Equations of Motion

Using the principles of conservation of momentum and conservation of angular momentum, come up with second order differential equations that describe the time evolution of the system.

Since we're working in the world of "big, slow" objects, we're going to apply the central rules of Newtonian mechanics here: *momentum is conserved, and angular momentum is conserved.* Mathematically, this means that for ANY system (including parts of a bigger system),

$$\frac{d\vec{p}}{dt} = \sum \vec{F}$$

and

$$\frac{d\vec{L}}{dt} = \sum \vec{\tau}$$

where $\vec{p}$ is the *momentum* of the system, $\vec{L}$ is the *angular momentum* of the system, $\sum \vec{F}$ is the sum of all the forces being applied to the system, and $\sum \vec{\tau}$ is the sum of all the torques applied to the system.

So let's apply these rules to each part we're dealing with here: the skateboarder, and the ramp.

For the skateboarder, the momentum of the skateboarder is simply the skateboarder's mass times his or her velocity:

$$\vec{p} = m\vec{v}$$

Now, the velocity of the skateboarder requires a little thought. In general, the velocity is the time derivative of the position. The position is given by

$$\vec{r} = r\cos\theta\hat{\imath} + r\sin\theta\hat{\jmath}$$

where $\hat{\imath}$ is a unit vector in the x direction, and $\hat{\jmath}$ is a unit vector in the y direction. So, we can find the velocity by taking a time derivative of this. We of course need to remember the product rule here!

$$\vec{v} = \frac{d\vec{r}}{dt} = (\dot{r}\cos\theta - r\dot{\theta}\sin\theta)\hat{\imath} + (\dot{r}\sin\theta + r\dot{\theta}\cos\theta)\hat{\jmath}$$

Now it's pretty obvious, if you look at it, that things are going to get ugly here with all these sines and cosines. Let's introduce some new notation: $\hat{r}$, which indicates a unit vector pointing along the ramp, and $\hat{\theta}$, which indicates a unit vector normal to the ramp. If we think through the trig here, we get

$$\hat{r} = \cos\theta\hat{\imath} + \sin\theta\hat{\jmath}$$

and

$$\hat{\theta} = -\sin\theta\hat{\imath} + \cos\theta\hat{\jmath}$$

Furthermore, if we think about derivatives here,

$$\frac{d\hat{r}}{dt} = \dot{\theta}\hat{\theta}$$

and

$$\frac{d\hat{\theta}}{dt} = -\dot{\theta}\hat{r}$$

So then we can write much more neatly:

$$\vec{r} = r\hat{r}$$

and

$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

. Ah, isn't that prettier!

OK, so now we're ready to apply the law of momentum conservation to the skateboarder. If we expand the derivative of the momentum (using product rule again on the velocity expression above), we get

$$\frac{d\vec{p}}{dt} = \frac{d(m\vec{v})}{dt} = m(\ddot{r}\hat{r} + 2\dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} - r\dot{\theta}^2\hat{r})$$

Now, the righthand side will be the sum of the forces applied to the skateboarder:

$$\sum \vec{F} = \vec{F}_G + \vec{F}_N$$

So how do we write these out? Well, $\vec{F}_G = -mg\hat{j}$, which is pretty simple. Unfortunately, we've written everything in terms of $\hat{\theta}$ and $\hat{r}$, so we need to write $\vec{F}_G$ this way too:

$$\vec{F}_G = -mg\cos\theta\hat{\theta} - mg\sin\theta\hat{r}$$

You'll have to draw yourself a few triangles to convince yourself of this.

$\vec{F}_N$ is a bit tougher. By inspection, it will only be in the $\hat{\theta}$ direction. It's tempting to say also that it has a magnitude that is simply $F_N = mg\cos\theta$. *But this would be very, very wrong!* Why? Well, the normal force is a constraint force – i.e., it is whatever it needs to be to keep the skateboarder from falling through the ramp. *If* the ramp were fixed, $mg\cos\theta$ would be right, because it would end up requiring that the net force was purely in the $\hat{r}$ direction. But since the ramp is not fixed, it is possible for the skateboarder to accelerate in the $\hat{\theta}$ direction too! So for now, we'll have to call F_N an unknown quantity.

Writing out the lefthand and righthand sides for the $\hat{r}$ components, we get the following differential equation:

$$m(\ddot{r} - r\dot{\theta}^2) = -mg\sin\theta$$

and for the $\hat{\theta}$ components, we obtain

$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = -mg\cos\theta + F_N$$

If we pause and look at these, in some ways you could say we have two equations and three unknowns: $\ddot{r}$, $th\ddot{e}ta$, and F_N. To get our third equation, we need to look at the other chunk in the system: the ramp.

Now, the ramp is a rigid body, which is constrained only to rotate. This means that the only interesting equation will be the angular momentum equation: the linear momentum of the ramp is ALWAYS zero. The angular momentum, on the other hand, will only contain a rotational component:

$$\vec{L} = I\dot{\theta}\hat{k}$$

where I is the moment of inertia of the ramp about its pivot, and $\hat{k}$ is a unit vector pointing out of the page (angular momentum points along the axis of rotation).

Calculating the derivative of this is pretty simple, since both I and $\hat{k}$ don't change in time:

$$\frac{d\vec{L}}{dt} = I\ddot{\theta}\hat{k}$$

Now, to calculate the RHS of the angular momentum conservation equation, we need to figure out the torque on the ramp. This is pretty easy too, as the only force that is being applied away from the pivot is the negative of the normal force. This is being applied perpendicular to the moment arm, so by inspection,

$$\sum \vec{\tau} = \vec{r} \times -\vec{F}_N = -rF_N\hat{k}$$

where $\vec{r}$ is the location of the skateboarder where the normal force is being applied.

Thus our angular momentum conservation equation becomes

$$I\ddot{\theta} = -rF_N$$

Now we're good to go: we have three equations, and three things we want to solve for. Manipulating algebraically, we obtain:

$$F_N = -\frac{I\ddot{\theta}}{r}$$

and

$$\ddot{\theta} = \frac{-mgr\cos\theta - 2mr\dot{r}\dot{\theta}}{mr^2 + I}$$

and finally

$$\ddot{r} = -g\sin\theta + r\dot{\theta}^2$$

Whew!

Of course, these equations only hold if the ramp free to move. Let's think about the other condition... We'll say that the ramp can rotate between θ_{min} and θ_{max}. Then, if θ drops below θ_{min}, then a couple of thing happen. First, it seems pretty clear that $\dot{\theta}$ would have to go to zero (i.e., the ramp would stop moving), and it also seems clear that the only way the ramp will start moving again is if $\ddot{\theta} > 0$ – i.e., if the skateboarder is in a location that leads to the ramp getting lifted. A similar situation applies when θ gets up to θ_{max} (although, of course, the condition for the ramp to move here is that $\ddot{\theta} < 0$).