E}[(X_{t+1} - P_{X_t,\ldots,X_2}(X_{t+1}))(X_1 - P_{X_t,\ldots,X_2}(X_1))] \\ &= r(t) + \phi_{t,t}^2 r(t) - 2\phi_{t,t} \underbrace{\mathrm{E}[X_{t+1}(X_1 - P_{X_t,\ldots,X_2}(X_1))]}_{=r(t)\phi_{t,t} \text{ by } (6.14)} \\ &= r(t)[1 - \phi_{tt}^2]. \end{aligned}$$

This gives the final part of the equation in Step 2 of the Levinson-Durbin algorithm.

Further references: Brockwell and Davis (1998), Chapter 5 and Fuller (1995), pages 82.

6.4.2 A proof based on symmetric Toeplitz matrices

We now give an alternative proof which is based on properties of the (symmetric) Toeplitz matrix. We use (6.8), which is a matrix equation where

$$\Sigma_t \begin{pmatrix} \phi_{t,1} \\ \vdots \\ \phi_{t,t} \end{pmatrix} = \underline{r}_t, \tag{6.15}$$

with

$$\Sigma_t = \begin{pmatrix} c(0) & c(1) & c(2) & \ldots & c(t-1) \\ c(1) & c(0) & c(1) & \ldots & c(t-2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c(t-1) & c(t-2) & \vdots & \vdots & c(0) \end{pmatrix} \qquad \text{and} \qquad \underline{r}_t = \begin{pmatrix} c(1) \\ c(2) \\ \vdots \\ c(t) \end{pmatrix}.$$

The proof is based on embedding $\underline{r}_{t-1}$ and Σ_{t-1} into Σ_{t-1} and using that $\Sigma_{t-1}\underline{\phi}_{t-1} = \underline{r}_{t-1}$.

To do this, we define the $(t-1) \times (t-1)$ matrix E_{t-1} which basically swops round all the elements in a vector

$$E_{t-1} = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & 0 & \ldots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ 1 & 0 & \vdots & 0 & 0 & 0 \end{pmatrix},$$

(recall we came across this swopping matrix in Section 4.3). Using the above notation, we have the interesting block matrix structure

$$\begin{aligned} \Sigma_t &= \begin{pmatrix} \Sigma_{t-1} & E_{t-1}\underline{r}_{t-1} \\ \underline{r}'_{t-1}E_{t-1} & c(0) \end{pmatrix} \\ \text{and } \underline{r}_t &= (\underline{r}'_{t-1}, c(t))'. \end{aligned}$$

Returning to the matrix equations in (6.15) and substituting the above into (6.15) we have

$$\Sigma_t\underline{\phi}_t = \underline{r}_t, \quad \Rightarrow \quad \begin{pmatrix} \Sigma_{t-1} & E_{t-1}\underline{r}_{t-1} \\ \underline{r}'_{t-1}E_{t-1} & c(0) \end{pmatrix} \begin{pmatrix} \underline{\phi}_{t-1,t} \\ \phi_{t,t} \end{pmatrix} = \begin{pmatrix} \underline{r}_{t-1} \\ c(t) \end{pmatrix},$$

where $\underline{\phi}'_{t-1,t} = (\phi_{1,t}, \ldots, \phi_{t-1,t})$. This leads to the two equations

$$\Sigma_{t-1}\underline{\phi}_{t-1,t} + E_{t-1}\underline{r}_{t-1}\phi_{t,t} = \underline{r}_{t-1} \tag{6.16}$$

$$\underline{r}'_{t-1}E_{t-1}\underline{\phi}_{t-1,t} + c(0)\phi_{t,t} = c(t). \tag{6.17}$$

We first show that equation (6.16) corresponds to the second equation in the Levinson-Durbin algorithm. Multiplying (6.16) by Σ_{t-1}^{-1}, and rearranging the equation we have

$$\underline{\phi}_{t-1,t} = \underbrace{\Sigma_{t-1}^{-1}\underline{r}_{t-1}}_{=\underline{\phi}_{t-1}} - \underbrace{\Sigma_{t-1}^{-1}E_{t-1}\underline{r}_{t-1}}_{=E_{t-1}\underline{\phi}_{t-1}}\phi_{t,t}.$$

Thus we have

$$\underline{\phi}_{t-1,t} = \underline{\phi}_{t-1} - \phi_{t,t}E_{t-1}\underline{\phi}_{t-1}. \qquad (6.18)$$

This proves the second equation in Step 2 of the Levinson-Durbin algorithm.

We now use (6.17) to obtain an expression for $\phi_{t,t}$, which is the first equation in Step 1. Substituting (6.18) into $\underline{\phi}_{t-1,t}$ of (6.17) gives

$$\underline{r}'_{t-1}E_{t-1}\left(\underline{\phi}_{t-1} - \phi_{t,t}E_{t-1}\underline{\phi}_{t-1}\right) + c(0)\phi_{t,t} = c(t). \qquad (6.19)$$

Thus solving for $\phi_{t,t}$ we have

$$\phi_{t,t} = \frac{c(t) - \underline{c}'_{t-1}E_{t-1}\underline{\phi}_{t-1}}{c(0) - \underline{c}'_{t-1}\underline{\phi}'_{t-1}}. \qquad (6.20)$$

Noting that $r(t) = c(0) - \underline{c}'_{t-1}\underline{\phi}'_{t-1}$. (6.20) is the first equation of Step 2 in the Levinson-Durbin equation.

Note from this proof we do not need that the (symmetric) Toeplitz matrix is positive semi-definite. See Pourahmadi (2001), Chapter 7.

6.4.3 Using the Durbin-Levinson to obtain the Cholesky decomposition of the precision matrix

We recall from Section 4.2.5 that by sequentially projecting the elements of random vector on the past elements in the vector gives rise to Cholesky decomposition of the inverse of the variance/co-variance (precision) matrix. This is exactly what was done in when we make the Durbin-Levinson algorithm. In other words,

$$\mathrm{var}\left(\begin{array}{c} \frac{X_1}{\sqrt{r(1)}} \\ \frac{X_1-\phi_{1,1}X_2}{\sqrt{r(2)}} \\ \vdots \\ \frac{X_n-\sum_{j=1}^{n-1}\phi_{n-1,j}X_{n-j}}{\sqrt{r(n)}} \end{array}\right) = I_n$$

Therefore, if $\Sigma_n = \mathrm{var}[\underline{X}_n]$, where $\underline{X}_n = (X_1, \ldots, X_n)$, then $\Sigma_n^{-1} = L_n D_n L_n'$, where

$$L_n = \begin{pmatrix} 1 & 0 & \ldots & \ldots & \ldots & 0 \\ -\phi_{1,1} & 1 & 0 & \ldots & \ldots & 0 \\ -\phi_{2,2} & -\phi_{2,1} & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ -\phi_{n-1,n-1} & -\phi_{n-1,n-2} & -\phi_{n-1,n-3} & \ldots & \ldots & 1 \end{pmatrix} \qquad (6.21)$$

and $D_n = \mathrm{diag}(r_1^{-1}, r_2^{-1}, \ldots, r_n^{-1})$.

6.5 Forecasting for ARMA processes

Given the autocovariance of any stationary process the Levinson-Durbin algorithm allows us to systematically obtain one-step predictors of second order stationary time series without directly inverting a matrix.

In this section we consider forecasting for a special case of stationary processes, the ARMA process. We will assume throughout this section that the parameters of the model are known.

We showed in Section 6.2 that if $\{X_t\}$ has an AR(p) representation and $t > p$, then the best linear predictor can easily be obtained using (6.4). Therefore, when $t > p$, there is no real gain in using the Levinson-Durbin for prediction of AR(p) processes. However, we do use it in Section 8.1.1 for recursively obtaining estimators of autoregressive parameters at increasingly higher orders.

Similarly if $\{X_t\}$ satisfies an ARMA(p, q) representation, then the prediction scheme can be simplified. Unlike the AR(p) process, which is p-Markovian, $P_{X_t,X_{t-1},\ldots,X_1}(X_{t+1})$ does involve all regressors $X_t, \ldots, X_1$. However, some simplifications are still posssible. To explain how, let us suppose that X_t satisfies the ARMA(p, q) representation

$$X_t - \sum_{j=1}^{p} \phi_i X_{t-j} = \varepsilon_t + \sum_{i=1}^{q} \theta_i \varepsilon_{t-i},$$

where $\{\varepsilon_t\}$ are iid zero mean random variables and the roots of $\phi(z)$ and $\theta(z)$ lie outside the unit circle. For the analysis below, we define the variables $\{W_t\}$, where $W_t = X_t$ for $1 \leq t \leq p$ and for $t > \max(p, q)$ let $W_t = \varepsilon_t + \sum_{i=1}^{q} \theta_i \varepsilon_{t-i}$ (which is the MA(q) part of the process). Since $X_{p+1} = \sum_{j=1}^{p} \phi_j X_{t+1-j} + W_{p+1}$ and so forth it is clear that $\overline{\text{sp}}(X_1, \ldots, X_t) = \overline{\text{sp}}(W_1, \ldots, W_t)$ (i.e. they are linear combinations of each other). We will show for $t > \max(p, q)$ that

$$X_{t+1|t} = P_{X_t,\ldots,X_1}(X_{t+1}) = \sum_{j=1}^{p} \phi_j X_{t+1-j} + \sum_{i=1}^{q} \theta_{t,i}(X_{t+1-i} - X_{t+1-i|t-i}), \tag{6.22}$$

for some $\theta_{t,i}$ which can be evaluated from the autocovariance structure. To prove the result we use the following steps:

$$\begin{aligned}
P_{X_t,\ldots,X_1}(X_{t+1}) &= \sum_{j=1}^{p} \phi_j \underbrace{P_{X_t,\ldots,X_1}(X_{t+1-j})}_{X_{t+1-j}} + \sum_{i=1}^{q} \theta_i P_{X_t,\ldots,X_1}(\varepsilon_{t+1-i}) \\
&= \sum_{j=1}^{p} \phi_j X_{t+1-j} + \sum_{i=1}^{q} \theta_i \underbrace{P_{X_t - X_{t|t-1},\ldots,X_2 - X_{2|1},X_1}(\varepsilon_{t+1-i})}_{=P_{W_t - W_{t|t-1},\ldots,W_2 - W_{2|1},W_1}(\varepsilon_{t+1-i})} \\
&= \sum_{j=1}^{p} \phi_j X_{t+1-j} + \sum_{i=1}^{q} \theta_i P_{W_t - W_{t|t-1},\ldots,W_2 - W_{2|1},W_1}(\varepsilon_{t+1-i}) \\
&= \sum_{j=1}^{p} \phi_j X_{t+1-j} + \sum_{i=1}^{q} \theta_i \underbrace{P_{W_{t+1-i} - W_{t+1-i|t-i},\ldots,W_t - W_{t|t-1}}(\varepsilon_{t+1-i})}_{\text{since } \varepsilon_{t+1-i} \text{is independent of } W_{t+1-i-j};j\geq 1} \\
&= \sum_{j=1}^{p} \phi_j X_{t+1-j} + \sum_{i=1}^{q} \theta_i \sum_{s=0}^{i-1} \underbrace{P_{W_{t+1-i+s} - W_{t+1-i+s|t-i+s}}(\varepsilon_{t+1-i})}_{\text{since } W_{t+1-i+s} - W_{t+1-i+s|t-i+s} \text{ are uncorrelated}} \\
&= \sum_{j=1}^{p} \phi_j X_{t+1-j} + \sum_{i=1}^{q} \theta_{t,i} \underbrace{(W_{t+1-i} - W_{t+1-i|t-i})}_{=X_{t+1-i} - X_{t+1-i|t-i}} \\
&= \sum_{j=1}^{p} \phi_j X_{t+1-j} + \sum_{i=1}^{q} \theta_{t,i}(X_{t+1-i} - X_{t+1-i|t-i}), \qquad (6.23)
\end{aligned}$$

this gives the desired result. Thus given the parameters $\{\theta_{t,i}\}$ is straightforward to construct the predictor $X_{t+1|t}$. It can be shown that $\theta_{t,i} \to \theta_i$ as $t \to \infty$ (see Brockwell and Davis (1998)), Chapter 5.

Example 6.5.1 (MA(q)) *In this case, the above result reduces to*

$$\widehat{X}_{t+1|t} = \sum_{i=1}^{q} \theta_{t,i} \left(X_{t+1-i} - \widehat{X}_{t+1-i|t-i} \right).$$

We now state a few results which will be useful later.

Lemma 6.5.1 *Suppose $\{X_t\}$ is a stationary time series with spectral density $f(\omega)$. Let $\boldsymbol{X}_t = (X_1, \ldots, X_t)$ and $\Sigma_t = \mathrm{var}(\boldsymbol{X}_t)$.*

(i) If the spectral density function is bounded away from zero (there is some $\gamma > 0$ such that $\inf_\omega f(\omega) > 0$), then for all t, $\lambda_{min}(\Sigma_t) \geq \gamma$ (where $\lambda_{\min}$ and $\lambda_{\max}$ denote the smallest and largest absolute eigenvalues of the matrix).

(ii) Further, $\lambda_{max}(\Sigma_t^{-1}) \leq \gamma^{-1}$.

(Since for symmetric matrices the spectral norm and the largest eigenvalue are the same, then $\|\Sigma_t^{-1}\|_{spec} \leq \gamma^{-1}$).

(iii) Analogously, $\sup_\omega f(\omega) \leq M < \infty$, then $\lambda_{\max}(\Sigma_t) \leq M$ (hence $\|\Sigma_t\|_{spec} \leq M$).

PROOF. See Chapter 9. □

Remark 6.5.1 *Suppose $\{X_t\}$ is an ARMA process, where the roots $\phi(z)$ and and $\theta(z)$ have absolute value greater than $1 + \delta_1$ and less than δ_2, then the spectral density $f(\omega)$ is bounded by $\mathrm{var}(\varepsilon_t)\frac{(1-\frac{1}{\delta_2})^{2p}}{(1-(\frac{1}{1+\delta_1})^{2p}} \leq f(\omega) \leq \mathrm{var}(\varepsilon_t)\frac{(1-(\frac{1}{1+\delta_1})^{2p}}{(1-\frac{1}{\delta_2})^{2p}}$. Therefore, from Lemma 6.5.1 we have that $\lambda_{\max}(\Sigma_t)$ and $\lambda_{\max}(\Sigma_t^{-1})$ is bounded uniformly over t.*

The prediction can be simplified if we make a simple approximation (which works well if t is relatively large). For $1 \leq t \leq \max(p, q)$, set $\widehat{X}_{t+1|t} = X_t$ and for $t > \max(p, q)$ we define the recursion

$$\widehat{X}_{t+1|t} = \sum_{j=1}^{p} \phi_j X_{t+1-j} + \sum_{i=1}^{q} \theta_i (X_{t+1-i} - \widehat{X}_{t+1-i|t-i}). \tag{6.24}$$

This approximation seems plausible, since in the exact predictor (6.23), $\theta_{t,i} \to \theta_i$. Note that this approximation is often used the case of prediction of other models too. We now derive a bound for this approximation. In the following proposition we show that the best linear predictor of X_{t+1} given $X_1, \ldots, X_t$, $X_{t+1|t}$, the approximating predictor $\widehat{X}_{t+1|t}$ and the best linear predictor given the infinite past, $X_t(1)$ are asymptotically equivalent. To do this we obtain expressions for $X_t(1)$ and $\widehat{X}_{t+1|t}$

$$X_t(1) \;=\; \sum_{j=1}^{\infty} b_j X_{t+1-j} (\text{ since } X_{t+1} = \sum_{j=1}^{\infty} b_j X_{t+1-j} + \varepsilon_{t+1}).$$

Furthermore, by iterating (6.24) backwards we can show that

$$\widehat{X}_{t+1|t} = \underbrace{\sum_{j=1}^{t-\max(p,q)} b_j X_{t+1-j}}_{\text{part of AR}(\infty)\text{ expansion}} + \sum_{j=1}^{\max(p,q)} \gamma_j X_j \tag{6.25}$$

where $|\gamma_j| \leq C\rho^t$, with $1/(1+\delta) < \rho < 1$ and the roots of $\theta(z)$ are outside $(1+\delta)$. We give a proof in the remark below.

Remark 6.5.2 *We prove (6.25) for the MA*(1) *model* $X_t = \theta X_{t-1} + \varepsilon_t$. *We recall that* $X_{t-1}(1) = \sum_{j=0}^{t-1}(-\theta)^j X_{t-j-1}$ *and*

$$\begin{aligned} \widehat{X}_{t|t-1} &= \theta\left(X_{t-1} - \widehat{X}_{t-1|t-2}\right) \\ \Rightarrow X_t - \widehat{X}_{t|t-1} &= -\theta\left(X_{t-1} - \widehat{X}_{t-1|t-2}\right) + X_t \\ &= \sum_{j=0}^{t-1}(-\theta)^j X_{t-j-1} + (-\theta)^t\left(X_1 - \widehat{X}_{1|0}\right). \end{aligned}$$

Thus we see that the first $(t-1)$ *coefficients of* $X_{t-1}(1)$ *and* $\widehat{X}_{t|t-1}$ *match. Indeed if we set* $\widehat{X}_{1|0} = 0$, *then*

$$\widehat{X}_{t|t-1} = \sum_{j=0}^{t-1}(-\theta)^j X_{t-j-1} = [(1+\theta B)]_{j=1}^{t} X_t$$

where $[\sum_{s=0}^{\infty} \alpha_s B^s]_{j=1}^t = \sum_{s=1}^t \alpha_s B^s$. *Hence the approximated predictor is simply the truncation of the AR*(∞) *representation of the MA*(1) $(1+\theta B)^{-1} X_t = \varepsilon_t$.

Next, we prove (6.25) for the ARMA$(1,2)$. *We first note that* $\overline{\text{sp}}(X_1, X_t, \ldots, X_t) = \overline{\text{sp}}(W_1, W_2, \ldots, W_t)$, *where* $W_1 = X_1$ *and for* $t \geq 2$ $W_t = \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2} + \varepsilon_t$. *The corresponding approximating predictor* *is defined as $\widehat{W}_{2|1} = W_1$, $\widehat{W}_{3|2} = W_2$ and for $t > 3$*

$$\widehat{W}_{t|t-1} = \theta_1[W_{t-1} - \widehat{W}_{t-1|t-2}] + \theta_2[W_{t-2} - \widehat{W}_{t-2|t-3}].$$

Note that by using (6.24), the above is equivalent to

$$\underbrace{\widehat{X}_{t+1|t} - \phi_1 X_t}_{\widehat{W}_{t+1|t}} = \theta_1 \underbrace{\left[X_t - \widehat{X}_{t|t-1}\right]}_{=(W_t - \widehat{W}_{t|t-1})} + \theta_2 \underbrace{\left[X_{t-1} - \widehat{X}_{t-1|t-2}\right]}_{=(W_{t-1} - \widehat{W}_{t-1|t-2})}.$$

By subtracting the above from W_{t+1} we have

$$W_{t+1} - \widehat{W}_{t+1|t} = -\theta_1(W_t - \widehat{W}_{t|t-1}) - \theta_2(W_{t-1} - \widehat{W}_{t-1|t-2}) + W_{t+1}. \qquad (6.26)$$

It is straightforward to rewrite $W_{t+1} - \widehat{W}_{t+1|t}$ as the matrix difference equation

$$\underbrace{\begin{pmatrix} W_{t+1} - \widehat{W}_{t+1|t} \\ W_t - \widehat{W}_{t|t-1} \end{pmatrix}}_{=\widehat{\underline{\varepsilon}}_{t+1}} = -\underbrace{\begin{pmatrix} \theta_1 & \theta_2 \\ -1 & 0 \end{pmatrix}}_{=Q} \underbrace{\begin{pmatrix} W_t - \widehat{W}_{t|t-1} \\ W_{t-1} - \widehat{W}_{t-1|t-2} \end{pmatrix}}_{=\widehat{\underline{\varepsilon}}_t} + \underbrace{\begin{pmatrix} W_{t+1} \\ 0 \end{pmatrix}}_{\underline{W}_{t+1}}$$

We now show that ε_{t+1} and $W_{t+1} - \widehat{W}_{t+1|t}$ lead to the same difference equation except for some initial conditions, it is this that will give us the result. To do this we write ε_t as function of $\{W_t\}$ (the irreducible condition). We first note that ε_t can be written as the matrix difference equation

$$\underbrace{\begin{pmatrix} \varepsilon_{t+1} \\ \varepsilon_t \end{pmatrix}}_{=\underline{\varepsilon}_{t+1}} = -\underbrace{\begin{pmatrix} \theta_1 & \theta_2 \\ -1 & 0 \end{pmatrix}}_{Q} \underbrace{\begin{pmatrix} \varepsilon_t \\ \varepsilon_{t-1} \end{pmatrix}}_{\underline{\varepsilon}_t} + \underbrace{\begin{pmatrix} W_{t+1} \\ 0 \end{pmatrix}}_{\underline{W}_{t+1}} \qquad (6.27)$$

Thus iterating backwards we can write

$$\varepsilon_{t+1} = \sum_{j=0}^{\infty} (-1)^j [Q^j]_{(1,1)} W_{t+1-j} = \sum_{j=0}^{\infty} \tilde{b}_j W_{t+1-j},$$

where $\tilde{b}_j = (-1)^j [Q^j]_{(1,1)}$ (noting that $\tilde{b}_0 = 1$) denotes the $(1,1)$th element of the matrix Q^j (note *we did something similar in Section 3.4.2). Furthermore the same iteration shows that*

$$\begin{aligned}\varepsilon_{t+1} &= \sum_{j=0}^{t-3}(-1)^j[Q^j]_{(1,1)}W_{t+1-j} + (-1)^{t-2}[Q^{t-2}]_{(1,1)}\varepsilon_3 \\ &= \sum_{j=0}^{t-3}\tilde{b}_j W_{t+1-j} + (-1)^{t-2}[Q^{t-2}]_{(1,1)}\varepsilon_3. \qquad (6.28)\end{aligned}$$

Therefore, by comparison we see that

$$\varepsilon_{t+1} - \sum_{j=0}^{t-3}\tilde{b}_j W_{t+1-j} = (-1)^{t-2}[Q^{t-2}\underline{\varepsilon}_3]_1 = \sum_{j=t-2}^{\infty}\tilde{b}_j W_{t+1-j}.$$

We now return to the approximation prediction in (6.26). Comparing (6.27) and (6.27) we see that they are almost the same difference equations. The only difference is the point at which the algorithm starts. $\underline{\varepsilon}_t$ *goes all the way back to the start of time. Whereas we have set initial values for* $\widehat{W}_{2|1} = W_1$, $\widehat{W}_{3|2} = W_2$, *thus* $\widehat{\underline{\varepsilon}}_3' = (W_3 - W_2, W_2 - W_1)$.*Therefore, by iterating both (6.27) and (6.27) backwards, focusing on the first element of the vector and using (6.28) we have*

$$\varepsilon_{t+1} - \widehat{\varepsilon}_{t+1} = \underbrace{(-1)^{t-2}[Q^{t-2}\underline{\varepsilon}_3]_1}_{=\sum_{j=t-2}^{\infty}\tilde{b}_j W_{t+1-j}} + (-1)^{t-2}[Q^{t-2}\widehat{\underline{\varepsilon}}_3]_1$$

We recall that $\varepsilon_{t+1} = W_{t+1} + \sum_{j=1}^{\infty}\tilde{b}_j W_{t+1-j}$ *and that* $\widehat{\varepsilon}_{t+1} = W_{t+1} - \widehat{W}_{t+1|t}$. *Substituting this into the above gives*

$$\widehat{W}_{t+1|t} - \sum_{j=1}^{\infty}\tilde{b}_j W_{t+1-j} = \sum_{j=t-2}^{\infty}\tilde{b}_j W_{t+1-j} + (-1)^{t-2}[Q^{t-2}\widehat{\underline{\varepsilon}}_3]_1.$$

Replacing W_t *with* $X_t - \phi_1 X_{t-1}$ *gives (6.25), where the* b_j *can be easily deduced from* $\tilde{b}_j$ *and* ϕ_1.

In summary we have three one-step ahead predictors

$$\begin{aligned}
X_{t+1|t} &= \sum_{j=1}^{p}\phi_j X_{t+1-j} + \sum_{i=1}^{q}\theta_{i,t}(X_{t+1-i} - \widehat{X}_{t+1-i|t-i}) = \sum_{s=1}^{t}\phi_{t,s}X_{t+1-s} \\
X_t(1) &= \sum_{j=1}^{p}\phi_j X_{t+1-j} + \sum_{i=1}^{q}\theta_i(X_{t+1-i} - X_{t-i}(1)) = \sum_{s=1}^{\infty}a_j X_{t+1-s} \\
\widehat{X}_{t+1|t} &= \sum_{j=1}^{p}\phi_j X_{t+1-j} + \sum_{i=1}^{q}\theta_i(X_{t+1-i} - \widehat{X}_{t-i+1|t-i}) = \sum_{s=1}^{t}a_j X_{t+1-s} + \underbrace{\sum_{s=1}^{\max(p,q)} b_s X_s}_{\text{due to initial conditions}} \quad .
\end{aligned}$$

These predictors will be very useful in deriving the approximate Gaussian likelihood for the ARMA model, see Section 8.2.2. We give a bound for the differences below.

Proposition 6.5.1 *Suppose $\{X_t\}$ is an ARMA process where the roots of $\phi(z)$ and $\theta(z)$ have roots which are greater in absolute value than $1+\delta$. Let $X_{t+1|t}$, $\widehat{X}_{t+1|t}$ and $X_t(1)$ be defined as in (6.23), (6.24) and (6.2) respectively. Then*

$$\mathrm{E}[X_{t+1|t} - \widehat{X}_{t+1|t}]^2 \leq K\rho^t, \tag{6.29}$$

$$\mathrm{E}[\widehat{X}_{t+1|t} - X_t(1)]^2 \leq K\rho^t \tag{6.30}$$

$$\left|\mathrm{E}[X_{t+1} - X_{t+1|t}]^2 - \sigma^2\right| \leq K\rho^t \tag{6.31}$$

for any $\frac{1}{1+\delta} < \rho < 1$ and $\mathrm{var}(\varepsilon_t) = \sigma^2$.

PROOF. The proof immediately follows from the more general Baxter's inequality below. However, we now give a separate proof. The proof of (6.29) becomes clear when we use the expansion $X_{t+1} = \sum_{j=1}^{\infty} b_j X_{t+1-j} + \varepsilon_{t+1}$, noting that by Lemma 3.5.1(iii), $|b_j| \leq C\rho^j$.

Evaluating the best linear predictor of X_{t+1} given $X_t, \ldots, X_1$, using the autoregressive expansion gives

$$
\begin{aligned}
X_{t+1|t} &= \sum_{j=1}^{\infty} b_j P_{X_t,\ldots,X_1}(X_{t+1-j}) + \underbrace{P_{X_t,\ldots,X_1}(\varepsilon_{t+1})}_{=0} \\
&= \underbrace{\sum_{j=1}^{t-\max(p,q)} b_j X_{t+1-j}}_{\widehat{X}_{t+1|t} - \sum_{j=1}^{\max(p,q)} \gamma_j X_j} + \sum_{j=t-\max(p,q)}^{\infty} b_j P_{X_t,\ldots,X_1}(X_{t-j+1}).
\end{aligned}
$$

Therefore by using (6.25) we see that the difference between the best linear predictor and $\widehat{X}_{t+1|t}$ is

$$
X_{t+1|t} - \widehat{X}_{t+1|t} \;=\; \sum_{j=-\max(p,q)}^{\infty} b_{t+j} P_{X_t,\ldots,X_1}(X_{-j+1}) + \sum_{j=1}^{\max(p,q)} \gamma_j X_j = I + II.
$$

By using (6.25), it is straightforward to show that the second term $\mathrm{E}[II^2] = \mathrm{E}[\sum_{j=1}^{\max(p,q)} \gamma_j X_{t-j}]^2 \leq C\rho^t$, therefore what remains is to show that $\mathrm{E}[II^2]$ attains a similar bound. As Zijuan pointed out, by definitions of projections, $\mathrm{E}[P_{X_t,\ldots,X_1}(X_{-j+1})^2] \leq \mathrm{E}[X_{-j+1}^2]$, which immediately gives the bound, instead we use a more convoluted proof. To obtain a bound, we first obtain a bound for $\mathrm{E}[P_{X_t,\ldots,X_1}(X_{-j+1})]^2$. Basic results in linear regression shows that

$$
P_{X_t,\ldots,X_1}(X_{-j+1}) = \boldsymbol{\beta}'_{j,t}\boldsymbol{X}_t, \tag{6.32}
$$

where $\boldsymbol{\beta}_{j,t} = \Sigma_t^{-1}\boldsymbol{r}_{t,j}$, with $\boldsymbol{\beta}'_{j,t} = (\beta_{1,j,t},\ldots,\beta_{t,j,t})$, $\boldsymbol{X}'_t = (X_1,\ldots,X_t)$, $\Sigma_t = \mathrm{E}(\boldsymbol{X}_t\boldsymbol{X}'_t)$ and $\boldsymbol{r}_{t,j} = \mathrm{E}(\boldsymbol{X}_t X_j)$. Substituting (6.32) into I gives

$$
\sum_{j=-\max(p,q)}^{\infty} b_{t+j} P_{X_t,\ldots,X_1}(X_{-j+1}) = \sum_{j=-\max(p,q)}^{\infty} b_{t+j}\boldsymbol{\beta}'_{j,t}\boldsymbol{X}_t = \Big(\sum_{j=t-\max(p,q)}^{\infty} b_j \boldsymbol{r}'_{t,j}\Big)\Sigma_t^{-1}\boldsymbol{X}_t. \tag{6.33}
$$

Therefore the mean squared error of I is

$$
\mathrm{E}[I^2] = \left(\sum_{j=-\max(p,q)}^{\infty} b_{t+j}\boldsymbol{r}'_{t,j}\right)\Sigma_t^{-1}\left(\sum_{j=-\max(p,q)}^{\infty} b_{t+j}\boldsymbol{r}_{t,j}\right).
$$

To bound the above we use the Cauchy schwarz inequality ($\|aBb\|_1 \leq \|a\|_2\|Bb\|_2$), the spectral norm inequality ($\|a\|_2\|Bb\|_2 \leq \|a\|_2\|B\|_{spec}\|b\|_2$) and Minkowiski's inequality ($\|\sum_{j=1}^{n} a_j\|_2 \leq$

$\sum_{j=1}^{n}\|a_j\|_2)$ we have

$$\mathrm{E}\left[I^2\right] \leq \Big\| \sum_{j=1}^{\infty} b_{t+j}\boldsymbol{r}_{t,j}'\Big\|_2^2 \|\Sigma_t^{-1}\|_{spec}^2 \leq \big(\sum_{j=1}^{\infty}|b_{t+j}|\cdot\|\boldsymbol{r}_{t,j}\|_2\big)^2\|\Sigma_t^{-1}\|_{spec}^2. \tag{6.34}$$

We now bound each of the terms above. We note that for all t, using Remark 6.5.1 that $\|\Sigma_t^{-1}\|_{spec} \leq K$ (for some constant K). We now consider $\boldsymbol{r}_{t,j}' = (\mathrm{E}(X_1X_{-j}),\ldots,\mathrm{E}(X_tX_{-j})) = (c(1-j),\ldots,c(t-j))$. By using (4.2) we have $|c(k)| \leq C\rho^k$, therefore

$$\|\boldsymbol{r}_{t,j}\|_2 \leq K(\sum_{r=1}^{t}\rho^{2(j+r)})^{1/2} \leq K\frac{\rho^j}{(1-\rho^2)^2}.$$

Substituting these bounds into (6.34) gives $\mathrm{E}\left[I^2\right] \leq K\rho^t$. Altogether the bounds for I and II give

$$\mathrm{E}(X_{t+1|t} - \widehat{X}_{t+1|t})^2 \leq K\frac{\rho^j}{(1-\rho^2)^2}.$$

Thus proving (6.29).

To prove (6.30) we note that

$$\mathrm{E}[X_t(1) - \widehat{X}_{t+1|t}]^2 = \mathrm{E}\left[\sum_{j=0}^{\infty} b_{t+j}X_{-j} + \sum_{j=t-\max(p,q)}^{t} b_jY_{t-j}\right]^2.$$

Using the above and that $b_{t+j} \leq K\rho^{t+j}$, it is straightforward to prove the result.

Finally to prove (6.31), we note that by Minkowski's inequality we have

$$\begin{aligned}&\left[\mathrm{E}\left(X_{t+1} - X_{t+1|t}\right)^2\right]^{1/2} \leq \\ &\underbrace{\left[\mathrm{E}\left(X_t - X_t(1)\right)^2\right]^{1/2}}_{=\sigma} + \underbrace{\left[\mathrm{E}\left(X_t(1) - \widehat{X}_{t+1|t}\right)^2\right]^{1/2}}_{\leq K\rho^{t/2} \text{ by (6.30)}} + \underbrace{\left[\mathrm{E}\left(\widehat{X}_{t+1|t} - X_{t+1|t}\right)^2\right]^{1/2}}_{\leq K\rho^{t/2} \text{ by (6.29)}}.\end{aligned}$$

Thus giving the desired result. □

6.6 Baxter's inequality for linear models

In this section we generalize the ideas outlied in the previous section on prediction for ARMA processes. We recall that

$$X_{t+1|t} = P_{X_t,\ldots,X_1}(X_{t+1}) = \sum_{j=1}^{t} \phi_{t,j} X_{t-j},$$

which is the best linear predictor given the finite past. However, often $\phi_{t,j}$ can be difficult to evaluate (usually with the Durbin-Levinson algorithm) in comparison to the AR(∞) parameters. Thus we define the above approximation

$$\widehat{X}_{t+1|t} = \sum_{j=1}^{t} \phi_j X_{t-j}.$$

How good an approximation $\widehat{X}_{t+1|t}$ is of $X_{t+1|t}$ is given by Baxter's inequality.

Theorem 6.6.1 (Baxter's inequality) *Suppose $\{X_t\}$ has an AR(∞) representation with parameters $\{\phi_j\}_{j=1}^{\infty}$ such that $\sum_{j=1}^{\infty} |\phi_j| < \infty$. Let $\{\phi_{n,j}\}_{t=1}^{n}$ denote the parameters of the parameters of the best linear predictor of X_{t+1} given $\{X_j\}_{j=1}^{t}$. Then if n is large enough we have*

$$\sum_{j=1}^{n} |\phi_{n,j} - \phi_j| \leq C \sum_{j=n+1}^{\infty} |\phi_j|$$

We note that since $\sum_{j=1}^{\infty} |\phi_j| < \infty$, then $\sum_{j=n+1}^{\infty} |\phi_j| \to 0$ as $n \to \infty$. Thus as n gets large

$$\sum_{j=1}^{n} |\phi_{n,j} - \phi_j| \approx 0.$$

We apply this result to measuring the difference between $X_{t+1|t}$ and $\widehat{X}_{t+1|t}$

$$\mathrm{E}|X_{t+1|t} - \widehat{X}_{t+1|t}| \leq \sum_{j=1}^{t} |\phi_{t,j} - \phi_j| \,\mathrm{E}|X_{t-j}| \leq \mathrm{E}|X_{t-j}| \sum_{j=1}^{t} |\phi_{t,j} - \phi_j| \leq C\mathrm{E}|X_t| \sum_{j=t+1}^{\infty} |\phi_j|.$$

Therefore the best linear predictor and its approximation are "close" for large t.

6.7 Forecasting for nonlinear models

In this section we consider forecasting for nonlinear models. The forecasts we construct, may not necessarily/formally be the best linear predictor, because the best linear predictor is based on minimising the mean squared error, which we recall from Chapter 5 requires the existence of the higher order moments. Instead our forecast will be the conditional expection of X_{t+1} given the past (note that we can think of it as the best linear predictor). Furthermore, with the exception of the ARCH model we will derive approximation of the conditional expectation/best linear predictor, analogous to the forecasting approximation for the ARMA model, $\widehat{X}_{t+1|t}$ (given in (6.24)).

6.7.1 Forecasting volatility using an ARCH(p) model

We recall the ARCH(p) model defined in Section 5.2

$$X_t = \sigma_t Z_t \qquad \sigma_t^2 = a_0 + \sum_{j=1}^{p} a_j X_{t-j}^2.$$

Using a similar calculation to those given in Section 5.2.1, we see that

$$\begin{aligned}
\mathrm{E}[X_{t+1}|X_t, X_{t-1}, \ldots, X_{t-p+1}] &= \mathrm{E}(Z_{t+1}\sigma_{t+1}|X_t, X_{t-1}, \ldots, X_{t-p+1}) = \underbrace{\sigma_{t+1}\mathrm{E}(Z_{t+1}|X_t, X_{t-1}, \ldots, X_{t-p+1})}_{\sigma_{t+1} \text{ function of } X_t, \ldots, X_{t-p+1}} \\
&= \sigma_{t+1} \underbrace{\mathrm{E}(Z_{t+1})}_{\text{by causality}} = 0 \cdot \sigma_{t+1} = 0.
\end{aligned}$$

In other words, past values of X_t have no influence on the expected value of X_{t+1}. On the other hand, in Section 5.2.1 we showed that

$$\mathrm{E}(X_{t+1}^2|X_t, X_{t-1}, \ldots, X_{t-p+1}) = \mathrm{E}(Z_{t+1}^2\sigma_{t+1}^2|X_t, X_{t-2}, \ldots, X_{t-p+1}) = \sigma_{t+1}^2\mathrm{E}[Z_{t+1}^2] = \sigma_{t+1}^2 = \sum_{j=1}^{p} a_j X_{t+1-j}^2,$$

thus X_t has an influence on the conditional mean squared/variance. Therefore, if we let $X_{t+k|t}$ denote the conditional variance of X_{t+k} given $X_t, \ldots, X_{t-p+1}$, it can be derived using the following recursion

$$\begin{aligned}
X^2_{t+1|t} &= \sum_{j=1}^{p} a_j X^2_{t+1-j} \\
X^2_{t+k|t} &= \sum_{j=k}^{p} a_j X^2_{t+k-j} + \sum_{j=1}^{k-1} a_j X^2_{t+k-j|k} \quad 2 \leq k \leq p \\
X^2_{t+k|t} &= \sum_{j=1}^{p} a_j X^2_{t+k-j|t} \qquad k > p.
\end{aligned}$$

6.7.2 Forecasting volatility using a GARCH$(1,1)$ model

We recall the GARCH$(1,1)$ model defined in Section 5.3

$$\sigma_t^2 = a_0 + a_1 X^2_{t-1} + b_1 \sigma^2_{t-1} = \left(a_1 Z^2_{t-1} + b_1\right) \sigma^2_{t-1} + a_0.$$

Similar to the ARCH model it is straightforward to show that $\mathrm{E}[X_{t+1}|X_t, X_{t-1}, \ldots] = 0$ (where we use the notation $X_t, X_{t-1}, \ldots$ to denote the infinite past or more precisely conditioned on the sigma algebra $\mathcal{F}_t = \sigma(X_t, X_{t-1}, \ldots)$). Therefore, like the ARCH process, our aim is to predict X_t^2.

We recall from Example 5.3.1 that if the GARCH the process is invertible (satisfied if $b < 1$), then

$$\mathrm{E}[X^2_{t+1}|X_t, X_{t-1}, \ldots] = \sigma^2_{t+1} = a_0 + a_1 X^2_{t-1} + b_1 \sigma^2_{t-1} = \frac{a_0}{1-b} + a_1 \sum_{j=0}^{\infty} b^j X^2_{t-j}. \qquad (6.35)$$

Of course, in reality we only observe the finite past $X_t, X_{t-1}, \ldots, X_1$. We can approximate $\mathrm{E}[X^2_{t+1}|X_t, X_{t-1}, \ldots, X_1]$ using the following recursion, set $\widehat{\sigma}^2_{1|0} = 0$, then for $t \geq 1$ let

$$\widehat{\sigma}^2_{t+1|t} = a_0 + a_1 X_t^2 + b_1 \widehat{\sigma}^2_{t|t-1}$$

(noting that this is similar in spirit to the recursive approximate one-step ahead predictor defined in (6.25)). It is straightforward to show that

$$\widehat{\sigma}^2_{t+1|t} = \frac{a_0(1 - b^{t+1})}{1-b} + a_1 \sum_{j=0}^{t-1} b^j X^2_{t-j},$$

taking note that this is not the same as $\mathrm{E}[X^2_{t+1}|X_t, \ldots, X_1]$ (if the mean square error existed $\mathrm{E}[X^2_{t+1}|X_t, \ldots, X_1]$ would give a smaller mean square error), but just like the ARMA process it will closely approximate it. Furthermore, from (6.35) it can be seen that $\widehat{\sigma}^2_{t+1|t}$ closely approximates σ^2_{t+1}

Exercise 6.3 *To answer this question you need* R `install.package("tseries")` *then remember* `library("garch")`*.*

(i) You will find the Nasdaq data from 4th January 2010 - 15th October 2014 on my website.

(ii) By taking log differences fit a GARCH(1,1) model to the daily closing data (ignore the adjusted closing value) from 4th January 2010 - 30th September 2014 (use the function `garch(x, order = c(1, 1))` *fit the GARCH$(1,1)$ model).*

(iii) Using the fitted GARCH$(1,1)$ model, forecast the volatility σ_t^2 from October 1st-15th (noting that no trading is done during the weekends). Denote these forecasts as $\sigma^2_{t|0}$. Evaluate $\sum_{t=1}^{11}\sigma^2_{t|0}$

(iv) Compare this to the actual volatility $\sum_{t=1}^{11}X_t^2$ (where X_t are the log differences).

6.7.3 Forecasting using a BL$(1, 0, 1, 1)$ model

We recall the Bilinear$(1, 0, 1, 1)$ model defined in Section 5.4

$$X_t \quad = \quad \phi_1 X_{t-1} + b_{1,1}X_{t-1}\varepsilon_{t-1} + \varepsilon_t.$$

Assuming invertibility, so that ε_t can be written in terms of X_t (see Remark 5.4.2):

$$\varepsilon_t = \sum_{j=0}^{\infty}\left((-b)^j \prod_{i=0}^{j-1} X_{t-1-j}\right)[X_{t-j} - \phi X_{t-j-1}],$$

it can be shown that

$$X_t(1) = \mathrm{E}[X_{t+1}|X_t, X_{t-1}, \ldots] = \phi_1 X_t + b_{1,1}X_t\varepsilon_t.$$

However, just as in the ARMA and GARCH case we can obtain an approximation, by setting $\widehat{X}_{1|0} = 0$ and for $t \geq 1$ defining the recursion

$$\widehat{X}_{t+1|t} = \phi_1 X_t + b_{1,1}X_t\left(X_t - \widehat{X}_{t|t-1}\right).$$

See **?** and **?** for further details.

Remark 6.7.1 (How well does $\widehat{X}_{t+1|t}$ approximate $X_t(1)$?) *We now derive conditions for $\widehat{X}_{t+1|t}$ to be a close approximation of $X_t(1)$ when t is large. We use a similar technique to that used in Remark 6.5.2.*

We note that $X_{t+1} - X_t(1) = \varepsilon_{t+1}$ (since a future innovation, ε_{t+1}, cannot be predicted). We will show that $X_{t+1} - \widehat{X}_{t+1|t}$ is 'close' to ε_{t+1}. Subtracting $\widehat{X}_{t+1|t}$ from X_{t+1} gives the recursion

$$X_{t+1} - \widehat{X}_{t+1|t} = -b_{1,1}(X_t - \widehat{X}_{t|t-1})X_t + (b\varepsilon_t X_t + \varepsilon_{t+1})\,. \qquad (6.36)$$

We will compare the above recursion to the recursion based on ε_{t+1}. Rearranging the bilinear equation gives

$$\varepsilon_{t+1} = -b\varepsilon_t X_t + \underbrace{(X_{t+1} - \phi_1 X_t)}_{=b\varepsilon_t X_t + \varepsilon_{t+1}}. \qquad (6.37)$$

We observe that (6.36) and (6.37) are almost the same difference equation, the only difference is that an initial value is set for $\widehat{X}_{1|0}$. This gives the difference between the two equations as

$$\varepsilon_{t+1} - [X_{t+1} - \widehat{X}_{t+1|t}] = (-1)^t b^t X_1 \prod_{j=1}^{t} \varepsilon_j + (-1)^t b^t [X_1 - \widehat{X}_{1|0}] \prod_{j=1}^{t} \varepsilon_j.$$

Thus if $b^t \prod_{j=1}^t \varepsilon_j \overset{a.s.}{\to} 0$ as $t \to \infty$, then $\widehat{X}_{t+1|t} \overset{\mathcal{P}}{\to} X_t(1)$ as $t \to \infty$. We now show that if $\mathrm{E}[\log|\varepsilon_t| < -\log|b|$, *then $b^t \prod_{j=1}^t \varepsilon_j \overset{a.s.}{\to} 0$. Since $b^t \prod_{j=1}^t \varepsilon_j$ is a product, it seems appropriate to take logarithms to transform it into a sum. To ensure that it is positive, we take absolutes and t-roots*

$$\log|b^t \prod_{j=1}^{t} \varepsilon_j|^{1/t} = \log|b| + \underbrace{\frac{1}{t}\sum_{j=1}^{t} \log|\varepsilon_j|}_{\textit{average of iid random variables}} .$$

Therefore by using the law of large numbers we have

$$\log|b^t \prod_{j=1}^{t} \varepsilon_j|^{1/t} = \log|b| + \frac{1}{t}\sum_{j=1}^{t} \log|\varepsilon_j| \overset{\mathcal{P}}{\to} \log|b| + \mathrm{E}\log|\varepsilon_0| = \gamma.$$

Thus we see that $|b^t \prod_{j=1}^t \varepsilon_j|^{1/t} \overset{a.s.}{\to} \exp(\gamma)$. In other words, $|b^t \prod_{j=1}^t \varepsilon_j| \approx \exp(t\gamma)$, which will only *converge to zero if* $\mathrm{E}[\log|\varepsilon_t| < -\log|b|$.

6.8 Nonparametric prediction

In this section we briefly consider how prediction can be achieved in the nonparametric world. Let us assume that $\{X_t\}$ is a stationary time series. Our objective is to predict X_{t+1} given the past. However, we don't want to make any assumptions about the nature of $\{X_t\}$. Instead we want to obtain a predictor of X_{t+1} given X_t which minimises the means squared error, $\mathrm{E}[X_{t+1} - g(X_t)]^2$. It is well known that this is conditional expectation $\mathrm{E}[X_{t+1}|X_t]$. (since $\mathrm{E}[X_{t+1} - g(X_t)]^2 = \mathrm{E}[X_{t+1} - \mathrm{E}(X_{t+1}|X_t)]^2 + \mathrm{E}[g(X_t) - \mathrm{E}(X_{t+1}|X_t)]^2$). Therefore, one can estimate

$$\mathrm{E}[X_{t+1}|X_t = x] = m(x)$$

nonparametrically. A classical estimator of $m(x)$ is the Nadaraya-Watson estimator

$$\widehat{m}_n(x) = \frac{\sum_{t=1}^{n-1} X_{t+1} K(\frac{x-X_t}{b})}{\sum_{t=1}^{n-1} K(\frac{x-X_t}{b})},$$

where $K : \mathbb{R} \to \mathbb{R}$ is a kernel function (see Fan and Yao (2003), Chapter 5 and 6). Under some 'regularity conditions' it can be shown that $\widehat{m}_n(x)$ is a consistent estimator of $m(x)$ and converges to $m(x)$ in mean square (with the typical mean squared rate $O(b^4 + (bn)^{-1})$). The advantage of going the non-parametric route is that we have not imposed any form of structure on the process (such as linear/(G)ARCH/Bilinear). Therefore, we do not run the risk of misspecifying the model A disadvantage is that nonparametric estimators tend to be a lot worse than parametric estimators (in Chapter **??** we show that parametric estimators have $O(n^{-1/2})$ convergence which is faster than the nonparametric rate $O(b^2 + (bn)^{-1/2})$). Another possible disavantage is that if we wanted to include more past values in the predictor, ie. $m(x_1, \ldots, x_d) = \mathrm{E}[X_{t+1}|X_t = x_1, \ldots, X_{t-p} = x_d]$ then the estimator will have an extremely poor rate of convergence (due to the curse of dimensionality).

A possible solution to the problem is to assume some structure on the nonparametric model, and define a semi-parametric time series model. We state some examples below:

(i) An additive structure of the type

$$X_t = \sum_{j=1}^{p} g_j(X_{t-j}) + \varepsilon_t$$

where $\{\varepsilon_t\}$ are iid random variables.

(ii) A functional autoregressive type structure

$$X_t = \sum_{j=1}^{p} g_j(X_{t-d})X_{t-j} + \varepsilon_t.$$

(iii) The semi-parametric GARCH$(1,1)$

$$X_t = \sigma_t Z_t, \qquad \sigma_t^2 = b\sigma_{t-1}^2 + m(X_{t-1}).$$

However, once a structure has been imposed, conditions need to be derived in order that the model has a stationary solution (just as we did with the fully-parametric models).

See **?**, **?**, **?**, **?**, **?** etc.

6.9 The Wold Decomposition

Section 6.3.1 nicely leads to the Wold decomposition, which we now state and prove. The Wold decomposition theorem, states that any stationary process, has something that appears close to an MA(∞) representation (though it is not). We state the theorem below and use some of the notation introduced in Section 6.3.1.

Theorem 6.9.1 *Suppose that $\{X_t\}$ is a second order stationary time series with a finite variance (we shall assume that it has mean zero, though this is not necessary). Then X_t can be uniquely expressed as*

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j} + V_t, \tag{6.38}$$

where $\{Z_t\}$ are uncorrelated random variables, with $\mathrm{var}(Z_t) = \mathrm{E}(X_t - X_{t-1}(1))^2$ (noting that $X_{t-1}(1)$ is the best linear predictor of X_t given $X_{t-1}, X_{t-2}, \ldots$) and $V_t \in \mathcal{X}_{-\infty} = \cap_{n=-\infty}^{-\infty} \mathcal{X}_n^{-\infty}$, where $\overline{\mathcal{X}_n^{-\infty}}$ is defined in (6.6).

PROOF. First let is consider the one-step ahead prediction of X_t given the infinite past, denoted $X_{t-1}(1)$. Since $\{X_t\}$ is a second order stationary process it is clear that $X_{t-1}(1) = \sum_{j=1}^{\infty} b_j X_{t-j}$, where the coefficients $\{b_j\}$ do not vary with t. For this reason $\{X_{t-1}(1)\}$ and $\{X_t - X_{t-1}(1)\}$ are second order stationary random variables. Furthermore, since $\{X_t - X_{t-1}(1)\}$ is uncorrelated with X_s for any $s \leq t$, then $\{X_s - X_{s-1}(1); s \in \mathbb{R}\}$ are uncorrelated random variables. Define $Z_s = X_s - X_{s-1}(1)$, and observe that Z_s is the one-step ahead prediction error. We recall from Section 6.3.1 that $X_t \in \overline{\text{sp}}((X_t - X_{t-1}(1)), (X_{t-1} - X_{t-2}(1)), \ldots) \oplus \bar{sp}(\mathcal{X}_{-\infty}) = \oplus_{j=0}^{\infty}\overline{\text{sp}}(Z_{t-j}) \oplus \bar{sp}(\mathcal{X}_{-\infty})$. Since the spaces $\oplus_{j=0}^{\infty}\overline{\text{sp}}(Z_{t-j})$ and $\overline{\text{sp}}(\mathcal{X}_{-\infty})$ are orthogonal, we shall first project X_t onto $\oplus_{j=0}^{\infty}\overline{\text{sp}}(Z_{t-j})$, due to orthogonality the difference between X_t and its projection will be in $\overline{\text{sp}}(\mathcal{X}_{-\infty})$. This will lead to the Wold decomposition.

First we consider the projection of X_t onto the space $\oplus_{j=0}^{\infty}\overline{\text{sp}}(Z_{t-j})$, which is

$$P_{Z_t,Z_{t-1},\ldots}(X_t) = \sum_{j=0}^{\infty}\psi_j Z_{t-j},$$

where due to orthogonality $\psi_j = cov(X_t, (X_{t-j} - X_{t-j-1}(1)))/\text{var}(X_{t-j} - X_{t-j-1}(1))$. Since $X_t \in \oplus_{j=0}^{\infty}\overline{\text{sp}}(Z_{t-j}) \oplus \bar{sp}(\mathcal{X}_{-\infty})$, the difference $X_t - P_{Z_t,Z_{t-1},\ldots}X_t$ is orthogonal to $\{Z_t\}$ and belongs in $\bar{sp}(\mathcal{X}_{-\infty})$. Hence we have

$$X_t = \sum_{j=0}^{\infty}\psi_j Z_{t-j} + V_t,$$

where $V_t = X_t - \sum_{j=0}^{\infty}\psi_j Z_{t-j}$ and is uncorrelated to $\{Z_t\}$. Hence we have shown (6.38). To show that the representation is unique we note that $Z_t, Z_{t-1}, \ldots$ are an orthogonal basis of $\overline{\text{sp}}(Z_t, Z_{t-1}, \ldots)$, which pretty much leads to uniqueness. □

Exercise 6.4 *Consider the process $X_t = A\cos(Bt + U)$ where A, B and U are random variables such that A, B and U are independent and U is uniformly distributed on $(0, 2\pi)$.*

(i) Show that X_t is second order stationary (actually it's stationary) and obtain its means and covariance function.

(ii) Show that the distribution of A and B can be chosen in such a way that $\{X_t\}$ has the same covariance function as the MA(1) process $Y_t = \varepsilon_t + \phi\varepsilon_t$ (where $|\phi| < 1$) (quite amazing).

(iii) Suppose A and B have the same distribution found in (ii).

(a) What is the best predictor of X_{t+1} given $X_t, X_{t-1}, \ldots$?

(b) What is the best linear predictor of X_{t+1} given $X_t, X_{t-1}, \ldots$?

It is worth noting that variants on the proof can be found in Brockwell and Davis (1998), Section 5.7 and Fuller (1995), page 94.

Remark 6.9.1 *Notice that the representation in (6.38) looks like an MA(∞) process. There is, however, a significant difference. The random variables $\{Z_t\}$ of an MA(∞) process are iid random variables and not just uncorrelated.*

We recall that we have already come across the Wold decomposition of some time series. In Section 4.4 we showed that a non-causal linear time series could be represented as a causal 'linear time series' with uncorrelated but dependent innovations. Another example is in Chapter 5, where we explored ARCH/GARCH process which have an AR and ARMA type representation. Using this representation we can represent ARCH and GARCH processes as the weighted sum of $\{(Z_t^2-1)\sigma_t^2\}$ which are uncorrelated random variables.

Remark 6.9.2 (Variation on the Wold decomposition) *In many technical proofs involving time series, we often use results related to the Wold decomposition. More precisely, we often decompose the time series in terms of an infinite sum of martingale differences. In particular, we define the sigma-algebra $\mathcal{F}_t = \sigma(X_t, X_{t-1}, \ldots)$, and suppose that $\mathrm{E}(X_t|\mathcal{F}_{-\infty}) = \mu$. Then by telescoping we can formally write X_t as*

$$X_t - \mu = \sum_{j=0}^{\infty} Z_{t,j}$$

where $Z_{t,j} = \mathrm{E}(X_t|\mathcal{F}_{t-j}) - \mathrm{E}(X_t|\mathcal{F}_{t-j-1})$. It is straightforward to see that $Z_{t,j}$ are martingale differences, and under certain conditions (mixing, physical dependence, your favourite dependence flavour etc) it can be shown that $\sum_{j=0}^{\infty}\|Z_{t,j}\|_p < \infty$ (where $\|\cdot\|_p$ is the pth moment). This means the above representation holds almost surely. Thus in several proofs we can replace $X_t - \mu$ by $\sum_{j=0}^{\infty} Z_{t,j}$. This decomposition allows us to use martingale theorems to prove results.

6.10 Kolmogorov's formula (theorem)

Suppose $\{X_t\}$ is a second order stationary time series. Kolmogorov's(-Szegö) theorem is an expression for the error in the linear prediction of X_t given the infinite past $X_{t-1}, X_{t-2}, \ldots$. It basically states that

$$\mathrm{E}\left[X_n - X_n(1)\right]^2 = \exp\left(\frac{1}{2\pi}\int_0^{2\pi}\log f(\omega)d\omega\right),$$

where f is the spectral density of the time series. Clearly from the definition we require that the spectral density function is bounded away from zero.

To prove this result we use (4.14);

$$\mathrm{var}[Y - \widehat{Y}] = \frac{\det(\Sigma)}{\det(\Sigma_{XX})}.$$

and Szegö's theorem (see, Gray's technical report, where the proof is given), which we state later on. Let $P_{X_1,\ldots,X_n}(X_{n+1}) = \sum_{j=1}^{n}\phi_{j,n}X_{n+1-j}$ (best linear predictor of X_{n+1} given $X_n,\ldots,X_1$). Then we observe that since $\{X_t\}$ is a second order stationary time series and using (4.14) we have

$$\mathrm{E}\left[X_{n+1} - \sum_{j=1}^{n}\phi_{n,j}X_{n+1-j}\right]^2 = \frac{\det(\Sigma_{n+1})}{\det(\Sigma_n)},$$

where $\Sigma_n = \{c(i-j); i,j = 0,\ldots,n-1\}$, and Σ_n is a non-singular matrix.

Szegö's theorem is a general theorem concerning Toeplitz matrices. Define the sequence of Toeplitz matrices $\Gamma_n = \{c(i-j); i,j = 0,\ldots,n-1\}$ and assume the Fourier transform

$$f(\omega) = \sum_{j\in\mathbb{Z}} c(j)\exp(ij\omega)$$

exists and is well defined ($\sum_j |c(j)|^2 < \infty$). Let $\{\gamma_{j,n}\}$ denote the Eigenvalues corresponding to Γ_n. Then for any function G we have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}G(\gamma_{j,n}) \to \int_0^{2\pi}G(f(\omega))d\omega.$$

To use this result we return to $\mathrm{E}[X_{n+1} - \sum_{j=1}^{n}\phi_{n,j}X_{n+1-j}]^2$ and take logarithms

$$\begin{aligned}\log \mathrm{E}[X_{n+1} - \sum_{j=1}^{n}\phi_{n,j}X_{n+1-j}]^2 &= \log\det(\Sigma_{n+1}) - \log\det(\Sigma_n) \\ &= \sum_{j=1}^{n+1}\log\gamma_{j,n+1} - \sum_{j=1}^{n}\log\gamma_{j,n}\end{aligned}$$

where the above is because $\det \Sigma_n = \prod_{j=1}^{n} \gamma_{j,n}$ (where $\gamma_{j,n}$ are the eigenvalues of Σ_n). Now we apply Szegö's theorem using $G(x) = \log(x)$, this states that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \log(\gamma_{j,n}) \to \int_0^{2\pi} \log(f(\omega)) d\omega.$$

thus for large n

$$\frac{1}{n+1} \sum_{j=1}^{n+1} \log \gamma_{j,n+1} \approx \frac{1}{n} \sum_{j=1}^{n} \log \gamma_{j,n}.$$

This implies that

$$\sum_{j=1}^{n+1} \log \gamma_{j,n+1} \approx \frac{n+1}{n} \sum_{j=1}^{n} \log \gamma_{j,n},$$

hence

$$\begin{aligned}
\log \mathrm{E}[X_{n+1} - \sum_{j=1}^{n} \phi_{n,j} X_{n+1-j}]^2 &= \log\det(\Sigma_{n+1}) - \log\det(\Sigma_n) \\
&= \sum_{j=1}^{n+1} \log \gamma_{j,n+1} - \sum_{j=1}^{n} \log \gamma_{j,n} \\
&\approx \frac{n+1}{n} \sum_{j=1}^{n} \log \gamma_{j,n} - \sum_{j=1}^{n} \log \gamma_{j,n} = \frac{1}{n} \sum_{j=1}^{n} \log \gamma_{j,n}.
\end{aligned}$$

Thus

$$\begin{aligned}
\lim_{n\to\infty} \log \mathrm{E}[X_{t+1} - \sum_{j=1}^{n} \phi_{n,j} X_{t+1-j}]^2 &= \lim_{n\to\infty} \log \mathrm{E}[X_{n+1} - \sum_{j=1}^{n} \phi_{n,j} X_{n+1-j}]^2 \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \log \gamma_{j,n} = \int_0^{2\pi} \log(f(\omega)) d\omega
\end{aligned}$$

and

$$\lim_{n\to\infty} \mathrm{E}[X_{t+1} - \sum_{j=1}^{n} \phi_{n,j} X_{t+1-j}]^2 = \exp\left(\int_0^{2\pi} \log(f(\omega)) d\omega\right).$$

This gives a rough outline of the proof. The precise proof can be found in Gray's technical report. There exists alternative proofs (given by Kolmogorov), see Brockwell and Davis (1998), Chapter 5.

This is the reason that in many papers the assumption

$$\int_0^{2\pi} \log f(\omega) d\omega > -\infty$$

is made. This assumption essentially ensures $X_t \notin \mathcal{X}_{-\infty}$.

Example 6.10.1 *Consider the AR(p) process* $X_t = \phi X_{t-1} + \varepsilon_t$ *(assume wlog that* $|\phi| < 1$*) where* $\mathrm{E}[\varepsilon_t] = 0$ *and* $\mathrm{var}[\varepsilon_t] = \sigma^2$*. We know that* $X_t(1) = \phi X_t$ *and*

$$\mathrm{E}[X_{t+1} - X_t(1)]^2 = \sigma^2.$$

We now show that

$$\exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log f(\omega) d\omega\right) = \sigma^2. \qquad (6.39)$$

We recall that the spectral density of the $AR(1)$ *is*

$$\begin{aligned} f(\omega) &= \frac{\sigma^2}{|1 - \phi e^{i\omega}|^2} \\ \Rightarrow \log f(\omega) &= \log \sigma^2 - \log |1 - \phi e^{i\omega}|^2. \end{aligned}$$

Thus

$$\frac{1}{2\pi}\int_0^{2\pi} \log f(\omega) d\omega = \underbrace{\frac{1}{2\pi}\int_0^{2\pi} \log \sigma^2 d\omega}_{=\log \sigma^2} - \underbrace{\frac{1}{2\pi}\int_0^{2\pi} \log |1 - \phi e^{i\omega}|^2 d\omega}_{=0}.$$

There are various ways to prove that the second term is zero. Probably the simplest is to use basic results in complex analysis. By making a change of variables $z = e^{i\omega}$ *we have*

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi} \log |1 - \phi e^{i\omega}|^2 d\omega &= \frac{1}{2\pi}\int_0^{2\pi} \log(1 - \phi e^{i\omega}) d\omega + \frac{1}{2\pi}\int_0^{2\pi} \log(1 - \phi e^{-i\omega}) d\omega \\ &= \frac{1}{2\pi}\int_0^{2\pi} \sum_{j=1}^{\infty} \left[\frac{\phi^j e^{ij\omega}}{j} + \frac{\phi^j e^{-ij\omega}}{j}\right] d\omega = 0. \end{aligned}$$

From this we immediately prove (6.39).

Chapter 7

Estimation of the mean and covariance

Prerequisites

- Some idea of what a cumulant is.

Objectives

- To derive the sample autocovariance of a time series, and show that this is a positive definite sequence.

- To show that the variance of the sample covariance involves fourth order cumulants, which can be unwielding to estimate in practice. But under linearity the expression for the variance greatly simplifies.

- To show that under linearity the correlation does not involve the fourth order cumulant. This is the Bartlett formula.

- To use the above results to construct a test for uncorrelatedness of a time series (the Portmanteau test). And understand how this test may be useful for testing for independence in various different setting. Also understand situations where the test may fail.

Here we summarize the Central limit theorems we will use in this chapter. The simplest is the case of iid random variables. The first is the classical central limit theorem. Suppose that $\{X_i\}$ are iid random variables with mean μ and variance $\sigma^2 < \infty$. Then

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(X_i - \mu) \xrightarrow{\mathcal{D}} \mathcal{N}(0,\sigma^2).$$

A small variant on the classical CLT is the case that $\{X_i\}$ are independent random variables (but not identically distributed). Suppose $\mathrm{E}[X_i] = \mu_i$, $\mathrm{var}[X_i] = \sigma_i^2 < \infty$ and for every $\varepsilon > 0$

$$\frac{1}{s_n^2}\sum_{i=1}^{n}\mathrm{E}\left((X_i - \mu_i)^2 I(s_n^{-1}|X_i - \mu_i| > \varepsilon)\right) \to 0$$

where $s_n^2 = \sum_{i=1}^{n}\sigma_i^2$, which is the variance of $\sum_{i=1}^{n}X_i$ (the above condition is called the Lindeberg condition). Then

$$\frac{1}{\sqrt{\sum_{i=1}^{n}\sigma_i^2}}\sum_{i=1}^{n}(X_i - \mu_i) \xrightarrow{\mathcal{D}} \mathcal{N}(0,1).$$

The Lindeberg condition looks unwieldy, however by using Chebyshev's and Hölder inequality it can be reduced to simple bounds on the moments.

Remark 7.0.1 (The aims of the Lindeberg condition) *The Lindeberg condition essential requires a uniform bound in the tails for all the random variables $\{X_i\}$ in the sum. For example, suppose X_i are t-distributed random variables where X_i is distributed with a t-distribution with $(2+i^{-1})$ degrees of freedom. We know that the number of df (which can be non-integer-valued) gets thicker the lower the df. Furthermore, $\mathrm{E}[X_i^2] < \infty$ only if X_i has a df greater than 2. Therefore, the second moments of X_i exists. But as i gets larger, X_i has thicker tails. Making it impossible (I believe) to find a uniform bound such that Lindeberg's condition is satisified.*

Note that the Lindeberg condition generalizes to the conditional Lindeberg condition when dealing with martingale differences.

We now state a generalisation of this central limit to triangular arrays. Suppose that $\{X_{t,n}\}$ are independent random variables with mean zero. Let $S_n = \sum_{t=1}^{n}X_{t,n}$ we assume that $\mathrm{var}[S_n] = \sum_{t=1}^{n}\mathrm{var}[X_{t,n}] = 1$. For example, in the case that $\{X_t\}$ are iid random variables and $S_n = \frac{1}{\sqrt{n}}\sum_{t=1}^{n}[X_t - \mu] = \sum_{t=1}^{n}X_{t,n}$, where $X_{t,n} = \sigma^{-1}n^{-1/2}(X_t - \mu)$. If for all $\varepsilon > 0$

$$\sum_{t=1}^{n}\mathrm{E}\left(X_{t,n}^2 I(|X_{t,n}| > \varepsilon)\right) \to 0,$$

then $S_n \overset{\mathcal{D}}{\to} \mathcal{N}(0,1)$.

7.1 An estimator of the mean

Suppose we observe $\{Y_t\}_{t=1}^n$, where

$$Y_t = \mu + X_t,$$

where μ is the finite mean, $\{X_t\}$ is a zero mean stationary time series with absolutely summable covariances ($\sum_k |\text{cov}(X_0, X_k)| < \infty$). Our aim is to estimate the mean μ. The most obvious estimator is the sample mean, that is $\bar{Y}_n = n^{-1}\sum_{t=1}^n Y_t$ as an estimator of μ.

7.1.1 The sampling properties of the sample mean

We recall from Example 2.3.1 that we obtained an expression for the sample mean. We showed that

$$\text{var}(\bar{Y}_n) \quad = \quad \frac{1}{n}\text{var}(X_0) + \frac{2}{n}\sum_{k=1}^{n}(\frac{n-k}{n})c(k).$$

Furthermore, if $\sum_k |c(k)| < \infty$, then in Example 2.3.1 we showed that

$$\text{var}(\bar{Y}_n) \quad = \quad \frac{1}{n}\text{var}(X_0) + \frac{2}{n}\sum_{k=1}^{\infty}c(k) + o(\frac{1}{n}).$$

Thus if the time series has sufficient decay in it's correlation structure a mean squared consistent estimator of the sample mean can be achieved. However, one drawback is that the dependency means that one observation will influence the next, and if the influence is positive (seen by a positive covariance), the resulting estimator may have a (much) larger variance than the iid case.

The above result does not require any more conditions on the process, besides second order stationarity and summability of its covariance. However, to obtain confidence intervals we require a stronger result, namely a central limit theorem for the sample mean. The above conditions are not enough to give a central limit theorem. To obtain a CLT for sums of the form $\sum_{t=1}^n X_t$ we need the following main ingredients:

(i) The variance needs to be finite.

(ii) The dependence between X_t decreases the further apart in time the observations. However, this is more than just the correlation, it really means the dependence.

The above conditions are satisfied by linear time series, if the cofficients ϕ_j decay sufficient fast. However, these conditions can also be verified for nonlinear time series (for example the (G)ARCH and Bilinear model described in Chapter 5).

We now state the asymptotic normality result for linear models.

Theorem 7.1.1 *Suppose that X_t is a linear time series, of the form $X_t = \sum_{j=-\infty}^{\infty} \psi_j \varepsilon_{t-j}$, where ε_t are iid random variables with mean zero and variance one, $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$ and $\sum_{j=-\infty}^{\infty} \psi_j \neq 0$. Let $Y_t = \mu + X_t$, then we have*

$$\sqrt{n}\left(\bar{Y}_n - \mu\right) = \mathcal{N}(0, \sigma^2)$$

where $\sigma^2 = \mathrm{var}(X_0) + 2\sum_{k=1}^{\infty} c(k)$.

PROOF. Later in this course we will give precise details on how to prove asymptotic normality of several different type of estimators in time series. However, we give a small flavour here by showing asymptotic normality of $\bar{Y}_n$ in the special case that $\{X_t\}_{t=1}^{n}$ satisfy an MA(q) model, then explain how it can be extended to MA(∞) processes.

The main idea of the proof is to transform/approximate the average into a quantity that we know is asymptotic normal. We know if $\{\epsilon_t\}_{t=1}^{n}$ are iid random variables with mean μ and variance one then

$$\sqrt{n}(\bar{\epsilon}_n - \mu) \overset{\mathcal{D}}{\rightarrow} \mathcal{N}(0, 1). \tag{7.1}$$

We aim to use this result to prove the theorem. Returning to $\bar{Y}_n$ by a change of variables ($s = t - j$)

we can show that

$$\begin{aligned}
\frac{1}{n}\sum_{t=1}^{n} Y_t &= \mu + \frac{1}{n}\sum_{t=1}^{n} X_t = \mu + \frac{1}{n}\sum_{t=1}^{n}\sum_{j=0}^{q}\psi_j\varepsilon_{t-j} \\
&= \mu + \frac{1}{n}\sum_{s=1}^{n-q}\varepsilon_s\left(\sum_{j=0}^{q}\psi_j\right) + \sum_{s=-q+1}^{0}\varepsilon_s\left(\sum_{j=q-s}^{q}\psi_j\right) + \sum_{s=n-q+1}^{n}\varepsilon_s\left(\sum_{j=0}^{n-s}\psi_j\right) \\
&= \mu + \frac{n-q}{n}\left(\sum_{j=0}^{q}\psi_j\right)\frac{1}{n-q}\sum_{s=1}^{n-q}\varepsilon_s + \frac{1}{n}\sum_{s=-q+1}^{0}\varepsilon_s\left(\sum_{j=q+s}^{q}\psi_j\right) + \frac{1}{n}\sum_{s=n-q+1}^{n}\varepsilon_s\left(\sum_{j=0}^{n-s}\psi_j\right) \\
&:= \mu + \frac{\Psi(n-q)}{n}\bar{\varepsilon}_{n-q} + E_1 + E_2, \qquad (7.2)
\end{aligned}$$

where $\Psi = \sum_{j=0}^{q}\psi_j$. It is straightforward to show that $\mathrm{E}|E_1| \leq Cn^{-1}$ and $\mathrm{E}|E_2| \leq Cn^{-1}$.

Finally we examine $\frac{\Psi(n-q)}{n}\bar{\varepsilon}_{n-q}$. We note that if the assumptions are not satisfied and $\sum_{j=0}^{q}\psi_j = 0$ (for example the process $X_t = \varepsilon_t - \varepsilon_{t-1}$), then

$$\frac{1}{n}\sum_{t=1}^{n} Y_t = \mu + \frac{1}{n}\sum_{s=-q+1}^{0}\varepsilon_s\left(\sum_{j=q-s}^{q}\psi_j\right) + \frac{1}{n}\sum_{s=n-q+1}^{n}\varepsilon_s\left(\sum_{j=0}^{n-s}\psi_j\right).$$

This is a degenerate case, since E_1 and E_2 only consist of a finite number of terms and thus if ε_t are non-Gaussian these terms will never be asymptotically normal. Therefore, in this case we simply have that $\frac{1}{n}\sum_{t=1}^{n} Y_t = \mu + O(\frac{1}{n})$ (this is why in the assumptions it was stated that $\Psi \neq 0$).

On the other hand, if $\Psi \neq 0$, then the dominating term in $\bar{Y}_n$ is $\bar{\varepsilon}_{n-q}$. From (7.1) it is clear that $\sqrt{n-q}\bar{\varepsilon}_{n-q} \xrightarrow{\mathcal{P}} \mathcal{N}(0,1)$ as $n \to \infty$. However, for finite q, $\sqrt{(n-q)/n} \xrightarrow{\mathcal{P}} 1$, therefore $\sqrt{n}\bar{\varepsilon}_{n-q} \xrightarrow{\mathcal{P}} \mathcal{N}(0,1)$. Altogether, substituting $\mathrm{E}|E_1| \leq Cn^{-1}$ and $\mathrm{E}|E_2| \leq Cn^{-1}$into (7.2) gives

$$\sqrt{n}\left(\bar{Y}_n - \mu\right) = \Psi\sqrt{n}\bar{\varepsilon}_{n-q} + O_p(\frac{1}{n}) \xrightarrow{\mathcal{P}} \mathcal{N}\left(0, \Psi^2\right).$$

With a little work, it can be shown that $\Psi^2 = \sigma^2$.

Observe that the proof simply approximated the sum by a sum of iid random variables. In the case that the process is a MA(∞) or linear time series, a similar method is used. More precisely, we have

$$\begin{aligned}
\sqrt{n}\left(\bar{Y}_n - \mu\right) &= \frac{1}{\sqrt{n}}\sum_{t=1}^{n}\sum_{j=0}^{\infty}\psi_j\varepsilon_{t-j} = \frac{1}{\sqrt{n}}\sum_{j=0}^{\infty}\psi_j\sum_{s=1-j}^{n-j}\varepsilon_s \\
&= \frac{1}{\sqrt{n}}\sum_{j=0}^{\infty}\psi_j\sum_{t=1}^{n}\varepsilon_t + R_n
\end{aligned}$$

where

$$\begin{aligned}
R_n &= \frac{1}{\sqrt{n}}\sum_{j=0}^{\infty}\psi_j\left(\sum_{s=1-j}^{n-j}\varepsilon_s - \sum_{s=1}^{n}\varepsilon_s\right) \\
&= \frac{1}{\sqrt{n}}\sum_{j=0}^{n}\psi_j\left(\sum_{s=1-j}^{0}\varepsilon_s - \sum_{s=n-j}^{n}\varepsilon_s\right) + \frac{1}{\sqrt{n}}\sum_{j=n+1}^{\infty}\psi_j\left(\sum_{s=1-j}^{n-j}\varepsilon_s - \sum_{s=1}^{n}\varepsilon_s\right) \\
&:= R_{n1} + R_{n2} + R_{n3} + R_{n4}.
\end{aligned}$$

We will show that $\mathrm{E}[R_{n,j}^2] = o(1)$ for $1 \leq j \leq 4$. We start with $R_{n,1}$

$$\begin{aligned}
\mathrm{E}[R_{n,1}^2] &= \frac{1}{n}\sum_{j_1,j_2=0}^{n}\psi_{j_1}\psi_{j_2}\mathrm{cov}\left(\sum_{s_1=1-j_1}^{0}\varepsilon_{s_1}, \sum_{s_2=1-j_2}^{0}\varepsilon_{s_2}\right) \\
&= \frac{1}{n}\sum_{j_1,j_2=0}^{n}\psi_{j_1}\psi_{j_2}\min[j_1-1, j_2-1] \\
&= \frac{1}{n}\sum_{j=0}^{n}\psi_j^2(j-1) + \frac{2}{n}\sum_{j_1=0}^{n}\psi_{j_1}, \sum_{j_2=0}^{j_1-1}\psi_{j_2}\min[j_2-1] \\
&\leq \frac{1}{n}\sum_{j=0}^{n}\psi_j^2(j-1) + \frac{2\Psi}{n}\sum_{j_1=0}^{n}|j_1\psi_{j_1}|.
\end{aligned}$$

Since $\sum_{j=0}^{\infty}|\psi_j| < \infty$ and, thus, $\sum_{j=0}^{\infty}|\psi_j|^2 < \infty$, then by dominated convegence $\sum_{j=0}^{n}[1-j/n]\psi_j \to \sum_{j=0}^{\infty}\psi_j$ and $\sum_{j=0}^{n}[1-j/n]\psi_j^2 \to \sum_{j=0}^{\infty}\psi_j^2$ as $n \to \infty$. This implies that $\sum_{j=0}^{n}(j/n)\psi_j \to 0$ and $\sum_{j=0}^{n}(j/n)\psi_j^2 \to 0$. Substituting this into the above bounds for $\mathrm{E}[R_{n,1}^2]$ we immediately obtain $\mathrm{E}[R_{n,1}^2] = o(1)$. Using the same argument we obtain the same bound for $R_{n,2}, R_{n,3}$ and $R_{n,4}$. Thus

$$\sqrt{n}\left(\bar{Y}_n - \mu\right) = \Psi\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\varepsilon_t + o_p(1)$$

and the result then immediately follows. □

Estimation of the so called long run variance (given in Theorem 7.1.1) can be difficult. There are various methods that can be used, such as estimating the spectral density function (which we define in Chapter 9) at zero. Another approach proposed in Lobato (2001) and Shao (2010) is to use the method of so called self-normalization which circumvents the need to estimate the long run mean, by privotalising the statistic.

7.2 An estimator of the covariance

Suppose we observe $\{Y_t\}_{t=1}^{n}$, to estimate the covariance we can estimate the covariance $c(k) = \text{cov}(Y_0, Y_k)$ from the the observations. A plausible estimator is

$$\hat{c}_n(k) = \frac{1}{n}\sum_{t=1}^{n-|k|}(Y_t - \bar{Y}_n)(Y_{t+|k|} - \bar{Y}_n), \tag{7.3}$$

since $\text{E}[(Y_t - \bar{Y}_n)(Y_{t+|k|} - \bar{Y}_n)] \approx c(k)$. Of course if the mean of Y_t is known to be zero ($Y_t = X_t$), then the covariance estimator is

$$\hat{c}_n(k) = \frac{1}{n}\sum_{t=1}^{n-|k|} X_t X_{t+|k|}. \tag{7.4}$$

The eagle-eyed amongst you may wonder why we don't use $\frac{1}{n-|k|}\sum_{t=1}^{n-|k|} X_t X_{t+|k|}$, when $\hat{c}_n(k)$ is a biased estimator, whereas $\frac{1}{n-|k|}\sum_{t=1}^{n-|k|} X_t X_{t+|k|}$ is not. However $\hat{c}_n(k)$ has some very nice properties which we discuss in the lemma below.

Lemma 7.2.1 *Suppose we define the empirical covariances*

$$\widehat{c}_n(k) = \begin{cases} \frac{1}{n}\sum_{t=1}^{n-|k|} X_t X_{t+|k|} & |k| \leq n-1 \\ 0 & \textit{otherwise} \end{cases}$$

then $\{\widehat{c}_n(k)\}$ is a positive definite sequence. Therefore, using Lemma 2.4.1 there exists a stationary time series $\{Z_t\}$ which has the covariance $\hat{c}_n(k)$.

PROOF. There are various ways to show that $\{\hat{c}_n(k)\}$ is a positive definite sequence. One method uses that the spectral density corresponding to this sequence is non-negative, we give this proof in Section 9.3.1.

Here we give an alternative proof. We recall a sequence is semi-positive definite if for any vector $\underline{a} = (a_1, \ldots, a_r)'$ we have

$$\sum_{k_1,k_2=1}^{r} a_{k_1} a_{k_2} \hat{c}_n(k_1 - k_2) = \sum_{k_1,k_2=1}^{n} a_{k_1} a_{k_2} \hat{c}_n(k_1 - k_2) = \underline{a}' \widehat{\Sigma}_n \underline{a} \geq 0$$

where

$$\widehat{\Sigma}_n = \begin{pmatrix} \hat{c}_n(0) & \hat{c}_n(1) & \hat{c}_n(2) & \ldots & \hat{c}_n(n-1) \\ \hat{c}_n(1) & \hat{c}_n(0) & \hat{c}_n(1) & \ldots & \hat{c}_n(n-2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \hat{c}_n(n-1) & \hat{c}_n(n-2) & \vdots & \vdots & \hat{c}_n(0) \end{pmatrix},$$

noting that $\hat{c}_n(k) = \frac{1}{n}\sum_{t=1}^{n-|k|} X_t X_{t+|k|}$. However, $\hat{c}_n(k) = \frac{1}{n}\sum_{t=1}^{n-|k|} X_t X_{t+|k|}$ has a very interesting construction, it can be shown that the above convariance matrix is $\widehat{\Sigma}_n = \mathbf{X}_n \mathbf{X}_n'$, where $\mathbf{X}_n$ is a $n \times 2n$ matrix with

$$\mathbf{X}_n = \begin{pmatrix} 0 & 0 & \ldots & 0 & X_1 & X_2 & \ldots & X_{n-1} & X_n \\ 0 & 0 & \ldots & X_1 & X_2 & \ldots & X_{n-1} & X_n & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ X_1 & X_2 & \ldots & X_{n-1} & X_n & 0 & \ldots & \ldots & 0 \end{pmatrix}$$

Using the above we have

$$\underline{a}' \widehat{\Sigma}_n \underline{a} = \underline{a}' \mathbf{X}_n \mathbf{X}_n' \underline{a} = \|\mathbf{X}' \underline{a}\|_2^2 \geq 0.$$

This this proves that $\{\hat{c}_n(k)\}$ is a positive definite sequence.

Finally, by using Theorem 2.4.1, there exists a stochastic process with $\{\hat{c}_n(k)\}$ as its autocovariance function. □

7.2.1 Asymptotic properties of the covariance estimator

The main reason we construct an estimator is either for testing or constructing a confidence interval for the parameter of interest. To do this we need the variance and distribution of the estimator. It is impossible to derive the finite sample distribution, thus we look at their asymptotic distribution.

Besides showing asymptotic normality, it is important to derive an expression for the variance.

In an ideal world the variance will be simple and will not involve unknown parameters. Usually in time series this will not be the case, and the variance will involve several (often an infinite) number of parameters which are not straightforward to estimate. Later in this section we show that the variance of the sample covariance can be extremely complicated. However, a substantial simplification can arise if we consider only the sample correlation (not variance) and assume linearity of the time series. This result is known as Bartlett's formula (you may have come across Maurice Bartlett before, besides his fundamental contributions in time series he is well known for proposing the famous Bartlett correction). This example demonstrates, how the assumption of linearity can really simplify problems in time series analysis and also how we can circumvent certain problems in which arise by making slight modifications of the estimator (such as going from covariance to correlation).

The following theorem gives the asymptotic sampling properties of the covariance estimator (7.3). One proof of the result can be found in Brockwell and Davis (1998), Chapter 8, Fuller (1995), but it goes back to Bartlett (indeed its called Bartlett's formula). We prove the result in Section 7.2.2.

Theorem 7.2.1 *Suppose $\{X_t\}$ is a mean zero <u>linear</u> stationary time series where*

$$X_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j \varepsilon_{t-j},$$

where $\sum_j |\psi_j| < \infty$, $\{\varepsilon_t\}$ are iid random variables with $\mathrm{E}(\varepsilon_t) = 0$ and $\mathrm{E}(\varepsilon_t^4) < \infty$. Suppose we observe $\{X_t : t = 1, \ldots, n\}$ and use (7.3) as an estimator of the covariance $c(k) = \mathrm{cov}(X_0, X_k)$. Define $\hat{\rho}_n(r) = \hat{c}_n(r)/\hat{c}_n(0)$ as the sample correlation. Then for each $h \in \{1, \ldots, n\}$

$$\sqrt{n}(\hat{\rho}_n(h) - \rho(h)) \overset{\mathcal{D}}{\to} \mathcal{N}(0, W_h) \qquad (7.5)$$

where $\hat{\rho}_n(h) = (\hat{\rho}_n(1), \ldots, \hat{\rho}_n(h))$, $\rho(h) = (\rho(1), \ldots, \rho(h))$ and

$$(W_h)_{ij} = \sum_{k=-\infty}^{\infty} \Big\{ \rho(k+i)\rho(k+j) + \rho(k-i)\rho(k+j) + 2\rho(i)\rho(j)\rho^2(k) \qquad (7.6$$

$$-2\rho(i)\rho(k)\rho(k+j) - 2\rho(j)\rho(k)\rho(k+i) \Big\} \qquad (7.7$$

Equation (7.6) is known as Bartlett's formula.

In Section 7.3 we apply the method for checking for correlation in a time series. We first show how the expression for the asymptotic variance is obtained.

7.2.2 Proof of Bartlett's formula

What are cumulants?

We first derive an expression for $\hat{c}_n(r)$ under the assumption that $\{X_t\}$ is a strictly stationary time series with finite fourth order moment, $\sum_k |c(k)| < \infty$ and for all $r_1, r_2 \in \mathbb{Z}$, $\sum_k |\kappa_4(r_1, k, k+r_2)| < \infty$ where $\kappa_4(k_1, k_2, k_3) = \text{cum}(X_0, X_{k_1}, X_{k_2}, X_{k_3})$.

It is reasonable to ask what cumulants are. Cumulants often crops up in time series. To understand what they are and why they are used, we focus the following discussion on just fourth order cumulants.

The joint cumulant of $X_t, X_{t+k_1}, X_{t+k_2}, X_{t+k_3}$ (denoted as $\text{cum}(X_t, X_{t+k_1}, X_{t+k_2}, X_{t+k_3})$) is the coefficient of $s_1 s_2 s_3 s_4$ in the power series expansion of

$$\log \mathrm{E}[e^{is_1 X_t + is_2 X_{t+k_1} + is_3 X_{t+k_2} + is_4 X_{t+k_4}}].$$

It looks very similar to the definition of moments. Indeed there is a one to one correpondence between the moments and the cumulants, which is why they arise in time series. However, there are important differences

- If X_t is independent of $X_{t+k_1}, X_{t+k_2}, X_{t+k_3}$ then

  $$\text{cum}\left(X_t, X_{t+k_1}, X_{t+k_2}, X_{t+k_3}\right) = 0.$$

  This is because the log of the corresponding characteristic function is

  $$\log \mathrm{E}[e^{is_1 X_t + is_2 X_{t+k_1} + is_3 X_{t+k_2} + is_4 X_{t+k_4}}] = \log \mathrm{E}[e^{is_1 X_t}] + \log[\mathrm{E}[e^{is_2 X_{t+k_1} + is_3 X_{t+k_2} + is_4 X_{t+k_4}}].$$

  Thus we see that the coefficient of $s_1 s_2 s_3 s_4$ in the above expansion is zero.

- If $X_t, X_{t+k_1}, X_{t+k_2}, X_{t+k_3}$ is multivariate Gaussian, then all cumulants higher than order 2 are zero.

Neither of the above two properties hold for moments.

We can see from the definition of the characteristic function, if the time series is strictly stationary then

$$\log \mathrm{E}[e^{is_1X_t+is_2X_{t+k_1}+is_3X_{t+k_2}+is_4X_{t+k_4}}] = \log \mathrm{E}[e^{is_1X_0+is_2X_{k_1}+is_3X_{k_2}+is_4X_{k_4}}].$$

Thus the cumulants are invariant to shift

$$\mathrm{cum}(X_t, X_{t+k_1}, X_{t+k_2}, X_{t+k_3}) = \mathrm{cum}(X_0, X_{k_1}, X_{k_2}, X_{k_3}) = \kappa_4(k_1, k_2, k_3).$$

Thus like the autocovariance functions for stationary processes it does not depend on t.

The cumulant is similar to the covariance in that

(a) The covariance measures the dependence between X_t and X_{t+k}. Note that $\mathrm{cov}[X_t, X_{t+k}] = \mathrm{cov}[X_{t+k}, X_t]$, hence the covariance is invariant to order (if the random variables are real). The covariance is the second order cumulant.

Like the covariance, the joint cumulant $\mathrm{cum}[X_t, X_{t+k_1}, X_{t+k_2}, X_{t+k_3}]$ is also invariant to order.

(b) The cumulant is measuring the dependence between $\mathrm{cum}[X_t, X_{t+k_1}, X_{t+k_2}, X_{t+k_3}]$ in "all directions". For example, suppose $\{X_t\}$ has zero mean then

$$\begin{aligned} & \mathrm{cum}[X_t, X_{t+k_1}, X_{t+k_2}, X_{t+k_3}] \\ = \; & \mathrm{E}[X_tX_{t+k_1}X_{t+k_2}X_{t+k_3}] - \mathrm{E}[X_tX_{t+k_1}]\mathrm{E}[X_{t+k_2}X_{t+k_3}] \\ & -\mathrm{E}[X_tX_{t+k_2}]\mathrm{E}[X_{t+k_1}X_{t+k_3}] - \mathrm{E}[X_tX_{t+k_3}]\mathrm{E}[X_{t+k_1}X_{t+k_2}]. \end{aligned} \qquad (7.8)$$

(c) In time series we usually assume that the covariance decays over time i.e. if $k > 0$

$$|\mathrm{cov}[X_t, X_{t+k}]| \leq \alpha(k)$$

where $\alpha(k)$ is a positive sequence such that $\alpha(k) \to 0$ as $k \to \infty$. We showed this result was true for linear time series. The same is true of cumulants i.e. assume $k_1 \leq k_2 \leq k_3$ then

$$|\mathrm{cum}[X_t, X_{t+k_1}, X_{t+k_2}, X_{t+k_3}]| \leq \alpha(k_1)\alpha(k_2 - k_1)\alpha(k_3 - k_2).$$

(d) Often in proofs we can the assumption $\sum_r |c(r)| < \infty$. An analogous assumption is $\sum_{k_1,k_2,k_3} |\kappa_4(k_1,k_2,k_3)| < \infty$.

Example 7.2.1 *For the causal AR(1) model $X_t = \phi X_{t-1} + \varepsilon_t$ (where $\{\varepsilon_t\}$ are iid random variables with finite fourth order cumulant κ_4) by using the MA(∞) representation (assuming $0 \leq k_1 \leq k_2 \leq k_3$) we have*

$$\begin{aligned} cum[X_t, X_{t+k_1}, X_{t+k_2}, X_{t+k_3}] &= \sum_{j_0,j_1,j_2,j_3=0}^{n} \phi^{j_0+j_1+j_2+j_3} \text{cum}\left[\varepsilon_{t-j_0}, \varepsilon_{t+k_1-j_1}, \varepsilon_{t+k_2-j_2}, \varepsilon_{t+k_3-j_3}\right] \\ & \kappa_4 \sum_{j=0}^{\infty} \phi^j \phi^{j+k_1} \phi^{j+k_2} \phi^{j+k_3} = \kappa_4 \phi^{k_1+k_2+k_3} \sum_{j=0}^{\infty} \phi^{4j}. \end{aligned}$$

Observe that the fourth order dependence decays as the lag increases.

The variance of the sample covariance in the case of strict stationarity

We will consider

$$\text{var}[\hat{c}_n(r)] = \frac{1}{n^2} \sum_{t,\tau=1}^{n-|r|} \text{cov}(X_t X_{t+r}, X_\tau X_{\tau+r}).$$

One approach for the analysis of $\text{cov}(X_t X_{t+r}, X_\tau X_{\tau+r})$ is to expand it in terms of expectations $\text{cov}(X_t X_{t+r}, X_\tau X_{\tau+r}) = \text{E}(X_t X_{t+r}, X_\tau X_{\tau+r}) - \text{E}(X_t X_{t+r})\text{E}(X_\tau X_{\tau+r})$, however it not clear how this will give $\text{var}[X_t X_{t+r}] = O(n^{-1})$. Instead we observe that $\text{cov}(X_t X_{t+r}, X_\tau X_{\tau+r})$ is the covariance of the product of random variables. This belongs to the general class of cumulants of products of random variables. We now use standard results on cumulants. The most important is that if X, Y, U and V are mean zero random variables, then

$$\text{cov}[XY, UV] = \text{cov}[X,U]\text{cov}[Y,V] + \text{cov}[X,V]\text{cov}[Y,U] + \text{cum}(X,Y,U,V)$$

(this result can be seen from (7.8)). This result can be generalized to higher order cumulants, see Brillinger (2001). Using this result we have

$$\begin{aligned}
&\mathrm{var}[\hat{c}_n(r)] \\
= & \frac{1}{n^2}\sum_{t,\tau=1}^{n-|r|}\big(\underbrace{\mathrm{cov}(X_t,X_\tau)}_{=c(t-\tau)\text{ by stationarity}}\quad\mathrm{cov}(X_{t+r},X_{\tau+r}) + \mathrm{cov}(X_t,X_{\tau+r})\mathrm{cov}(X_{t+r},X_\tau) + cum(X_t,X_{t+r},X_\tau,X_{\tau+r})\big) \\
= & \frac{1}{n^2}\sum_{t,\tau=1}^{n-|r|}\big[c(t-\tau)^2 + c(t-\tau-r)c(t+r-\tau) + k_4(r,\tau-t,\tau+r-t)\big] \\
:= & I + II + III,
\end{aligned}$$

where the above is due to strict stationarity of the time series. We analyse the above term by term. Either (i) by changing variables and letting $k = t-\tau$ and thus changing the limits of the summand in an appropriate way or (ii) observing that $\sum_{t,\tau=1}^{n-|r|} c(t-\tau)^2$ is the sum of the elements in the Toeplitz matrix

$$\begin{pmatrix} c(0)^2 & c(1)^2 & \dots & c(n-|r|-1)^2 \\ c(-1)^2 & c(0)^2 & \dots & c(n-|r|-2)^2 \\ \vdots & \vdots & \ddots & \vdots \\ c(-(n-|r|-1))^2 & c(-(n-|r|-2))^2 & \dots & c(0)^2 \end{pmatrix},$$

(noting that $c(-k) = c(k)$) the sum I can be written as

$$I = \frac{1}{n^2}\sum_{t,\tau=1}^{n-|r|} c(t-\tau)^2 = \frac{1}{n^2}\sum_{k=-(n-|r|-1)}^{(n-|r|-1)} c(k)^2 \sum_{t=1}^{n-|r|-|k|} 1 = \frac{1}{n}\sum_{k=-(n-|r|)}^{n-|r|}\left(\frac{n-|r|-|k|}{n}\right)c(k)^2.$$

For all k, $(1-|k|/n)c(k)^2 \to c(k)^2$ and $|\sum_{k=-(n-|r|)}^{n-|r|}(1-|k|/n)c(k)^2| \le \sum_k c(k)^2$, thus by dominated convergence (see Appendix A) $\sum_{k=-(n-|r|)}^{n}(1-|k|/n)c(k)^2 \to \sum_{k=-\infty}^{\infty} c(k)^2$. This gives

$$I = \frac{1}{n}\sum_{k=-\infty}^{\infty} c(k)^2 + o(\frac{1}{n}).$$

Using a similar argument we can show that

$$II = \frac{1}{n}\sum_{k=-\infty}^{\infty} c(k+r)c(k-r) + o(\frac{1}{n}).$$

To derive the limit of III, again we use a change of variables to give

$$III = \frac{1}{n}\sum_{k=-(n-|r|)}^{n-|r|}\left(\frac{n-|r|-|k|}{n}\right)k_4(r,k,k+r).$$

To bound III we note that for all k, $(1-|k|/n)k_4(r,k,k+r) \to k_4(r,k,k+r)$ and $|\sum_{k=-(n-|r|)}^{n-|r|}(1-|k|/n)k_4(r,k,k+r)| \leq \sum_k |k_4(r,k,k+r)|$, thus by dominated convergence we have $\sum_{k=-(n-|r|)}^{n}(1-|k|/n)k_4(r,k,k+r) \to \sum_{k=-\infty}^{\infty} k_4(r,k,k+r)$. This gives

$$III = \frac{1}{n}\sum_{k=-\infty}^{\infty}\kappa_4(r,k,k+r) + o(\frac{1}{n}).$$

Therefore altogether we have

$$n\text{var}[\hat{c}_n(r)] = \sum_{k=-\infty}^{\infty} c(k)^2 + \sum_{k=-\infty}^{\infty} c(k+r)c(k-r) + \sum_{k=-\infty}^{\infty}\kappa_4(r,k,k+r) + o(1).$$

Using similar arguments we obtain

$$n\text{cov}[\hat{c}_n(r_1),\hat{c}_n(r_2)] = \sum_{k=-\infty}^{\infty} c(k)c(k+r_1-r_2) + \sum_{k=-\infty}^{\infty} c(k-r_1)c(k+r_2) + \sum_{k=-\infty}^{\infty}\kappa_4(r_1,k,k+r_2) + o(1).$$

We observe that the covariance of the covariance estimator contains both covariance and cumulants terms. Thus if we need to estimate them, for example to construct confidence intervals, this can be difficult. However, there does exist methods for estimating the variance, the most popular is to use a block type bootstrap, another method is to exploit properties of Fourier transforms and use the method of orthogonal samples (see Subba Rao (2017)).

Below that under linearity the above fourth order cumulant term has a simpler form.

The covariance of the sample covariance under linearity

We recall that

$$\sum_{k=-\infty}^{\infty} c(k+r_1-r_2)c(k) + \sum_{k=-\infty}^{\infty} c(k-r_1)c(k+r_2) + \sum_{k=-\infty}^{\infty}\kappa_4(r_1,k,k+r_2) + o(1) \quad = \quad T_1 + T_2 + T_3 + o(1).$$

We now show that under linearity, T_3 (the fourth order cumulant) has a much simpler form. Let us suppose that the time series is linear

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j \varepsilon_{t-j}$$

where $\sum_j |\psi_j| < \infty$, $\{\varepsilon_t\}$ are iid, $\mathrm{E}(\varepsilon_t) = 0$, $\mathrm{var}(\varepsilon_t) = \sigma^2$ and $\kappa_4 = \mathrm{cum}_4(\varepsilon_t)$. Then T_3 is

$$\begin{aligned} T_3 &= \sum_{k=-\infty}^{\infty} \mathrm{cum}\left(\sum_{j_1=-\infty} \psi_{j_1}\varepsilon_{-j_1}, \sum_{j_2=-\infty} \psi_{j_2}\varepsilon_{r_1-j_2}, \sum_{j_3=-\infty} \psi_{j_3}\varepsilon_{k-j_3}, \sum_{j_4=-\infty} \psi_{j_4}\varepsilon_{k+r_2-j_1} \right) \\ &= \sum_{k=-\infty}^{\infty} \sum_{j_1,\ldots,j_4=-\infty} \psi_{j_1}\psi_{j_2}\psi_{j_3}\psi_{j_4} \mathrm{cum}\left(\varepsilon_{-j_1}, \varepsilon_{r_1-j_2}, \varepsilon_{k-j_3}, \varepsilon_{k+r_2-j_1}\right). \end{aligned}$$

Standard results in cumulants, state that if any a variable is independent of all the others (see Section 7.2.2), then $\mathrm{cum}[Y_1, Y_2, \ldots, Y_n] = 0$. Applying this result to $\mathrm{cum}\left(\varepsilon_{-j_1}, \varepsilon_{r_1-j_2}, \varepsilon_{k-j_3}, \varepsilon_{k+r_2-j_1}\right)$ reduces T_3 to

$$T_3 = \kappa_4 \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \psi_j \psi_{j-r_1} \psi_{j-k} \psi_{j-r_2-k}.$$

Using a change of variables $j_1 = j$ and $j_2 = j - k$ we have

$$\kappa_4 \Big(\sum_{j_1=-\infty}^{\infty} \psi_j \psi_{j-r_1} \Big) \Big(\sum_{j_2=-\infty}^{\infty} \psi_{j_2} \psi_{j_2-r_2} \Big) = \kappa_4 \frac{c(r_1)}{\sigma^2} \frac{c(r_2)}{\sigma^2},$$

recalling the covariance of a linear process in Lemma 4.1.1 (and assuming $\mathrm{var}[\varepsilon_t] = \sigma^2$).

Altogether this gives

$$n\mathrm{cov}[\hat{c}_n(r_1), \hat{c}_n(r_2)] = \sum_{k=-\infty}^{\infty} c(k)c(k+r_1-r_2) + \sum_{k=-\infty}^{\infty} c(k-r_1)c(k+r_2) + \frac{\kappa_4}{\sigma^4} c(r_1)c(r_2) + o(1). \quad (7.9)$$

Thus in the case of linearity our expression for the variance is simpler, and the only difficult parameter to estimate of κ_4.

7.2.3 The sampling variance of the sample correlation under linearity

A suprisingly twist in the story is that (7.9) can be reduced further, if we are interested in estimating the correlation rather than the covariance. We recall the sample correlation is

$$\hat{\rho}_n(r) = \frac{\hat{c}_n(r)}{\hat{c}_n(0)},$$

which is an estimator of $\rho(r) = c(r)/c(0)$.

Lemma 7.2.2 (Bartlett's formula) *Suppose $\{X_t\}$ is a linear time series, where $\sum_j |\psi(j)| < \infty$. Then the variance of the distribution of $\hat{\rho}_n(r)$ is*

$$\sum_{k=-\infty}^{\infty} \{\rho(k+r)^2 + \rho(k-r)\rho(k+r) + 2\rho(r)^2\rho^2(k) - 4\rho(r)\rho(k)\rho(k+r)\}.$$

PROOF. We use that

$$\mathrm{var}[\hat{c}_n(r)] = O(n^{-1}) \text{ and } \mathrm{E}[\hat{c}_n(r)] = \left(\frac{n-|r|}{n}\right) c(r).$$

Thus for fixed r

$$|\hat{c}_n(0) - c(0)| = O_p(n^{-1/2}), \quad |\hat{c}_n(r) - c(r)| = O_p(n^{-1/2}) \text{ and } |\hat{c}_n(0) - c(0)|\,|\hat{c}_n(r) - c(r)| = O_p(n^{-1}).$$

By making a Taylor expansion of $\hat{c}_n(0)^{-1}$ about $c(0)^{-1}$ we have

$$\begin{aligned}
\hat{\rho}_n(r) - \rho(r) &= \frac{\hat{c}_n(r)}{\hat{c}_n(0)} - \frac{c(r)}{c(0)} \\
&= \frac{[\hat{c}_n(r) - c(r)]}{c(0)} - [\hat{c}_n(0) - c(0)]\frac{\hat{c}_n(r)}{c(0)^2} + \underbrace{[\hat{c}_n(0) - c(0)]^2 \frac{\hat{c}_n(r)}{\hat{c}_n(0)^3}}_{=O(n^{-1})} \qquad (7.10) \\
&= \frac{[\hat{c}_n(r) - c(r)]}{c(0)} - [\hat{c}_n(0) - c(0)]\frac{c(r)}{c(0)^2} + \underbrace{2\,[\hat{c}_n(0) - c(0)]^2 \frac{\hat{c}_n(r)}{\bar{c}_n(0)^3} - [\hat{c}_n(0) - c(0)]\frac{[\hat{c}_n(r) - c(r)]}{c(0)^2}}_{O(n^{-1})} \\
&:= A_n + O_p(n^{-1}), \qquad (7.11)
\end{aligned}$$

where $\bar{c}_n(0)$ lies between $\hat{c}_n(0)$ and $c(0)$ and

$$A_n = \frac{[\hat{c}_n(r) - c(r)]}{c(0)} - [\hat{c}_n(0) - c(0)]\,\frac{c(r)}{c(0)^2}.$$

Thus the dominating term in $\hat{\rho}_n(r) - \rho(r)$ is A_n, which is of order $O(n^{-1/2})$ (by (7.9)). Thus the limiting distribution of $\hat{\rho}_n(r)-\rho(r)$ is determined by A_n and the variance of the limiting distribution is also determined by A_n. It is straightforward to show that

$$n\mathrm{var}[A_n] = n\frac{\mathrm{var}[\hat{c}_n(r)]}{c(0)^2} - 2n\mathrm{cov}[\hat{c}_n(r), \hat{c}_n(0)]\frac{c(r)^2}{c(0)^3} + n\mathrm{var}[\hat{c}_n(0)]\frac{c(r)^2}{c(0)^4}. \tag{7.12}$$

By using (7.9) we have

$$\begin{aligned}
& n\mathrm{var}\begin{pmatrix} \hat{c}_n(r) \\ \hat{c}_n(0) \end{pmatrix} \\
= & \begin{pmatrix} \sum_{k=-\infty}^{\infty} c(k)^2 + \sum_{k=-\infty}^{\infty} c(k)c(k-r) + \kappa_4 c(r)^2 & 2\sum_{k=-\infty}^{\infty} c(k)c(k-r) + \kappa_4 c(r)c(0) \\ 2\sum_{k=-\infty}^{\infty} c(k)c(k-r) + \kappa_4 c(r)c(0) & \sum_{k=-\infty}^{\infty} c(k)^2 + \sum_{k=-\infty}^{\infty} c(k)c(k-r) + \kappa_4 c(0)^2 \end{pmatrix} \\
& +o(1).
\end{aligned}$$

Substituting the above into (7.12) gives us

$$\begin{aligned}
n\mathrm{var}[A_n] \;=\; & \left(\sum_{k=-\infty}^{\infty} c(k)^2 + \sum_{k=-\infty}^{\infty} c(k)c(k-r) + \kappa_4 c(r)^2\right)\frac{1}{c(0)^2} - \\
& 2\left(2\sum_{k=-\infty}^{\infty} c(k)c(k-r) + \kappa_4 c(r)c(0)\right)\frac{c(r)^2}{c(0)^3} + \\
& \left(\sum_{k=-\infty}^{\infty} c(k)^2 + \sum_{k=-\infty}^{\infty} c(k)c(k-r) + \kappa_4 c(0)^2\right)\frac{c(r)^2}{c(0)^4} + o(1).
\end{aligned}$$

Focusing on the fourth order cumulant terms, we see that these cancel, which gives the result. □

To prove Theorem 7.2.1, we simply use the Lemma 7.2.2 to obtain an asymptotic expression for the variance, then we use A_n to show asymptotic normality of $\hat{c}_n(r)$ (under linearity).

Exercise 7.1 *Under the assumption that $\{X_t\}$ are iid random variables show that $\hat{c}_n(1)$ is asymptotically normal.*

Hint: Let $m = n/(B+1)$ and partition the sum $\sum_{k=1}^{n-1} X_t X_{t+1}$ as follows

$$\begin{aligned}
\sum_{t=1}^{n-1} X_t X_{t+1} &= \sum_{t=1}^{B} X_t X_{t+1} + X_{B+1}X_{B+2} + \sum_{t=B+2}^{2B+1} X_t X_{t+1} + X_{2B+2}X_{2B+3} + \\
&\quad \sum_{t=2B+3}^{3B+2} X_t X_{t+1} + X_{3B+3}X_{3B+4} + \sum_{t=3B+4}^{4B+3} X_t X_{t+1} + \ldots \\
&= \sum_{j=0}^{m-1} U_{m,j} + \sum_{j=0}^{m-1} X_{(j+1)(B+1)} X_{(j+1)(B+1)+1}
\end{aligned}$$

where $U_{m,j} = \sum_{t=j(B+1)+1}^{j(B+1)+B} X_t X_{t+1}$. Show that the second term in the above summand is asymptotically negligible and show that the classical CLT for triangular arrays can be applied to the first term.

Exercise 7.2 *Under the assumption that $\{X_t\}$ is a MA(1) process, show that $\widehat{c}_n(1)$ is asymptotically normal.*

Exercise 7.3 *The block bootstrap scheme is a commonly used method for estimating the finite sample distribution of a statistic (which includes its variance). The aim in this exercise is to see how well the bootstrap variance approximates the finite sample variance of a statistic.*

(i) In `R` *write a function to calculate the autocovariance $\widehat{c}_n(1) = \frac{1}{n}\sum_{t=1}^{n-1} X_t X_{t+1}$.*

Remember the function is defined as `cov1 = function(x){...}`

(ii) Load the library boot `library("boot")` *into* `R`*. We will use the block bootstrap, which partitions the data into blocks of lengths l and then samples from the blocks n/l times to construct a new bootstrap time series of length n. For each bootstrap time series the covariance is evaluated and this is done R times. The variance is calculated based on these R bootstrap estimates.*

You will need to use the function `tsboot(tseries,statistic,R=100,l=20,sim="fixed")`*. tseries refers to the original data, statistic to the function you wrote in part (i) (which should only be a function of the data), R=is the number of bootstrap replications and l is the length of the block.*

Note that `tsboot(tseries,statistic,R=100,l=20,sim="fixed")$t` *will be vector of length $R = 100$ which will contain the bootstrap statistics, you can calculate the variance of this vector.*

(iii) Simulate the $AR(2)$ *time series* $arima.sim(list(order = c(2,0,0), ar = c(1.5,-0.75)), n = 128)$ *500 times. For each realisation calculate the sample autocovariance at lag one and also the bootstrap variance.*

(iv) Calculate the mean of the bootstrap variances and also the mean squared error (compared with the empirical variance), how does the bootstrap perform?

(iv) Play around with the bootstrap block length l*. Observe how the block length can influence the result.*

Remark 7.2.1 *The above would appear to be a nice trick, but there are two major factors that lead to the cancellation of the fourth order cumulant term*

- *Linearity of the time series*
- *Ratio between* $\hat{c}_n(r)$ *and* $\hat{c}_n(0)$*.*

Indeed this is not a chance result, in fact there is a logical reason why this result is true (and is true for many statistics, which have a similar form - commonly called ratio statistics). It is easiest explained in the Fourier domain. If the estimator can be written as

$$\frac{1}{n}\frac{\sum_{k=1}^{n}\phi(\omega_k)I_n(\omega_k)}{\frac{1}{n}\sum_{k=1}^{n}I_n(\omega_k)},$$

where $I_n(\omega)$ *is the periodogram, and* $\{X_t\}$ *is a linear time series, then we will show later that the asymptotic distribution of the above has a variance which is only in terms of the covariances* <u>*not*</u> *higher order cumulants. We prove this result in Section 10.5.*

7.3 Checking for correlation in a time series

Bartlett's formula if commonly used to check by 'eye; whether a time series is uncorrelated (there are more sensitive tests, but this one is often used to construct CI in for the sample autocovariances in several statistical packages). This is an important problem, for many reasons:

- Given a data set, we need to check whether there is dependence, if there is we need to analyse it in a different way.

- Suppose we fit a linear regression to time series data. We may to check whether the residuals are actually uncorrelated, else the standard errors based on the assumption of uncorrelatedness would be unreliable.

- We need to check whether a time series model is the appropriate model. To do this we fit the model and estimate the residuals. If the residuals appear to be uncorrelated it would seem likely that the model is correct. If they are correlated, then the model is inappropriate. For example, we may fit an AR(1) to the data, estimate the residuals ε_t, if there is still correlation in the residuals, then the AR(1) was not the correct model, since $X_t - \hat{\phi}X_{t-1}$ is still correlated (which it would not be, if it were the correct model).

We now apply Theorem 7.2.1 to the case that the time series are iid random variables. Suppose $\{X_t\}$ are iid random variables, then it is clear that it is trivial example of a (not necessarily Gaussian) linear process. We use (7.3) as an estimator of the autocovariances.

To derive the asymptotic variance of $\{\hat{c}_n(r)\}$, we recall that if $\{X_t\}$ are iid then $\rho(k) = 0$ for $k \neq 0$. Thus by making a Taylor expansion (and noting that $c(k)$=0) (and/or using (7.6)) we see that

$$\sqrt{n}\hat{\boldsymbol{\rho}}_n = \frac{\sqrt{n}}{c(0)}\hat{\mathbf{c}}_n + o_p(1) \overset{\mathcal{D}}{\to} \mathcal{N}(0, W_h),$$

where $\widehat{\boldsymbol{\rho}}_n = (\widehat{\rho}_n(1), \ldots, \widehat{\rho}_n(m))$, similar with $\widehat{\mathbf{c}}_n$ and

$$(W_h)_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

In other words, $\sqrt{n}\hat{\boldsymbol{\rho}}_n \overset{\mathcal{D}}{\to} \mathcal{N}(0, I_h)$. Hence the sample autocovariances at different lags are asymptotically uncorrelated and have variance one. This allows us to easily construct error bars for the sample autocovariances under the assumption of independence. If the vast majority of the sample autocovariance lie inside the error bars there is not enough evidence to suggest that the data is a realisation of a iid random variables (often called a white noise process). An example of the empirical ACF and error bars is given in Figure 7.1. We see that the empirical autocorrelations of the realisation from iid random variables all lie within the error bars. In contrast in Figure 7.2 we give a plot of the sample ACF of an AR(2). We observe that a large number of the sample autocorrelations lie outside the error bars.

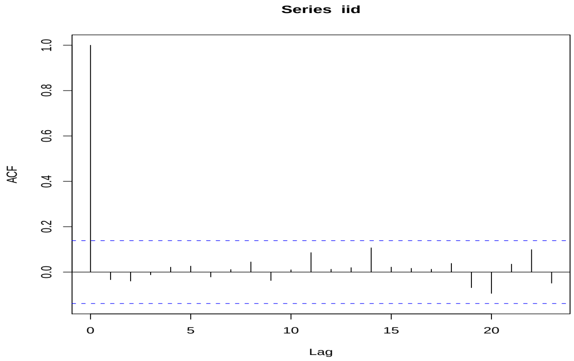


Figure 7.1: The sample ACF of an iid sample with error bars (sample size $n = 200$).

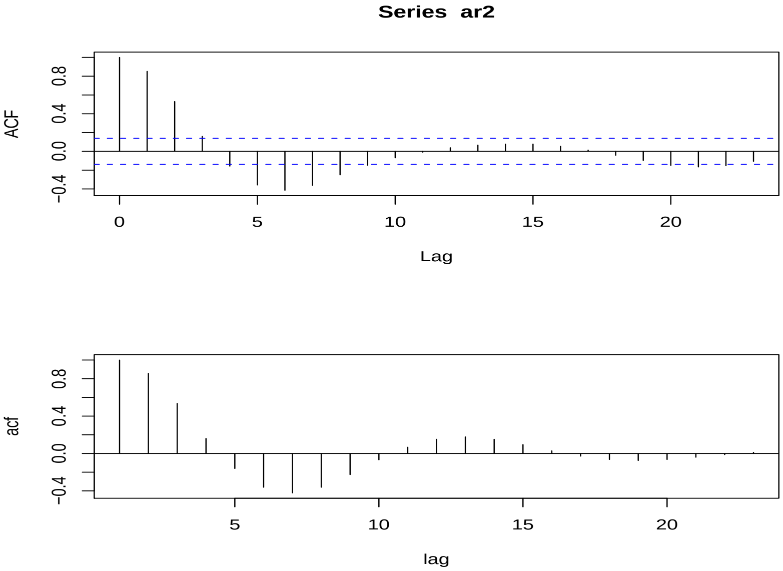


Figure 7.2: Top: The sample ACF of the AR(2) process $X_t = 1.5X_{t-1} + 0.75X_{t-2} + \varepsilon_t$ with error bars $n = 200$. Bottom: The true ACF.

Of course, simply checking by eye means that we risk misconstruing a sample coefficient that lies outside the error bars as meaning that the time series is correlated, whereas this could simply be a false positive (due to multiple testing). To counter this problem, we construct a test statistic for testing uncorrelatedness. We test the hypothesis $H_0 : c(r) = 0$ for all r against H_A : at least one $c(r) \neq 0$.

A popular method for measuring correlation is to use the squares of the sample correlations

$$\mathcal{S}_h = n \sum_{r=1}^{h} |\widehat{\rho}_n(r)|^2. \tag{7.13}$$

Since under the null $\sqrt{n}(\hat{\rho}_n(h) - \rho(h)) \overset{\mathcal{D}}{\to} \mathcal{N}(0, I)$, under the null $\mathcal{S}_h$ asymptotically will have a χ^2-distribution with h degrees of freedom, under the alternative it will be a non-central (generalised) chi-squared. The non-centrality is what makes us reject the null if the alternative of correlatedness is true. This is known as the Box-Pierce (or Portmanteau) test. The Ljung-Box test is a variant on the Box-Pierce test and is defined as

$$\mathcal{S}_h = n(n+2) \sum_{r=1}^{h} \frac{|\hat{\rho}_n(r)|^2}{n-r}. \tag{7.14}$$

Again under the null of no correlation, asymptotically, $\mathcal{S}_h \overset{\mathcal{D}}{\to} \chi^2_h$. Generally, the Ljung-Box test is suppose to give more reliable results than the Box-Pierce test.

Of course, one needs to select h. In general, we do not have to use large h since most correlations will arise when the lag is small, However the choice of h will have an influence on power. If h is too large the test will loose power (since the mean of the chi-squared grows as $h \to \infty$), on the other hand choosing h too small may mean that certain correlations at higher lags are missed. How to selection h is discussed in several papers, see for example Escanciano and Lobato (2009).

7.3.1 The robust Portmanteau test

One disadvantage of the Box-Pierce/Portmanteau test described above is that it assumes under the null that the time series is *independent* not just uncorrelated. Even though the test statistic involves the sample covariance and cannot test for independence. Hence it cannot be used to test for uncorrelatedness in financial data, since this type of data may be uncorrelated but still dependent.

To understand the impact that dependence has on the test we simulate from the iid standard normal χ^2_s-df and an ARCH(1) model $X_t = \sigma_t Z_t$ where $\sigma_t^2 = 0.5 + 0.6X_{t-1}^2$ (which is uncorrelated but dependent), sample size $n = 300$. These three time series are uncorrelated. 200 replications were made. In the simulations we set $h = 2$ and use the test statistic

$$\mathcal{S}_2 = n\sum_{r=1}^{2}\left|\frac{\frac{1}{n}\sum_{t=1}^{n-|r|}(X_t-\bar{X})(X_{t+r}-\bar{X})}{\frac{1}{n}\sum_{t=1}^{n}(X_t-\bar{X})^2}\right|^2,$$

where under the null $\mathcal{S}_2 \overset{\mathcal{D}}{\to} \chi^2_2$. A Quantile-Quantile plot of $\mathcal{S}_2$ against a the quantiles of a chi-square distribution in the case that $\{X_t\}$ are iid normal and chi-squared are given in Figure 7.3. We observe that chi-square distribution approximates relatively well the finite sampling distribution of $\mathcal{S}_2$. The proportion of rejections at the 5% level are given in Table 7.1. We see that the test appears to keep the stated 5% level.

IID normal	6%
IID chi-square	5%

Table 7.1: Proportion of rejections under the null hypothesis. Test done at the 5% level over 200 replications.

Now let us return to the ARCH(1). We recall from Section 5.2.1 that the ARCH process is uncorrelated. Thus the null hypothesis $H_0 : c(r) = 0$ for all r holds. We simulate from the ARCH model 200 times (sample size $n = 300$) and evaluate $\mathcal{S}_2$. A QQplot of $\mathcal{S}_2$ against the quantiles of a chi-square distribution are given in the left hand side plot in Figure 7.4. We observe that it deviates massively from a chi-square distribution and from Table 7.2 the proportion of rejections at the 5% level is 26%. In other words, there are far too many false positive despite the time series being uncorrelated. The reason that $\mathcal{S}_2$ is not asymptotically a chi-square is because the sampling variance of $\widehat{\rho}_n(r)$ is not one when the time series is dependent. More precisely, the simple pivotal result $\sqrt{n}(\widehat{\rho}_n(h) - \rho(h)) \overset{\mathcal{D}}{\to} \mathcal{N}(0, I)$ does not hold. Remember that pesky fourth cumulant term, it makes an appearance when the process is uncorrelated but dependent. We recall that under the null of no correlation we have

$$\mathrm{cov}\left(\sqrt{n}\widehat{c}_n(r_1), \sqrt{n}\widehat{c}_n(r_2)\right) = \begin{cases} \sum_k \kappa_4(r_1, k, k+r_2) & r_1 \neq r_2 \\ c(0)^2 + \sum_k \kappa_4(r, k, k+r) & r_1 = r_2 = (r) \end{cases}$$

Thus even under asymptotic normality of $\widehat{c}_n(r)$ the marginal distribution of $\widehat{c}_n(r)/(0)$ (if the time

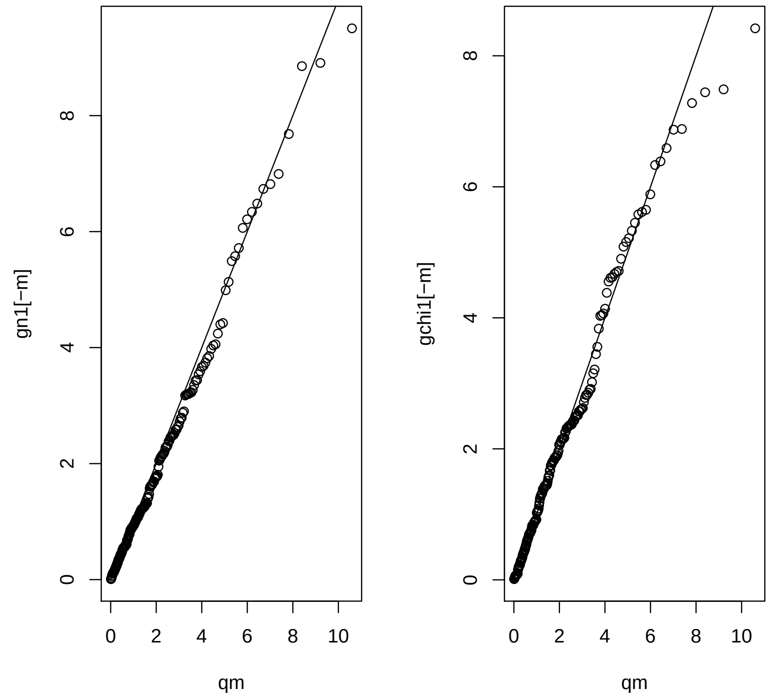


Figure 7.3: Plot of $\mathcal{S}_2$ against the quantiles of a chi-square distribution with 2df. Left: $\{X_t\}$ are iid normal. Right: $\{X_t\}$ are chi-squared.

series is uncorrelated but not necessarily independent) is

$$\sqrt{n}\frac{\widehat{c}_n(r)}{c(0)} \stackrel{\mathcal{D}}{\to} \mathcal{N}\left(0, 1 + c(0)^{-2}\sum_k \kappa_4(r,k,k+r)\right).$$

Thus even ignoring the dependence between $\{\widehat{c}_n(r)\}_r$ the Box-Pierce test does not correctly estimate the variance. Usually, $\sum_k \kappa_4(r,k,k+r)$ is positive and which means the Box-Pierce test underestimates the variance, which results in the quantiles of $\mathcal{S}_2$ in Figure 7.4 being far larger than the chi square quantiles (which gives the over rejection under the null hypothesis).

However, there is an important subset of uncorrelated time series, which are dependent, where a slight modification of the Box-Pierce test does give reliable results. This subset includes the aforementioned ARCH process and is a very useful test in financial applications. ARCH and GARCH processes are uncorrelated time series which are martigale differences. That is $\{X_t\}$ is called a martingale difference if

$$\mathrm{E}(X_t|X_{t-1}, X_{t-2}, X_{t-3}, \ldots) = 0.$$

Martingale differences include independent random variables as a special case. Clearly, from this definition $\{X_t\}$ is uncorrelated since for $r > 0$ and by using the definition of a martingale difference we have

$$\begin{aligned} \mathrm{cov}(X_t, X_{t+r}) &= \mathrm{E}(X_t X_{t+r}) - \mathrm{E}(X_t)\mathrm{E}(X_{t+r}) \\ &= \mathrm{E}(X_t \mathrm{E}(X_{t+r}|X_t)) - \mathrm{E}(\mathrm{E}(X_t|X_{t-1}))\mathrm{E}(\mathrm{E}(X_{t+r}|X_{t+r-1})) = 0. \end{aligned}$$

Thus a martingale difference sequence is an uncorrelated sequence. However, it has some useful properties that just incorrelated random variables do not have. For example, if $t_1 \neq t_2$ and suppose for simplicity $t_2 + r_2 > t_2, t_1, t_1 + r_1$. Then

$$\mathrm{E}(X_{t_1} X_{t_1+r_2} X_{t_2} X_{t_2+r_2}) = \mathrm{E}(X_{t_1} X_{t_1+r_2} X_{t_2} | \mathrm{E}(X_{t_2+r_2}|X_{t_1}, X_{t_1+r_2}, X_{t_2})) = 0$$

and if $r_1 \neq r_2$ (assume $r_2 > r_1$) by the same argument

$$\mathrm{E}(X_t X_{t+r_2} X_t X_{t+r_2}) = \mathrm{E}(X_t^2, X_{t+r_1} | \mathrm{E}(X_{t+r_2}|X_t^2, X_{t+r_1}) = 0.$$

The above two results can be used to show that the variance of $\widehat{c}_n(r)$ (under the assumption that the time series martingale differences) has a very simple form

$$\begin{aligned} \mathrm{var}\left(\sqrt{n}\widehat{c}_n(r)\right) &= \frac{1}{n} \sum_{t_1,t_2=1}^{n} \mathrm{cov}\left(X_{t_1} X_{t_1+r}, X_{t_2} X_{t_2+r}\right) \\ &= \frac{1}{n} \sum_{t_1,t_2=1}^{n} \mathrm{E}\left(X_{t_1} X_{t_1+r} X_{t_2} X_{t_2+r}\right) = \frac{1}{n} \sum_{t=1}^{n} \mathrm{E}\left(X_t^2 X_{t+r}^2\right) \end{aligned}$$

and if $r_1 \neq r_2$ then $\mathrm{cov}(\widehat{c}_n(r_1), \widehat{c}_n(r_2)) = 0$. Let $\sigma_r^2 = \frac{1}{n}\sum_{t=1}^{n} \mathrm{E}\left(X_t^2 X_{t+r}^2\right)$. Then we have that under the null hypothesis (and suitable conditions to ensure normality) that

$$\sqrt{n}\begin{pmatrix} \widehat{c}_n(1)/\sigma_1 \\ \widehat{c}_n(2)/\sigma_2 \\ \vdots \\ \widehat{c}_n(L)/\sigma_L \end{pmatrix} \overset{\mathcal{D}}{\to} \mathcal{N}\left(0, I_L\right).$$

It is straightforward to estimate the σ_r^2 with

$$\widehat{\sigma}_r^2 = \frac{1}{n}\sum_{t=1}^{n} X_t^2 X_{t+r}^2.$$

Thus a similar squared distance as the Box-Pierce test is used to define the Robust Portmanteau test, which is defined as

$$\mathcal{R}_h = n\sum_{r=1}^{h} \frac{|\widehat{c}_n(r)|^2}{\sigma_r^2}.$$

Under the null hypothesis asymptotically $\mathcal{R}_h \overset{\mathcal{D}}{\to} \chi_h^2$. To see how this test performs in the right hand plot in Figure 7.4 we give the quantile quantile plot of $\mathcal{R}_h$ against the chi-squared distribution. We observe that it lies pretty much on the $x = y$ line. Moreover, the test results at the 5% level are given in Table 7.2. We observe that it is close to the stated 5% level and performs far better than the classical Box-Pierce test.

ARCH Box-Pierce	26%
ARCH Robust Portmanteau	4.5%

Table 7.2: Proportion of rejections under the null hypothesis. Test done at the 5% level over 200 replications.

7.4 Checking for partial correlation

We recall that the partial correlation of a stationary time series at lag t is given by the last coefficient of the best linear predictor of X_{m+1} given $\{X_j\}_{j=1}^m$ i.e. ϕ_m where $\widehat{X}_{m+1|m} = \sum_{j=1}^m \phi_j X_{m+1-j}$. Thus ϕ_m can be estimated using the Yule-Walker estimator or least squares (more of this later) and the

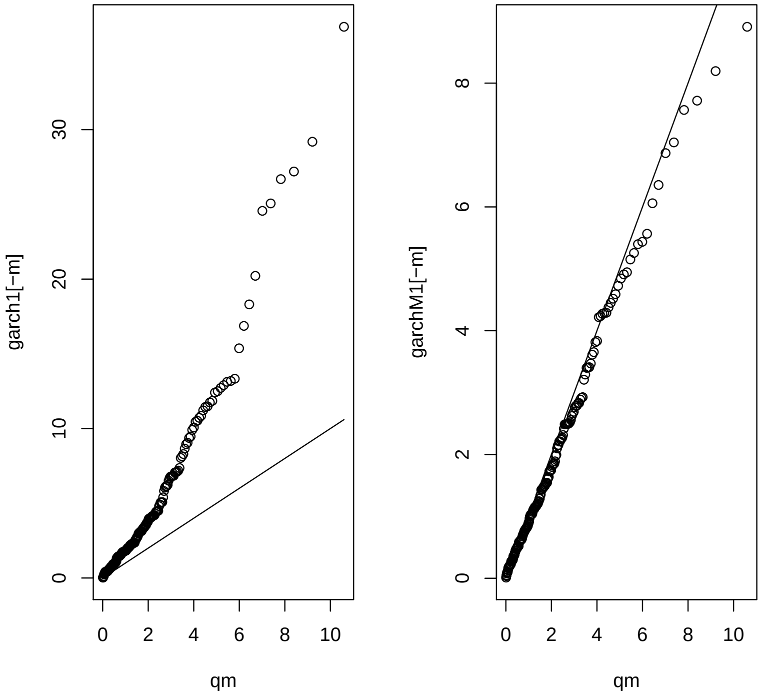


Figure 7.4: Using ARCH(1) time series over 200 replications Left: $\mathcal{S}_2$ against the quantiles of a chi-square distribution with 2df for an ARCH process. Right: $\mathcal{R}_2$ against the quantiles of a chi-square distribution with 2df for an ARCH process.

sampling properties of the estimator are determined by the sampling properties of the estimator of an AR(m) process. We state these now. We assume $\{X_t\}$ is a AR(p) time series of the form

$$X_t = \sum_{j=1}^{p} \phi_j X_{t-j} + \varepsilon_t$$

where $\{\varepsilon_t\}$ are iid random variables with mean zero and variance σ^2. Suppose an AR(m) model is fitted to the data using the Yule-Walker estimator, we denote this estimator as $\widehat{\boldsymbol{\phi}}_m = \widehat{\Sigma}_m^{-1} \underline{r}_m$. Let $\widehat{\phi}_m = (\widehat{\phi}_{m1}, \ldots, \widehat{\phi}_{mm})$, the estimator of the partial correlation at lag m is $\widehat{\phi}_{mm}$. Assume $m \geq p$. Then by using Theorem 8.2.1 (see also Theorem 8.1.2, Brockwell and Davis (1998)) we have

$$\sqrt{n}\left(\widehat{\boldsymbol{\phi}}_m - \boldsymbol{\phi}_m\right) \overset{\mathcal{P}}{\to} N(0, \sigma^2 \Sigma_m^{-1}).$$

where $\boldsymbol{\phi}_m$ are the true parameters. If $m > p$, then $\boldsymbol{\phi}_m = (\phi_1, \ldots, \phi_p, 0, \ldots, 0)$ and the last coefficient has the marginal distribution

$$\sqrt{n}\widehat{\phi}_{mm} \overset{\mathcal{P}}{\to} N(0, \sigma^2 \Sigma^{mm}).$$

Since $m > p$, we can obtain a closed for expression for Σ^{mm}. By using Remark 4.3.1 we have $\Sigma^{mm} = \sigma^{-2}$, thus

$$\sqrt{n}\widehat{\phi}_{mm} \overset{\mathcal{P}}{\to} N(0, 1).$$

Therefore, for lags $m > p$ the partial correlations will be asymptotically pivotal. The errors bars in the partial correlations are $[-1.96n^{-1/2}, 1.96n^{-1/2}]$ and these can be used as a guide in determining the order of the autoregressive process (note there will be dependence between the partial correlation at different lags).

This is quite a surprising result and very different to the behaviour of the sample autocorrelation function of an MA(p) process.

Exercise 7.4

(a) *Simulate a mean zero invertible MA*(1) *process (use Gaussian errors). Use a reasonable sample size (say* $n = 200$*). Evaluate the sample correlation at lag 2,* $\widehat{rho}_n(2)$*. Note the sample correlation at lag two is estimating 0. Do this 500 times.*

- *Calculate of proportion of sample covariances* $|\widehat{\rho}_n(2)| > 1.96/\sqrt{n}$

- *Make a QQplot of* $\widehat{\rho}_n(2)/\sqrt{n}$ *against a standard normal distribution. What do you observe?*

(b) Simulate a causal, stationary AR(1) *process (use Gaussian errors). Use a reasonable sample size (say* $n = 200$*). Evaluate the sample partial correlation at lag 2,* $\widehat{\phi}_n(2)$*. Note the sample partial correlation at lag two is estimating 0. Do this 500 times.*

- *Calculate of proportion of sample partial correlations* $|\widehat{\phi}_n(2)| > 1.96/\sqrt{n}$

- *Make a QQplot of* $\widehat{\phi}_n(2)/\sqrt{n}$ *against a standard normal distribution. What do you observe?*

7.5 Checking for Goodness of fit

To check for adequency of a model, after fitting a model to the data the sample correlation of the estimated residuals is evaluated. If there appears to be no correlation in the estimated residuals (so the residuals are near uncorrelated) then the model is determined to adequately fit the data.

Consider the general model

$$X_t = g(Y_t, \theta) + \varepsilon_t$$

where $\{\varepsilon_t\}$ are iid random variables and ε_t is independent of $Y_t, Y_{t-1}, \ldots$. Note Y_t can be a vector, such as $Y_{t-1} = (X_{t-1}, X_{t-2}, \ldots, X_{t-p})$ and examples of models which satisfy the above include the AR(p) process. We will assume that $\{X_t, Y_t\}$ is a stationary ergodic process. Further to simplify the discussion we will assume that θ is univariate, it is straightforward to generalize the discussion below to the multivariate case.

Let $\widehat{\theta}$ denote the least squares estimator of θ i.e.

$$\widehat{\theta} = \arg\min \sum_{t=1}^{n} \left(X_t - g(Y_t, \theta)\right)^2. \tag{7.15}$$

Using the "usual" Taylor expansion methods (and assuming all the usual conditions are satisfied, such as $|\widehat{\theta} - \theta| = O_p(n^{-1/2})$ etc) then it can be shown that

$$\sqrt{n}\left(\widehat{\theta} - \theta\right) = \mathcal{I}^{-1} \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \varepsilon_t \frac{\partial g(Y_t, \theta)}{\partial \theta} + o_p(1) \text{ where } \mathcal{I} = \mathrm{E}\left(\frac{\partial g(Y_t, \theta)}{\partial \theta}\right)^2.$$

$\{\varepsilon_t \frac{\partial g(Y_t,\theta)}{\partial\theta}\}$ are martingale differences, which is why $\sqrt{n}\left(\widehat{\theta}-\theta\right)$ is asymptotically normal, but more of this in the next chapter. Let $\mathcal{L}_n(\theta)$ denote the least squares criterion. Note that the above is true because

$$\frac{\partial \mathcal{L}_n(\theta)}{\partial\theta} = -2\sum_{t=1}^{n}\left[X_t - g(Y_t,\theta)\right]\frac{\partial g(Y_t,\theta)}{\partial\theta}$$

and

$$\frac{\partial^2 \mathcal{L}_n(\theta)}{\partial\theta^2} = -2\sum_{t=1}^{n}\left[X_t - g(Y_t,\theta)\right]\frac{\partial^2 g(Y_t,\theta)}{\partial\theta^2} + 2\sum_{t=1}^{n}\left(\frac{\partial g(Y_t,\theta)}{\partial\theta}\right)^2,$$

thus at the true parameter, θ,

$$\frac{1}{n}\frac{\partial^2 \mathcal{L}_n(\theta)}{\partial\theta^2} \overset{\mathcal{P}}{\to} 2\mathcal{I}.$$

Based on (7.15) we estimate the residuals using

$$\widehat{\varepsilon}_t = X_t - g(Y_t,\widehat{\theta})$$

and the sample correlation with $\widehat{\rho}(r) = \widehat{c}(r)/\widehat{c}(0)$ where

$$\widehat{c}(r) = \frac{1}{n}\sum_{t=1}^{n-|r|}\sum_{t}\widehat{\varepsilon}_t\widehat{\varepsilon}_{t+r}.$$

Often it is (wrongly) assumed that one can simply apply the results in Section 7.3 when checking for adequacy of the model. That is make an ACF plot of $\widehat{\rho}(r)$ and use $[-n^{-1/2}, n^{1/2}]$ as the error bars. However, since the parameters have been estimated the size of the error bars will change. In particular, under the null that the model is correct we will show that

$$\sqrt{n}\widehat{\rho}(r) = \mathcal{N}\left(0, \underbrace{1}_{\text{iid part}} - \underbrace{\frac{\sigma^2}{c(0)}\mathcal{J}_r\mathcal{I}^{-1}\mathcal{J}_r}_{\text{due to parameter estimation}}\right)$$

where $c(0) = \mathrm{var}[X_t]$, $\sigma^2 = \mathrm{var}(\varepsilon_t)$ and $\mathcal{J}_r = \mathrm{E}[\frac{\partial g(Y_{t+r},\theta)}{\partial\theta}\varepsilon_t]$ and $\mathcal{I} = \mathrm{E}\left(\frac{\partial g(Y_t,\theta)}{\partial\theta}\right)^2$ (see, for example, Li (1992)). Thus the error bars under the null are

$$\left[\pm\left(\frac{1}{\sqrt{n}}\left[1-\frac{\sigma^2}{c(0)}\mathcal{J}_r\mathcal{I}^{-1}\mathcal{J}_r\right]\right)\right].$$

Estimation of the parameters means the inclusion of the term $\frac{\sigma^2}{c(0)}\mathcal{J}_r\mathcal{I}^{-1}\mathcal{J}_r$. If the lag r is not too small then $\mathcal{J}_r$ will be close to zero and the $[\pm 1/\sqrt{n}]$ approximation is fine, but for small r, $\mathcal{J}_r\mathcal{I}^{-1}\mathcal{J}_r$ can be large and positive, thus the error bars, $\pm n^{-1/2}$, are too wide. Thus one needs to be a little cautious when interpreting the $\pm n^{-1/2}$ error bars. Note if there is no dependence between ε_t and Y_{t+r} then using the usual error bars is fine.

Remark 7.5.1 *The fact that the error bars get narrower after fitting a model to the data seems a little strange. However, it is far from unusual. One explanation is that the variance of the estimated residuals tend to be less than the true residuals (since the estimated residuals contain less information about the process than the true residuals). The most simplest example are iid observations $\{X_i\}_{i=1}^{n}$ with mean μ and variance σ^2. The variance of the "estimated residual" $X_i-\bar{X}$ is $(n-1)\sigma^2/n$.*

We now derive the above result (using lots of Taylor expansions). By making a Taylor expansion similar to (7.10) we have

$$\sqrt{n}\left[\widehat{\rho}_n(r)-\rho(r)\right]\sqrt{n}\frac{[\hat{c}_n(r)-c(r)]}{c(0)}-\sqrt{n}\left[\hat{c}_n(0)-c(0)\right]\frac{c(r)}{c(0)^2}+O_p(n^{-1/2}).$$

However, under the "null" that the correct model was fitted to the data we have $c(r)=0$ for $|r|>0$, this gives

$$\sqrt{n}\widehat{\rho}_n(r)=\sqrt{n}\frac{\widehat{c}_n(r)}{c(0)}+o_p(1),$$

thus the sampling properties of $\widehat{\rho}_n(r)$ are determined by $\widehat{c}_n(r)$, and we focus on this term. It is easy to see that

$$\sqrt{n}\widehat{c}_n(r)=\frac{1}{\sqrt{n}}\sum_{t=1}^{n-r}\left(\varepsilon_t+g(\theta,Y_t)-g(\widehat{\theta},Y_t)\right)\left(\varepsilon_{t+r}+g(\theta,Y_{t+r})-g(\widehat{\theta},Y_{t+r})\right).$$

Heuristically, by expanding the above, we can see that

$$\sqrt{n}\widehat{c}_n(r) \approx \frac{1}{\sqrt{n}}\sum_{t=1}^{n-r}\varepsilon_t\varepsilon_{t+r} + \frac{1}{\sqrt{n}}\sum_{t=1}^{n}\varepsilon_{t+r}\left(g(\theta,Y_t)-g(\widehat{\theta},Y_t)\right) + \frac{1}{\sqrt{n}}\sum_{t=1}^{n}\varepsilon_t\left(g(\theta,Y_{t+r})-g(\widehat{\theta},Y_{t+r})\right),$$

then by making a Taylor expansion of $g(\widehat{\theta},\cdot)$ about $g(\theta,\cdot)$ (to take $(\widehat{\theta}-\theta)$ out of the sum)

$$\begin{aligned}
\sqrt{n}\widehat{c}_n(r) &\approx \frac{1}{\sqrt{n}}\sum_{t=1}^{n-r}\varepsilon_t\varepsilon_{t+r} + \frac{(\widehat{\theta}-\theta)}{\sqrt{n}}\left[\sum_{t=1}^{n}\varepsilon_{t+r}\frac{\partial g(\theta,Y_t)}{\partial\theta}+\varepsilon_t\frac{\partial g(\theta,Y_{t+r})}{\partial\theta}\right]+o_p(1)\\
&= \frac{1}{\sqrt{n}}\sum_{t=1}^{n-r}\varepsilon_t\varepsilon_{t+r} + \sqrt{n}(\widehat{\theta}-\theta)\frac{1}{n}\left[\sum_{t=1}^{n}\varepsilon_{t+r}\frac{\partial g(\theta,Y_t)}{\partial\theta}+\varepsilon_t\frac{\partial g(\theta,Y_{t+r})}{\partial\theta}\right]+o_p(1).
\end{aligned}$$

We make this argument precise below. Making a Taylor expansion we have

$$\begin{aligned}
\sqrt{n}\widehat{c}_n(r) &= \frac{1}{\sqrt{n}}\sum_{t=1}^{n-r}\left(\varepsilon_t-(\widehat{\theta}-\theta)\frac{\partial g(\theta,Y_t)}{\partial\theta}+\frac{(\widehat{\theta}-\theta)^2}{2}\frac{\partial^2 g(\bar{\theta}_t,Y_t)}{\partial\theta^2}\right)\times\\
&\quad\left(\varepsilon_{t+r}-(\widehat{\theta}-\theta)\frac{\partial g(\theta,Y_{t+r})}{\partial\theta}+\frac{(\widehat{\theta}-\theta)^2}{2}\frac{\partial^2 g(\bar{\theta}_{t+r},Y_{t+r})}{\partial\theta^2}\right)\\
&= \sqrt{n}\widetilde{c}_n(r)-\sqrt{n}(\widehat{\theta}-\theta)\frac{1}{n}\sum_{t=1}^{n-r}\left(\varepsilon_t\frac{\partial g(\theta,Y_{t+r})}{\partial\theta}+\varepsilon_{t+r}\frac{\partial g(\theta,Y_t)}{\partial\theta}\right)+O_p(n^{-1/2})(7.16)
\end{aligned}$$

where θ_t lies between $\widehat{\theta}$ and θ and

$$\widetilde{c}_n(r)=\frac{1}{n}\sum_{t=1}^{n-r}\varepsilon_t\varepsilon_{t+r}.$$

We recall that by using ergodicity we have

$$\frac{1}{n}\sum_{t=1}^{n-r}\left(\varepsilon_t\frac{\partial g(\theta,Y_{t+r})}{\partial\theta}+\varepsilon_{t+r}\frac{\partial g(\theta,Y_t)}{\partial\theta}\right)\overset{\text{a.s.}}{\rightarrow}\mathrm{E}\left(\varepsilon_t\frac{\partial g(\theta,Y_{t+r})}{\partial\theta}\right)=\mathcal{J}_r,$$

where we use that ε_{t+r} and $\frac{\partial g(\theta,Y_t)}{\partial\theta}$ are independent. Subsituting this into (7.16) gives

$$\begin{aligned}
\sqrt{n}\widehat{c}_n(r) &= \sqrt{n}\widetilde{c}_n(r)-\sqrt{n}(\widehat{\theta}-\theta)\mathcal{J}_r+o_p(1)\\
&= \sqrt{n}\widetilde{c}_n(r)-\mathcal{I}^{-1}\mathcal{J}_r\underbrace{\frac{1}{\sqrt{n}}\sum_{t=1}^{n-r}\frac{\partial g(Y_t,\theta)}{\partial\theta}\varepsilon_t}_{=-\frac{\sqrt{n}}{2}\frac{\partial\mathcal{L}_n(\theta)}{\partial\theta}}+o_p(1).
\end{aligned}$$

Asymptotic normality of $\sqrt{n}\widehat{c}_n(r)$ can be shown by showing asymptotic normality of the bivariate vector $\sqrt{n}(\widetilde{c}_n(r), \frac{\partial \mathcal{L}_n(\theta)}{\partial \theta})$. Therefore all that remains is to obtain the asymptotic variance of the above (which will give the desired result);

$$\text{var}\left[\sqrt{n}\widetilde{c}_n(r) + \frac{\sqrt{n}}{2}\mathcal{I}^{-1}\mathcal{J}_r\frac{\partial \mathcal{L}_n(\theta)}{\partial \theta}\right]$$

$$\underbrace{\text{var}\left(\sqrt{n}\widetilde{c}_n(r)\right)}_{=1} + 2\mathcal{I}^{-1}\mathcal{J}_r\text{cov}\left(\sqrt{n}\widetilde{c}_n(r), \frac{\sqrt{n}}{2}\frac{\partial \mathcal{L}_n(\theta)}{\partial \theta}\right) + \mathcal{I}^{-2}\mathcal{J}_r^2\text{var}\left(\frac{\sqrt{n}}{2}\frac{\partial \mathcal{L}_n(\theta)}{\partial \theta}\right) \quad (7.17)$$

We evaluate the two covariance above;

$$\text{cov}\left(\sqrt{n}\widetilde{c}_n(r), -\frac{\sqrt{n}}{2}\frac{\partial \mathcal{L}_n(\theta)}{\partial \theta}\right) = \frac{1}{n}\sum_{t_1,t_2=1}^{n-r}\left[\text{cov}\left\{\varepsilon_{t_1}\varepsilon_{t_1+r}, \varepsilon_{t_2}\frac{\partial g(Y_{t_2},\theta)}{\partial \theta}\right\}\right]$$

$$= \frac{1}{n}\sum_{t_1,t_2=1}^{n-r}\left[\text{cov}\left\{\varepsilon_{t_1}, \varepsilon_{t_2}\right\}\text{cov}\left\{\varepsilon_{t_1+r}, \frac{\partial g(Y_{t_2},\theta)}{\partial \theta}\right\} + \text{cov}\left\{\varepsilon_{t_1+r}, \varepsilon_{t_2}\right\}\text{cov}\left\{\varepsilon_{t_1}, \frac{\partial g(Y_{t_2},\theta)}{\partial \theta}\right\}\right.$$

$$\left.+\text{cum}\left\{\varepsilon_{t_1}, \varepsilon_{t_1+r}, \varepsilon_{t_2}, \frac{\partial g(Y_{t_2},\theta)}{\partial \theta}\right\}\right] = \sigma^2\text{E}\left[\varepsilon_t\frac{\partial g(Y_{t+r},\theta)}{\partial \theta}\right] = \sigma^2\mathcal{J}_r.$$

Similarly we have

$$\text{var}\left(\frac{\sqrt{n}}{2}\frac{\partial \mathcal{L}_n(\theta)}{\partial \theta}\right) = \frac{1}{n}\sum_{t_1,t_2=1}^{n}\text{cov}\left(\varepsilon_{t_1}\frac{\partial g(Y_{t_1},\theta)}{\partial \theta}, \varepsilon_{t_2}\frac{\partial g(Y_{t_2},\theta)}{\partial \theta}\right) = \sigma^2\text{E}\left(\frac{\partial g(Y_{t_1},\theta)}{\partial \theta}\right)^2 = \sigma^2\mathcal{I}.$$

Substituting the above into (7.17) gives the asymptotic variance of $\sqrt{n}\widehat{c}(r)$ to be

$$1 - \sigma^2\mathcal{J}_r\mathcal{I}^{-1}\mathcal{J}_r.$$

Thus we obtain the required result

$$\sqrt{n}\widehat{\rho}(r) = \mathcal{N}\left(0, 1 - \frac{\sigma^2}{c(0)}\mathcal{J}_r\mathcal{I}^{-1}\mathcal{J}_r\right).$$

7.6 Long range dependence (long memory) versus changes in the mean

A process is said to have long range dependence if the autocovariances are not absolutely summable, i.e. $\sum_k |c(k)| = \infty$. A nice historical background on long memory is given in this paper.

From a practical point of view data is said to exhibit long range dependence if the autocovariances do not decay very fast to zero as the lag increases. Returning to the Yahoo data considered in Section 5.1.1 we recall that the ACF plot of the absolute log differences, given again in Figure 7.5 appears to exhibit this type of behaviour. However, it has been argued by several authors that

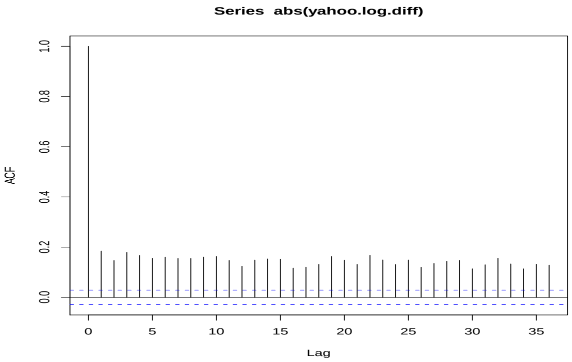


Figure 7.5: ACF plot of the absolute of the log differences.

the 'appearance of long memory' is really because of a time-dependent mean has not been corrected for. Could this be the reason we see the 'memory' in the log differences?

We now demonstrate that one must be careful when diagnosing long range dependence, because a slow/none decay of the autocovariance could also imply a time-dependent mean that has not been corrected for. This was shown in Bhattacharya et al. (1983), and applied to econometric data in Mikosch and Stărică (2000) and Mikosch and Stărică (2003). A test for distinguishing between long range dependence and change points is proposed in Berkes et al. (2006).

Suppose that Y_t satisfies

$$Y_t = \mu_t + \varepsilon_t,$$

where $\{\varepsilon_t\}$ are iid random variables and the mean μ_t depends on t. We observe $\{Y_t\}$ but do not know the mean is changing. We want to evaluate the autocovariance function, hence estimate the autocovariance at lag k using

$$\hat{c}_n(k) = \frac{1}{n}\sum_{t=1}^{n-|k|}(Y_t - \bar{Y}_n)(Y_{t+|k|} - \bar{Y}_n).$$

Observe that $\bar{Y}_n$ is not really estimating the mean but the average mean! If we plotted the empirical

ACF $\{\hat{c}_n(k)\}$ we would see that the covariances do not decay with time. However the true ACF would be zero and at all lags but zero. The reason the empirical ACF does not decay to zero is because we have not corrected for the time dependent mean. Indeed it can be shown that

$$
\begin{aligned}
\hat{c}_n(k) &= \frac{1}{n}\sum_{t=1}^{n-|k|}(Y_t-\mu_t+\mu_t-\bar{Y}_n)(Y_{t+|k|}-\mu_{t+k}+\mu_{t+k}-\bar{Y}_n) \\
&\approx \frac{1}{n}\sum_{t=1}^{n-|k|}(Y_t-\mu_t)(Y_{t+|k|}-\mu_{t+k})+\frac{1}{n}\sum_{t=1}^{n-|k|}(\mu_t-\bar{Y}_n)(\mu_{t+k}-\bar{Y}_n) \\
&\approx \underbrace{c(k)}_{\text{true autocovariance=0}} + \underbrace{\frac{1}{n}\sum_{t=1}^{n-|k|}(\mu_t-\bar{Y}_n)(\mu_{t+k}-\bar{Y}_n)}_{\text{additional term due to time-dependent mean}}
\end{aligned}
$$

Expanding the second term and assuming that $k << n$ and $\mu_t \approx \mu(t/n)$ (and is thus smooth) we have

$$
\begin{aligned}
&\frac{1}{n}\sum_{t=1}^{n-|k|}(\mu_t-\bar{Y}_n)(\mu_{t+k}-\bar{Y}_n) \\
\approx\ & \frac{1}{n}\sum_{t=1}^{n}\mu_t^2-\left(\frac{1}{n}\sum_{t=1}^{n}\mu_t\right)^2+o_p(1) \\
=\ & \frac{1}{n^2}\sum_{s=1}^{n}\sum_{t=1}^{n}\mu_t^2-\left(\frac{1}{n}\sum_{t=1}^{n}\mu_t\right)^2+o_p(1) \\
=\ & \frac{1}{n^2}\sum_{s=1}^{n}\sum_{t=1}^{n}\mu_t\left(\mu_t-\mu_s\right)=\frac{1}{n^2}\sum_{s=1}^{n}\sum_{t=1}^{n}\left(\mu_t-\mu_s\right)^2+\underbrace{\frac{1}{n^2}\sum_{s=1}^{n}\sum_{t=1}^{n}\mu_s\left(\mu_t-\mu_s\right)}_{=-\frac{1}{n^2}\sum_{s=1}^{n}\sum_{t=1}^{n}\mu_t(\mu_t-\mu_s)} \\
=\ & \frac{1}{n^2}\sum_{s=1}^{n}\sum_{t=1}^{n}\left(\mu_t-\mu_s\right)^2+\frac{1}{2n^2}\sum_{s=1}^{n}\sum_{t=1}^{n}\mu_s\left(\mu_t-\mu_s\right)-\frac{1}{2n^2}\sum_{s=1}^{n}\sum_{t=1}^{n}\mu_t\left(\mu_t-\mu_s\right) \\
=\ & \frac{1}{n^2}\sum_{s=1}^{n}\sum_{t=1}^{n}\left(\mu_t-\mu_s\right)^2+\frac{1}{2n^2}\sum_{s=1}^{n}\sum_{t=1}^{n}\left(\mu_s-\mu_t\right)\left(\mu_t-\mu_s\right)=\frac{1}{2n^2}\sum_{s=1}^{n}\sum_{t=1}^{n}\left(\mu_t-\mu_s\right)^2.
\end{aligned}
$$

Therefore

$$
\frac{1}{n}\sum_{t=1}^{n-|k|}(\mu_t-\bar{Y}_n)(\mu_{t+k}-\bar{Y}_n)\approx\frac{1}{2n^2}\sum_{s=1}^{n}\sum_{t=1}^{n}\left(\mu_t-\mu_s\right)^2.
$$

Thus we observe that the sample covariances are positive and don't tend to zero for large lags. This gives the false impression of long memory.

It should be noted if you study a realisation of a time series with a large amount of dependence, it is unclear whether what you see is actually a stochastic time series or an underlying trend. This makes disentangling a trend from data with a large amount of correlation extremely difficult.

Chapter 8

Parameter estimation

Prerequisites

- The Gaussian likelihood.

Objectives

- To be able to derive the Yule-Walker and least squares estimator of the AR parameters.
- To understand what the quasi-Gaussian likelihood for the estimation of ARMA models is, and how the Durbin-Levinson algorithm is useful in obtaining this likelihood (in practice). Also how we can approximate it by using approximations of the predictions.
- Understand that there exists alternative methods for estimating the ARMA parameters, which exploit the fact that the ARMA can be written as an AR(∞).

We will consider various methods for estimating the parameters in a stationary time series. We first consider estimation parameters of an AR and ARMA process. It is worth noting that we will look at maximum likelihood estimators for the AR and ARMA parameters. The maximum likelihood will be constructed as if the observations were Gaussian. However, these estimators 'work' both when the process is Gaussian is also non-Gaussian. In the non-Gaussian case, the likelihood simply acts as a contrast function (and is commonly called the quasi-likelihood). In time series, often the distribution of the random variables is unknown and the notion of 'likelihood' has little meaning. Instead we seek methods that give good estimators of the parameters, meaning that they are consistent and as close to efficiency as possible without placing too many assumption on the distribution. We need to 'free' ourselves from the notion of likelihood acting as a likelihood (and attaining the Crámer-Rao lower bound).

8.1 Estimation for Autoregressive models

Let us suppose that $\{X_t\}$ is a zero mean stationary time series which satisfies the AR(p) representation

$$X_t = \sum_{j=1}^{p} \phi_j X_{t-j} + \varepsilon_t,$$

where $\mathrm{E}(\varepsilon_t) = 0$ and $\mathrm{var}(\varepsilon_t) = \sigma^2$ and the roots of the characteristic polynomial $1 - \sum_{j=1}^{p} \phi_j z^j$ lie outside the unit circle. We will assume that the AR(p) is **causal** (the techniques discussed here will not consistently estimate the parameters in the case that the process is non-causal, they will only consistently estimate the corresponding causal model). Our aim in this section is to construct estimator of the AR parameters $\{\phi_j\}$. We will show that in the case that $\{X_t\}$ has an AR(p) representation the estimation is relatively straightforward, and the estimation methods all have properties which are asymptotically equivalent to the Gaussian maximum estimator.

The Yule-Walker estimator is based on the Yule-Walker equations derived in (4.4) (Section 4.1.4).

8.1.1 The Yule-Walker estimator

We recall that the Yule-Walker equation state that if an AR process is causal, then for $i > 0$ we have

$$\mathrm{E}(X_t X_{t-i}) \;=\; \sum_{j=1}^{p} \phi_j \mathrm{E}(X_{t-j} X_{t-i}), \Rightarrow c(i) = \sum_{j=1}^{p} \phi_j c(i-j). \tag{8.1}$$

Putting the cases $1 \leq i \leq p$ together we can write the above as

$$\underline{r}_p = \Sigma_p \underline{\phi}_p, \tag{8.2}$$

where $(\Sigma_p)_{i,j} = c(i-j)$, $(\underline{r}_p)_i = c(i)$ and $\underline{\phi}_p' = (\phi_1, \ldots, \phi_p)$. Thus the autoregressive parameters solve these equations. It is important to observe that $\underline{\phi}_p = (\phi_1, \ldots, \phi_p)$ minimise the mean squared error

$$\mathrm{E}[X_{t+1} - \sum_{j=1}^{p} \phi_j X_{t+1-j}]^2,$$

(see Section 6.3).

The Yule-Walker equations inspire the method of moments estimator called the Yule-Walker estimator. We use (8.2) as the basis of the estimator. It is clear that $\hat{\underline{r}}_p$ and $\hat{\Sigma}_p$ are estimators of $\underline{r}_p$ and Σ_p where $(\hat{\Sigma}_p)_{i,j} = \hat{c}_n(i-j)$ and $(\hat{\underline{r}}_p)_i = \hat{c}_n(i)$. Therefore we can use

$$\hat{\underline{\phi}}_p = \hat{\Sigma}_p^{-1} \hat{\underline{r}}_p, \tag{8.3}$$

as an estimator of the AR parameters $\underline{\phi}'_p = (\phi_1, \ldots, \phi_p)$. We observe that if p is large this involves inverting a large matrix. However, we can use the Durbin-Levinson algorithm to calculate $\hat{\underline{\phi}}_p$ by recursively fitting lower order AR processes to the observations and increasing the order. This way an explicit inversion can be avoided. We detail how the Durbin-Levinson algorithm can be used to estimate the AR parameters below.

Step 1 Set $\hat{\phi}_{1,1} = \hat{c}_n(1)/\hat{c}_n(0)$ and $\hat{r}_n(2) = 2\hat{c}_n(0) - 2\hat{\phi}_{1,1}\hat{c}_n(1)$.

Step 2 For $2 \leq t \leq p$, we define the recursion

$$\begin{aligned} \hat{\phi}_{t,t} &= \frac{\hat{c}_n(t) - \sum_{j=1}^{t-1} \hat{\phi}_{t-1,j}\hat{c}_n(t-j)}{\hat{r}_n(t)} \\ \hat{\phi}_{t,j} &= \hat{\phi}_{t-1,j} - \hat{\phi}_{t,t}\hat{\phi}_{t-1,t-j} \qquad 1 \leq j \leq t-1, \\ \text{and } \hat{r}_n(t+1) &= \hat{r}_n(t)(1 - \hat{\phi}_{t,t}^2). \end{aligned}$$

Step 3 We recall from (6.12) that $\phi_{t,t}$ is the partial correlation between X_{t+1} and X_1, therefore $\hat{\phi}_{tt}$ are estimators of the partial correlation between X_{t+1} and X_1.

As mentioned in Step 3, the Yule-Walker estimators have the useful property that the partial correlations can easily be evaluated within the procedure. This is useful when trying to determine the order of the model to fit to the data. In Figure 8.1 we give the partial correlation plot corresponding to Figure 7.1. Notice that only the first two terms are outside the error bars. This rightly suggests the time series comes from an autoregressive process of order two.

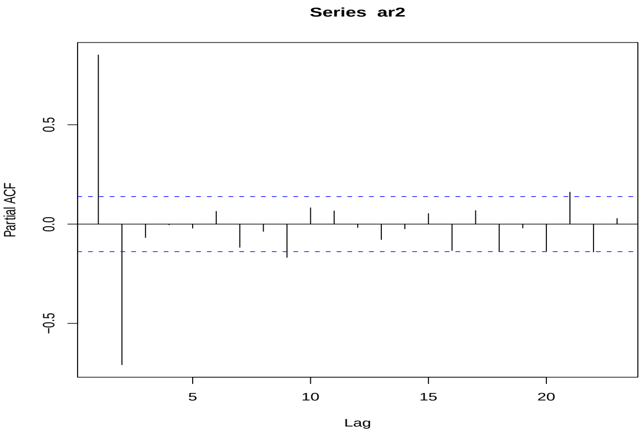


Figure 8.1: Top: The sample partial autocorrelation plot of the AR(2) process $X_t = 1.5X_{t-1} + 0.75X_{t-2} + \varepsilon_t$ with error bars $n = 200$.

In previous chapters it was frequently alluded to that the autocovariance is "blind" to non-causality and that any estimator based on estimating the covariance will always be estimating the causal solution. We now show that the Yule-Walker estimator has the property that the parameter estimates $\{\hat{\phi}_j; j = 1, \ldots, p\}$ correspond to a causal AR(p), in other words, the roots corresponding to $\hat{\phi}(z) = 1 - \sum_{j=1}^{p} \widehat{\phi}_j z^j$ lie outside the unit circle. A non-causal solution cannot arise.

This is because the sample autocovariances $\{\widehat{c}_n(r)\}$ form a positive semi-definite sequence (see Lemma 7.2.1). thus there exists a time series a stationary time series $\{Z_t\}$ with $\{\widehat{c}_n(r)\}$ as its autocovariance. Define the vector $\underline{Z}_{p+1} = (Z_1, \ldots, Z_{p+1})$ where $\text{var}[\underline{Z}]_{p+1} = (\hat{\Sigma}_{p+1})_{i,j} = \hat{c}_n(i-j)$, using this and the following result it follows that $\{\widehat{\phi}_j; j = 1, \ldots, p\}$ corresponds to a causal AR process.

Remark 8.1.1 (Fitting an AR(1) using the Yule-Walker) *We generalize this idea to general AR(p) models below. However, it is straightforward to show that the Yule-Walker estimator of the AR(1) parameter will always be less than or equal to one. We recall that*

$$\widehat{\phi}_{YW} = \frac{\sum_{t=1}^{n-1} X_t X_{t+1}}{\sum_{t=1}^{n} X_t^2}.$$

By using Cauchy-Schwarz we have

$$\begin{aligned}
|\widehat{\phi}_{YW}| &\leq \frac{\sum_{t=1}^{n-1}|X_tX_{t+1}|}{\sum_{t=1}^{n}X_t^2} \leq \frac{[\sum_{t=1}^{n-1}X_t^2]^{1/2}[\sum_{t=1}^{n-1}X_{t+1}^2]}{\sum_{t=1}^{n}X_t^2}^{1/2} \\
&\leq \frac{[\sum_{t=1}^{n}X_t^2]^{1/2}[\sum_{t=0}^{n-1}X_{t+1}^2]^{1/2}}{\sum_{t=1}^{n}X_t^2} = 1.
\end{aligned}$$

We use a similar idea below, but the proof hinges on the fact that the sample covariances forms a positive semi-definite sequence.

An alternative proof using that $\{\widehat{c}_n(r)\}$ is the ACF of a stationary time series $\{Z_t\}$. Then

$$\widehat{\phi}_{YW} = \frac{\widehat{c}_n(1)}{\widehat{c}_n(0)} = \frac{\mathrm{cov}(Z_t,Z_{t+1})}{\mathrm{var}(Z_t)} = \frac{\mathrm{cov}(Z_t,Z_{t+1})}{\sqrt{\mathrm{var}(Z_t)\mathrm{var}(Z_{t+1})}},$$

which is a correlation and thus lies between $[-1,1]$.

Lemma 8.1.1 *Let us suppose $\underline{Z}_{p+1} = (Z_1,\ldots,Z_{p+1})$ is a zero mean random vector, where* $\mathrm{var}[\underline{Z}]_{p+1} = (\Sigma_{p+1})_{i,j} = c_n(i-j)$ *(which is* **Toeplitz***). Let $Z_{p+1|p}$ be the best linear predictor of Z_{p+1} given $Z_p,\ldots,Z_1$, where $\underline{\phi}_p = (\phi_1,\ldots,\phi_p) = \Sigma_p^{-1}\underline{r}_p$ are the coefficients corresponding to the best linear predictor. Then the roots of the corresponding characteristic polynomial $\phi(z) = 1-\sum_{j=1}^{p}\phi_j z^j$ lie outside the unit circle.*

PROOF. The proof is based on the following facts:

(i) Any sequence $\{\phi_j\}_{j=1}^p$ has the following reparameterisation. There exists parameters $\{a_j\}_{j=1}^p$ and λ such that $a_1 = 1$, for $2 \leq j \leq p-2$, $a_j - \lambda a_{j-1} = \phi_j$ and $\lambda a_p = \phi_p$. Using $\{a_j\}_{j=1}^p$ and λ, for rewrite the linear combination $\{Z_j\}_{j=1}^{p+1}$ as

$$Z_{p+1} - \sum_{j=1}^{p}\phi_j Z_{p+1-j} = \sum_{j=1}^{p}a_j Z_{p+1-j} - \lambda\sum_{j=1}^{p}a_j Z_{p-j}.$$

(ii) If $\underline{\phi}_p = (\phi_1,\ldots,\phi_p)' = \Sigma_p^{-1}\underline{r}_p$, then $\underline{\phi}_p$ minimises the mean square error i.e. for any $\{b_j\}_{j=1}^p$

$$\mathrm{E}_{\Sigma_{p+1}}\left(Z_{p+1} - \sum_{j=1}^{p}\phi_j Z_{p+1-j}\right)^2 \leq \mathrm{E}_{\Sigma_{p+1}}\left(Z_{p+1} - \sum_{j=1}^{p}b_j Z_{p+1-j}\right)^2 \qquad (8.4)$$

where $\Sigma_{p+1} = \mathrm{var}[\underline{Z}_{p+1}]$ and $\underline{Z}_{p+1} = (Z_{p+1},\ldots,Z_1)$.

We use these facts to prove the result. Our objective is to show that the roots of $\phi(B) = 1 - \sum_{j=1}^{p}\phi_j B^j$ lie outside the unit circle. Using (i) we factorize $\phi(B) = (1-\lambda B)a(B)$ where $a(B) = \sum_{j=1}^{p} a_j B^j$. Suppose by contraction $|\lambda| > 1$ (thus at least one root of $\phi(B)$ lies inside the unit circle). We will show if this were true, then by the Toeplitz nature of Σ_{p+1}, $\underline{\phi}_p = (\phi_1,\ldots,\phi_p)$ cannot be the best linear predictor.

Let

$$Y_{p+1} = \sum_{j=1}^{p} a_j B^j Z_{t+2} = \sum_{j=1}^{p} a_j Z_{p+2-j} \text{ and } Y_p = BY_{p+1} = B\sum_{j=1}^{p} a_j B^j Z_{t+2} = \sum_{j=1}^{p} a_j Z_{p+1-j}.$$

By (i) is clear that $Z_{p+1} - \sum_{j=1}^{p}\phi_j Z_{p+1-j} = Y_{p+1} - \lambda Y_p$. Furthermore, since $\{\phi_j\}$ minimises the mean squared error in (8.4), then λY_p must be the best linear predictor of Y_{p+1} given Y_p i.e. λ must minimise the mean squared error

$$\lambda = \arg\min_{\beta} \mathrm{E}_{\Sigma_{p+1}}\left(Y_{p+1} - \beta Y_p\right)^2,$$

that is $\lambda = \frac{\mathrm{E}[Y_{p+1}Y_p]}{\mathrm{E}[Y_p^2]}$. However, we now show that $|\frac{\mathrm{E}[Y_{p+1}Y_p]}{\mathrm{E}[Y_p^2]}| \leq 1$ which leads to a contradiction.

We recall that Y_{p+1} is a linear combination of a stationary sequence, thus BY_{p+1} has the same variance as Y_{p+1}. I.e. $\mathrm{var}(Y_{p+1}) = \mathrm{var}(Y_p)$. It you want to see the exact calculation, then

$$\begin{aligned}\mathrm{E}[Y_p^2] &= \mathrm{var}[Y_p] = \sum_{j_1,j_2=1}^{p} a_{j_1}a_{j_2}\mathrm{cov}[Y_{p+1-j_1}, Y_{p+1-j_2}] = \sum_{j_1,j_2=1}^{p} a_{j_1}a_{j_2}c(j_1-j_2) \\ &= \mathrm{var}[Y_{p+1}] = \mathrm{E}[Y_{p+1}^2].\end{aligned}$$

In other words, since Σ_{p+1} is a Toeplitz matrix, then $\mathrm{E}[Y_p^2] = \mathrm{E}[Y_{p+1}^2]$ and

$$\lambda = \frac{\mathrm{E}[Y_{p+1}Y_p]}{(\mathrm{E}[Y_p^2]\mathrm{E}[Y_{p+1}^2])^{1/2}}.$$

This means λ measures the correlation between Y_p and Y_{p+1} and must be less than or equal to one. Thus leading to a contradiction.

Observe this proof only works when Σ_{p+1} is a Toeplitz matrix. If it is not we do not have $\mathrm{E}[Y_p^2] = \mathrm{E}[Y_{p+1}^2]$ and that λ can be intepretated as the correlation. □

From the above result we can immediately see that the Yule-Walker estimators of the AR(p) coefficients yield a causal solution. Since the autocovariance estimators $\{\widehat{c}_n(r)\}$ form a positive semi-definite sequence, there exists a vector $\underline{Y}_p$ where $\mathrm{var}_{\widehat{\Sigma}_{p+1}}[\underline{Y}_{p+1}] = \widehat{\Sigma}_{p+1}$ with $(\widehat{\Sigma}_{p+1}) = \widehat{c}_n(i-j)$, thus by the above lemma we have that $\widehat{\Sigma}_p^{-1}\widehat{\underline{r}}_p$ are the coefficients of a Causal AR process.

We now consider the least squares estimator, which can either be defined in its right or can be considered as the conditional Gaussian likelihood. Unlike the Yule-Walker estimator, there is not guarantee that the characteristic function of the estimator will have roots within the unit circle.

8.1.2 The Gaussian maximum likelihood and conditional likelihood

Our object here is to obtain the maximum likelihood estimator of the AR(p) parameters. We recall that the maximum likelihood estimator is the parameter which maximises the joint density of the observations. Since the log-likelihood often has a simpler form, we will focus on the log-likelihood. We note that the Gaussian MLE is constructed as if the observations $\{X_t\}$ were Gaussian, though it is not necessary that $\{X_t\}$ is Gaussian when doing the estimation. In the case that the innovations are not Gaussian, the estimator may be less efficient (may not obtain the Cramer-Rao lower bound) then the likelihood constructed as if the distribution were known.

Suppose we observe $\{X_t; t = 1, \ldots, n\}$ where X_t are observations from an AR(p) process. Let us suppose for the moment that the innovations of the AR process are Gaussian, this implies that $\underline{X}_n = (X_1, \ldots, X_n)$ is a n-dimension Gaussian random vector, with the corresponding log-likelihood

$$\mathcal{L}_n(\underline{a}) \quad = \quad -\log|\Sigma_n(\underline{a})| - \mathbf{X}_n'\Sigma_n(\underline{a})^{-1}\mathbf{X}_n, \tag{8.5}$$

where $\Sigma_n(\underline{a})$ the variance covariance matrix of $\mathbf{X}_n$ constructed as if $\mathbf{X}_n$ came from an AR process with parameters $\underline{a}$. Of course, in practice, the likelihood in the form given above is impossible to maximise. Therefore we need to rewrite the likelihood in a more tractable form.

We now derive a tractable form of the likelihood under the assumption that the innovations come from an arbitrary distribution. To construct the likelihood, we use the method of conditioning, to write the likelihood as the product of conditional likelihoods. In order to do this, we derive the conditional distribution of X_{t+1} given $X_{t-1}, \ldots, X_1$. We first note that the AR(p) process is p-Markovian (if it is causal), therefore if $t \geq p$ all the information about X_{t+1} is contained in the past p observations, therefore

$$\mathbb{P}(X_{t+1} \leq x | X_t, X_{t-1}, \ldots, X_1) = \mathbb{P}(X_{t+1} \leq x | X_t, X_{t-1}, \ldots, X_{t-p+1}), \tag{8.6}$$

by causality. Since the Markov property applies to the distribution function it also applies to the density

$$f(X_{t+1}|X_t,\ldots,X_1) = f(X_{t+1}|X_t,\ldots,X_{t-p+1}).$$

By using the (8.6) we have

$$\mathbb{P}(X_{t+1}\leq x|X_t,\ldots,X_1) = \mathbb{P}(X_{t+1}\leq x|X_t,\ldots,X_1) = \mathbb{P}_{\varepsilon}(\varepsilon \leq x - \sum_{j=1}^{p} a_j X_{t+1-j}), \tag{8.7}$$

where $\mathbb{P}_{\varepsilon}$ denotes the distribution of the innovation. Differentiating $\mathbb{P}_{\varepsilon}$ with respect to X_{t+1} gives

$$f(X_{t+1}|X_t,\ldots,X_{t-p+1}) = \frac{\partial \mathbb{P}_{\varepsilon}(\varepsilon \leq X_{t+1} - \sum_{j=1}^{p} a_j X_{t+1-j})}{\partial X_{t+1}} = f_{\varepsilon}\left(X_{t+1} - \sum_{j=1}^{p} a_j X_{t+1-j}\right). \tag{8.8}$$

Example 8.1.1 (AR(1)) *To understand why (8.6) is true consider the simple case that $p = 1$ (AR(1) with $|\phi| < 1$). Studying the conditional probability gives*

$$\begin{aligned}\mathbb{P}(X_{t+1}\leq x_{t+1}|X_t = x_t,\ldots,X_1 = x_1) &= \mathbb{P}(\underbrace{\phi X_t + \varepsilon_t \leq x_{t+1}}_{\textit{all information contained in } X_t} |X_t = x_t,\ldots,X_1 = x_1) \\ &= \mathbb{P}_{\varepsilon}(\varepsilon_t \leq x_{t+1} - \phi x_t) = \mathbb{P}(X_{t+1}\leq x_{t+1}|X_t = x_t),\end{aligned}$$

where $\mathbb{P}_{\varepsilon}$ denotes the distribution function of the innovation ε.

Using (8.8) we can derive the joint density of $\{X_t\}_{t=1}^{n}$. By using conditioning we obtain

$$\begin{aligned}f(X_1,X_2,\ldots,X_n) &= f(X_1,\ldots,X_p)\prod_{t=p}^{n-1} f(X_{t+1}|X_t,\ldots,X_1) \quad \text{(by repeated conditioning)} \\ &= f(X_1,\ldots,X_p)\prod_{t=p}^{n-1} f(X_{t+1}|X_t,\ldots,X_{t-p+1}) \quad \text{(by the Markov property)} \\ &= f(X_1,\ldots,X_p)\prod_{t=p}^{n-1} f_{\varepsilon}(X_{t+1} - \sum_{j=1}^{p} a_j X_{t+1-j}) \quad \text{(by (8.8))}.\end{aligned}$$

Therefore the log likelihood is

$$\underbrace{\log f(X_1, X_2, \ldots, X_n)}_{\text{Full log-likelihood } \mathcal{L}_n(\underline{a};\underline{X}_n)} = \underbrace{\log f(X_1, \ldots, X_p)}_{\text{initial observations}} + \underbrace{\sum_{t=p}^{n-1} \log f_\varepsilon(X_{t+1} - \sum_{j=1}^{p} a_j X_{t+1-j})}_{\text{conditional log-likelihood}=L_n(\underline{a};\underline{X}_n)}.$$

In the case that the sample sizes are large $n >> p$, the contribution of initial observations $\log f(X_1, \ldots, X_p)$ is minimal and the conditional log-likelihood and full log-likelihood are asymptotically equivalent.

So far we have not specified the distribution of ε. From now on we shall assume that it is Gaussian. In the case that ε is Gaussian, $\log f(X_1, \ldots, X_p)$ is multivariate normal with mean zero (since we are assuming, for convenience, that the time series has zero mean) and variance Σ_p. We recall that $\Sigma_p(\underline{a})$ is a Toeplitz matrix whose covariance is determined by the AR parameters $\underline{a}$, see (4.7). The maximum likelihood estimator is

$$\widehat{\underline{\phi}}_n = \arg\max_{\underline{a}\in\Theta} \left[-\log|\Sigma_p(\underline{a})| - \underline{X}_p'\Sigma_p(\underline{a})^{-1}\underline{X}_p + \underbrace{L_n(\underline{a};\underline{X})}_{\text{conditional likelihood}} \right]. \tag{8.9}$$

As can be seen from (4.7), the coefficients are 'buried' within the covariance. By constraining the parameter space, we ensure the estimator correspond to a causal AR process. However, it is clear that despite having the advantage that it attains the Crámer-Rao lower bound in the case that the innovations are Gaussian, it not simple to evaluate.

On the other hand the conditional log-likelihood the form

$$L_n(\underline{a};\underline{X}) = -(n-p)\log\sigma^2 - \frac{1}{\sigma^2}\sum_{t=p}^{n-1}\left(X_{t+1} - \sum_{j=1}^{p} a_j X_{t+1-j}\right)^2,$$

is straightforward to maximise, since it is simply the least squares estimator. That is $\widetilde{\underline{\phi}}_p = \arg\max L_n(\underline{a};\underline{X})$ were

$$\widetilde{\underline{\phi}}_p = \tilde{\Sigma}_p^{-1}\tilde{\underline{r}}_p,$$

with $(\widetilde{\Sigma}_p)_{i,j} = \frac{1}{n-p}\sum_{t=p+1}^{n} X_{t-i}X_{t-j}$ and $(\widetilde{\underline{r}}_n)_i = \frac{1}{n-p}\sum_{t=p+1}^{n} X_t X_{t-i}$.

Remark 8.1.2 (A comparison of the Yule-Walker and least squares estimators) *Comparing* *the least squares estimator* $\widetilde{\underline{\phi}}_p = \widetilde{\Sigma}_p^{-1}\widetilde{\underline{r}}_p$ *with the Yule-Walker estimator* $\widehat{\underline{\phi}}_p = \widehat{\Sigma}_p^{-1}\widehat{\underline{r}}_p$ *we see that they are very similar. The difference lies in* $\widetilde{\Sigma}_p$ *and* $\widehat{\Sigma}_p$ *(and the corresponding* $\widetilde{\underline{r}}_p$ *and* $\widehat{\underline{r}}_p$*). We see that* $\widehat{\Sigma}_p$ *is a Toeplitz matrix, defined entirely by the positive definite sequence* $\widehat{c}_n(r)$*. On the other hand,* $\widetilde{\Sigma}_p$ *is not a Toeplitz matrix, the estimator of* $c(r)$ *changes subtly at each row. This means that the proof given in Lemma 8.1.1 cannot be applied to the least squares estimator as it relies on the matrix* Σ_{p+1} *(which is a combination of* Σ_p *and* $\underline{r}_p$*) being Toeplitz (thus stationary). Thus the characteristic polynomial corresponding to the least squares estimator will not necessarily have roots which lie outside the unit circle.*

Example 8.1.2 (Toy Example) *To illustrate the difference between the Yule-Walker and least squares estimator (at least for example samples) consider the rather artifical example that the time series consists of two observations* X_1 *and* X_2 *(we will assume the mean is zero). We fit an* $AR(1)$ *model to the data, the least squares estimator of the* $AR(1)$ *parameter is*

$$\widehat{\phi}_{LS} = \frac{X_1X_2}{X_1^2}$$

whereas the Yule-Walker estimator of the $AR(1)$ *parameter is*

$$\widehat{\phi}_{YW} = \frac{X_1X_2}{X_1^2 + X_2^2}.$$

It is clear that $\widehat{\phi}_{LS} < 1$ *only if* $X_2 < X_1$*. On the other hand* $\widehat{\phi}_{YW} < 1$*. Indeed since* $(X_1 - X_2)^2 > 0$*, we see that* $\widehat{\phi}_{YW} \leq 1/2$*.*

Remark 8.1.3 (Shortcomings of the Yule-Walker estimator) *It is worth bearing in mind that the Yule-Walker is sometimes criticized for having a larger bias than the least squares estimator when the sample size is small. However, if the process is a stationary, this difference is negligible if the sample size is relatively large.*

One major issue is when the process is not stable (either unit root of explosive) i.e. nonstationary. In this case the least squares estimator can consistently estimate the parameters, whereas the Yule-Walker estimator will always force the estimators to be stable, even when the are not (and this cannot consistently estimate the parameters).

Exercise 8.1 *(i) In* `R` *you can estimate the AR parameters using ordinary least squares (*`ar.ols`*), yule-walker (*`ar.yw`*) and (Gaussian) maximum likelihood (*`ar.mle`*).*

Simulate the causal AR(2) model $X_t = 1.5X_{t-1} - 0.75X_{t-2} + \varepsilon_t$ using the routine `arima.sim` *(which gives Gaussian realizations) and also innovations which from a t-distribution with 4df. Use the sample sizes $n = 100$ and $n = 500$ and compare the three methods through a simulation study.*

(ii) Use the ℓ_1-norm defined as

$$L_n(\phi) = \sum_{t=p+1}^{t} \left| X_t - \sum_{j=1}^{p} \phi_j X_{t-j} \right|,$$

with $\hat{\phi}_n = \arg\min L_n(\phi)$ to estimate the AR(p) parameters.

You may need to use a Quantile Regression package to minimise the ℓ_1 norm. I suggest using the package `quantreg` *and the function* `rq` *where we set $\tau = 0.5$ (the median).*

Note that so far we have only considered estimation of causal AR(p) models. Breidt et. al. (2001) propose a method for estimating parameters of a non-causal AR(p) process (see page 18).

8.1.3 Sampling properties

Both the Yule-Walker and least squares estimator have the same asymptotic sampling properties (under the assumption of stationarity). Suppose that $\{X_t\}$ has a causal AR(p) representation

$$X_t = \sum_{j=1}^{p} \phi_j X_{t-j} + \varepsilon_t$$

where $\{\varepsilon_t\}$ are iid random variables with $\mathrm{var}[\varepsilon_t] = \sigma^2$ and $\mathrm{E}[|\varepsilon_t|^{2+\delta}] < \infty$ for some $\delta > 0$. Suppose the AR(p) model is fitted to the time series, using either least squares or Yule-Walker estimator. We denote this estimator as $\widehat{\phi}_p$. Then

$$\sqrt{n}(\underline{\widehat{\phi}} - \underline{\phi}) \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \sigma^2 \Sigma_p^{-1}\right),$$

where $\Sigma_p = \mathrm{E}[\underline{X}_p \underline{X}_p']$ and $\underline{X}_p = (X_1, \ldots, X_p)$.

Remark 8.1.4 *We note that the assumption $\mathrm{E}|\varepsilon_t^{2+\delta}| < \infty$ implies that $\mathrm{E}[|X_t|^{2+\delta}] < \infty$. In the proof below we use the stronger assumption $\mathrm{E}(\varepsilon_t^4) < \infty$ to make the proof easier to follow.*

Tools to prove the result: Martingale central limit theorem

We summarize the result, see Billingsley (1995) Hall and Heyde (1980) (Theorem 3.2 and Corollary 3.1) for the details.

Definition 8.1.1 *The random variables $\{Z_t\}$ are called martingale differences if*

$$\mathrm{E}(Z_t|Z_{t-1}, Z_{t-2}, \ldots) = 0.$$

The sequence $\{\mathcal{S}_n\}_n$, where

$$\mathcal{S}_n = \sum_{t=1}^{n} Z_t$$

are called martingales if $\{Z_t\}$ are martingale differences. Observe that $\mathrm{E}[\mathcal{S}_n|\mathcal{S}_{n-1}] = \mathcal{S}_{n-1}$.

Remark 8.1.5 (Martingales and covariances) *We observe that if $\{Z_t\}$ are martingale differences then*

$$\mathrm{E}[Z_t] = \mathrm{E}[\mathrm{E}[Z_t|mathcalF_{t-1}]] = 0,$$

where $\mathcal{F}_s = \sigma(Z_s, Z_{s-1}, \ldots)$ and for $t > s$ and

$$\mathrm{cov}(Z_s, Z_t) = \mathrm{E}(Z_s Z_t) = \mathrm{E}\big(\mathrm{E}(Z_s Z_t|\mathcal{F}_s)\big) = \mathrm{E}\big(Z_s \mathrm{E}(Z_t|\mathcal{F}_s)\big) = \mathrm{E}(Z_s \times 0) = 0.$$

Hence martingale differences are uncorrelated.

Example 8.1.3 *Suppose that $X_t = \phi X_{t-1} + \varepsilon_t$, where $\{\varepsilon_t\}$ are iid r.v. with $\mathrm{E}(\varepsilon_t) = 0$ and $|\phi| < 1$. Then $\{\varepsilon_t X_{t-1}\}_t$ are martingale differences. To see why note that*

$$\begin{aligned}
\mathrm{E}\left[\varepsilon_t X_{t-1}|\varepsilon_{t-j}X_{t-j-1}; j \geq 1\right] &= \mathrm{E}\left[\mathrm{E}\left(\varepsilon_t X_{t-1}|\varepsilon_{t-j}; j \geq 1\right)|\varepsilon_{t-j}X_{t-j-1}; j \geq 1\right] \\
&= \mathrm{E}\left[X_{t-1}\mathrm{E}\left(\varepsilon_t|\varepsilon_{t-j}; j \geq 1\right)|\varepsilon_{t-j}X_{t-j-1}; j \geq 1\right] = 0, a.s
\end{aligned}$$

since $\sigma(\varepsilon_{t-j}X_{t-j-1}; j \geq 1) \subseteq \sigma(\varepsilon_{t-j}; j \geq 1)$. In general, if X_t is a causal time series then $\{\varepsilon_t X_{t-j}\}_t$ are martingale differences ($j > 0$).

Let

$$S_n = \frac{1}{n}\sum_{t=1}^{n} Z_t, \tag{8.10}$$

and $\mathcal{F}_t = \sigma(Z_t, Z_{t-1}, \ldots)$, $\mathrm{E}(Z_t|\mathcal{F}_{t-1}) = 0$ and $\mathrm{E}(Z_t^2) < \infty$. We shall show asymptotic normality of $\sqrt{n}(S_n - \mathrm{E}(S_n))$. The reason for normalising by $\sqrt{n}$, is that $(S_n - \mathrm{E}(S_n)) \overset{\mathcal{P}}{\to} 0$ as $n \to \infty$, hence in terms of distributions it converges towards the point mass at zero. Therefore we need to increase the magnitude of the difference. If it can show that $\mathrm{var}(S_n) = O(n^{-1})$, then $\sqrt{n}(S_n - \mathrm{E}(\mathcal{S}_0) = O(1)$.

Theorem 8.1.1 *Let S_n be defined as in (11.16). Further suppose*

$$\frac{1}{n}\sum_{t=1}^{n} Z_t^2 \overset{\mathcal{P}}{\to} \sigma^2, \tag{8.11}$$

where σ^2 is a finite constant, for all $\eta > 0$,

$$\frac{1}{n}\sum_{t=1}^{n} \mathrm{E}(Z_t^2 I(|Z_t| > \eta\sqrt{n})|\mathcal{F}_{t-1}) \overset{\mathcal{P}}{\to} 0, \tag{8.12}$$

(this is known as the conditional Lindeberg condition) and

$$\frac{1}{n}\sum_{t=1}^{n} \mathrm{E}(Z_t^2|\mathcal{F}_{t-1}) \overset{\mathcal{P}}{\to} \sigma^2. \tag{8.13}$$

Then we have

$$n^{1/2} S_n \overset{\mathcal{D}}{\to} \mathcal{N}(0, \sigma^2). \tag{8.14}$$

Remark 8.1.6 (The conditional likelihood and martingales) *It is interesting to note that the derivative of conditional log-likelihood of a time series at the true parameter* **is** *a martingale so long as the likelihood is correctly specified. In other works, using that*

$$\log f(X_n, \ldots, X_1|X_1;\theta) \quad = \quad \sum_{t=2}^{n} \log f(X_t|X_{t-1}, \ldots, X_1;\theta),$$

then $\frac{\partial \log f(X_t|X_{t-1},\ldots,X_1;\theta)}{\partial\theta}$ is a martingale difference. To see why, note that if we can take the *derivative outside the integral then*

$$\begin{aligned}
& \mathrm{E}\left[\frac{\partial \log f(X_t|X_{t-1},\ldots,X_1;\theta)}{\partial\theta}|X_{t-1},\ldots,X_1\right] \\
= & \int \frac{\partial \log f(X_t|X_{t-1},\ldots,X_1;\theta)}{\partial\theta} f(X_t|X_{t-1},\ldots,X_1;\theta)dX_t \\
= & \int \frac{\partial f(X_t|X_{t-1},\ldots,X_1;\theta)}{\partial\theta}dX_t = \frac{\partial}{\partial\theta}\int f(X_t|X_{t-1},\ldots,X_1;\theta)dX_t = 0.
\end{aligned}$$

Asymptotic normality of the least squares estimator of the AR(1) parameter

In this section we show asymptotic normality of the least squares estimator of the AR(1), where $\hat{\phi}_n = \arg\max \mathcal{L}_n(a)$ and

$$\mathcal{L}_n(a) = \frac{1}{n-1}\sum_{t=2}^{n}(X_t - aX_{t-1})^2.$$

The first and the second derivative (at the true parameter) is

$$\begin{aligned}
\nabla\mathcal{L}_n(a)\rfloor_{a=\phi} &= \frac{-2}{n-1}\sum_{t=2}^{n} X_{t-1}\underbrace{(X_t-\phi X_{t-1})}_{=\varepsilon_t} = \frac{-2}{n-1}\sum_{t=2}^{n}X_{t-1}\varepsilon_t \\
\text{and } \nabla^2\mathcal{L}_n(a) &= \frac{2}{n-1}\sum_{t=2}^{n}X_{t-1}^2 \text{ (does not depend on unknown parameters).}
\end{aligned}$$

Thus it is clear that

$$(\widehat{\phi}_n - \phi) = -\left(\nabla^2\mathcal{L}_n\right)^{-1}\nabla\mathcal{L}_n(\phi). \tag{8.15}$$

Since $\{X_t^2\}$ are ergodic random variables, by using the ergodic theorem we have $\nabla^2\mathcal{L}_n \overset{\text{a.s.}}{\to} 2\mathrm{E}(X_0^2)$. This, together with (8.15), implies

$$\begin{aligned}
\sqrt{n}(\widehat{\phi}_n - \phi) &= \frac{\sum_{t=2}^{n}X_tX_{t-1}}{\sum_{t=2}^{n}X_{t-1}^2} - \phi \\
&= \frac{\sum_{t=2}^{n}X_{t-1}(X_t-\phi X_{t-1})}{\sum_{t=2}^{n}X_{t-1}^2} = \frac{\sum_{t=2}^{n}X_{t-1}\varepsilon_t}{\sum_{t=2}^{n}X_{t-1}^2} \\
&= -\underbrace{\left(\nabla^2\mathcal{L}_n\right)^{-1}}_{\overset{\text{a.s.}}{\to}(2\mathrm{E}(X_0^2))^{-1}}\sqrt{n}\nabla\mathcal{L}_n(\phi) = -\Sigma_1^{-1}\sqrt{n}S_n + O_p(n^{-1/2}),
\end{aligned}$$

where $S_n = \frac{1}{n-1}\sum_{t=2}^{n} X_{t-1}\varepsilon_t$. Thus to show asymptotic normality of $\sqrt{n}(\widehat{\phi}_n - \phi)$, will need only show asymptotic normality of $\sqrt{n}S_n$. S_n is the sum of martingale differences, since $\mathrm{E}(X_{t-1}\varepsilon_t|X_{t-1}) = X_{t-1}\mathrm{E}(\varepsilon_t|X_{t-1}) = X_{t-1}\mathrm{E}(\varepsilon_t) = 0$, therefore we apply the martingale central limit theorem (summarized in the previous section).

To show that $\sqrt{n}S_n$ is asymptotically normal, we need to verify conditions (8.11)-(8.13). We note in our example that $Z_t := X_{t-1}\varepsilon_t$, and that the series $\{X_{t-1}\varepsilon_t\}_t$ is an ergodic process (this simply means that sample means converge almost surely to their expectation, so it is a great tool to use). Furthermore, since for any function g, $\mathrm{E}(g(X_{t-1}\varepsilon_t)|\mathcal{F}_{t-1}) = \mathrm{E}(g(X_{t-1}\varepsilon_t)|X_{t-1})$, where $\mathcal{F}_{t-1} = \sigma(X_{t-1}, X_{t-2}, \ldots)$ we need only to condition on X_{t-1} rather than the entire sigma-algebra $\mathcal{F}_{t-1}$. To simplify the notation we let $S_n = \frac{1}{n}\sum_{t=1}^{n}\varepsilon_t X_{t-1}$ (included an extra term here).

Verification of conditions

C1 : By using the ergodicity of $\{X_{t-1}\varepsilon_t\}_t$ we have

$$\frac{1}{n}\sum_{t=1}^{n} Z_t^2 = \frac{1}{n}\sum_{t=1}^{n} X_{t-1}^2\varepsilon_t^2 \xrightarrow{\mathcal{P}} \mathrm{E}(X_{t-1}^2)\underbrace{\mathrm{E}(\varepsilon_t^2)}_{=1} = \sigma^2 c(0).$$

C2 : We now verify the conditional Lindeberg condition.

$$\frac{1}{n}\sum_{t=1}^{n}\mathrm{E}(Z_t^2 I(|Z_t| > \eta\sqrt{n})|\mathcal{F}_{t-1}) = \frac{1}{n-1}\sum_{t=1}^{n}\mathrm{E}(X_{t-1}^2\varepsilon_t^2 I(|X_{t-1}\varepsilon_t| > \eta\sqrt{n})|X_{t-1}).$$

We now use the Cauchy-Schwartz inequality for conditional expectations to split $X_{t-1}^2\varepsilon_t^2$ and $I(|X_{t-1}\varepsilon_t| > \varepsilon)$ (see the conditional Hölder inequality). We recall that the Cauchy-Schwartz inequality for conditional expectations is $\mathrm{E}(X_t Z_t|\mathcal{G}) \leq [\mathrm{E}(X_t^2|\mathcal{G})\mathrm{E}(Z_t^2|\mathcal{G})]^{1/2}$ almost surely. Therefore

$$\begin{aligned}
& \frac{1}{n}\sum_{t=1}^{n}\mathrm{E}(Z_t^2 I(|Z_t| > \varepsilon\sqrt{n})|\mathcal{F}_{t-1}) \quad \text{(use the conditional Cauchy-Schwartz to split these terms)} \\
\leq\ & \frac{1}{n}\sum_{t=1}^{n}\left\{\mathrm{E}(X_{t-1}^4\varepsilon_t^4|X_{t-1})\mathrm{E}(I(|X_{t-1}\varepsilon_t| > \eta\sqrt{n})^2|X_{t-1})\right\}^{1/2} \\
\leq\ & \frac{1}{n}\sum_{t=1}^{n} X_{t-1}^2\mathrm{E}(\varepsilon_t^4)^{1/2}\left\{\mathrm{E}(I(|X_{t-1}\varepsilon_t| > \eta\sqrt{n})^2|X_{t-1})\right\}^{1/2}, \qquad (8.16)
\end{aligned}$$

almost surely. We note that rather than use the conditional Cauchy-Schwartz inequality we can use a generalisation of it called the conditional Hölder inequality. The Hölder inequality states that if $p^{-1}+q^{-1}=1$, then $\mathrm{E}(XY|\mathcal{F}) \leq \{\mathrm{E}(X^p|\mathcal{F})\}^{1/p}\{\mathrm{E}(Y^q|\mathcal{F})\}^{1/q}$ almost surely. The advantage of using this inequality is that one can reduce the moment assumptions on X_t.

Returning to (8.16), and studying $\mathrm{E}(I(|X_{t-1}\varepsilon_t| > \varepsilon)^2|X_{t-1})$ we use that $\mathrm{E}(I(A)^2) = \mathrm{E}(I(A)) = \mathbb{P}(A)$ and the Chebyshev inequality to show

$$\begin{aligned}
\mathrm{E}\left(I(|X_{t-1}\varepsilon_t| > \eta\sqrt{n})^2|X_{t-1}\right) &= \mathrm{E}\left(I(|X_{t-1}\varepsilon_t| > \eta\sqrt{n})|X_{t-1}\right) \\
&= \mathrm{E}\left(I\left(|\varepsilon_t| > \frac{\eta\sqrt{n}}{X_{t-1}}\right)|X_{t-1}\right) \\
&= P_\varepsilon\left(|\varepsilon_t| > \frac{\eta\sqrt{n}}{X_{t-1}}\right) \leq \frac{X_{t-1}^2\mathrm{var}(\varepsilon_t)}{\eta^2 n}. \qquad (8.17)
\end{aligned}$$

Substituting (8.17) into (8.16) we have

$$\begin{aligned}
\frac{1}{n}\sum_{t=1}^{n}\mathrm{E}\left(Z_t^2 I(|Z_t| > \eta\sqrt{n})|\mathcal{F}_{t-1}\right) &\leq \frac{1}{n}\sum_{t=1}^{n} X_{t-1}^2\mathrm{E}(\varepsilon_t^4)^{1/2}\left\{\frac{X_{t-1}^2\mathrm{var}(\varepsilon_t)}{\eta^2 n}\right\}^{1/2} \\
&\leq \frac{\mathrm{E}(\varepsilon_t^4)^{1/2}}{\eta n^{3/2}}\sum_{t=1}^{n}|X_{t-1}|^3\mathrm{E}(\varepsilon_t^2)^{1/2} \\
&\leq \frac{\mathrm{E}(\varepsilon_t^4)^{1/2}\mathrm{E}(\varepsilon_t^2)^{1/2}}{\eta n^{1/2}}\frac{1}{n}\sum_{t=1}^{n}|X_{t-1}|^3.
\end{aligned}$$

If $\mathrm{E}(\varepsilon_t^4) < \infty$, then $\mathrm{E}(X_t^4) < \infty$, therefore by using the ergodic theorem we have $\frac{1}{n}\sum_{t=1}^{n}|X_{t-1}|^3 \overset{\text{a.s.}}{\to} \mathrm{E}(|X_0|^3)$. Since almost sure convergence implies convergence in probability we have

$$\frac{1}{n}\sum_{t=1}^{n}\mathrm{E}(Z_t^2 I(|Z_t| > \eta\sqrt{n})|\mathcal{F}_{t-1}) \leq \underbrace{\frac{\mathrm{E}(\varepsilon_t^4)^{1/2}\mathrm{E}(\varepsilon_t^2)^{1/2}}{\eta n^{1/2}}}_{\to 0}\underbrace{\frac{1}{n}\sum_{t=1}^{n}|X_{t-1}|^3}_{\overset{\mathcal{P}}{\to}\mathrm{E}(|X_0|^3)} \overset{\mathcal{P}}{\to} 0.$$

Hence condition (8.12) is satisfied.

C3 : Finally, we need to verify that

$$\frac{1}{n}\sum_{t=1}^{n}\mathrm{E}(Z_t^2|\mathcal{F}_{t-1}) \overset{\mathcal{P}}{\to} \sigma^2.$$

Since $\{X_t\}_t$ is an ergodic sequence we have

$$\begin{aligned}
& \frac{1}{n}\sum_{t=1}^{n}\mathrm{E}(Z_t^2|\mathcal{F}_{t-1}) = \frac{1}{n}\sum_{t=1}^{n}\mathrm{E}(X_{t-1}^2\varepsilon_t^2|X_{t-1}) \\
= \quad & \frac{1}{n}\sum_{t=1}^{n}X_{t-1}^2\mathrm{E}(\varepsilon_t^2|X_{t-1}) = \mathrm{E}(\varepsilon_t^2)\underbrace{\frac{1}{n}\sum_{t=1}^{n}X_{t-1}^2}_{\overset{\text{a.s.}}{\to}\mathrm{E}(X_0^2)} \overset{\text{a.s.}}{\to} \mathrm{E}(\varepsilon^2)\mathrm{E}(X_0^2) = \sigma^2\Sigma_1,
\end{aligned}$$

hence we have verified condition (8.13).

Altogether conditions C1-C3 imply that

$$\sqrt{n}\nabla\mathcal{L}_n(\phi) = \frac{1}{\sqrt{n}}\sum_{t=1}^{n}X_{t-1}\varepsilon_t \overset{\mathcal{D}}{\to} \mathcal{N}(0,\sigma^2\Sigma_1). \tag{8.18}$$

Therefore

$$\sqrt{n}(\widehat{\phi}_n - \phi) = \underbrace{\left(\frac{1}{2}\nabla^2\mathcal{L}_n\right)^{-1}}_{\overset{\text{a.s.}}{\to}(\mathrm{E}(X_0^2))^{-1}} \underbrace{\sqrt{n}S_n}_{\overset{\mathcal{D}}{\to}\mathcal{N}(0,\sigma^2 c(0))} . \tag{8.19}$$

Using that $\mathrm{E}(X_0^2) = c(0)$, this implies that

$$\sqrt{n}(\hat{\phi}_n - \phi) \overset{\mathcal{D}}{\to} \mathcal{N}(0,\sigma^2\Sigma_1^{-1}). \tag{8.20}$$

Thus we have derived the limiting distribution of $\hat{\phi}_n$.

Remark 8.1.7 *We recall that*

$$(\hat{\phi}_n - \phi) = -\left(\nabla^2\mathcal{L}_n\right)^{-1}\nabla\mathcal{L}_n(\phi) = \frac{\frac{1}{n-1}\sum_{t=2}^{n}\varepsilon_t X_{t-1}}{\frac{1}{n-1}\sum_{t=2}^{n}X_{t-1}^2}, \tag{8.21}$$

and that $\mathrm{var}(\frac{1}{n-1}\sum_{t=2}^{n}\varepsilon_t X_{t-1}) = \frac{1}{n-1}\sum_{t=2}^{n}\mathrm{var}(\varepsilon_t X_{t-1}) = O(\frac{1}{n})$. *This implies*

$$(\hat{\phi}_n - \phi) = O_p(n^{-1/2}).$$

Indeed the results also holds almost surely

$$(\hat{\phi}_n - \phi) = O(n^{-1/2}). \tag{8.22}$$

The same result is true for autoregressive processes of arbitrary finite order. That is

$$\sqrt{n}(\underline{\hat{\phi}}_n - \underline{\phi}) \overset{\mathcal{D}}{\to} \mathcal{N}(0, \sigma^2 \Sigma_p^{-1}). \tag{8.23}$$

8.2 Estimation for ARMA models

Let us suppose that $\{X_t\}$ satisfies the ARMA representation

$$X_t - \sum_{i=1}^{p} \phi_i X_{t-i} = \varepsilon_t + \sum_{j=1}^{q} \theta_j \varepsilon_{t-j},$$

and $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_q)$, $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_p)$ and $\sigma^2 = \text{var}(\varepsilon_t)$. We will suppose for now that p and q are known. The objective in this section is to consider various methods for estimating these parameters.

8.2.1 The Gaussian maximum likelihood estimator

We now derive the Gaussian maximum likelihood estimator (GMLE) to estimate the parameters $\underline{\theta}$ and $\underline{\phi}$. Let $\underline{X}_n' = (X_1, \ldots, X_n)$. The criterion (the GMLE) is constructed as if $\{X_t\}$ were Gaussian, but this need not be the case. The likelihood is similar to the likelihood given in (8.5), but just as in the autoregressive case it can be not directly maximised, i.e.

$$L_n(\phi, \theta, \sigma) \quad = \quad -\log\det(\Sigma_n(\phi, \theta, \sigma)) - \underline{X}_n' \Sigma_n(\phi, \theta, \sigma)^{-1} \underline{X}_n, \tag{8.24}$$

where $\Sigma_n(\phi, \theta, \sigma)$ the variance covariance matrix of $\underline{X}_n$. However, the above can be written in a tractable way by using conditioning

$$L_n(\phi, \theta, \sigma) \quad = \quad \log f(X_1; \theta) + \sum_{t=1}^{n-1} \log f(X_{t+1} | X_t, \ldots, X_1; \theta) \quad \text{(by repeated conditioning)}.$$

Note that $f(X_{t+1} | X_1, \ldots, X_t, \theta)$ is the conditional density of X_{t+1} given $X_1, \ldots, X_t$ under Gaussianity. Thus we need to obtain the conditional mean and conditional variance. By using (6.23), if $t \leq \max(p, q)$ then

$$\text{E}[X_t | X_{t-1}, \ldots, X_1, \theta, \phi] = X_{t|t-1}(\phi, \theta) = \sum_{j=1}^{p} \phi_{t,j}(\phi, \theta) X_{t+1-j}.$$

if $t > \max(p,q)$, then

$$\mathrm{E}[X_t|X_{t-1},\ldots,X_1,\theta,\phi] = X_{t|t-1}(\phi,\theta) = \sum_{j=1}^{p}\phi_j X_{t-j} + \sum_{i=1}^{q}\theta_{t,i}(\phi,\theta)(X_{t+1-i} - X_{t+1-i|t-i}(\phi,\theta))$$

and Section 6.4

$$\mathrm{E}(X_t - X_{t|t-1}(\phi,\theta)|\phi,\theta)^2 = r(t;\phi,\theta,\sigma).$$

Note that $r(t;\phi,\theta,\sigma)$ can be evaluated using the Durbin-Levinson algorithm. Since the likelihood is constructed as if X_t were Gaussian, then $X_t - X_{t|t-1}(\phi,\theta)$ is *independent* of $X_{t-1},\ldots,X_1$ (this is not true for other distributions). This implies that

$$\begin{aligned}\mathrm{var}[X_t|X_{t-1},\ldots,X_1,\phi,\theta,\sigma] &= \mathrm{E}\left[\left(X_t - X_{t|t-1}(\phi,\theta)\right)^2|\phi,\theta,X_{t-1},\ldots,X_1\right] \\ &= \mathrm{E}\left[X_t - X_{t|t-1}(\phi,\theta)|\phi,\theta\right]^2 = r(t;\phi,\theta,\sigma).\end{aligned}$$

Thus the conditional density for $t > \max(p,q)$

$$\log f(X_t|X_{t-1},\ldots,X_1;\theta,\phi) \propto -\log r(t;\phi,\theta,\sigma) - \frac{(X_t - X_{t|t-1}(\phi,\theta))^2}{r(t;\phi,\theta,\sigma)}.$$

Substituting this into $L_n(\phi,\theta,\sigma)$ gives

$$\begin{aligned}L_n(\phi,\theta,\sigma) &= -\sum_{t=1}^{n}\log r(t;\phi,\theta,\sigma) - \frac{X_1^2}{r(1;\phi,\theta,\sigma)} - \sum_{t=1}^{\max(p,q)-1}\frac{(X_{t+1} - \sum_{j=1}^{t}\phi_{t+1,j}(\phi,\theta)X_{t+1-j})^2}{r(t+1;\phi,\theta,\sigma)} \\ &\quad - \sum_{t=\max(p,q)}^{n-1}\frac{(X_{t+1} - \sum_{j=1}^{p}\phi_j X_{t+1-j} - \sum_{i=1}^{q}\theta_{t,i}(\theta,\phi)(X_{t+1-i} - X_{t+1-i|t-i}(\phi,\theta))^2}{r(t+1;\phi,\theta,\sigma)}.\end{aligned}$$

An alternative derivation of the above is to use the Cholesky decomposition of $\Sigma_n(\phi,\theta,\sigma)$ (see Section 6.4.3, equation (6.21)). For each set of parameters ϕ, θ and σ^2, $r(t+1;\phi,\theta,\sigma)$ and $X_{t+1-i|t-i}(\phi,\theta)$ can be evaluated. Thus maximum likelihood estimator are the parameters $\widehat{\theta}_n, \widehat{\phi}_n, \widehat{\sigma}_n^2 = \arg\max L_n(\phi,\theta,\sigma)$.

8.2.2 The approximate likelihood

We can obtain an approximation to the log-likelihood which can simplify the estimation scheme. We recall in Section 6.5 we approximated $X_{t+1|t}$ with $\widehat{X}_{t+1|t}$. This motivates the approximation where we replace $X_{t+1|t}$ in $\widehat{L}_n(\phi,\theta,\sigma)$ with $\hat{X}_{t+1|t}$, where $\hat{X}_{t+1|t}$ is defined in (6.25) and $r(t,\phi,\theta,\sigma^2)$ with σ^2 to give the approximate Gaussian log-likelihood

$$\begin{aligned}\widehat{L}_n(\phi,\theta,\sigma) &= -\sum_{t=1}^{n}\log\sigma^2 - \sum_{t=2}^{n}\frac{[X_t-\widehat{X}_{t|t-1}(\phi,\theta)]^2}{\sigma^2}\\ &= -\sum_{t=1}^{n}\log\sigma^2 - \sum_{t=2}^{n}\frac{[(\theta(B)^{-1}\phi(B))_{[t]}X_t]^2}{\sigma^2}\end{aligned}$$

where $(\theta(B)^{-1}\phi(B))_{[t]}$ denotes the approximation of the polynomial in B to the tth order. The approximate likelihood greatly simplifies the estimation scheme because the derivatives (which is the main tool used in the maximising it) can be easily obtained. To do this we note that

$$\begin{aligned}\frac{d}{d\theta_i}\frac{\phi(B)}{\theta(B)}X_t &= -\frac{B^i\phi(B)}{\theta(B)^2}X_t = -\frac{\phi(B)}{\theta(B)^2}X_{t-i}\\ \frac{d}{d\phi_j}\frac{\phi(B)}{\theta(B)}X_t &= -\frac{B^j}{\theta(B)^2}X_t = -\frac{1}{\theta(B)^2}X_{t-j}\end{aligned} \tag{8.25}$$

therefore

$$\frac{d}{d\theta_i}\left(\frac{\phi(B)}{\theta(B)}X_t\right)^2 = -2\left(\frac{\phi(B)}{\theta(B)}X_t\right)\left(\frac{\phi(B)}{\theta(B)^2}X_{t-i}\right) \text{ and } \frac{d}{d\phi_j}\left(\frac{\phi(B)}{\theta(B)}X_t\right)^2 = -2\left(\frac{\phi(B)}{\theta(B)}X_t\right)\left(\frac{1}{\theta(B)^2}X_{t-j}\right). \tag{8.26}$$

Substituting this into the approximate likelihood gives the derivatives

$$\begin{aligned}\frac{\partial\widehat{L}}{\partial\theta_i} &= -\frac{2}{\sigma^2}\sum_{t=2}^{n}\left[\left(\theta(B)^{-1}\phi(B)\right)_{[t]}X_t\right]\left[\left(\frac{\phi(B)}{\theta(B)^2}\right)_{[t-i]}X_{t-i}\right]\\ \frac{\partial\widehat{L}}{\partial\phi_j} &= -\frac{2}{\sigma^2}\sum_{t=1}^{n}\left[\left(\theta(B)^{-1}\phi(B)\right)_{[t]}X_t\right]\left[\left(\frac{1}{\theta(B)}\right)_{[t-j]}X_{t-j}\right]\\ \frac{\partial\widehat{L}}{\partial\sigma^2} &= \frac{1}{\sigma^2}-\frac{1}{n\sigma^4}\sum_{t=1}^{n}\left[\left(\theta(B)^{-1}\phi(B)\right)_{[t]}X_t\right]^2.\end{aligned} \tag{8.27}$$

We then use the Newton-Raphson scheme to maximise the approximate likelihood.

It should be mentioned that such approximations are very common in time series, though as with all approximation, they can be slightly different. Lütkepohl (2005), Section 12.2 gives a very similar approximation, but uses the variance/covariance matrix of the time series $\underline{X}_n$ as the basis of the approximation. By approximating the variance/covariance he finds an approximation of the likelihood.

8.2.3 Sampling properties

It can be shown that the approximate likelihood is close to the actual true likelihood and asymptotically both methods are equivalent.

Theorem 8.2.1 *Let us suppose that X_t has a causal and invertible ARMA representation*

$$X_t - \sum_{j=1}^{p} \phi_j X_{t-j} = \varepsilon_t + \sum_{i=1}^{q} \theta_i \varepsilon_{t-i}$$

where $\{\varepsilon_t\}$ are iid random variables with mean zero and $\mathrm{var}[\varepsilon_t] = \sigma^2$ (we do not assume Gaussianity). Then the (quasi)-Gaussian

$$\sqrt{n} \begin{pmatrix} \hat{\underline{\phi}}_n - \underline{\phi} \\ \hat{\underline{\theta}}_n - \underline{\theta} \end{pmatrix} \overset{\mathcal{D}}{\to} \mathcal{N}(0, \Lambda^{-1}),$$

with

$$\Lambda = \begin{pmatrix} \mathrm{E}(\mathbf{U}_t \mathbf{U}_t') & \mathrm{E}(\mathbf{V}_t \mathbf{U}_t') \\ \mathrm{E}(\mathbf{U}_t \mathbf{V}_t') & \mathrm{E}(\mathbf{V}_t \mathbf{V}_t') \end{pmatrix}$$

and $\mathbf{U}_t = (U_t, \ldots, U_{t-p+1})$ and $\mathbf{V}_t = (V_t, \ldots, V_{t-q+1})$, where $\{U_t\}$ and $\{V_t\}$ are autoregressive processes which satisfy $\phi(B)U_t = \varepsilon_t$ and $\theta(B)V_t = \varepsilon_t$.

We do not give the proof in this section, however it is possible to understand where this result comes from. We recall that that the maximum likelihood and the approximate likelihood are asymptotically equivalent. They are both approximations of the unobserved likelihood

$$\widetilde{L}_n(\boldsymbol{\theta}) = -\sum_{t=1}^{n} \log \sigma^2 - \sum_{t=2}^{n-1} \frac{[X_{t+1} - X_t(1; \boldsymbol{\theta})]^2}{\sigma^2} = -\sum_{t=1}^{n} \log \sigma^2 - \sum_{t=2}^{n-1} \frac{[\theta(B)^{-1}\phi(B)X_{t+1}]^2}{\sigma^2},$$

where $\boldsymbol{\theta} = (\phi, \theta, \sigma^2)$. This likelihood is infeasible in the sense that it cannot be maximised since the finite past $X_0, X_1, \ldots$ is unobserved, however is a very convenient tool for doing the asymptotic analysis. Using Lemma 6.5.1 we can show that all three likelihoods L_n, $\widehat{L}_n$ and $\widetilde{L}_n$ are all asymptotically equivalent. Therefore, to obtain the asymptotic sampling properties of L_n or $\widehat{L}_n$ we can simply consider the unobserved likelihood $\widetilde{L}_n$.

To show asymptotic normality (we assume here that the estimators are consistent) we need to consider the first and second derivative of $\widetilde{L}_n$ (since the asymptotic properties are determined by Taylor expansions). In particular we need to consider the distribution of $\frac{\partial \widetilde{L}_n}{\partial \boldsymbol{\theta}}$ at its true parameters and the expectation of $\frac{\partial^2 \widetilde{L}_n}{\partial \boldsymbol{\theta}^2}$ at it's true parameters. We note that by using (8.26) we have

$$\begin{aligned} \frac{\partial \widetilde{L}}{\partial \theta_i} &= -\frac{2}{\sigma^2}\sum_{t=1}^{n}\left[\left(\theta(B)^{-1}\phi(B)\right)X_t\right]\left[\left(\frac{\phi(B)}{\theta(B)^2}\right)X_{t-i}\right] \\ \frac{\partial \widetilde{L}}{\partial \phi_j} &= -\frac{2}{\sigma^2}\sum_{t=1}^{n}\left[\left(\theta(B)^{-1}\phi(B)\right)X_t\right]\left[\left(\frac{1}{\theta(B)}\right)X_{t-j}\right] \qquad (8.28) \end{aligned}$$

Since we are considering the derivatives at the true parameters we observe that $\left(\theta(B)^{-1}\phi(B)\right)X_t = \varepsilon_t$,

$$\frac{\phi(B)}{\theta(B)^2}X_{t-i} = \frac{\phi(B)}{\theta(B)^2}\frac{\theta(B)}{\phi(B)}\varepsilon_{t-i} = \frac{1}{\theta(B)}\varepsilon_{t-i} = V_{t-i}$$

and

$$\frac{1}{\theta(B)}X_{t-j} = \frac{1}{\theta(B)}\frac{\theta(B)}{\phi(B)}\varepsilon_{t-j} = \frac{1}{\phi(B)}\varepsilon_{t-j} = U_{t-j}.$$

Thus $\phi(B)U_t = \varepsilon_t$ and $\theta(B)V_t = \varepsilon_t$ are autoregressive processes (compare with theorem). This means that the derivative of the unobserved likelihood can be written as

$$\frac{\partial \widetilde{L}}{\partial \theta_i} = -\frac{2}{\sigma^2}\sum_{t=1}^{n}\varepsilon_t U_{t-i} \text{ and } \frac{\partial \widetilde{L}}{\partial \phi_j} = -\frac{2}{\sigma^2}\sum_{t=1}^{n}\varepsilon_t V_{t-j} \qquad (8.29)$$

Note that by causality ε_t, U_{t-i} and V_{t-j} are independent. Again like many of the other estimators we have encountered this sum is 'mean-like' so can show normality of it by using a central limit theorem designed for dependent data. Indeed we can show asymptotically normality of $\{\frac{\partial \widetilde{L}}{\partial \theta_i}; i = 1, \ldots, q\}$, $\{\frac{\partial \widetilde{L}}{\partial \phi_j}; j = 1, \ldots, p\}$ and their linear combinations using the Martingale central limit theorem, see Theorem 3.2 (and Corollary 3.1), Hall and Heyde (1980) - note that one can also use m-dependence. Moreover, it is relatively straightforward to show that $n^{-1/2}(\frac{\partial \widetilde{L}}{\partial \theta_i}, \frac{\partial \widetilde{L}}{\partial \phi_j})$ has the limit variance matrix

Δ. Finally, by taking second derivative of the likelihood we can show that $\mathrm{E}[n^{-1}\frac{\partial^2 \widehat{L}}{\partial \boldsymbol{\theta}^2}] = \Delta$. Thus giving us the desired result.

8.2.4 The Hannan-Rissanen AR(∞) expansion method

The methods detailed above require good initial values in order to begin the maximisation (in order to prevent convergence to a local maximum).

We now describe a simple method first propose in Hannan and Rissanen (1982) and An et al. (1982). It is worth bearing in mind that currently the 'large p small n problem' is a hot topic. These are generally regression problems where the sample size n is quite small but the number of regressors p is quite large (usually model selection is of importance in this context). The methods proposed by Hannan involves expanding the ARMA process (assuming invertibility) as an $AR(\infty)$ process and estimating the parameters of the $AR(\infty)$ process. In some sense this can be considered as a regression problem with an infinite number of regressors. Hence there are some parallels between the estimation described below and the 'large p, small n problem'.

As we mentioned in Lemma 3.5.1, if an ARMA process is invertible it is can be represented as

$$X_t = \sum_{j=1}^{\infty} b_j X_{t-j} + \varepsilon_t. \tag{8.30}$$

The idea behind Hannan's method is to estimate the parameters $\{b_j\}$, then estimate the innovations ε_t, and use the estimated innovations to construct a multiple linear regression estimator of the ARMA paramters $\{\theta_i\}$ and $\{\phi_j\}$. Of course in practice we cannot estimate all parameters $\{b_j\}$ as there are an infinite number of them. So instead we do a type of sieve estimation where we only estimate a finite number and let the number of parameters to be estimated grow as the sample size increases. We describe the estimation steps below:

(i) Suppose we observe $\{X_t\}_{t=1}^{n}$. Recalling (8.30), will estimate $\{b_j\}_{j=1}^{p_n}$ parameters. We will suppose that $p_n \to \infty$ as $n \to \infty$ and $p_n << n$ (we will state the rate below).

    We use Yule-Walker to estimate $\{b_j\}_{j=1}^{p_n}$, where

$$\underline{\hat{b}}_{p_n} = \hat{\Sigma}_{p_n}^{-1} \underline{\hat{r}}_{p_n},$$

where

$$(\hat{\Sigma}_{p_n})_{i,j} = \frac{1}{n}\sum_{t=1}^{n-|i-j|}(X_t - \bar{X})(X_{t+|i-j|} - \bar{X}) \text{ and } (\hat{\underline{r}}_{p_n})_j = \frac{1}{n}\sum_{t=1}^{n-|j|}(X_t - \bar{X})(X_{t+|j|} - \bar{X}).$$

(ii) Having estimated the first $\{b_j\}_{j=1}^{p_n}$ coefficients we estimate the residuals with

$$\tilde{\varepsilon}_t = X_t - \sum_{j=1}^{p_n} \hat{b}_{j,n} X_{t-j}.$$

(iii) Now use as estimates of $\underline{\phi}_0$ and $\underline{\theta}_0$ $\tilde{\underline{\phi}}_n, \tilde{\underline{\theta}}_n$ where

$$\tilde{\underline{\phi}}_n, \tilde{\underline{\theta}}_n = \arg\min \sum_{t=p_n+1}^{n}(X_t - \sum_{j=1}^{p}\phi_j X_{t-j} - \sum_{i=1}^{q}\theta_i \tilde{\varepsilon}_{t-i})^2.$$

We note that the above can easily be minimised. In fact

$$(\widetilde{\underline{\phi}}_n, \widetilde{\underline{\theta}}_n) = \widetilde{\mathcal{R}}_n^{-1}\widetilde{\underline{s}}_n$$

where

$$\tilde{\mathcal{R}}_n = \frac{1}{n}\sum_{t=\max(p,q)}^{n} \widetilde{\underline{Y}}_t \widetilde{\underline{Y}}_t' \quad \text{and} \quad \widetilde{\underline{s}}_n = \frac{1}{n}\sum_{t=\max(p,q)}^{n} \widetilde{\underline{Y}}_t X_t,$$

$\widetilde{\underline{Y}}_t' = (X_{t-1}, \ldots, X_{t-p}, \widetilde{\varepsilon}_{t-1}, \ldots, \widetilde{\varepsilon}_{t-q})$.

8.3 The quasi-maximum likelihood for ARCH processes

In this section we consider an estimator of the parameters $\underline{a}_0 = \{a_j : j = 0, \ldots, p\}$ given the observations $\{X_t : t = 1, \ldots, N\}$, where $\{X_t\}$ is a ARCH(p) process. We use the conditional log-likelihood to construct the estimator. We will assume throughout that $\mathrm{E}(Z_t^2) = 1$ and $\sum_{j=1}^{p}\alpha_j = \rho < 1$.

We now construct an estimator of the ARCH parameters based on $Z_t \sim \mathcal{N}(0,1)$. It is worth mentioning that despite the criterion being constructed under this condition it is not necessary that the innovations Z_t are normally distributed. In fact in the case that the innovations are not normally distributed but have a finite fourth moment the estimator is still good. This is why it is called the quasi-maximum likelihood , rather than the maximum likelihood (similar to the how the GMLE estimates the parameters of an ARMA model regardless of whether the innovations are Gaussian or not).

Let us suppose that Z_t is Gaussian. Since $Z_t = X_t/\sqrt{a_0 + \sum_{j=1}^{p} a_j X_{t-j}^2}$, $\mathrm{E}(X_t|X_{t-1},\ldots,X_{t-p}) = 0$ and $\mathrm{var}(X_t|X_{t-1},\ldots,X_{t-p}) = a_0 + \sum_{j=1}^{p} a_j X_{t-j}^2$, then the log density of X_t given $X_{t-1},\ldots,X_{t-p}$ is

$$\log(a_0 + \sum_{j=1}^{p} a_j X_{t-j}^2) + \frac{X_t^2}{a_0 + \sum_{j=1}^{p} a_j X_{t-j}^2}.$$

Therefore the conditional log density of $X_{p+1}, X_{p+2}, \ldots, X_n$ given $X_1,\ldots,X_p$ is

$$\sum_{t=p+1}^{n}\left(\log(a_0 + \sum_{j=1}^{p} a_j X_{t-j}^2) + \frac{X_t^2}{a_0 + \sum_{j=1}^{p} a_j X_{t-j}^2}\right).$$

This inspires the the conditional log-likelihood

$$\mathcal{L}_n(\underline{\alpha}) = \frac{1}{n-p}\sum_{t=p+1}^{n}\left(\log(\alpha_0 + \sum_{j=1}^{p} \alpha_j X_{t-j}^2) + \frac{X_t^2}{\alpha_0 + \sum_{j=1}^{p} \alpha_j X_{t-j}^2}\right).$$

To obtain the estimator we define the parameter space

$$\Theta = \{\underline{\alpha} = (\alpha_0,\ldots,\alpha_p) : \sum_{j=1}^{p} \alpha_j \leq 1, 0 < c_1 \leq \alpha_0 \leq c_2 < \infty, c_1 \leq \alpha_j\}$$

and assume the true parameters lie in its interior $\underline{a} = (a_0,\ldots,a_p) \in Int(\Theta)$. We let

$$\hat{\underline{a}}_n = \arg\min_{\underline{\alpha}\in\Theta} \mathcal{L}_n(\underline{\alpha}). \tag{8.31}$$

The method for estimation of GARCH parameters parallels the approximate likelihood ARMA estimator given in Section 8.2.1.

Exercise 8.2 *The objective of this question is to estimate the parameters of a random autoregressive process of order one*

$$X_t = (\phi + \xi_t)\, X_{t-1} + \varepsilon_t,$$

where, $|\phi| < 1$ and $\{\xi_t\}_t$ and $\{\varepsilon_t\}_t$ are zero mean iid random variables which are independent of *each other, with* $\sigma_{\xi}^2 = \text{var}[\xi_t]$ *and* $\sigma_{\varepsilon}^2 = \text{var}[\varepsilon_t]$.

Suppose that $\{X_t\}_{t=1}^n$ *is observed. We will assume for parts (a-d) that* ξ_t *and* ε_t *are Gaussian random variables. In parts (b-c) the objective is to construct an initial value estimator which is easy to obtain but not optimal in (d) to obtain the maximum likelihood estimator.*

(a) What is the conditional expectation (best predictor) of X_t *given the past?*

(b) Suppose that $\{X_t\}_{t=1}^n$ *is observed. Use your answer in part (a) to obtain an explicit expression for estimating* ϕ.

(c) Define residual as $\xi_t X_{t-1} + \varepsilon_t$. *Use your estimator in (b) to estimate the residuals.*

Evaluate the variance of $\xi_t X_{t-1} + \varepsilon_t$ *conditioned on* X_{t-1}. *By using the estimated residuals explain how the conditional variance can be used to obtain an explicit expression for estimating* σ_{ξ}^2 *and* σ_{ε}^2.

(d) By conditioning on X_1 *obtain the log-likelihood of* $X_2, \ldots, X_n$ *under the assumption of Guassianity of* ξ_t *and* ε_t. *Explain the role that (b) and (c) plays in your maximisation algorithm.*

(e) **Bonus question (only attempt if you really want to)**

Show that the expectation of the conditional log-likelihood is maximised at the true parameters *(*$\phi_0, \sigma_{0,\xi}^2$ *and* $\sigma_{0,\varepsilon}^2$*) even when* ξ_t *and* ε_t *are not Gaussian.*

Hint: You may want to use that the function $g(x) = -\log x + x$ *is minimum at* $x = 1$ *where* $g(1) = 1$ *and let*

$$x = \frac{\sigma_{0,\varepsilon}^2 + \sigma_{0,\xi}^2 X_{t-1}^2}{\sigma_{\varepsilon}^2 + \sigma_{\xi}^2 X_{t-1}^2}.$$

Chapter 9

Spectral Representations

Prerequisites

- Knowledge of complex numbers.
- Have some idea of what the covariance of a complex random variable (we do define it below).
- Some idea of a Fourier transform (a review is given in Section A.3).

Objectives

- Know the definition of the spectral density.
- The spectral density is always non-negative and this is a way of checking that a sequence is actually non-negative definite (is a autocovariance).
- The DFT of a second order stationary time series is almost uncorrelated.
- The spectral density of an ARMA time series, and how the roots of the characteristic polynomial of an AR may influence the spectral density function.
- There is no need to understand the proofs of either Bochner's (generalised) theorem or the spectral representation theorem, just know what these theorems are. However, you should know the proof of Bochner's theorem in the simple case that $\sum_r |rc(r)| < \infty$.

9.1 How we have used Fourier transforms so far

We recall in Section 1.3.4 that we considered models of the form

$$X_t = A\cos(\omega t) + B\sin(\omega t) + \varepsilon_t \qquad t = 1,\ldots,n. \tag{9.1}$$

where ε_t are iid random variables with mean zero and variance σ^2 and ω is unknown. We estimated the frequency ω by taking the Fourier transform $J_n(\omega) = \frac{1}{\sqrt{n}}\sum_{t=1}^{n} X_t e^{it\omega}$ and using as an estimator of ω, the value which maximised $|J_n(\omega)|^2$. As the sample size grows the peak (which corresponds the frequency estimator) grows in size. Besides the fact that this corresponds to the least squares estimator of ω, we note that

$$\begin{aligned}
\frac{1}{\sqrt{n}}J_n(\omega_k) &= \frac{1}{2\pi n}\sum_{t=1}^{n} X_t \exp(it\omega_k) \\
&= \underbrace{\frac{1}{2\pi n}\sum_{t=1}^{n}\mu(\frac{t}{n})\exp(it\omega_k)}_{=O(1)} + \underbrace{\frac{1}{2\pi n}\sum_{t=1}^{n}\varepsilon_t \exp(it\omega_k)}_{=O_p(n^{-1/2}) \text{ compare with} \frac{1}{n}\sum_{t=1}^{n}\varepsilon_t}
\end{aligned} \tag{9.2}$$

where $\omega_k = \frac{2\pi k}{n}$, is an estimator the the Fourier transform of the deterministic mean at frequency k. In the case that the mean is simply the sin function, there is only one frequency which is non-zero. A plot of one realization ($n = 128$), periodogram of the realization, periodogram of the iid noise and periodogram of the sin function is given in Figure 9.1. Take careful note of the scale (y-axis), observe that the periodogram of the sin function dominates the the periodogram of the noise (magnitudes larger). We can understand why from (9.2), where the asymptotic rates are given and we see that the periodogram of the deterministic signal is estimating $n\times$Fourier coefficient, whereas the periodgram of the noise is $O_p(1)$. However, this is an asymptotic result, for small samples sizes you may not see such a big difference between deterministic mean and the noise. Next look at the periodogram of the noise we see that it is very erratic (we will show later that this is because it is an **inconsistent** estimator of the spectral density function), however, despite the erraticness, the amount of variation overall frequencies seems to be same (there is just one large peak - which could be explained by the randomness of the periodogram).

Returning again to Section 1.3.4, we now consider the case that the sin function has been

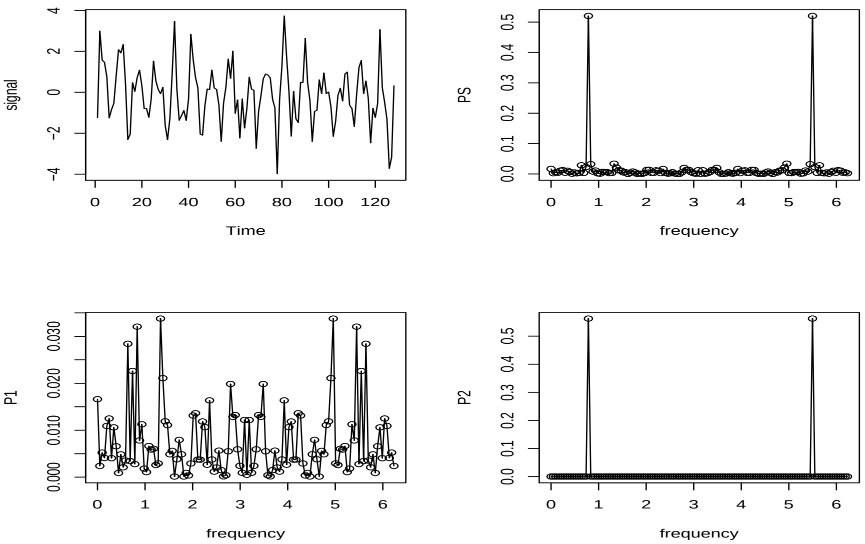
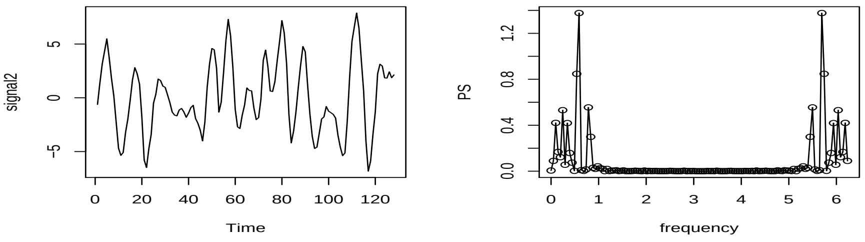


Figure 9.1: Top Left: Realisation of (1.6) $(2\sin(\frac{2\pi t}{8}))$ with iid noise, Top Right: Periodogram of sin + noise. Bottom Left: Periodogram of just the noise. Bottom Right: Periodogram of just the sin function.

corrupted by colored noise, which follows an AR(2) model

$$\varepsilon_t = 1.5\varepsilon_{t-1} - 0.75\varepsilon_{t-2} + \epsilon_t. \tag{9.3}$$

A realisation and the corresponding periodograms are given in Figure 9.2. The results are different to the iid case. The peak in the periodogram no longer corresponds to the period of the sin function. From the periodogram of the just the AR(2) process we observe that it erratic, just as in the iid case, however, there appears to be varying degrees of variation over the frequencies (though this is not so obvious in this plot). We recall from Chapters 2 and 3, that the AR(2) process has a pseudo-period, which means the periodogram of the colored noise will have pronounced peaks which correspond to the frequencies around the pseudo-period. It is these pseudo-periods which are dominating the periodogram, which is giving a peak at frequency that does not correspond to the sin function. However, asymptotically the rates given in (9.2) still hold in this case too. In other words, for large enough sample sizes the DFT of the signal should dominate the noise. To see that this is the case, we increase the sample size to $n = 1024$, a realisation is given in Figure 9.3. We see that the period corresponding the sin function dominates the periodogram. Studying the periodogram of just the AR(2) noise we see that it is still erratic (despite the large sample size), but we also observe that the variability clearly changes over frequency.



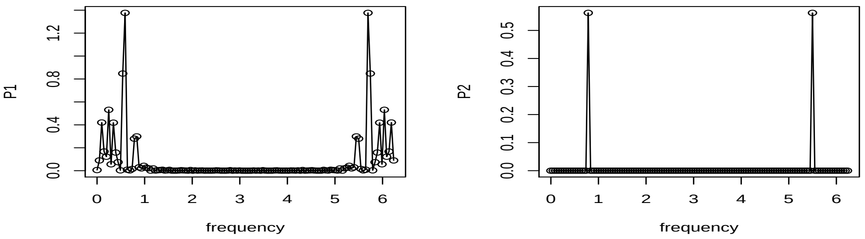


Figure 9.2: Top Left: Realisation of (1.6) $(2\sin(\frac{2\pi t}{8}))$ with AR(2) noise ($n = 128$), Top Right: Periodogram. Bottom Left: Periodogram of just the AR(2) noise. Bottom Right: Periodogram of the sin function.

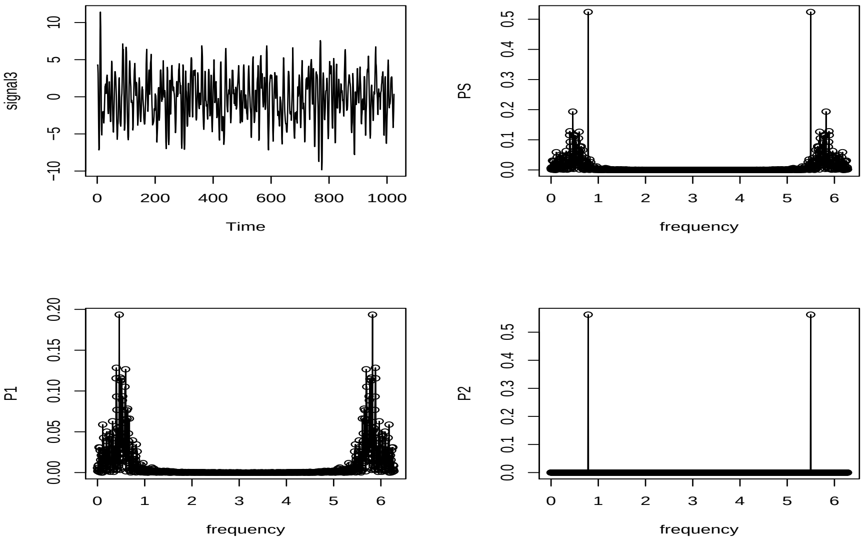


Figure 9.3: Top Left: Realisation of (1.6) $(2\sin(\frac{2\pi t}{8}))$ with AR(2) noise ($n = 1024$), Top Right: Periodogram. Bottom Left: Periodogram of just the AR(2) noise. Bottom Right: Periodogram of the sin function.

From now on we focus on the constant mean stationary time series (eg. iid noise and the AR(2)) (where the mean is either constant or zero). As we have observed above, the periodogram is the absolute square of the discrete Fourier Transform (DFT), where

$$J_n(\omega_k) = \frac{1}{\sqrt{2\pi n}} \sum_{t=1}^{n} X_t \exp(it\omega_k). \tag{9.4}$$

This is simply a (linear) transformation of the data, thus it easily reversible by taking the inverse DFT

$$X_t = \frac{\sqrt{2\pi}}{\sqrt{n}} \sum_{t=1}^{n} J_n(\omega_k) \exp(-it\omega_k). \tag{9.5}$$

Therefore, just as one often analyzes the log transform of data (which is also an invertible transform), one can analyze a time series through its DFT.

In Figure 9.4 we give plots of the periodogram of an iid sequence and AR(2) process defined in equation (9.3). We recall from Chapter 3, that the periodogram is an **inconsistent** estimator of the spectral density function $f(\omega) = (2\pi)^{-1}\sum_{r=-\infty}^{\infty} c(r)\exp(ir\omega)$ and a plot of the spectral density function corresponding to the iid and AR(2) process defined in (**??**). We will show later that by inconsistent estimator we mean that $\mathrm{E}[|J_n(\omega_k)|^2] = f(\omega_k) + O(n^{-1})$ but $\mathrm{var}[|J_n(\omega_k)|^2] \nrightarrow 0$ as $n \to \infty$. this explains why the general 'shape' of $|J_n(\omega_k)|^2$ looks like $f(\omega_k)$ but $|J_n(\omega_k)|^2$ is extremely erratic and variable.

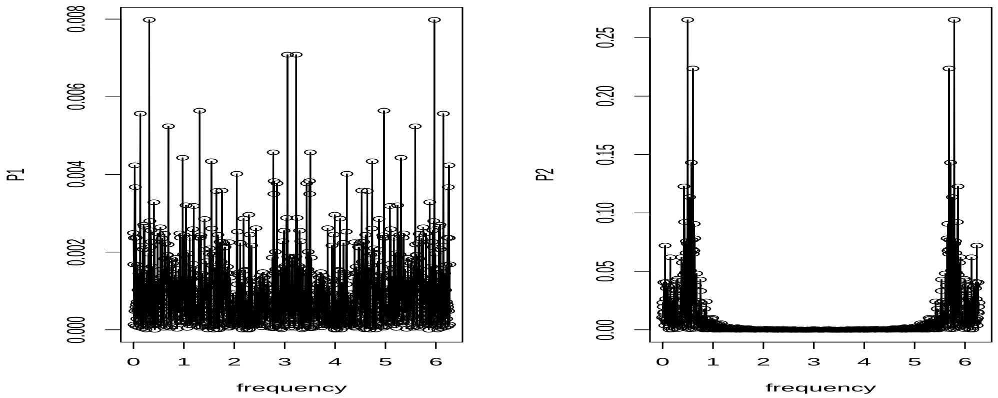


Figure 9.4: Left: Periodogram of iid noise. Right: Periodogram of AR(2) process.

Remark 9.1.1 (Properties of the spectral density function) *The spectral density function was first introduced in in Section 2.4. We recall that given an autoregressive process $\{c(k)\}$, the*

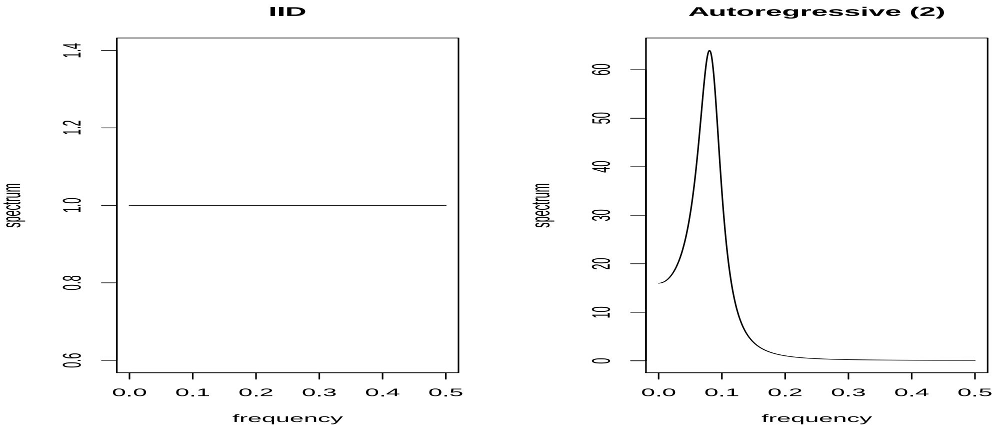


Figure 9.5: Left: Spectral density of iid noise. Right: Spectral density of AR(2), note that the interval $[0,1]$ corresponds to $[0,2\pi]$ in Figure 9.5

spectral density is defined as

$$f(\omega) = \frac{1}{2\pi}\sum_{r=-\infty}^{\infty} c(r)\exp(2\pi i r).$$

And visa versa, given the spectral density we can recover the autocovariance via the inverse transform $c(r) = \int_0^{2\pi} f(\omega)\exp(-2\pi i r\omega)d\omega$. *We recall from Section 2.4 that the spectral density function can be used to construct a valid autocovariance function since only a sequence whose Fourier transform is real and positive can be positive definite.*

In Section 6.5 we used the spectral density function to define conditions under which the variance covariance matrix of a stationary time series had minimum and maximim eigenvalues. Now from the discussion above we observe that the variance of the DFT is approximately the spectral density function (note that for this reason the spectral density is sometimes called the power spectrum).

We now collect some of the above observations, to summarize some of the basic properties of the DFT:

(i) We note that $\overline{J_n(\omega_k)} = J_n(\omega_{n-k})$, therefore, all the information on the time series is contain in the first $n/2$ frequencies $\{J_n(\omega_k); k = 1, \ldots, n/2\}$.

(ii) If the time series $\mathrm{E}[X_t] = \mu$ and $k \neq 0$ then

$$\mathrm{E}[J_n(\omega_k)] = \frac{1}{\sqrt{n}}\sum_{t=1}^{n}\mu\exp(it\omega_k) = 0.$$

If $k = 0$ then

$$\mathrm{E}[J_n(\omega_0)] = \frac{1}{\sqrt{n}}\sum_{t=1}^{n}\mu = \sqrt{n}\mu.$$

In other words, the mean of the DFT (at non-zero frequencies) is zero regardless of whether the time series has a zero mean (it just needs to have a constant mean).

(iii) However, unlike the original stationary time series, we observe that the variance of the DFT depends on frequency (unless it is a white noise process) and that for $k \neq 0$, $\mathrm{var}[J_n(\omega_k)] = \mathrm{E}[|J_n(\omega_k)|^2] = f(\omega_k) + O(n^{-1})$.

The focus of this chapter will be on properties of the spectral density function (proving some of the results we stated previously) and on the so called Cramer representation (or spectral representation) of a second order stationary time series. However, before we go into these results (and proofs) we give one final reason why the analysis of a time series is frequently done by transforming to the frequency domain via the DFT. Above we showed that there is a one-to-one correspondence between the DFT and the original time series, below we show that the DFT almost decorrelates the stationary time series. In other words, one of the main advantages of working within the frequency domain is that we have transformed a correlated time series into something that it almost uncorrelated (this also happens to be a heuristic reason behind the spectral representation theorem).

9.2 The 'near' uncorrelatedness of the Discrete Fourier Transform

Let $\underline{X}_n = \{X_t; t = 1, \ldots, n\}$ and $\Sigma_n = \mathrm{var}[\underline{X}_n]$. It is clear that $\Sigma_n^{-1/2}\underline{X}_n$ is an uncorrelated sequence. This means to formally decorrelate $\underline{X}_n$ we need to know $\Sigma_n^{-1/2}$. However, if X_t is a second order stationary time series, something curiously, remarkable happens. The DFT, almost uncorrelates the $\underline{X}_n$. The implication of this is extremely useful in time series, and we shall be using this transform in estimation in Chapter 10.

We start by defining the Fourier transform of $\{X_t\}_{t=1}^{n}$ as

$$J_n(\omega_k) = \frac{1}{\sqrt{2\pi n}}\sum_{t=1}^{n} X_t \exp(ik\frac{2\pi t}{n})$$

where the frequences $\omega_k = 2\pi k/n$ are often called the fundamental, Fourier frequencies.

Lemma 9.2.1 *Suppose $\{X_t\}$ is a second order stationary time series, where $\sum_r |rc(r)| < \infty$. Then we have*

$$\mathrm{cov}(J_n(\frac{2\pi k_1}{n}), J_n(\frac{2\pi k_2}{n})) = \begin{cases} f(\frac{2\pi k}{n}) + O(\frac{1}{n}) & k_1 = k_2 \\ O(\frac{1}{n}) & 1 \leq_1 \neq k_2 \leq n/2 \end{cases}$$

where $f(\omega) = \frac{1}{2\pi}\sum_{r=-\infty}^{\infty} c(r)\exp(ir\omega)$. If one wants to consider the real and imaginary parts of $J_n(\omega_k)$ then

$$\mathrm{cov}(J_{n,C}(\frac{2\pi k_1}{n}), J_{n,C}(\frac{2\pi k_2}{n})) = \begin{cases} f(\frac{2\pi k}{n}) + O(\frac{1}{n}) & k_1 = k_2 \\ O(\frac{1}{n}) & 1 \leq k_1 \neq k_2 \leq n/2 \end{cases}$$

$$\mathrm{cov}(J_{n,S}(\frac{2\pi k_1}{n}), J_{n,S}(\frac{2\pi k_2}{n})) = \begin{cases} f(\frac{2\pi k}{n}) + O(\frac{1}{n}) & k_1 = k_2 \\ O(\frac{1}{n}) & 1 \leq k_1 \neq k_2 \leq n/2 \end{cases}$$

and $\mathrm{cov}[J_{n,C}(\frac{2\pi k_1}{n}), J_{n,S}(\frac{2\pi k_2}{n})] = O(n^{-1})$ for $1 \leq k_1, k_2 \leq n/2$, where

$$J_{n,C}(\omega_k) = \frac{1}{\sqrt{2\pi n}}\sum_{t=1}^{n} X_t \cos(t\omega_k), \quad J_{n,S}(\omega_k) = \frac{1}{\sqrt{2\pi n}}\sum_{t=1}^{n} X_t \sin(t\omega_k).$$

In the sections below we give two proofs for the same result.

We note that the principle reason behind both proofs is that

$$\sum_{t=1}^{n} \exp\left(it\frac{2\pi j}{n}\right) = \begin{cases} 0 & j \neq n\mathbb{Z} \\ n & j \in \mathbb{Z} \end{cases}. \tag{9.6}$$

9.2.1 'Seeing' the decorrelation in practice

We evaluate the DFT using the following piece of code (note that we do not standardize by $\sqrt{2\pi}$)

```
dft <- function(x){
              n=length(x)
              dft <- fft(x)/sqrt(n)
              return(dft)
```

```
    }
```

We have shown above that $\{J_n(\omega_k)\}_k$ are close to uncorrelated and have variance close to $f(\omega_k)$. This means that the ratio $J_n(\omega_k)/f(\omega_k)^{1/2}$ are close to uncorrelated with variance close to one. Let us treat

$$Z_k = \frac{J_n(\omega_k)}{\sqrt{f(\omega_k)}},$$

as the transformed random variables, noting that $\{Z_k\}$ is complex, our aim is to show that the acf corresponding to $\{Z_k\}$ is close to zero. Of course, in practice we do not know the spectral density function f, therefore we estimate it using the piece of code (where `test` is the time series)

```
k<-kernel("daniell",6)
temp2 <-spec.pgram(test,k, taper=0, log = "no")$spec
n <- length(temp2)
temp3 <- c(temp2[c(1:n)],temp2[c(n:1)])
```

`temp3` simply takes a local average of the periodogram about the frequency of interest (however it is worth noting that `spec.pgram` does not do precisely this, which can be a bit annoying). In Section 10.3 we explain why this is a **consistent** estimator of the spectral density function. Notice that we also double the length, because the estimator `temp2` only gives estimates in the interval $[0, \pi]$. Thus our estimate of $\{Z_k\}$, which we denote as $\widehat{Z}_k = J_n(\omega_k)/\widehat{f}_n(\omega_k)^{1/2}$ is

```
temp1 <- dft(test);  temp4 <- temp1/sqrt(temp3)
```

We want to evaluate the covariance of $\{\widehat{Z}_k\}$ over various lags

$$\widehat{C}_n(r) = \frac{1}{n}\sum_{k=1}^{n} \widehat{Z}_k \overline{\widehat{Z}}_{k+r} = \frac{1}{n}\sum_{k=1}^{n} \frac{J_n(\omega_k)\overline{J_n(\omega_{k+r})}}{\sqrt{\widehat{f}_n(\omega_k)\widehat{f}_n(\omega_{k+r})}}$$

To speed up the evaluation, we use we can exploit the speed of the FFT, Fast Fourier Transform.

A plot of the AR(2) model

$$\varepsilon_t = 1.5\varepsilon_{t-1} - 0.75\varepsilon_{t-2} + \epsilon_t.$$

together with the real and imaginary parts of its DFT autocovariance is given in Figure 9.6. We observe that most of the correlations lie between $[-1.96, 1.96]$ (which corresponds to the 2.5% limits of a standard normal). Note that the 1.96 corresponds to the 2.5% limits, however this bound only holds if the time series is Gaussian. If the time series is non-Gaussian some corrections have to be made (see Dwivedi and Subba Rao (2011) and Jentsch and Subba Rao (2014)).

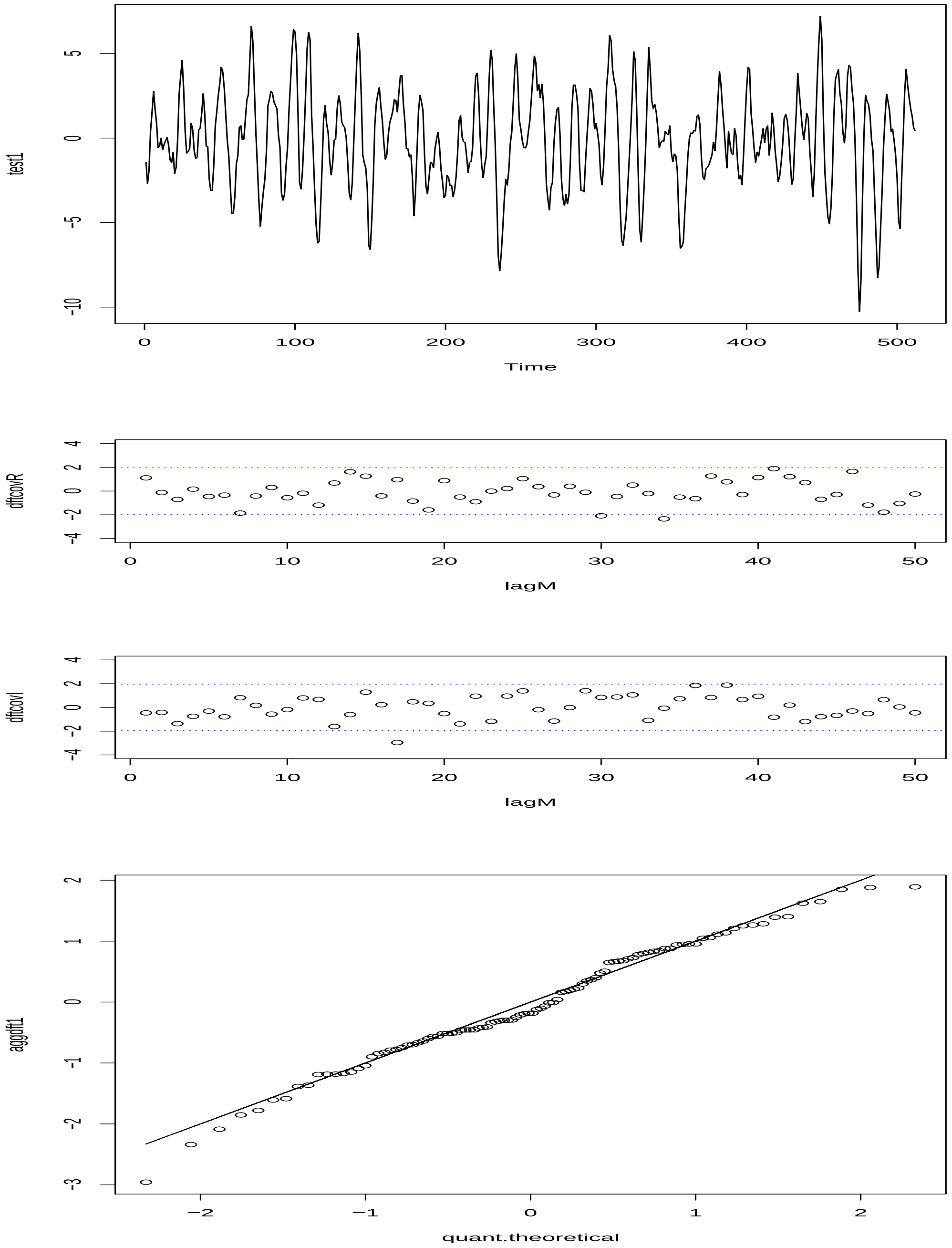


Figure 9.6: Top: Realization. Middle: Real and Imaginary of $\sqrt{n}\widehat{C}_n(r)$ plotted against the 'lag' r. Bottom: QQplot of the real and imaginary $\sqrt{n}\widehat{C}_n(r)$ against a standard normal.

Exercise 9.1 *(a) Simulate an* $AR(2)$ *process and run the above code using the sample size*

- *(i)* $n = 64$ *(however use* `k<-kernel("daniell",3)`*)*
- *(ii)* $n = 128$ *(however use* `k<-kernel("daniell",4)`*)*

Does the 'near decorrelation property' hold when the sample size is very small. Explain your answer by looking at the proof of the lemma.

(b) Simulate a piecewise stationary time series (this is a simple example of a nonstationary time series) by stringing two stationary time series together. One example is

```
ts1 = arima.sim(list(order=c(2,0,0), ar = c(1.5, -0.75)), n=128);
ts2 = arima.sim(list(order=c(1,0,0), ar = c(0.7)), n=128)
test = c(ts1/sd(ts1),ts2/sd(ts2))
```

Make a plot of this time series. Calculate the DFT covariance of this time series, what do you observe in comparison to the stationary case?

9.2.2 Proof 1 of Lemma 9.2.1: By approximating Toeplitz with Circulant matrices

Let $\underline{X}_n' = (X_n, \ldots, X_1)$ and F_n be the Fourier transformation matrix $(F_n)_{s,t} = n^{-1/2}\Omega_n^{(s-1)(t-1)} = n^{-1/2}\exp(\frac{2i\pi(s-1)(t-1)}{n})$ (note that $\Omega_n = \exp(\frac{2\pi}{n})$). It is clear that $F_n\underline{X}_n = (J_n(\omega_0), \ldots, J_n(\omega_{n-1}))'$. We now prove that $F_n\underline{X}_n$ is almost an uncorrelated sequence.

The first proof will be based on approximating the symmetric Toeplitz variance matrix of $\underline{X}_n$ with a circulant matrix, which has well know eigen values and functions. We start by considering the variance of $F_n\underline{X}_n$, $\text{var}(F_n\underline{X}_n) = F_n\Sigma_n\overline{F}_n$, and our aim is to show that it is almost a diagonal. We first recall that if Σ_n were a circulant matrix, then $F_n\underline{X}_n$ would be uncorrelated since F_n is the eigenmatrix of any circulant matrix. This is not the case. However, the upper right hand side and the lower left hand side of Σ_n can *approximated* by circulant matrices - this is the trick in showing the 'near' uncorrelatedness. Studying Σ_n

$$\Sigma_n = \begin{pmatrix} c(0) & c(1) & c(2) & \dots & c(n-1) \\ c(1) & c(0) & c(1) & \dots & c(n-2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c(n-1) & c(n-2) & \vdots & c(1) & c(0) \end{pmatrix}$$

we observe that it can be written as the sum of two circulant matrices, plus some error, that we will bound. That is, we define the two circulant matrices

$$C_{1n} = \begin{pmatrix} c(0) & c(1) & c(2) & \dots & c(n-1) \\ c(n-1) & c(0) & c(1) & \dots & c(n-2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c(1) & c(2) & \vdots & c(n-1) & c(0) \end{pmatrix}$$

and

$$C_{2n} = \begin{pmatrix} 0 & c(n-1) & c(n-2) & \dots & c(1) \\ c(1) & 0 & c(n-1) & \dots & c(2) \\ c(2) & c(1) & 0 & \dots & c(3) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c(n-1) & c(n-2) & \vdots & c(1) & 0 \end{pmatrix}$$

We observe that the upper right hand sides of C_{1n} and Σ_n match and the lower left and sides of C_{2n} and Σ_n match. As the above are circulant their eigenvector matrix is F_n (note that $F_n^{-1} = \overline{F}'_n$). Furthermore, the eigenvalues matrix of C_{n1} is

$$\text{diag}\left(\sum_{j=0}^{n-1} c(j), \sum_{j=0}^{n-1} c(j)\Omega_n^j, \dots, \sum_{j=0}^{n-1} c(j)\Omega_n^{(t-1)j}\right),$$

whereas the eigenvalue matrix of C_{n2} is

$$\begin{aligned}
& \text{diag}\left(\sum_{j=1}^{n-1} c(j), \sum_{j=1}^{n-1} c(n-j)\Omega_n^j, \ldots, \sum_{j=1}^{n-1} c(n-j)\Omega_n^{(t-1)j}\right) \\
= & \text{diag}\left(\sum_{j=1}^{n-1} c(j), \sum_{j=1}^{n-1} c(j)\Omega_n^{-j}, \ldots, \sum_{j=1}^{n-1} c(j)\Omega_n^{-(t-1)j}\right),
\end{aligned}$$

More succinctly, the kth eigenvalues of C_{n1} and C_{n2} are $\lambda_{k1} = \sum_{j=0}^{n-1} c(j)\Omega_n^{j(k-1)}$ and $\lambda_{k2} = \sum_{j=1}^{n-1} c(j)\Omega_n^{-j(k-1)}$. Observe that $\lambda_{k1} + \lambda_{k2} = \sum_{|j|\leq(n-1)} c(j)e^{\frac{2\pi j}{n}} \approx f(\omega_j)$, thus the sum of these eigenvalues approximate the spectral density function.

We now show that under the condition $\sum_r |rc(r)| < \infty$ we have

$$F_n\Sigma_n\overline{F}'_n - F_n\big(C_{n1} + C_{n2}\big)\overline{F}'_n = O\left(\frac{1}{n}\right)\mathcal{I}, \qquad (9.7)$$

where $\mathcal{I}$ is a $n \times n$ matrix of ones. To show the above we consider the differences element by element. Since the upper right hand sides of C_{n1} and Σ_n match and the lower left and sides of C_{n2} and Σ_n match, the above difference is

$$\begin{aligned}
& \left|\Big(F_n\Sigma_n\overline{F}'_n - F_n\big(C_{n1} + C_{n2}\big)\overline{F}_n\Big)_{(s,t)}\right| \\
= & \left|\underline{e}_s\Sigma_n\underline{\overline{e}}'_t - \underline{e}_s C_{n1}\underline{\overline{e}}'_t - \underline{e}_s C_{n2}\underline{\overline{e}}'_t\right| \leq \frac{2}{n}\sum_{r=1}^{n-1}|rc(r)| = O(\frac{1}{n}).
\end{aligned}$$

Thus we have shown (9.7). Therefore, since F_n is the eigenvector matrix of C_{n1} and C_{n2}, altogether we have

$$F_n\big(C_{n1} + C_{n2}\big)\overline{F}_n = \text{diag}\left(f_n(0), f_n(\frac{2\pi}{n}), \ldots, f_n(\frac{2\pi(n-1)}{n})\right),$$

where $f_n(\omega) = \sum_{r=-(n-1)}^{n-1} c(r)\exp(ij\omega)$. Altogether this gives

$$\text{var}(F_n\underline{X}_n) = F_n\Sigma_n\overline{F}_n = \begin{pmatrix} f_n(0) & 0 & \ldots & 0 & 0 \\ 0 & f_n(\frac{2\pi}{n}) & \ldots & 0 & 0 \\ \vdots & \ddots & \ddots & \ldots & \vdots \\ 0 & \ldots & \ldots & 0 & f_n(\frac{2\pi(n-1)}{n})) \end{pmatrix} + O(\frac{1}{n})\begin{pmatrix} 1 & 1 & \ldots & 1 & 1 \\ 1 & 1 & \ldots & 1 & 1 \\ \vdots & \ddots & \ddots & \ldots & \vdots \\ 1 & \ldots & \ldots & 1 & 1 \end{pmatrix}.$$

Finally, we note that since $\sum_r |rc(r)| < \infty$

$$|f_n(\omega) - f(\omega)| \leq \sum_{|r|>n} |c(r)| \leq \frac{1}{n} \sum_{|r|>n} |rc(r)| = O(n^{-1}), \tag{9.8}$$

which gives the required result.

Remark 9.2.1 *Note the eigenvalues of a matrix are often called the spectrum and that above calculation shows that spectrum of* $\mathrm{var}[\underline{X}_n]$ *is close to* $f(\omega_n)$*, which may be a one reason why* $f(\omega)$ *is called the spectral density (the reason for density probably comes from the fact that* f *is positive).*

These ideas can also be used for inverting Toeplitz matrices (see Chen et al. (2006)).

9.2.3 Proof 2 of Lemma 9.2.1: Using brute force

A more hands on proof is to just calculate $\mathrm{cov}(J_n(\frac{2\pi k_1}{n}), J_n(\frac{2\pi k_2}{n}))$. The important aspect of this proof is that if we can isolate the exponentials than we can use (9.6). It is this that gives rise to the near uncorrelatedness property. Remember also that $\exp(i\frac{2\pi}{n}jk) = \exp(ij\omega_k) = \exp(ik\omega_j)$, hence we can interchange between the two notations.

We note that $\mathrm{cov}(A, B) = \mathrm{E}(A\overline{B}) - \mathrm{E}(A)\mathrm{E}(\overline{B})$, thus we have

$$\mathrm{cov}\left(J_n(\frac{2\pi k_1}{n}), J_n(\frac{2\pi k_2}{n})\right) = \frac{1}{n}\sum_{t,\tau=1}^{n} \mathrm{cov}(X_t, X_\tau)\exp\left(i(tk_1 - \tau k_2)\frac{2\pi}{n}\right)$$

Now change variables with $r = t - \tau$, this gives (for $0 \leq k_1, k_2 < n$)

$$\begin{aligned}
&\mathrm{cov}\left(J_n(\frac{2\pi k_1}{n}), J_n(\frac{2\pi k_2}{n})\right) \\
= \;& \frac{1}{n}\sum_{r=-(n-1)}^{n-1} c(r)\exp\left(-ir\frac{2\pi k_2}{n}\right)\sum_{t=1}^{n-|r|}\exp\left(\frac{2\pi it(k_1-k_2)}{n})\right) \\
= \;& \sum_{r=-(n-1)}^{n-1} c(r)\exp\left(ir\frac{2\pi k_2}{n}\right)\underbrace{\frac{1}{n}\sum_{t=1}^{n}\exp\left(\frac{2\pi it(k_1-k_2)}{n}\right)}_{\delta_{k_1}(k_2)} + R_n,
\end{aligned}$$

where

$$R_n = \frac{1}{n}\sum_{r=-(n-1)}^{n-1} c(r)\exp\left(-ir\frac{2\pi k_2}{n}\right)\sum_{t=n-|r|+1}^{n}\exp\left(\frac{2\pi it(k_1-k_2)}{n})\right)$$

Thus $|R_n| \leq \frac{1}{n}\sum_{|r|\leq n}|rc(r)| = O(n^{-1})$ Finally by using (9.8) we obtain the result. $\square$

Exercise 9.2 *The the above proof (in Section 9.2.3) uses that $\sum_r |rc(r)| < \infty$. What bounds do we obtain if we relax this assumption to $\sum_r |c(r)| < \infty$?*

9.2.4 Heuristics

In this section we summarize some spectral properties. We do this by considering the DFT of the data $\{J_n(\omega_k)\}_{k=1}^n$. It is worth noting that to calculate $\{J_n(\omega_k)\}_{k=1}^n$ is computationally very fast and requires only $O(n\log n)$ computing operations (see Section A.5, where the Fast Fourier Transform is described).

The spectral (Cramer's) representation theorem

We observe that for any sequence $\{X_t\}_{t=1}^n$ that it can be written as the inverse transform for $1 \leq t \leq n$

$$X_t = \frac{1}{\sqrt{n}}\sum_{k=1}^{n} J_n(\omega_k)\exp(-it\omega_k), \tag{9.9}$$

which can be written as an integral

$$X_t = \sum_{k=2}^{n}\exp(-it\omega_k)\left[Z_n(\omega_k) - Z_n(\omega_{k-1})\right] = \int_0^{2\pi}\exp(-it\omega)dZ_n(\omega), \tag{9.10}$$

where $Z_n(\omega) = \frac{1}{\sqrt{n}}\sum_{k=1}^{\lfloor \frac{\omega}{2\pi}n\rfloor} J_n(\omega_k)$.

The second order stationary property of X_t means that the DFT $J_n(\omega_k)$ is close to an uncorrelated sequence or equivalently the process $Z_n(\omega)$ has near 'orthogonal' increments, meaning that for any two non-intersecting intervals $[\omega_1,\omega_2]$ and $[\omega_3,\omega_4]$ that $Z_n(\omega_2) - Z_n(\omega_1)$ and $Z_n(\omega_4) - Z_n(\omega_3)$. The spectral representation theorem generalizes this result, it states that for any second order stationary time series $\{X_t\}$ there exists an a process $\{Z(\omega); \omega \in [0,2\pi]\}$ where for all $t \in \mathbb{Z}$

$$X_t = \int_0^{2\pi}\exp(-it\omega)dZ(\omega) \tag{9.11}$$

and $Z(\omega)$ has orthogonal increments, meaning that for any two non-intersecting intervals $[\omega_1,\omega_2]$ and $[\omega_3,\omega_4]$ $\mathrm{E}[Z(\omega_2) - Z(\omega_1)][Z(\omega_2) - Z(\omega_1)] = 0$.

We now explore the relationship between the DFT with the orthogonal increment process. Using (9.11) we see that

$$
\begin{aligned}
J_n(\omega_k) &= \frac{1}{\sqrt{2\pi n}} \sum_{t=1}^{n} X_t \exp(it\omega_k) = \frac{1}{\sqrt{2\pi n}} \int_0^{2\pi} \left(\sum_{t=1}^{n} \exp(it[\omega_k - \omega]) \right) dZ(\omega) \\
&= \frac{1}{\sqrt{2\pi n}} \int_0^{2\pi} \left(e^{i(n+1)(\omega_k - \omega_0)/2} D_{n/2}(\omega_k - \omega) \right) dZ(\omega),
\end{aligned}
$$

where $D_{n/2}(x) = \sin[((n+1)/2)x]/\sin(x/2)$ is the Dirichlet kernel (see Priestley (1983), page 419). We recall that the Dirichlet kernel limits to the Dirac-delta function, therefore very crudely speaking we observe that the DFT is an approximation of the orthogonal increment localized about ω_k (though mathematically this is not strictly correct).

Bochner's theorem

This is a closely related result that is stated in terms of the so called spectral distribution. First the heuristics. We see that from Lemma 9.2.1 that the DFT $J_n(\omega_k)$, is close to uncorrelated. Using this and inverse Fourier transforms we see that for $1 \leq t, \tau \leq n$ we have

$$
\begin{aligned}
c(t-\tau) = \mathrm{cov}(X_t, X_\tau) &= \frac{1}{n} \sum_{k_1=1}^{n} \sum_{k_2=1}^{n} \mathrm{cov}\left(J_n(\omega_{k_1}), J_n(\omega_{k_2})\right) \exp(-it\omega_{k_1} + i\tau\omega_{k_2}) \\
&\approx \frac{1}{n} \sum_{k=1}^{n} \mathrm{var}(J_n(\omega_k)) \exp(-i(t-\tau)\omega_k). \qquad (9.12)
\end{aligned}
$$

Let $F_n(\omega) = \frac{1}{n}\sum_{k=1}^{\lfloor \frac{\omega}{2\pi} n \rfloor} \mathrm{var}[J_n(\omega_k)]$, then the above can be written as

$$
c(t-\tau) \approx \int_0^{2\pi} \exp(-i(t-\tau)\omega) dF_n(\omega),
$$

where we observe that $F_n(\omega)$ is a positive function which in non-decreasing over ω. Bochner's theorem is an extension of this is states that for any autocovariance function $\{c(k)\}$ we have the representation

$$
c(t-\tau) = \int_0^{2\pi} \exp(-i(t-\tau)\omega) f(\omega) d\omega = \int_0^{2\pi} \exp(-i(t-\tau)\omega) dF(\omega).
$$

where $F(\omega)$ is a positive non-decreasing bounded function. Moreover, $F(\omega) = \mathrm{E}(|Z(\omega)|^2)$. We note that if the spectral density function exists (which is only true if $\sum_r |c(r)|^2 < \infty$) then $F(\omega) =$

$\int_0^\omega f(\lambda)d\lambda$.

Remark 9.2.2 *The above results hold for both linear and nonlinear time series, however, in the case that X_t has a linear representation*

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j \varepsilon_{t-j},$$

then X_t has the particular form

$$X_t = \int A(\omega)\exp(-ik\omega)dZ(\omega), \tag{9.13}$$

where $A(\omega) = \sum_{j=-\infty}^{\infty} \psi_j \exp(ij\omega)$ and $Z(\omega)$ is an orthogonal increment process, but in addition $\mathrm{E}(|dZ(\omega)|^2) = d\omega$ ie. the variance of increments do not vary over frequency (as this varying has been absorbed by $A(\omega)$, since $F(\omega) = |A(\omega)|^2$).

We mention that a more detailed discussion on spectral analysis in time series is give in Priestley (1983), Chapters 4 and 6, Brockwell and Davis (1998), Chapters 4 and 10, Fuller (1995), Chapter 3, Shumway and Stoffer (2006), Chapter 4. In many of these references they also discuss tests for periodicity etc (see also Quinn and Hannan (2001) for estimation of frequencies etc.).

9.3 The spectral density and spectral distribution

9.3.1 The spectral density and some of its properties

Finally, having made ourselves familiar with the DFT and the spectral density function we can prove Theorem 2.4.2, which relates the autocovariance with the positiveness of its Fourier transform. In the following lemma we consider absolutely summable autocovariances, in a later theorem (called Bochner's theorem) we show that any valid autocovariance has this representation.

Theorem 9.3.1 (Positiveness of the spectral density) *Suppose the coefficients $\{c(k)\}$ are absolutely summable (that is $\sum_k |c(k)| < \infty$). Then the sequence $\{c(k)\}$ is positive semi-definite if an only if the function $f(\omega)$, where*

$$f(\omega) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} c(k)\exp(ik\omega)$$

is nonnegative. Moreover

$$c(k) = \int_0^{2\pi} \exp(-ik\omega) f(\omega) d\omega. \tag{9.14}$$

It is worth noting that f is called the spectral density corresponding to the covariances $\{c(k)\}$.

PROOF. We first show that if $\{c(k)\}$ is a non-negative definite sequence, then $f(\omega)$ is a nonnegative function. We recall that since $\{c(k)\}$ is non-negative then for any sequence $\underline{x} = (x_1, \ldots, x_N)$ (real or complex) we have $\sum_{s,t=1}^{n} x_s c(s-t) \bar{x}_s \geq 0$ (where $\bar{x}_s$ is the complex conjugate of x_s). Now we consider the above for the particular case $\underline{x} = (\exp(i\omega), \ldots, \exp(in\omega))$. Define the function

$$f_n(\omega) = \frac{1}{2\pi n} \sum_{s,t=1}^{n} \exp(is\omega) c(s-t) \exp(-it\omega).$$

Thus by definition $f_n(\omega) \geq 0$. We note that $f_n(\omega)$ can be rewritten as

$$f_n(\omega) = \frac{1}{2\pi} \sum_{k=-(n-1)}^{(n-1)} \left(\frac{n-|k|}{n} \right) c(k) \exp(ik\omega).$$

Comparing $f(\omega) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} c(k) \exp(ik\omega)$ with $f_n(\omega)$ we see that

$$\begin{aligned} \big| f(\omega) - f_n(\omega) &\leq \frac{1}{2\pi} \big| \sum_{|k| \geq n} c(k) \exp(ik\omega) \big| + \frac{1}{2\pi} \big| \sum_{k=-(n-1)}^{(n-1)} \frac{|k|}{n} c(k) \exp(ik\omega) \big| \\ &:= I_n + II_n. \end{aligned}$$

Since $\sum_{k=-\infty}^{\infty} |c(k)| < \infty$ it is clear that $I_n \to 0$ as $n \to \infty$. Using Lemma A.1.1 we have $II_n \to 0$ as $n \to \infty$. Altogether the above implies

$$\big| f(\omega) - f_n(\omega) \big| \to 0 \quad \text{as } n \to \infty. \tag{9.15}$$

Now it is clear that since for all n, $f_n(\omega)$ are nonnegative functions, the limit f must be nonnegative (if we suppose the contrary, then there must exist a sequence of functions $\{f_{n_k}(\omega)\}$ which are not necessarily nonnegative, which is not true). Therefore we have shown that if $\{c(k)\}$ is a nonnegative definite sequence, then $f(\omega)$ is a nonnegative function.

We now show the converse, that is the Fourier coefficients of any non-negative ℓ_2 function $f(\omega) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} c(k) \exp(ik\omega)$, is a positive semi-definite sequence. Writing $c(k) = \int_0^{2\pi} f(\omega) \exp(ik\omega) d\omega$

we substitute this into Definition 2.4.1 to give

$$\sum_{s,t=1}^{n} x_s c(s-t)\bar{x}_s = \int_0^{2\pi} f(\omega)\{\sum_{s,t=1}^{n} x_s \exp(i(s-t)\omega)\bar{x}_s\}d\omega = \int_0^{2\pi} f(\omega)\left|\sum_{s=1}^{n} x_s \exp(is\omega)\right|^2 d\omega \geq 0.$$

Hence we obtain the desired result. □

The above theorem is very useful. It basically gives a simple way to check whether a sequence $\{c(k)\}$ is non-negative definite or not (hence whether it is a covariance function - recall Theorem 2.4.1). See Brockwell and Davis (1998), Corollary 4.3.2 or Fuller (1995), Theorem 3.1.9, for alternative explanations.

Example 9.3.1 *Consider the empirical covariances (here we gives an alternative proof to Remark 7.2.1) defined in Chapter 7*

$$\hat{c}_n(k) = \begin{cases} \frac{1}{n}\sum_{t=1}^{n-|k|} X_t X_{t+|k|} & |k| \leq n-1 \\ 0 & \textit{otherwise} \end{cases},$$

we give an alternative proof to Lemma 7.2.1 to show that $\{\hat{c}_n(k)\}$ is non-negative definite sequence. To show that the sequence we take the Fourier transform of $\hat{c}_n(k)$ and use Theorem 9.3.1. The Fourier transform of $\{\hat{c}_n(k)\}$ is

$$\sum_{k=-(n-1)}^{(n-1)} \exp(ik\omega)\hat{c}_n(k) = \sum_{k=-(n-1)}^{(n-1)} \exp(ik\omega)\frac{1}{n}\sum_{t=1}^{n-|k|} X_t X_{t+|k|} = \frac{1}{n}\big|\sum_{t=1}^{n} X_t \exp(it\omega)\big| \geq 0.$$

Since the above is non-negative, this means that $\{\hat{c}_n(k)\}$ is a non-negative definite sequence.

We now state a useful result which relates the largest and smallest eigenvalue of the variance of a stationary process to the smallest and largest values of the spectral density (we recall we used this in Lemma 6.5.1).

Lemma 9.3.1 *Suppose that $\{X_k\}$ is a stationary process with covariance function $\{c(k)\}$ and spectral density $f(\omega)$. Let $\Sigma_n = \mathrm{var}(\underline{X}_n)$, where $\underline{X}_n = (X_1, \ldots, X_n)$. Suppose $\inf_\omega f(\omega) \geq m > 0$ and $\sup_\omega f(\omega) \leq M < \infty$ Then for all n we have*

$$\lambda_{\min}(\Sigma_n) \geq \inf_\omega f(\omega) \quad \textit{and} \quad \lambda_{\max}(\Sigma_n) \leq \sup_\omega f(\omega).$$

PROOF. Let $\underline{e}_1$ be the eigenvector with smallest eigenvalue λ_1 corresponding to Σ_n. Then using $c(s-t) = \int f(\omega)\exp(i(s-t)\omega)d\omega$ we have

$$\begin{aligned}
\lambda_{\min}(\Sigma_n) &= \underline{e}_1'\Sigma_n\underline{e}_1 = \sum_{s,t=1}^{n} \bar{e}_{s,1}c(s-t)e_{t,1} = \int f(\omega)\sum_{s,t=1}^{n}\bar{e}_{s,1}\exp(i(s-t)\omega)e_{t,1}d\omega = \\
&= \int_0^{2\pi} f(\omega)\left|\sum_{s=1}^{n} e_{s,1}\exp(is\omega)\right|^2 d\omega \geq \inf_{\omega} f(\omega)\int_0^{2\pi}\left|\sum_{s=1}^{n} e_{s,1}\exp(is\omega)\right|^2 d\omega = \inf_{\omega} f(\omega),
\end{aligned}$$

since by definition $\int |\sum_{s=1}^{n} e_{s,1}\exp(is\omega)|^2 d\omega = \sum_{s=1}^{n}|e_{s,1}|^2 = 1$ (using Parseval's identity). Using a similar method we can show that $\lambda_{\max}(\Sigma_n) \leq \sup f(\omega)$. □

We now state a version of the above result which requires weaker conditions on the autocovariance function (only that they decay to zero).

Lemma 9.3.2 *Suppose the covariance $\{c(k)\}$ decays to zero as $k \to \infty$, then for all n, $\Sigma_n = \mathrm{var}(\underline{X}_n)$ is a non-singular matrix (Note we do not require the stronger condition the covariances are absolutely summable).*

PROOF. See Brockwell and Davis (1998), Proposition 5.1.1. □

9.3.2 The spectral distribution and Bochner's (Hergoltz) theorem

Theorem 9.3.1 hinges on the result that $f_n(\omega) = \sum_{r=-(n-1)}^{(n-1)}(1-|r|/n)e^{ir\omega}$ has a well defined pointwise limit as $n \to \infty$, this only holds when the sequence $\{c(k)\}$ is absolutely summable. Of course this may not always be the case. An extreme example is the time series $X_t = Z$. Clearly this is a stationary time series and its covariance is $c(k) = \mathrm{var}(Z) = 1$ for all k. In this case the autocovariance sequence $\{c(k) = 1; k \in \mathbb{Z}\}$, is not absolutely summable, hence the representation of the covariance in Theorem 9.3.1 does not apply. The reason is because the Fourier transform of the infinite sequence $\{c(k) = 1; k \in \mathbb{Z}\}$ is not well defined (clearly $\{c(k) = 1\}_k$ does not belong to ℓ_1).

However, we now show that Theorem 9.3.1 can be generalised to include <u>all</u> non-negative definite sequences and stationary processes, by considering the spectral distribution rather than the spectral density.

Theorem 9.3.2 *A function $\{c(k)\}$ is non-negative definite sequence if and only if*

$$c(k) = \int_0^{2\pi} \exp(-ik\omega) dF(\omega), \tag{9.16}$$

where $F(\omega)$ is a right-continuous (this means that $F(x+h) \to F(x)$ as $0 < h \to 0$), non-decreasing, non-negative, bounded function on $[-\pi, \pi]$ (hence it has all the properties of a distribution and it can be consider as a distribution - it is usually called the spectral distribution). This representation is unique.

This is a very constructive result. It shows that the Fourier coefficients of any distribution function form a non-negative definite sequence, and thus, if $c(k) = c(-k)$ (hence is symmetric) correspond to the covarance function of a random process. In Figure 9.7 we give two distribution functions. the top plot is continuous and smooth, therefore it's derivative will exist, be positive and belong to ℓ_2. So it is clear that its Fourier coefficients form a non-negative definite sequence. The interesting aspect of Thereom 9.3.2 is that the Fourier coefficients corresponding to the distribution function in the second plot also forms a non-negative definite sequence even though the derivative of this distribution function does not exist. However, this sequence will not belong to ℓ_2 (ie. the correlations function will not decay to zero as the lag grows).

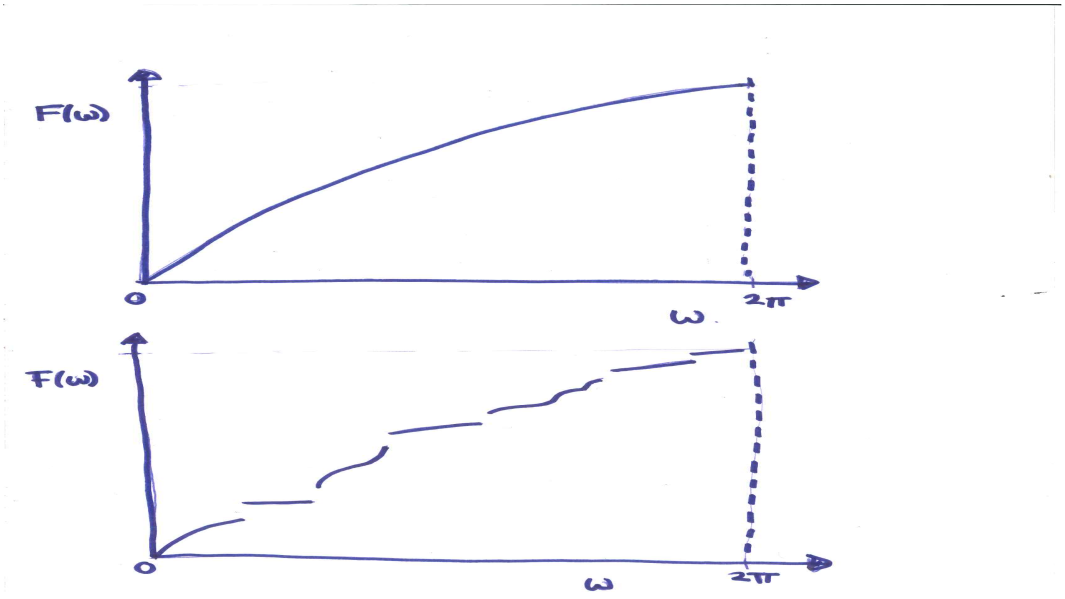


Figure 9.7: Both plots are of non-decreasing functions, hence are valid distribution functions. The top plot is continuous and smooth, thus its derivative (the spectral density function) exists. Whereas the bottom plot is not (spectral density does not exist).

PROOF of Theorem 9.3.2. We first show that if $\{c(k)\}$ is non-negative definite sequence, then we can write $c(k) = \int_0^{2\pi} \exp(ik\omega)dF(\omega)$, where $F(\omega)$ is a distribution function.

To prove the result we adapt some of the ideas used to prove Theorem 9.3.1. As in the proof of Theorem 9.3.1 define the (nonnegative) function

$$f_n(\omega) = \mathrm{var}[J_n(\omega)] = \frac{1}{2\pi n}\sum_{s,t=1}^{n} \exp(is\omega)c(s-t)\exp(-it\omega) = \frac{1}{2\pi}\sum_{k=-(n-1)}^{(n-1)} \left(\frac{n-|k|}{n}\right) c(k)\exp(ik\omega).$$

If $\{c(k)\}$ is not absolutely summable, the limit of $f_n(\omega)$ is no longer well defined. Instead we consider its integral, which will always be a distribution function (in the sense that it is nondecreasing and bounded). Let us define the function $F_n(\omega)$ whose derivative is $f_n(\omega)$, that is

$$F_n(\omega) = \int_0^{\omega} f_n(\lambda)d\lambda = \frac{\omega}{2\pi}c(0) + \frac{2}{2\pi}\sum_{r=1}^{n-1}\left(1-\frac{r}{n}\right)c(r)\frac{\sin(\omega r)}{r} \qquad 0 \leq \lambda \leq 2\pi.$$

Since $f_n(\lambda)$ is nonnegative, $F_n(\omega)$ is a nondecreasing function. Furthermore it is bounded since

$$F_n(2\pi) = \int_0^{2\pi} f_n(\lambda)d\lambda = c(0).$$

Hence F_n satisfies all properties of a distribution and can be treated as a distribution function. This means that we can apply Helly's theorem to the sequence $\{F_n\}_n$. We first recall that if $\{x_n\}$ are real numbers defined on a compact set $X \subset \mathbb{R}$, then there exists a subsequence $\{x_{n_m}\}_m$ which has a limit in the set X (this is called the Bolzano-Weierstrass theorem). An analogous result exists for measures, this is called Helly's theorem (see Ash (1972), page 329). It states that for any sequence of distributions $\{G_n\}$ defined on $[0, 2\pi]$, were $G_n(0) = 0$ and $\sup_n G_n(2\pi) < M < \infty$, there exists a subsequence $\{n_m\}_m$ where $G_{n_m}(x) \to G(x)$ as $m \to \infty$ for each $x \in [0, 2\pi]$ at which G is continuous. Furthermore, since $G_{n_m}(x) \to G(x)$ (pointwise as $m \to \infty$), this implies (see Varadhan, Theorem 4.1 for equivalent forms of convergence) that for any bounded sequence h we have that

$$\int h(x)dG_{n_m}(x) \to \int h(x)dG(x) \qquad \text{as } m \to \infty.$$

We now apply this result to $\{F_n\}_n$. Using Helly's theorem there exists a subsequence of distributions

$\{F_{n_m}\}_m$ which has a pointwise limit F. Thus for any bounded function h we have

$$\int h(x)dF_{n_m}(x) \to \int h(x)dF(x) \qquad \text{as } m \to \infty. \tag{9.17}$$

We focus on the function $h(x) = \exp(-ik\omega)$. It is clear that for every k and n we have

$$\int_0^{2\pi} \exp(-ik\omega)dF_n(\omega) = \int_0^{2\pi} \exp(ik\omega)f_n(\omega)d\omega = \begin{cases} (1 - \frac{|k|}{n})c(k) & |k| \leq n \\ 0 & |k| \geq n \end{cases} \tag{9.18}$$

Define the sequence

$$d_{n,k} = \int_0^{2\pi} \exp(ik\omega)dF_n(\omega) = \left(1 - \frac{|k|}{n}\right)c(k).$$

We observe that for fixed k, $\{d_{n,k}; n \in \mathbb{Z}\}$ is a Cauchy sequence, where

$$d_{n,k} \to d_k = c(k) \tag{9.19}$$

as $n \to \infty$.

Now we use (9.17) and focus on the convergent subsequence $\{n_m\}_m$. By using (9.17) we have

$$d_{n_m,k} = \int \exp(-ikx)dF_{n_m}(x) \to \int \exp(-ikx)dF(x) \qquad \text{as } m \to \infty$$

and by (9.19) $d_{n_m,k} \to c(k)$ as $m \to \infty$. Thus

$$c(k) = \int \exp(-ikx)dF(x).$$

This gives the first part of the assertion.

To show the converse, that is $\{c(k)\}$ is a non-negative definite sequence when $c(k)$ is defined as $c(k) = \int \exp(ik\omega)dF(\omega)$, we use the same method given in the proof of Theorem 9.3.1, that is

$$\begin{aligned} \sum_{s,t=1}^{n} x_s c(s-t)\bar{x}_s &= \int_0^{2\pi} \{\sum_{s,t=1}^{n} x_s \exp(-i(s-t)\omega)\bar{x}_s\}dF(\omega) \\ &= \int_0^{2\pi} \left|\sum_{s=1}^{n} x_s \exp(-is\omega)\right|^2 dF(\omega) \geq 0, \end{aligned}$$

since $F(\omega)$ is a distribution.

Finally, if $\{c(k)\}$ were absolutely summable, then we can use Theorem 9.3.1 to write $c(k) = \int_0^{2\pi} \exp(-ik\omega)dF(\omega)$, where $F(\omega) = \int_0^{\omega} f(\lambda)d\lambda$ and $f(\lambda) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} c(k)\exp(ik\omega)$. By using Theorem 9.3.1 we know that $f(\lambda)$ is nonnegative, hence $F(\omega)$ is a distribution, and we have the result. □

Example 9.3.2 *Using the above we can construct the spectral distribution for the (rather silly) time series $X_t = Z$. Let $F(\omega) = 0$ for $\omega < 0$ and $F(\omega) = \mathrm{var}(Z)$ for $\omega \geq 0$ (hence F is the step function). Then we have*

$$\mathrm{cov}(X_t, X_{t+k}) = \mathrm{var}(Z) = \int \exp(-ik\omega)dF(\omega).$$

Example 9.3.3 *Consider the second order stationary time series*

$$X_t = U_1\cos(\lambda t) + U_2\sin(\lambda t),$$

where U_1 and U_2 are iid random variables with mean zero and variance σ^2 and λ the frequency. It can be shown that

$$\mathrm{cov}(X_t, X_{t+k}) = \frac{\sigma^2}{2}\left[\exp(i\lambda k) + \exp(-i\lambda k)\right].$$

Observe that this covariance does not decay with the lag k. Then

$$\mathrm{cov}(X_t, X_{t+k}) = \mathrm{var}(Z) = \int_0^{2\pi} \exp(-ik\omega)dF(\omega).$$

where

$$F(\omega) = \begin{cases} 0 & \omega < -\lambda \\ \sigma^2/2 & -\lambda \leq \omega < \lambda \\ \sigma^2 & \lambda \geq \omega. \end{cases}$$

9.4 The spectral representation theorem

We now state the spectral representation theorem and give a rough outline of the proof.

Theorem 9.4.1 *If $\{X_t\}$ is a second order stationary time series with mean zero, and spectral distribution $F(\omega)$, and the spectral distribution function is $F(\omega)$, then there exists a right continuous,* *orthogonal increment process* $\{Z(\omega)\}$ *(that is* $\mathrm{E}[(Z(\omega_1)-Z(\omega_2)\overline{(Z(\omega_3)-Z(\omega_4))}] = 0$*, when the intervals* $[\omega_1,\omega_2]$ *and* $[\omega_3,\omega_4]$ *do not overlap) such that*

$$X_t = \int_0^{2\pi} \exp(-it\omega)dZ(\omega), \tag{9.20}$$

where for $\omega_1 \geq \omega_2$, $\mathrm{E}|Z(\omega_1)-Z(\omega_2)|^2 = F(\omega_1)-F(\omega_2)$ *(noting that* $F(0)=0$*). (One example of a right continuous, orthogonal increment process is Brownian motion, though this is just one example, and usually* $Z(\omega)$ *will be far more general than Brownian motion).*

Heuristically we see that (9.20) is the decomposition of X_t in terms of frequencies, whose amplitudes are orthogonal. In other words X_t is decomposed in terms of frequencies $\exp(it\omega)$ which have the orthogonal amplitudes $dZ(\omega) \approx (Z(\omega+\delta)-Z(\omega))$.

Remark 9.4.1 *Note that so far we have not defined the integral on the right hand side of (9.20). It is known as a stochastic integral. Unlike many deterministic functions (functions whose derivative exists), one cannot really suppose* $dZ(\omega) \approx Z'(\omega)d\omega$*, because usually a typical realisation of* $Z(\omega)$ *will not be smooth enough to differentiate. For example, it is well known that Brownian is quite 'rough', that is a typical realisation of Brownian motion satisfies* $|B(t_1,\bar{\omega})-B(t_2,\bar{\omega})| \leq K(\bar{\omega})|t_1-t_t|^{\gamma}$*, where* $\bar{\omega}$ *is a realisation and* $\gamma \leq 1/2$*, but in general* γ *will not be larger. The integral* $\int g(\omega)dZ(\omega)$ *is well defined if it is defined as the limit (in the mean squared sense) of discrete sums. More precisely, let* $Z_n(\omega) = \sum_{k=1}^{n} Z(\omega_k)I_{\omega_{n_k-1},\omega_{n_k}}(\omega) = \sum_{k=1}^{\lfloor n\omega/2\pi\rfloor}[Z(\omega_k)-Z(\omega_{k-1})]$*, then*

$$\int g(\omega)dZ_n(\omega) = \sum_{k=1}^{n} g(\omega_k)\{Z(\omega_k)-Z(\omega_{k-1})\}.$$

The limit of $\int g(\omega)dZ_n(\omega)$ *as* $n\to\infty$ *is* $\int g(\omega)dZ(\omega)$ *(in the mean squared sense, that is* $\mathrm{E}[\int g(\omega)dZ(\omega) - \int g(\omega)dZ_n(\omega)]^2$*). Compare this with our heuristics in equation (9.10).*

For a more precise explanation, see Parzen (1959), Priestley (1983), Sections 3.6.3 and Section 4.11, page 254, and Brockwell and Davis (1998), Section 4.7. For a very good review of elementary stochastic calculus see Mikosch (1999).

A very elegant explanation on the different proofs of the spectral representation theorem is given in Priestley (1983), Section 4.11. We now give a rough outline of the proof using the functional theory approach.

Rough PROOF of the Spectral Representation Theorem To prove the result we first define two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, where $\mathcal{H}_1$ one contains deterministic functions and $\mathcal{H}_2$ contains random variables.

First we define the space

$$\mathcal{H}_1 = \overline{\text{sp}}\{e^{it\omega}; t \in \mathbb{Z}\}$$

with inner-product

$$\langle f, g \rangle = \int_0^{2\pi} f(x)\overline{g(x)}dF(x) \tag{9.21}$$

(and of course distance $\langle f-g, f-g \rangle = \int_0^{2\pi} |f(x)-g(x)|^2 dF(x)$) it is clear that this inner product is well defined because $\langle f, f \rangle \geq 0$ (since F is a measure). It can be shown (see Brockwell and Davis (1998), page 144) that $\mathcal{H}_1 = \left\{g; \int_0^{2\pi} |g(\omega)|^2 dF(\omega) < \infty\right\}$ [1]. We also define the space

$$\mathcal{H}_2 = \overline{\text{sp}}\{X_t; t \in \mathbb{Z}\}$$

with inner-product $\text{cov}(X,Y) = \text{E}[X\overline{Y}] - \text{E}[X]\text{E}[\overline{Y}]$.

Now let us define the linear mapping $T : \mathcal{H}_1 \to \mathcal{H}_2$

$$T(\sum_{j=1}^{n} a_j \exp(ik\omega)) = \sum_{j=1}^{n} a_j X_k, \tag{9.22}$$

for any n (it is necessary to show that this can be extended to infinite n, but we won't do so here). We will shown that T defines an isomorphism (ie. it is a one-to-one linear mapping that preserves norm). To show that it is a one-to-one mapping see Brockwell and Davis (1998), Section 4.7. It is clear that it is linear, there all that remains is to show that the mapping preserves inner-product. Suppose $f, g \in \mathcal{H}_1$, then there exists coefficients $\{f_j\}$ and $\{g_j\}$ such that $f(x) = \sum_j f_j \exp(ij\omega)$ and $g(x) = \sum_j g_j \exp(ij\omega)$. Hence by definition of T in (9.22) we have

$$\langle Tf, Tg \rangle \quad = \quad \text{cov}(\sum_j f_j X_j, \sum_j g_j X_j) = \sum_{j_1,j_2} f_{j_1}\overline{g_{j_2}}\text{cov}(X_{j_1}, X_{j_2}) \tag{9.23}$$

[1] Roughly speaking it is because all continuous functions on $[0, 2\pi]$ are dense in $L_2([0, 2\pi], \mathcal{B}, F)$ (using the metric $\|f - g\| = \langle f-g, f-g \rangle$ and the limit of Cauchy sequences). Since all continuous function can be written as linear combinations of the Fourier basis, this gives the result.

Now by using Bochner's theorem (see Theorem 9.3.2) we have

$$\langle Tf, Tg\rangle \;=\; \int_0^{2\pi} \big(\sum_{j_1,j_2} f_{j_1}\overline{g_{j_2}}\exp(i(j_1-j_2)\omega)\big)dF(\omega) = \int_0^{2\pi} f(x)\overline{g(x)}dF(x) = \langle f, g\rangle. \tag{9.24}$$

Hence $< Tf, Tg >=< f, g >$, so the inner product is preserved (hence T is an isometry).

Altogether this means that T defines an isomorphism betwen $\mathcal{H}_1$ and $\mathcal{H}_2$. Therefore all functions which are in $\mathcal{H}_1$ have a corresponding random variable in $\mathcal{H}_2$ which has similar properties.

For all $\omega \in [0, 2\pi]$, it is clear that the identity functions $I_{[0,\omega]}(x) \in \mathcal{H}_1$. Thus we define the random function $\{Z(\omega); 0 \le \omega \le 2\pi\}$, where $T(I_{[0,\omega]}(\cdot)) = Z(\omega) \in \mathcal{H}_2$ (since T is an isomorphism). Since that mapping T is linear we observe that

$$T(I_{[\omega_1,\omega_2]}) = T(I_{[0,\omega_1]} - I_{[0,\omega_2]}) = T(I_{[0,\omega_1]}) - T(I_{[0,\omega_2]}) = Z(\omega_1) - Z(\omega_2).$$

Moreover, since T preserves the norm for any non-intersecting intervals $[\omega_1, \omega_2]$ and $[\omega_3, \omega_4]$ we have

$$\begin{aligned}\mathrm{cov}\left((Z(\omega_1) - Z(\omega_2), (Z(\omega_3) - Z(\omega_4))\right) &= \langle T(I_{[\omega_1,\omega_2]}), T(I_{[\omega_3,\omega_4]})\rangle = \langle I_{[\omega_1,\omega_2]}, I_{[\omega_3,\omega_4]}\rangle \\ &= \int I_{[\omega_1,\omega_2]}(\omega) I_{[\omega_3,\omega_4]}(\omega)dF(\omega) = 0.\end{aligned}$$

Therefore by construction $\{Z(\omega); 0 \le \omega \le 2\pi\}$ is an orthogonal increment process, where

$$\begin{aligned}\mathrm{E}|Z(\omega_2) - Z(\omega_1)|^2 &= < T(I_{[\omega_1,\omega_2]}), T(I_{[\omega_1,\omega_2]}) >=< I_{[\omega_1,\omega_2]}, I_{[\omega_1,\omega_2]} > \\ &= \int_0^{2\pi} I_{[\omega_1,\omega_2]} dF(\omega) = \int_{\omega_1}^{\omega_2} dF(\omega) = F(\omega_2) - F(\omega_1).\end{aligned}$$

Having defined the two spaces which are isomorphic and the random function $\{Z(\omega); 0 \le \omega \le 2\pi\}$ and function $I_{[0,\omega]}(x)$ which have orthogonal increments, we can now prove the result. Since $dI_{[0,\omega]}(s) = \delta_\omega(s)ds$, where $\delta_\omega(s)$ is the dirac delta function, any function $g \in L_2[0, 2\pi]$ can be represented as

$$g(\omega) = \int_0^{2\pi} g(s)dI_{[\omega,2\pi]}(s).$$

Thus for $g(\omega) = \exp(-it\omega)$ we have

$$\exp(-it\omega) = \int_0^{2\pi} \exp(-its) dI_{[\omega,2\pi]}(s).$$

Therefore

$$\begin{aligned} T(\exp(-it\omega)) &= T\left(\int_0^{2\pi} \exp(-its) dI_{[\omega,2\pi]}(s)\right) = \int_0^{2\pi} \exp(-its) T[dI_{[\omega,2\pi]}(s)] \\ &= \int_0^{2\pi} \exp(-its) dT[I_{[\omega,2\pi]}(s)], \end{aligned}$$

where the mapping goes inside the integral due to the linearity of the isomorphism. Using that $I_{[\omega,2\pi]}(s) = I_{[0,s]}(\omega)$ we have

$$T(\exp(-it\omega)) \quad = \quad \int_0^{2\pi} \exp(-its) dT[I_{[0,s]}(\omega)].$$

By definition we have $T(I_{[0,s]}(\omega)) = Z(s)$ which we substitute into the above to give

$$X_t \quad = \quad \int_0^{2\pi} \exp(-its) dZ(s),$$

which gives the required result.

Note that there are several different ways to prove this result. □

It is worth taking a step back from the proof and see where the assumption of stationarity crept in. By Bochner's theorem we have that

$$c(t-\tau) = \int \exp(-i(t-\tau)\omega) dF(\omega),$$

where F is a distribution. We use F to define the space $\mathcal{H}_1$, the mapping T (through $\{\exp(ik\omega)\}_k$), the inner-product and thus the isomorphism. However, it was the construction of the orthogonal random functions $\{Z(\omega)\}$ that was instrumental. The main idea of the proof was that there are functions $\{\phi_k(\omega)\}$ and a distribution H such that all the covariances of the stochastic process $\{X_t\}$ can be written as

$$\mathrm{E}(X_t X_\tau) = c(t,\tau) = \int_0^{2\pi} \phi_t(\omega)\overline{\phi_\tau(\omega)} dH(\omega),$$

where H is a measure. As long as the above representation exists, then we can define two spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ where $\{\phi_k\}$ is the basis of the functional space $\mathcal{H}_1$ and it contains all functions f such that $\int |f(\omega)|^2 dH(\omega) < \infty$ and $\mathcal{H}_2$ is the random space defined by $\overline{\text{sp}}(X_t; t \in \mathbb{Z})$. From here we can define an isomorphism $T : \mathcal{H}_1 \to \mathcal{H}_2$, where for all functions $f(\omega) = \sum_k f_k \phi_k(\omega) \in \mathcal{H}_1$

$$T(f) = \sum_k f_k X_k \in \mathcal{H}_2.$$

An important example is $T(\phi_k) = X_k$. Now by using the same arguments as those in the proof above we have

$$X_t = \int \phi_t(\omega) dZ(\omega)$$

where $\{Z(\omega)\}$ are orthogonal random functions and $\text{E}|Z(\omega)|^2 = H(\omega)$. We state this result in the theorem below (see Priestley (1983), Section 4.11).

Theorem 9.4.2 (General orthogonal expansions) *Let $\{X_t\}$ be a time series (not necessarily second order stationary) with covariance $\{\text{E}(X_t X_\tau) = c(t,s)\}$. If there exists a sequence of functions $\{\phi_k(\cdot)\}$ which satisfy for all k*

$$\int_0^{2\pi} |\phi_k(\omega)|^2 dH(\omega) < \infty$$

and the covariance admits the representation

$$c(t,s) = \int_0^{2\pi} \phi_t(\omega)\overline{\phi_s(\omega)} dH(\omega), \tag{9.25}$$

where H is a distribution then for all t we have the representation

$$X_t = \int \phi_t(\omega) dZ(\omega) \tag{9.26}$$

where $\{Z(\omega)\}$ are orthogonal random functions and $\text{E}|Z(\omega)|^2 = H(\omega)$. On the other hand if X_t has the representation (9.26), then $c(s,t)$ admits the representation (9.25).

Remark 9.4.2 *We mention that the above representation applies to both stationary and nonstationary time series. What makes the exponential functions $\{\exp(ik\omega)\}$ special is if a process is* *stationary then the representation of* $c(k) := \mathrm{cov}(X_t, X_{t+k})$ *in terms of exponentials is guaranteed:*

$$c(k) = \int_0^{2\pi} \exp(-ik\omega)dF(\omega). \tag{9.27}$$

Therefore there always exists an orthogonal random function $\{Z(\omega)\}$ *such that*

$$X_t = \int \exp(-it\omega)dZ(\omega).$$

Indeed, whenever the exponential basis is used in the definition of either the covariance or the process $\{X_t\}$*, the resulting process will always be second order stationary.*

Brockwell and Davis (1998), Proposition 4.8.2 states an interesting consequence of the spectral representation theorem. Suppose that $\{X_t\}$ is a second order stationary time series with spectral distribution $F(\omega)$. If $F(\omega)$ has a discontinuity at λ_0, then X_t almost surely has the representation

$$X_t = \int_0^{2\pi} e^{it\omega}dZ(\omega) + e^{it\lambda_0}\left(Z(\lambda_0^+) - Z(\lambda_0^-)\right)$$

where $Z(\lambda_0^-)$ and $Z(\lambda_0^+)$ denote the left and right limit. This result means that discontinuities in the spectral distribution mean that the corresponding time series contains a deterministic sinusoid functions i.e.

$$X_t = A\cos(\lambda_0 t) + B\sin(\lambda_0 t) + \varepsilon_t$$

where ε_t is a stationary time series. We came across this “feature” in Section 1.3.4. If the spectral distribution contains a discontinuity, then “formally” the spectral density (which is the derivative of the spectral distribution) is the dirac-delta function at the discontinuity. The periodogram is a “crude” (inconsistent) estimator of the spectral density function, however it captures the general features of the underlying spectral density. Look at Figures 1.10-1.12, observe that there is a large peak corresponding the deterministic frequency and that this peak grows taller as the sample size n grows. This large peak is limiting to the dirac delta function.

Finally we state Brockwell and Davis (1998), Proposition 4.9.1, which justifies our use of the DFT. Brockwell and Davis (1998), Proposition 4.9.1 states that if $\{X_t\}$ is a second order stationary time series with spectral distribution F and ν_1 and ν_2 are continuity points of F then

$$\frac{1}{2\pi}\sum_{|t|\leq n} X_t \int_{\nu_1}^{\nu_2} \exp(it\omega)d\omega \to Z(\nu_2) - Z(\nu_1),$$

where the convergence is in mean squared.

Let $\omega_k = 2\pi k/n$, then using this result we have

$$\frac{1}{2\pi\sqrt{n}}\sum_{|t|\leq n} X_t \exp(it\omega_k) \approx \sqrt{n}\sum_{|t|\leq n} X_t \int_{\omega_k}^{\omega_{k+1}} \exp(it\omega)d\omega \approx \sqrt{n}\left[Z(\omega_{k+1}) - Z(\omega_k)\right],$$

without the scaling factor $\sqrt{n}$, the above would limit to zero. Thus as claimed previously, the DFT estimates the "increments".

9.5 The spectral density functions of MA, AR and ARMA models

We obtain the spectral density function for MA(∞) processes. Using this we can easily obtain the spectral density for ARMA processes. Let us suppose that $\{X_t\}$ satisfies the representation

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j \varepsilon_{t-j} \tag{9.28}$$

where $\{\varepsilon_t\}$ are iid random variables with mean zero and variance σ^2 and $\sum_{j=-\infty}^{\infty}|\psi_j| < \infty$. We recall that the covariance of above is

$$c(k) = \mathrm{E}(X_t X_{t+k}) = \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+k}. \tag{9.29}$$

Since $\sum_{j=-\infty}^{\infty}|\psi_j| < \infty$, it can be seen that

$$\sum_k |c(k)| \leq \sum_k \sum_{j=-\infty}^{\infty} |\psi_j|\cdot|\psi_{j+k}| < \infty.$$

Hence by using Theorem 9.3.1, the spectral density function of $\{X_t\}$ is well defined. There are several ways to derive the spectral density of $\{X_t\}$, we can either use (9.29) and $f(\omega) = \frac{1}{2\pi}\sum_k c(k)\exp(ik\omega)$ or obtain the spectral representation of $\{X_t\}$ and derive $f(\omega)$ from the spectral representation. We prove the results using the latter method.

9.5.1 The spectral representation of linear processes

Since $\{\varepsilon_t\}$ are iid random variables, using Theorem 9.4.1 there exists an orthogonal random function $\{Z(\omega)\}$ such that

$$\varepsilon_t = \int_0^{2\pi} \exp(-it\omega) dZ_\varepsilon(\omega).$$

Since $\mathrm{E}(\varepsilon_t) = 0$ and $\mathrm{E}(\varepsilon_t^2) = \sigma^2$ multiplying the above by ε_t, taking expectations and noting that due to the orthogonality of $\{Z_\varepsilon(\omega)\}$ we have $\mathrm{E}(dZ_\varepsilon(\omega_1)\overline{dZ_\varepsilon(\omega_2)}) = 0$ unless $\omega_1 = \omega_2$ we have that $\mathrm{E}(|dZ_\varepsilon(\omega)|^2) = \sigma^2 d\omega$, hence $f_\varepsilon(\omega) = (2\pi)^{-1}\sigma^2$.

Using the above we obtain the following spectral representation for $\{X_t\}$, where X_t is a linear time series

$$X_t = \sum_{j=\infty}^{\infty} \psi_j \varepsilon_{t-j} \int_0^{2\pi} \left\{ \sum_{j=-\infty}^{\infty} \psi_j \exp(ij\omega) \right\} \exp(-it\omega) dZ_\varepsilon(\omega).$$

Hence

$$X_t = \int_0^{2\pi} A(\omega) \exp(-it\omega) dZ_\varepsilon(\omega) = \int_0^{2\pi} \exp(-it\omega) dZ_X(\omega) \tag{9.30}$$

where $A(\omega) = \sum_{j=-\infty}^{\infty} \psi_j \exp(ij\omega)$ and $Z_X(\omega) = A(\omega) Z_\varepsilon(\omega)$. We note that this is the unique spectral representation of X_t.

Definition 9.5.1 (The Cramer Representation) *We mention that the representation in (9.30) of a stationary process is usually called the Cramer representation of a stationary process, where*

$$X_t = \int_0^{2\pi} A(\omega) \exp(-it\omega) dZ(\omega),$$

where $\{Z(\omega) : 0 \leq \omega \leq 2\pi\}$ are orthogonal functions.

Exercise 9.3 *(i) Suppose that $\{X_t\}$ has an $MA(1)$ representation $X_t = \theta\varepsilon_t + \varepsilon_{t-1}$. What is its Cramer's representation?*

(ii) Suppose that $\{X_t\}$ has a causal $AR(1)$ representation $X_t = \phi X_{t-1} + \varepsilon_t$. What is its Cramer's

representation?

9.5.2 The spectral density of a linear process

Multiplying (9.30) by X_{t+k} and taking expectations gives

$$\mathrm{E}(X_t X_{t+k}) = c(k) = \int_0^{2\pi} A(\omega_1)A(-\omega_2)\exp(-i(t+k)\omega_1 + it\omega_2)\mathrm{E}(dZ(\omega_1)d\overline{Z(\omega_2)}).$$

Due to the orthogonality of $\{Z(\omega)\}$ we have $\mathrm{E}(dZ(\omega_1)d\overline{Z(\omega_2)}) = 0$ unless $\omega_1 = \omega_2$, altogether this gives

$$\mathrm{E}(X_t X_{t+k}) = c(k) = \int_0^{2\pi} |A(\omega)|^2 \exp(-ik\omega)\mathrm{E}(|dZ(\omega)|^2) = \int_0^{2\pi} f(\omega)\exp(-ik\omega)d\omega,$$

where $f(\omega) = \frac{\sigma^2}{2\pi}|A(\omega)|^2$. Comparing the above with (9.14) we see that $f(\cdot)$ is the spectral density function.

The spectral density function corresponding to the linear process defined in (9.28) is

$$f(\omega) = \frac{\sigma^2}{2\pi}|\sum_{j=-\infty}^{\infty}\psi_j \exp(-ij\omega)|^2.$$

Remark 9.5.1 (An alternative, more hands on proof) *An alternative proof which avoids the Cramer representation is to use that the acf of a linear time series is $c(r) = \sigma^2\sum_k \psi_j\psi_{j+r}$ (see Lemma 4.1.1). Thus by definition the spectral density function is*

$$\begin{aligned} f(\omega) &= \frac{1}{2\pi}\sum_{r=-\infty}^{\infty} c(r)\exp(ir\omega) \\ &= \frac{\sigma^2}{2\pi}\sum_{r=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}\psi_j\psi_{j+r}\exp(ir\omega). \end{aligned}$$

Now make a change of variables $s = j + r$ this gives

$$f(\omega) = \frac{\sigma^2}{2\pi}\sum_{s=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}\psi_j\psi_s\exp(i(s-j)\omega) = \frac{\sigma^2}{2\pi}\left|\sum_{j=-\infty}^{\infty}\psi_j e^{ij\omega}\right|^2 = \frac{\sigma^2}{2\pi}|A(\omega)|^2.$$

Example 9.5.1 *Let us suppose that $\{X_t\}$ is a stationary $ARMA(p,q)$ time series (not necessarily* *invertible or causal), where*

$$X_t - \sum_{j=1}^{p} \psi_j X_{t-j} = \sum_{j=1}^{q} \theta_j \varepsilon_{t-j},$$

$\{\varepsilon_t\}$ *are iid random variables with* $\mathrm{E}(\varepsilon_t) = 0$ *and* $\mathrm{E}(\varepsilon_t^2) = \sigma^2$*. Then the spectral density of* $\{X_t\}$ *is*

$$f(\omega) = \frac{\sigma^2}{2\pi} \frac{|1 + \sum_{j=1}^{q} \theta_j \exp(ij\omega)|^2}{|1 - \sum_{j=1}^{q} \phi_j \exp(ij\omega)|^2}$$

We note that because the ARMA is the ratio of trignometric polynomials, this is known as a rational spectral density.

Remark 9.5.2 *The roots of the characteristic function of an AR process will have an influence on the location of peaks in its corresponding spectral density function. To see why consider the AR(2) model*

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \varepsilon_t,$$

where $\{\varepsilon_t\}$ *are iid random variables with zero mean and* $\mathrm{E}(\varepsilon^2) = \sigma^2$*. Suppose the roots of the characteristic polynomial* $\phi(B) = 1 - \phi_1 B - \phi_2 B^2$ *lie outside the unit circle and are complex conjugates where* $\lambda_1 = r\exp(i\theta)$ *and* $\lambda_2 = r\exp(-i\theta)$*. Then the spectral density function is*

$$\begin{aligned} f(\omega) &= \frac{\sigma^2}{|1 - r\exp(i(\theta - \omega))|^2 |1 - r\exp(i(-\theta - \omega)|^2} \\ &= \frac{\sigma^2}{[1 + r^2 - 2r\cos(\theta - \omega)][1 + r^2 - 2r\cos(\theta + \omega)]}. \end{aligned}$$

If $r > 0$*, the* $f(\omega)$ *is maximum when* $\omega = \theta$*, on the other hand if,* $r < 0$ *then the above is maximum when* $\omega = \theta - \pi$*. Thus the peaks in* $f(\omega)$ *correspond to peaks in the pseudo periodicities of the time series and covariance structure (which one would expect), see Section 4.1.2. How pronounced these peaks are depend on how close* r *is to one. The close* r *is to one the larger the peak. We can generalise the above argument to higher order Autoregressive models, in this case there may be multiple peaks. In fact, this suggests that the larger the number of peaks, the higher the order of the AR model that should be fitted.*

9.5.3 Approximations of the spectral density to AR and MA spectral densities

In this section we show that the spectral density

$$f(\omega) = \frac{1}{2\pi}\sum_{r=-\infty}^{\infty} c(r)\exp(ir\omega)$$

can be approximated to any order by the spectral density of an AR(p) or MA(q) process.

We do this by truncating the infinite number of covariances by a finite number, however, this does not necessarily lead to a positive definite spectral density. This can easily be proven by noting that

$$\widetilde{f}_m(\omega) = \sum_{r=-m}^{m} c(r)\exp(ir\omega) = \int_0^{2\pi} f(\lambda)D_m(\omega-\lambda)d\lambda,$$

where $D_m(\lambda) = \sin[(n+1/2)\lambda]/\sin(\lambda/2)$. Observe that $D_m(\cdot)$ can be negative, which means that $\widetilde{f}_m(\omega)$ can be negative despite f being positive.

Example 9.5.2 *Consider the AR*(1) *process* $X_t = 0.75X_{t-1} + \varepsilon_t$ *where* $\mathrm{var}[\varepsilon_t] = 1$. *In Lemma 4.1.1 we showed that the autcovariance corresponding to this model is* $c(r) = [1-0.75^2]^{-1}0.75^{|r|}$.

Let us define a process whose autocorrelation is $\tilde{c}(0) = [1-0.75^2]^{-1}$, $c(1) = c(-1) = [1-0.75^2]^{-1}0.75$ *and* $\tilde{c}(r) = 0$ *for* $|r| > 1$. *The 'spectral density' of this process is*

$$\widetilde{f}_m(\omega) = \frac{1}{1-0.75^2}\left(1 + 2\times\frac{3}{4}\cos[\omega]\right).$$

It is clear that this function can be zero for some values of ω. *This means that* $\{\tilde{c}(r)\}$ *is not a well defined covariance function, hence there does not exist a time series with this covariance structure. In other words, simply truncating an autocovariance is not enough to guarantee that it positive definite sequence.*

Instead we consider a slight variant on this and define

$$\frac{1}{2\pi}\sum_{r=-m}^{m}\left(1-\frac{|r|}{m}\right)c(r)\exp(ir\omega)$$

which is positive.

Remark 9.5.3 *We note that f_m is known as a Cesáro sum because it can be written as*

$$f_m(\omega) \quad = \quad \frac{1}{2\pi}\sum_{r=-m}^{m}\left(1-\frac{|r|}{m}\right)c(r)\exp(ir\omega) = \frac{1}{m}\sum_{n=0}^{m}\widetilde{f}_n(\omega), \tag{9.31}$$

where $\widetilde{f}_n(\cdot) = \frac{1}{2\pi}\sum_{r=-n}^{n}c(r)\exp(ir\omega)$. Strangely, there is no guarantee that the truncated Fourier transform $\widetilde{f}_n$ is not negative, however $f_n(\cdot)$ is definitely positive. There are are a few ways to prove this:

(i) The first method we came across previously, $\mathrm{var}[J_n(\omega)] = f_n(\omega)$*, it is clear that using this construction* $\inf_\omega f_n(\omega) \geq 0$.

(ii) By using (9.31) we can write $f_m(\cdot)$ as

$$f_m(\omega) = \int_0^{2\pi} f(\lambda)F_m(\omega-\lambda)d\lambda,$$

where $F_m(\lambda) = \frac{1}{m}\sum_{r=-m}^{m}D_r(\lambda) = \frac{1}{m}\left(\frac{\sin(n\lambda/2)}{\sin(\lambda/2)}\right)^2$ and $D_r(\lambda) = \sum_{j=-r}^{r}\exp(ij\omega)$ (these are the Fejer and Dirichlet kernels respectively). Since both f and F_m are positive, then f_m has to be positive.

The Cesaro sum is special in the sense that

$$\sup_\omega |f_m(\omega) - f(\omega)| \to 0, \qquad \text{as } m \to \infty. \tag{9.32}$$

Thus for a large enough m, $f_m(\omega)$ will be within δ of the spectral density f. Using this we can prove the results below.

Lemma 9.5.1 *Suppose that $\sum_r |c(r)| < \infty$, f is the spectral density of the covariances and* $\inf_{\omega\in[0,2\pi]} f(\omega) > 0$. *Then for every $\delta > 0$, there exists a m such that $|f(\omega) - f_m(\omega)| < \delta$ and $f_m(\omega) = \sigma^2|\psi(\omega)|^2$, where $\psi(\omega) = \sum_{j=0}^{m}\psi_j\exp(ij\omega)$. Thus we can approximate the spectral density of f with the spectral density of a MA.*

PROOF. We show that there exists an MA(m) which has the spectral density $f_m(\omega)$, where f_m is defined in (9.31). Thus by (9.32) we have the result.

Before proving the result we note that if a "polynomial" is of the form

$$p(z) = a_0 + \sum_{j=1}^{m} a_j\left(z + z^{-1}\right)$$

then it has the factorization $p(z) = C\prod_{j=1}^{m}[1-\lambda_j z][1-\lambda_j^{-1}z]$, where λ_j is such that $|\lambda_j| < 1$. Furthermore, if $\{a_j\}_{j=0}^{m}$ are real and $z^m p(z)$ has no roots on the unit circle, then the coefficients of the polynomial $\prod_{j=1}^{m}[1-\lambda_j z]$ are real. The above claims are true because

(i) To prove that $p(z) = C\prod_{j=1}^{m}[1-\lambda_j z][1-\lambda_j^{-1}z]$, we note that $z^m p(z)$ is a $2m$-order polynomial. Thus it can be factorized. If there exists a root λ whose inverse is not a root, then the resulting polynomial will have not have the symmetric structure.

(ii) By the complex conjugate theorem, since $z^m p(z)$ has real coefficients, then its complex roots must be conjugates. Moreover, since no roots lie on the unit circle, then no conjugates lie on the unit circle. Thus the coefficients of $\prod_{j=1}^{m}[1-\lambda_j z]$ are real (if it did lie on the unit circle, then we can distribute the two roots between the two polynomials).

Thus setting $z = e^{i\omega}$

$$\sum_{r=-m}^{m} a_r \exp(ir\omega) \quad = \quad C\prod_{j=1}^{m}[1-\lambda_j \exp(i\omega)]\left[1-\lambda_j^{-1}\exp(-i\omega)\right].$$

for some finite constant C. We use the above result. Since $\inf f_m(\omega) > 0$ and setting $a_r = [1-|r|n^{-1}]c(r)$, we can write f_m as

$$\begin{aligned} f_m(\omega) &= K\left[\prod_{j=1}^{m}(1-\lambda_j^{-1}\exp(i\omega))\right]\left[\prod_{j=1}^{m}(1-\lambda_j\exp(-i\omega))\right] \\ &= A(\omega)A(-\omega) = |A(\omega)|^2, \end{aligned}$$

where

$$A(z) = \prod_{j=1}^{m}(1-\lambda_j^{-1}z).$$

Since $A(z)$ is an mth order polynomial where all the roots are greater than 1, we can always construct an MA(m) process which has $A(z)$ as its 'transfer' function. Thus there exists an MA(m) process which has $f_m(\omega)$ as its spectral density function. □

Remark 9.5.4 *(i) The above result requires that* $\inf_\omega f(\omega) > 0$, *in order to ensure that* $f_m(\omega)$ *is strictly positive. This assumption can be relaxed (and the proof becomes a little more complicated), see Brockwell and Davis (1998), Theorem 4.4.3.*

(ii)

Lemma 9.5.2 *Suppose that $\sum_r |c(r)| < \infty$ and f is corresponding the spectral density function where $\inf_\omega f(\omega) > 0$. Then for every $\delta > 0$, there exists a m such that $|f(\omega) - g_m(\omega)| < \delta$ and $g_m(\omega) = \sigma^2|\phi(\omega)^{-1}|^2$, where $\phi(\omega) = \sum_{j=0}^m \phi_j \exp(ij\omega)$ and the roots of $\phi(z)$ lie outside the unit circle. Thus we can approximate the spectral density of f with the spectral density of a causal autoregressive process.*

PROOF. We first note that we can write

$$\big|f(\omega) - g_m(\omega)\big| \;=\; f(\omega)|g_m(\omega)^{-1} - f(\omega)^{-1}|g_m(\omega).$$

Since $f(\cdot) \in L_2$ and is bounded away from zero, then $f^{-1} \in L_2$ and we can write f^{-1} as

$$f^{-1}(\omega) = \sum_{r=\infty}^{\infty} d_r \exp(ir\omega),$$

where d_r are the Fourier coefficients of f^{-1}. Since f is positive and symmetric, then f^{-1} is positive and symmetric such that $f^{-1}(\omega) = \sum_{r=-\infty}^{\infty} d_r e^{ir\omega}$ <u>and</u> $\{d_r\}$ is a positive definite symmetric sequence. Thus we can define the positive function g_m where

$$g_m^{-1}(\omega) = \sum_{|r|\leq m} \left(1 - \frac{|r|}{m}\right) d_r \exp(ir\omega)$$

and is such that $|g_m^{-1}(\omega) - f^{-1}(\omega)| < \delta$, which implies

$$\big|f(\omega) - g_m(\omega)\big| \leq [\sum_r |c(r)|]^2\delta.$$

Now we can apply the same arguments to prove to Lemma 9.5.1 we can show that g_m^{-1} can be factorised as $g_m^{-1}(\omega) = C|\phi_m(\omega)|^2$ (where ϕ_m is an mth order polynomial whose roots lie outside the unit circle). Thus $g_m(\omega) = C|\phi_m(\omega)|^{-2}$ and we obtain the desired result. □

9.6 Higher order spectrums

We recall that the covariance is a measure of linear dependence between two random variables. Higher order cumulants are a measure of higher order dependence. For example, the third order cumulant for the zero mean random variables X_1, X_2, X_3 is

$$\mathrm{cum}(X_1, X_2, X_3) = \mathrm{E}(X_1X_2X_3)$$

and the fourth order cumulant for the zero mean random variables X_1, X_2, X_3, X_4 is

$$\mathrm{cum}(X_1, X_2, X_3, X_4) = \mathrm{E}(X_1X_2X_3X_4) - \mathrm{E}(X_1X_2)\mathrm{E}(X_3X_4) - \mathrm{E}(X_1X_3)\mathrm{E}(X_2X_4) - \mathrm{E}(X_1X_4)\mathrm{E}(X_2X_3).$$

From the definition we see that if X_1, X_2, X_3, X_4 are independent then $\mathrm{cum}(X_1, X_2, X_3) = 0$ and $\mathrm{cum}(X_1, X_2, X_3, X_4) = 0$.

Moreover, if X_1, X_2, X_3, X_4 are Gaussian random variables then $\mathrm{cum}(X_1, X_2, X_3) = 0$ and $\mathrm{cum}(X_1, X_2, X_3, X_4) = 0$. Indeed all cumulants higher than order two is zero. This comes from the fact that cumulants are the coefficients of the power series expansion of the logarithm of the characteristic function of $\{X_t\}$, which is

$$g_X(t) = i \underbrace{\underline{\mu}'}_{\text{mean}} \underline{t} - \frac{1}{2}\underline{t}' \underbrace{\Sigma}_{\text{cumulant}} \underline{t}.$$

Since the spectral density is the Fourier transform of the covariance it is natural to ask whether one can define the higher order spectral density as the fourier transform of the higher order cumulants. This turns out to be the case, and the higher order spectra have several interesting properties. Let us suppose that $\{X_t\}$ is a stationary time series (notice that we are assuming it is strictly stationary and not second order). Let $\kappa_3(t,s) = \mathrm{cum}(X_0, X_t, X_s)$, $\kappa_3(t,s,r) = \mathrm{cum}(X_0, X_t, X_s, X_r)$ and $\kappa_q(t_1, \ldots, t_{q-1}) = cum(X_0, X_{t_1}, \ldots, X_{t_q})$ (noting that like the covariance the higher order cumulants are invariant to shift). The third, fourth and the general qth order spectras is defined as

$$\begin{aligned}
f_3(\omega_1, \omega_2) &= \sum_{s=-\infty}^{\infty}\sum_{t=-\infty}^{\infty} \kappa_3(s,t)\exp(is\omega_1 + it\omega_2) \\
f_4(\omega_1, \omega_2, \omega_3) &= \sum_{s=-\infty}^{\infty}\sum_{t=-\infty}^{\infty}\sum_{r=-\infty}^{\infty} \kappa_4(s,t,r)\exp(is\omega_1 + it\omega_2 + ir\omega_3) \\
f_q(\omega_1, \omega_2, \ldots, \omega_{q-1}) &= \sum_{t_1,\ldots,t_{q-1}=-\infty}^{\infty} \kappa_q(t_1, t_2, \ldots, t_{q-1})\exp(it_1\omega_1 + it_2\omega_2 + \ldots + it_{q-1}\omega_{q-1}).
\end{aligned}$$

Example 9.6.1 (Third and Fourth order spectral density of a linear process) *Let us sup-*

pose that $\{X_t\}$ satisfies

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j \varepsilon_{t-j}$$

where $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$, $\mathrm{E}(\varepsilon_t) = 0$ and $\mathrm{E}(\varepsilon_t^4) < \infty$. Let $A(\omega) = \sum_{j=-\infty}^{\infty} \psi_j \exp(ij\omega)$. Then it is straightforward to show that

$$\begin{aligned} f(\omega) &= \sigma^2 |A(\omega)|^2 \\ f_3(\omega_1, \omega_2) &= \kappa_3 A(\omega_1) A(\omega_2) A(-\omega_1 - \omega_2) \\ f_4(\omega_1, \omega_2, \omega_3) &= \kappa_4 A(\omega_1) A(\omega_2) A(\omega_3) A(-\omega_1 - \omega_2 - \omega_3), \end{aligned}$$

where $\kappa_3 = \mathrm{cum}(\varepsilon_t, \varepsilon_t, \varepsilon_t)$ and $\kappa_4 = \mathrm{cum}(\varepsilon_t, \varepsilon_t, \varepsilon_t, \varepsilon_t)$.

We see from the example, that unlike the spectral density, the higher order spectras are not necessarily positive or even real.

A review of higher order spectra can be found in Brillinger (2001). Higher order spectras have several applications especially in nonlinear processes, see Subba Rao and Gabr (1984). We will consider one such application in a later chapter.

Using the definition of the higher order spectrum we can now generalise Lemma 9.2.1 to higher order cumulants (see Brillinger (2001), Theorem 4.3.4).

Proposition 9.6.1 *$\{X_t\}$ is a strictly stationary time series, where for all $1 \leq i \leq q-1$ we have $\sum_{t_1,\ldots,t_{q-1}=\infty}^{\infty} |(1+t_i)\kappa_q(t_1, \ldots, t_{q-1})| < \infty$ (note that this is simply a generalization of the covariance assumption $\sum_r |rc(r)| < \infty$). Then we have*

$$\begin{aligned} \mathrm{cum}(J_n(\omega_{k_1}), \ldots, J_n(\omega_{k_q})) &= \frac{1}{n^{q/2}} f_q(\omega_{k_2}, \ldots, \omega_{k_q}) \sum_{j=1}^{n} \exp(ij(\omega_{k_1} - \ldots - \omega_{k_q})) + O(\frac{1}{n^{q/2}}) \\ &= \begin{cases} \frac{1}{n^{(q-1)/2}} f_q(\omega_{k_2}, \ldots, \omega_{k_q}) + O(\frac{1}{n^{q/2}}) & \sum_{i=1}^{q} k_i = n\mathbb{Z} \\ O(\frac{1}{n^{q/2}}) & \textit{otherwise} \end{cases} \end{aligned}$$

where $\omega_{k_i} = \frac{2\pi k_i}{n}$.

9.7 Extensions

9.7.1 The spectral density of a time series with randomly missing observations

Let us suppose that $\{X_t\}$ is a second order stationary time series. However $\{X_t\}$ is not observed at everytime point and there are observations missing, thus we only observe X_t at $\{\tau_k\}_k$. Thus what is observed is $\{X_{\tau_k}\}$. The question is how to deal with this type of data. One method was suggested in **?**. He suggested that the missingness mechanism $\{\tau_k\}$ be modelled stochastically. That is define the random process $\{Y_t\}$ which only takes the values $\{0,1\}$, where $Y_t = 1$ if X_t is observed, but $Y_t = 0$ if X_t is not observed. Thus we observe $\{X_tY_t\}_t = \{X_{t_k}\}$ and also $\{Y_t\}$ (which is the time points the process is observed). He also suggests modelling $\{Y_t\}$ as a stationary process, which is independent of $\{X_t\}$ (thus the missingness mechanism and the time series are independent).

The spectral densities of $\{X_tY_t\}$, $\{X_t\}$ and $\{Y_t\}$ have an interest relationship, which can be exploited to estimate the spectral density of $\{X_t\}$ given estimators of the spectral densities of $\{X_tY_t\}$ and $\{X_t\}$ (which we recall are observed). We first note that since $\{X_t\}$ and $\{Y_t\}$ are stationary, then $\{X_tY_t\}$ is stationary, furthermore

$$\begin{aligned}
\mathrm{cov}(X_tY_t, X_\tau Y_\tau) &= \mathrm{cov}(X_t, X_\tau)\mathrm{cov}(Y_t, Y_\tau) + \mathrm{cov}(X_t, Y_\tau)\mathrm{cov}(Y_t, X_\tau) + cum(X_t, Y_t, X_\tau, Y_\tau) \\
&= \mathrm{cov}(X_t, X_\tau)\mathrm{cov}(Y_t, Y_\tau) = c_X(t-\tau)c_Y(t-\tau)
\end{aligned}$$

where the above is due to independence of $\{X_t\}$ and $\{Y_t\}$. Thus the spectral density of $\{X_tY_t\}$ is

$$\begin{aligned}
f_{XY}(\omega) &= \frac{1}{2\pi}\sum_{r=-\infty}^{\infty} \mathrm{cov}(X_0Y_0, X_rY_r)\exp(ir\omega) \\
&= \frac{1}{2\pi}\sum_{r=-\infty}^{\infty} c_X(r)c_Y(r)\exp(ir\omega) \\
&= \int f_X(\lambda)f_Y(\omega-\lambda)d\omega,
\end{aligned}$$

where $f_X(\lambda) = \frac{1}{2\pi}\sum_{r=-\infty}^{\infty} c_X(r)\exp(ir\omega)$ and $f_Y(\lambda) = \frac{1}{2\pi}\sum_{r=-\infty}^{\infty} c_Y(r)\exp(ir\omega)$ are the spectral densities of the observations and the missing process.

Chapter 10

Spectral Analysis

Prerequisites

- The Gaussian likelihood.
- The approximation of a Toeplitz by a Circulant (covered in previous chapters).

Objectives

- The DFTs are close to uncorrelated but have a frequency dependent variance (under stationarity).
- The DFTs are asymptotically Gaussian.
- For a linear time series the DFT is almost equal to the transfer function times the DFT of the innovations.
- The periodograms is the square of the DFT, whose expectation is approximately equal to the spectral density. Smoothing the periodogram leads to an estimator of the spectral density as does truncating the covariances.
- The Whittle likelihood and how it is related to the Gaussian likelihood.
- Understand that many estimator can be written in the frequency domain.
- Calculating the variance of an estimator.

10.1 The DFT and the periodogram

In the previous section we motivated transforming the stationary time series $\{X_t\}$ into it's discrete Fourier transform

$$\begin{aligned} J_n(\omega_k) &= \frac{1}{\sqrt{2\pi n}}\sum_{t=1}^{n} X_t \exp(ik\frac{2\pi t}{n}) \\ &= \Big(\frac{1}{\sqrt{2\pi n}}\sum_{t=1}^{n} X_t \cos(k\frac{2\pi t}{n}) + i\frac{1}{\sqrt{2\pi n}}\sum_{t=1}^{n} X_t \sin(k\frac{2\pi t}{n})\Big) \qquad k = 0,\ldots,n/2 \end{aligned}$$

(frequency series) as an alternative way of analysing the time series. Since there is a one-to-one mapping between the two, nothing is lost by making this transformation. Our principle reason for using this transformation is given in Lemma 9.2.1, where we showed that $\{J_n(\omega_k)\}_{n=1}^{n/2}$ is an almost uncorrelated series. However, there is a cost to the uncorrelatedness property, that is unlike the original stationary time series $\{X_t\}$, the variance of the DFT varies over the frequencies, and the variance is the spectral density at that frequency. We summarise this result below, but first we recall the definition of the spectral density function

$$f(\omega) = \frac{1}{2\pi}\sum_{r=-\infty}^{\infty} c(r)\exp(ir\omega) \qquad \omega \in [0,2\pi]. \tag{10.1}$$

We summarize some of the results derived in Chapter 9 here.

Lemma 10.1.1 *Suppose that $\{X_t\}$ is a zero second order stationary time series, where $\mathrm{cov}(X_0, X_r) = c(r)$ and $\sum_r |c(r)| < \infty$. Define $\omega_k = \frac{2\pi k}{n}$. Then*

(i)

$$|J_n(\omega)|^2 = \frac{1}{2\pi}\sum_{r=-(n-1)}^{n-1} \hat{c}_n(r)\exp(ir\omega), \tag{10.2}$$

where $\hat{c}_n(r)$ is the sample autocovariance.

(ii) for $k \neq 0$ we have $\mathrm{E}[J_n(\omega_k)] = 0$,

$$\big|\mathrm{E}(|J_n(\omega)|^2) - f(\omega)\big| \leq \frac{1}{2\pi}\big(\sum_{|r|\geq n}|c(r)| + \frac{1}{n}\sum_{|r|\leq n}|rc(r)|\big) \to 0 \tag{10.3}$$

as $n \to \infty$,

(iii)

$$\mathrm{cov}\left[J_n(\frac{2\pi k_1}{n}), J_n(\frac{2\pi k_2}{n})\right] = \begin{cases} f(\frac{2\pi k}{n}) + o(1) & k_1 = k_2 \\ o(1) & k_1 \neq k_2 \end{cases}$$

where $f(\omega)$ is the spectral density function defined in (10.1). Under the stronger condition $\sum_r |rc(r)| < \infty$ the $o(1)$ above is replaced with $O(n^{-1})$.

In addition if we have higher order stationarity (or strict stationarity), then we also can find expressions for the higher order cumulants of the DFT (see Proposition 9.6.1).

It should be noted that even if the mean of the stationary time series $\{X_t\}$ is not zero (ie. $\mathrm{E}(X_t) = \mu \neq 0$), so long as $\omega_k \neq 0$ $\mathrm{E}(J_n(\omega_k)) = 0$ (even without centering X_t, with $X_t - \bar{X}$).

Since there is a one-to-one mapping between the observations and the DFT, it is not surprising that classical estimators can be written in terms of the DFT. For example, the sample covariance can be rewritten in terms of the DFT

$$\widehat{c}_n(r) + \widehat{c}_n(n-r) = \frac{1}{n}\sum_{k=1}^{n} |J_n(\omega_k)|^2 \exp(-ir\omega_k). \tag{10.4}$$

(see Appendix A.3(iv)). Since $\widehat{c}_n(n-r) = \frac{1}{n}\sum_{t=|n-r|}^{n} X_t X_{t+|n-r|}$, for small r (relative to T) this term is negligible, and gives

$$\widehat{c}_n(r) \approx \frac{1}{n}\sum_{k=1}^{n} |J_n(\omega_k)|^2 \exp(-ir\omega_k). \tag{10.5}$$

The modulo square of the DFT plays such an important role in time series analysis that it has it's own name, the periodogram, which is defined as

$$I_n(\omega) = |J_n(\omega)|^2 = \frac{1}{2\pi}\sum_{r=-(n-1)}^{n-1} \hat{c}_n(r)\exp(ir\omega). \tag{10.6}$$

By using Lemma 10.1.1 or Theorem 9.6.1 we have $\mathrm{E}(I_n(\omega)) = f(\omega) + O(\frac{1}{n})$. Moreover, (10.4) belongs to a general class of integrated mean periodogram estimators which have the form

$$A(\phi, I_n) = \frac{1}{n}\sum_{k=1}^{n} I_n(\omega_k)\phi(\omega_k). \tag{10.7}$$

Replacing the sum by an integral and the periodogram by its limit, it is clear that these are estimators of the integrated spectral density

$$A(f,\phi) = \int_0^{2\pi} f(\omega)\phi(\omega)d\omega.$$

Before we consider these estimators (in Section 10.5). We analyse some of the properties of the DFT.

10.2 Distribution of the DFT and Periodogram under linearity

An interesting aspect of the DFT, is that under certain conditions the DFT is asymptotically normal. We can heuristically justify this by noting that the DFT is a (weighted) sample mean. In fact at frequency zero, it is the sample mean ($J_n(0) = \sqrt{\frac{n}{2\pi}}\bar{X}$). In this section we prove this result, and a similar result for the periodogram. We do the proof under linearity of the time series, that is

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j \varepsilon_{t-j},$$

however the result also holds for nonlinear time series (but is beyond this course).

The DFT of the innovations $J_\varepsilon(\omega_k) = \frac{1}{\sqrt{2\pi n}}\sum_{t=1}^{n} \varepsilon_t e^{it\omega_k}$ is a very simple object to deal with it. First the DFT is an orthogonal transformation and the orthogonal transformation of iid random variables leads to uncorrelated random variables. In other words, $\{J_\varepsilon(\omega_k)\}$ is completely uncorrelated as are its real and imaginary parts. Secondly, if $\{\varepsilon_t\}$ are Gaussian, then $\{J_\varepsilon(\omega_k)\}$ are independent and Gaussian. Thus we start by showing the DFT of a linear time series is approximately equal to the DFT of the innovations multiplied by the transfer function. This allows us to transfer results regarding $J_\varepsilon(\omega_k)$ to $J_n(\omega_k)$.

We will use the assumption that $\sum_j |j^{1/2}\psi_j| < \infty$, this is a slightly stronger assumption than $\sum_j |\psi_j| < \infty$ (which we worked under in Chapter 3).

Lemma 10.2.1 *Let us suppose that $\{X_t\}$ satisfy $X_t = \sum_{j=-\infty}^{\infty} \psi_j \varepsilon_t$, where $\sum_{j=-\infty}^{\infty} |j^{1/2}\psi_j| < \infty$, and $\{\varepsilon_t\}$ are iid random variables with mean zero and variance σ^2. Let*

$$J_\varepsilon(\omega) = \frac{1}{\sqrt{2\pi n}} \sum_{t=1}^{n} \varepsilon_t \exp(it\omega).$$

Then we have

$$J_n(\omega) \;\; = \;\; \big\{\sum_j \psi_j \exp(ij\omega)\big\} J_\varepsilon(\omega) + Y_n(\omega), \qquad (10.8)$$

where $Y_n(\omega) = \frac{1}{\sqrt{2\pi n}} \sum_j \psi_j \exp(ij\omega) U_{n,j}$, *with* $U_{n,j} = \sum_{t=1-j}^{n-j} \exp(it\omega)\varepsilon_t - \sum_{t=1}^{n} \exp(it\omega)\varepsilon_t$ *and* $\mathrm{E}(Y_n(\omega))^2 \leq (\frac{1}{n^{1/2}} \sum_{j=-\infty}^{\infty} |\psi_j| \min(|j|, n)^{1/2})^2 = O(\frac{1}{n})$.

PROOF. We note that

$$\begin{aligned}
J_n(\omega) &= \sum_{j=-\infty}^{\infty} \psi_j \exp(ij\omega) \frac{1}{\sqrt{2\pi n}} \sum_{t=1}^{n} \varepsilon_{t-j} \exp(it\omega) \\
&= \sum_{j=-\infty}^{\infty} \psi_j \exp(ij\omega) \frac{1}{\sqrt{2\pi n}} \sum_{s=1-j}^{n-j} \varepsilon_s \exp(is\omega) \\
&= \left(\frac{1}{\sqrt{2\pi n}} \sum_j \psi_j \exp(ij\omega) \right) J_\varepsilon(\omega) + \underbrace{\sum_j \psi_j \exp(ij\omega) \left[\sum_{t=1-j}^{n-j} \exp(it\omega)\varepsilon_t - \sum_{t=1}^{n} \exp(it\omega)\varepsilon_t \right]}_{=Y_n(\omega)}.
\end{aligned}$$

We will show that $Y_n(\omega)$ is negligible with respect to the first term. We decompose $Y_n(\omega)$ into three terms

$$\begin{aligned}
Y_n(\omega) &= \frac{1}{\sqrt{2\pi n}} \sum_{j=-\infty}^{-n} \psi_j e^{ij\omega} \underbrace{\left[\sum_{t=1-j}^{n-j} \exp(it\omega)\varepsilon_t - \sum_{t=1}^{n} \exp(it\omega)\varepsilon_t \right]}_{\text{no terms in common}} + \\
& \quad \frac{1}{\sqrt{2\pi n}} \sum_{j=-n}^{n} \psi_j e^{ij\omega} \underbrace{\left[\sum_{t=1-j}^{n-j} \exp(it\omega)\varepsilon_t - \sum_{t=1}^{n} \exp(it\omega)\varepsilon_t \right]}_{(n-j) \text{ terms in common, } 2j \text{ terms not in common}} + \\
& \quad \frac{1}{\sqrt{2\pi n}} \sum_{j=n+1}^{\infty} \psi_j e^{ij\omega} \underbrace{\left[\sum_{t=1-j}^{n-j} \exp(it\omega)\varepsilon_t - \sum_{t=1}^{n} \exp(it\omega)\varepsilon_t \right]}_{\text{no terms in common}} \\
&= I + II + II.
\end{aligned}$$

If we took the expectation of the absolute of $Y_n(\omega)$ we find that we require the condition $\sum_j |j\psi_j| < \infty$ (and we don't exploit independence of the innovations). However, by evaluating $\mathrm{E}|Y_n(\omega)|^2$ we exploit to independence of $\{\varepsilon_t\}$, ie.

$$\begin{aligned}
[\mathrm{E}(I^2)]^{1/2} &\leq \frac{1}{\sqrt{2\pi n}}\sum_{j=-\infty}^{-n}|\psi_j|\left[\mathrm{E}\left(\sum_{t=1-j}^{n-j}\exp(it\omega)\varepsilon_t-\sum_{t=1}^{n}\exp(it\omega)\varepsilon_t\right)^2\right]^{1/2} \\
&\leq \frac{1}{\sqrt{2\pi n}}\sum_{j=-\infty}^{-n}|\psi_j|\left[2n\sigma^2\right]^{1/2} \leq \frac{1}{\sqrt{2\pi n}}\sum_{j=-\infty}^{-n}|j^{1/2}\psi_j| \leq \frac{1}{\sqrt{2\pi n}}\sum_{j=-\infty}^{\infty}|j^{1/2}\psi_j|
\end{aligned}$$

similarly, $III = O(n^{-1/2})$ and

$$\begin{aligned}
[\mathrm{E}(I^2)]^{1/2} &\leq \frac{1}{\sqrt{2\pi n}}\sum_{j=-n}^{n}|\psi_j|\left[\mathrm{E}\left(\sum_{t=1-j}^{n-j}\exp(it\omega)\varepsilon_t-\sum_{t=1}^{n}\exp(it\omega)\varepsilon_t\right)^2\right]^{1/2} \\
&\leq \frac{1}{\sqrt{2\pi n}}\sum_{j=-n}^{-n}|\psi_j|\left[2j\sigma^2\right]^{1/2} \leq \frac{1}{\sqrt{2\pi n}}\sum_{j=-\infty}^{-n}|j^{1/2}\psi_j| \leq \frac{1}{\sqrt{2\pi n}}\sum_{j=-\infty}^{\infty}|j^{1/2}\psi_j|.
\end{aligned}$$

Thus we obtain the desired result. □

The above shows that under linearity and the condition $\sum_j |j^{1/2}\psi_j| < \infty$ we have

$$J_n(\omega) = \big\{\sum_j \psi_j \exp(ij\omega)\big\}J_\varepsilon(\omega) + O_p(\frac{1}{\sqrt{n}}). \qquad (10.9)$$

This implies that the distribution of $J_n(\omega)$ is determined by the DFT of the innovations $J_\varepsilon(\omega)$. We generalise the above result to the periodogram.

Lemma 10.2.2 *Let us suppose that $\{X_t\}$ is a linear time series $X_t = \sum_{j=-\infty}^{\infty}\psi_j\varepsilon_{t-j}$, where $\sum_{j=-\infty}^{\infty}|j^{1/2}\psi_j| < \infty$, and $\{\varepsilon_t\}$ are iid random variables with mean zero, variance σ^2 $\mathrm{E}(\varepsilon_t^4) < \infty$. Then we have*

$$I_n(\omega) = \left|\sum_j \psi_j \exp(ij\omega)\right|^2 |J_\varepsilon(\omega)|^2 + R_n(\omega), \qquad (10.10)$$

where $\mathrm{E}(\sup_\omega |R_n(\omega)|) = O(\frac{1}{n})$.

PROOF. See Priestley (1983), Theorem 6.2.1 or Brockwell and Davis (1998), Theorem 10.3.1. □

To summarise the above result, for a general linear process $X_t = \sum_{j=-\infty}^{\infty} \psi_j \varepsilon_{t-j}$ we have

$$I_n(\omega) = |\sum_j \psi_j \exp(ij\omega)|^2 |J_\varepsilon(\omega)|^2 + O_p(\frac{1}{n}) = 2\pi f(\omega) I_\varepsilon(\omega) + O_p(\frac{1}{n}), \tag{10.11}$$

where we assume w.l.o.g. that $\mathrm{var}(\varepsilon_t) = 1$ and $f(\omega) = \frac{1}{2\pi} |\sum_j \psi_j \exp(ij\omega)|^2$ is the spectral density of $\{X_t\}$.

The asymptotic normality of $J_n(\omega)$ follows from asymptotic normality of $J_\varepsilon(\omega)$, which we prove in the following proposition.

Proposition 10.2.1 *Suppse $\{\varepsilon_t\}$ are iid random variables with mean zero and variance σ^2. We define $J_\varepsilon(\omega) = \frac{1}{\sqrt{2\pi n}} \sum_{t=1}^n \varepsilon_t \exp(it\omega)$ and $I_\varepsilon(\omega) = \frac{1}{2\pi n} \big| \sum_{t=1}^n \varepsilon_t \exp(it\omega) \big|^2$. Then we have*

$$\underline{J}_\varepsilon(\omega) = \begin{pmatrix} \Re J_\varepsilon(\omega) \\ \Im J_\varepsilon(\omega) \end{pmatrix} \overset{\mathcal{D}}{\rightarrow} \mathcal{N}\left(0, \frac{\sigma^2}{2(2\pi)} I_2\right), \tag{10.12}$$

where I_2 is the identity matrix. Furthermore, for any finite m

$$(\underline{J}_\varepsilon(\omega_{k_1})', \ldots, \underline{J}_\varepsilon(\omega_{k_m})') \overset{\mathcal{D}}{\rightarrow} \mathcal{N}\left(0, \frac{\sigma^2}{2(2\pi)} I_{2m}\right), \tag{10.13}$$

$I_\varepsilon(\omega)/\sigma^2 \overset{\mathcal{D}}{\rightarrow} \chi^2(2)/2$ (which is equivalent to the exponential distribution with mean one) and

$$\mathrm{cov}(|J_\varepsilon(\omega_j)|^2, |J_\varepsilon(\omega_k)|^2) = \begin{cases} \frac{\kappa_4}{(2\pi)^2 n} & j \neq k \\ \frac{\kappa_4}{(2\pi)^2 n} + \frac{2\sigma^4}{(2\pi)^2} & j = k \end{cases} \tag{10.14}$$

where $\omega_j = 2\pi j/n$ and $\omega_k = 2\pi k/n$ (and $j, k \neq 0$ or n).

PROOF. We first show (10.15). We note that $\Re(J_\varepsilon(\omega_k)) = \frac{1}{\sqrt{2\pi n}} \sum_{t=1}^n \alpha_{t,n}$ and $\Im(J_\varepsilon(\omega_k)) = \frac{1}{\sqrt{2\pi n}} \sum_{t=1}^n \beta_{t,n}$ where $\alpha_{t,n} = \varepsilon_t \cos(2k\pi t/n)$ and $\beta_{t,n} = \varepsilon_t \sin(2k\pi t/n)$. We note that $\Re(J_\varepsilon(\omega_k)) = \frac{1}{\sqrt{2\pi n}} \sum_{t=1}^n \alpha_{t,n}$ and $\Im(J_\varepsilon(\omega_k)) = \frac{1}{\sqrt{2\pi n}} \sum_{t=1}^n \beta_{t,n}$ are the weighted sum of iid random variables, hence $\{\alpha_{t,n}\}$ and $\{\beta_{t,n}\}$ are martingale differences. Therefore, to show asymptotic normality, we will use the martingale central limit theorem with the Cramer-Wold device to show that (10.15). We note that since $\{\alpha_{t,n}\}$ and $\{\beta_{t,n}\}$ are independent random variables we an prove the same result using a CLT for independent, non-identically distributed variables. However, for practice we will use a martingale CLT. To prove the result we need to verify the three conditions of the martingale

CLT. First we consider the conditional variances

$$\begin{aligned}
\frac{1}{2\pi n}\sum_{t=1}^{n}\mathrm{E}\big(|\alpha_{t,n}|^{2}\big|\varepsilon_{t-1},\varepsilon_{t-2},\ldots,\varepsilon_{1}\big) &= \frac{1}{2\pi n}\sum_{t=1}^{n}\cos(2k\pi t/n)^{2}\varepsilon_{t}^{2}\overset{\mathcal{P}}{\rightarrow}\frac{\sigma^{2}}{2\pi}\\
\frac{1}{2\pi n}\sum_{t=1}^{n}\mathrm{E}\big(|\beta_{t,n}|^{2}\big|\varepsilon_{t-1},\varepsilon_{t-2},\ldots,\varepsilon_{1}\big) &= \frac{1}{2\pi n}\sum_{t=1}^{n}\sin(2k\pi t/n)^{2}\varepsilon_{t}^{2}\overset{\mathcal{P}}{\rightarrow}\frac{\sigma^{2}}{2\pi}\\
\frac{1}{2\pi n}\sum_{t=1}^{n}\mathrm{E}\big(\alpha_{t,n}\beta_{t,n}\big|\varepsilon_{t-1},\varepsilon_{t-2},\ldots,\varepsilon_{1}\big) &= \frac{1}{2\pi n}\sum_{t=1}^{n}\cos(2k\pi t/n)\sin(2k\pi t/n)\varepsilon_{t}^{2}\overset{\mathcal{P}}{\rightarrow}0,
\end{aligned}$$

where the above follows from basic calculations using the mean and variance of the above. Finally we need to verify the Lindeberg condition, we only verify it for $\frac{1}{\sqrt{2\pi n}}\sum_{t=1}^{n}\alpha_{t,n}$, the same argument holds true for $\frac{1}{\sqrt{2\pi n}}\sum_{t=1}^{n}\beta_{t,n}$. We note that for every $\epsilon>0$ we have

$$\frac{1}{2\pi n}\sum_{t=1}^{n}\mathrm{E}\big(|\alpha_{t,n}|^{2}I(|\alpha_{t,n}|\geq 2\pi\sqrt{n}\epsilon)\big|\varepsilon_{t-1},\varepsilon_{t-2},\ldots\big)=\frac{1}{2\pi n}\sum_{t=1}^{n}\mathrm{E}\left[|\alpha_{t,n}|^{2}I(|\alpha_{t,n}|\geq 2\pi\sqrt{n}\epsilon)\right].$$

By using $|\alpha_{t,n}|=|\cos(2\pi t/n)\varepsilon_{t}|\leq|\varepsilon_{t}|$ the above can be bounded by

$$\begin{aligned}
&\frac{1}{2\pi n}\sum_{t=1}^{n}\mathrm{E}\big[|\alpha_{t,n}|^{2}I(|\alpha_{t,n}|\geq 2\pi\sqrt{n}\epsilon)\big]\\
\leq\;&\frac{1}{2\pi n}\sum_{t=1}^{n}\mathrm{E}\big[|\varepsilon_{t}|^{2}I(|\varepsilon_{t}|\geq 2\pi\sqrt{n}\epsilon)\big]=\mathrm{E}\big[|\varepsilon_{t}|^{2}I(|\varepsilon_{t}|\geq 2\pi\sqrt{n}\epsilon)\big]\overset{\mathcal{P}}{\rightarrow}0\quad\text{as}\quad n\rightarrow\infty,
\end{aligned}$$

the above is true because $\mathrm{E}(\varepsilon_{t}^{2})<\infty$. Hence we have verified Lindeberg condition and we obtain (10.15). The proof of (10.13) is similar, hence we omit the details. Because $I_{\varepsilon}(\omega)=\Re(J_{\varepsilon}(\omega))^{2}+\Im(J_{\varepsilon}(\omega))^{2}$, from (10.15) we have $I_{\varepsilon}(\omega)/\sigma^{2}\sim\chi^{2}(2)/2$ (which is the same as an exponential with mean one).

To prove (10.14) we can either derive it from first principles or by using Proposition 9.6.1. Here we do it from first principles. We observe

$$\mathrm{cov}(I_{\varepsilon}(\omega_{j}),I_{\varepsilon}(\omega_{k}))\;=\;\frac{1}{(2\pi)^{2}n^{2}}\sum_{k_{1}}\sum_{k_{2}}\sum_{t_{1}}\sum_{t_{2}}\mathrm{cov}(\varepsilon_{t_{1}}\varepsilon_{t_{1}+k_{1}},\varepsilon_{t_{2}}\varepsilon_{t_{2}+k_{2}}).$$

Expanding the covariance gives

$$\begin{aligned}
\mathrm{cov}(\varepsilon_{t_{1}}\varepsilon_{t_{1}+k_{1}},\varepsilon_{t_{2}}\varepsilon_{t_{2}+k_{2}}) &= \mathrm{cov}(\varepsilon_{t_{1}},\varepsilon_{t_{2}+k_{2}})\mathrm{cov}(\varepsilon_{t_{2}},\varepsilon_{t_{1}+k_{1}})+\mathrm{cov}(\varepsilon_{t_{1}},\varepsilon_{t_{2}})\mathrm{cov}(\varepsilon_{t_{1}+k_{1}},\varepsilon_{t_{2}+k_{2}})+\\
&\quad\mathrm{cum}(\varepsilon_{t_{1}},\varepsilon_{t_{1}+k_{1}},\varepsilon_{t_{2}},\varepsilon_{t_{2}+k_{2}}).
\end{aligned}$$

Since $\{\varepsilon_t\}$ are iid random variables, for most t_1, t_2, k_1 and k_2 the above covariance is zero. The exceptions are when $t_1 = t_2$ and $k_1 = k_2$ or $t_1 = t_2$ and $k_1 = k_2 = 0$ or $t_1 - t_2 = k_1 = -k_2$. Counting all these combinations we have

$$\mathrm{cov}(|J_\varepsilon(\omega_j)|^2, |J_\varepsilon(\omega_k)|^2) = \frac{2\sigma^4}{(2\pi)^2 n^2}\sum_k\sum_t\sum_t \exp(ik(\omega_j - \omega_k)) + \frac{1}{(2\pi)^2 n^2}\sum_t \kappa_4$$

where $\sigma^2 = \mathrm{var}(\varepsilon_t)$ and $\kappa_4 = \mathrm{cum}_4(\varepsilon) = \mathrm{cum}(\varepsilon_t, \varepsilon_t, \varepsilon_t, \varepsilon_t)$. We note that for $j \neq k$, $\sum_t \exp(ik(\omega_j - \omega_k)) = 0$ and for $j = k$, $\sum_t \exp(ik(\omega_j - \omega_k)) = n$, substutiting this into $\mathrm{cov}(|J_\varepsilon(\omega_j)|^2, |J_\varepsilon(\omega_k)|^2)$ gives us the desired result. □

By using (10.9) the following result follows immediately from Lemma 10.2.1, equation (10.15).

Corollary 10.2.1 *Let us suppose that $\{X_t\}$ is a linear time series $X_t = \sum_{j=-\infty}^{\infty}\psi_j\varepsilon_{t-j}$, where $\sum_{j=-\infty}^{\infty}|j^{1/2}\psi_j| < \infty$, and $\{\varepsilon_t\}$ are iid random variables with mean zero, variance σ^2 $\mathrm{E}(\varepsilon_t^4) < \infty$. Then we have*

$$\begin{pmatrix} \Re J_n(\omega) \\ \Im J_n(\omega) \end{pmatrix} \overset{\mathcal{D}}{\to} \mathcal{N}\left(0, \frac{1}{2}f(\omega)I_2\right), \qquad (10.15)$$

Using (10.11) we see that $I_n(\omega) \approx f(\omega)|J_\varepsilon(\omega)|^2$. This suggest that most of the properties which apply to $|J_\varepsilon(\omega)^2$ also apply to $I_n(\omega)$. Indeed in the following theorem we show that the asympototic distribution of $I_n(\omega)$ is exponential with asymptotic mean $f(\omega)$ and variance $f(\omega)^2$ (unless $\omega = 0$ in which case it is $2f(\omega)^2$).

By using Lemma 10.2.1 we now generalise Proposition 10.2.1 to linear processes. We show that just like the DFT the Periodogram is also 'near uncorrelated' at different frequencies. This result will be useful when motivating and deriving the sampling of the spectral density estimator in Section 10.3.

Theorem 10.2.1 *Suppose $\{X_t\}$ is a linear time series $X_t = \sum_{j=-\infty}^{\infty}\psi_j\varepsilon_{t-j}$, where $\sum_{j=-\infty}^{\infty}|j^{1/2}\psi_j| < \infty$ with $\mathrm{E}[\varepsilon_t] = 0$, $\mathrm{var}[\varepsilon_t] = \sigma^2$ and $\mathrm{E}[\varepsilon_t^4] < \infty$. Let $I_n(\omega)$ denote the periodogram associated with $\{X_1, \ldots, X_n\}$ and $f(\cdot)$ be the spectral density. Then*

(i) If $f(\omega) > 0$ for all $\omega \in [0, 2\pi]$ and $0 < \omega_1, \ldots, \omega_m < \pi$, then

$$\big(I_n(\omega_1)/f(\omega_1), \ldots, I_n(\omega_m)/f(\omega_m)\big)$$

converges in distribution (as $n \to \infty$) to a vector of independent exponential distributions with mean one.

(ii) Furthermore, for $\omega_j = \frac{2\pi j}{n}$ and $\omega_k = \frac{2\pi k}{n}$ we have

$$\mathrm{cov}(I_n(\omega_k), I_n(\omega_j)) = \begin{cases} 2f(\omega_k)^2 + O(n^{-1/2}) & \omega_j = \omega_k = 0 \text{ or } \pi \\ f(\omega_k)^2 + O(n^{-1/2}) & 0 < \omega_j = \omega_k < \pi \\ O(n^{-1}) & \omega_j \neq \omega_k \end{cases}$$

where the bound is uniform in ω_j and ω_k.

Remark 10.2.1 (Summary of properties of the periodogram) *(i) The periodogram is non-negative and is an asymptotically an unbiased estimator of the spectral density (when $\sum_j |\psi_j| < \infty$).*

(ii) It symmetric about zero, $I_n(\omega) = I_n(\omega + \pi)$, like the spectral density function.

(iii) At the fundemental frequencies $\{I_n(\omega_j)\}$ are asymptotically uncorrelated.

(iv) If $0 < \omega < \pi$, $I_n(\omega)$ is asymptotically exponentially distributed with mean $f(\omega)$.

It should be mentioned that Theorem 10.2.1 also holds for several nonlinear time series too.

10.3 Estimating the spectral density function

There are several explanations as to why the raw periodogram can not be used as an estimator of the spectral density function, despite its mean being approximately equal to the spectral density. One explanation is a direct consequence of Theorem 10.2.1, where we showed that the distribution of the periodogram standardized with the spectral density function is an exponential distribution, from here it is clear it will not converge to the mean, however large the sample size. An alternative, explanation is that the periodogram is the Fourier transform of the autocovariances estimators at n different lags. Typically the variance for each covariance $\hat{c}_n(k)$ will be about $O(n^{-1})$, thus, roughly speaking, the variance of $I_n(\omega)$ will be the sum of these n $O(n^{-1})$ variances which leads to a variance of $O(1)$, this clearly does not converge to zero.

Both these explanation motivate estimators of the spectral density function, which turn out to be the same. It is worth noting that Parzen (1957) first proposed a consistent estimator of the spectral density. These results not only lead to a revolution in spectral density estimation but also the usual density estimation that you may have encountered in nonparametric statistics (one of the first papers on density estimation is Parzen (1962)).

We recall that $J_n(\omega_k)$ are zero mean uncorrelated random variables whose variance is almost equal to $f(\omega_k)$. This means that $\mathrm{E}|J_n(\omega_k)|^2 = \mathrm{E}[I_n(\omega_n)] \approx f(\omega_k)$.

Remark 10.3.1 (Smoothness of the spectral density) *We observe that*

$$f^{(s)}(\omega) = \frac{1}{(2\pi)} \sum_{r \in \mathbb{Z}} (ir)^s c(r) \exp(ir\omega).$$

Therefore, the smoothness of the spectral density function is determined by finiteness of $\sum_r |r^s c(r)|$, in other words how fast the autocovariance function converges to zero. We recall that the acf of ARMA processes decay exponential fast to zero, thus f is extremely smooth (all derivatives exist).

Assuming that the autocovariance function converges to zero sufficiently fast f will slowly vary over frequency. Furthermore, using Theorem 10.2.1, we know that $\{I_n(\omega_k)\}$ are close to uncorrelated and $I_n(\omega_k)/f(\omega_k)$ is $2^{-1}\chi^2_2$. Therefore we can write $I_n(\omega_k)$ as

$$\begin{aligned} I_n(\omega_k) &= \mathrm{E}(I_n(\omega_k)) + [I_n(\omega_k) - \mathrm{E}(I_n(\omega_k))] \\ &\approx f(\omega_k) + f(\omega_k)U_k, \qquad k = 1, \ldots, n, \end{aligned} \tag{10.16}$$

where $\{U_k\}$ is sequence of mean zero and constant variance almost uncorrelated random variables.

We recall (10.16) resembles the usual nonparametric equation (function plus noise) often considered in nonparametric statistics.

Remark 10.3.2 (Nonparametric Kernel estimation) *Let us suppose that we observe Y_i where*

$$Y_i = g\left(\frac{i}{n}\right) + \varepsilon_i \qquad 1 \leq i \leq n,$$

and $\{\varepsilon_i\}$ are iid random variables and $g(\cdot)$ is a 'smooth' function. The kernel density estimator of $\widehat{g}_n(\frac{i}{n})$

$$\widehat{g}_n\left(\frac{j}{n}\right) = \sum_i \frac{1}{bn} W\left(\frac{j-i}{bn}\right) Y_i,$$

where $W(\cdot)$ is a smooth kernel function of your choosing, such as the Gaussian kernel, etc.

This suggest that to estimate the spectral density we could use a local weighted average of $\{I_n(\omega_k)\}$. Equation (10.16) motivates the following nonparametric estimator of $f(\omega)$

$$\widehat{f}_n(\omega_j) = \sum_k \frac{1}{bn} W\left(\frac{j-k}{bn}\right) I_n(\omega_k), \tag{10.17}$$

where $W(\cdot)$ is a spectral window which satisfies $\int W(x)dx = 1$ and $\int W(x)^2 dx < \infty$.

Example 10.3.1 (Spectral windows) *Here we give examples of spectral windows (see Section 6.2.3, page 437 in Priestley (1983)).*

(i) The Daniell spectral Window is the local average

$$W(x) = \begin{cases} 1/2 & |x| \leq 1 \\ 0 & |x| > 1 \end{cases}$$

This window leads to the estimator

$$\widehat{f}_n(\omega_j) = \frac{1}{bn} \sum_{k=j-bn/2}^{j+bn/2} I_n(\omega_k).$$

A plot of the periodgram, spectral density and different estimators (using Daniell kernel with $bn = 2$ and $bn = 10$) of the $AR(2)$ process $X_t = 1.5X_{t-1} - 0.75X_{t-2} + \varepsilon_t$ is given in Figure 10.1. We observe that too small b leads to undersmoothing but too large b leads to over smoothing of features. There are various methods for selecting the bandwidth, one commonly method based on the Kullbach-Leibler criterion is proposed in Beltrao and Bloomfield (1987).

(ii) The Bartlett-Priestley spectral Window

$$W(x) = \begin{cases} \frac{3}{4}\left(1 - x^2\right) & |x| \leq 1 \\ 0 & |x| > 1 \end{cases}$$

This spectral window was designed to reduce the mean squared error of the spectral density estimator (under certain smoothness conditions).

The above estimator was constructed within the frequency domain. We now consider a spectral density estimator constructed within the 'time domain'. We do this by considering the periodogram

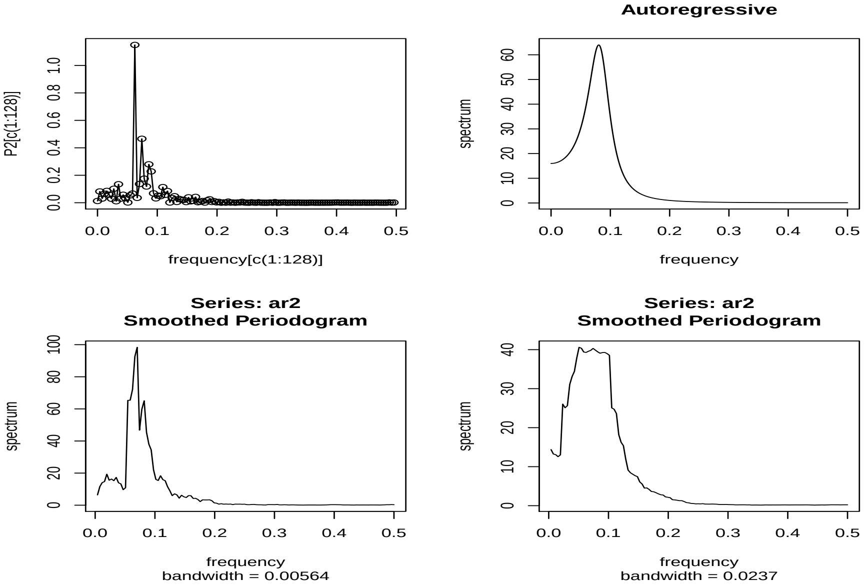


Figure 10.1: Using a realisation of the AR(2): $X_t = 1.5X_{t-1} - 0.75X_{t-2} + \varepsilon_t$ where $n = 256$. Top left: Periodogram, Top Right: True spectral density function. Bottom left: Spectral density estimator with $bn = 2$ and Bottom right: Spectral density estimator with $bn = 10$.

from an alternative angle. We recall that

$$I_n(\omega) = \frac{1}{2\pi}\sum_{k=-(n-1)}^{n-1} \hat{c}_n(k)\exp(ik\omega),$$

thus it is the sum of n autocovariance estimators. This is a type of sieve estimator (a nonparametric function estimator which estimates the coefficients/covariances in a series expansion). But as we explained above, this estimator is not viable because it uses too many coefficient estimators. Since the true coefficients/covariances decay to zero for large lags, this suggests that we do not use all the sample covariances in the estimator, just some of them. Hence a viable estimator of the spectral density is the truncated autocovariance estimator

$$\widetilde{f}_n(\omega) = \frac{1}{2\pi}\sum_{k=-m}^{m} \hat{c}_n(k)\exp(ik\omega), \qquad (10.18)$$

or a generalised version of this which down weights the sample autocovariances at larger lags

$$\widetilde{f}_n(\omega) = \frac{1}{2\pi}\sum_{k=-(n-1)}^{n-1} \lambda\left(\frac{k}{m}\right)\hat{c}_n(k)\exp(ik\omega), \qquad (10.19)$$

where $\lambda(\cdot)$ is the so called lag window. The estimators (10.17) and (10.19) are very conceptionally similar, this can be understood if we rewrite $\hat{c}_n(r)$ in terms the periodogram $\hat{c}_n(r) = \int_0^{2\pi} I_n(\omega)\exp(-ir\omega)d\omega$, and transforming (10.19) back into the frequency domain

$$\widetilde{f}_n(\omega) \quad = \quad \frac{1}{2\pi}\int I_n(\lambda) \sum_{k=-(n-1)}^{n-1} \lambda(\frac{k}{m})\exp(ik(\omega-\lambda))d\lambda = \frac{1}{2\pi}\int I_n(\lambda)W_m(\omega-\lambda)d\omega, \tag{10.20}$$

where $W_m(\omega) = \frac{1}{2\pi}\sum_{k=-(n-1)}^{n-1}\lambda(\frac{k}{m})\exp(ik\omega)$.

Example 10.3.2 (Examples of Lag windows) *Here we detail examples of lag windows.*

(i) <u>*Truncated Periodogram lag Window*</u> $\lambda(u) = I_{[-1,1]}(u)$*, where* $\{\lambda(k/m)\}$ *corresponds to*

$$W_m(x) = \frac{1}{2\pi}\sum_{k=-m}^{m} e^{ik\omega} = \frac{1}{2\pi}\frac{\sin[(m+1/2)x]}{\sin(x/2)},$$

which is the Dirichlet kernel.

Note that the Dirchlet kernel can be negative, thus we can see from (10.20) that $\widetilde{f}_n$ *can be negative. Which is one potential drawback of this estimator (see Example 9.5.2).*

(ii) <u>*The Bartlett lag Window*</u> $\lambda(x) = (1-|x|)I_{[-1,1]}(x)$*, where* $\{\lambda(k/m)\}$ *corresponds to*

$$W_m(x) = \frac{1}{2\pi}\sum_{k=-m}^{m}\left(1-\frac{||k}{m}\right)e^{ik\omega} = \frac{1}{2\pi n}\left(\frac{\sin(nx/2)}{\sin(x/2)}\right)^2$$

which is the Fejer kernel. We can immediately see that one advantage of the Bartlett window is that it corresponds to a spectral density estimator which is positive.

Note that in the case that $m = n$ *(the sample size), the truncated periodogram window estimator corresponds to* $\sum_{|r|\leq n} c(r)e^{ir\omega}$ *and the Bartlett window estimator corresponds to* $\sum_{|r|\leq n}[1-|r|/n]c(r)e^{ir\omega}$*.*

$W_m(\cdot)$ and $\frac{1}{b}W(\frac{\cdot}{b})$ (defined in (10.17)) cannot not be the same function, but they share many of the same characteristics. In particular,

$$\begin{aligned} W_m(\omega) \quad &= \quad \sum_{k=-(n-1)}^{n-1}\lambda\left(\frac{k}{m}\right)\exp(ik\omega) = m\sum_{k=-(m-1)}^{m-1}\frac{1}{m}\lambda\left(\frac{k}{m}\right)\exp\left(i\frac{k}{m}\cdot m\omega\right) \\ &= \quad m\frac{1}{m}\sum_{k=-(m-1)}^{m-1}\lambda\left(\omega_k\right)\exp\left(i\omega_k(m\omega)\right), \end{aligned}$$

where $\omega_k = k/n$. By using (A.2) and (A.3) (in the appendix), we can approximate the sum by the integral and obtain

$$W_m(\omega) \quad = \quad mW(m\omega) + O(1), \qquad \text{where } W(\omega) = \int \lambda(x)\exp(i\omega)dx.$$

Therefore

$$\widetilde{f}_n(\omega) \quad \approx \quad m\int I_n(\lambda)K(m(\omega-\lambda))d\omega.$$

Comparing with $\widehat{f}_n$ and $\widetilde{f}_n(\omega)$ we see that m plays the same role as b^{-1}. Furthermore, we observe $\sum_k \frac{1}{bn}W(\frac{j-k}{bn})I(\omega_k)$ is the sum of about nb $I(\omega_k)$ terms. The equivalent for $W_m(\cdot)$, is that it has the 'spectral' width n/m. In other words since $\tilde{f}_n(\omega) = \frac{1}{2\pi}\sum_{k=-(n-1)}^{n-1}\lambda(\frac{k}{M})\hat{c}_n(k)\exp(ik\omega) = \frac{1}{2\pi}\int mI_n(\lambda)W(M(\omega-\lambda))d\omega$, it is the sum of about n/m terms.

We now analyze the sampling properties of the spectral density estimator. It is worth noting that the analysis is very similar to the analysis of nonparametric kernel regression estimator $\widehat{g}_n(\frac{j}{n}) = \frac{1}{bn}\sum_i W(\frac{j-i}{bn})Y_i$, where $Y_i = g(\frac{i}{n}) + g(\frac{i}{n})\varepsilon_i$ and $\{\varepsilon_i\}$ are iid random variables. This is because the periodogram $\{I_n(\omega)\}_k$ is 'near uncorrelated'. However, still some care needs to be taken in the proof to ensure that the errors in the near uncorrelated term does not build up.

Theorem 10.3.1 *Suppose $\{X_t\}$ satisfy $X_t = \sum_{j=-\infty}^{\infty}\psi_j\varepsilon_{t-j}$, where $\sum_{j=-\infty}^{\infty}|j\psi_j| < \infty$ and $\mathrm{E}(\varepsilon_t^4) < \infty$. Let $\hat{f}_n(\omega)$ be the spectral estimator defined in (10.17). Then*

$$\left|\mathrm{E}(\hat{f}_n(\omega_j)) - f(\omega_j)\right| \leq C\left(\frac{1}{n} + b\right) \tag{10.21}$$

and

$$\mathrm{var}[\hat{f}_n(\omega_j)] \to \begin{cases} \frac{1}{bn}f(\omega_j)^2 & 0 < \omega_j < \pi \\ \frac{2}{bn}f(\omega_j)^2 & \omega_j = 0 \textit{ or } \pi \end{cases}, \tag{10.22}$$

$bn \to \infty$, $b \to 0$ as $n \to \infty$.

PROOF. The proof of both (10.21) and (10.22) are based on the spectral window $W(x/b)$ becoming narrower as $b \to 0$, hence there is increasing localisation as the sample size grows (just like nonparametric regression).

We first note that by using Lemma 4.1.1(ii) we have $\sum_r |rc(r)| < \infty$, thus $|f'(\omega)| \leq \sum_r |rc(r)| <$

∞. Hence f is continuous with a bounded first derivative.

To prove (10.21) we take expections

$$\begin{aligned}
\left|\mathrm{E}(\hat{f}_n(\omega_j)) - f(\omega_j)\right| &= \left|\sum_k \frac{1}{bn} W\left(\frac{k}{bn}\right)\left\{\mathrm{E}\big[I(\omega_{j-k})\big] - f(\omega_j)\right\}\right| \\
&= \sum_k \frac{1}{bn}\left|W\left(\frac{k}{bn}\right)\right| \left|\mathrm{E}\big[I(\omega_{j-k})\big] - f(\omega_{j-k})\right| + \sum_k \frac{1}{bn}\left|W\left(\frac{k}{bn}\right)\right| |f(\omega_j) - f(\omega_{j-k})| \\
&:= I + II.
\end{aligned}$$

Using Lemma 10.1.1 we have

$$\begin{aligned}
I &= \sum_k \frac{1}{bn}\left|W\left(\frac{k}{bn}\right)|\right| \left|\mathrm{E}\big(I(\omega_{j-k})\big) - f(\omega_{j-k})\right| \\
&\leq C\left(\frac{1}{bn}\sum_k |W(\frac{k}{bn})|\right)\left(\sum_{|k|\geq n} |c(k)| + \frac{1}{n}\sum_{|k|\leq n} |kc(k)|\right) = O(\frac{1}{n}).
\end{aligned}$$

To bound II we use that $|f(\omega_1) - f(\omega_2)| \leq \sup|f'(\omega)| \cdot |\omega_1 - \omega_2|$, this gives

$$II = \big|\sum_k \frac{1}{bn} K(\frac{k}{bn})\big\{f(\omega_j) - f(\omega_{j-k})\big\}\big| = O(b).$$

Altogether this gives $I = O(n^{-1})$ and $II = O(b)$ as $bn \to \infty$, $b \to 0$ and $n \to \infty$. The above two bounds mean give (10.21).

We will use Theorem 10.2.1 to prove (10.22). We first assume that $j \neq 0$ or n. To prove the result we use that

$$\begin{aligned}
&\mathrm{cov}(|J_n(\omega_{k_1})|^2, |J_n(\omega_{k_2})|^2) = \\
&[f(\omega_{k_1})I(k_1 = k_2) + O(\frac{1}{n})]^2 + [f(\omega_{k_1})I(k_1 = n - k_2) + O(\frac{1}{n})][f(\omega_{k_1})I(n - k_1 = k_2) + O(\frac{1}{n})] \\
&+[\frac{1}{n}f_4(\omega_1, -\omega_1, \omega_2) + O(\frac{1}{n^2})].
\end{aligned}$$

where the above follows from Proposition 9.6.1. This gives

$$
\begin{aligned}
& \mathrm{var}(\widehat{f}_n(\omega_j)) \\
= \;& \sum_{k_1,k_2} \frac{1}{(bn)^2} W\left(\frac{j-k_1}{bn}\right) W\left(\frac{j-k_2}{bn}\right) \mathrm{cov}(I(\omega_{k_1}), I(\omega_{k_1})) \\
= \;& \sum_{k_1,k_2} \frac{1}{(bn)^2} W\left(\frac{j-k_1}{bn}\right) W\left(\frac{j-k_2}{bn}\right) \\
& \Bigg(\big[f(\omega_{k_1}) I(k_1 = k_2) + O(\frac{1}{n})\big]^2 + \big[f(\omega_{k_1}) I(k_1 = n-k_2) + O(\frac{1}{n})][f(\omega_{k_1}) I(n-k_1 = k_2) + O(\frac{1}{n})\big] \\
& + [\frac{1}{n} f_4(\omega_1, -\omega_1, \omega_2) + O(\frac{1}{n^2})] \Bigg) \\
= \;& \sum_{k=1}^{n} \frac{1}{(bn)^2} W\left(\frac{j-k_1}{bn}\right)^2 f(\omega_k^2) \\
& + \sum_{k=1}^{n} \frac{1}{(bn)^2} W\left(\frac{j-k_1}{bn}\right) W\left(\frac{j-(n-k_1)}{bn}\right) f(\omega_k^2) + O(\frac{1}{n}) \\
= \;& \frac{1}{2\pi nb} \int \frac{1}{b} W\left(\frac{\omega_j - \omega}{b}\right)^2 f(\omega)^2 d\omega + \frac{1}{2\pi nb} \underbrace{\int \frac{1}{b} W\left(\frac{\omega_j - 2\pi + \omega}{b}\right) W\left(\frac{\omega_j - \omega}{b}\right) f(\omega) d\omega}_{\to 0} + O(\frac{1}{n}) \\
= \;& \frac{1}{2\pi nb} f(\omega_j)^2 \int \frac{1}{b} W\left(\frac{\omega}{b}\right)^2 d\omega + O(\frac{1}{n})
\end{aligned}
$$

where the above is using the Riemann integral. A similar proof can be used to prove the case $j = 0$ or n. □

The above result means that the mean squared error of the estimator

$$
\mathrm{E}\big[\hat{f}_n(\omega_j) - f(\omega_j)\big]^2 \to 0,
$$

where $bn \to \infty$ and $b \to 0$ as $n \to \infty$. Moreover

$$
\mathrm{E}\big[\hat{f}_n(\omega_j) - f(\omega_j)\big]^2 \;=\; O\left(\frac{1}{bn} + b^2\right).
$$

Remark 10.3.3 (The distribution of the spectral density estimator) *Using that the periodogram $I_n(\omega)/f(\omega)$ is asymptotically exponentially distributed and uncorrelated at the fundemental frequencies, we can heuristically deduce the limiting distribution of $\hat{f}_n(\omega)$. Here we consider the* *distribution with the rectangular spectral window*

$$\hat{f}_n(\omega_j) \;=\; \frac{1}{bn}\sum_{k=j-bn/2}^{j+bn/2} I(\omega_k).$$

Since $I(\omega_k)/f(\omega_k)$ are approximately $\chi^2(2)/2$, then since the sum $\sum_{k=j-bn/2}^{j+bn/2} I(\omega_k)$ is taken over a local neighbourhood of ω_j, we have that $f(\omega_j)^{-1}\sum_{k=j-bn/2}^{j+bn/2} I(\omega_k)$ is approximately $\chi^2(2bn)/2$.

We note that when bn is large, then $\chi^2(2bn)/2$ is close to normal. Hence

$$\sqrt{bn}\hat{f}_n(\omega_j) \approx N(f(\omega_j), f(\omega_j)^2).$$

Using this these asymptotic results, we can construct confidence intervals for $f(\omega_j)$.

In general, to prove normality of $\hat{f}_n$ we rewrite it as a quadratic form, from this asymptotic normality can be derived, where

$$\sqrt{bn}\hat{f}_n(\omega_j) \approx N\bigg(f(\omega_j), f(\omega_j)^2\int W(u)^2du\bigg).$$

The variance of the spectral density estimator is simple to derive by using Proposition 9.6.1. The remarkable aspect is that the variance of the spectral density does not involve (asymptotically) the fourth order cumulant (as it is off lower order).

10.4 The Whittle Likelihood

In Chapter 7 we considered various methods for estimating the parameters of an ARMA process. The most efficient method (in terms of Fisher efficiency), when the errors are Gaussian is the Gaussian maximum likelihood estimator. This estimator was defined in the time domain, but it is interesting to note that a very similar estimator which is asymptotically equivalent to the GMLE estimator can be defined within the frequency domain. We start by using heuristics to define the Whittle likelihood. We then show how it is related to the Gaussian maximum likelihood.

To motivate the method let us return to the Sunspot data considered in Exercise 5.1. The

Periodogram and the spectral density corresponding to the best fitting autoregressive model,

$$\begin{aligned} f(\omega) &= (2\pi)^{-1}\Big|1 - 1.1584e^{i\omega} - 0.3890e^{i2\omega} - 0.1674e^{i3\omega} - 0.1385e^{i4\omega} - 0.1054e^{i5\omega} - 0.0559e^{i6\omega} - \\ &\quad 0.0049e^{i7\omega} - 0.0572e^{i8\omega} - 0.2378e^{\omega}\Big|^{-2}, \end{aligned}$$

is given in Figure 10.2. We see that the spectral density of the best fitting AR process closely follows the shape of the periodogram (the DFT modulo square). This means that indirectly the autoregressive estimator (Yule-Walker) chose the AR parameters which best fitted the shape of the periodogram. The Whittle likelihood estimator, that we describe below, does this directly. By selecting the parametric spectral density function which best fits the periodogram. The Whittle likelihood measures the distance between $I_n(\omega)$ and the parametric spectral density function using the Kullbach-Leibler criterion

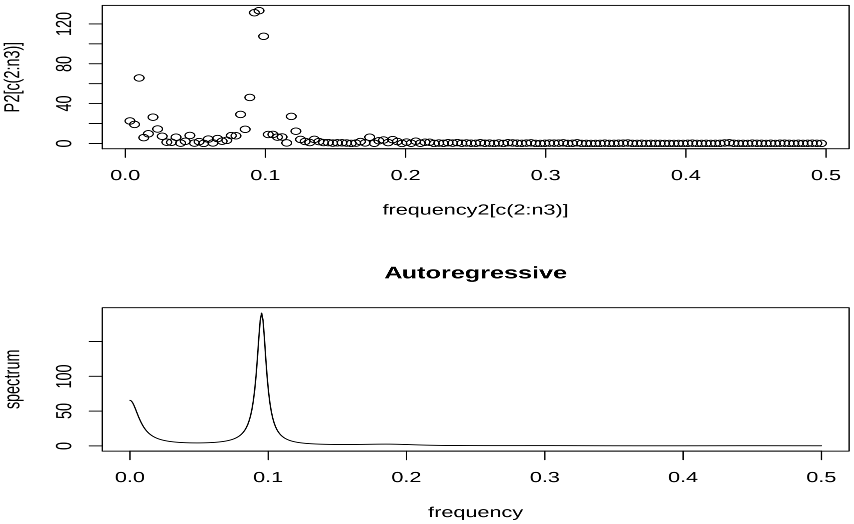


Figure 10.2: The periodogram of sunspot data (with the mean removed, which is necessary to prevent a huge peak at zero) and the spectral density of the best fitting AR model.

$$L_n^w(\boldsymbol{\theta}) = \sum_{k=1}^{n}\left(\log f_{\boldsymbol{\theta}}(\omega_k) + \frac{I_n(\omega_k)}{f_{\boldsymbol{\theta}}(\omega_k)}\right), \quad \omega_k = \frac{2\pi k}{n},$$

and the parametric model which minimises this 'distance' is used as the estimated model. The choice of this criterion over the other distance criterions may appear to be a little arbitrary, however there are several reasons why this is considered a good choice. Below we give some justifications as to why this criterion is the prefered one.

First let us suppose that we observe $\{X_t\}_{t=1}^{n}$, where X_t satisfies the ARMA representation

$$X_t = \sum_{j=1}^{p} \phi_j X_{t-j} + \sum_{j=1}^{q} \psi_j \varepsilon_{t-j} + \varepsilon_t,$$

and $\{\varepsilon_t\}$ are iid random variables. We will assume that $\{\phi_j\}$ and $\{\psi_j\}$ are such that the roots of their corresponding characteristic polynomial are greater than $1 + \delta$. Let $\boldsymbol{\theta} = (\underline{\phi}, \underline{\theta})$. As we mentioned in Section 9.2 if $\sum_r |rc(r)| < \infty$, then

$$\mathrm{cov}(J_n(\omega_{k_1}), J_n(\omega_{k_2})) = \left(\begin{array}{cc} f(\omega_{k_1}) + O(\frac{1}{n}) & k_1 = k_2 \\ O(\frac{1}{n}) & k_1 \neq k_2, \end{array} \right).$$

where

$$f(\omega) = \frac{\sigma^2 |1 + \sum_{j=1}^{q} \theta_j \exp(ij\omega)|^2}{2\pi |1 + \sum_{j=1}^{p} \phi_j \exp(ij\omega)|^2}.$$

In other words, if the time series satisfies an ARMA presentation the DFT is 'near' uncorrelated, its mean is zero and its variance has a well specified parametric form. Using this information we can define a criterion for estimating the parameters. We motivate this criterion through the likelihood, however there are various other methods for motivating the criterion for example the Kullbach-Leibler criterion is an alternative motivation, we comment on this later on.

If the innovations are Gaussian then $\Re J_n(\omega)$ and $\Im J_n(\omega)$ are also Gaussian, thus by using above we approximately have

$$\mathcal{J}_n = \left(\begin{array}{c} \Re J_n(\omega_1) \\ \Im J_n(\omega_1) \\ \vdots \\ \Re J_n(\omega_{n/2}) \\ \Im J_n(\omega_{n/2}) \end{array} \right) \sim \mathcal{N}(0, \mathrm{diag}(f(\omega_1), f(\omega_1), \ldots, f(\omega_{n/2}), f(\omega_{n/2}))).$$

In the case that the innovations are not normal then, by Corollary 10.2.1, the above holds asymptotically for a finite number of frequencies. Here we construct the likelihood under normality of the innovations, however, this assumption is <u>not</u> required and is only used to motivate the construction.

Since $\mathcal{J}_n$ is normally distributed random vector with mean zero and 'approximate' diagonal matrix variance matrix $\text{diag}(f(\omega_1), \ldots, f(\omega_n))$, the negative log-likelihood of $\mathcal{J}_n$ is approximately

$$L_n^w(\boldsymbol{\theta}) = \sum_{k=1}^{n} \bigg(\log |f_{\boldsymbol{\theta}}(\omega_k)| + \frac{|J_X(\omega_k)|^2}{f_{\boldsymbol{\theta}}(\omega_k)} \bigg).$$

To estimate the parameter we would choose the $\underline{\theta}$ which minimises the above criterion, that is

$$\widehat{\boldsymbol{\theta}}_n^w = \arg\min_{\boldsymbol{\theta} \in \Theta} L_n^w(\boldsymbol{\theta}), \tag{10.23}$$

where Θ consists of all parameters where the roots of the corresponding characteristic polynomial have absolute value greater than $(1 + \delta)$ (note that under this assumption all spectral densities corresponding to these parameters will be bounded away from zero).

Example 10.4.1 *Fitting an ARMA$(1,1)$ model to the data To fit an ARMA model to the data using the Whittle likelihood we use the criterion*

$$L_n^w(\boldsymbol{\theta}) = \sum_{k=1}^{n/2} \left(\log \frac{\sigma^2 |1 + \theta e^{i\omega_k}|^2}{2\pi |1 - \phi e^{i\omega_k}|} + I_n(\omega_k) \frac{2\pi |1 - \phi e^{i\omega_k}|^2}{\sigma^2 |1 + \theta e^{i\omega_k}|^2} \right).$$

By differentiating L_n^ω with respect to ϕ, σ^2 and θ we solve these three equations (usually numerically), this gives us the Whittle likelihood estimators.

Whittle (1962) showed that the above criterion is an approximation of the GMLE. The correct proof is quite complicated and uses several matrix approximations due to Grenander and Szegö (1958). Instead we give a heuristic proof which is quite enlightening.

Returning the the Gaussian likelihood for the ARMA process, defined in (8.24), we rewrite it as

$$L_n(\boldsymbol{\theta}) = -\big(\det |R_n(\boldsymbol{\theta})| + \mathbf{X}_n' R_n(\boldsymbol{\theta})^{-1} \mathbf{X}_n\big) = -\big(\det |R_n(f_{\boldsymbol{\theta}})| + \mathbf{X}_n' R_n(f_{\boldsymbol{\theta}})^{-1} \mathbf{X}_n\big), \tag{10.24}$$

where $R_n(f_{\boldsymbol{\theta}})_{s,t} = \int f_{\boldsymbol{\theta}}(\omega) \exp(i(s-t)\omega) d\omega$ and $\mathbf{X}_n' = (X_1, \ldots, X_n)$. We now show that $L_n(\boldsymbol{\theta}) \approx -L_n^w(\boldsymbol{\theta})$.

Lemma 10.4.1 *Suppose that $\{X_t\}$ is a stationary ARMA time series with absolutely summable* *covariances and $f_{\boldsymbol{\theta}}(\omega)$ is the corresponding spectral density function. Then*

$$\det|R_n(f_{\boldsymbol{\theta}})| + \mathbf{X}_n' R_n(f_{\boldsymbol{\theta}})^{-1}\mathbf{X}_n = \sum_{k=1}^{n}\left(\log|f_{\boldsymbol{\theta}}(\omega_k)| + \frac{|J_n(\omega_k)|^2}{f_{\boldsymbol{\theta}}(\omega_k)}\right) + O(1),$$

for large n.

PROOF. There are various ways to precisely prove this result. All of them show that the Toeplitz matrix can in some sense be approximated by a circulant matrix. This result uses Szegö's identity (Grenander and Szegö (1958)). The main difficulty in the proof is showing that $R_n(f_{\boldsymbol{\theta}})^{-1} \approx U_n(f_{\boldsymbol{\theta}}^{-1})$, where $U_n(f_{\boldsymbol{\theta}}^{-1})_{s,t} = \int f_{\boldsymbol{\theta}}(\omega)^{-1}\exp(i(s-t)\omega)d\omega$. An interesting derivation is given in Brockwell and Davis (1998), Section 10.8. The main ingredients in the proof are:

1. For a sufficiently large m, $R_n(f_{\boldsymbol{\theta}})^{-1}$ can be approximated by $R_n(g_m)^{-1}$, where g_m is the spectral density of an mth order autoregressive process (this follows from Lemma 9.5.2), and showing that

$$\begin{aligned}\underline{X}_n' R_n(f_{\boldsymbol{\theta}})^{-1}\underline{X}_n - \underline{X}_n' R_n(g_m)^{-1}\underline{X}_n &= \underline{X}_n'\left[R_n(f_{\boldsymbol{\theta}})^{-1} - R_n(g_m)^{-1}\right]\underline{X}_n \\ &= \underline{X}_n' R_n(g_m)^{-1}\left[R_n(g_m) - R_n(f_{\boldsymbol{\theta}})\right]R_n(f_{\boldsymbol{\theta}}^{-1})\underline{X}_n \to 0.\end{aligned}$$

2. From Section 4.3.1, we recall if g_m is the spectral density of an AR(m) process, then for $n >> m$, $R_n(g_m)^{-1}$ will be bandlimited with most of its rows a shift of the other (thus with the exception of the first m and last m rows it is close to circulant).

3. We approximate $R_n(g_m)^{-1}$ with a circulant matrix, showing that

$$\underline{X}_n'\left[R_n(g_m)^{-1} - C_n(g_m^{-1})\right]\underline{X}_n \to 0,$$

   where $C_n(g_m^{-2})$ is the corresponding circulant matrix (where for $0 < |i-j| \leq m$ and either i or j is greater than m, $(C_n(g^{-1}))_{ij} = 2\sum_{k=|i-j|}^{m}\phi_{m,k}\phi_{m,k-|i-j|+1} - \phi_{m,|i-j|}$) with the eigenvalues $\{g_m(\omega_k)^{-1}\}_{k=1}^{n}$.

4. These steps show that

$$\underline{X}_n'\left[R_n(f_{\boldsymbol{\theta}})^{-1} - U_n(g_m^{-1})\right]\underline{X}_n \to 0$$

   as $m \to \infty$ as $n \to \infty$, which gives the result.

□

Remark 10.4.1 (A heuristic derivation) *We give a heuristic proof. Using the results in Section 9.2 we have see that $R_n(f_{\boldsymbol{\theta}})$ can be approximately written in terms of the eigenvalue and eigenvectors of the circulant matrix associated with $R_n(f_{\boldsymbol{\theta}})$, that is*

$$R_n(f_{\boldsymbol{\theta}}) \approx F_n \Delta(f_{\boldsymbol{\theta}}) \bar{F_n} \quad \textit{thus} \quad R_n(f_{\boldsymbol{\theta}})^{-1} \approx \bar{F_n} \Delta(f_{\boldsymbol{\theta}})^{-1} F_n, \tag{10.25}$$

where $\Delta(f_{\boldsymbol{\theta}}) = diag(f_{\boldsymbol{\theta}}^{(n)}(\omega_1), \ldots, f_{\boldsymbol{\theta}}^{(n)}(\omega_n))$, $f_{\boldsymbol{\theta}}^{(n)}(\omega) = \sum_{j=-(n-1)}^{(n-1)} c_{\boldsymbol{\theta}}(k)\exp(ik\omega) \to f_{\boldsymbol{\theta}}(\omega)$ and $\omega_k = 2\pi k/n$. Basic calculations give

$$\mathbf{X}_n \bar{F_n} = (J_n(\omega_1), \ldots, J_n(\omega_n)). \tag{10.26}$$

Substituting (10.26) and (10.25) into (10.27) yields

$$\frac{1}{n}L_n(\boldsymbol{\theta}) \;\approx\; -\frac{1}{n}\sum_{k=1}^{n}\left(\log f_{\boldsymbol{\theta}}(\omega_k) + \frac{|J_n(\omega_k)|^2}{f_{\boldsymbol{\theta}}(\omega_k)}\right) = \frac{1}{n}L^w(\boldsymbol{\theta}). \tag{10.27}$$

Hence using the approximation in (10.25) leads to a heuristic equivalence between the Whittle and Gaussian likelihood.

Lemma 10.4.2 (Consistency) *Suppose that $\{X_t\}$ is a causal ARMA process with parameters $\boldsymbol{\theta}$ whose roots lie outside the $(1+\delta)$-circle (where $\delta > 0$ is arbitrary). Let $\widehat{\boldsymbol{\theta}}^w$ be defined as in (10.23) and suppose that $\mathrm{E}(\varepsilon_t^4) < \infty$. Then we have*

$$\widehat{\boldsymbol{\theta}}^w \overset{\mathcal{P}}{\to} \boldsymbol{\theta}.$$

PROOF. To show consistency we need to show pointwise convergence and equicontinuity of $\frac{1}{n}\mathcal{L}_n$. Let

$$L^w(\boldsymbol{\theta}) = \frac{1}{2\pi}\int_0^{2\pi}\left(\log f_{\boldsymbol{\theta}}(\omega) + \frac{f_{\boldsymbol{\theta}_0}(\omega)}{f_{\boldsymbol{\theta}}(\omega)}\right)d\omega.$$

It is straightforward to show that $\mathrm{E}(\frac{1}{n}\mathcal{L}_n^w(\boldsymbol{\theta})) \to \tilde{\mathcal{L}_n}(\boldsymbol{\theta})$. Next we evaluate the variance, to do this we use Proposition 9.6.1 and obtain

$$\text{var}\left[\frac{1}{n}L_n^w(\boldsymbol{\theta})\right] = \frac{1}{n^2}\sum_{k_1,k_2=1}^{n}\frac{1}{f_{\boldsymbol{\theta}}(\omega_{k_1})f_{\boldsymbol{\theta}}(\omega_{k_2})}\text{cov}(|J_n(\omega_{k_1})|^2,|J_n(\omega_{k_2})|^2) = O(\frac{1}{n}).$$

Thus we have

$$\frac{1}{n}L_n^w(\boldsymbol{\theta}) \overset{\mathcal{P}}{\to} L^w(\boldsymbol{\theta}).$$

To show equicontinuity we apply the mean value theorem to $\frac{1}{n}L_n^w$. We note that because the parameters $(\underline{\phi},\underline{\theta}) \in \Theta$, have characteristic polynomial whose roots are greater than $(1+\delta)$ then $f_{\boldsymbol{\theta}}(\omega)$ is bounded away from zero (there exists a $\delta^* > 0$ where $\inf_{\omega,\boldsymbol{\theta}} f_{\boldsymbol{\theta}}(\omega) \geq \delta^*$). Hence it can be shown that there exists a random sequence $\{\mathcal{K}_n\}$ such that $|\frac{1}{n}L_n^w(\boldsymbol{\theta}_1) - \frac{1}{n}L_n^w(\boldsymbol{\theta}_2))| \leq \mathcal{K}_n(\|\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2\|)$ and $\mathcal{K}_n$ converges almost surely to a finite constant as $n \to \infty$. Therefore $\frac{1}{n}L_n$ is stochastically equicontinuous. Since the parameter space Θ is compact, the three standard conditions are satisfied and we have consistency of the Whittle estimator. □

To show asymptotic normality we note that $\frac{1}{n}L_n^w(\boldsymbol{\theta})$ can be written as a quadratic form

$$\frac{1}{n}L_n^w(\boldsymbol{\theta}) = \int_0^{2\pi}\log f_{\boldsymbol{\theta}}(\omega_k) + \frac{1}{n}\sum_{r=-(n-1)}^{n-1} d_n(r;\boldsymbol{\theta})\sum_{k=1}^{n-|r|}X_kX_{k+r}$$

where

$$d_n(r;\boldsymbol{\theta}) = \frac{1}{n}\sum_{k=1}^{n} f_{\boldsymbol{\theta}}(\omega_k)^{-1}\exp(ir\omega_k).$$

Using the above quadratic form and it's derivatives wrt $\boldsymbol{\theta}$ one can show normality of the Whittle likelihood under various dependence conditions on the time series. Using this result, in the following theorem we show asymptotic normality of the Whittle estimator. Note, this result not only applies to linear time series, but several types of nonlinear time series too.

Theorem 10.4.1 *Let us suppose that $\{X_t\}$ is a strictly stationary time series with a sufficient dependence structure (such as linearity, mixing at a certain rate, etc.) with spectral density function*

$f_{\boldsymbol{\theta}}(\omega)$ and $\mathrm{E}|X_t^4| < \infty$. Let

$$L_n^w(\boldsymbol{\theta}) = \sum_{k=1}^{n}\bigg(\log|f_{\boldsymbol{\theta}}(\omega_k)| + \frac{|J_n(\omega_k)|^2}{f_{\boldsymbol{\theta}}(\omega_k)}\bigg),$$

$$\widehat{\boldsymbol{\theta}}_n = \arg\min_{\boldsymbol{\theta}\in\Theta} L_n^w(\boldsymbol{\theta}) \qquad \boldsymbol{\theta} = \arg\min_{\boldsymbol{\theta}\in\Theta} L^w(\boldsymbol{\theta})$$

Then we have

$$\sqrt{n}\big(\widehat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}\big) \overset{\mathcal{D}}{\to} \mathcal{N}(0, 2V^{-1} + V^{-1}WV^{-1})$$

where

$$\begin{aligned} V &= \frac{1}{2\pi}\int_0^{2\pi}\bigg(\frac{\nabla_{\boldsymbol{\theta}} f_{\boldsymbol{\theta}}(\omega)}{f_{\boldsymbol{\theta}}(\omega)}\bigg)\bigg(\frac{\nabla_{\boldsymbol{\theta}} f_{\boldsymbol{\theta}}(\omega)}{f_{\boldsymbol{\theta}}(\omega)}\bigg)' d\omega \\ W &= \frac{2}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}\big(\nabla_{\boldsymbol{\theta}} f_{\boldsymbol{\theta}}(\omega_1)^{-1}\big)\big(\nabla_{\boldsymbol{\theta}} f_{\boldsymbol{\theta}}(\omega_2)^{-1}\big)' f_{4,\boldsymbol{\theta}_0}(\omega_1, -\omega_1, \omega_2) d\omega_1 d\omega_2, \end{aligned}$$

and $f_{4,\boldsymbol{\theta}_0}(\omega_1, \omega_2, \omega_3)$ is the fourth order spectrum of $\{X_t\}$.

We now apply the above result to the case of linear time series. We now show that in this case, in the fourth order cumulant term, W, falls out. This is due to the following lemma.

Lemma 10.4.3 *Suppose that the spectral density has the form $f(\omega) = \sigma^2|1 + \sum_{j=1}^{\infty}\psi_j\exp(ij\omega)|^2$ and $\inf f(\omega) > 0$. Then we have*

$$\frac{1}{2\pi}\int_0^{2\pi}\log f(\omega)d\omega = \log\sigma^2$$

PROOF. Since $f(z)$ is non-zero for $|z| \leq 1$, then $\log f(z)$ has no poles in $\{z; |z| \leq 1\}$. Thus we have

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi}\log f(\omega)d\omega &= \frac{1}{2\pi}\int_0^{2\pi}\log\sigma^2 d\omega + \frac{1}{2\pi}\int_0^{2\pi}\log|1 + \sum_{j=1}^{\infty}\psi_j\exp(ij\omega)|^2 d\omega \\ = & \frac{1}{2\pi}\int_0^{2\pi}\log\sigma^2 d\omega + \frac{1}{2\pi}\int_{|z|=1}\log|1 + \sum_{j=1}^{\infty}\psi_j z|^2 dz \\ = & \frac{1}{2\pi}\int_0^{2\pi}\log\sigma^2 d\omega. \end{aligned}$$

An alternative proof is that since $f(z)$ is analytic and does not have any poles for $|z| \leq 1$, then

$\log f(z)$ is also analytic in the region $|z| \leq 1$, thus for $|z| \leq 1$ we have the power series expansion $\log|1 + \sum_{j=1}^{\infty} \psi_j \exp(ij\omega)|^2 = \sum_{j=1}^{\infty} b_j z^j$ (a Taylor expansion about $\log 1$). Using this we have

$$\frac{1}{2\pi}\int_0^{2\pi} \log|1 + \sum_{j=1}^{\infty} \psi_j \exp(ij\omega)|^2 d\omega = \frac{1}{2\pi}\int_0^{2\pi} \sum_{j=1}^{\infty} b_j \exp(ij\omega) d\omega$$

$$= \frac{1}{2\pi}\sum_{j=1}^{\infty} b_j \int_0^{2\pi} \exp(ij\omega) d\omega = 0,$$

and we obtain the desired result. □

Lemma 10.4.4 *Suppose that $\{X_t\}$ is a linear ARMA time series $X_t - \sum_{j=1}^{p} \phi_j X_{t-j} = \sum_{i=1}^{q} \theta_i \varepsilon_{t-i} + \varepsilon_t$, where $\mathrm{E}[\varepsilon_t] = 0$, $\mathrm{var}[\varepsilon_t] = \sigma^2$ and $\mathrm{E}[\varepsilon_t^4] < \infty$. Let $\boldsymbol{\theta} = (\{\phi_j, \theta_j\})$, then we have $W = 0$ and*

$$\sqrt{n}\left(\widehat{\boldsymbol{\theta}}_n^w - \boldsymbol{\theta}\right) \xrightarrow{\mathcal{D}} \mathcal{N}(0, 2V^{-1}).$$

PROOF. The result follows from Theorem 10.4.1, however we need to show that in the case of linearity that $W = 0$.

We use Example 9.6.1 for linear processes to give $f_{4,\boldsymbol{\theta}}(\omega_1, \omega_1, -\omega_2) = \kappa_4 |A(\omega_1)|^2 |A(\omega_2)|^2 = \frac{\kappa_4}{\sigma^4} f(\omega_1) f(\omega_2)$. Substituting this into W gives

$$\begin{aligned}
W &= \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} \left(\nabla_{\boldsymbol{\theta}} f_{\boldsymbol{\theta}}(\omega_1)^{-1}\right)\left(\nabla_{\boldsymbol{\theta}} f_{\boldsymbol{\theta}}(\omega_2)^{-1}\right)' f_{4,\boldsymbol{\theta}_0}(\omega_1, -\omega_1, \omega_2) d\omega_1 d\omega_2 \\
&= \frac{\kappa_4}{\sigma^4}\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{\nabla_{\boldsymbol{\theta}} f_{\boldsymbol{\theta}}(\omega)}{f_{\boldsymbol{\theta}}(\omega)^2} f_{\boldsymbol{\theta}}(\omega) d\omega\right)^2 = \frac{\kappa_4}{\sigma^4}\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{\nabla_{\boldsymbol{\theta}} f_{\boldsymbol{\theta}}(\omega)}{f_{\boldsymbol{\theta}}(\omega)} d\omega\right)^2 \\
&= \frac{\kappa_4}{\sigma^4}\left(\frac{1}{2\pi}\int_0^{2\pi} \nabla_{\boldsymbol{\theta}} \log f_{\boldsymbol{\theta}}(\omega) d\omega\right)^2 \\
&= \frac{\kappa_4}{\sigma^4}\left(\frac{1}{2\pi}\nabla_{\boldsymbol{\theta}}\int_0^{2\pi} \log f_{\boldsymbol{\theta}}(\omega) d\omega\right)^2 = \frac{\kappa_4}{\sigma^4}\left(\nabla_{\boldsymbol{\theta}} \log\frac{\sigma^2}{2\pi}\right)^2 = 0,
\end{aligned}$$

where by using Lemma 10.4.3 we have $\int_0^{2\pi} \log f_{\boldsymbol{\theta}}(\omega) d\omega = 2\pi \log\frac{\sigma^2}{2\pi}$ and since $\boldsymbol{\theta}$ does not include σ^2 we obtain the above. Hence for linear processes the higher order cumulant does not play an asymptotic role in the variance thus giving the result. □

On first appearances there does not seem to be a connection between the Whittle likelihood and the sample autocorrelation estimator defined in Section 7.2.1. However, we observe that the variance of both estimators, under linearity, do not contain the fourth order cumulant (even for non-Gaussian linear time series). In Section 10.5 we explain there is a connection between the two, and it is this connection that explains away this fourth order cumulant term.

Remark 10.4.2 *Under linearity, the GMLE and the Whittle likelihood are asymptotically equivalent, therefore they have the same asymptotic distributions. The GMLE has the asymptotic distribution* $\sqrt{n}(\underline{\hat{\phi}}_n - \underline{\phi}, \underline{\hat{\theta}}_n - \underline{\theta}) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \Lambda^{-1})$*, where*

$$\Lambda = \begin{pmatrix} \mathrm{E}(U_t U_t') & \mathrm{E}(V_t U_t') \\ \mathrm{E}(U_t V_t') & \mathrm{E}(V_t V_t') \end{pmatrix}$$

and $\{U_t\}$ *and* $\{V_t\}$ *are autoregressive processes which satisfy* $\phi(B)U_t = \varepsilon_t$ *and* $\theta(B)V_t = \varepsilon_t$*.*

By using the similar derivatives to those given in (8.25) we can show that

$$\begin{pmatrix} \mathrm{E}(U_t U_t') & \mathrm{E}(V_t U_t') \\ \mathrm{E}(U_t V_t') & \mathrm{E}(V_t V_t') \end{pmatrix} = \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{\nabla_{\boldsymbol{\theta}} f_{\boldsymbol{\theta}}(\omega)}{f_{\boldsymbol{\theta}}(\omega)}\right)\left(\frac{\nabla_{\boldsymbol{\theta}} f_{\boldsymbol{\theta}}(\omega)}{f_{\boldsymbol{\theta}}(\omega)}\right)' d\omega.$$

10.5 Ratio statistics in Time Series

We recall from (10.4) that the covariance can be written as a general periodogram mean which has the form

$$A(\phi, I_n) = \frac{1}{n}\sum_{k=1}^{n} I_n(\omega_k)\phi(\omega_k). \tag{10.28}$$

The variance of this statistic is

$$\begin{aligned} \mathrm{var}(A(\phi, I_n)) &= \frac{1}{n^2}\sum_{k_1,k_2=1}^{n} \phi(\omega_{k_1})\overline{\phi(\omega_{k_1})}\mathrm{cov}(|J_n(\omega_{k_1})|^2, |J_n(\omega_{k_2})|^2) \\ &= \frac{1}{n^2}\sum_{k_1,k_2=1}^{n} \phi(\omega_{k_1})\overline{\phi(\omega_{k_1})}\Big[\mathrm{cov}(J_n(\omega_{k_1}), J_n(\omega_{k_2}))\mathrm{cov}(\overline{J_n(\omega_{k_1})}, \overline{J_n(\omega_{k_2})}) \\ &\quad +\mathrm{cov}(J_n(\omega_{k_1}), \overline{J_n(\omega_{k_2})})\mathrm{cov}(\overline{J_n(\omega_{k_1})}, J_n(\omega_{k_2})) \\ &\quad +cum(J_n(\omega_{k_1}), \overline{J_n(\omega_{k_2})}, J_n(\omega_{k_2}), \overline{J_n(\omega_{k_2})}\Big]. \end{aligned} \tag{10.29}$$

By using Proposition 9.6.1 we have

$$\begin{aligned}
&\text{cov}(|J_n(\omega_{k_1})|^2, |J_n(\omega_{k_2})|^2) = \\
&\left[f(\omega_{k_1})I(k_1 = k_2) + O\left(\frac{1}{n}\right)\right]^2 + \left[f(\omega_{k_1})I(k_1 = n - k_2) + O\left(\frac{1}{n}\right)\right]\left[f(\omega_{k_1})I(n - k_1 = k_2) + O\left(\frac{1}{n}\right)\right] \\
&+\frac{1}{n}f_4(\omega_1, -\omega_1, \omega_2) + O\left(\frac{1}{n^2}\right). \qquad (10.30)
\end{aligned}$$

Substituting (10.30) into (10.29) the above gives

$$\begin{aligned}
&\text{var}(A(\phi, I_n)) \\
= \;& \frac{1}{n^2}\sum_{k=1}^{n}|\phi(\omega_k)|^2 f(\omega_k)^2 + \frac{1}{n^2}\sum_{k=1}^{n}\phi(\omega_k)\overline{\phi(\omega_{n-k})}f(\omega_k)^2 \\
&+\frac{1}{n^3}\sum_{k_1,k_2=1}^{n}\phi(\omega_{k_1})\overline{\phi(\omega_{k_2})}f_4(\omega_{k_1}, -\omega_{k_1}, \omega_{k_2}) + O(\frac{1}{n^2}) \\
= \;& \frac{1}{n}\int_0^{2\pi}|\phi(\omega)|^2 f(\omega)^2 d\omega + \frac{1}{n}\int_0^{2\pi}\phi(\omega)\overline{\phi(2\pi - \omega)}f(\omega)^2 d\omega \\
&+\frac{1}{n}\int_0^{2\pi}\int_0^{2\pi}\phi(\omega_1)\overline{\phi(\omega_2)}f_4(\omega_1, -\omega_1, \omega_2)d\omega_1 d\omega_2 + O(\frac{1}{n^2}), \qquad (10.31)
\end{aligned}$$

where f_4 is the fourth order cumulant of $\{X_t\}$. From above we see that unless ϕ satisfies some special conditions, $\text{var}(A(\phi, I_n))$ contains the fourth order spectrum, which can be difficult to estimate. There are bootstrap methods which can be used to estimate the variance or finite sample distribution, but simple bootstrap methods, such as the frequency domain bootstrap, cannot be applied to $A(\phi, I_n)$, since it is unable to capture the fourth order cumulant structure. However, in special cases the fourth order structure is disappears, we consider this case below and then discuss how this case can be generalised.

Lemma 10.5.1 *Suppose $\{X_t\}$ is a linear time series, with spectral density $f(\omega)$. Let $A(\phi, I_n)$ be defined as in (10.28) and suppose the condition*

$$A(\phi, f) = \int \phi(\omega)f(\omega)d\omega = 0 \qquad (10.32)$$

holds, then

$$\text{var}(A(\phi, I_n)) = \frac{1}{n}\int_0^{2\pi}|\phi(\omega)|^2 f(\omega)^2 d\omega + \frac{1}{n}\int_0^{2\pi}\phi(\omega)\overline{\phi(2\pi - \omega)}f(\omega)^2 d\omega.$$

PROOF. By using (10.31) we have

$$\begin{aligned}
& \mathrm{var}(A(\phi, I_n)) \\
= & \frac{1}{n}\int_0^{2\pi}|\phi(\omega)|^2 f(\omega)^2 d\omega + \frac{1}{n}\int_0^{2\pi}\phi(\omega)\overline{\phi(2\pi-\omega)}f(\omega)^2 d\omega \\
& \frac{1}{n}\int_0^{2\pi}\int_0^{2\pi}\phi(\omega_1)\overline{\phi(\omega_2)}f_4(\omega_1,-\omega_1,\omega_2) + O(\frac{1}{n^2}).
\end{aligned}$$

But under linearity $f_4(\omega_1,-\omega_1,\omega_2) = \frac{\kappa_4}{\sigma^4}f(\omega_1)f(\omega_2)$, substituting this into the above gives

$$\begin{aligned}
& \mathrm{var}(A(\phi, I_n)) \\
= & \frac{1}{n}\int_0^{2\pi}|\phi(\omega)|^2 f(\omega)^2 d\omega + \frac{1}{n}\int_0^{2\pi}\phi(\omega)\overline{\phi(2\pi-\omega)}f(\omega)^2 d\omega \\
& \frac{\kappa_4}{\sigma^4}\frac{1}{n}\int_0^{2\pi}\int_0^{2\pi}\phi(\omega_1)\overline{\phi(\omega_2)}f(\omega_1)f(\omega_2)d\omega_1 d\omega_2 + O\left(\frac{1}{n^2}\right) \\
= & \frac{1}{n}\int_0^{2\pi}|\phi(\omega)|^2 f(\omega)^2 d\omega + \frac{1}{n}\int_0^{2\pi}\phi(\omega)\overline{\phi(2\pi-\omega)}f(\omega)^2 d\omega \\
& +\frac{\kappa_4}{\sigma^4}\frac{1}{n}\left|\int_0^{2\pi}\phi(\omega)f(\omega)d\omega\right|^2 + O\left(\frac{1}{n^2}\right).
\end{aligned}$$

Since $\int \phi(\omega)f(\omega)d\omega = 0$ we have the desired result. □

Example 10.5.1 (The Whittle likelihood) *Let us return to the Whittle likelihood in the case of linearity. In Lemma 10.4.4 we showed that the fourth order cumulant term does not play a role in the variance of the ARMA estimator. We now show that condition (10.32) holds.*

Consider the partial derivative of the Whittle likelihood

$$\nabla_{\boldsymbol{\theta}} L_n^w(\boldsymbol{\theta}) = \sum_{k=1}^n \left(\frac{\nabla_{\boldsymbol{\theta}} f_{\boldsymbol{\theta}}(\omega_k)}{f_{\boldsymbol{\theta}}(\omega_k)} - \frac{I_n(\omega_k)}{f_{\boldsymbol{\theta}}(\omega_k)^2}\nabla_{\boldsymbol{\theta}} f_{\boldsymbol{\theta}}(\omega_k)\right).$$

To show normality we consider the above at the true parameter $\boldsymbol{\theta}$, this gives

$$\nabla_{\boldsymbol{\theta}} L_n^w(\boldsymbol{\theta}) = \sum_{k=1}^n \left(\frac{\nabla_{\boldsymbol{\theta}} f_{\boldsymbol{\theta}}(\omega_k)}{f_{\boldsymbol{\theta}}(\omega_k)} - \frac{I_n(\omega_k)}{f_{\boldsymbol{\theta}}(\omega_k)^2}\nabla_{\boldsymbol{\theta}} f_{\boldsymbol{\theta}}(\omega_k)\right).$$

Only the second term of the above is random, therefore it is only this term that yields the variance. Let

$$A(f_{\boldsymbol{\theta}}^{-2}\nabla_{\boldsymbol{\theta}} f_{\boldsymbol{\theta}}, I_n) = \frac{1}{n}\sum_{k=1}^n \frac{I_n(\omega_k)}{f_{\boldsymbol{\theta}}(\omega_k)^2}\nabla_{\boldsymbol{\theta}} f_{\boldsymbol{\theta}}(\omega_k).$$

To see whether this term satisfies the conditions of Lemma 10.5.1 we evaluate

$$\begin{aligned} A(f_{\boldsymbol{\theta}}^{-2}\nabla_{\boldsymbol{\theta}}f_{\boldsymbol{\theta}}, f_{\boldsymbol{\theta}}) &= \int_0^{2\pi} \frac{f_{\boldsymbol{\theta}}(\omega)}{f_{\boldsymbol{\theta}}(\omega)^2}\nabla_{\boldsymbol{\theta}}f_{\boldsymbol{\theta}}(\omega) \\ &= \int_0^{2\pi} \nabla_{\boldsymbol{\theta}} \log f_{\boldsymbol{\theta}}(\omega) \\ &= \nabla_{\boldsymbol{\theta}} \int_0^{2\pi} \log f_{\boldsymbol{\theta}}(\omega) = \nabla_{\boldsymbol{\theta}} \frac{1}{2\pi}\int_0^{2\pi} \log f_{\boldsymbol{\theta}}(\omega) d\omega = 0, \end{aligned}$$

by using Lemma 10.4.3. Thus we see that the derivative of the Whittle likelihood satisfies the condition (10.32). Therefore the zero cumulant term is really due to this property. □

The Whittle likelihood is a rather special example. However we now show that any statistic of the form $A(\phi, I_n)$ can be transformed such that the resulting transformed statistic satisfies condition (10.32). To find the suitable transformation we recall from Section 7.2.1 that the variance of $\widehat{c}_n(r)$ involves the fourth order cumulant, but under linearity the sample correlation $\widehat{\rho}_n(r) = \widehat{c}_n(r)/\widehat{c}_n(0)$ does given not. Returning to the frequency representation of the autocovariance given in (10.5) we observe that

$$\widehat{\rho}_n(r) \quad = \quad \frac{1}{\widehat{c}_n(0)}\frac{1}{n}\sum_{k=1}^{n/2} I_n(\omega_k)\exp(ir\omega_k) \approx \frac{1}{\widehat{c}_n(0)}\frac{1}{n}\sum_{k=1}^{n} I_n(\omega_k)\exp(ir\omega_k),$$

(it does not matter whether we sum over n or $n/2$ for the remainder of this section we choose the case of summing over n). Motivated by this example we define the so called 'ratio' statistic

$$\widetilde{A}(\phi, I_n) = \frac{1}{n}\sum_{k=1}^{n} \frac{I_n(\omega_k)\phi(\omega_k)}{\widehat{c}_n(0)} = \frac{1}{n}\sum_{k=1}^{n} \frac{I_n(\omega_k)\phi(\omega_k)}{\hat{F}_n(2\pi)}, \tag{10.33}$$

where $\hat{F}_n(2\pi) = \frac{1}{n}\sum_{k=1}^{n} I_n(\omega_k) = \frac{1}{n}\sum_{t=1}^{n} X_t^2 = \widehat{c}_n(0)$. We show in the following lemma that $\widetilde{A}(\phi, I_n)$ can be written in a form that 'almost' satisfies condition (10.32).

Lemma 10.5.2 *Let us suppose that $\widetilde{A}(\phi, I_n)$ satisfies (10.33) and*

$$\widetilde{A}(\phi, f) = \frac{1}{n}\sum_{k=1}^{n} \frac{f(\omega_k)\phi(\omega_k)}{F_n(2\pi)},$$

where $F_n(2\pi) = \frac{1}{n}\sum_{j=1}^{n} f(\omega_k)$. *Then we can represent* $\tilde{A}(\phi, I_n)$ *as*

$$\widetilde{A}(\phi, I_n) - \widetilde{A}(\phi, f) = \frac{1}{F(2\pi)\hat{F}_n(2\pi)}\frac{1}{n}\sum_{k=1}^{n}\psi_n(\omega_k)I_n(\omega_k),$$

where

$$\psi_n(\omega_k) = \phi(\omega_k)F_n(2\pi) - \frac{1}{n}\sum_{j=1}^{n}\phi(\omega_j)f(\omega_j) \qquad \text{and} \qquad \frac{1}{n}\sum_{k=1}^{n}\psi(\omega_k)f(\omega_k) = 0. \tag{10.34}$$

PROOF. Basic algebra gives

$$\begin{aligned}
\tilde{A}(\phi, I_n) - \tilde{A}(\phi, f) &= \frac{1}{n}\sum_{k=1}^{n}\left(\frac{\phi(\omega_k)I_n(\omega_k)}{\hat{F}_n(2\pi)} - \frac{\phi(\omega_k)f(\omega_k)}{F_n(2\pi)}\right) \\
&= \frac{1}{n}\sum_{k=1}^{n}\left(\frac{\phi(\omega_k)F_n(2\pi)I_n(\omega_k) - \phi(\omega_k)\hat{F}_n(2\pi)f(\omega_k)}{F_n(2\pi)\hat{F}_n(2\pi)}\right) \\
&= \frac{1}{n}\sum_{k=1}^{n}\left(\phi(\omega_k)F_n(2\pi) - \frac{1}{n}\sum_{k=1}^{n}\phi(\omega_k)f(\omega_k)\right)\frac{I_n(\omega_k)}{F_n(2\pi)\hat{F}_n(2\pi)} \\
&= \frac{1}{n}\sum_{k=1}^{n}\frac{\psi(\omega_k)I_n(\omega_k)}{F_n(2\pi)\hat{F}_n(2\pi)},
\end{aligned}$$

where $F_n(2\pi)$ and ψ are defined as above. To show (10.34), again we use basic algebra to give

$$\begin{aligned}
\frac{1}{n}\sum_{k=1}^{n}\psi(\omega_k)f(\omega_k) &= \frac{1}{n}\sum_{k=1}^{n}\left(\phi(\omega)F_n(2\pi) - \frac{1}{n}\sum_{j=1}^{n}\phi(\omega_j)f(\omega_j)\right)f(\omega_k) \\
&= \frac{1}{n}\sum_{k=1}^{n}\phi(\omega_k)f(\omega_k)F_n(2\pi) - \frac{1}{n}\sum_{k=1}^{n}\phi(\omega_k)f(\omega_k)\frac{1}{n}\sum_{j=1}^{n}f(\omega_j) = 0.
\end{aligned}$$

□

From the lemma above we see that $\tilde{A}(\phi, I_n) - \tilde{A}(\phi, f)$ almost seems to satisfy the conditions in Lemma 10.5.1, the only difference is the random term $\widehat{c}_n(0) = \widehat{F}_n(2\pi)$ in the denominator. We now show that that we can replace $\widehat{F}_n(2\pi)$ with it's limit and that error is asymptotically negligible. Let

$$\tilde{A}(\phi, I_n) - \tilde{A}(\phi, f) = \frac{1}{F_n(2\pi)\hat{F}_n(2\pi)}\frac{1}{n}\sum_{k=1}^{n}\psi_n(\omega_k)I_n(\omega_k) := \widetilde{B}(\psi, I_n)$$

and

$$B(\psi_n, I_n) = \frac{1}{F_n(2\pi)^2}\frac{1}{n}\sum_{k=1}^{n}\psi(\omega_k)I_n(\omega_k).$$

By using the mean value theorem (basically the Delta method) and expanding $\widetilde{B}(\psi_n, I_n)$ about $B(\psi_n, I_n)$ (noting that $B(\phi_n, f) = 0$) gives

$$\begin{aligned} & \widetilde{B}(\psi, I_n) - B(\psi, I_n) \\ = \ & \underbrace{\big(\hat{F}_n(2\pi) - F_n(2\pi)\big)}_{O_p(n^{-1/2})}\frac{1}{\overline{F_n(2\pi)}^3}\underbrace{\frac{1}{n}\sum_{k=1}^{n}\psi_n(\omega_k)I_n(\omega_k)}_{O_p(n^{-1/2})} = O_p(\frac{1}{n}), \end{aligned}$$

where $\overline{F}_n(2\pi)$ lies between $F_n(2\pi)$ and $\widehat{F}_n(2\pi)$. Therefore the limiting distribution variance of $\tilde{A}(\phi, I_n) - \tilde{A}(\phi, f)$ is determined by

$$\widetilde{A}(\phi, I_n) - \widetilde{A}(\phi, f) = B(\psi_n, I_n) + O_p(n^{-1/2}).$$

$B(\psi_n, I_n)$ does satisfy the conditions in (10.32) and the lemma below immediately follows.

Lemma 10.5.3 *Suppose that $\{X_t\}$ is a linear time series, then*

$$\mathrm{var}(B(\psi_n, I_n)) = \frac{1}{n}\int_0^{2\pi}|\psi(\omega)|^2 f(\omega)^2 d\omega + \frac{1}{n}\int_0^{2\pi}\psi(\omega)\overline{\psi(2\pi-\omega)}f(\omega)^2 d\omega + O(\frac{1}{n^2}),$$

where

$$\psi(\omega) = \phi(\omega)F(2\pi) - \frac{1}{2\pi}\int_0^{2\pi}\phi(\omega)f(\omega)d\omega.$$

Therefore, the limiting variance of $\widetilde{A}(\phi, I_n)$ is

$$\frac{1}{n}\int_0^{2\pi}|\psi(\omega)|^2 f(\omega)^2 d\omega + \frac{1}{n}\int_0^{2\pi}\psi(\omega)\overline{\psi(2\pi-\omega)}f(\omega)^2 d\omega + O(\frac{1}{n^2}).$$

This is a more elegant explanation as to why under linearity the limiting variance of the correlation estimator does not contain the fourth order cumulant term. It also allows for a general class of statistics.

Remark 10.5.1 (Applications) *As we remarked above, many statistics can be written as a ratio statistic. The advantage of this is that the variance of the limiting distribution is only in terms of the spectral densities, and not any other higher order terms (which are difficult to estimate). Another perk is that simple schemes such as the frequency domain bootstrap can be used to estimate the finite sample distributions of statistics which satisfy the assumptions in Lemma 10.5.1 or is a ratio statistic (so long as the underlying process is linear), see Dahlhaus and Janas (1996) for the details. The frequency domain bootstrap works by constructing the DFT from the data* $\{J_n(\omega)\}$ *and dividing by the square root of either the nonparametric estimator of* f *or a parametric estimator, ie.* $\{J_n(\omega)/\sqrt{\hat{f}_n(\omega)}\}$, *these are close to constant variance random variables.* $\{\hat{J}_{\varepsilon}(\omega_k) = J_n(\omega_k)/\sqrt{\hat{f}_n(\omega_k)}\}$ *is bootstrapped, thus* $J_n^*(\omega_k) = \hat{J}_{\varepsilon}^*(\omega_k)\sqrt{\hat{f}_n(\omega_k)}$ *is used as the bootstrap DFT. This is used to construct the bootstrap estimator, for example*

- *The Whittle likelihood estimator.*
- *The sample correlation.*

With these bootstrap estimators we can construct an estimator of the finite sample distribution.

The nature of frequency domain bootstrap means that the higher order dependence structure is destroyed, eg. $cum^*(J_n^*(\omega_{k_1}), J_n^*(\omega_{k_2}), \ldots, J_n^*(\omega_{k_r})) = 0$ *(where* cum^* *is the cumulant with respect to the bootstrap measure) if all the* k_i*s that are not the same. However, we know from Proposition 9.6.1 that for the actual DFT this is not the case, there is still some 'small' dependence, which can add up. Therefore, the frequency domain bootstrap is unable to capture any structure beyond the second order. This means for a linear time series which is not Gaussian the frequency domain bootstrap cannot approximate the distribution of the sample covariance (since it is asymptotically with normal with a variance which contains the forth order cumulant), but it can approximate the finite sample distribution of the correlation.*

Remark 10.5.2 (Estimating κ_4 in the case of linearity) *Suppose that* $\{X_t\}$ *is a linear time series*

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j \varepsilon_{t-j},$$

with $\mathrm{E}(\varepsilon_t) = 0$, $\mathrm{var}(\varepsilon_t) = \sigma^2$ *and* $cum_4(\varepsilon_t) = \kappa_4$. *Then we can use the spectral density estimator to estimate* κ_4 *without any additional assumptions on* $\{X_t\}$ *(besides linearity). Let* $f(\omega)$ *denote the* *spectral density of $\{X_t\}$ and $g_2(\omega)$ the spectral density of $\{X_t^2\}$, then it can be shown that*

$$\kappa_4 = \frac{2\pi g_2(0) - 4\pi \int_0^{2\pi} f(\omega)^2 d\omega}{\left(\int_0^{2\pi} f(\omega) d\omega\right)^2}.$$

Thus by estimating f and g_2 we can estimate κ_4.

Alternatively, we can use the fact that for linear time series, the fourth order spectral density $f_4(\omega_1, \omega_2, \omega_3) = \kappa_4 A(\omega_1)A(\omega_2)A(\omega_3)A(-\omega_1 - \omega_2 - \omega_3)$. Thus we have

$$\kappa_4 = \frac{\sigma^4 f_4(\omega_1, -\omega_1, \omega_2)}{f(\omega_1) f(\omega_2)}.$$

This just demonstrates, there is no unique way to solve a statistical problem!

10.6 Goodness of fit tests for linear time series models

As with many other areas in statistics, we often want to test the appropriateness of a model. In this section we briefly consider methods for validating whether, say an ARMA(p, q), is the appropriate model to fit to a time series. One method is to fit the model to the data and the estimate the residuals and conduct a Portmanteau test (see Section 4, equation (7.13)) on the estimated residuals. It can be shown that if model fitted to the data is the correct one, the estimated residuals behave almost like the true residuals in the model and the Portmanteau test statistic

$$\mathcal{S}_h = n \sum_{r=1}^{h} |\hat{\rho}_n(r)|^2,$$

where $\hat{\rho}_n(r) = \hat{c}_n(r)/\hat{c}_n(0)$

$$\hat{c}_n(r) = \frac{1}{n} \sum_{t=1}^{n-|r|} \hat{\varepsilon}_t \hat{\varepsilon}_{t+r}$$

should be asymptotically a chi-squared. An alternative (but somehow equivalent) way to do the test, is through the DFTs. We recall if the time series is linear then (10.11) is true, thus

$$\frac{I_X(\omega)}{f_{\boldsymbol{\theta}}(\omega)} = |J_\varepsilon(\omega)|^2 + o_p(1).$$

Therefore, if we fit the correct model to the data we would expect that

$$\frac{I_X(\omega)}{f_{\hat{\boldsymbol{\theta}}}(\omega)} = |J_{\varepsilon}(\omega)|^2 + o_p(1).$$

where $\hat{\boldsymbol{\theta}}$ are the model parameter estimators. Now $|J_{\varepsilon}(\omega)|^2$ has the special property that not only is it almost uncorrelated at various frequencies, but it is constant over all the frequencies. Therefore, we would expect that

$$\frac{1}{2\pi\sqrt{n}}\sum_{k=1}^{n/2}\big(\frac{I_X(\omega)}{f_{\hat{\boldsymbol{\theta}}}(\omega)} - 2\big) \overset{\mathcal{D}}{\to} N(0,1)$$

Thus, as an alternative to the goodness fit test based on the portmanteau test statistic we can use the above as a test statistic, noting that under the alternative the mean would be different.

Chapter 11

Consistency and and asymptotic normality of estimators

In the previous chapter we considered estimators of several different parameters. The hope is that as the sample size increases the estimator should get 'closer' to the parameter of interest. When we say closer we mean to converge. In the classical sense the sequence $\{x_k\}$ converges to x ($x_k \to x$), if $|x_k - x| \to 0$ as $k \to \infty$ (or for every $\varepsilon > 0$, there exists an n where for all $k > n$, $|x_k - x| < \varepsilon$). Of course the estimators we have considered are random, that is for every $\omega \in \Omega$ (set of all out comes) we have an different *estimate*. The natural question to ask is what does convergence mean for random sequences.

11.1 Modes of convergence

We start by defining different modes of convergence.

Definition 11.1.1 (Convergence)

- **Almost sure convergence** *We say that the sequence $\{X_t\}$ converges almost sure to μ, if there exists a set $M \subset \Omega$, such that $\mathbb{P}(M) = 1$ and for every $\omega \in N$ we have*

$$X_t(\omega) \to \mu.$$

In other words for every $\varepsilon > 0$, there exists an $N(\omega)$ such that

$$|X_t(\omega) - \mu| < \varepsilon, \tag{11.1}$$

for all $t > N(\omega)$. Note that the above definition is very close to classical convergence. We denote $X_t \to \mu$ almost surely, as $X_t \overset{a.s.}{\to} \mu$.

An equivalent definition, in terms of probabilities, is for every $\varepsilon > 0$ $X_t \overset{a.s.}{\to} \mu$ if

$$P(\omega; \cap_{m=1}^{\infty} \cup_{t=m}^{\infty} \{|X_t(\omega) - \mu| > \varepsilon\}) = 0.$$

It is worth considering briefly what $\cap_{m=1}^{\infty} \cup_{t=m}^{\infty} \{|X_t(\omega) - \mu| > \varepsilon\}$ means. If $\cap_{m=1}^{\infty} \cup_{t=m}^{\infty}$ $\{|X_t(\omega) - \mu| > \varepsilon\} \neq \oslash$, then there exists an $\omega^* \in \cap_{m=1}^{\infty} \cup_{t=m}^{\infty} \{|X_t(\omega) - \mu| > \varepsilon\}$ such that for some infinite sequence $\{k_j\}$, we have $|X_{k_j}(\omega^*) - \mu| > \varepsilon$, this means $X_t(\omega^*)$ does not converge to μ. Now let $\cap_{m=1}^{\infty} \cup_{t=m}^{\infty} \{|X_t(\omega) - \mu| > \varepsilon\} = A$, if $P(A) = 0$, then for 'most' ω the sequence $\{X_t(\omega)\}$ converges.

- **Convergence in mean square**

  We say $X_t \to \mu$ in mean square (or L_2 convergence), if $\mathrm{E}(X_t - \mu)^2 \to 0$ as $t \to \infty$.

- **Convergence in probability**

  Convergence in probability cannot be stated in terms of realisations $X_t(\omega)$ but only in terms of probabilities. X_t is said to converge to μ in probability (written $X_t \overset{\mathcal{P}}{\to} \mu$) if

$$P(|X_t - \mu| > \varepsilon) \to 0, \quad t \to \infty.$$

  Often we write this as $|X_t - \mu| = o_p(1)$.

  If for any $\gamma \geq 1$ we have

$$\mathrm{E}(X_t - \mu)^{\gamma} \to 0 \quad t \to \infty,$$

  then it implies convergence in probability (to see this, use Markov's inequality).

- **Rates of convergence***:*

  (i) Suppose $a_t \to 0$ as $t \to \infty$. We say the stochastic process $\{X_t\}$ is $|X_t - \mu| = O_p(a_t)$,

if the sequence $\{a_t^{-1}|X_t - \mu|\}$ is bounded in probability (this is defined below). We see from the definition of boundedness, that for all t, the distribution of $a_t^{-1}|X_t - \mu|$ should mainly lie within a certain interval.

(ii) *We say the stochastic process $\{X_t\}$ is $|X_t - \mu| = o_p(a_t)$, if the sequence $\{a_t^{-1}|X_t - \mu|\}$ converges in probability to zero.*

Definition 11.1.2 (Boundedness) (i) **Almost surely bounded** *If the random variable X is almost surely bounded, then for a positive sequence $\{e_k\}$, such that $e_k \to \infty$ as $k \to \infty$ (typically $e_k = 2^k$ is used), we have*

$$P(\omega; \{\cup_{k=1}^{\infty}\{|X(\omega)| \leq e_k\}\}) = 1.$$

Usually to prove the above we consider the complement

$$P((\omega; \{\cup_{k=1}^{\infty}\{|X| \leq e_k\}\})^c) = 0.$$

Since $(\cup_{k=1}^{\infty}\{|X| \leq e_k\})^c = \cap_{k=1}^{\infty}\{|X| > e_k\} \subset \cap_{k=1}^{\infty} \cup_{m=k}^{\infty} \{|X| > e_k\}$, to show the above we show

$$P(\omega : \{\cap_{k=1}^{\infty} \cup_{m=k}^{\infty} \{|X(\omega)| > e_k\}\}) = 0. \tag{11.2}$$

We note that if $(\omega : \{\cap_{k=1}^{\infty} \cup_{m=k}^{\infty} \{|X(\omega)| > e_k\}\}) \neq \oslash$, then there exists a $\omega^* \in \Omega$ and an infinite subsequence k_j, where $|X(\omega^*)| > e_{k_j}$, hence $X(\omega^*)$ is not bounded (since $e_k \to \infty$). To prove (11.2) we usually use the Borel Cantelli Lemma. This states that if $\sum_{k=1}^{\infty} P(A_k) < \infty$, the events $\{A_k\}$ occur only finitely often with probability one. Applying this to our case, if we can show that $\sum_{m=1}^{\infty} P(\omega : \{|X(\omega)| > e_m|\}) < \infty$, then $\{|X(\omega)| > e_m|\}$ happens only finitely often with probability one. Hence if $\sum_{m=1}^{\infty} P(\omega : \{|X(\omega)| > e_m|\}) < \infty$, then $P(\omega : \{\cap_{k=1}^{\infty} \cup_{m=k}^{\infty} \{|X(\omega)| > e_k\}\}) = 0$ and X is a bounded random variable.

It is worth noting that often we choose the sequence $e_k = 2^k$, in this case $\sum_{m=1}^{\infty} P(\omega : \{|X(\omega)| > e_m|\}) = \sum_{m=1}^{\infty} P(\omega : \{\log|X(\omega)| > \log 2^k|\}) \leq C\mathrm{E}(\log|X|)$. Hence if we can show that $\mathrm{E}(\log|X|) < \infty$, then X is bounded almost surely.

b

(ii) **Sequences which are bounded in probability** *A sequence is bounded in probability,* *written $X_t = O_p(1)$, if for every $\varepsilon > 0$, there exists a $\delta(\varepsilon) < \infty$ such that $P(|X_t| \geq \delta(\varepsilon)) < \varepsilon$. Roughly speaking this means that the sequence is only extremely large with a very small probability. And as the 'largeness' grows the probability declines.*

11.2 Sampling properties

Often we will estimate the parameters by maximising (or minimising) a criterion. Suppose we have the criterion $\mathcal{L}_n(a)$ (eg. likelihood, quasi-likelihood, Kullback-Leibler etc) we use as an estimator of a_0, $\hat{a}_n$ where

$$\hat{a}_n = \arg\max_{a \in \Theta} \mathcal{L}_n(a)$$

and Θ is the parameter space we do the maximisation (minimisation) over. Typically the true parameter a should maximise (minimise) the 'limiting' criterion $\mathcal{L}$.

If this is to be a good estimator, as the sample size grows the estimator should converge (in some sense) to the parameter we are interesting in estimating. As we discussed above, there are various modes in which we can measure this convergence (i) almost surely (ii) in probability and (iii) in mean squared error. Usually we show either (i) or (ii) (noting that (i) implies (ii)), in time series its usually quite difficult to show (iii).

Definition 11.2.1 *(i) An estimator $\hat{a}_n$ is said to be almost surely consistent estimator of a_0, if there exists a set $M \subset \Omega$, where $\mathbb{P}(M) = 1$ and for all $\omega \in M$ we have*

$$\hat{a}_n(\omega) \to a.$$

(ii) An estimator $\hat{a}_n$ is said to converge in probability to a_0, if for every $\delta > 0$

$$P(|\hat{a}_n - a| > \delta) \to 0 \quad T \to \infty.$$

To prove either (i) or (ii) usually involves verifying two main things, pointwise convergence and equicontinuity.

11.3 Showing almost sure convergence of an estimator

We now consider the general case where $\mathcal{L}_n(a)$ is a 'criterion' which we maximise. Let us suppose we can write $\mathcal{L}_n$ as

$$\mathcal{L}_n(a) = \frac{1}{n}\sum_{t=1}^{n} \ell_t(a), \tag{11.3}$$

where for each $a \in \Theta$, $\{\ell_t(a)\}_t$ is a ergodic sequence. Let

$$\mathcal{L}(a) = \mathrm{E}(\ell_t(a)), \tag{11.4}$$

we assume that $\mathcal{L}(a)$ is continuous and has a unique maximum in Θ. We define the estimator $\hat{\alpha}_n$ where $\hat{\alpha}_n = \arg\min_{a\in\Theta} \mathcal{L}_n(a)$.

Definition 11.3.1 (Uniform convergence) *$\mathcal{L}_n(a)$ is said to almost surely converge uniformly to $\mathcal{L}(a)$, if*

$$\sup_{a\in\Theta} |\mathcal{L}_n(a) - \mathcal{L}(a)| \overset{a.s.}{\rightarrow} 0.$$

In other words there exists a set $M \subset \Omega$ where $P(M) = 1$ and for every $\omega \in M$,

$$\sup_{a\in\Theta} |\mathcal{L}_n(\omega, a) - \mathcal{L}(a)| \rightarrow 0.$$

Theorem 11.3.1 (Consistency) *Suppose that $\hat{a}_n = \arg\max_{a\in\Theta} \mathcal{L}_n(a)$ and $a_0 = \arg\max_{a\in\Theta} \mathcal{L}(a)$ is the unique maximum. If $\sup_{a\in\Theta} |\mathcal{L}_n(a) - \mathcal{L}(a)| \overset{a.s.}{\rightarrow} 0$ as $n \rightarrow \infty$ and $\mathcal{L}(a)$ has a unique maximum. Then Then $\hat{a}_n \overset{a.s.}{\rightarrow} a_0$ as $n \rightarrow \infty$.*

PROOF. We note that by definition we have $\mathcal{L}_n(a_0) \leq \mathcal{L}_n(\hat{a}_n)$ and $\mathcal{L}(\hat{a}_n) \leq \mathcal{L}(a_0)$. Using this inequality we have

$$\mathcal{L}_n(a_0) - \mathcal{L}(a_0) \leq \mathcal{L}_n(\hat{a}_n) - \mathcal{L}(a_0) \leq \mathcal{L}_n(\hat{a}_n) - \mathcal{L}(\hat{a}_n).$$

Therefore from the above we have

$$|\mathcal{L}_n(\hat{a}_T) - \mathcal{L}(a_0)| \leq \max\left\{|\mathcal{L}_n(a_0) - \mathcal{L}(a_0)|, |\mathcal{L}_n(\hat{a}_T) - \mathcal{L}(\hat{a}_n)|\right\} \leq \sup_{a\in\Theta} |\mathcal{L}_n(a) - \mathcal{L}(a)|.$$

Hence since we have uniform converge we have $|\mathcal{L}_n(\hat{a}_n) - \mathcal{L}(a_0)| \overset{\text{a.s.}}{\to} 0$ as $n \to \infty$. Now since $\mathcal{L}(a)$ has a unique maximum, we see that $|\mathcal{L}_n(\hat{a}_n) - \mathcal{L}(a_0)| \overset{\text{a.s.}}{\to} 0$ implies $\hat{a}_n \overset{\text{a.s.}}{\to} a_0$. □

We note that directly establishing uniform convergence is not easy. Usually it is done by assuming the parameter space is compact and showing point wise convergence and stochastic equicontinuity, these three facts imply uniform convergence. Below we define stochastic equicontinuity and show consistency under these conditions.

Definition 11.3.2 *The sequence of stochastic functions $\{f_n(a)\}_n$ is said to be stochastically equicontinuous if there exists a set $M \in \Omega$ where $P(M) = 1$ and for every $\omega \in M$ and and $\varepsilon > 0$, there exists a δ and such that for every $\omega \in M$*

$$\sup_{|a_1 - a_2| \leq \delta} |f_n(\omega, a_1) - f_n(\omega, a_2)| \leq \varepsilon,$$

for all $n > N(\omega)$.

A sufficient condition for stochastic equicontinuity of $f_n(a)$ (which is usually used to prove equicontinuity), is that $f_n(a)$ is in some sense Lipschitz continuous. In other words,

$$\sup_{a_1, a_2 \in \Theta} |f_n(a_1) - f_n(a_2)| < K_n \|a_1 - a_2\|,$$

where k_n is a random variable which converges to a finite constant as $n \to \infty$ ($K_n \overset{a.s.}{\to} K_0$ as $n \to \infty$). To show that this implies equicontinuity we note that $K_n \overset{a.s.}{\to} K_0$ means that for every $\omega \in M$ ($P(M) = 1$) and $\gamma > 0$, we have $|K_n(\omega) - K_0| < \gamma$ for all $n > N(\omega)$. Therefore if we choose $\delta = \varepsilon/(K_0 + \gamma)$ we have

$$\sup_{|a_1 - a_2| \leq \varepsilon/(K_0+\gamma)} |f_n(\omega, a_1) - f_n(\omega, a_2)| < \varepsilon,$$

for all $n > N(\omega)$.

In the following theorem we state sufficient conditions for almost sure uniform convergence. It is worth noting this is the Arzela-Ascoli theorem for random variables.

Theorem 11.3.2 (The stochastic Ascoli Lemma) *Suppose the parameter space Θ is compact, for every $a \in \Theta$ we have $\mathcal{L}_n(a) \overset{a.s.}{\to} \mathcal{L}(a)$ and $\mathcal{L}_n(a)$ is stochastic equicontinuous. Then $\sup_{a \in \Theta} |\mathcal{L}_n(a) - \mathcal{L}(a)| \overset{a.s.}{\to} 0$ as $n \to \infty$.*

We use the theorem below.

Corollary 11.3.1 *Suppose that $\hat{a}_n = \arg\max_{a\in\Theta} \mathcal{L}_n(a)$ and $a_0 = \arg\max_{a\in\Theta} \mathcal{L}(a)$, moreover $\mathcal{L}(a)$ has a unique maximum. If*

(i) We have point wise convergence, that is for every $a \in \Theta$ we have $\mathcal{L}_n(a) \overset{a.s.}{\to} \mathcal{L}(a)$.

(ii) The parameter space Θ is compact.

(iii) $\mathcal{L}_n(a)$ is stochastic equicontinuous.

then $\hat{a}_n \overset{a.s.}{\to} a_0$ as $n \to \infty$.

PROOF. By using Theorem 11.3.2 three assumptions imply that $|\sup_{\theta\in\Theta} ||\mathcal{L}_n(\theta) - \mathcal{L}(\theta)| \to 0$, thus by using Theorem 11.3.1 we obtain the result.

We prove Theorem 11.3.2 in the section below, but it can be omitted on first reading.

11.3.1 Proof of Theorem 11.3.2 (The stochastic Ascoli theorem)

We now show that stochastic equicontinuity and almost pointwise convergence imply uniform convergence. We note that on its own, pointwise convergence is a much weaker condition than uniform convergence, since for pointwise convergence the rate of convergence can be different for each parameter.

Before we continue a few technical points. We recall that we are assuming almost pointwise convergence. This means for each parameter $a \in \Theta$ there exists a set $N_a \in \Omega$ (with $P(N_a) = 1$) such that for all $\omega \in N_a$ $\mathcal{L}_n(\omega, a) \to \mathcal{L}(a)$. In the following lemma we unify this set. That is show (using stochastic equicontinuity) that there exists a set $N \in \Omega$ (with $P(N) = 1$) such that for all $\omega \in N$ $\mathcal{L}_n(\omega, a) \to \mathcal{L}(a)$.

Lemma 11.3.1 *Suppose the sequence $\{\mathcal{L}_n(a)\}_n$ is stochastically equicontinuous and also pointwise convergent (that is $\mathcal{L}_n(a)$ converges almost surely to $\mathcal{L}(a)$), then there exists a set $M \in \Omega$ where $P(\bar{M}) = 1$ and for every $\omega \in \bar{M}$ and $a \in \Theta$ we have*

$$|\mathcal{L}_n(\omega, a) - \mathcal{L}(a)| \to 0.$$

PROOF. Enumerate all the rationals in the set Θ and call this sequence $\{a_i\}_i$. Since we have almost sure convergence, this implies for every a_i there exists a set M_{a_i} where $P(M_{a_i}) = 1$ and for every

$\omega \in M_{a_i}$ we have $|\mathcal{L}_T(\omega, a_i) - \mathcal{L}(a_i)| \to 0$. Define $M = \cap M_{a_i}$, since the number of sets is countable $P(M) = 1$ and for every $\omega \in M$ and a_i we have $\mathcal{L}_n(\omega, a_i) \to \mathcal{L}(a_i)$.

Since we have stochastic equicontinuity, there exists a set $\tilde{M}$ where $P(\tilde{M}) = 1$ and for every $\omega \in \tilde{M}$, $\{\mathcal{L}_n(\omega, \cdot)\}$ is equicontinuous. Let $\bar{M} = \tilde{M} \cap \{\cap M_{a_i}\}$, we will show that for all $a \in \Theta$ and $\omega \in \bar{M}$ we have $\mathcal{L}_n(\omega, a) \to \mathcal{L}(a)$. By stochastic equicontinuity for every $\omega \in \bar{M}$ and $\varepsilon/3 > 0$, there exists a $\delta > 0$ such that

$$\sup_{|b_1 - b_2| \leq \delta} |\mathcal{L}_n(\omega, b_1) - \mathcal{L}_n(\omega, b_2)| \leq \varepsilon/3, \tag{11.5}$$

for all $n > N(\omega)$. Furthermore by definition of $\bar{M}$ for every rational $a_j \in \Theta$ and $\omega \in \bar{N}$ we have

$$|\mathcal{L}_n(\omega, a_i) - \mathcal{L}(a_i)| \leq \varepsilon/3, \tag{11.6}$$

where $n > N'(\omega)$. Now for any given $a \in \Theta$, there exists a rational a_i such that $\|a - a_j\| \leq \delta$. Using this, (11.5) and (11.6) we have

$$|\mathcal{L}_n(\omega, a) - \mathcal{L}(a)| \leq |\mathcal{L}_n(\omega, a) - \mathcal{L}_n(\omega, a_i)| + |\mathcal{L}_n(\omega, a_i) - \mathcal{L}(a_i)| + |\mathcal{L}(a) - \mathcal{L}(a_i)| \leq \varepsilon,$$

for $n > \max(N(\omega), N'(\omega))$. To summarise for every $\omega \in \bar{M}$ and $a \in \Theta$, we have $|\mathcal{L}_n(\omega, a) - \mathcal{L}(a)| \to 0$. Hence we have pointwise covergence for every realisation in $\bar{M}$. □

We now show that equicontinuity implies uniform convergence.

Proof of Theorem 11.3.2. Using Lemma 11.3.1 we see that there exists a set $\bar{M} \in \Omega$ with $P(\bar{M}) = 1$, where $\mathcal{L}_n$ is equicontinuous and also pointwise convergent. We now show uniform convergence on this set. Choose $\varepsilon/3 > 0$ and let δ be such that for every $\omega \in \bar{M}$ we have

$$\sup_{|a_1 - a_2| \leq \delta} |\mathcal{L}_T(\omega, a_1) - \mathcal{L}_T(\omega, a_2)| \leq \varepsilon/3, \tag{11.7}$$

for all $n > n(\omega)$. Since Θ is compact it can be divided into a finite number of open sets. Construct the sets $\{O_i\}_{i=1}^p$, such that $\Theta \subset \cup_{i=1}^p O_i$ and $\sup_{x,y,i} \|x - y\| \leq \delta$. Let $\{a_i\}_{i=1}^p$ be such that $a_i \in O_i$. We note that for every $\omega \in \bar{M}$ we have $\mathcal{L}_n(\omega, a_i) \to \mathcal{L}(a_i)$, hence for every $\varepsilon/3$, there exists an $n_i(\omega)$ such that for all $n > n_i(\omega)$ we have $|\mathcal{L}_T(\omega, a_i) - \mathcal{L}(a_i)| \leq \varepsilon/3$. Therefore, since p is finite (due to compactness), there exists a $\tilde{n}(\omega)$ such that

$$\max_{1\leq i\leq p}|\mathcal{L}_n(\omega,a_i)-\mathcal{L}(a_i)|\leq\varepsilon/3,$$

for all $n>\tilde{n}(\omega)=\max_{1\leq i\leq p}(n_i(\omega))$. For any $a\in\Theta$, choose the i, such that open set O_i such that $a\in O_i$. Using (11.7) we have

$$|\mathcal{L}_T(\omega,a)-\mathcal{L}_T(\omega,a_i)|\leq\varepsilon/3,$$

for all $n>n(\omega)$. Altogether this gives

$$|\mathcal{L}_T(\omega,a)-\mathcal{L}(a)|\leq|\mathcal{L}_T(\omega,a)-\mathcal{L}_T(\omega,a_i)|+|\mathcal{L}_T(\omega,a_i)-\mathcal{L}(a_i)|+|\mathcal{L}(a)-\mathcal{L}(a_i)|\leq\varepsilon,$$

for all $n\geq\max(n(\omega),\tilde{n}(\omega))$. We observe that $\max(n(\omega),\tilde{n}(\omega))$ and $\varepsilon/3$ does not depend on a, therefore for all $n\geq\max(n(\omega),\tilde{n}(\omega))$ and we have $\sup_a|\mathcal{L}_n(\omega,a)-\mathcal{L}(a)|<\varepsilon$. This gives for every $\omega\in\bar{M}$ ($\mathbb{P}(\bar{M})=1$), $\sup_a|\mathcal{L}_n(\omega,a)-\mathcal{L}(a)|\to 0$, thus we have almost sure uniform convergence. □

11.4 Toy Example: Almost sure convergence of the least squares estimator for an AR(p) process

In Chapter **??** we will consider the sampling properties of many of the estimators defined in Chapter 7. However to illustrate the consistency result above we apply it to the least squares estimator of the autoregressive parameters.

To simply notation we only consider estimator for $AR(1)$ models. Suppose that X_t satisfies $X_t=\phi X_{t-1}+\varepsilon_t$ (where $|\phi|<1$). To estimate ϕ we use the least squares estimator defined below. Let

$$\mathcal{L}_n(a)=\frac{1}{n-1}\sum_{t=2}^{n}(X_t-aX_{t-1})^2, \qquad (11.8)$$

we use $\hat{\phi}_n$ as an estimator of ϕ, where

$$\hat{\phi}_n \;=\; \arg\min_{a\in\Theta}\mathcal{L}_T(a), \qquad (11.9)$$

where $\Theta = [-1, 1]$.

How can we show that this is consistent?

- In the case of least squares for AR processes, $\hat{a}_T$ has the explicit form

$$\hat{\phi}_n = \frac{\frac{1}{n-1}\sum_{t=2}^{n} X_t X_{t-1}}{\frac{1}{n-1}\sum_{t=1}^{T-1} X_t^2}.$$

  By just applying the ergodic theorem to the numerator and denominator we get $\hat{\phi}_n \overset{\text{a.s.}}{\to} \phi$. It is worth noting, that unlike the Yule-Walker estimator $\left|\frac{\frac{1}{n-1}\sum_{t=2}^{n} X_t X_{t-1}}{\frac{1}{n-1}\sum_{t=1}^{n-1} X_t^2}\right| < 1$ is not necessarily true.

- Here we will tackle the problem in a rather artifical way and assume that it does not have an explicit form and instead assume that $\hat{\phi}_n$ is obtained by minimising $\mathcal{L}_n(a)$ using a numerical routine.

- In order to derive the sampling properties of $\hat{\phi}_n$ we need to directly study the least squares criterion $\mathcal{L}_n(a)$. We will do this now in the least squares case.

We will first show almost sure convergence, which will involve repeated use of the ergodic theorem. We will then demonstrate how to show convergence in probability. We look at almost sure convergence as its easier to follow. Note that almost sure convergence implies convergence in probability (but the converse is not necessarily true).

The first thing to do it let

$$\ell_t(a) = (X_t - aX_{t-1})^2.$$

Since $\{X_t\}$ is an ergodic process (recall Example **??**(ii)) by using Theorem **??** we have for a, that $\{\ell_t(a)\}_t$ is an ergodic process. Therefore by using the ergodic theorem we have

$$\mathcal{L}_n(a) = \frac{1}{n-1}\sum_{t=2}^{n} \ell_t(a) \overset{\text{a.s.}}{\to} \mathrm{E}(\ell_0(a)).$$

In other words for every $a \in [-1, 1]$ we have that $\mathcal{L}_n(a) \overset{\text{a.s.}}{\to} \mathrm{E}(\ell_0(a))$ (almost sure pointwise convergence).

Since the parameter space $\Theta = [-1, 1]$ is compact and a is the unique minimum of $\ell(\cdot)$ in the parameter space, all that remains is to show show stochastic equicontinuity. From this we deduce almost sure uniform convergence.

To show stochastic equicontinuity we expand $\mathcal{L}_T(a)$ and use the mean value theorem to obtain

$$\mathcal{L}_n(a_1) - \mathcal{L}_n(a_2) = \nabla \mathcal{L}_T(\bar{a})(a_1 - a_2), \tag{11.10}$$

where $\bar{a} \in [\min[a_1, a_2], \max[a_1, a_2]]$ and

$$\nabla \mathcal{L}_n(\bar{a}) = \frac{-2}{n-1} \sum_{t=2}^{n} X_{t-1}(X_t - \bar{a}X_{t-1}).$$

Because $\bar{a} \in [-1, 1]$ we have

$$|\nabla \mathcal{L}_n(\bar{a})| \leq \mathcal{D}_n, \text{ where } \mathcal{D}_n = \frac{2}{n-1} \sum_{t=2}^{n} (|X_{t-1}X_t| + X_{t-1}^2).$$

Since $\{X_t\}_t$ is an ergodic process, then $\{|X_{t-1}X_t| + X_{t-1}^2\}$ is an ergodic process. Therefore, if $\text{var}(\varepsilon_t) < \infty$, by using the ergodic theorem we have

$$\mathcal{D}_n \overset{\text{a.s.}}{\to} 2\text{E}(|X_{t-1}X_t| + X_{t-1}^2).$$

Let $\mathcal{D} := 2\text{E}(|X_{t-1}X_t| + X_{t-1}^2)$. Therefore there exists a set $M \in \Omega$, where $\mathbb{P}(M) = 1$ and for every $\omega \in M$ and $\varepsilon > 0$ we have

$$|\mathcal{D}_T(\omega) - \mathcal{D}| \leq \delta^*,$$

for all $n > N(\omega)$. Substituting the above into (11.10) we have

$$|\mathcal{L}_n(\omega, a_1) - \mathcal{L}_n(\omega, a_2)| \quad \leq \quad \mathcal{D}_n(\omega)|a_1 - a_2| \leq (\mathcal{D} + \delta^*)|a_1 - a_2|,$$

for all $n \geq N(\omega)$. Therefore for every $\varepsilon > 0$, there exists a $\delta := \varepsilon/(\mathcal{D} + \delta^*)$ such that

$$\sup_{|a_1 - a_2| \leq \varepsilon/(\mathcal{D}+\delta^*)} |\mathcal{L}_n(\omega, a_1) - \mathcal{L}_n(\omega, a_2)| \quad \leq \quad \varepsilon,$$

for all $n \geq N(\omega)$. Since this is true for all $\omega \in M$ we see that $\{\mathcal{L}_n(a)\}$ is stochastically equicontinuous.

Theorem 11.4.1 *Let $\hat{\phi}_n$ be defined as in (11.9). Then we have $\hat{\phi}_n \overset{a.s.}{\to} \phi$.*

PROOF. Since $\{\mathcal{L}_n(a)\}$ is almost sure equicontinuous, the parameter space $[-1,1]$ is compact and we have pointwise convergence of $\mathcal{L}_n(a) \overset{\text{a.s.}}{\to} \mathcal{L}(a)$, by using Theorem 11.3.1 we have that $\hat{\phi}_n \overset{\text{a.s.}}{\to} a$, where $a = \min_{a\in\Theta} \mathcal{L}(a)$. Finally we need to show that $a = \phi$. Since

$$\mathcal{L}(a) = \mathrm{E}(\ell_0(a)) = -\mathrm{E}(X_1 - aX_0)^2,$$

we see by differentiating $\mathcal{L}(a)$ with respect to a, that it is minimised at $a = \mathrm{E}(X_0X_1)/\mathrm{E}(X_0^2)$, hence $a = \mathrm{E}(X_0X_1)/\mathrm{E}(X_0^2)$. To show that this is ϕ, we note that by the Yule-Walker equations

$$X_t = \phi X_{t-1} + \epsilon_t \quad \Rightarrow \quad \mathrm{E}(X_tX_{t-1}) = \phi \mathrm{E}(X_{t-1}^2) + \underbrace{\mathrm{E}(\epsilon_t X_{t-1})}_{=0}.$$

Therefore $\phi = \mathrm{E}(X_0X_1)/\mathrm{E}(X_0^2)$, hence $\hat{\phi}_n \overset{\text{a.s.}}{\to} \phi$. □

We note that by using a very similar methods we can show strong consistency of the least squares estimator of the parameters in an AR(p) model.

11.5 Convergence in probability of an estimator

We described above almost sure (strong) consistency ($\hat{a}_T \overset{\text{a.s.}}{\to} a_0$). Sometimes its not possible to show strong consistency (eg. when ergodicity cannot be verified). As an alternative, weak consistency where $\hat{a}_T \overset{\mathcal{P}}{\to} a_0$ (convergence in probability), is shown. This requires a weaker set of conditions, which we now describe:

(i) The parameter space Θ should be compact.

(ii) Probability pointwise convergence: for every $a \in \Theta$ $\mathcal{L}_n(a) \overset{\mathcal{P}}{\to} \mathcal{L}(a)$.

(iii) The sequence $\{\mathcal{L}_n(a)\}$ is equicontinuous in probability. That is for every $\epsilon > 0$ and $\eta > 0$ there exists a δ such that

$$\lim_{n\to\infty} \mathbb{P}\left(\sup_{|a_1-a_2|\le\delta} |\mathcal{L}_n(a_1) - \mathcal{L}_n(a_2)| > \epsilon \right) < \eta. \tag{11.11}$$

If the above conditions are satisified we have $\hat{a}_T \overset{\mathcal{P}}{\to} a_0$.

Verifying conditions (ii) and (iii) may look a little daunting but by using Chebyshev's (or Markov's) inequality it can be quite straightforward. For example if we can show that for every $a \in \Theta$

$$\mathrm{E}(\mathcal{L}_n(a) - \mathcal{L}(a))^2 \to 0 \quad T \to \infty.$$

Therefore by applying Chebyshev's inequality we have for every $\varepsilon > 0$ that

$$P(|\mathcal{L}_n(a) - \mathcal{L}(a)| > \varepsilon) \leq \frac{\mathrm{E}(\mathcal{L}_n(a) - \mathcal{L}(a))^2}{\varepsilon^2} \to 0 \quad T \to \infty.$$

Thus for every $a \in \Theta$ we have $\mathcal{L}_n(a) \overset{\mathcal{P}}{\to} \mathcal{L}(a)$.

To show (iii) we often use the mean value theorem $\mathcal{L}_n(a)$. Using the mean value theorem we have

$$|\mathcal{L}_n(a_1) - \mathcal{L}_n(a_2)| \leq \sup_a \|\nabla_a \mathcal{L}_n(a)\|_2 \|a_1 - a_2\|.$$

Now if we can show that $\sup_n \mathrm{E} \sup_a \|\nabla_a \mathcal{L}_n(a)\|_2 < \infty$ (in other words it is uniformly bounded in probability over n) then we have the result. To see this observe that

$$\begin{aligned} \mathbb{P}\left(\sup_{|a_1 - a_2| \leq \delta} |\mathcal{L}_n(a_1) - \mathcal{L}_n(a_2)| > \epsilon\right) &\leq \mathbb{P}\left(\sup_{a \in \Omega} \|\nabla_a \mathcal{L}_n(a)\|_2 |a_1 - a_2| > \epsilon\right) \\ &\leq \frac{\sup_n \mathrm{E}(|a_1 - a_2| \sup_{a \in \Omega} \|\nabla_a \mathcal{L}_n(a)\|_2)}{\epsilon}. \end{aligned}$$

Therefore by a careful choice of $\delta > 0$ we see that (11.11) is satisfied (and we have equicontinuity in probability).

11.6 Asymptotic normality of an estimator

Once consistency of an estimator has been shown this paves the way to showing normality. To make the derivations simple we will assume that θ is univariate (this allows to easily use Taylor expansion) . We will assume that that the third derivative of the contrast function, $\mathcal{L}_n(\theta)$, exists, its expectation is bounded and it's variance converges to zero as $n \to \infty$. If this is the case we have have the following result

Lemma 11.6.1 *Suppose that the third derivative of the contrast function* $\mathcal{L}_n(\theta)$ *exists, for* $k = 0,1,2$ $\mathrm{E}(\frac{\partial^k \mathcal{L}_n(\theta)}{\partial theta^k}) = \frac{\partial^k \mathcal{L}}{\partial \theta^k}$ *and* $\mathrm{var}(\frac{\partial^k \mathcal{L}_n(\theta)}{\partial theta^k}) \to 0$ *as* $n \to \infty$ *and* $\frac{\partial^3 \mathcal{L}_n(\theta)}{\partial theta^3}$ *is bounded by a random variable* Z_n *which is independent of* n *where* $\mathrm{E}(Z_n) < \infty$ *and* $\mathrm{var}(Z_n) \to 0$. *Then we have*

$$(\hat{\theta}_n - \theta_0) = V(\theta)^{-1} \frac{\partial \mathcal{L}_n(\theta)}{\partial \theta}_{\theta=\theta_0} + o_p(1) \frac{\partial \mathcal{L}_n(\theta)}{\partial \theta}_{\theta=\theta_0},$$

where $V(\theta_0) = \frac{\partial^2 \mathcal{L}(\theta)}{\partial \theta^2}_{\theta_0}$.

PROOF. By the mean value theorem we have

$$\frac{\partial \mathcal{L}_n(\theta)}{\partial \theta}_{\theta=\theta_0} = \frac{\partial \mathcal{L}_n(\theta)}{\partial \theta}_{\theta=\hat{\theta}_n} - (\hat{\theta}_n - \theta_0) \frac{\partial^2 \mathcal{L}_n(\theta)}{\partial \theta^2}_{\theta=\bar{\theta}_n} = -(\hat{\theta}_n - \theta_0) \frac{\partial^2 \mathcal{L}_n(\theta)}{\partial \theta^2}_{\theta=\bar{\theta}_n} \quad (11.12)$$

where $\bar{\theta}_n$ lies between θ_0 and $\hat{\theta}_n$. We first study $\frac{\partial^2 \mathcal{L}_n(\theta)}{\partial \theta^2}_{\theta=\bar{\theta}_n}$. By using the man value theorem we have

$$\frac{\partial^2 \mathcal{L}_n(\theta)}{\partial \theta^2}_{\theta=\bar{\theta}_n} = \frac{\partial^2 \mathcal{L}_n(\theta)}{\partial \theta^2}_{\theta_0} + (\bar{\theta}_n - \theta_0) \frac{\partial^2 \mathcal{L}_n(\theta)}{\partial \theta^2}_{\theta=\tilde{\theta}_n}$$

where $\tilde{\theta}_n$ lies between θ_0 and $\bar{\theta}_n$. Since $\frac{\partial^2 \mathcal{L}_n(\theta)}{\partial \theta^2}_{\theta_0} \to \frac{\partial^2 \mathcal{L}(\theta)}{\partial \theta^2}_{\theta_0} = V(\theta_0)$, under the stated assumptions we have

$$\big| \frac{\partial^2 \mathcal{L}}{\partial \theta^2}_{\theta=\bar{\theta}_n} - V(\theta_0) \big| \leq |\bar{\theta}_n - \theta_0| \big| \frac{\partial^2 \mathcal{L}_n(\theta)}{\partial \theta^2}_{\theta=\tilde{\theta}_n} \big| \leq |\bar{\theta}_n - \theta_0| W_n.$$

Therefore, by consistency of the estimator it is clear that $\frac{\partial^2 \mathcal{L}}{\partial \theta^2}_{\theta=\bar{\theta}_n} \xrightarrow{\mathcal{P}} V(\theta_0)$. Substituting this into (11.12) we have

$$\frac{\partial \mathcal{L}}{\partial \theta}_{\theta=\theta_0} = -(\hat{\theta}_n - \theta_0)(V(\theta_0) + o(1)),$$

since $V(\theta_0)$ is bounded away from zero we have $[\frac{\partial^2 \mathcal{L}}{\partial \theta^2}_{\theta=\bar{\theta}_n}]^{-1} = V(\theta_0)^{-1} + o_p(1)$ and we obtain the desired result. □

The above result means that the distribution of $(\hat{\theta}_n - \theta_0)$ is determined by $\frac{\partial \mathcal{L}}{\partial \theta}_{\theta=\theta_0}$. In the following section we show to show asymptotic normality of $\frac{\partial \mathcal{L}}{\partial \theta}_{\theta=\theta_0}$.

11.6.1 Martingale central limit theorem

Remark 11.6.1 *We recall that*

$$(\hat{\phi}_n - \phi) = -\left(\nabla^2\mathcal{L}_n\right)^{-1}\nabla\mathcal{L}_n(\phi) = \frac{\frac{-2}{n-1}\sum_{t=2}^{n}\varepsilon_t X_{t-1}}{\frac{2}{n-1}\sum_{t=2}^{n}X_{t-1}^2}, \tag{11.13}$$

and that $\mathrm{var}(\frac{-2}{n-1}\sum_{t=2}^{n}\varepsilon_t X_{t-1}) = \frac{-2}{n-1}\sum_{t=2}^{n}\mathrm{var}(\varepsilon_t X_{t-1}) = O(\frac{1}{n})$. *This implies*

$$(\hat{\phi}_n - \phi) = O_p(n^{-1/2}).$$

Indeed the results also holds almost surely

$$(\hat{\phi}_n - \phi) = O(n^{-1/2}). \tag{11.14}$$

The same result is true for autoregressive processes of arbitrary finite order. That is

$$\sqrt{n}(\hat{\underline{\phi}}_n - \underline{\phi}) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \mathrm{E}(\Gamma_p)^{-1}\sigma^2). \tag{11.15}$$

11.6.2 Example: Asymptotic normality of the weighted periodogram

Previously we have discussed the weight peiodogram, here we show normality of it, in the case that the time series X_t is zero mean linear time series (has the representation $X_t = \sum_j \psi_j \varepsilon_{t-j}$). Recalling Lemma 10.2.2 we have

$$\begin{aligned} A(\phi, I_n) &= \frac{1}{n}\sum_{k=1}^{n}\phi(\omega_k)I_n(\omega_k) \\ &= \frac{1}{n}\sum_{k=1}^{n}\phi(\omega_k)|A(\omega_k)^2|I_\varepsilon(\omega_k) + o(\frac{1}{n}). \end{aligned}$$

Therefore we will show asymptotic normality of $\frac{1}{n}\sum_{k=1}^{n}\phi(\omega_k)|A(\omega_k)^2|I_\varepsilon(\omega_k)$, which will give asymptotic normality of $A(\phi, I_n)$. Expanding $|I_\varepsilon(\omega_k)$ and substituting this into
$\frac{1}{n}\sum_{k=1}^{n}\phi(\omega_k)|A(\omega_k)^2|I_\varepsilon(\omega_k)$ gives

$$\frac{1}{n}\sum_{k=1}^{n}\phi(\omega_k)|A(\omega_k)^2|I_\varepsilon(\omega_k) = \frac{1}{n}\sum_{t,\tau=1}^{n}\varepsilon_t\varepsilon_\tau\frac{1}{n}\sum_{k=1}^{n}\phi(\omega_k)|A(\omega_k)^2\exp(i\omega_k(t-\tau)) = \frac{1}{n}\sum_{t,\tau=1}^{n}\varepsilon_t\varepsilon_\tau g_n(t-\tau)$$

where

$$g_n(t-\tau) = \frac{1}{n}\sum_{k=1}^{n}\phi(\omega_k)|A(\omega_k)^2\exp(i\omega_k(t-\tau)) = \frac{1}{2\pi}\int_0^{2\pi}\phi(\omega)|A(\omega)|^2\exp(i\omega(t-\tau))d\omega + O(\frac{1}{n^2}),$$

(the rate for the derivative exchange is based on assuming that the second derivatives of $A(\omega)$ and ϕ exist and $\phi(0) = \phi(2\pi)$). We can rewrite $\frac{1}{n}\sum_{t,\tau=1}^{n}\varepsilon_t\varepsilon_\tau g_n(t-\tau)$ as

$$\begin{aligned}
& \frac{1}{n}\sum_{t,\tau=1}^{n}[\varepsilon_t\varepsilon_\tau - \mathrm{E}(\varepsilon_t\varepsilon_\tau)]g_n(t-\tau) \\
= \quad & \frac{1}{n}\sum_{t=1}^{n}\Big([(\varepsilon_t^2 - \mathrm{E}(\varepsilon_t^2)]g_n(0) + \varepsilon_t\big(\sum_{\tau<t}\varepsilon_\tau[g_n(t-\tau) - g_n(\tau-t)]\big)\Big) \\
:= \quad & \frac{1}{n}\sum_{t=1}^{n}Z_{t,n}
\end{aligned}$$

where it is straightforward to show that $\{Z_{t,n}\}$ are the sum of martingale differences. Thus we can show that

$$\frac{1}{\sqrt{n}}\sum_{t,\tau=1}^{n}\varepsilon_t\varepsilon_\tau g_n(t-\tau) - \mathrm{E}\big(\frac{1}{\sqrt{n}}\sum_{t,\tau=1}^{n}\varepsilon_t\varepsilon_\tau g_n(t-\tau)\big) = \frac{1}{\sqrt{n}}\sum_{t=1}^{n}Z_{t,n}$$

satisfies the conditions of the martingale central limit theorem, which gives asymptotic normality of $\frac{1}{n}\sum_{t,\tau=1}^{n}\varepsilon_t\varepsilon_\tau g_n(t-\tau)$ and thus $A(\phi, I_n)$.

In the remainder of this chapter we obtain the sampling properties of the ARMA estimators defined in Sections 8.2.1 and 8.2.4.

11.7 Asymptotic properties of the Hannan and Rissanen estimation method

In this section we will derive the sampling properties of the Hannan-Rissanen estimator. We will obtain an almost sure rate of convergence (this will be the only estimator where we obtain an almost sure rate). Typically obtaining only sure rates can be more difficult than obtaining probabilistic rates, moreover the rates can be different (worse in the almost sure case). We now illustrate why that is with a small example. Suppose $\{X_t\}$ are iid random variables with mean zero and variance one. Let $S_n = \sum_{t=1}^{n} X_t$. It can easily be shown that

$$\text{var}(S_n) = \frac{1}{n} \text{ therefore } S_n = O_p(\frac{1}{\sqrt{n}}). \tag{11.16}$$

However, from the law of iterated logarithm we have for any $\varepsilon > 0$

$$P(S_n \geq (1+\varepsilon)\sqrt{2n\log\log n} \text{ infinitely often}) = 0 P(S_n \geq (1-\varepsilon)\sqrt{2n\log\log n} \text{ infinitely often}) = 1 \tag{11.17}$$

Comparing (11.16) and (11.17) we see that for any given trajectory (realisation) most of the time $\frac{1}{n}S_n$ will be within the $O(\frac{1}{\sqrt{n}})$ bound but there will be excursions above when it to the $O(\frac{\log\log n}{\sqrt{n}}$ bound. In other words we cannot say that $\frac{1}{n}S_n = (\frac{1}{\sqrt{n}})$ almost surely, but we can say that This basically means that

$$\frac{1}{n}S_n = O(\frac{\sqrt{2\log\log n}}{\sqrt{n}}) \text{ almost surely.}$$

Hence the probabilistic and the almost sure rates are (slightly) different. Given this result is true for the average of iid random variables, it is likely that similar results will hold true for various estimators.

In this section we derive an almost sure rate for Hannan-Rissanen estimator, this rate will be determined by a few factors (a) an almost sure bound similar to the one derived above (b) the increasing number of parameters p_n (c) the bias due to estimating only a finite number of parameters when there are an infinite number in the model.

We first recall the algorithm:

(i) Use least squares to estimate $\{b_j\}_{j=1}^{p_n}$ and define

$$\hat{\mathbf{b}}_n = \hat{R}_n^{-1}\hat{\mathbf{r}}_n, \tag{11.18}$$

where $\hat{\mathbf{b}}_n' = (\hat{b}_{1,n}, \ldots, \hat{b}_{p_n,n})$,

$$\hat{R}_n = \sum_{t=p_n+1}^{n} \mathbf{X}_{t-1}\mathbf{X}_{t-1}' \quad \hat{\mathbf{r}}_n = \sum_{t=p_n+1}^{T} X_t\mathbf{X}_{t-1}$$

and $\mathbf{X}_{t-1}' = (X_{t-1}, \ldots, X_{t-p_n})$.

(ii) Estimate the residuals with

$$\tilde{\varepsilon}_t = X_t - \sum_{j=1}^{p_n} \hat{b}_{j,n} X_{t-j}.$$

(iii) Now use as estimates of $\underline{\phi}_0$ and $\underline{\theta}_0$ $\tilde{\underline{\phi}}_n, \tilde{\underline{\theta}}_n$ where

$$\tilde{\underline{\phi}}_n, \tilde{\underline{\theta}}_n = \arg\min \sum_{t=p_n+1}^{n} (X_t - \sum_{j=1}^{p} \phi_j X_{t-j} - \sum_{i=1}^{q} \theta_i \tilde{\varepsilon}_{t-i})^2. \tag{11.19}$$

We note that the above can easily be minimised. In fact

$$(\tilde{\underline{\phi}}_n, \tilde{\underline{\theta}}_n) = \tilde{\mathcal{R}}_n^{-1} \tilde{\mathbf{s}}_n$$

where

$$\tilde{\mathcal{R}}_n = \frac{1}{n} \sum_{t=p_n+1}^{n} \tilde{\mathbf{Y}}_t \tilde{\mathbf{Y}}_t \quad \tilde{\mathbf{s}}_n = \frac{1}{T} \sum_{t=p_n+1}^{n} \tilde{\mathbf{Y}}_t X_t,$$

$\tilde{\mathbf{Y}}_t' = (X_{t-1}, \ldots, X_{t-p}, \tilde{\varepsilon}_{t-1}, \ldots, \tilde{\varepsilon}_{t-q})$. Let $\hat{\varphi}_n = (\tilde{\phi}_n, \tilde{\theta}_n)$.

We observe that in the second stage of the scheme where the estimation of the ARMA parameters are done, it is important to show that the empirical residuals are close to the true residuals. That is $\tilde{\varepsilon}_t = \varepsilon_t + o(1)$. We observe that from the definition of $\tilde{\varepsilon}_t$, this depends on the rate of convergence of the AR estimators $\hat{b}_{j,n}$

$$\begin{aligned}
\tilde{\varepsilon}_t &= X_t - \sum_{j=1}^{p_n} \hat{b}_{j,n} X_{t-j} \\
&= \varepsilon_t + \sum_{j=1}^{p_n} (\hat{b}_{j,n} - b_j) X_{t-j} - \sum_{j=p_n+1}^{\infty} b_j X_{t-j}.
\end{aligned} \tag{11.20}$$

Hence

$$\big|\hat{\varepsilon}_t - \varepsilon_t\big| \leq \big| \sum_{j=1}^{p_n} (\hat{b}_{j,n} - b_j) X_{t-j} \big| + \big| \sum_{j=p_n+1}^{\infty} b_j X_{t-j} \big|. \tag{11.21}$$

Therefore to study the asymptotic properties of $\tilde{\boldsymbol{\varphi}} = \hat{\underline{\phi}}_n, \hat{\underline{\theta}}_n$ we need to

- Obtain a rate of convergence for $\sup_j |\hat{b}_{j,n} - b_j|$.

- Obtain a rate for $|\hat{\varepsilon}_t - \varepsilon_t|$.

- Use the above to obtain a rate for $\tilde{\boldsymbol{\varphi}}_n = (\hat{\underline{\phi}}_n, \hat{\underline{\theta}}_n)$.

We first want to obtain the uniform rate of convergence for $\sup_j |\hat{b}_{j,n} - b_j|$. Deriving this is technically quite challanging. We state the rate in the following theorem, an outline of the proof can be found in Section 11.7.1. The proofs uses results from mixingale theory which can be found in Chapter B.

Theorem 11.7.1 *Suppose that $\{X_t\}$ is from an ARMA process where the roots of the true characteristic polynomials $\phi(z)$ and $\theta(z)$ both have absolute value greater than $1+\delta$. Let $\hat{\mathbf{b}}_n$ be defined as in (11.18), then we have almost surely*

$$\|\hat{\mathbf{b}}_n - \mathbf{b}_n\|_2 = O\left(p_n^2\sqrt{\frac{(\log\log n)^{1+\gamma}\log n}{n}} + \frac{p_n^3}{n} + p_n\rho^{p_n}\right)$$

for any $\gamma > 0$.

PROOF. See Section 11.7.1.

Corollary 11.7.1 *Suppose the conditions in Theorem 11.7.1 are satisfied. Then we have*

$$\left|\tilde{\varepsilon}_t - \varepsilon_t\right| \leq p_n \max_{1\leq j\leq p_n} |\hat{b}_{j,n} - b_j| Z_{t,p_n} + K\rho^{p_n} Y_{t-p_n}, \tag{11.22}$$

where $Z_{t,p_n} = \frac{1}{p_n}\sum_{t=1}^{p_n}|X_{t-j}|$ and $Y_t = \sum_{t=1}^{p_n}\rho^j|X_t|$,

$$\frac{1}{n}\sum_{t=p_n+1}^{n}\left|\tilde{\varepsilon}_{t-i}X_{t-j} - \varepsilon_{t-i}X_{t-j}\right| = O(p_nQ(n) + \rho^{p_n}) \tag{11.23}$$

$$\frac{1}{n}\sum_{t=p_n+1}^{n}\left|\tilde{\varepsilon}_{t-i}\tilde{\varepsilon}_{t-j} - \varepsilon_{t-i}\tilde{\varepsilon}_{t-j}\right| = O(p_nQ(n) + \rho^{p_n}) \tag{11.24}$$

where $Q(n) = p_n^2\sqrt{\frac{(\log\log n)^{1+\gamma}\log n}{n}} + \frac{p_n^3}{n} + p_n\rho^{p_n}$.

PROOF. Using (11.21) we immediately obtain (11.22).

To obtain (11.23) we use (11.21) to obtain

$$\begin{aligned}
\frac{1}{n}\sum_{t=p_n+1}^{n}\big|\tilde{\varepsilon}_{t-i}X_{t-j}-\varepsilon_{t-i}X_{t-j}\big| \leq \frac{1}{n}\sum_{t=p_n+1}^{n}|X_{t-j}|\big|\tilde{\varepsilon}_{t-i}-\varepsilon_{t-i}\big| \\
\leq \quad O(p_nQ(n))\frac{1}{n}\sum_{t=p_n+1}^{n}|X_t||Z_{t,p_n}|+O(\rho^{p_n})\frac{1}{n}\sum_{t=p_n+1}^{n}|X_t||Y_{t-p_n}| \\
= \quad O(p_nQ(n)+\rho^{p_n}).
\end{aligned}$$

To prove (11.24) we use a similar method, hence we omit the details. □

We apply the above result in the theorem below.

Theorem 11.7.2 *Suppose the assumptions in Theorem 11.7.1 are satisfied. Then*

$$\big\|\tilde{\boldsymbol{\varphi}}_n-\boldsymbol{\varphi}_0\big\|_2 \quad = \quad O\left(p_n^3\sqrt{\frac{(\log\log n)^{1+\gamma}\log n}{n}}+\frac{p_n^4}{n}+p_n^2\rho^{p_n}\right).$$

for any $\gamma>0$, where $\tilde{\boldsymbol{\varphi}}_n=(\underline{\tilde{\phi}}_n,\underline{\tilde{\theta}}_n)$ and $\boldsymbol{\varphi}_0=(\underline{\phi}_0,\underline{\theta}_0)$.

PROOF. We note from the definition of $\tilde{\boldsymbol{\varphi}}_n$ that

$$\big(\tilde{\boldsymbol{\varphi}}_n-\boldsymbol{\varphi}_0\big) \quad = \quad \tilde{\mathcal{R}}_n^{-1}\big(\tilde{\mathbf{s}}_n-\tilde{\mathcal{R}}_n\tilde{\boldsymbol{\varphi}}_0\big).$$

Now in the $\tilde{\mathcal{R}}_n$ and $\tilde{\mathbf{s}}_n$ we replace the estimated residuals $\tilde{\varepsilon}_n$ with the true unobserved residuals. This gives us

$$\big(\tilde{\boldsymbol{\varphi}}_n-\boldsymbol{\varphi}_0\big) \quad = \quad \mathcal{R}_n^{-1}\big(\mathbf{s}_n-\mathcal{R}_n\boldsymbol{\varphi}_0\big)+(\mathcal{R}_n^{-1}\mathbf{s}_n-\tilde{\mathcal{R}}_n^{-1}\tilde{\mathbf{s}}_n) \tag{11.25}$$

$$\mathcal{R}_n=\frac{1}{n}\sum_{t=\max(p,q)}^{n}\mathbf{Y}_t\mathbf{Y}_t \quad \mathbf{s}_n=\frac{1}{n}\sum_{t=\max(p,q)}^{n}\mathbf{Y}_tX_t,$$

$\mathbf{Y}_t'=(X_{t-1},\ldots,X_{t-p},\varepsilon_{t-1},\ldots,\varepsilon_{t-q})$ (recalling that $\mathbf{Y}_t'=(X_{t-1},\ldots,X_{t-p},\tilde{\varepsilon}_{t-1},\ldots,\tilde{\varepsilon}_{t-q})$. The error term is

$$(\mathcal{R}_n^{-1}\mathbf{s}_n-\tilde{\mathcal{R}}_n^{-1}\tilde{\mathbf{s}}_n)=\mathcal{R}_n^{-1}(\tilde{\mathcal{R}}_n-\mathcal{R}_n)\tilde{\mathcal{R}}_n^{-1}\mathbf{s}_n+\tilde{\mathcal{R}}_n^{-1}(\mathbf{s}_n-\tilde{\mathbf{s}}_n).$$

Now, almost surely $\mathcal{R}_n^{-1},\tilde{\mathcal{R}}_n^{-1}=O(1)$ (if $\mathrm{E}(\mathcal{R}_n)$ is non-singular). Hence we only need to obtain a bound for $\tilde{\mathcal{R}}_n - \mathcal{R}_n$ and $\mathbf{s}_n - \tilde{\mathbf{s}}_n$. We recall that

$$\tilde{\mathcal{R}}_n - \mathcal{R}_n = \frac{1}{n}\sum_{t=p_n+1}(\tilde{\mathbf{Y}}_t\tilde{\mathbf{Y}}_t' - \mathbf{Y}_t\mathbf{Y}_t'),$$

hence the terms differ where we replace the estimated $\tilde{\varepsilon}_t$ with the true ε_t, hence by using (11.23) and (11.24) we have almost surely

$$|\tilde{\mathcal{R}}_n - \mathcal{R}_n| = O(p_nQ(n) + \rho^{p_n}) \text{ and } |\tilde{\mathbf{s}}_n - \mathbf{s}_n| = O(p_nQ(n) + \rho^{p_n}).$$

Therefore by substituting the above into (11.26) we obtain

$$\big(\tilde{\boldsymbol{\varphi}}_n - \boldsymbol{\varphi}_0\big) \quad = \quad \mathcal{R}_n^{-1}\big(\mathbf{s}_n - \mathcal{R}_n\boldsymbol{\varphi}_0\big) + O(p_nQ(n) + \rho^{p_n}). \qquad (11.26)$$

Finally using straightforward algebra it can be shown that

$$\mathbf{s}_n - \mathcal{R}_n\boldsymbol{\varphi}_n = \frac{1}{n}\sum_{t=\max(p,q)}^{n}\varepsilon_t\mathbf{Y}_t.$$

By using Theorem 11.7.3, below, we have $\mathbf{s}_n - \mathcal{R}_n\boldsymbol{\varphi}_n = O((p+q)\sqrt{\frac{(\log\log n)^{1+\gamma}\log n}{n}})$. Substituting the above bound into (**??**), and noting that $O(Q(n))$ dominates $O(\sqrt{\frac{(\log\log n)^{1+\gamma}\log n}{n}})$ gives

$$\big\|\tilde{\boldsymbol{\varphi}}_n - \boldsymbol{\varphi}_n\big\|_2 \quad = \quad O\left(p_n^3\sqrt{\frac{(\log\log n)^{1+\gamma}\log n}{n}} + \frac{p_n^4}{n} + p_n^2\rho^{p_n}\right)$$

and the required result. □

11.7.1 Proof of Theorem 11.7.1 (A rate for $\|\hat{\mathbf{b}}_T - \mathbf{b}_T\|_2$)

We observe that

$$\hat{\mathbf{b}}_n - \mathbf{b}_n = R_n^{-1}\big(\hat{\mathbf{r}}_n - \hat{R}_n\mathbf{b}_n\big) + \big(\hat{R}_n^{-1} - R_n^{-1}\big)\big(\hat{\mathbf{r}}_n - \hat{R}_n\mathbf{b}_n\big)$$

where $\mathbf{b}$, R_n and $\mathbf{r}_n$ are deterministic, with $\mathbf{b}_n = (b_1 \ldots, b_{p_n})$, $(R_n)_{i,j} = \mathrm{E}(X_i X_j)$ and $(\mathbf{r}_n)_i = \mathrm{E}(X_0 X_{-i})$. Evaluating the Euclidean distance we have

$$\|\hat{\mathbf{b}}_n - \mathbf{b}_n\|_2 \leq \|R_n^{-1}\|_{spec}\big\|\hat{\mathbf{r}}_n - \hat{R}_n\mathbf{b}_n\big\|_2 + \|R_n^{-1}\|_{spec}\|\hat{R}_n^{-1}\|_{spec}\big\|\hat{R}_n - R_n\big\|_2\big\|\hat{\mathbf{r}}_n - \hat{R}_n\mathbf{b}_n\big\|_2, \tag{11.27}$$

where we used that $\hat{R}_n^{-1} - \hat{R}_n^{-1} = \hat{R}_n^{-1}(R_n - \hat{R}_n)R_n^{-1}$ and the norm inequalities. Now by using Lemma 6.5.1 we have $\lambda_{min}(R_n^{-1}) > \delta/2$ for all T. Thus our aim is to obtain almost sure bounds for $\|\hat{\mathbf{r}}_n - \hat{R}_n\mathbf{b}_n\|_2$ and $\|\hat{R}_n - R_n\|_2$, which requires the lemma below.

Theorem 11.7.3 *Let us suppose that $\{X_t\}$ has an ARMA representation where the roots of the characteristic polynomials $\phi(z)$ and $\theta(z)$ lie are greater than $1+\delta$. Then*

(i)

$$\frac{1}{n}\sum_{t=r+1}^{n} \varepsilon_t X_{t-r} = O\Big(\sqrt{\frac{(\log\log n)^{1+\gamma}\log n}{n}}\Big) \tag{11.28}$$

(ii)

$$\frac{1}{n}\sum_{t=\max(i,j)}^{n} X_{t-i}X_{t-j} = O\Big(\sqrt{\frac{(\log\log n)^{1+\gamma}\log n}{n}}\Big). \tag{11.29}$$

for any $\gamma > 0$.

PROOF. The result is proved in Chapter B.2. □

To obtain the bounds we first note that if the there wasn't an MA component in the ARMA process, in other words $\{X_t\}$ was an AR(p) process with $p_n \geq p$, then $\hat{\mathbf{r}}_n - \hat{R}_n\mathbf{b}_n = \frac{1}{n}\sum_{t=p_n+1}^{n}\varepsilon_t X_{t-r}$, which has a mean zero. However because an ARMA process has an AR(∞) representation and we are only estimating the first p_n parameters, there exists a 'bias' in $\hat{\mathbf{r}}_n - \hat{R}_n\mathbf{b}_n$. Therefore we obtain the decomposition

$$\begin{aligned}(\hat{\mathbf{r}}_n - \hat{R}_n\mathbf{b}_n)_r &= \frac{1}{n}\sum_{t=p_n+1}^{n}\big(X_t - \sum_{j=1}^{\infty} b_j X_{t-j}\big)X_{t-r} + \frac{1}{n}\sum_{t=p_n+1}^{n}\sum_{j=p_n+1}^{\infty} b_j X_{t-j}X_{t-r} && (11.30)\\ &= \underbrace{\frac{1}{n}\sum_{t=p_n+1}^{n}\varepsilon_t X_{t-r}}_{\text{stochastic term}} + \underbrace{\frac{1}{n}\sum_{t=p_n+1}^{n}\sum_{j=p_n+1}^{\infty} b_j X_{t-j}X_{t-r}}_{\text{bias}} && (11.31)\end{aligned}$$

Therefore we can bound the bias with

$$\left|(\hat{\mathbf{r}}_n - \hat{R}_n\mathbf{b}_n)_r - \frac{1}{n}\sum_{t=p_n+1}^{n}\varepsilon_t X_{t-r}\right| \leq K\rho^{p_n}\frac{1}{n}\sum_{t=1}^{n}|X_{t-r}|\sum_{j=1}^{\infty}\rho^j|X_{t-p_n-j}|. \tag{11.32}$$

Let $Y_t = \sum_{j=1}^{\infty}\rho^j|X_{t-j}$ and $S_{n,k,r} = \frac{1}{n}\sum_{t=1}^{n}|X_{t-r}|\sum_{j=1}^{\infty}\rho^j|X_{t-k-j}|$. We note that $\{Y_t\}$ and $\{X_t\}$ are ergodic sequences. By applying the ergodic theorm we can show that for a fixed k and r, $S_{n,k,r} \overset{\text{a.s.}}{\to} \mathrm{E}(X_{t-r}Y_{t-k})$. Hence $S_{n,k,r}$ are almost surely bounded sequences and

$$\rho^{p_n}\frac{1}{n}\sum_{t=1}^{n}|X_{t-r}|\sum_{j=1}^{\infty}\rho^j|X_{t-p_n-j}| = O(\rho^{p_n}).$$

Therefore almost surely we have

$$\|\hat{\mathbf{r}}_n - \hat{R}_n\mathbf{b}_n\|_2 = \|\frac{1}{n}\sum_{t=p_n+1}^{n}\varepsilon_t\mathbf{X}_{t-1}\|_2 + O(p_n\rho^{p_n}).$$

Now by using (11.28) we have

$$\|\hat{\mathbf{r}}_n - \hat{R}_n\mathbf{b}_n\|_2 = O\left(p_n\left\{\sqrt{\frac{(\log\log n)^{1+\gamma}\log n}{n}} + \rho^{p_n}\right\}\right). \tag{11.33}$$

This gives us a rate for $\hat{\mathbf{r}}_n - \hat{R}_n\mathbf{b}_n$. Next we consider $\hat{R}_n$. It is clear from the definition of $\hat{R}_n$ that almost surely we have

$$\begin{aligned}
(\hat{R}_n)_{i,j} - \mathrm{E}(X_iX_j) &= \frac{1}{n}\sum_{t=p_n+1}^{n}X_{t-i}X_{t-j} - \mathrm{E}(X_iX_j)\\
&= \frac{1}{n}\sum_{t=\min(i,j)}^{n}[X_{t-i}X_{t-j} - \mathrm{E}(X_iX_j)] - \frac{1}{n}\sum_{t=\min(i,j)}^{p_n}X_{t-i}X_{t-j} + \frac{\min(i,j)}{n}\mathrm{E}(X_iX_j)\\
&= \frac{1}{n}\sum_{t=\min(i,j)}^{T}[X_{t-i}X_{t-j} - \mathrm{E}(X_iX_j)] + O(\frac{p_n}{n}).
\end{aligned}$$

Now by using (11.29) we have almost surely

$$|(\hat{R}_n)_{i,j} - \mathrm{E}(X_iX_j)| = O(\frac{p_n}{n} + \sqrt{\frac{(\log\log n)^{1+\gamma}\log n}{n}}).$$

Therefore we have almost surely

$$\|\hat{R}_n - R_n\|_2 = O\left(p_n^2\left\{\frac{p_n}{n} + \sqrt{\frac{(\log\log n)^{1+\gamma}\log n}{n}}\right\}\right). \tag{11.34}$$

We note that by using (11.27), (11.33) and (11.34) we have

$$\|\hat{\mathbf{b}}_n - \mathbf{b}_n\|_2 \leq \|R_n^{-1}\|_{spec}\|\hat{R}_n^{-1}\|_{spec} O\left(p_n^2\sqrt{\frac{(\log\log n)^{1+\gamma}\log n}{n}} + \frac{p_n^2}{n} + p_n\rho^{p_n}\right).$$

As we mentioned previously, because the spectrum of X_t is bounded away from zero, $\lambda_{min}(R_n)$ is bounded away from zero for all T. Moreover, since $\lambda_{min}(\hat{R}_n) \geq \lambda_{min}(R_n) - \lambda_{max}(\hat{R}_n - R_n) \geq \lambda_{min}(R_n) - tr((\hat{R}_n - R_n)^2)$, which for a large enough n is bounded away from zero. Hence we obtain almost surely

$$\|\hat{\mathbf{b}}_n - \mathbf{b}_n\|_2 = O\left(p_n^2\sqrt{\frac{(\log\log n)^{1+\gamma}\log n}{n}} + \frac{p_n^3}{n} + p_n\rho^{p_n}\right), \tag{11.35}$$

thus proving Theorem 11.7.1 for any $\gamma > 0$.

11.8 Asymptotic properties of the GMLE

Let us suppose that $\{X_t\}$ satisfies the ARMA representation

$$X_t - \sum_{i=1}^{p}\phi_i^{(0)}X_{t-i} = \varepsilon_t + \sum_{j=1}^{q}\theta_j^{(0)}\varepsilon_{t-j}, \tag{11.36}$$

and $\boldsymbol{\theta}_0 = (\theta_1^{(0)}, \ldots, \theta_q^{(0)})$, $\boldsymbol{\phi}_0 = (\phi_1^{(0)}, \ldots, \phi_p^{(0)})$ and $\sigma_0^2 = \mathrm{var}(\varepsilon_t)$. In this section we consider the sampling properties of the GML estimator, defined in Section 8.2.1. We first recall the estimator. We use as an estimator of $(\underline{\theta}_0, \underline{\phi}_0)$, $\hat{\boldsymbol{\phi}}_n = (\hat{\underline{\theta}}_n, \hat{\underline{\phi}}_n, \hat{\boldsymbol{\sigma}}_n) = \arg\min_{(\underline{\theta},\underline{\phi})\in\Theta} L_n(\underline{\phi}, \underline{\theta}, \sigma)$, where

$$\frac{1}{n}L_n(\underline{\phi}, \underline{\theta}, \sigma) \quad = \quad \frac{1}{n}\sum_{t=1}^{n-1}\log r_{t+1}(\sigma, \underline{\phi}, \underline{\theta}) + \frac{1}{n}\sum_{t=1}^{n-1}\frac{(X_{t+1} - X_{t+1|t}^{(\phi,\theta)})^2}{r_{t+1}(\sigma, \underline{\phi}, \underline{\theta})}. \tag{11.37}$$

To show consistency and asymptotic normality we will use the following assumptions.

Assumption 11.8.1 *(i) X_t is both invertible and causal.*

(ii) *The parameter space should be such that all $\phi(z)$ and $\theta(z)$ in the parameter space have roots whose absolute value is greater than $1+\delta$. $\phi_0(z)$ and $\theta_0(z)$ belong to this space.*

Assumption 11.8.1 means for for some finite constant K and $\frac{1}{1+\delta} \leq \rho < 1$, we have $|\phi(z)^{-1}| \leq K\sum_{j=0}^{\infty}|\rho^j||z^j|$ and $|\phi(z)^{-1}| \leq K\sum_{j=0}^{\infty}|\rho^j||Z^j|$.

To prove the result, we require the following approximations of the GML. Let

$$\tilde{X}_{t+1|t,\ldots}^{(\phi,\theta)} = \sum_{j=1}^{t} b_j(\underline{\phi},\underline{\theta})X_{t+1-j}. \tag{11.38}$$

This is an approximation of the one-step ahead predictor. Since the likelihood is constructed from the one-step ahead predictors, we can approximated the likelihood $\frac{1}{n}L_n(\underline{\phi},\underline{\theta},\sigma)$ with the above and define

$$\frac{1}{n}\tilde{\mathcal{L}}_n(\underline{\phi},\underline{\theta},\sigma) \;\; = \;\; \log\sigma^2 + \frac{1}{n\sigma^2}\sum_{t=1}^{T-1}(X_{t+1} - \tilde{X}_{t+1|t,\ldots}^{(\phi,\theta)})^2. \tag{11.39}$$

We recall that $\tilde{X}_{t+1|t,\ldots}^{(\phi,\theta)}$ was derived from $X_{t+1|t,\ldots}^{(\phi,\theta)}$ which is the one-step ahead predictor of X_{t+1} given $X_t, X_{t-1}, \ldots$, this is

$$X_{t+1|t,\ldots}^{(\phi,\theta)} = \sum_{j=1}^{\infty} b_j(\underline{\phi},\underline{\theta})X_{t+1-j}. \tag{11.40}$$

Using the above we define a approximation of $\frac{1}{n}L_n(\underline{\phi},\underline{\theta},\sigma)$ which in practice cannot be obtained (since the infinite past of $\{X_t\}$ is not observed). Let us define the criterion

$$\frac{1}{n}\mathcal{L}_n(\underline{\phi},\underline{\theta},\sigma) \;\; = \;\; \log\sigma^2 + \frac{1}{n\sigma^2}\sum_{t=1}^{T-1}(X_{t+1} - X_{t+1|t,\ldots}^{(\phi,\theta)})^2. \tag{11.41}$$

In practice $\frac{1}{n}\mathcal{L}_n(\underline{\phi},\underline{\theta},\sigma)$ can not be evaluated, but it proves to be a convenient tool in obtaining the sampling properties of $\hat{\boldsymbol{\phi}}_n$. The main reason is because $\frac{1}{n}\mathcal{L}_n(\underline{\phi},\underline{\theta},\sigma)$ is a function of $\{X_t\}$ and $\{X_{t+1|t,\ldots}^{(\phi,\theta)} = \sum_{j=1}^{\infty} b_j(\underline{\phi},\underline{\theta})X_{t+1-j}\}$ both of these are ergodic (since the ARMA process is ergodic when its roots lie outside the unit circle and the roots of $\phi, \theta \in \Theta$ are such that they lie outside the unit circle). In contrast looking at $L_n(\underline{\phi},\underline{\theta},\sigma)$, which is comprised of $\{X_{t+1|t}\}$, which not an ergodic random variable because X_{t+1} is the best linear predictor of X_{t+1} given $X_t, \ldots, X_1$ (see the number of elements in the prediction changes with t). Using this approximation really simplifies the proof, though it is possible to prove the result without using these approximations.

First we obtain the result for the estimators $\hat{\varphi}_n^* = (\underline{\theta}_n^*, \underline{\phi}_n^*, \hat{\sigma}_n) = \arg\min_{(\underline{\theta},\underline{\phi})\in\Theta} \mathcal{L}_n(\underline{\phi}, \underline{\theta}, \sigma)$ and then show the same result can be applied to $\hat{\varphi}_n$.

Proposition 11.8.1 *Suppose $\{X_t\}$ is an ARMA process which satisfies (11.36), and Assumption 11.8.1 is satisfied. Let $X_{t+1|t}^{(\phi,\theta)}$, $\tilde{X}_{t+1|t,\ldots}^{(\phi,\theta)}$ and $X_{t+1|t,\ldots}^{(\phi,\theta)}$ be the predictors defined in (**??**), (11.38) and (11.40), obtained using the parameters $\phi = \{\phi_j\}$ and $\theta = \{\theta_i\}$, where the roots the corresponding characteristic polynomial $\phi(z)$ and $\theta(z)$ have absolute value greater than $1+\delta$. Then*

$$\big|X_{t+1|t}^{(\phi,\theta)} - \tilde{X}_{t+1|t,\ldots}^{(\phi,\theta)}\big| \leq \frac{\rho^t}{1-\rho}\sum_{i=1}^{t}\rho^i |X_i|, \qquad (11.42)$$

$$\mathrm{E}(X_{t+1|t}^{(\phi,\theta)} - \tilde{X}_{t+1|t,\ldots}^{(\phi,\theta)})^2 \leq K\rho^t, \qquad (11.43)$$

$$\big|\tilde{X}_{t+1|t,\ldots}(1) - X_{t+1|t,\ldots}\big| = \big|\sum_{j=t+1}^{\infty} b_j(\phi,\theta)X_{t+1-j}\big| \leq K\rho^t \sum_{j=0}^{\infty}\rho^j|X_{-j}|, \qquad (11.44)$$

$$\mathrm{E}(X_{t+1|t,\ldots}^{(\phi,\theta)} - \tilde{X}_{t+1|t,\ldots}^{(\phi,\theta)})^2 \leq K\rho^t \qquad (11.45)$$

and

$$|r_t(\sigma, \underline{\phi}, \underline{\theta}) - \sigma^2| \leq K\rho^t \qquad (11.46)$$

for any $1/(1+\delta) < \rho < 1$ and K is some finite constant.

PROOF. The proof follows closely the proof of Proposition 11.8.1. First we define a separate ARMA process $\{Y_t\}$, which is driven by the parameters θ and ϕ (recall that $\{X_t\}$ is drive by the parameters θ_0 and ϕ_0). That is Y_t satisfies $Y_t - \sum_{j=1}^{p}\phi_j Y_{t-j} = \varepsilon_t + \sum_{j=1}^{q}\theta_j\varepsilon_{t-j}$. Recalling that $X_{t+1|t}^{\phi,\theta}$ is the best linear predictor of X_{t+1} given $X_t, \ldots, X_1$ and the variances of $\{Y_t\}$ (noting that it is the process driven by θ and ϕ), we have

$$X_{t+1|t}^{\phi,\theta} \;=\; \sum_{j=1}^{t} b_j(\phi,\theta)X_{t+1-j} + \big(\sum_{j=t+1}^{\infty} b_j(\phi,\theta)\boldsymbol{r}_{t,j}'(\phi,\theta)\Sigma_t(\phi,\theta)^{-1}\big)\boldsymbol{X}_t, \qquad (11.47)$$

where $\Sigma_t(\phi,\theta)_{s,t} = \mathrm{E}(Y_sY_t)$, $(\boldsymbol{r}_{t,j})_i = \mathrm{E}(Y_{t-i}Y_{-j})$ and $\boldsymbol{X}'_t = (X_t,\ldots,X_1)$. Therefore

$$X^{\phi,\theta}_{t+1|t} - \tilde{X}_{t+1|t,\ldots} = \big(\sum_{j=t+1}^{\infty} b_j\boldsymbol{r}'_{t,j}\Sigma_t(\phi,\theta)^{-1}\big)\boldsymbol{X}_t.$$

Since the largest eigenvalue of $\Sigma_t(\phi,\theta)^{-1}$ is bounded (see Lemma 6.5.1) and $|(\boldsymbol{r}_{t,j})_i| = |\mathrm{E}(Y_{t-i}Y_{-j})| \leq K\rho^{|t-i+j|}$ we obtain the bound in (11.42). Taking expectations, we have

$$\mathrm{E}(X^{\phi,\theta}_{t+1|t} - \tilde{X}^{\phi,\theta}_{t+1|t,\ldots})^2 = \big(\sum_{j=t+1}^{\infty} b_j\boldsymbol{r}'_{t,j}\big)\Sigma_t(\phi,\theta)^{-1}\Sigma_t(\phi_0,\theta_0)\Sigma_t(\phi,\theta)^{-1}\big(\sum_{j=t+1}^{\infty} b_{t+j}\boldsymbol{r}_{t,j}\big).$$

Now by using the same arguments given in the proof of (6.29) we obtain (11.43).

To prove (11.45) we note that

$$\mathrm{E}(\tilde{X}_{t+1|t,\ldots}(1) - X_{t+1|t,\ldots})^2 = \mathrm{E}\big(\sum_{j=t+1}^{\infty} b_j(\phi,\theta)X_{t+1-j}\big)^2 = \mathrm{E}\big(\sum_{j=1}^{\infty} b_{t+j}(\phi,\theta)X_{-j}\big)^2,$$

now by using (3.24), we have $|b_{t+j}(\phi,\theta)| \leq K\rho^{t+j}$, for $\frac{1}{1+\delta} < \rho < 1$, and the bound in (11.44). Using this we have $\mathrm{E}(\tilde{X}_{t+1|t,\ldots}(1) - X_{t+1|t,\ldots})^2 \leq K\rho^t$, which proves the result. □

Using $\varepsilon_t = X_t - \sum_{j=1}^{\infty} b_j(\boldsymbol{\phi}_0,\boldsymbol{\theta}_0)X_{t-j}$ and substituting this into $\mathcal{L}_n(\boldsymbol{\phi},\boldsymbol{\theta},\sigma)$ gives

$$\begin{aligned}
\frac{1}{n}\mathcal{L}_n(\boldsymbol{\phi},\boldsymbol{\theta},\sigma) &= \log\sigma^2 + \frac{1}{n\sigma^2}\big(X_t - \sum_{j=1}^{\infty} b_j(\underline{\phi},\underline{\theta})X_{t+1-j}\big)^2 \\
&= \frac{1}{n}\mathcal{L}_n(\underline{\phi},\underline{\theta},\sigma)\log\sigma^2 + \frac{1}{n\sigma^2}\sum_{t=1}^{T-1}\big\{\theta(B)^{-1}\phi(B)X_t\big\}\big\{\theta(B)^{-1}\phi(B)X_t\big\} \\
&= \log\sigma^2 + \frac{1}{n\sigma^2}\sum_{t=1}^{n}\varepsilon_t^2 + \frac{2}{n}\sum_{t=1}^{n}\varepsilon_t\big(\sum_{j=1}^{\infty} b_j(\boldsymbol{\phi},\boldsymbol{\theta})X_{t-j}\big) \\
&\quad + \frac{1}{n}\sum_{t=1}^{n}\big(\sum_{j=1}^{\infty}(b_j(\boldsymbol{\phi},\boldsymbol{\theta}) - b_j(\boldsymbol{\phi}_0,\boldsymbol{\theta}_0))X_{t-j}\big)^2.
\end{aligned}$$

Remark 11.8.1 (Derivatives involving the Backshift operator) *Consider the transformation*

$$\frac{1}{1-\theta B}X_t = \sum_{j=0}^{\infty}\theta^jB^jX_t = \sum_{j=0}^{\infty}\theta^jX_{t-j}.$$

Suppose we want to differentiate the above with respect to θ, there are two ways this can be done. Either differentiate $\sum_{j=0}^{\infty}\theta^jX_{t-j}$ with respect to θ or differentiate $\frac{1}{1-\theta B}$ with respect to θ. In other *words*

$$\frac{d}{d\theta}\frac{1}{1-\theta B}X_t = \frac{-B}{(1-\theta B)^2}X_t = \sum_{j=0}^{\infty} j\theta^{j-1}X_{t-j}.$$

Often it is easier to differentiate the operator. Suppose that $\theta(B) = 1 + \sum_{j=1}^{p}\theta_j B^j$ *and* $\phi(B) = 1 - \sum_{j=1}^{q}\phi_j B^j$*, then we have*

$$\begin{aligned}\frac{d}{d\theta_j}\frac{\phi(B)}{\theta(B)}X_t &= -\frac{B^j\phi(B)}{\theta(B)^2}X_t = -\frac{\phi(B)}{\theta(B)^2}X_{t-j}\\ \frac{d}{d\phi_j}\frac{\phi(B)}{\theta(B)}X_t &= -\frac{B^j}{\theta(B)^2}X_t = -\frac{1}{\theta(B)^2}X_{t-j}.\end{aligned}$$

Moreover in the case of squares we have

$$\frac{d}{d\theta_j}(\frac{\phi(B)}{\theta(B)}X_t)^2 = -2(\frac{\phi(B)}{\theta(B)}X_t)(\frac{\phi(B)}{\theta(B)^2}X_{t-j}), \qquad \frac{d}{d\phi_j}(\frac{\phi(B)}{\theta(B)}X_t)^2 = -2(\frac{\phi(B)}{\theta(B)}X_t)(\frac{1}{\theta(B)^2}X_{t-j}).$$

Using the above we can easily evaluate the gradient of $\frac{1}{n}\mathcal{L}_n$

$$\begin{aligned}\frac{1}{n}\nabla_{\theta_i}\mathcal{L}_n(\boldsymbol{\phi},\boldsymbol{\theta},\sigma) &= -\frac{2}{\sigma^2}\sum_{t=1}^{n}(\theta(B)^{-1}\phi(B)X_t)\frac{\phi(B)}{\theta(B)^2}X_{t-i}\\ \frac{1}{n}\nabla_{\phi_j}\mathcal{L}_n(\boldsymbol{\phi},\boldsymbol{\theta},\sigma) &= -\frac{2}{n\sigma^2}\sum_{t=1}^{n}(\theta(B)^{-1}\phi(B)X_t)\frac{1}{\theta(B)}X_{t-j}\\ \frac{1}{n}\nabla_{\sigma^2}\mathcal{L}_n(\boldsymbol{\phi},\boldsymbol{\theta},\sigma) &= \frac{1}{\sigma^2} - \frac{1}{n\sigma^4}\sum_{t=1}^{n}\big(X_t - \sum_{j=1}^{\infty}b_j(\boldsymbol{\phi},\boldsymbol{\theta})X_{t-j}\big)^2. \qquad (11.48)\end{aligned}$$

Let $\nabla = (\nabla_{\phi_i}, \nabla_{\theta_j}, \nabla_{\sigma^2})$. We note that the second derivative $\nabla^2\mathcal{L}_n$ can be defined similarly.

Lemma 11.8.1 *Suppose Assumption 11.8.1 holds. Then*

$$\sup_{\underline{\phi},\underline{\theta}\in\Theta}\|\frac{1}{n}\nabla\mathcal{L}_n\|_2 \leq KS_n \qquad \sup_{\underline{\phi},\underline{\theta}\in\Theta}\|\frac{1}{n}\nabla^3\mathcal{L}_n\|_2 \leq KS_n \qquad (11.49)$$

for some constant K*,*

$$S_n = \frac{1}{n}\sum_{r_1,r_2=0}^{\max(p,q)}\sum_{t=1}^{n}Y_{t-r_1}Y_{t-r_2} \qquad (11.50)$$

where

$$Y_t = K \sum_{j=0}^{\infty} \rho^j \cdot |X_{t-j}|.$$

for any $\frac{1}{(1+\delta)} < \rho < 1$.

PROOF. The proof follows from the the roots of $\phi(z)$ and $\theta(z)$ having absolute value greater than $1+\delta$. □

Define the expectation of the likelihood $\mathcal{L}(\boldsymbol{\phi}, \boldsymbol{\theta}, \sigma)) = \mathrm{E}(\frac{1}{n}\mathcal{L}_n(\boldsymbol{\phi}, \boldsymbol{\theta}, \sigma))$. We observe

$$\mathcal{L}(\boldsymbol{\phi}, \boldsymbol{\theta}, \sigma)) = \log \sigma^2 + \frac{\sigma_0^2}{\sigma_2} + \frac{1}{\sigma^2}\mathrm{E}(Z_t(\boldsymbol{\phi}, \boldsymbol{\theta})^2)$$

where

$$Z_t(\boldsymbol{\phi}, \boldsymbol{\theta}) = \sum_{j=1}^{\infty}(b_j(\boldsymbol{\phi}, \boldsymbol{\theta}) - b_j(\boldsymbol{\phi}_0, \boldsymbol{\theta}_0))X_{t-j}$$

Lemma 11.8.2 *Suppose that Assumption 11.8.1 are satisfied. Then for all* $\underline{\theta}, \underline{\phi}, \theta \in \Theta$ *we have*

(i) $\frac{1}{n}\nabla^i \mathcal{L}_n(\boldsymbol{\phi}, \boldsymbol{\theta}, \sigma)) \overset{a.s.}{\to} \nabla^i \mathcal{L}(\boldsymbol{\phi}, \boldsymbol{\theta}, \sigma))$ *for* $i = 0, 1, 2, 3$.

(ii) Let S_n *defined in (11.50), then* $S_n \overset{a.s.}{\to} \mathrm{E}(\sum_{r_1, r_2=0}^{\max(p,q)} \sum_{t=1}^{n} Y_{t-r_1} Y_{t-r_2})$.

PROOF. Noting that the ARMA process $\{X_t\}$ are ergodic random variables, then $\{Z_t(\boldsymbol{\phi}, \boldsymbol{\theta})\}$ and $\{Y_t\}$ are ergodic random variables, the result follows immediately from the Ergodic theorem.

We use these results in the proofs below.

Theorem 11.8.1 *Suppose that Assumption 11.8.1 is satisfied. Let* $(\hat{\underline{\theta}}_n^*, \hat{\underline{\phi}}_n^*, \hat{\sigma}_n^*) = \arg\min \mathcal{L}_n(\underline{\theta}, \underline{\phi}, \sigma)$ *(noting the practice that this cannot be evaluated). Then we have*

(i) $(\hat{\underline{\theta}}_n^*, \hat{\underline{\phi}}_n^*, \hat{\sigma}_n^*) \overset{a.s.}{\to} (\underline{\theta}_0, \underline{\phi}_0, \sigma_0)$.

(ii) $\sqrt{n}(\hat{\underline{\theta}}_n^* - \underline{\theta}_0, \hat{\underline{\phi}}_n^* - \underline{\theta}_0) \overset{\mathcal{D}}{\to} \mathcal{N}(0, \sigma_0^2 \Lambda^{-1})$, *where*

$$\Lambda = \begin{pmatrix} \mathrm{E}(U_t U_t') & \mathrm{E}(V_t U_t') \\ \mathrm{E}(U_t V_t') & \mathrm{E}(V_t V_t') \end{pmatrix}$$

and $\{U_t\}$ *and* $\{V_t\}$ *are autoregressive processes which satisfy* $\phi_0(B)U_t = \varepsilon_t$ *and* $\theta_0(B)V_t = \varepsilon_t$.

PROOF. We prove the result in two stages below. □

PROOF of Theorem 11.8.1(i) We will first prove Theorem 11.8.1(i). Noting the results in Section 11.3, to prove consistency we recall that we must show (a) the $(\underline{\phi}_0, \underline{\theta}_0, \sigma_0)$ is the unique minimum of $\mathcal{L}(\cdot)$ (b) pointwise convergence $\frac{1}{T}\mathcal{L}(\boldsymbol{\phi}, \boldsymbol{\theta}, \sigma)) \overset{\text{a.s.}}{\to} \mathcal{L}(\boldsymbol{\phi}, \boldsymbol{\theta}, \sigma))$ and (b) stochastic equicontinuity (as defined in Definition 11.3.2). To show that $(\underline{\phi}_0, \underline{\theta}_0, \sigma_0)$ is the minimum we note that

$$\mathcal{L}(\boldsymbol{\phi}, \boldsymbol{\theta}, \sigma)) - \mathcal{L}(\boldsymbol{\phi}_0, \boldsymbol{\theta}_0, \sigma_0)) = \log(\frac{\sigma^2}{\sigma_0^2}) + \frac{\sigma^2}{\sigma_0^2} - 1 + \mathrm{E}(Z_t(\boldsymbol{\phi}, \boldsymbol{\theta})^2).$$

Since for all positive x, $\log x + x - 1$ is a positive function and $\mathrm{E}(Z_t(\boldsymbol{\phi}, \boldsymbol{\theta})^2) = \mathrm{E}(\sum_{j=1}^{\infty}(b_j(\boldsymbol{\phi}, \boldsymbol{\theta}) - b_j(\boldsymbol{\phi}_0, \boldsymbol{\theta}_0))X_{t-j})^2$ is positive and zero at $(\underline{\phi}_0, \underline{\theta}_0, \sigma_0)$ it is clear that $\boldsymbol{\phi}_0, \boldsymbol{\theta}_0, \sigma_0$ is the minimum of $\mathcal{L}$. We will assume for now it is the unique minimum. Pointwise convergence is an immediate consequence of Lemma 11.8.2(i). To show stochastic equicontinuity we note that for any $\boldsymbol{\varphi}_1 = (\boldsymbol{\phi}_1, \boldsymbol{\theta}_1, \sigma_1)$ and $\boldsymbol{\varphi}_2 = (\boldsymbol{\phi}_2, \boldsymbol{\theta}_2, \sigma_2)$ we have by the mean value theorem

$$\mathcal{L}_n(\boldsymbol{\phi}_1, \boldsymbol{\theta}_1, \sigma_1) - \mathcal{L}_n(\boldsymbol{\phi}_2, \boldsymbol{\theta}_2, \sigma_2)) = (\boldsymbol{\varphi}_1 - \boldsymbol{\varphi}_2)\nabla\mathcal{L}_n(\bar{\boldsymbol{\phi}}, \bar{\boldsymbol{\theta}}, \bar{\sigma}).$$

Now by using (11.49) we have

$$\mathcal{L}_n(\boldsymbol{\phi}_1, \boldsymbol{\theta}_1, \sigma_1) - \mathcal{L}_n(\boldsymbol{\phi}_2, \boldsymbol{\theta}_2, \sigma_2)) \leq S_T\|(\boldsymbol{\phi}_1 - \boldsymbol{\phi}_2), (\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2), (\sigma_1 - \sigma_2)\|_2.$$

By using Lemma 11.8.2(ii) we have $S_n \overset{\text{a.s.}}{\to} \mathrm{E}(\sum_{r_1,r_2=0}^{\max(p,q)}\sum_{t=1}^{n} Y_{t-r_1}Y_{t-r_2})$, hence $\{S_n\}$ is almost surely bounded. This implies that $\mathcal{L}_n$ is equicontinuous. Since we have shown pointwise convergence and equicontinuity of $\mathcal{L}_n$, by using Corollary 11.3.1, we almost sure convergence of the estimator. Thu proving (i). □

PROOF of Theorem 11.8.1(ii) We now prove Theorem 11.8.1(i) using the Martingale central limit theorem (see Billingsley (1995) and Hall and Heyde (1980)) in conjunction with the Cramer-Wold device (see Theorem 8.1.1).

Using the mean value theorem we have

$$\left(\hat{\boldsymbol{\varphi}}_n^* - \boldsymbol{\varphi}_0\right) = \nabla^2\mathcal{L}_n^*(\bar{\boldsymbol{\varphi}}_n)^{-1}\nabla\mathcal{L}_n^*(\boldsymbol{\phi}_0, \boldsymbol{\theta}_0, \sigma_0)$$

where $\hat{\boldsymbol{\varphi}}_n^* = (\hat{\boldsymbol{\phi}}_n^*, \hat{\boldsymbol{\theta}}_n^*, \hat{\sigma}_n^*)$, $\boldsymbol{\varphi}_0 = (\boldsymbol{\phi}_0, \boldsymbol{\theta}_0, \sigma_0)$ and $\bar{\boldsymbol{\varphi}}_n = \bar{\boldsymbol{\phi}}, \bar{\boldsymbol{\theta}}, \bar{\sigma}$ lies between $\hat{\boldsymbol{\varphi}}_n^*$ and $\boldsymbol{\varphi}_0$.

Using the same techniques given in Theorem 11.8.1(i) and Lemma 11.8.2 we have pointwise convergence and equicontinuity of $\nabla^2\mathcal{L}_n$. This means that $\nabla^2\mathcal{L}_n(\bar{\varphi}_n) \overset{\text{a.s.}}{\to} \mathrm{E}(\nabla^2\mathcal{L}_n(\boldsymbol{\phi}_0,\boldsymbol{\theta}_0,\sigma_0)) = \frac{1}{\sigma^2}\Lambda$ (since by definition of $\bar{\varphi}_n$ $\bar{\boldsymbol{\varphi}}_n \overset{\text{a.s.}}{\to} \boldsymbol{\varphi}_0$). Therefore by applying Slutsky's theorem (since Λ is nonsingular) we have

$$\nabla^2\mathcal{L}_n(\bar{\boldsymbol{\varphi}}_n)^{-1} \overset{\text{a.s.}}{\to} \sigma^2\Lambda^{-1}. \tag{11.51}$$

Now we show that $\nabla\mathcal{L}_n(\boldsymbol{\varphi}_0)$ is asymptotically normal. By using (11.48) and replacing $X_{t-i} = \phi_0(B)^{-1}\theta_0(B)\varepsilon_{t-i}$ we have

$$\begin{aligned}
\frac{1}{n}\nabla_{\theta_i}\mathcal{L}_n(\boldsymbol{\phi}_0,\boldsymbol{\theta}_0,\sigma_0) &= \frac{2}{\sigma^2 n}\sum_{t=1}^{n}\varepsilon_t\frac{(-1)}{\theta_0(B)}\varepsilon_{t-i} = \frac{-2}{\sigma^2 n}\sum_{t=1}^{n}\varepsilon_t V_{t-i} \quad i=1,\ldots,q \\
\frac{1}{n}\nabla_{\phi_j}\mathcal{L}_n(\boldsymbol{\phi}_0,\boldsymbol{\theta}_0,\sigma_0) &= \frac{2}{\sigma^2 n}\sum_{t=1}^{n}\varepsilon_t\frac{1}{\phi_0(B)}\varepsilon_{t-j} = \frac{2}{\sigma^2 n}\sum_{t=1}^{T}\varepsilon_t U_{t-j} \quad j=1,\ldots,p \\
\frac{1}{n}\nabla_{\sigma^2}L_n(\boldsymbol{\phi}_0,\boldsymbol{\theta}_0,\sigma_0) &= \frac{1}{\sigma^2} - \frac{1}{\sigma^4 n}\sum_{t=1}^{T}\varepsilon^2 = \frac{1}{\sigma^4 n}\sum_{t=1}^{T}(\sigma^2-\varepsilon^2),
\end{aligned}$$

where $U_t = \frac{1}{\phi_0(B)}\varepsilon_t$ and $V_t = \frac{1}{\theta_0(B)}\varepsilon_t$. We observe that $\frac{1}{n}\nabla\mathcal{L}_n$ is the sum of vector martingale differences. If $\mathrm{E}(\varepsilon_t^4) < \infty$, it is clear that $\mathrm{E}((\varepsilon_t U_{t-j})^4) = \mathrm{E}((\varepsilon_t^4)\mathrm{E}(U_{t-j})^4) < \infty$, $\mathrm{E}((\varepsilon_t V_{t-i})^4) = \mathrm{E}((\varepsilon_t^4)\mathrm{E}(V_{t-i})^4) < \infty$ and $\mathrm{E}((\sigma^2-\varepsilon_t^2)^2) < \infty$. Hence Lindeberg's condition is satisfied (see the proof given in Section 8.1.3, for why this is true). Hence we have

$$\sqrt{n}\nabla\mathcal{L}_n(\boldsymbol{\phi}_0,\boldsymbol{\theta}_0,\sigma_0) \overset{\mathcal{D}}{\to} \mathcal{N}(0,\Lambda).$$

Now by using the above and (11.51) we have

$$\sqrt{n}\big(\hat{\boldsymbol{\varphi}}_n^* - \boldsymbol{\varphi}_0\big) = \sqrt{n}\nabla^2\mathcal{L}_n(\bar{\boldsymbol{\varphi}}_n)^{-1}\nabla\mathcal{L}_n(\boldsymbol{\varphi}_0) \Rightarrow \quad \sqrt{n}\big(\hat{\boldsymbol{\varphi}}_n^* - \boldsymbol{\varphi}_0\big) \overset{\mathcal{D}}{\to} \mathcal{N}(0,\sigma^4\Lambda^{-1}).$$

Thus we obtain the required result. □

The above result proves consistency and asymptotically normality of $(\underline{\hat{\theta}}_n^*, \underline{\hat{\phi}}_n^*, \hat{\sigma}_n^*)$, which is based on $\mathcal{L}_n(\underline{\theta},\underline{\phi},\sigma)$, which in practice is impossible to evaluate. However we will show below that the gaussian likelihood, $L_n(\underline{\theta},\underline{\phi},\sigma)$ and is derivatives are sufficiently close to $\mathcal{L}_n(\underline{\theta},\underline{\phi},\sigma)$ such that the estimators $(\underline{\hat{\theta}}_n^*, \underline{\hat{\phi}}_n^*, \hat{\sigma}_n^*)$ and the GMLE, $(\underline{\hat{\theta}}_n, \underline{\hat{\phi}}_n, \hat{\sigma}_n) = \arg\min L_n(\underline{\theta},\underline{\phi},\sigma)$ are asymptotically equivalent. We use Lemma 11.8.1 to prove the below result.

Proposition 11.8.2 *Suppose that Assumption 11.8.1 hold and $L_n(\underline{\theta},\underline{\phi},\sigma)$, $\tilde{\mathcal{L}}_n(\underline{\theta},\underline{\phi},\sigma)$ and $\mathcal{L}_n(\underline{\theta},\underline{\phi},\sigma)$ are defined as in (11.37), (11.39) and (11.41) respectively. Then we have for all $(\underline{\theta},\underline{\phi}) \in Theta$ we have almost surely*

$$\sup_{(\boldsymbol{\phi},\boldsymbol{\theta},\sigma)} \frac{1}{n}|\nabla^{(k)}\tilde{\mathcal{L}}(\boldsymbol{\phi},\boldsymbol{\theta},\sigma) - \nabla^k L_n(\boldsymbol{\phi},\boldsymbol{\theta},\sigma)| = O(\frac{1}{n}) \qquad \sup_{(\boldsymbol{\phi},\boldsymbol{\theta},\sigma)} \frac{1}{n}|\tilde{\mathcal{L}}_n(\boldsymbol{\phi},\boldsymbol{\theta},\sigma) - \mathcal{L}(\boldsymbol{\phi},\boldsymbol{\theta},\sigma)| = O(\frac{1}{n}),$$

for $k = 0, 1, 2, 3$.

PROOF. The proof of the result follows from (11.42) and (11.44). We show that result for $\sup_{(\boldsymbol{\phi},\boldsymbol{\theta},\sigma)} \frac{1}{n}|\tilde{\mathcal{L}}(\boldsymbol{\phi},\boldsymbol{\theta},\sigma) - L_n(\boldsymbol{\phi},\boldsymbol{\theta},\sigma)|$, a similar proof can be used for the rest of the result.

Let us consider the difference

$$\mathcal{L}_n(\underline{\phi},\underline{\theta}) - L_n(\underline{\phi},\underline{\theta}) = \frac{1}{n}(I_n + II_n + III_n),$$

where

$$\begin{aligned} I_n &= \sum_{t=1}^{n-1}\big\{r_t(\underline{\phi},\underline{\theta},\sigma) - \sigma^2\big\}, \quad II_n = \sum_{t=1}^{n-1}\frac{1}{r_t(\underline{\phi},\underline{\theta},\sigma)}(X_{t+1}^{(\phi,\theta)} - X_{t+1|t}^{(\phi,\theta)})^2 \\ III_n &= \sum_{t=1}^{n-1}\frac{1}{\sigma^2}\big\{2X_{t+1}(X_{t+1|t}^{(\phi,\theta)} - \tilde{X}_{t+1|t,\ldots}^{(\phi,\theta)}) + ((X_{t+1|t}^{(\phi,\theta)})^2 - (\tilde{X}_{t+1|t,\ldots}^{(\phi,\theta)})^2)\big\}. \end{aligned}$$

Now we recall from Proposition 11.8.1 that

$$\big|X_{t+1|t}^{(\phi,\theta)} - \tilde{X}_{t+1|t,\ldots}^{(\phi,\theta)}\big| \leq K \cdot V_t \frac{\rho^t}{(1-\rho)}$$

where $V_t = \sum_{i=1}^t \rho^i |X_i|$. Hence since $\mathrm{E}(X_t^2) < \infty$ and $\mathrm{E}(V_t^2) < \infty$ we have that $\sup_n \mathrm{E}|I_n| < \infty$, $\sup_n \mathrm{E}|II_n| < \infty$ and $\sup_n \mathrm{E}|III_n| < \infty$. Hence the sequence $\{|I_n + II_n + III_n|\}_n$ is almost surely bounded. This means that almost surely

$$\sup_{\underline{\phi},\underline{\theta},\sigma} \big|\mathcal{L}_n(\underline{\phi},\underline{\theta}) - L_n(\underline{\phi},\underline{\theta})\big| = O(\frac{1}{n}).$$

Thus giving the required result. □

Now by using the above proposition the result below immediately follows.

Theorem 11.8.2 *Let $(\hat{\underline{\theta}},\hat{\underline{\phi}}) = \arg\min L_T(\underline{\theta},\underline{\phi},\sigma)$ and $(\tilde{\underline{\theta}},\tilde{\underline{\phi}}) = \arg\min \tilde{L}_T(\underline{\theta},\underline{\phi},\sigma)$*

(i) $(\underline{\hat{\theta}}, \underline{\hat{\phi}}) \overset{a.s.}{\to} (\underline{\theta}_0, \underline{\phi}_0)$ *and* $(\underline{\tilde{\theta}}, \underline{\tilde{\phi}}) \overset{a.s.}{\to} (\underline{\theta}_0, \underline{\phi}_0)$.

(ii) $\sqrt{T}(\underline{\hat{\theta}}_T - \underline{\theta}_0, \underline{\hat{\phi}}_T - \underline{\theta}_0) \overset{\mathcal{D}}{\to} \mathcal{N}(0, \sigma_0^4 \Lambda^{-1})$

and $\sqrt{T}(\underline{\tilde{\theta}}_T - \underline{\theta}_0, \underline{\tilde{\phi}}_T - \underline{\theta}_0) \overset{\mathcal{D}}{\to} \mathcal{N}(0, \sigma_0^4 \Lambda^{-1})$.

PROOF. The proof follows immediately from Proposition 11.8.1. □

Appendix A

Background

A.1 Some definitions and inequalities

- Some norm definitions.

  The norm of an object, is a postive numbers which measure the 'magnitude' of that object. Suppose $\underline{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, then we define $\|\underline{x}\|_1 = \sum_{j=1}^{n} |x_j|$ and $\|\underline{x}\|_2 = (\sum_{j=1}^{n} |x_j^2|)^{1/2}$ (this is known as the Euclidean norm). There are various norms for matrices, the most popular is the spectral norm $\|\cdot\|_{spec}$: let A be a matrix, then $\|A\|_{spec} = \lambda_{max}(AA')$, where λ_{max} denotes the largest eigenvalue.

- $\mathbb{Z}$ denotes the set of a integers $\{\ldots, -1, 0, 1, 2, \ldots\}$. $\mathbb{R}$ denotes the real line $(-\infty, \infty)$.

- Complex variables.

  $i = \sqrt{-1}$ and the complex variable $z = x + iy$, where x and y are real.

  Often the radians representation of a complex variable is useful. If $z = x + iy$, then it can also be written as $r\exp(i\theta)$, where $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}(y/x)$.

  If $z = x + iy$, its complex conjugate is $\bar{z} = x - iy$.

- The roots of a rth order polynomial $a(z)$, are those values $\lambda_1, \ldots, \lambda_r$ where $a(\lambda_i) = 0$ for $i = 1, \ldots, r$.

- Let $\lambda(A)$ denote the spectral radius of the the matrix A (the largest eigenvalue in absolute terms). Then for any matrix norm $\|A\|$ we have $\lim_{j\to\infty} \|A^j\|^{1/j} = \lambda(A)$ (see Gelfand's formula). Suppose $\lambda(A) < 1$, then Gelfand's formula implies that for any $\lambda(A) < \rho < 1$, there exists a constant, C, (which only depends A and ρ), such that $\|A^j\| \leq C_{A,\rho}\rho^j$.

- The mean value theorem.

  This basically states that if the partial derivative of the function $f(x_1, x_2, \ldots, x_n)$ has a bounded in the domiain Ω, then for $\underline{x} = (x_1, \ldots, x_n)$ and $\underline{y} = (y_1, \ldots, y_n)$

  $$f(x_1, x_2, \ldots, x_n) - f(y_1, y_2, \ldots, y_n) = \sum_{i=1}^{n}(x_i - y_i)\frac{\partial f}{\partial x_i}\rfloor_{\underline{x}=\underline{x}^*}$$

  where $\underline{x}^*$ lies somewhere between $\underline{x}$ and $\underline{y}$.

- The Taylor series expansion.

  This is closely related to the mean value theorem and a second order expansion is

  $$f(x_1, x_2, \ldots, x_n) - f(y_1, y_2, \ldots, y_n) = \sum_{i=1}^{n}(x_i - y_i)\frac{\partial f}{\partial x_i} + \sum_{i,j=1}^{n}(x_i - y_i)(x_j - y_j)\frac{\partial f^2}{\partial x_i \partial x_j}\rfloor_{\underline{x}=\underline{x}^*}$$

- Partial Fractions.

  We use the following result mainly for obtaining the MA(∞) expansion of an AR process.

  Suppose that $|g_i| > 1$ for $1 \leq i \leq n$. Then if $g(z) = \prod_{i=1}^{n}(1 - z/g_i)^{r_i}$, the inverse of $g(z)$ satisfies

  $$\frac{1}{g(z)} = \sum_{i=1}^{n}\Big\{\sum_{j=1}^{r_i}\frac{g_{i,j}}{(1 - \frac{z}{g_i})^j}\Big\},$$

  where $g_{i,j} = ...$ Now we can make a polynomial series expansion of $(1 - \frac{z}{g_i})^{-j}$ which is valid for all $|z| \leq 1$.

- Dominated convergence.

  Suppose a sequence of functions $f_n(x)$ is such that pointwise $f_n(x) \to f(x)$ and for all n and x, $|f_n(x)| \leq g(x)$, then $\int f_n(x)dx \to \int f(x)dx$ as $n \to \infty$.

  We use this result all over the place to exchange infinite sums and expectations. For example, if $\sum_{j=1}^{\infty}|a_j|\mathrm{E}(|Z_j|)<\infty$, then by using dominated convergence we have

$$\mathrm{E}(\sum_{j=1}^{\infty}a_jZ_j)=\sum_{j=1}^{\infty}a_j\mathrm{E}(Z_j).$$

- Dominated convergence can be used to prove the following lemma. A more hands on proof is given below the lemma.

Lemma A.1.1 *Suppose $\sum_{k=-\infty}^{\infty}|c(k)|<\infty$, then we have*

$$\frac{1}{n}\sum_{k=-(n-1)}^{(n-1)}|kc(k)|\rightarrow 0$$

as $n\rightarrow\infty$. Moreover, if $\sum_{k=-\infty}^{\infty}|kc(k)|<\infty$, then $\frac{1}{n}\sum_{k=-(n-1)}^{(n-1)}|kc(k)|=O(\frac{1}{n})$.

PROOF. The proof is straightforward in the case that $\sum_{k=\infty}^{\infty}|kc(k)|<\infty$ (the second assertion), in this case $\sum_{k=-(n-1)}^{(n-1)}\frac{|k|}{n}|c(k)|=O(\frac{1}{n})$. The proof is slightly more tricky in the case that $\sum_{k=\infty}^{\infty}|c(k)|<\infty$. First we note that since $\sum_{k=-\infty}^{\infty}|c(k)|<\infty$ for every $\varepsilon>0$ there exists a N_ε such that for all $n\geq N_\varepsilon$, $\sum_{|k|\geq n}|c(k)|<\varepsilon$. Let us suppose that $n>N_\varepsilon$, then we have the bound

$$\begin{aligned}\frac{1}{n}\sum_{k=-(n-1)}^{(n-1)}|kc(k)| &\leq \frac{1}{n}\sum_{k=-(N_\varepsilon-1)}^{(N_\varepsilon-1)}|kc(k)|+\frac{1}{n}\sum_{N_\varepsilon\leq|k|\leq n}|kc(k)|\\ &\leq \frac{1}{2\pi n}\sum_{k=-(N_\varepsilon-1)}^{(N_\varepsilon-1)}|kc(k)|+\varepsilon.\end{aligned}$$

Hence if we keep N_ε fixed we see that $\frac{1}{n}\sum_{k=-(N_\varepsilon-1)}^{(N_\varepsilon-1)}|kc(k)|\rightarrow 0$ as $n\rightarrow\infty$. Since this is true for all ε (for different thresholds N_ε) we obtain the required result. □

- Cauchy Schwarz inequality.

In terms of sequences it is

$$|\sum_{j=1}^{\infty}a_jb_j|\leq(\sum_{j=1}^{\infty}a_j^2)^{1/2}(\sum_{j=1}^{\infty}b_j^2)^{1/2}$$

. For integrals and expectations it is

$$\mathrm{E}|XY| \leq \mathrm{E}(X^2)^{1/2}\mathrm{E}(Y^2)^{1/2}$$

- Holder's inequality.

  This is a generalisation of the Cauchy Schwarz inequality. It states that if $1 \leq p, q \leq \infty$ and $p + q = 1$, then

  $$\mathrm{E}|XY| \leq \mathrm{E}(|X|^p)^{1/p}\mathrm{E}(|Y|^q)^{1/q}$$

  . A similar results is true for sequences too.

- Martingale differences. Let $\mathcal{F}_t$ be a sigma-algebra, where $X_t, X_{t-1}, \ldots \in \mathcal{F}_t$. Then $\{X_t\}$ is a sequence of martingale differences if $\mathrm{E}(X_t|\mathcal{F}_{t-1}) = 0$.

- Minkowski's inequality.

  If $1 < p < \infty$, then

  $$(\mathrm{E}(\sum_{i=1}^{n} X_i)^p)^{1/p} \leq \sum_{i=1}^{n}(\mathrm{E}(|X_i|^p))^{1/p}.$$

- Doob's inequality.

  This inequality concerns martingale differences. Let $\mathcal{S}_n = \sum_{t=1}^{n} X_t$, then

  $$\mathrm{E}(\sup_{n \leq N} |\mathcal{S}_n|^2) \leq \mathrm{E}(\mathcal{S}_N^2).$$

- Burkhölder's inequality.

  Suppose that $\{X_t\}$ are martingale differences and define $S_n = \sum_{k=1}^{n} X_t$. For any $p \geq 2$ we have

  $$\{\mathrm{E}(S_n^p)\}^{1/p} \leq \big(2p\sum_{k=1}^{n}\mathrm{E}(X_k^p)^{2/p}\big)^{1/2}.$$

  An application, is to the case that $\{X_t\}$ are identically distributed random variables, then we have the bound $\mathrm{E}(S_n^p) \leq \mathrm{E}(X_0^p)^2(2p)^{p/2}n^{p/2}$.

  It is worthing noting that the Burkhölder inequality can also be defined for $p < 2$ (see

Davidson (1994), pages 242). It can also be generalised to random variables $\{X_t\}$ which are not necessarily martingale differences (see Dedecker and Doukhan (2003)).

- Riemann-Stieltjes Integrals.

  In basic calculus we often use the basic definition of the Riemann integral, $\int g(x)f(x)dx$, and if the function $F(x)$ is continuous and $F'(x) = f(x)$, we can write $\int g(x)f(x)dx = \int g(x)dF(x)$. There are several instances where we need to broaden this definition to include functions F which are not continuous everywhere. To do this we define the Riemann-Stieltjes integral, which coincides with the Riemann integral in the case that $F(x)$ is continuous.

  $\int g(x)dF(x)$ is defined in a slightly different way to the Riemann integral $\int g(x)f(x)dx$. Let us first consider the case that $F(x)$ is the step function $F(x) = \sum_{i=1}^{n} a_i I_{[x_{i-1},x_i]}$, then $\int g(x)dF(x)$ is defined as $\int g(x)dF(x) = \sum_{i=1}^{n}(a_i - a_{i-1})g(x_i)$ (with $a_{-1} = 0$). Already we see the advantage of this definition, since the derivative of the step function is not well defined at the jumps. As most functions can be written as the limit of step functions ($F(x) = \lim_{k\infty} F_k(x)$, where $F_k(x) = \sum_{i=1}^{n_k} a_{i,n_k} I_{[x_{i_{k-1}-1},x_{i_k}]}$), we define $\int g(x)dF(x) = \lim_{k\to\infty} \sum_{i=1}^{n_k}(a_{i,n_k} - a_{i-1,n_k})g(x_{i_k})$.

  In statistics, the function F will usually be non-decreasing and bounded. We call such functions distributions.

Theorem A.1.1 (Helly's Theorem) *Suppose that $\{F_n\}$ are a sequence of distributions with $F_n(-\infty) = 0$ and $\sup_n F_n(\infty) \leq M < \infty$. There exists a distribution F, and a subsequence F_{n_k} such that for each $x \in \mathbb{R}$ $F_{n_k} \to F$ and F is right continuous.*

A.2 Martingales

Definition A.2.1 *A sequence $\{X_t\}$ is said to be a martingale difference if $\mathrm{E}[X_t|\mathcal{F}_{t-1}]$, where $\mathcal{F}_{t=1} = \sigma(X_{t-1}, X_{t-2}, \ldots)$. In other words, the best predictor of X_t given the past is simply zero.*

Martingales are very useful when proving several results, including central limit theorems.

Martingales arise naturally in several situations. We now show that if correct likelihood is used (not the quasi-case), then the gradient of the conditional log likelihood evaluated at the true parameter is the sum of martingale differences. To see why, let $\mathcal{B}_T = \sum_{t=2}^{T} \log f_\theta(X_t|X_{t-1}, \ldots, X_1)$

be the conditonal log likelihood and $\mathcal{C}_T(\theta)$ its derivative, where

$$\mathcal{C}_T(\theta) = \sum_{t=2}^{T} \frac{\partial \log f_\theta(X_t|X_{t-1},\ldots,X_1)}{\partial\theta}.$$

We want to show that $\mathcal{C}_T(\theta_0)$ is the sum of martingale differences. By definition if $\mathcal{C}_T(\theta_0)$ is the sum of martingale differences then

$$\mathrm{E}\left(\frac{\partial \log f_\theta(X_t|X_{t-1},\ldots,X_1)}{\partial\theta}\rfloor_{\theta=\theta_0} \middle| X_{t-1}, X_{t-2},\ldots,X_1\right) = 0,$$

we will show this. Rewriting the above in terms of integrals and exchanging derivative with integral we have

$$\begin{aligned}
& \mathrm{E}\left(\frac{\partial \log f_\theta(X_t|X_{t-1},\ldots,X_1)}{\partial\theta}\rfloor_{\theta=\theta_0} \middle| X_{t-1}, X_{t-2},\ldots,X_1\right) \\
= & \int \frac{\partial \log f_\theta(x_t|X_{t-1},\ldots,X_1)}{\partial\theta}\rfloor_{\theta=\theta_0} f_{\theta_0}(x_t|X_{t-1},\ldots,X_1)dx_t \\
= & \int \frac{1}{f_{\theta_0}(x_t|X_{t-1},\ldots,X_1)} \frac{\partial f_\theta(x_t|X_{t-1},\ldots,X_1)}{\partial\theta}\rfloor_{\theta=\theta_0} f_{\theta_0}(x_t|X_{t-1},\ldots,X_1)dx_t \\
= & \frac{\partial}{\partial\theta}\left(\int f_\theta(x_t|X_{t-1},\ldots,X_1)dx_t\right)\rfloor_{\theta=\theta_0} = 0.
\end{aligned}$$

Therefore $\{\frac{\partial \log f_\theta(X_t|X_{t-1},\ldots,X_1)}{\partial\theta}\rfloor_{\theta=\theta_0}\}_t$ are a sequence of martingale differences and $\mathcal{C}_t(\theta_0)$ is the sum of martingale differences (hence it is a martingale).

A.3 The Fourier series

The Fourier transform is a commonly used tool. We recall that $\{\exp(2\pi ij\omega); j\in\mathbb{Z}\}$ is an orthogonal basis of the space $L^2[0,1]$. In other words, if $f\in L^2[0,1]$ (ie, $\int_0^2 f(\omega)^2 d\omega < \infty$) then

$$f_n(u) = \sum_{j=-n}^{n} c_j e^{iju2\pi} \qquad c_j = \int_0^1 f(u)\exp(i2\pi ju)du,$$

where $\int |f(u)-f_n(u)|^2 du \to 0$ as $n\to\infty$. Roughly speaking, if the function is continuous then we can say that

$$f(u) = \sum_{j\in\mathbb{Z}} c_j e^{iju}.$$

An important property is that $f(u) \equiv$constant iff $c_j = 0$ for all $j \neq 0$. Moreover, for all $n \in \mathbb{Z}$ $f(u+n) = f(u)$ (hence f is periodic).

Some relations:

(i) **Discrete Fourier transforms of finite sequences**

It is straightforward to show (by using the property $\sum_{j=1}^{n} \exp(i2\pi k/n) = 0$ for $k \neq 0$) that if

$$d_k = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} x_j \exp(i2\pi jk/n),$$

then $\{x_r\}$ can be recovered by inverting this transformation

$$x_r = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} d_k \exp(-i2\pi rk/n),$$

(ii) **Fourier sums and integrals**

Of course the above only has meaning when $\{x_k\}$ is a finite sequence. However suppose that $\{x_k\}$ is a sequence which belongs to ℓ_2 (that is $\sum_k x_k^2 < \infty$), then we can define the function

$$f(\omega) = \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} x_k \exp(ik\omega),$$

where $\int_0^{2\pi} f(\omega)^2 d\omega = \sum_k x_k^2$, and we we can recover $\{x_k\}$ from $f(\omega)$, through

$$x_k = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(\omega) \exp(-ik\omega).$$

(iii) **Convolutions**. Let us suppose that $\sum_k |a_k|^2 < \infty$ and $\sum_k |b_k|^2 < \infty$ and we define the Fourier transform of the sequences $\{a_k\}$ and $\{b_k\}$ as $A(\omega) = \frac{1}{\sqrt{2\pi}} a_k \exp(ik\omega)$ and $B(\omega) = \frac{1}{\sqrt{2\pi}} \sum_k b_k \exp(ik\omega)$ respectively. Then

$$\begin{aligned} \sum_{j=-\infty}^{\infty} a_j b_{k-j} &= \int_0^{2\pi} A(\omega)B(-\omega)\exp(-ik\omega)d\omega \\ \sum_{j=-\infty}^{\infty} a_j b_j \exp(ij\omega) &= \int_0^{2\pi} A(\lambda)B(\omega-\lambda)d\lambda. \end{aligned} \tag{A.1}$$

The proof of the above follows from

$$\begin{aligned}
\sum_{j=-\infty}^{\infty} a_j b_j \exp(ij\omega) &= \sum_{r=-\infty}^{\infty} \int_0^{2\pi}\int_0^{2\pi} A(\lambda_1)B(\lambda_2)\exp(-ir(\lambda_1+\lambda_2))\exp(ij\omega) \\
&= \int\int A(\lambda_1)B(\lambda_2)\underbrace{\sum_{r=-\infty}^{\infty}\exp(ir(\omega-\lambda_1-\lambda_2))\,d\lambda_1 d\lambda_2}_{=\delta_\omega(\lambda_1+\lambda_2)} \\
&= \int_0^{2\pi} A(\lambda)B(\omega-\lambda)d\lambda.
\end{aligned}$$

(iv) **Using the DFT to calculate convolutions**. Our objective is calculate $\sum_{j=k}^{n} a_j b_{j-s}$ for all $s = 0,\ldots,n-1$ in as few computing computing operations. This is typically done via the DFT. Examples in time series where this is useful is in calculating the sample autocovariance function.

Suppose we have two sequences $\underline{a} = (a_1,\ldots,a_n)$ and $\underline{b} = (b_1,\ldots,b_n)$. Let $A_n(\omega_{k,n}) = \sum_{j=1}^{n} a_j \exp(ij\omega_{k,n})$ and $B_n(\omega_{k,n}) = \sum_{j=1}^{n} b_j \exp(ij\omega_{k,n})$ where $\omega_{k,n} = 2\pi k/n$. It is straightforward to show that

$$\frac{1}{n}\sum_{k=1}^{n} A_n(\omega_{k,n})\overline{B_n(\omega_{k,n})}\exp(-is\omega_{k,n}) = \sum_{j=s}^{n} a_j b_{j-s} + \sum_{j=1}^{s-1} a_j b_{j-s+n},$$

this is very fast to compute (requiring only $O(n\log n)$ operations using first the FFT and then inverse FFT). The only problem is that we don't want the second term.

By padding the sequences and defining $A_n(\omega_{k,2n}) = \sum_{j=1}^{n} a_j \exp(ij\omega_{k,2n}) = \sum_{j=1}^{2n} a_j \exp(ij\omega_{k,2n})$, with $\omega_{k,2n} = 2\pi k/2n$ (where we set $a_j = 0$ for $j > 0$) and analogously $B_n(\omega_{k,2n}) = \sum_{j=1}^{n} b_j \exp(ij\omega_{k,2n})$, we are able to remove the second term. Using the same calculations we have

$$\frac{1}{n}\sum_{k=1}^{2n} A_n(\omega_{k,2n})\overline{B_n(\omega_{k,2n})}\exp(-is\omega_{k,2n}) = \sum_{j=s}^{n} a_j b_{j-s} + \underbrace{\sum_{j=1}^{s-1} a_j b_{j-s+2n}}_{=0}.$$

This only requires $O(2n\log(2n))$ operations to compute the convolution for all $0 \leq k \leq n-1$.

(v) **The Poisson Summation Formula** Suppose we do not observe the entire function and observe a sample from it, say $f_{t,n} = f(\frac{t}{n})$ we can use this to estimate the Fourier coefficient

c_j via the Discrete Fourier Transform:

$$c_{j,n} = \frac{1}{n}\sum_{t=1}^{n} f(\frac{t}{n})\exp(ij\frac{2\pi t}{n}).$$

The *Poisson Summation formula* is

$$c_{j,n} = c_j + \sum_{k=1}^{\infty} c_{j+kn} + \sum_{k=1}^{\infty} c_{j-kn},$$

which we can prove by replacing $f(\frac{t}{n})$ with $\sum_{j\in\mathbb{Z}} c_j e^{ij2\pi t/n}$. In other words, $c_{j,n}$ cannot disentangle frequency $e^{ij\omega}$ from it's harmonics $e^{i(j+n)\omega}$ (this is *aliasing*).

(vi) **Error in the DFT** By using the Poisson summation formula we can see that

$$|c_{j,n} - c_j| \leq \sum_{k=1}^{\infty} |c_{j+kn}| + \sum_{k=1}^{\infty} |c_{j-kn}|$$

It can be shown that if a function $f(\cdot)$ is $(p+1)$ times differentiable with bounded derivatives or that $f^p(\cdot)$ is bounded and piecewise montonic then the corresponding Fourier coefficients satisfy

$$|c_j| \leq C|j|^{-(p+1)}.$$

Using this result and the Poisson summation formula we can show that for $|j| \leq n/2$ that if if a function $f(\cdot)$ is $(p+1)$ times differentiable with bounded derivatives or that $f^p(\cdot)$ is piecewise montonic and $p \geq 1$ then

$$|c_{j,n} - c_j| \leq Cn^{-(p+1)}, \tag{A.2}$$

where C is some finite constant. However, we cannot use this result in the case that f is bounded and piecewise monotone, however it can still be shown that

$$|c_{j,n} - c_j| \leq Cn^{-1}, \tag{A.3}$$

see Section 6.3, page 189, Briggs and Henson (1997).

A.4 Application of Burkholder's inequality

There are two inequalities (one for $1 < p \leq 2$). Which is the following:

Theorem A.4.1 *Suppose that Y_k are martingale differences and that $S_n = \sum_{j=1}^{n} Y_k$, then for $0 < q \leq 2$*

$$\mathrm{E}|S_n|^q \leq 2 \sum_{j=1}^{n} \mathrm{E}(X_k^q), \tag{A.4}$$

See for example Davidson (p. 242, Theorem 15.17).

And one for ($p \geq 2$), this is the statement for the Burkölder inequality:

Theorem A.4.2 *Suppose $\{S_i : \mathcal{F}_i\}$ is a martingale and $1 < p < \infty$. Then there exists constants C_1, C_2 depending only on p such that*

$$C_1 \mathrm{E}\left(\sum_{i=1}^{m} X_i^2\right)^{p/2} \leq \mathrm{E}|S_n|^p \leq C_2 \mathrm{E}\left(\sum_{i=1}^{m} X_i^2\right)^{p/2}. \tag{A.5}$$

An immediately consequence of the above for $p \geq 2$ is the following corollary (by using Hölder's inequality):

Corollary A.4.1 *Suppose $\{S_i : \mathcal{F}_i\}$ is a martingale and $2 \leq p < \infty$. Then there exists constants C_1, C_2 depending only on p such that*

$$\|S_n\|_p^E \leq \left(C_2^{2/p} \sum_{i=1}^{m} \|X_i^2\|_{p/2}^E\right)^{1/2}. \tag{A.6}$$

PROOF. By using the right hand side of (A.5) we have

$$\begin{aligned} \left\{\mathrm{E}|S_n|^p\right\}^{1/p} &\leq \left[\left(C_2 \mathrm{E}\left(\sum_{i=1}^{m} X_i^2\right)^{p/2}\right)^{2/p}\right]^{1/2} \\ &= \left[C_2^{2/p} \left\|\sum_{i=1}^{m} X_i^2\right\|_{p/2}^{E}\right]^{1/2}. \end{aligned} \tag{A.7}$$

By using Hölder inequality we have

$$\left\{\mathrm{E}|S_n|^p\right\}^{1/p} \quad \leq \quad \left[C_2^{2/p}\sum_{i=1}^{m}\|X_i^2\|_{p/2}^E\right]^{1/2}. \tag{A.8}$$

Thus we have the desired result. □

We see the value of the above result in the following application. Suppose $S_n = \frac{1}{n}\sum_{k=1}^{n} X_k$ and $\|X_k\|_p^E \leq K$. Then we have

$$\begin{aligned}\mathrm{E}\left(\frac{1}{n}\sum_{k=1}^{n} X_k\right)^p &\leq \left[\frac{1}{n}C_2^{2/p}\sum_{k=1}^{n}\|X_k^2\|_{p/2}^E\right]^{p/2} \\ &\leq \frac{C_2}{n^p}\left[\sum_{k=1}^{n}\|X_k^2\|_{p/2}^E\right]^{p/2} \leq \frac{C_2}{n^p}\left[\sum_{k=1}^{n} K^2\right]^{p/2} = O(\frac{1}{n^{p/2}}).\end{aligned} \tag{A.9}$$

Below is the result that that Moulines et al (2004) use (they call it the generalised Burkholder inequality) the proof can be found in Dedecker and Doukhan (2003). Note that it is for $p \geq 2$, which I forgot to state in what I gave you.

Lemma A.4.1 *Suppose* $\{\phi_k : \ k = 1, 2, \ldots\}$ *is a stochastic process which satisfies* $\mathrm{E}(\phi_k) = 0$ *and* $\mathrm{E}(\phi_k^p) < \infty$ *for some* $p \geq 2$. *Let* $\mathcal{F}_k = \sigma(\phi_k, \phi_{k-1}, \ldots)$. *Then we have that*

$$\left\|\sum_{k=1}^{s}\phi_k\right\|_p^E \quad \leq \quad \left(2p\sum_{k=1}^{s}\|\phi_k\|_p^E\sum_{j=k}^{s}\|\mathrm{E}(\phi_j|\mathcal{F}_k)\|_p^E\right)^{1/2}. \tag{A.10}$$

We note if $\sum_{j=k}^{s}\|\mathrm{E}(\phi_j|\mathcal{F}_k)\|_p^E < \infty$, then we (A.11) is very similar to (A.6), and gives the same rate as (A.9).

But I think one can obtain something similar for $1 \leq p \leq 2$. I think the below is correct.

Lemma A.4.2 *Suppose* $\{\phi_k : \ k = 1, 2, \ldots\}$ *is a stochastic process which satisfies* $\mathrm{E}(\phi_k) = 0$ *and* $\mathrm{E}(\phi_k^q) < \infty$ *for some* $1 < q \leq 2$. *Let* $\mathcal{F}_k = \sigma(\phi_k, \phi_{k-1}, \ldots)$. *Further, we suppose that there exists a* $0 < \rho < 1$, *and* $0 < K < \infty$ *such that* $\|\mathrm{E}(\phi_t|\mathcal{F}_{t-j})\|_q < K\rho^j$. *Then we have that*

$$\left\|\sum_{k=1}^{s}a_k\phi_k\right\|_q^E \quad \leq \quad \frac{K^*}{1-\rho}\left(\sum_{k=1}^{s}|a_k|^q\right)^{1/q}, \tag{A.11}$$

where K^* is a finite constant.

PROOF. Let $\mathrm{E}_j(\phi_k) = \mathrm{E}(\phi_k|\mathcal{F}_{k-j})$. We note that by definition $\{\phi_k\}$ is a mixingale (see, for example, Davidson (1997), chapter 16), therefore amost surely ϕ_k satisfies the representation

$$\phi_k = \sum_{j=0}^{\infty}[\mathrm{E}_{k-j}(\phi_k) - \mathrm{E}_{k-j-1}(\phi_k)]. \tag{A.12}$$

By substituting the above into the sum $\sum_{k=1}^{s} a_k\phi_k$ we obtain

$$\sum_{k=1}^{s} a_k\phi_k = \sum_{k=1}^{s}\sum_{j=0}^{\infty}[\mathrm{E}_{k-j}(\phi_k) - \mathrm{E}_{k-j-1}(\phi_k)] = \sum_{j=0}^{\infty}\left(\sum_{k=1}^{s}[\mathrm{E}_{k-j}(\phi_k) - \mathrm{E}_{k-j-1}(\phi_k)]\right). \tag{A.13}$$

Keeping j constant, we see that $\{\mathrm{E}_{k-j}(\phi_k) - \mathrm{E}_{k-j-1}(\phi_k)\}_k$ is a martingale sequence. Hence $\sum_{k=1}^{s}[\mathrm{E}_{k-j}(\phi_k) - \mathrm{E}_{k-j-1}(\phi_k)]$ is the sum of martingale differences. This implies we can apply (A.4) to (A.13), and get

$$\begin{aligned}\left\|\sum_{k=1}^{s} a_k\phi_k\right\|_q^E &\leq \sum_{j=0}^{\infty}\left\|\sum_{k=1}^{s}|a_k|[\mathrm{E}_{k-j}(\phi_k) - \mathrm{E}_{k-j-1}(\phi_k)]\right\|_q^E \\ &\leq \sum_{j=0}^{\infty}\left(2\sum_{k=1}^{s}|a_k|(\|\mathrm{E}_{k-j}(\phi_k) - \mathrm{E}_{k-j-1}(\phi_k)\|_q^E)^q\right)^{1/q}\end{aligned}$$

Under the stated assumption $\|\mathrm{E}_{k-j}(\phi_k) - \mathrm{E}_{k-j-1}(\phi_k)\|_q^E \leq 2K\rho^j$. Substituting this inequality into the above gives

$$\left\|\sum_{k=1}^{s} a_k\phi_k\right\|_q^E \leq \sum_{j=0}^{\infty}\left(2\sum_{k=1}^{s}|a_k|^q(2K\rho^j)^q\right)^{1/q} \leq 2^{1+1/q}K\sum_{j=0}^{\infty}\rho^j\left(\sum_{k=1}^{s}|a_k|^q\right)^{1/q}.$$

Thus we obtain the desired result. □

A.5 The Fast Fourier Transform (FFT)

The Discrete Fourier transform is used widely in several disciplines. Even in areas its use may not be immediately obvious (such as inverting Toeplitz matrices) it is still used because it can be evalated in a speedy fashion using what is commonly called the fast fourier transform (FFT). It is an algorithm which simplifies the number of computing operations required to compute the Fourier transform of a sequence of data. Given that we are in the age of 'big data' it is useful to learn what one of most popular computing algorithms since the 60s actually does.

Recalling the notation in Section 9.2.2 the Fourier transform is the linear transformation

$$F_n \underline{X}_n = (J_n(\omega_0), \ldots, J_n(\omega_{n-1})).$$

If this was done without any using any tricks this requires $O(n^2)$ computing operations. By using some neat factorizations, the fft reduces this to $n \log n$ computing operations.

To prove this result we will ignore the standardization factor $(2\pi n)^{-1/2}$ and consider just the Fourier transform

$$d(\omega_{k,n}) = \underbrace{\sum_{t=1}^{n} x_t \exp\left(it\omega_{k,n}\right)}_{k \text{ different frequencies}},$$

where $\omega_{k,n} = \frac{2\pi k}{n}$. Here we consider the proof for general n, later in Example A.5.1 we consider the specific case that $n = 2^m$. Let us assume that n is not a prime (if it is then we simply pad the vector with one zero and increase the length to $n+1$), then it can be factorized as $n = pq$. Using these factors we write t as $t = t_1 p + t\text{mod}p$ where t_1 is some integer value that lies between 0 to $q-1$ and $t_0 = t\text{mod}p$ lies between 0 to $p-1$. Substituting this into $d(\omega_k)$ gives

$$\begin{aligned} d(\omega_k) &= \sum_{t=1}^{n} x_t \exp\left[i(t_1 p + t\text{mod}p)\omega_{k,n}\right] \\ &= \sum_{t_0=0}^{p-1} \sum_{t_1=0}^{q-1} x_{t_1 p + t_0} \exp\left[i(t_1 p + t_0)\omega_{k,n}\right] = \sum_{t_0=0}^{p-1} \exp\left[it_0 \omega_{k,n}\right] \sum_{t_1=0}^{q-1} x_{t_1 p + t_0} \exp\left[it_1 p \omega_{k,n}\right] \end{aligned}$$

It is straightforward to see that $t_1 p \omega_{k,n} = \frac{2\pi t_1 p k}{n} = \frac{2\pi t_1 k}{q} = t_1 \omega_{k,q}$ and that $\exp(it_1 p \omega_{k,n}) =$

$\exp(it_1\omega_{k,q}) = \exp(it_1\omega_{k\text{mod}q,q})$. This means $d(\omega_k)$ can be simplified as

$$\begin{aligned} d(\omega_k) &= \sum_{t_0=0}^{p-1} \exp\left[it_0\omega_{k,n}\right] \sum_{t_1=0}^{q-1} x_{t_1p+t_0} \exp\left[it_1\omega_{k\text{mod}q,q}\right] \\ &= \sum_{t_0=0}^{p-1} \exp\left[it_0\omega_{k,n}\right] \underbrace{\sum_{t_1=0}^{q-1} x_{t_1p+t_0} \exp\left[it_1\omega_{k_0,q}\right]}_{\text{embedded Fourier transform}} \\ &= \sum_{t_0=0}^{p-1} \exp\left[it_0\omega_{k,n}\right] \underbrace{A(t_0, k\text{mod}q)}_{q\text{ frequencies}}, \end{aligned}$$

where $k_0 = k\text{mod}q$ can take values from $0, \ldots, q-1$. Thus to evaluate $d(\omega_k)$ we need to evaluate $A(t_0, k\text{mod}q)$ for $0 \leq t_0 \leq p-1$, $0 \leq k_0 \leq q-1$. To evaluate $A(t_0, k\text{mod}q)$ requires q computing operations, to evaluate it for all t_0 and $k\text{mod}q$ requires pq^2 operations. Note, the key is that less frequencies need to be evaluated when calculating $A(t_0, k\text{mod}q)$, in particular q frequencies rather than N. After evaluating $\{A(t_0, k_0); 0 \leq t_0 \leq p-1, 0 \leq k_0 \leq q-1\}$ we then need to take the Fourier transform of this over t_0 to evaluate $d(\omega_k)$ which is p operations and this needs to be done n times (to get all $\{d(\omega_k)\}_k$) this leads to np. Thus in total this leads to

$$\underbrace{p^2q}_{\text{evaluation of all A}} + \underbrace{np}_{\text{evaluation of the transforms of A}} = pq^2 + pn = n(q+p). \tag{A.14}$$

Observe that $n(p+q)$ is a lot smaller than n^2.

Looking back at the above calculation we observe that q^2 operations were required to calculate $A(t_0, k\text{mod}q) = A(t_0, k_0)$ for all $0 \leq k_0 \leq q-1$. However $A(t_0, k_0)$ is a Fourier transform

$$A(t_0, k_0) = \sum_{t_1=0}^{q-1} x_{t_1p+t_0} \exp\left[it_1\omega_{k_0,q}\right].$$

Therefore, we can use the same method as was used above to reduce this number. To do this we need to factorize q into $p = p_1 q_1$ and using the above method we can write this as

$$\begin{aligned} A(t_0, k_0) &= \sum_{t_2=0}^{p_1-1} \sum_{t_3=0}^{q_1-1} x_{(t_2+t_3 p_1)p+t_0} \exp\left[i(t_2 + t_3 p_1)\omega_{k_0,q}\right] \\ &= \sum_{t_2=0}^{p_1-1} \exp\left[it_2\omega_{k_0,q}\right] \sum_{t_3=0}^{q_1-1} x_{(t_2+t_3 p_1)p+t_0} \exp\left[it_3 p_1 \omega_{k_0,q}\right] \\ &= \sum_{t_2=0}^{p_1-1} \exp\left[it_2\omega_{k_0,q}\right] \sum_{t_3=0}^{q_1-1} x_{(t_2+t_3 p_1)p+t_0} \exp\left[it_3 \omega_{k_0 \text{mod} q_1, q_1}\right]. \end{aligned}$$

We note that $k_0 \text{mod} q_1 = (k \text{mod}(p_1 q_1) \text{mod} q_1) = k \text{mod} q_1$, substituting this into the above we have

$$\begin{aligned} A(t_0, k_0) &= \sum_{t_2=0}^{p_1-1} \exp\left[it_2\omega_{k_0,q}\right] \sum_{t_3=0}^{q_1-1} x_{(t_2+t_3 p_1)p+t_0} \exp\left[it_3 \omega_{k_0 \text{mod} q_1, q_1}\right] \\ &= \sum_{t_2=0}^{p_1-1} \exp\left[it_2\omega_{k_0,q}\right] \underbrace{A(t_0, t_2, k_0 \text{mod} q_1)}_{q_1 \text{ frequencies}}. \end{aligned}$$

Thus we see that q_1 computing operations are required to calculate $A(t_0, t_2, k_0 \text{mod} q_1)$ and to calculate $A(t_0, t_2, k \text{mod} q_1)$ for all $0 \leq t_2 \leq p_1 - 1$ and $0 \leq k \text{mod} q_1 \leq q_1 - 1$ requires in total $q_1^2 p_1$ computing operations. After evaluating $\{A(t_0, t_2, k_0 \text{mod} q_1); 0 \leq t_2 \leq q_2 - 1, 0 \leq k \text{mod} q_1 \leq q_1 - 1\}$ we then need to take its Fourier transform over t_2 to evaluate $A(t_0, k_0)$, which is p_1 operations. Thus in total to evaluate $A(t_0, k_0)$ over all k_0 we require $q_1^2 p_1 + p_1 q$ operations. Thus we have reduced the number of computing operations for $A(t_0, k_0)$ from q^2 to $q(p_1 + q_1)$, substituting this into (A.14) gives the total number of computing operations to calculate $\{d(\omega_k)\}$

$$pq(p_1 + q_1) + np = n(p + p_1 + q_1).$$

In general the same idea can be used to show that given the prime factorization of $n = \prod_{s=1}^{m} p_s$, then the number of computing operations to calculate the DFT is $n(\sum_{s=1}^{m} p_s)$.

Example A.5.1 *Let us suppose that $n = 2^m$ then we can write $d(\omega_k)$ as*

$$\begin{aligned}
d(\omega_k) = \sum_{t=1}^{n} x_t \exp(it\omega_k) &= \sum_{t=1}^{n/2} X_{2t}\exp(i2t\omega_k) + \sum_{t=0}^{(n/2)-1} X_{2t+1}\exp(i(2t+1)\omega_k) \\
&= \sum_{t=1}^{n/2} X_{2t}\exp(i2t\omega_k) + \exp(i\omega_k)\sum_{t=0}^{(n/2)-1} X_{2t+1}\exp(i2t\omega_k) \\
&= A(0, k\text{mod}(n/2)) + \exp(i\omega_k)A(1, k\text{mod}(n/2)),
\end{aligned}$$

since $\sum_{t=1}^{n/2} X_{2t}\exp(i2t\omega_k)$ and $\sum_{t=1}^{n/2} X_{2t+1}\exp(i2t\omega_k)$ are the Fourier transforms of $\{X_t\}$ on a coarser scale, therefore we can only identify the frequencies on a coarser scale. It is clear from the above that the evaluation of $A(0, k\text{mod}(n/2))$ for $0 \leq k\text{mod}(n/2) \leq n/2$ requires $(n/2)^2$ operations and same for $A(1, k\text{mod}(n/2))$. Thus to evaluate both $A(0, k\text{mod}(n/2))$ and $A(1, k\text{mod}(n/2))$ requires $2(n/2)^2$ operations. Then taking the Fourier transform of these two terms over all $0 \leq k \leq n-1$ is an additional $2n$ operations leading to

$$2(n/2)^2 + 2n = n^2/2 + 2n \text{ operations } < n^2.$$

We can continue this argument and partition

$$\begin{aligned}
A(0, k\text{mod}(n/2)) &= \sum_{t=1}^{n/2} X_{2t}\exp(i2t\omega_k) \\
&= \sum_{t=1}^{n/4} X_{4t}\exp(i4t\omega_k) + \exp(i2\omega_k)\sum_{t=0}^{(n/4)-1} X_{4t+2}\exp(i4t\omega_k).
\end{aligned}$$

Using the same argument as above the calculation of this term over all k requires $2(n/4)^2 + 2(n/2) = n^2/8 + n$ operations. The same decomposition applies to $A(1, k\text{mod}(n/2))$. Thus calculation of both terms over all k requires $2[n^2/8 + n] = n^2/4 + 2n$ operations. In total this gives

$$(n^2/4 + 2n + 2n)\text{operations.}$$

Continuing this argument gives $mn = n\log_2 n$ operations, which is the often cited rate.

Typically, if the sample size is not of order 2^m zeros are added to the end of the sequence (called padding) to increase the length to 2^m.

Appendix B

Mixingales

In this section we prove some of the results stated in the previous sections using mixingales.

We first define a mixingale, noting that the definition we give is not the most general definition.

Definition B.0.1 (Mixingale) *Let $\mathcal{F}_t = \sigma(X_t, X_{t-1}, \ldots)$, $\{X_t\}$ is called a mixingale if it satisfies*

$$\rho_{t,k} = \left\{\mathrm{E}\Big(\mathrm{E}(X_t|\mathcal{F}_{t-k}) - \mathrm{E}(X_t)\Big)^2\right\}^{1/2},$$

where $\rho_{t,k} \to 0$ as $k \to \infty$. We note if $\{X_t\}$ is a stationary process then $\rho_{t,k} = \rho_k$.

Lemma B.0.1 *Suppose $\{X_t\}$ is a mixingale. Then $\{X_t\}$ almost surely satisfies the decomposition*

$$X_t = \sum_{j=0}^{\infty}\Big\{\mathrm{E}(X_t|\mathcal{F}_{t-j-1}) - \mathrm{E}(X_t|\mathcal{F}_{t-j-1})\Big\}. \tag{B.1}$$

PROOF. We first note that by using a telescoping argument that

$$X_t - \mathrm{E}(X_t) = \sum_{k=0}^{m}\big\{\mathrm{E}(X_t|\mathcal{F}_{t-k}) - \mathrm{E}(X_t|\mathcal{F}_{t-k-1})\big\} + \big\{\mathrm{E}(X_t|\mathcal{F}_{t-m-1}) - \mathrm{E}(X_t)\big\}.$$

By definition of a martingale $\mathrm{E}\big(\mathrm{E}(X_t|\mathcal{F}_{t-m-1}) - \mathrm{E}(X_t)\big)^2 \to 0$ as $k \to \infty$, hence the remainder term in the above expansion becomes negligable as $m \to \infty$ and we have almost surely

$$\begin{aligned} & X_t - \mathrm{E}(X_t) \\ = & \sum_{k=0}^{\infty}\big\{\mathrm{E}(X_t|\mathcal{F}_{t-k}) - \mathrm{E}(X_t|\mathcal{F}_{t-k-1})\big\}. \end{aligned}$$

Thus giving the required result. □

We observe that (B.1) resembles the Wold decomposition. The difference is that the Wolds decomposition decomposes a stationary process into elements which are the errors in the best linear predictors. Whereas the result above decomposes a process into sums of martingale differences.

It can be shown that functions of several ARCH-type processes are mixingales (where $\rho_{t,k} \leq K\rho^k$ ($rho < 1$)), and Subba Rao (2006) and Dahlhaus and Subba Rao (2007) used these properties to obtain the rate of convergence for various types of ARCH parameter estimators. In a series of papers, Wei Biao Wu considered properties of a general class of stationary processes which satisfied Definition B.0.1, where $\sum_{k=1}^{\infty} \rho_k < \infty$.

In Section B.2 we use the mixingale property to prove Theorem 11.7.3. This is a simple illustration of how useful mixingales can be. In the following section we give a result on the rate of convergence of some random variables.

B.1 Obtaining almost sure rates of convergence for some sums

The following lemma is a simple variant on a result proved in Móricz (1976), Theorem 6.

Lemma B.1.1 *Let $\{\mathcal{S}_T\}$ be a random sequence where* $\mathrm{E}(\sup_{1\leq t\leq T}|\mathcal{S}_t|^2) \leq \phi(T)$ *and $\{phi(t)\}$ is a monotonically increasing sequence where $\phi(2^j)/\phi(2^{j-1}) \leq K < \infty$ for all j. Then we have almost surely*

$$\frac{1}{T}\mathcal{S}_T = O\Big(\frac{\sqrt{\phi(T)(\log T)(\log\log T)^{1+\delta}}}{T}\Big).$$

PROOF. The idea behind the proof is to that we find a subsequence of the natural numbers and define a random variables on this subsequence. This random variable, should 'dominate' (in some sense) S_T. We then obtain a rate of convergence for the subsequence (you will see that for the subsequence its quite easy by using the Borel-Cantelli lemma), which, due to the dominance, can be transfered over to S_T. We make this argument precise below.

Define the sequence $V_j = \sup_{t\leq 2^j}|S_t|$. Using Chebyshev's inequality we have

$$P(V_j > \varepsilon) \leq \frac{\phi(2^j)}{\varepsilon}.$$

Let $\varepsilon(t) = \sqrt{\phi(t)(\log\log t)^{1+\delta}\log t}$. It is clear that

$$\sum_{j=1}^{\infty} P(V_j > \varepsilon(2^j)) \leq \sum_{j=1}^{\infty} \frac{C\phi(2^j)}{\phi(2^j)(\log j)^{1+\delta} j} < \infty,$$

where C is a finite constant. Now by Borel Cantelli, this means that almost surely $V_j \leq \varepsilon(2^j)$. Let us now return to the orginal sequence S_T. Suppose $2^{j-1} \leq T \leq 2^j$, then by definition of V_j we have

$$\frac{S_T}{\varepsilon(T)} \leq \frac{V_j}{\varepsilon(2^{j-1})} \overset{a.s}{\leq} \frac{\varepsilon(2^j)}{\varepsilon(2^{j-1})} < \infty$$

under the stated assumptions. Therefore almost surely we have $S_T = O(\varepsilon(T))$, which gives us the required result. □

We observe that the above result resembles the law of iterated logarithms. The above result is very simple and nice way of obtaining an almost sure rate of convergence. The main problem is obtaining bounds for $\mathrm{E}(\sup_{1\leq t\leq T}|\mathcal{S}_t|^2)$. There is on exception to this, when $\mathcal{S}_t$ is the sum of martingale differences then one can simply apply Doob's inequality, where $\mathrm{E}(\sup_{1\leq t\leq T}|\mathcal{S}_t|^2) \leq \mathrm{E}(|\mathcal{S}_T|^2)$. In the case that S_T is not the sum of martingale differences then its not so straightforward. However if we can show that S_T is the sum of mixingales then with some modifications a bound for $\mathrm{E}(\sup_{1\leq t\leq T}|\mathcal{S}_t|^2)$ can be obtained. We will use this result in the section below.

B.2 Proof of Theorem 11.7.3

We summarise Theorem 11.7.3 below.

Theorem 1 *Let us suppose that $\{X_t\}$ has an ARMA representation where the roots of the characteristic polynomials $\phi(z)$ and $\theta(z)$ lie are greater than $1+\delta$. Then*

(i)

$$\frac{1}{n}\sum_{t=r+1}^{n} \varepsilon_t X_{t-r} = O(\sqrt{\frac{(\log\log n)^{1+\gamma}\log n}{n}}) \qquad (B.2)$$

(ii)

$$\frac{1}{n}\sum_{t=\max(i,j)}^{n} X_{t-i}X_{t-j} = O(\sqrt{\frac{(\log\log n)^{1+\gamma}\log n}{n}}). \qquad (B.3)$$

for any $\gamma > 0$.

By using Lemma **??**, and that $\sum_{t=r+1}^{n} \varepsilon_t X_{t-r}$ is the sum of martingale differences, we prove Theorem 11.7.3(i) below.

PROOF of Theorem 11.7.3. We first observe that $\{\varepsilon_t X_{t-r}\}$ are martingale differences, hence we can use Doob's inequality to give $\mathrm{E}(\sup_{r+1 \leq s \leq T}(\sum_{t=r+1}^{s} \varepsilon_t X_{t-r})^2) \leq (T-r)\mathrm{E}(\varepsilon_t^2)\mathrm{E}(X_t^2)$. Now we can apply Lemma **??** to obtain the result. □

We now show that

$$\frac{1}{T}\sum_{t=\max(i,j)}^{T} X_{t-i}X_{t-j} = O(\sqrt{\frac{(\log\log T)^{1+\delta}\log T}{T}}).$$

However the proof is more complex, since $\{X_{t-i}X_{t-j}\}$ are not martingale differences and we cannot directly use Doob's inequality. However by showing that $\{X_{t-i}X_{t-j}\}$ is a mixingale we can still show the result.

To prove the result let $\mathcal{F}_t = \sigma(X_t, X_{t-1}, \ldots)$ and $\mathcal{G}_t = \sigma(X_{t-i}X_{t-j}, X_{t-1-i}X_{t-j-i}, \ldots)$. We observe that if $i > j$, then $\mathcal{G}_t \subset \mathcal{F}_{t-i}$.

Lemma B.2.1 *Let $\mathcal{F}_t = \sigma(X_t, X_{t-1}, \ldots)$ and suppose X_t comes from an ARMA process, where the roots are greater than $1+\delta$. Then if $\mathrm{E}(\varepsilon_t^4) < \infty$ we have*

$$\mathrm{E}\big(\mathrm{E}(X_{t-i}X_{t-j}|\mathcal{F}_{t-\min(i,j)-k}) - \mathrm{E}(X_{t-i}X_{t-j})\big)^2 \leq C\rho^k.$$

PROOF. By expanding X_t as an MA(∞) process we have

$$\begin{aligned} & \mathrm{E}(X_{t-i}X_{t-j}|\mathcal{F}_{t-\min(i,j)-k}) - \mathrm{E}(X_{t-i}X_{t-j}) \\ = & \sum_{j_1,j_2=0}^{\infty} a_{j_1}a_{j_2}\big\{\mathrm{E}(\varepsilon_{t-i-j_1}\varepsilon_{t-j-j_2}|\mathcal{F}_{t-k-\min(i,j)}) - \mathrm{E}(\varepsilon_{t-i-j_1}\varepsilon_{t-j-j_2})\big\}. \end{aligned}$$

Now in the case that $t-i-j_1 > t-k-\min(i,j)$ and $t-j-j_2 > t-k-\min(i,j)$, $\mathrm{E}(\varepsilon_{t-i-j_1}\varepsilon_{t-j-j_2}|\mathcal{F}_{t-k-\min(i,j)}) = \mathrm{E}(\varepsilon_{t-i-j_1}\varepsilon_{t-j-j_2})$. Now by considering when $t-i-j_1 \leq t-k-\min(i,j)$ or $t-j-j_2 \leq t-k-\min(i,j)$ we have have the result. □

Lemma B.2.2 *Suppose $\{X_t\}$ comes from an ARMA process. Then* *(i) The sequence* $\{X_{t-i}X_{t-j}\}_t$ *satisfies the mixingale property*

$$\mathrm{E}\big(\mathrm{E}(X_{t-i}X_{t-j}|\mathcal{F}_{t-\min(i,j)-k}) - \mathrm{E}(X_{t-i}X_{t-j}|\mathcal{F}_{t-k-1})\big)^2 \leq K\rho^k, \tag{B.4}$$

and almost surely we can write $X_{t-i}X_{t-j}$ *as*

$$X_{t-i}X_{t-j} - \mathrm{E}(X_{t-i}X_{t-j}) = \sum_{k=0}^{\infty}\sum_{t=\min(i,j)}^{n} V_{t,k} \tag{B.5}$$

where $V_{t,k} = \mathrm{E}(X_{t-i}X_{t-j}|\mathcal{F}_{t-k-\min(i,j)}) - \mathrm{E}(X_{t-i}X_{t-j}|\mathcal{F}_{t-k-\min(i,j)-1})$*, are martingale differences.*

(ii) Furthermore $\mathrm{E}(V_{t,k}^2) \leq K\rho^k$ *and*

$$\mathrm{E}\big\{\sup_{\min(i,j)\leq s\leq n}\big(\sum_{t=\min(i,j)}^{s}\{X_{t-i}X_{t-j} - \mathrm{E}(X_{t-i}X_{t-j})\}\big)^2\big\} \leq Kn, \tag{B.6}$$

where K *is some finite constant.*

PROOF. To prove (i) we note that by using Lemma B.2.1 we have (B.4). To prove (B.5) we use the same telescoping argument used to prove Lemma B.0.1.

To prove (ii) we use the above expansion to give

$$\begin{aligned}
& \mathrm{E}\big\{\sup_{\min(i,j)\leq s\leq n}\big(\sum_{t=\min(i,j)}^{s}\{X_{t-i}X_{t-j} - \mathrm{E}(X_{t-i}X_{t-j})\}\big)^2\big\} \qquad \text{(B.7)} \\
= \ & \mathrm{E}\big\{\sup_{\min(i,j)\leq s\leq n}\big(\sum_{k=0}^{\infty}\sum_{t=\min(i,j)}^{s} V_{t,k}\big)^2\big\} \\
= \ & \mathrm{E}\big\{\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sup_{\min(i,j)\leq s\leq n}|\sum_{t=\min(i,j)}^{s} V_{t,k_1}| \times |\sum_{t=\min(i,j)}^{s} V_{t,k_2}|\big\} \\
= \ & \Big(\sum_{k=0}^{\infty}\Big\{\mathrm{E}\Big(\sup_{\min(i,j)\leq s\leq n}|\sum_{t=\min(i,j)}^{s} V_{t,k_1}|^2\Big)\Big\}^{1/2}\Big)^2
\end{aligned}$$

Now we see that $\{V_{t,k}\}_t = \{\mathrm{E}(X_{t-i}X_{t-j}|\mathcal{F}_{t-k-\min(i,j)}) - \mathrm{E}(X_{t-i}X_{t-j}|\mathcal{F}_{t-k-\min(i,j)-1})\}_t$, therefore $\{V_{t,k}\}_t$ are also martingale differences. Hence we can apply Doob's inequality to $\mathrm{E}\big\{\sup_{\min(i,j)\leq s\leq n}\big(\sum_{t=\min(i,j)}^{s} V_{t,k}\big)$

and by using (B.4) we have

$$\mathrm{E}\Big\{\sup_{\min(i,j)\leq s\leq n}\Big(\sum_{t=\min(i,j)}^{s} V_{t,k}\Big)^{2}\Big\} \leq \mathrm{E}\Big(\sum_{t=\min(i,j)}^{n} V_{t,k}\Big)^{2} = \sum_{t=\min(i,j)}^{n} \mathrm{E}(V_{t,k}^{2}) \leq K\cdot n\rho^{k}.$$

Therefore now by using (B.7) we have

$$\mathrm{E}\Big\{\sup_{\min(i,j)\leq s\leq n}\Big(\sum_{t=\min(i,j)}^{s}\{X_{t-i}X_{t-j}-\mathrm{E}(X_{t-i}X_{t-j})\}\Big)^{2}\Big\} \leq Kn.$$

Thus giving (B.6). □

We now use the above to prove Theorem 11.7.3(ii).

PROOF of Theorem 11.7.3(ii). To prove the result we use (B.6) and Lemma B.1.1. □

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