Minimum Perimeter-Sum Bipartition Implementation

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Abstract

The Minimum Perimeter-Sum Bipartition problem is stated as follows: Given set of points in the Euclidean plane, partition it into two subsets such that the sum of perimeters of their convex hulls is as small as possible. In this text, we describe our implementation of the algorithm by Abrahamsen et al [1] and provide analyses on its behaviour given different inputs.

1 Introduction

The Minimum Perimeter-Sum Bipartition problem is the problem of partitioning a set of points in the two-dimensional plane into two disjoint subsets P_1 and P_2 such that the sum of the perimeters of $\mathrm{CH}(P_1)$ and $\mathrm{CH}(P_2)$ is minimized, where $\mathrm{CH}(P)$ is the convex hull of the set of points P. In 2017, Abrahamsen et al [1] devised an algorithm that solves this problem in time $O(n \log^4 n)$, n being the number of points in the input.

1.1 The Scope of this Project

Our goal has been to assess the effects of various inputs on the running time of an implementation of the algorithm presented in [1].

We have not implemented the entire algorithm, but have concentrated on the subroutines we have identified to impose the greatest impact on the running time. We have failed to implement one of the important subroutines—the one we call `combine_convex_hulls`—within the specified time bound. We have not been able to identify a way that allows us to withhold this bound in practice. We will briefly discuss the issue by the end of this report.

We have thus implemented a number of the subroutines, and give an analysis of how different input sets affect the performance of the algorithm, and discuss our method of constructing such input sets.

Our implementation is publicly available [5].

2 Background

We here give a summary of the algorithm solving the Minimum Perimeter-Sum Bipartition Problem.

Abrahamsen et al. [1] shows that, for any optimal partition $\{P_1^*, P_2^*\}$ of the input points, the following holds:

Either the outer angle between the inner tangents of the convex hulls $CH(P_1^*)$ and $CH(P_2^*)$ (see illustration) is at least $\pi/6$, or the smallest distance between two points in P_1^* and P_2^* is at least $1/250 \cdot \min(per(P_1^*), per(P_2^*))$.

$\mathrm{per}(P)$ denotes the length of the perimeter of the convex hull of P. In figure Figure 1, we show the two inner tangents of two polygons L_1 and L_2 of the polygons, with the outer angle B in-between them.

In the remainder of the discussion, the angle B is referred to as the separation angle.

This immediately splits our approach into two cases, one where we assume the separation angle to be at least $\pi/6$, and another where we assume the distance between the sets to be "large".

We solve the problem with each of these assumptions in turn, and choose the best of the results as our answer.

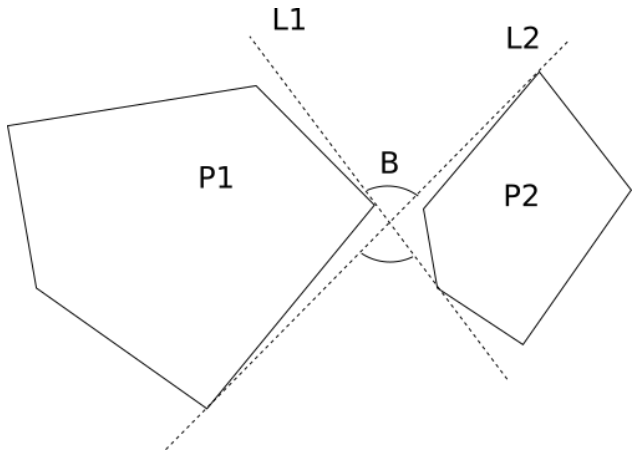


Figure 1: The inner tangents L_1 and L_2 and their outer angle B

2.1 Algorithm Assuming Large Separation Angle of Solution

In the case where the separation angle is larger than $\pi/6$, we make the following observation: Consider the set of seven angles $\Phi = \{\pi \cdot i/7 \,|\, i \in [0, 7)\}$. There must exist a line separating the optimal subsets, whose angle is among those seven. Thus, we will approach as follows:

For each of the angles $\phi \in \Phi$, we use a line l_ϕ having angle ϕ in a Graham's scan; that is, we treat l_ϕ as a scanning line in a modified version of Graham's algorithm: When first scanning from one side to the other, we compute the "upper" hull, as usual, but we also make sure to store the length of the upper hull ending at each intermediate point. The same is done for the the lower hull when scanning the other way, and the process is repeated switching the direction of scan for upper and lower hull.

Now, each point has four lengths associated with it (from being the end point for the upper and lower hull from each of the two scanning directions), which makes it easy for us to compute the total length of any convex hull ending at each point in constant time. Scanning through the points again, repeatedly using the two points immediately to each side of the scanning line and computing the length two hulls ending at these points, we can find the best possible partition with separating line of angle ϕ.

Repeating this approach for each of the seven angles, we will, by assumption of large separation angle, find the best partition overall.

This approach uses Graham's algorithm a constant number of times, and thus has a complexity of $O(n \log n)$.

2.2 Algorithm Assuming Large Separation Distance of Solution

Now, we must find the best possible partition assuming the distance between the two subsets is large.

We first treat the special case where one of the subsets is a single point p. Some thought reveal that p must lie on the convex hull of the set of input points P. We will thus go through each of the points v_i at the boundary of $\mathrm{CH}(P)$ in turn and compute the perimeter $\mathrm{per}(P \setminus \{v_i\}$.

We do this by constructing each triangle Δ_i formed by three consecutive points on the boundary v_{i-1}, v_i and v_{i+1}, and find the set P_i of the points contained inside Δ_i, excluding v_i, but including v_{i-1} and v_{i+1}. Note now that each point can belong to a maximum of two triangles. According to [1], doing this is "easy" in $O(n \log n)$ time, without specifying how. We have chosen to compute the set P_i in the following way:

- Construct a suitable center point c not contained in any triangle *We find this by taking the crossing between two diagonals*
- Sort the points by their angle around the center point, such that a chosen first boundary point v_0 has angle 0
- Initialize $t_0 = \Delta_0$, $t_1 = \Delta_1$
- Iterate over each point p_i:
  - If p_i is the end point of t_0, shift t_0 and t_1 to the next Δ
  - If p_i is inside t_0 or t_1 (or both), put it in the triangle's respective set

Finally, having found the sets P_i, we compute the convex hulls $\mathrm{CH}(P_i)$, and can find the perimeters $\mathrm{per}(P \setminus \{v_i\})$ as $\mathrm{per}(P \setminus \{v_i\}) = \mathrm{per}(P) - |v_{i-1}v_i| - |v_iv_{i+1}| + \mathrm{per}(P_i) - |v_{i-1}v_{i+1}|$.

Thus, finding the optimal partition where one of the sets is a single point, can be done in $O(n \log n)$ time.

Now we have reduced the problem to the following: Partition the set P into two non-trivial sets with smallest sum of perimeters.

[1] proceeds at this point to compute a set S of $O(n)$ axis-aligned squares which is certain to contain a so-called *good* square, a square in the plane that fully contains one of the optimal subsets, and such that one edge of the square is no larger than 18 times the perimeter of that optimal subset.

We will not go into detail as to how the set S is computed, as the technique uses a compressed quad tree [6], which is well beyond what there is place for here given reasonable prerequisites. We also find it unlikely to be a bottleneck for our analysis, which we will discuss later. The algorithm runs in $O(n \log n)$ time. For the remainder of this discussion, we will assume that we have found the set S of squares, among which at least one is *good*.

What we will do in the following, is to intersect each of the $O(n)$ square σ_i in S with a constant number of lines l_j, each line representing the boundary of a half plane h_j. For each such half plane, we will find the set of points within the square, and also within the half plane: $P \cap \sigma_i \cap h_j$.

From the way we construct these half planes (which we will review shortly), and from the assumption of large separation distance, if σ_i is a good square, the set $P \cap \sigma_i \cap h_j$ will be one of the optimal sets for some half plane h_j (Lemma 8 in [1]). We will thus compute all such sets in intersections of a half plane and a square, and pick the one optimizing the perimeter lengths. The remainder of this section will assess how we proceed to compute these set intersections and the corresponding perimeter lengths.

The half planes are constructed as follows: $4 \cdot 18/\frac{1}{250} + 1 = 18001$ points are placed evenly around the perimeter of the square. The number is chosen such that the distance between two neighboring points is less than the separation distance between the optimal sets. Now, we create the set of lines connecting every pair of these points. We thus create a set of $O(18001^2)$ lines, which are the boundaries of the half planes h_j.

The next question we have to answer, is how to find the set $P \cap \sigma_i \cap h_j$, and also $\mathrm{per}(P \cap \sigma_i \cap h_j)$ fast. Notice that this is a query for a set of points in a rectangular axis-align region, then cut against a half plane. A normal two-level range tree [2] can be used for query for sets contained in axis-align rectangles, and adding one more level for the half plane axis (perpendicular to the half plane boundary) ensures that we can now also easily find a set of points on a specific side of the half plane.

Thus, for each of the half planes constructed, we also construct a range tree. Now, note that for any square, all the lines we constructed for that square will have its counterpart lines in other squares with the same angle and relative placement within the square. Since the sorting of the lowest level of the range tree will only depend on the angle of the respective half plane, we will only need to construct a constant number of range trees, one for each half plane angle.

As mentioned, when finding the set $P \cap \sigma_i \cap h_j$ with a range tree query, we are also interested in finding its perimeter. We can do this by augmenting each node of the lowest level trees in the range tree to contain the convex hull of all nodes in the subtree rooted at that node. This can be done recursively, in the sense that the convex hull of a node can be computed by combining the hulls at its two children. We have implemented

this by computing the outer tangents (so that the two polygons lie on the same side of each tangent) using [8], and their touch points on both polygons, and tracing the original hulls, but upon hitting a touch point, switches to the corresponding touch point of the other polygon. We remark that [8] directly computes the touch point indices in the polygons, and not the tangent lines themselves.

This construction gives us a range tree that contains convex hulls at its nodes in $O(n \log^3 n)$ time and using $O(n \log^3 n)$ space. Now. a general query region of the form $\sigma_i \cap h_j$ will in $O(\log^3 n)$ time return $O(\log^3 n)$ nodes from the range tree, each covering a disjoint subset of the points in the query region. We will also make a few more disjoint queries into the range tree, in order to find $P \setminus (\sigma_i \cap h_j)$. We will need up to five more queries of the same form as the first.

Now we have found a candidate partition, the set of points within $\sigma_i \cap h_j$ and the set of points outside. We have now a collection of $O(\log^3 n)$ convex hulls representing each of the region. In the last part of this section, we show how to compute the combined hull and perimeter length of each of those collections.

The general approach to combine a set of convex hulls into one, is to scan over them using Graham's scan. We first construct a new set of polygons in the following way: Create two "events" for each polygon, one corresponding to its first (leftmost) point, and the second to its last point (rightmost) point. Sort this list, and traverse it. Imagine this as a scan line scanning over the polygons left to right. We keep track of all the currently "active" polygons in a balanced binary tree (e.g. a red-black tree), sorted on their y-coordinate (which is well defined between "active" polygons, as the they are disjoint). Whenever a the scan line hits a new polygon, it is inserted in the tree, and likewise deleted when the scan line passes it. Whenever the top node changes, (and the tree was non-empty before the change), a polygon is constructed by taking a vertical slice of the previous top polygon from this point of change to the previous point of change. Doing this until the scan line has passed all polygons, we have created a new set of polygons, each occupying disjoint ranges in the x-axes.

Now, we will run something very similar to a Graham's scan, going through the polygons from left to right (an ordering that is now well defined). When considering a new polygon to add to the hull, we check the (upper, outer) tangent between the two previous polygons and the one between the previous polygon and the new one, using the tangent algorithm in [8]. Comparing these two tangents, if they make a right turn, we add the new polygon in our list. If not, we discard the last polygon already added, and try inserting the new polygon again.

In the end, having determined which polygons belong to the hull and not, we find the hull length itself using the point indices given to us by the tangent computation. Note that we here only have described how to compute the length of the upper part of the hull. Computing the length of the lower part is analogous.

The paper [1] describes this algorithm to take $k \log m$ time, k being the number of hulls, m the total number of points in the hulls. Thus, since we will use this procedure on sets containing at most $O(\log^3 n)$ hulls, we can conclude that each such "hull combining call" takes time $O(\log^4 n)$.

2.3 Summary of Algorithm

For convenience, we summarize the main steps of the algorithm here. The highest level view is that we compute many candidate solutions and pick the best one. We compute the candidate solutions in the following ways:

- Do several Graham scans in seven different directions, to find the best bipartition with separation angle greater than $\pi/6$.
- Compute set of points contained in each triangle formed by three consecutive points on the convex hull $\mathrm{CH}(P)$ of all input points, and use this to compute length of the convex hull of $P \setminus v_i$ for each corner v_i of $\mathrm{CH}(P)$. This finds the best partition where one of the sets is trivial.
- Construct $O(n)$ squares, and intersect each of them with $O(1)$ lines. Precompute $O(1)$ three-level range trees searchable in x- and y-axis and in the direction perpendicular to one of the $O(1)$ lines. For each square and each line (which we think of the line as a two complementary half planes), find the set of points in the intersection of the square and the half plane by a query into the appropriate range tree,

and similarily, find the rest of the points in P by more queries, retrieving $O(\log^3 n)$ hulls. Combine the hulls belonging to each of these two sets in $O(\log^4)$ time, and find their lengths. This finds the best partition where the sets have a "large" separation distance.

The outline of the program flow is show in Figure 2.

3 Implementation

3.1 Perturbation

As a preprocessing step of our input points, we perform an operation inspired by *symbolic perturbation* [3] on the input points.

We want to avoid degeneracies in our input set. That is, we want to avoid having three distinct point lying on the same line. Such colinear points form special edge cases to several of our algorithms, which would increase the complexity of our implementation significantly.

We have chosen a very simple approach to address this that has worked well in our analysis. Each point p_i is displaced by a small vector ϵ_i drawn from a normal distribution. Note that the sum of perimeters of the optimal bipartition cannot change by more than a constant times $\max|\epsilon_i|$, and that the difference of the sum of perimeters in an optimal partition and a suboptimal one is non-infinitesimal. This means that choosing ϵ_i small enough will ensure that the optimal bipartition we find is indeed optimal when using the original coordinates of the points as well.

Naturally, this step is not a deterministic solution to the problem of degeneracy, but suffices in our analysis under controlled conditions. Real implementations would do wisely in making this process deterministic.

3.2 Hierarchy of Subroutines

Here follows a description of the most important subroutines found in the algorithm. Refer to figure 2 for an overview of the relations between them.

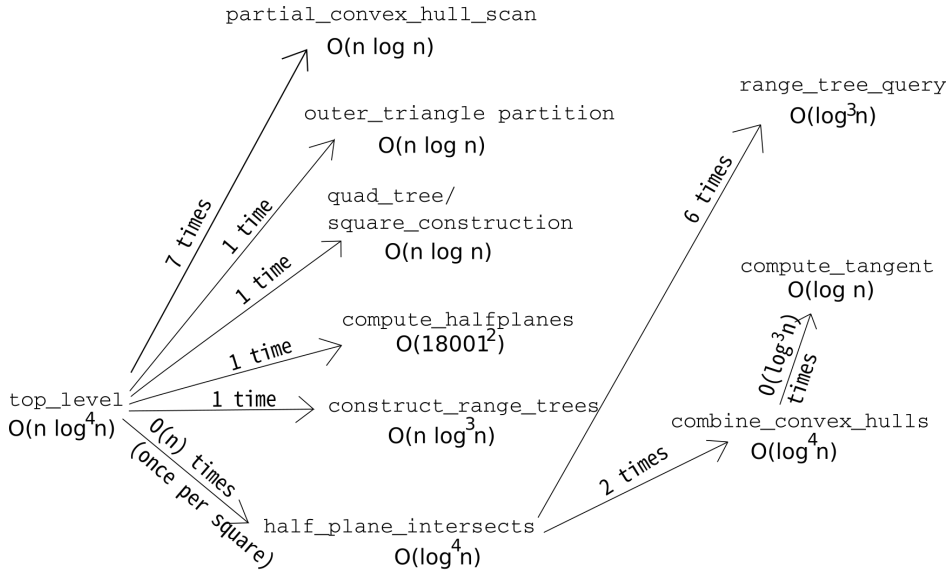


Figure 2: Structural hierarchy of subroutines

3.2.1 Partial Convex Hull Scan

For a given angle ϕ and a set of points P, this subroutine computes the upper and lower halves of the convex hull of P, scanning through the nodes like in Graham's scan, but along a line with angle ϕ to the x-axis. This allows us to quickly compute convex hulls of a bipartition of P separated by a line perpendicular to the scan direction. This operation takes $O(n \log n)$ time for a given angle, n being the size of P.

3.2.2 Outer Triangle Partitioning

For a given set of points P, computes all triangles formed by three successive points on the set's convex hull, and finds and stores the subset of the points contained in each such triangle. This allows us to quickly compute the length of the convex hull of $P \setminus p$, where p is a point on the convex hull of P. This operation takes $O(n \log n)$ time, where $n = |P|$.

3.2.3 Quad Tree Construction

Each node in such a quad tree represents a square in the plane. Each node has four children, representing the regions of the four equal squares that make up their parent's region. The construction of a compressed quad tree over n points takes $O(n \log n)$ time, given an appropriate computational model is used.

3.2.4 Range Tree Construction and Search

The range tree we construct here has three levels. The first level sorts the points on the x-axis, the subsequent level sorts the points at each node by the y-axis. The lowest level sort the points by a given direction in the plane. This allows us to easily find points that are restricted to a given rectangular axis-aligned region in the plane, and that are also inside a specified half-plane. This range tree also compute the convex hull of each node in the lowest level of the range tree. The range tree construction takes $O(n \log^3 n)$, n being the number of points over which we construct the tree.

3.2.5 Tangent Computation

For two polygons A and B, computes their tangents using the algorithm provided in [8]. We use this to determine which polygons belongs to the hull in the *convex hull combining* subroutine. The algorithm uses $O(\log(a + b))$ time, where a and b are the number of points on the boundary of A and B respectively.

3.2.6 Convex Hull Combining

For a given set of convex hulls $\mathcal{C}$, finds the convex hull of all points on all the hulls. We use this to combine the convex hulls found from a query in the range tree described above, including its length, so that we can evaluate the perimeter. The running time for this subroutine is $O(k \log m)$, k being the number of convex polygons, and m the number of points on the hulls in total.

3.3 Choosing Subroutines to Implement

As mentioned in the introduction, we have decided to only implement a subset of the subroutines. The subroutines we have decided to cover are: `construct_range_tree(s)`, `range_tree_query`, and `combine_convex_hull`. In addition, we have included the implementation of `compute_tangent` directly adapted from [8], as this serves as a component to other subroutines.

We have chosen these subroutines because their individual performances are likely to have a huge impact on the total performance in practice. This conclusion is drawn on the basis that they are run a large constant number of times, in the order of 18001^2. We leave a note on how to optimize this in section **??**, but still leaving a large number of iterations.

We have also implemented `outer_triangle_partition`, which is a subroutine we had to design ourselves.

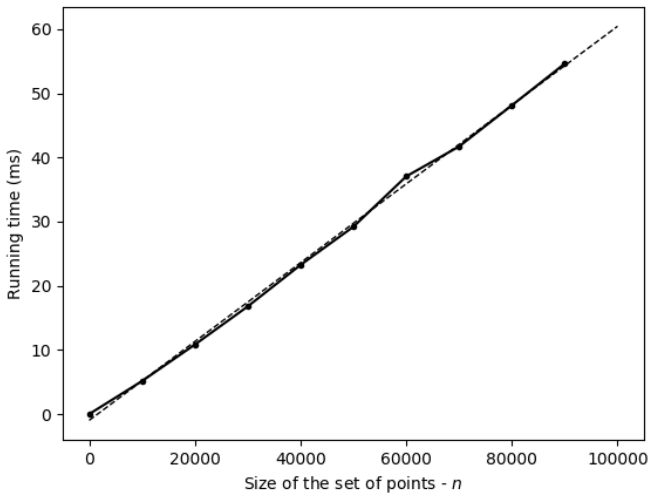
4 Performance and Input Design

4.1 Empirical Running Times

Recall from Figure 2 the structural hierarchy of this algorithm. We are mainly interested in the leaf entries of the dependency tree, as they constitute the basic subroutines of the algorithm. In this section, we perform empirical performance testing on the basic subroutines we have implemented, and discuss the relationship between the theoretical bounds outlined in [1] and our empirical results. We primarily investigate how their running times scale with respect to the size of input. The detailed hardware specifications and testing methodology are listed in Appendix A.

4.1.1 Outer Triangle Partitioning

The outer triangle partitioning, as described in 3.2.2, is an $O(n \log n)$ operation, where n denotes the size of the input set of points. The outer triangle partitioning is tested on point sets of size $n = 10^4 k + 1$ for each $k \in [0, 9]$. For each k, the set of points is generated by taking n random samples of the unit disk under uniform distribution. The results are presented in Figure 3. We derive a line of best fit $y = \alpha(n \log n) + \beta$ with $\alpha = 6.142 \times 10^{-4}$ and $\beta = -9.784 \times 10^{-1}$. This gives an empirical complexity of $t \sim 6.142 \times 10^{-4} n \log n - 9.784 \times 10^{-1}$, where t denotes the running time. The empirical running time matches the theoretical bound of $O(n \log n)$.



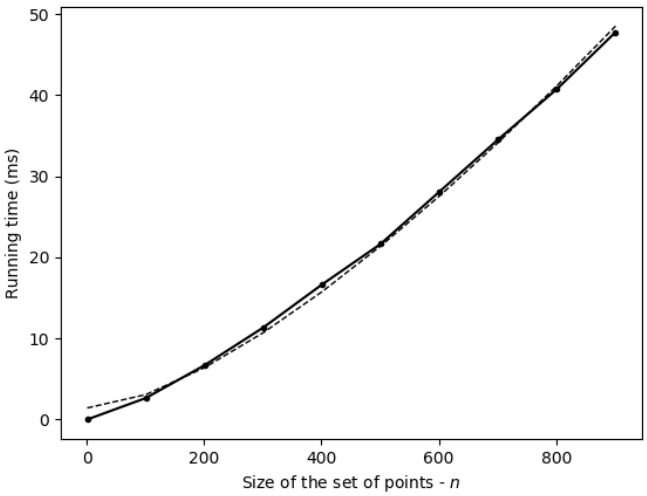
Size n	Running time (ms)
1	0.002
10001	5.145
20001	10.831
30001	16.776
40001	23.247
50001	29.197
60001	37.049
70001	41.707
80001	48.124
90001	54.627

Figure 3: Performance of the outer triangle partitioning

4.1.2 Range Tree Construction and Search

Both range tree construction and search are tested on point sets of size $n = 10^2 k + 1$ for each $k \in [0, 9]$. For each k, the set of points is generated by taking n random samples of the unit square under uniform distribution.

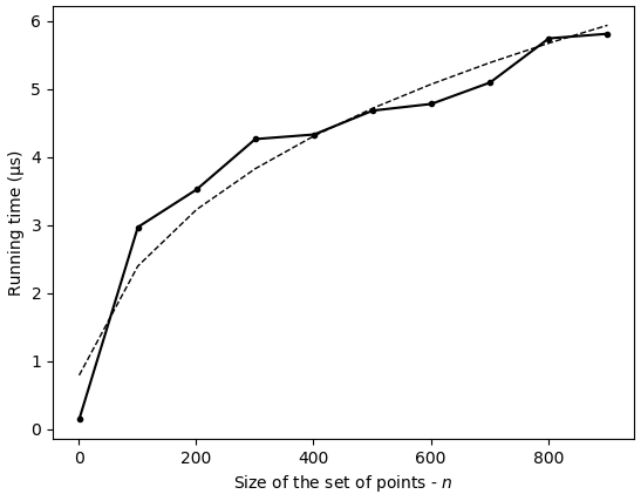
Range tree construction. As described in 3.2.4, the range tree construction is an $O(n \log^3 n)$ operation, where n denotes the size of the input set of points. The results are presented in Figure 4. We derive a line of best fit $y = \alpha n \log^3 n + \beta$ with $\alpha = 1.660 \times 10^{-4}$ and $\beta = 1.429 \times 10^0$. This gives an empirical complexity of $t \sim 1.660 \times 10^{-4} n \log^3 n + 1.429 \times 10^0$, where t denotes the running time. The empirical running time matches the theoretical bound of $O(n \log^3 n)$.



Size n	Running time (ms)
1	0.003
101	2.653
201	6.700
301	11.351
401	16.652
501	21.668
601	28.102
701	34.542
801	40.757
901	47.785

Figure 4: Performance of the range tree construction

Range tree search. The range tree search is an $O(\log^3 n)$ operation. The results are presented in Figure 5. We derive a line of best fit $y = \alpha \log^3 n + \beta$ with $\alpha = 1.633 \times 10^{-2}$ and $\beta = 7.932 \times 10^{-1}$. This gives an empirical complexity of $t \sim 1.633 \times 10^{-2} \log^3 n + 7.932 \times 10^{-1}$, where t denotes the running time. The empirical running time matches the theoretical bound of $O(\log^3 n)$.



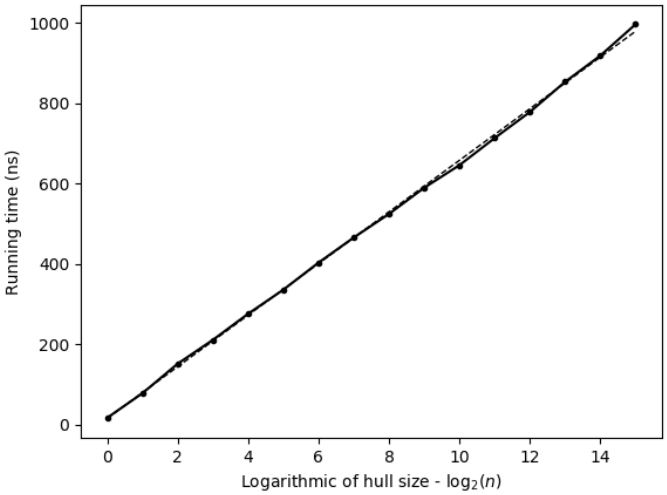
Size n	Running time (μs)
1	0.146
101	2.970
201	3.523
301	4.263
401	4.330
501	4.682
601	4.780
701	5.095
801	5.473
901	5.810

Figure 5: Performance of the range tree search

4.1.3 Tangent Computation

The tangent computation, as described in 3.2.5, is an $O(\log(a+b))$ subroutine, where a and b are the sizes of each polygon respectively. The tangent computation subroutine is tested on polygons of size $n = 2^k$ for each $k \in [0, 15]$. For each k, the two polygons of size 2^k are independently generated by taking 2^k random

samples of the unit circle under uniform distribution. The results are presented in Figure 6. Empirically, the running time of the tangent computation is directly proportional to $\log_2(n)$. From the data, we derive a line of best fit $y = \alpha k + \beta$ with $\alpha = 64.273$ and $\beta = 14.939$. This gives an empirical complexity of $t \sim 64.273 \log_2(n) + 14.939$, where t denotes the running time. The empirical running time matches the theoretical bound of $O(\log(a + b))$.



$\log_2(n)$	Running time (ns)
0	16.191
1	77.376
2	150.99
3	210.307
4	275.398
5	335.235
6	402.821
7	465.305
8	524.074
9	589.302
10	645.533
11	713.073
12	778.048
13	853.159
14	918.426
15	996.516

Figure 6: Performance of the tangent computation

4.2 Designing Adversarial Inputs

4.2.1 Identifying the Performance Bottleneck

In this section, we aim to identify the characteristics of "adversarial inputs", that is, inputs that impose a worst-case scenario on one or more subroutines of the algorithm. Recall from Figure 2 that the $O(n \log^4)$ overall running time of this algorithm is produced by $O(n)$ calls to the $O(\log^4 n)$ subroutine of `half_plane_intersects`, which can be further dissected into $O(\log^3 n)$ calls to the $O(\log n)$ tangent computation subroutine (`compute_tangent`). Therefore, the critical chain for acheiving this $O(n \log^4 n)$ is as Figure 7.

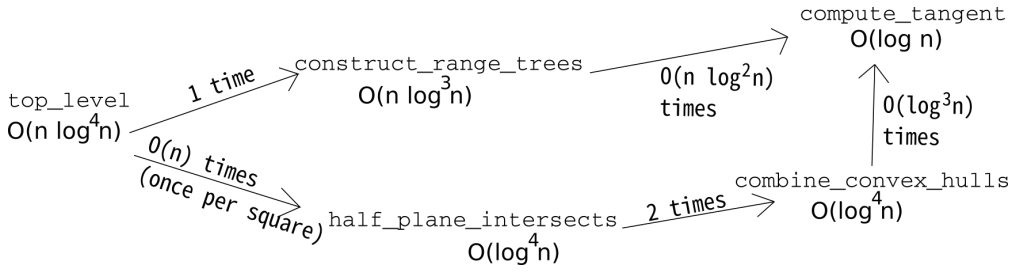


Figure 7: The critical chain for acheiving $O(n \log^4 n)$ running time

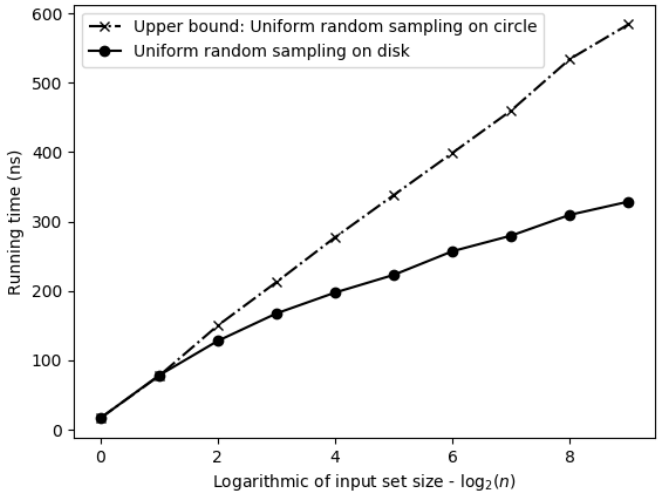
In the theoretical analysis by Abrahamsem et al. [1], the $O(\log^4 n)$ running time bound is given as a function of the input size n. Given a fixed n, the running time bound does not include any characterization of the distribution of the set of input points. Nonetheless, observe that in the tangent computation subroutine,

the running time is a function of the sizes of each polygon, rather than the number of points contained within each polygon. Therefore, it is necessary for us to investigate how the distribution of points within a point set impacts the performance of the tangent computation subroutine, as well as the entire algorithm.

To be concrete, given a process of generating a point set P, we are interested in the expected complexity of $\mathrm{CH}(P)$. We first formalize the process of generating a point set. Given a bounded convex set $\mathcal{R} \subset \mathbb{R}^2$, as well as the number of points n. A point set $P_{\mathcal{R}}^D$ is generated via n independent random samples of $\mathcal{R}$ under distribution D. Let $h_{\mathcal{R}}(n)^D$ denote the expected size of $\mathrm{CH}(P)$. Efron [4] showed that if the expected area of $\mathrm{CH}(P)$ is at least $(1 - f(n))\mathrm{Area}(\mathcal{R})$, where $1 \geq f(n) \geq 0$ for $n \geq 0$, then the expected size of $\mathrm{CH}(P)$ is at most $nf(n/2)$ under uniform distribution. Based on this result, Har-Peled [7] showed that if $\mathcal{R}$ resembles a convex polygon with k sides, then $h_{\mathcal{R}}^U = O(k \log n)$, where U denotes a uniform distribution over $\mathcal{R}$. And if $\mathcal{R}$ resembles a disk, then $h_{\mathcal{R}}^U = O(n^{1/3})$. On a more practical note, Raynaud [9], via integral estimation, showed that under $D \equiv N$ a normal distribution over $\mathcal{R}$, $h_{\mathcal{R}}^N = O((\log(n)^{1/2})$. Based on these findings, we will attempt to design a worst-case input for a fixed point-set size n selected via independent uniform samples over a convex region. We will then compare this result with the point set selected under normal distribution. For completeness, we will also include the extreme case, where all points lie on the convex hull of the point set.

4.2.2 Uniform Random Sampling on a Disk

We test the tangent computation subroutine with input sets derived by uniform random sampling on a disk. The tangent computation subroutine is tested on input set size $n = 2^k$ for each $k \in [0, 9]$. For each k, the two point sets of size 2^k are independently generated by taking 2^k random samples on a unit disk under uniform distribution. The results are presented in Figure 8. Observe that sampling on a disk exhibits a sublinear pattern with respect to $\log_2(n)$, as compared to the linear pattern exhibited by sampling directly on a circle. This observation is qualitatively in line with the $O(n^{1/3})$ expected hull size by Har-Peled [7].



$\log_2(n)$	Running time (ns)
0	16.371
1	78.192
2	127.893
3	167.572
4	197.670
5	223.183
6	257.281
7	279.767
8	309.755
9	328.822

Figure 8: Performance of uniform random sampling on disk

4.2.3 Uniform Random Sampling in a Polygon

Next, we test the tangent computation subroutine with input sets derived by uniform random sampling in the area enclosed by a polygon. The tangent computation subroutine is tested on input size $n = 2^k$ for each $k \in [0, 9]$. For each k, the two point sets of size 2^k are independently generated by taking 2^k random samples

in the area enclosed by a polygon under uniform distribution. For this test, regular polygons of 3, 4 and 5 sides are used. The results are presented in Figure 9 and Table 1. Observe that sampling in a polygon exhibits a sublinear pattern with respect to $\log_2(n)$. Addtionally, the running time increases as the number of sides γ increases. These findings are in line with the $O(\gamma \log n)$ expected hull size by Har-Peled [7].

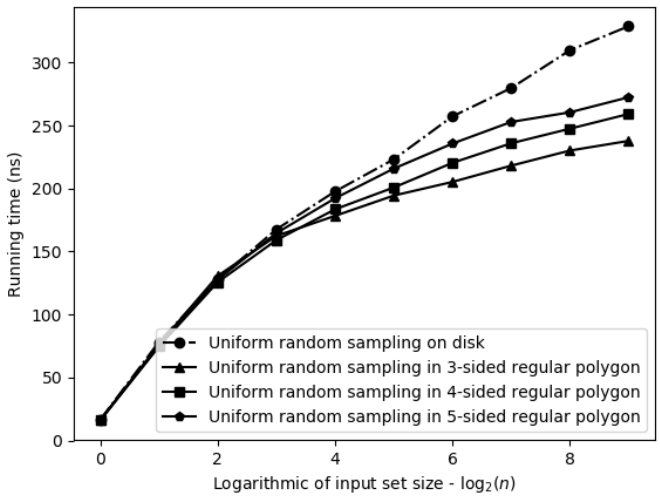


Figure 9: Performance of uniform random sampling in polygons

$\log_2(n)$	3-sided time (ns)	4-sided time (ns)	5-sided time (ns)
0	15.965	15.952	16.088
1	76.272	74.553	75.002
2	130.562	125.597	127.57
3	162.382	158.997	164.573
4	178.309	183.507	192.230
5	194.411	200.815	215.828
6	205.253	220.388	235.682
7	218.142	235.914	252.868
8	230.136	247.495	260.496
9	237.814	259.173	272.446

Table 1: Performance statistics of uniform random sampling in polygons

5 Implementation Difficulties

After working with this algorithm, we have identified a couple of key issues concerning the practical use of the algorithm.

5.1 RAM Usage

As we described, the algorithm requires us (at least if we wish to withhold the running time) to build and keep about 18000^2 range trees in memories, one for each pair of 18001 points, each such tree of size $O(nlog^3n)$. An optimization that minimizes this number a little, is to exploit that many of the lines will

be parallel: Since the range trees computed over the lines only depend on the angles of the lines, we can avoid many duplicate trees by pruning the ones corresponding to equal angles. A simple computation (see `/count_unique_line_orientations` in [5]) showed that the number of remaining unique line orientations was 63,431,423, which still imposes a huge memory consumption for large n.

5.2 The Convex Hull Combining Procedure

As mentioned before, we use a subroutine taking in k convex hulls consisting of m points in total. The paper claims that this subroutine runs in $O(k \log m)$ time. Our implementation of the routine, however, uses $O(m + k \log m)$ time, which shifts the total running time of the algorithm from $O(n \log^4 n)$ to $O(n^2)$ in the worst case.

The main problem is that our implementation computes each vertically sliced polygons explicitly, which automatically ramps up the running time to $O(m)$. We have failed to identify a proper way of implicitly storing these new polygons in a way that is suitable for the subsequent tangent computation we desire.

6 Conclusion

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Appendix A Specifications and Testing Methodology

A.1 Testing Environment

All performance testings in this report are completed on a single 4.7GHz physical thread on an `x86-64` processor. The memory in use is of DDR4 at 3200MHz (PC4 25600). Testing is completed under a virtualized instance of Arch Linux distribution via VT-d, with Linux kernel version `4.19.8` and `gcc` version `8.2.1 20181127`.

A.2 Testing Methodology

The testing methodologies applied for each subroutine are listed as follows:

A.2.1 Outer Triangle Partitioning

In performance testing, given target size n, the point set of size n is generated as follows:

1. Define the center of the sampling region as $(0, 0)$.
2. Define the sampling region to be a disk of radius 500.
3. Points are selected via uniform sampling over the disk region via the discard method.

For each given n, we generate 10^2 random point sets. For each point set, the outer triangle partitioning operation is performed once. The result is obtained via averaging these samples.

A.2.2 Range Tree Construction and Search

In performance testing, given target size n, the point set of size n is generated as follows:

1. Define the sampling region as a square of side length 1000 defined by the following vertices: $(500, 500)$, $(500, -500)$, $(-500, 500)$, $(-500, -500)$.
2. Points are selected via uniform random sampling over the sampling region.

Additionally, the following definitions are used:

1. The vector perpendicular to the x-y separator halfplane (`halfplane_dir`) is defined to be $(1, 1)$.
2. The region of the range tree query is defined to be the square on the following vertices: $(500, 500)$, $(500, -500)$, $(-500, 500)$, $(-500, -500)$.
3. The comparator point is defined to be $(0, 0)$.

For the construction test, we generate 10^1 random point sets. For each point set, the range tree construction operation is performed once. For the search test, we generate 10^1 random point sets. For each point set, the range tree construction operation is performed 10^5 times. The result is obtained via averaging in each category.

A.2.3 Tangent Computation

In performance testing, given convex hull complexity k, the pair of polygons is generated as follows:

1. Define the center of the left polygon as $(-1000, 0)$, and the center of the right polygon as $(1000, 0)$.
2. For each polygon, define a circle of radius 500 with its respective center.
3. For each polygon, generate $\theta \in [0, 2\pi)$ via random sampling. Find the point on the polygon's circle that corresponds to θ. Repeat this process k times. A size k point set via independent uniform sampling on the circle is thus generated. Observe that the convex hull of this point set would have size exactly k.

We generate 10^3 random pairs of polygons following this methodology. For each pair of polygons, the tangent computation is performed 10^6 times. The result is obtained via averaging these samples.

In input analysis, given convex point set size n and distribution D, the pair of point sets is generated as follows:

1. Define the center of the left sampling region as $(-1000, 0)$, and the center of the right sampling region as $(1000, 0)$.
2. For each sampling region, we identify a bounding box of the sampling region, and apply the distribution D on the bounding box, and generate the point sets via the discard method. Specifically,
   (a) If the sampling region is a regular polygon, we define its radius, that is, the distance from its center to any of its vertices, to be 500.
   (b) If the sampling region is a disk, we define its radius to be 500.
   (c) If the sampling region is a circle (for the degenerate extreme case), sampling is performed in the same way as in performance testing.

We generate 10^3 random pairs of point sets following this methodology. For each pair of point sets, the tangent computation is performed 10^5 times. The result is obtained via averaging these samples.