The Power Method Algorithm

1 Introduction

The **power method** (also called **power iteration**) is a classic iterative algorithm for computing the dominant eigenvalue and its corresponding eigenvector of a matrix. Given a square matrix $A \in \mathbb{R}^{n \times n}$, the method finds the eigenvalue with the largest absolute value, denoted λ_1, and its associated eigenvector $\mathbf{v}_1$.

2 Algorithm Statement

2.1 Initialization

- Choose an initial vector $\mathbf{x}^{(0)} \in \mathbb{R}^n$ such that $\mathbf{x}^{(0)} \neq \mathbf{0}$.
- Typically, $\mathbf{x}^{(0)}$ is chosen randomly or as $\mathbf{x}^{(0)} = (1, 1, \ldots, 1)^T$.
- Normalize: $\mathbf{x}^{(0)} \leftarrow \frac{\mathbf{x}^{(0)}}{\|\mathbf{x}^{(0)}\|}$ (optional but recommended).
- Set convergence tolerance $\epsilon > 0$ and maximum iterations $k_{\max}$.

2.2 Iteration Step

For $k = 0, 1, 2, \ldots$ until convergence:

1. **Matrix-vector multiplication:**

$$\mathbf{y}^{(k+1)} = A\mathbf{x}^{(k)} \tag{1}$$

2. **Normalization:**

$$\mathbf{x}^{(k+1)} = \frac{\mathbf{y}^{(k+1)}}{\|\mathbf{y}^{(k+1)}\|} \tag{2}$$

   where $\|\cdot\|$ denotes a vector norm (typically the Euclidean norm $\|\cdot\|_2$).

3. **Eigenvalue estimate:** Compute the Rayleigh quotient

$$\lambda^{(k+1)} = \frac{(\mathbf{x}^{(k+1)})^T A \mathbf{x}^{(k+1)}}{(\mathbf{x}^{(k+1)})^T \mathbf{x}^{(k+1)}} = (\mathbf{x}^{(k+1)})^T A \mathbf{x}^{(k+1)} \tag{3}$$

   where the second equality holds if $\mathbf{x}^{(k+1)}$ is normalized.

   Alternatively, if the largest eigenvalue is known to be positive, one can use:

$$\lambda^{(k+1)} = \|\mathbf{y}^{(k+1)}\| = \|A\mathbf{x}^{(k)}\| \tag{4}$$

2.3 Convergence Criteria

The iteration terminates when one of the following conditions is met:

1. **Eigenvalue convergence:**
$$|\lambda^{(k+1)} - \lambda^{(k)}| < \epsilon \tag{5}$$

2. **Eigenvector convergence:**
$$\|\mathbf{x}^{(k+1)} - \mathbf{x}^{(k)}\| < \epsilon \tag{6}$$
or
$$\|\mathbf{x}^{(k+1)} + \mathbf{x}^{(k)}\| < \epsilon \tag{7}$$
(the second condition accounts for sign ambiguity in eigenvectors).

3. **Residual criterion:**
$$\|A\mathbf{x}^{(k+1)} - \lambda^{(k+1)}\mathbf{x}^{(k+1)}\| < \epsilon \tag{8}$$

4. **Maximum iterations reached:** $k \geq k_{\max}$

The most commonly used criterion is the eigenvalue convergence test (1) or the residual criterion (3).

3 Formal Algorithm

Algorithm 1 Power Method

Require: Matrix $A \in \mathbb{R}^{n \times n}$, tolerance $\epsilon > 0$, max iterations $k_{\max}$
Ensure: Dominant eigenvalue λ and eigenvector $\mathbf{x}$

Choose initial vector $\mathbf{x}^{(0)} \in \mathbb{R}^n$, $\mathbf{x}^{(0)} \neq \mathbf{0}$
$\mathbf{x}^{(0)} \leftarrow \mathbf{x}^{(0)} / \|\mathbf{x}^{(0)}\|$
$\lambda^{(0)} \leftarrow 0$
for $k = 0, 1, 2, \ldots, k_{\max} - 1$ **do**
    $\mathbf{y}^{(k+1)} \leftarrow A\mathbf{x}^{(k)}$
    $\mathbf{x}^{(k+1)} \leftarrow \mathbf{y}^{(k+1)} / \|\mathbf{y}^{(k+1)}\|$
    $\lambda^{(k+1)} \leftarrow (\mathbf{x}^{(k+1)})^T A \mathbf{x}^{(k+1)}$
    if $|\lambda^{(k+1)} - \lambda^{(k)}| < \epsilon$ **then**
        return $\lambda^{(k+1)}, \mathbf{x}^{(k+1)}$
    end if
end for
return $\lambda^{(k_{\max})}, \mathbf{x}^{(k_{\max})}$ ▷ Max iterations reached

4 Convergence Properties

Theorem 1 (Convergence of Power Method)**.** *Let $A \in \mathbb{R}^{n \times n}$ be diagonalizable with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that*
$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n| \tag{9}$$
and corresponding eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ that form a basis for $\mathbb{R}^n$. If the initial vector $\mathbf{x}^{(0)}$ has a nonzero component in the direction of $\mathbf{v}_1$, then the power method converges:
$$\lambda^{(k)} \to \lambda_1 \quad \textit{and} \quad \mathbf{x}^{(k)} \to \pm\mathbf{v}_1 \quad \textit{as } k \to \infty \tag{10}$$

Remark 1 (Rate of Convergence)**.** *The rate of convergence is determined by the ratio* $\left|\frac{\lambda_2}{\lambda_1}\right|$*. Specifically, the error decreases approximately as:*

$$Error^{(k)} \approx C \left|\frac{\lambda_2}{\lambda_1}\right|^k \tag{11}$$

where C *is a constant depending on the initial vector. The convergence is* ***linear*** *with rate* $\left|\frac{\lambda_2}{\lambda_1}\right|$*.*

Remark 2 (Conditions for Convergence)**.** *The power method requires:*

- *The matrix* A *must have a unique dominant eigenvalue:* $|\lambda_1| > |\lambda_2|$
- *The initial vector must have a nonzero component in the direction of the dominant eigenvector*
- *The matrix need not be symmetric*

5 Numerical Example

Consider a 5×5 symmetric matrix with eigenvalues $\{10, 5, 3, 1, 0.5\}$. Applying the power method with a random initial vector demonstrates the convergence behavior shown in Figure **??**.

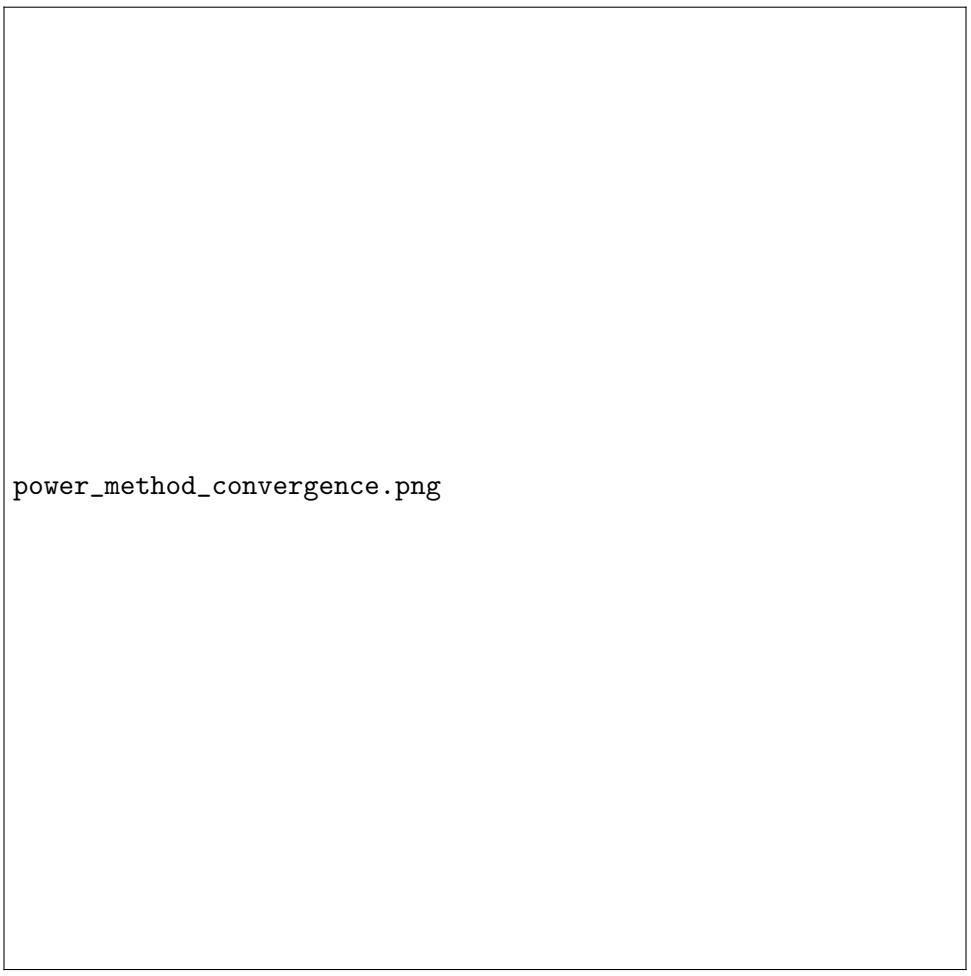

Figure 1: Convergence of the power method. The error in the eigenvalue estimate decreases exponentially, exhibiting linear convergence with rate $|\lambda_2/\lambda_1| = 5/10 = 0.5$.

The method successfully computes the dominant eigenvalue $\lambda_1 = 10$ to high accuracy within a few dozen iterations.

6 Variants and Extensions

Several variants of the power method address specific computational needs:

- **Inverse Power Method:** Applies the power method to A^{-1} to find the smallest eigenvalue
- **Shifted Inverse Power Method:** Uses $(A - \mu I)^{-1}$ to find the eigenvalue closest to a shift μ
- **Rayleigh Quotient Iteration:** Adaptive shift strategy for faster convergence
- **Orthogonal Iteration:** Generalizes to find multiple dominant eigenvalues and eigenvectors

7 Computational Complexity

Each iteration of the power method requires:

- One matrix-vector multiplication: $O(n^2)$ operations for dense matrices, potentially $O(n)$ for sparse matrices
- One vector normalization: $O(n)$ operations
- One Rayleigh quotient computation: $O(n^2)$ operations for dense matrices

The total cost per iteration is $O(n^2)$ for dense matrices and potentially much less for sparse matrices, making the power method particularly attractive for large sparse eigenvalue problems.