Math 436 (Lesieutre)
Homework 4
February 16, 2022

Problem 1. Consider the function $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (2x + axy, -y)$. For what values of a is this a linear map? Prove that your answer is correct. (You should also prove it's not linear for those values of a where you think it's not linear.)

Problem 2. Consider the map $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(x, y) = (2x + y, -x + y)$.

a) Find the matrix for T with respect to the standard basis.

b) Find the matrix for T with respect to the basis $(1, 1)$, $(1, -1)$ (use this basis for both copies of $\mathbb{R}^2$).

Problem 3. Define a map $T : \mathcal{P}_3(\mathbb{R}) \to \mathcal{P}_2(\mathbb{R})$ by setting $T(f) = xf'' + 2f'$. Give bases for both spaces, and find the matrix for T with respect to these bases. Is the map injective? Justify your answer.

Problem 4. Suppose that $V = W = \mathbb{R}^2$. Consider the subset

$$\Upsilon = \{T \in \mathcal{L}(V, W) : T \text{ is not injective.}\}$$

Is a subspace of $\mathcal{L}(V, W)$? Justify your answer (either prove or disprove it).

Problem 5. Suppose that U, V and W are three vector spaces.

a) Suppose that $S : U \to V$ and $T : V \to W$ are both injective. Prove that the composition $TS : U \to W$ is also injective.

b) Suppose that S is surjective and T is injective. Must $TS : U \to W$ be injective? Justify your answer.

HW 4 Allen Gao

1. to be a linear map, let $T: V \to W$

it has to follow $T(v_1+v_2) = Tv_1 + Tv_2$

$T(nv_1) = nT(v_1)$

$T(n(x,y)) = T(xn, yn) = (2xn + \underline{an^2xy}, -yn)$

$= n(2x + anxy, -y) \neq n(2x + axy, -y)$

as we can see above, this transformation is not linear as we see an^2xy term in above linear transformation, $\therefore$ a has to be 0 for T to be a linear transformation which:

$(2xn, -yn) = n(2x, -y)$ is linear trans

$T(v_1+v_2)$ (with $a=0$): $V_1 = (x_1, y_1)$ $V_2 (x_2, y_2)$

$T((x_1+x_2), (y_1+y_2)) = (2(x_1+x_2), -(y_1+y_2))$

$= (2x_1 + 2x_2, -y_1 - y_2)$

$= (2x_1, -y_1) + (2x_2, -y_2)$

$= T(v_1) + T(v_2)$

thus, it is a linear trans. with $a = 0$

2. with standard basis: $v_1 = (1,0)$ $v_2 = (0,1)$

(a) $T(v_1) = T(1,0) = (2,-1)$

$T(v_2) = T(0,1) = (1,1)$

$T = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$

(b) $v_1 = (1,1)$ $v_2 = (1,-1)$

$T(v_1) = T(1,1) = (3,0)$

$T(v_2) = T(1,-1) = (1,-2)$

$T = \begin{bmatrix} \frac{3}{2} & \frac{3}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix}$

$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} x_1 = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 1 & 1 & 3 \\ 0 & 2 & 3 \end{pmatrix}$

$\begin{pmatrix} 1 & 0 & \frac{3}{2} \\ 0 & 1 & \frac{3}{2} \end{pmatrix}$

$T(v_1) = \frac{3}{2}v_1 + \frac{3}{2}v_2$

$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} x_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \end{pmatrix}$

$\begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{3}{2} \end{pmatrix}$

$T(v_2) = -\frac{1}{2}v_1 + \frac{3}{2}v_2$

$) = (2x + y, -x + y).$

$12x^2 + 6x + 2$

3. $\boxed{\begin{array}{l} \text{basis for } P_3(\mathbb{R}) = \{1, x, x^2, x^3\} \\ \text{basis for } P_2(\mathbb{R}) = (1, x, x^2) \end{array}}$

$T(f) = xf'' + 2f'$

$T(1) = 0 + 0 = 0$

$T(x) = 0 + 2 = 2$

$T(x^2) = 2x + 4x = 6x$

$T(x^3) = 6x^2 + 6x^2 = 12x^2$

$$T = \begin{array}{c} x^2 \\ x \\ 1 \end{array} \begin{bmatrix} 12 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

to show T is injective, $Null(T) = \{0\}$

$Tv_1 = 0$

v_1 can be $f = a$ a_1 as a constant

$\therefore \dim(Null(T)) = 1$

$$\begin{bmatrix} 12 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ a \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

not injective.

4. $\because T$ is not injective, $\therefore Null(T) \neq 0$

$T: V \to W$ and $V = W = \mathbb{R}^2$

$T(x, y) = (x, y)$ when T isn't injective, so $null(T) \neq \{0\}$

however, $Tv_1 = 0$

only if $v_1 = 0$

$\therefore Null(T) = 0$ $\therefore \gamma = \{0\}$

γ has $0 \in \gamma$

γ is closed under scalar multi: $v \cdot 0 = 0 \in \gamma$

γ is closed under addition: $0 + 0 = 0 \in \gamma$

γ is a subset of $\mathcal{L}(V, W)$

5. By the theorem learned in class

(a) $M(TS) = M(T) \cdot M(S)$

$\because T: V \to W$ $S: U \to V$ and both of them are injective $null(T) = Null(S) = \{0\}$

$\therefore$ by the transitivity of matrix multiplication.

$ST: U \to W$ and $null(ST) = \{0\}$ $\because null(T) = Null(S) = \{0\}$

ST is also injective.

(b) if S is surjective and $\dim(U) > \dim(V)$, then by the fundamental theorem, $Null(S) \neq \{0\}$ not injective.

if $S: U \to V$ not injective, then $ST: U \to W$ isn't injective.

$\therefore$ in conclusion, if $\dim(U) > \dim(V)$, ST will not be injective, if $\dim(U) = \dim(V)$, ST is still injective.