CHAPTER 1

Introduction

In this first chapter we provide a foundation for your study of differential equations in several different ways. First, we use two problems to illustrate some of the basic ideas that we will return to, and elaborate upon, frequently throughout the remainder of the book. Later, to provide organizational structure for the book, we indicate several ways of classifying differential equations.

The study of differential equations has attracted the attention of many of the world's greatest mathematicians during the past three centuries. On the other hand, it is important to recognize that differential equations remains a dynamic field of inquiry today, with many interesting open questions. We outline some of the major trends in the historical development of the subject and mention a few of the outstanding mathematicians who have contributed to it. Additional biographical information about some of these contributors will be highlighted at appropriate times in later chapters.

1.1 Some Basic Mathematical Models; Direction Fields

Before embarking on a serious study of differential equations (for example, by reading this book or major portions of it), you should have some idea of the possible benefits to be gained by doing so. For some students the intrinsic interest of the subject itself is enough motivation, but for most it is the likelihood of important applications to other fields that makes the undertaking worthwhile.

Many of the principles, or laws, underlying the behavior of the natural world are statements or relations involving rates at which things happen. When expressed in mathematical terms, the relations are equations and the rates are derivatives. Equations containing derivatives are **differential equations**. Therefore, to understand and to investigate problems involving the motion of fluids, the flow of current in electric circuits, the dissipation of heat in solid objects, the propagation and detection of seismic waves, or the increase or decrease of populations, among many others, it is necessary to know something about differential equations.

A differential equation that describes some physical process is often called a **mathematical model** of the process, and many such models are discussed throughout this book. In this section we begin with two models leading to equations that are easy to solve. It is noteworthy that even the simplest differential equations provide useful models of important physical processes.

EXAMPLE 1 A Falling Object

Suppose that an object is falling in the atmosphere near sea level. Formulate a differential equation that describes the motion.

Solution:

We begin by introducing letters to represent various quantities that may be of interest in this problem. The motion takes place during a certain time interval, so let us use t to denote time. Also, let us use v to represent the velocity of the falling object. The velocity will presumably change with time, so we think of v as a function of t; in other words, t is the independent variable and v is the dependent variable. The choice of units of measurement is somewhat arbitrary, and there is nothing in the statement of the problem to suggest appropriate units, so we are free to make any choice that seems reasonable. To be specific, let us measure time t in seconds and velocity v in meters/second. Further, we will assume that v is positive in the downward direction—that is, when the object is falling.

The physical law that governs the motion of objects is **Newton's second law**, which states that the mass of the object times its acceleration is equal to the net force on the object. In mathematical terms this law is expressed by the equation

$$F = ma, \tag{1}$$

where m is the mass of the object, a is its acceleration, and F is the net force exerted on the object. To keep our units consistent, we will measure m in kilograms, a in meters/second^2, and F in newtons. Of course, a is related to v by $a = dv/dt$, so we can rewrite equation (1) in the form

$$F = m\frac{dv}{dt}. \tag{2}$$

Next, consider the forces that act on the object as it falls. Gravity exerts a force equal to the weight of the object, or mg, where g is the acceleration due to gravity. In the units we have chosen, g has been determined experimentally to be approximately equal to 9.8 m/s^2 near the earth's surface.

There is also a force due to air resistance, or drag, that is more difficult to model. This is not the place for an extended discussion of the drag force; suffice it to say that it is often assumed that the drag is proportional to the velocity, and we will make that assumption here. Thus the drag force has the magnitude γv, where γ is a constant called the drag coefficient. The numerical value of the drag coefficient varies widely from one object to another; smooth streamlined objects have much smaller drag coefficients than rough blunt ones. The physical units for γ are mass/time, or kg/s for this problem; if these units seem peculiar, remember that γv must have the units of force, namely, kg·m/s^2.

In writing an expression for the net force F, we need to remember that gravity always acts in the downward (positive) direction, whereas, for a falling object, drag acts in the upward (negative) direction, as shown in Figure 1.1.1. Thus

$$F = mg - \gamma v \tag{3}$$

and equation (2) then becomes

$$m\frac{dv}{dt} = mg - \gamma v. \tag{4}$$

Differential equation (4) is a mathematical model for the velocity v of an object falling in the atmosphere near sea level. Note that the model contains the three constants m, g, and γ. The constants m and γ depend very much on the particular object that is falling, and they are usually different for different objects. It is common to refer to them as parameters, since they may take on a range of values during the course of an experiment. On the other hand, g is a physical constant, whose value is the same for all objects.

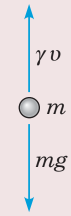

FIGURE 1.1.1 Free-body diagram of the forces on a falling object.

To solve equation (4), we need to find a function $v = v(t)$ that satisfies the equation. It is not hard to do this, and we will show you how in the next section. For the present, however, let us see what we can learn about solutions without actually finding any of them. Our task is simplified slightly if we assign numerical values to m and γ, but the procedure is the same regardless of which values we choose. So, let us suppose that $m = 10$ kg and $\gamma = 2$ kg/s. Then equation (4) can be rewritten as

$$\frac{dv}{dt} = 9.8 - \frac{v}{5}. \tag{5}$$

EXAMPLE 2 A Falling Object (continued)

Investigate the behavior of solutions of equation (5) without solving the differential equation.

Solution:

First let us consider what information can be obtained directly from the differential equation itself. Suppose that the velocity v has a certain given value. Then, by evaluating the right-hand side of differential equation (5), we can find the corresponding value of dv/dt. For instance, if $v = 40$, then $dv/dt = 1.8$. This means that the slope of a solution $v = v(t)$ has the value 1.8 at any point where $v = 40$. We can display this information graphically in the tv-plane by drawing short line segments with slope 1.8 at several points on the line $v = 40$. (See Figure 1.1.2(a)). Similarly, when $v = 50$, then $dv/dt = -0.2$, and when $v = 60$, then $dv/dt = -2.2$, so we draw line segments with slope -0.2 at several points on the line $v = 50$ (see Figure 1.1.2(b)) and line segments with slope -2.2 at several points on the line $v = 60$ (see Figure 1.1.2(c)). Proceeding in the same way with other values of v we create what is called a **direction field**, or a **slope field**. The direction field for differential equation (5) is shown in Figure 1.1.3.

Remember that a solution of equation (5) is a function $v = v(t)$ whose graph is a curve in the tv-plane. The importance of Figure 1.1.3 is that each line segment is a tangent line to one of these solution curves. Thus, even though we have not found any solutions, and no graphs of solutions appear in the figure, we can nonetheless draw some qualitative conclusions about the behavior of solutions. For instance, if v is less than a certain critical value, then all the line segments have positive slopes, and the speed of the falling object increases as it falls. On the other hand, if v is greater than the critical value, then the line segments have negative slopes, and the falling object slows down as it falls. What is this critical value of v that separates objects whose speed is increasing from those whose speed is decreasing? Referring again to equation (5), we ask what value of v will cause dv/dt to be zero. The answer is $v = (5)(9.8) = 49$ m/s.

In fact, the constant function $v(t) = 49$ is a solution of equation (5). To verify this statement, substitute $v(t) = 49$ into equation (5) and observe that each side of the equation is zero. Because it does not change with time, the solution $v(t) = 49$ is called an **equilibrium solution**. It is the solution that corresponds to a perfect balance between gravity and drag. In Figure 1.1.3 we show the equilibrium solution $v(t) = 49$ superimposed on the direction field. From this figure we can draw another conclusion, namely, that all other solutions seem to be converging to the equilibrium solution as t increases. Thus, in this context, the equilibrium solution is often called the **terminal velocity**.

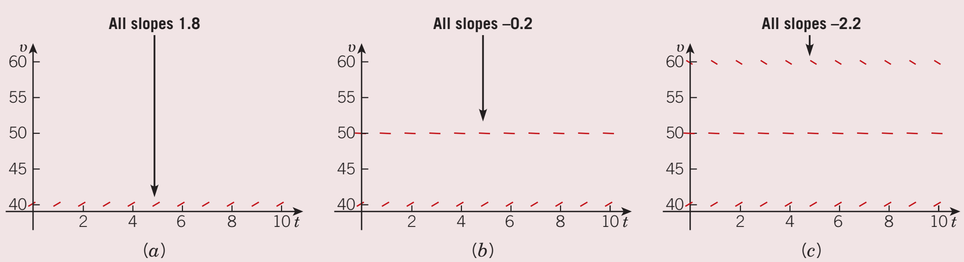

FIGURE 1.1.2 Assembling a direction field for equation (5): $dv/dt = 9.8 - v/5$. (a) when $v = 40$, $dv/dt = 1.8$, (b) when $v = 50$, $dv/dt = -0.2$, and (c) when $v = 60$, $dv/dt = -2.2$.

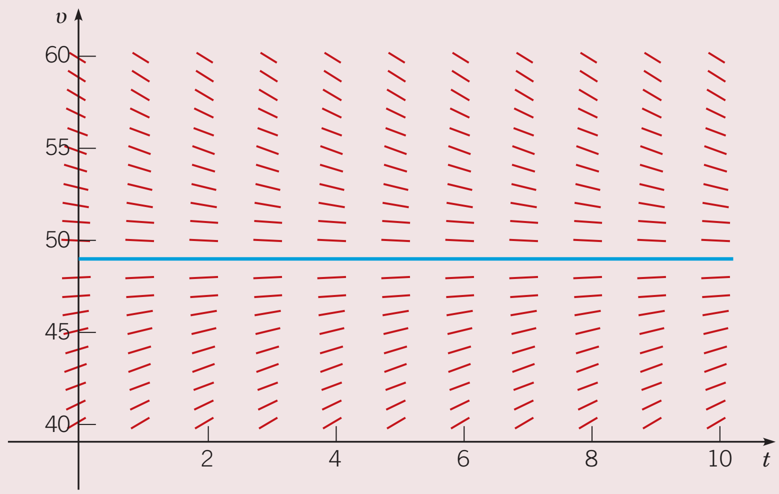

FIGURE 1.1.3 Direction field and equilibrium solution for equation (5): $dv/dt = 9.8 - v/5$.

The approach illustrated in Example 2 can be applied equally well to the more general differential equation (4), where the parameters m and γ are unspecified positive numbers. The results are essentially identical to those of Example 2. The equilibrium solution of equation (4) is the constant solution $v(t) = mg/\gamma$. Solutions below the equilibrium solution increase with time, and those above it decrease with time. As a result, we conclude that all solutions approach the equilibrium solution as t becomes large.

Direction Fields. Direction fields are valuable tools in studying the solutions of differential equations of the form

$$\frac{dy}{dt} = f(t, y), \tag{6}$$

where f is a given function of the two variables t and y, sometimes referred to as the **rate function**. A direction field for equations of the form (6) can be constructed by evaluating f at each point of a rectangular grid. At each point of the grid, a short line segment is drawn whose slope is the value of f at that point. Thus each line segment is tangent to the graph of the solution passing through that point. A direction field drawn on a fairly fine grid gives a good picture of the overall behavior of solutions of a differential equation. Usually a grid consisting of a few hundred points is sufficient. The construction of a direction field is often a useful first step in the investigation of a differential equation.

Two observations are worth particular mention. First, in constructing a direction field, we do not have to solve equation (6); we just have to evaluate the given function $f(t, y)$ many times. Thus direction fields can be readily constructed even for equations that may be quite difficult to solve. Second, repeated evaluation of a given function and drawing a direction field are tasks for which a computer or other computational or graphical aid are well suited. All the direction fields shown in this book, such as the one in Figures 1.1.2 and 1.1.3, were computer generated.

Field Mice and Owls. Now let us look at another, quite different example. Consider a population of field mice that inhabit a certain rural area. In the absence of predators we assume that the mouse population increases at a rate proportional to the current population. This assumption is not a well-established physical law (as Newton's law of motion is in Example 1), but it is a common initial hypothesis[1] in a study of population growth. If we denote time by t and the mouse population at time t by $p(t)$, then the assumption about population growth can be expressed by the equation

$$\frac{dp}{dt} = rp, \tag{7}$$

[1] A better model of population growth is discussed in Section 2.5.

where the proportionality factor r is called the **rate constant** or **growth rate**. To be specific, suppose that time is measured in months and that the rate constant r has the value 0.5/month. Then the two terms in equation (7) have the units of mice/month.

Now let us add to the problem by supposing that several owls live in the same neighborhood and that they kill 15 field mice per day. To incorporate this information into the model, we must add another term to the differential equation (7), so that it becomes

$$\frac{dp}{dt} = \frac{p}{2} - 450. \tag{8}$$

Observe that the predation term is -450 rather than -15 because time is measured in months, so the monthly predation rate is needed.

EXAMPLE 3

Investigate the solutions of differential equation (8) graphically.

Solution:

A direction field for equation (8) is shown in Figure 1.1.4. For sufficiently large values of p it can be seen from the figure, or directly from equation (8) itself, that dp/dt is positive, so that solutions increase. On the other hand, if p is small, then dp/dt is negative and solutions decrease. Again, the critical value of p that separates solutions that increase from those that decrease is the value of p for which dp/dt is zero. By setting dp/dt equal to zero in equation (8) and then solving for p, we find the equilibrium solution $p(t) = 900$, for which the growth term and the predation term in equation (8) are exactly balanced. The equilibrium solution is also shown in Figure 1.1.4.

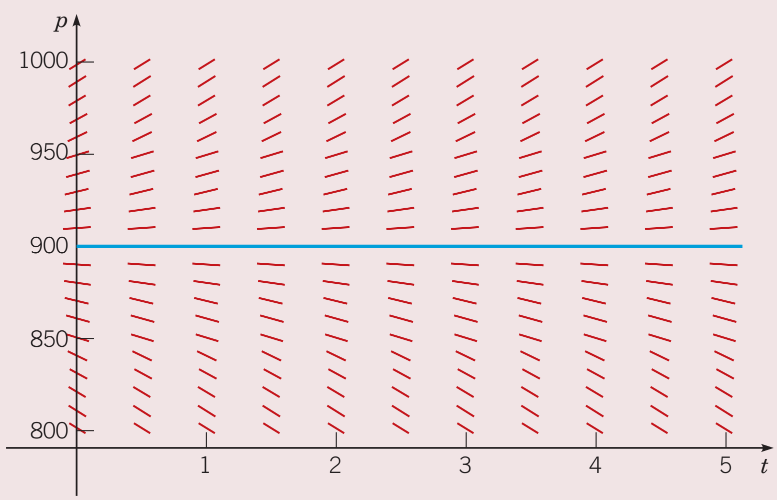

FIGURE 1.1.4 Direction field (red) and equilibrium solution (blue) for equation (8): $dp/dt = p/2 - 450$.

Comparing Examples 2 and 3, we note that in both cases the equilibrium solution separates increasing from decreasing solutions. In Example 2 other solutions converge to, or are attracted by, the equilibrium solution, so that after the object falls long enough, an observer will see it moving at very nearly the equilibrium velocity. On the other hand, in Example 3 other solutions diverge from, or are repelled by, the equilibrium solution. Solutions behave very differently depending on whether they start above or below the equilibrium solution. As time passes, an observer might see populations either much larger or much smaller than the equilibrium population, but the equilibrium solution itself will not, in practice, be observed. In both problems, however, the equilibrium solution is very important in understanding how solutions of the given differential equation behave.

A more general version of equation (8) is

$$\frac{dp}{dt} = rp - k, \tag{9}$$

where the growth rate r and the predation rate k are positive constants that are otherwise unspecified. Solutions of this more general equation are very similar to those of equation (8). The equilibrium solution of equation (9) is $p(t) = k/r$. Solutions above the equilibrium solution increase, while those below it decrease.

You should keep in mind that both of the models discussed in this section have their limitations. The model (5) of the falling object is valid only as long as the object is falling freely, without encountering any obstacles. If the velocity is large enough, the assumption that the frictional resistance is linearly proportional to the velocity has to be replaced with a nonlinear approximation (see Problem 28). The population model (8) eventually predicts negative numbers of mice (if $p < 900$) or enormously large numbers (if $p > 900$). Both of these predictions are unrealistic, so this model becomes unacceptable after a fairly short time interval.

Constructing Mathematical Models. In applying differential equations to any of the numerous fields in which they are useful, it is necessary first to formulate the appropriate differential equation that describes, or models, the problem being investigated. In this section we have looked at two examples of this modeling process, one drawn from physics and the other from ecology. In constructing future mathematical models yourself, you should recognize that each problem is different, and that successful modeling cannot be reduced to the observance of a set of prescribed rules. Indeed, constructing a satisfactory model is sometimes the most difficult part of the problem. Nevertheless, it may be helpful to list some steps that are often part of the process:

1. Identify the independent and dependent variables and assign letters to represent them. Often the independent variable is time.
2. Choose the units of measurement for each variable. In a sense the choice of units is arbitrary, but some choices may be much more convenient than others. For example, we chose to measure time in seconds for the falling-object problem and in months for the population problem.
3. Articulate the basic principle that underlies or governs the problem you are investigating. This may be a widely recognized physical law, such as Newton's law of motion, or it may be a more speculative assumption that may be based on your own experience or observations. In any case, this step is likely not to be a purely mathematical one, but will require you to be familiar with the field in which the problem originates.
4. Express the principle or law in step 3 in terms of the variables you chose in step 1. This may be easier said than done. It may require the introduction of physical constants or parameters (such as the drag coefficient in Example 1) and the determination of appropriate values for them. Or it may involve the use of auxiliary or intermediate variables that must then be related to the primary variables.
5. If the units agree, then your equation at least is dimensionally consistent, although it may have other shortcomings that this test does not reveal.
6. In the problems considered here, the result of step 4 is a single differential equation, which constitutes the desired mathematical model. Keep in mind, though, that in more complex problems the resulting mathematical model may be much more complicated, perhaps involving a system of several differential equations, for example.

Historical Background, Part I: Newton, Leibniz, and the Bernoullis. Without knowing something about differential equations and methods of solving them, it is difficult to appreciate the history of this important branch of mathematics. Further, the development of differential equations is intimately interwoven with the general development of mathematics and cannot be separated from it. Nevertheless, to provide some historical perspective, we indicate here some of the major trends in the history of the subject and identify the most prominent early contributors. The rest of the historical background in this section focuses on the earliest contributors from the seventeenth century. The story continues at the end of Section 1.2 with an overview of the contributions of Euler and other eighteenth-century (and early-nineteenth-century) mathematicians. More recent advances, including the use of computers and other technologies, are summarized at the end of Section 1.3. Additional historical information is contained in footnotes scattered throughout the book and in the references listed at the end of the chapter.

The subject of differential equations originated in the study of calculus by Isaac Newton (1643–1727) and Gottfried Wilhelm Leibniz (1646–1716) in the seventeenth century. Newton grew up in the English countryside, was educated at Trinity College, Cambridge, and became Lucasian Professor of Mathematics there in 1669. His epochal discoveries of calculus and of the fundamental laws of mechanics date to 1665. They were circulated privately among his friends, but Newton was extremely sensitive to criticism and did not begin to publish his results until 1687 with the appearance of his most famous book *Philosophiae Naturalis Principia Mathematica*. Although Newton did relatively little work in differential equations as such, his development of the calculus and elucidation of the basic principles of mechanics provided a basis for their applications in the eighteenth century, most notably by Euler (see Historical Background, Part II in Section 1.2). Newton identified three forms of first-order differential equations: $dy/dx = f(x)$, $dy/dx = f(y)$, and $dy/dx = f(x, y)$. For the latter equation he developed a method of solution using infinite series when $f(x, y)$ is a polynomial in x and y. Newton's active research in mathematics ended in the early 1690s, except for the solution of occasional "challenge problems" and the revision and publication of results obtained much earlier. He was appointed Warden of the British Mint in 1696 and resigned his professorship a few years later. He was knighted in 1705 and, upon his death in 1727, became the first scientist buried in Westminster Abbey.

Leibniz was born in Leipzig, Germany, and completed his doctorate in philosophy at the age of 20 at the University of Altdorf. Throughout his life he engaged in scholarly work in several different fields. He was mainly self-taught in mathematics, since his interest in this subject developed when he was in his twenties. Leibniz arrived at the fundamental results of calculus independently, although a little later than Newton, but was the first to publish them, in 1684. Leibniz was very conscious of the power of good mathematical notation and was responsible for the notation dy/dx for the derivative and for the integral sign. He discovered the method of separation of variables (Section 2.2) in 1691, the reduction of homogeneous equations to separable ones (Section 2.2, Problem 29) in 1691, and the procedure for solving first-order linear equations (Section 2.1) in 1694. He spent his life as ambassador and adviser to several German royal families, which permitted him to travel widely and to carry on an extensive correspondence with other mathematicians, especially the Bernoulli brothers. In the course of this correspondence many problems in differential equations were solved during the latter part of the seventeenth century.

The Bernoulli brothers, Jakob (1654–1705) and Johann (1667–1748), of Basel, Switzerland did much to develop methods of solving differential equations and to extend the range of their applications. Jakob became professor of mathematics at Basel in 1687, and Johann was appointed to the same position upon his brother's death in 1705. Both men were quarrelsome, jealous, and frequently embroiled in disputes, especially with each other. Nevertheless, both also made significant contributions to several areas of mathematics. With the aid of calculus, they solved a number of problems in mechanics by formulating them as differential equations. For example, Jakob Bernoulli solved the differential equation $y' = \left(a^3/(b^2 y - a^3)\right)^{1/2}$ (see Problem 9 in Section 2.2) in 1690 and, in the same paper, first used the term "integral" in the modern sense. In 1694 Johann Bernoulli was able to solve the equation $dy/dx = y/(ax)$ (see Problem 10 in Section 2.2). One problem that both brothers solved, and that led to much friction between them, was the **brachistochrone problem**. The brachistochrone problem was also solved by Leibniz, Newton, and the Marquis de l'Hôpital. It is said, perhaps apocryphally, that Newton learned of the problem late in the afternoon of a tiring day at the Mint and solved it that evening after dinner. He published the solution anonymously, but upon seeing it, Johann Bernoulli exclaimed, "Ah, I know the lion by his paw."

Daniel Bernoulli (1700–1782), son of Johann, migrated to St. Petersburg, Russia, as a young man to join the newly established St. Petersburg Academy, but returned to Basel in 1733 as professor of botany and, later, of physics. His interests were primarily in partial differential equations and their applications. For instance, it is his name that is associated with the Bernoulli equation in fluid mechanics. He was also the first to encounter the functions that a century later became known as Bessel functions (Section 5.7).

Problems

In each of Problems 1 through 6, draw a direction field for the given differential equation. Based on the direction field, determine the behavior of y as $t \to \infty$. If this behavior depends on the initial value of y at $t = 0$, describe the dependency.

G **1.** $y' = 4 - 2y$

G **2.** $y' = 2y - 4$

G **3.** $y' = -1 - 4y$

G **4.** $y' = 1 + 4y$

G **5.** $y' = 3 + 2y$

G **6.** $y' = y + 3$

In each of Problems 7 through 10, write down a differential equation of the form $dy/dt = ay + b$ whose solutions have the required behavior as $t \to \infty$.

7. All solutions approach $y = 3/4$.

8. All other solutions diverge from $y = 3$.

9. All other solutions diverge from $y = 1/3$.

10. All solutions approach $y = -1/2$.

In each of Problems 11 through 14, draw a direction field for the given differential equation. Based on the direction field, determine the behavior of y as $t \to \infty$. If this behavior depends on the initial value of y at $t = 0$, describe this dependency. Note that in these problems the equations are not of the form $y' = ay + b$, and the behavior of their solutions is somewhat more complicated than for the equations in the text.

G **11.** $y' = y(3 - y)$

G **12.** $y' = -y(4 - y)$

G **13.** $y' = y^4$

G **14.** $y' = y(y - 1)^2$

Consider the following list of differential equations, some of which produced the direction fields shown in Figures 1.1.5 through 1.1.10. In each of Problems 15 through 20, identify the differential equation that corresponds to the given direction field.

a. $y' = 2 + y$
b. $y' = y(y - 3)$
c. $y' = -2 - y$
d. $y' = 2 - y$
e. $y' = y - 2$
f. $y' = y(3 - y)$

15. The direction field of Figure 1.1.5.

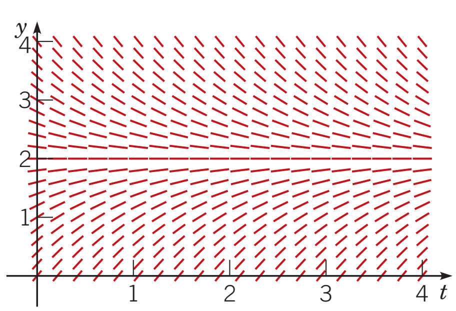

FIGURE 1.1.5 Problem 15.

16. The direction field of Figure 1.1.6.

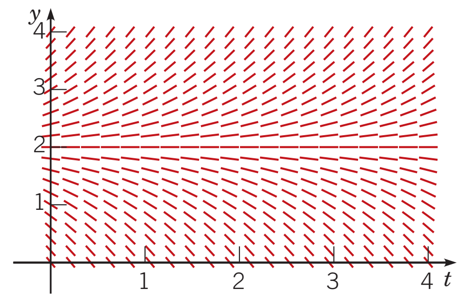

FIGURE 1.1.6 Problem 16.

17. The direction field of Figure 1.1.7.

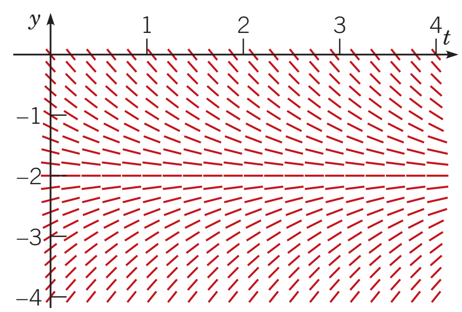

FIGURE 1.1.7 Problem 17.

18. The direction field of Figure 1.1.8.

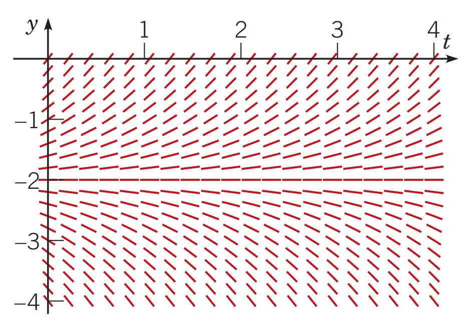

FIGURE 1.1.8 Problem 18.

19. The direction field of Figure 1.1.9.

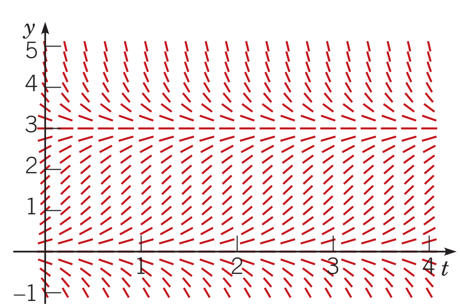

FIGURE 1.1.9 Problem 19.

20. The direction field of Figure 1.1.10.

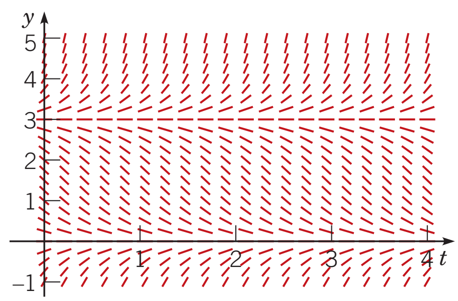

FIGURE 1.1.10 Problem 20.

In each of Problems 21 through 24, draw a direction field for the given differential equation. Based on the direction field, determine the behavior of y as $t \to \infty$. If this behavior depends on the initial value of y at $t = 0$, describe this dependency. Note that the right-hand sides of these equations depend on t as well as y; therefore, their solutions can exhibit more complicated behavior than those in the text.

G **21.** $y' = -3 + t - y$

G **22.** $y' = te^{-2t} - 2y$

G **23.** $y' = 3t - 1 - y^2$

G **24.** $y' = \frac{1}{6}y^3 - y - \frac{1}{3}t^2$

25. A spherical raindrop evaporates at a rate proportional to its surface area. Write a differential equation for the volume of the raindrop as a function of time.

26. A pond initially contains 1,000,000 L of water and an unknown amount of an undesirable chemical. Water containing 0.01 g of this chemical per liter flows into the pond at a rate of 300 L/h. The mixture flows out at the same rate, so the amount of water in the pond remains constant. Assume that the chemical is uniformly distributed throughout the pond.

a. Write a differential equation for the amount of chemical in the pond at any time.

b. How much of the chemical will be in the pond after a very long time? Does this limiting amount depend on the amount that was present initially?

27. Newton's law of cooling states that the temperature of an object changes at a rate proportional to the difference between the temperature of the object itself and the temperature of its surroundings (the ambient air temperature in most cases). Suppose that the ambient temperature is 20°C and that the rate constant is 0.05 $(\text{min})^{-1}$. Write a differential equation for the temperature of the object at any time. Note that the differential equation is the same whether the temperature of the object is above or below the ambient temperature.

N **28.** For small, slowly falling objects, the assumption made in the text that the drag force is proportional to the velocity is a good one. For larger, more rapidly falling objects, it is more accurate to assume that the drag force is proportional to the square of the velocity.[2]

a. Write a differential equation for the velocity of a falling object of mass m if the magnitude of the drag force is proportional to the square of the velocity and its direction is opposite to that of the velocity.

b. Determine the limiting velocity after a long time.

c. If $m = 10$ kg, find the drag coefficient so that the limiting velocity is 49 m/s.

N **d.** Using the date in part c, draw a direction field and compare it with Figure 1.1.3.

29. A certain drug is being administered intravenously to a hospital patient. Fluid containing 5 mg/cm^3 of the drug enters the patient's bloodstream at a rate of 100 cm^3/h. The drug is absorbed by body tissues or otherwise leaves the bloodstream at a rate proportional to the amount present, with a rate constant of 0.4h^{-1}.

a. Assuming that the drug is always uniformly distributed throughout the bloodstream, write a differential equation for the amount of the drug that is present in the bloodstream at any time.

b. How much of the drug is present in the bloodstream after a long time?

[2]See Lyle N. Long and Howard Weiss, "The Velocity Dependence of Aerodynamic Drag: A Primer for Mathematicians," *American Mathematical Monthly 106* (1999), 2, pp. 127–135.

1.2 Solutions of Some Differential Equations

In the preceding section we derived the differential equations

$$m\frac{dv}{dt} = mg - \gamma v \tag{1}$$

and

$$\frac{dp}{dt} = rp - k. \tag{2}$$

Equation (1) models a falling object, and equation (2) models a population of field mice preyed on by owls. Both of these equations are of the general form

$$\frac{dy}{dt} = ay - b, \tag{3}$$

where a and b are given constants. We were able to draw some important qualitative conclusions about the behavior of solutions of equations (1) and (2) by considering the associated direction fields. To answer questions of a quantitative nature, however, we need to find the solutions themselves, and we now investigate how to do that.

EXAMPLE 1 Field Mice and Owls

Consider the equation

$$\frac{dp}{dt} = 0.5p - 450, \tag{4}$$

which describes the interaction of certain populations of field mice and owls (see equation (8) of Section 1.1). Find solutions of this equation.

Solution:

To solve equation (4), we need to find functions $p(t)$ that, when substituted into the equation, reduce it to an obvious identity. Here is one way to proceed. First, rewrite equation (4) in the form

$$\frac{dp}{dt} = \frac{p - 900}{2}, \tag{5}$$

or, if $p \neq 900$,

$$\frac{dp/dt}{p - 900} = \frac{1}{2}. \tag{6}$$

By the chain rule the left-hand side of equation (6) is the derivative of $\ln|p - 900|$ with respect to t, so we have

$$\frac{d}{dt}\ln|p - 900| = \frac{1}{2}. \tag{7}$$

Then, by integrating both sides of equation (7), we obtain

$$\ln|p - 900| = \frac{t}{2} + C, \tag{8}$$

where C is an arbitrary constant of integration. Therefore, by taking the exponential of both sides of equation (8), we find that

$$|p - 900| = e^{t/2+C} = e^C e^{t/2}, \tag{9}$$

or

$$p - 900 = \pm e^C e^{t/2}, \tag{10}$$

and finally

$$p = 900 + ce^{t/2}, \tag{11}$$

where $c = \pm e^C$ is also an arbitrary (nonzero) constant. Note that the constant function $p = 900$ is also a solution of equation (5) and that it is contained in the expression (11) if we allow c to take the value zero. Graphs of equation (11) for several values of c are shown in Figure 1.2.1.

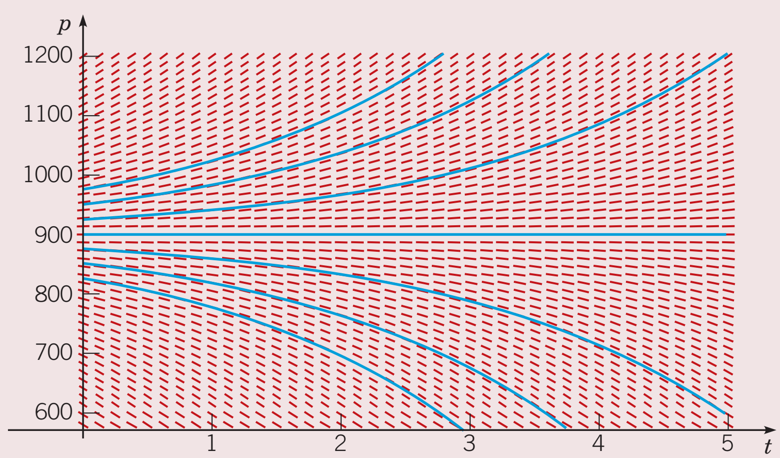

FIGURE 1.2.1 Graphs of $p = 900 + ce^{t/2}$ for several values of c. Each blue curve is a solution of $dp/dt = 0.5p - 450$.

Note that they have the character inferred from the direction field in Figure 1.1.4. For instance, solutions lying on either side of the equilibrium solution $p = 900$ tend to diverge from that solution.

In Example 1 we found infinitely many solutions of the differential equation (4), corresponding to the infinitely many values that the arbitrary constant c in equation (11) might have. This is typical of what happens when you solve a differential equation. The solution process involves an integration, which brings with it an arbitrary constant, whose possible values generate an infinite family of solutions.

Frequently, we want to focus our attention on a single member of the infinite family of solutions by specifying the value of the arbitrary constant. Most often, we do this indirectly by specifying instead a point that must lie on the graph of the solution. For example, to determine the constant c in equation (11), we could require that the population have a given value at a certain time, such as the value 850 at time $t = 0$. In other words, the graph of the solution must pass through the point $(0, 850)$. Symbolically, we can express this condition as

$$p(0) = 850. \tag{12}$$

Then, substituting $t = 0$ and $p = 850$ into equation (11), we obtain

$$850 = 900 + c.$$

Hence $c = -50$, and by inserting this value into equation (11), we obtain the desired solution, namely,

$$p = 900 - 50e^{t/2}. \tag{13}$$

The additional condition (12) that we used to determine c is an example of an **initial condition**. The differential equation (4) together with the initial condition (12) forms an **initial value problem**.

Now consider the more general problem consisting of the differential equation (3)

$$\frac{dy}{dt} = ay - b$$

and the initial condition

$$y(0) = y_0, \tag{14}$$

where y_0 is an arbitrary initial value. We can solve this problem by the same method as in Example 1. If $a \neq 0$ and $y \neq b/a$, then we can rewrite equation (3) as

$$\frac{dy/dt}{y - \frac{b}{a}} = a. \tag{15}$$

By integrating both sides, we find that

$$\ln\left|y(t) - \frac{b}{a}\right| = at + C, \tag{16}$$

where C is an arbitrary constant. Then, taking the exponential of both sides of equation (16) and solving for y, we obtain

$$y(t) = \frac{b}{a} + ce^{at}, \tag{17}$$

where $c = \pm e^C$ is also an arbitrary constant. Observe that $c = 0$ corresponds to the equilibrium solution $y(t) = b/a$. Finally, the initial condition (14) requires that $c = y_0 - (b/a)$, so the solution of the initial value problem (3), (14) is

$$y(t) = \frac{b}{a} + \left(y_0 - \frac{b}{a}\right)e^{at}. \tag{18}$$

For $a \neq 0$ the expression (17) contains all possible solutions of equation (3) and is called the **general solution**.[3] The geometric representation of the general solution (17) is an infinite family of curves called **integral curves**. Each integral curve is associated with a particular

[3] If $a = 0$, then the solution of equation (3) is not given by equation (17). We leave it to you to find the general solution in this case.

value of c and is the graph of the solution corresponding to that value of c. Satisfying an initial condition amounts to identifying the integral curve that passes through the given initial point.

To relate the solution (18) to equation (2), which models the field mouse population, we need only replace a by the growth rate r and replace b by the predation rate k; we assume that $r > 0$ and $k > 0$. Then the solution (18) becomes

$$p(t) = \frac{k}{r} + \left(p_0 - \frac{k}{r}\right)e^{rt}, \tag{19}$$

where p_0 is the initial population of field mice. The solution (19) confirms the conclusions reached on the basis of the direction field and Example 1. If $p_0 = k/r$, then from equation (19) it follows that $p(t) = k/r$ for all t; this is the constant, or equilibrium, solution. If $p_0 \neq k/r$, then the behavior of the solution depends on the sign of the coefficient $p_0 - k/r$ of the exponential term in equation (19). If $p_0 > k/r$, then p grows exponentially with time t; if $p_0 < k/r$, then p decreases and becomes zero (at a finite time), corresponding to extinction of the field mouse population. Negative values of p, while possible for the expression (19), make no sense in the context of this particular problem.

To put the falling-object equation (1) in the form (3), we must identify a with $-\gamma/m$ and b with $-g$. Observe that assuming $\gamma > 0$ and $m > 0$ implies that $a < 0$ and $b < 0$. Making these substitutions in the solution (18), we obtain

$$v(t) = \frac{mg}{\gamma} + \left(v_0 - \frac{mg}{\gamma}\right)e^{-\gamma t/m}, \tag{20}$$

where v_0 is the initial velocity. Again, this solution confirms the conclusions reached in Section 1.1 on the basis of a direction field. There is an equilibrium, or constant, solution $v(t) = mg/\gamma$, and all other solutions tend to approach this equilibrium solution. The speed of convergence to the equilibrium solution is determined by the exponent $-\gamma/m$. Thus, for a given mass m, the velocity approaches the equilibrium value more rapidly as the drag coefficient γ increases.

EXAMPLE 2 A Falling Object (continued)

Suppose that, as in Example 2 of Section 1.1, we consider a falling object of mass $m = 10$ kg and drag coefficient $\gamma = 2$ kg/s. Then the equation of motion (1) becomes

$$\frac{dv}{dt} = 9.8 - \frac{v}{5}. \tag{21}$$

Suppose this object is dropped from a height of 300 m. Find its velocity at any time t. How long will it take to fall to the ground, and how fast will it be moving at the time of impact?

Solution:

The first step is to state an appropriate initial condition for equation (21). The word “dropped” in the statement of the problem suggests that the object starts from rest, that is, its initial velocity is zero, so we will use the initial condition

$$v(0) = 0. \tag{22}$$

The solution of equation (21) can be found by substituting the values of the coefficients into the solution (20), but we will proceed instead to solve equation (21) directly. First, rewrite the equation as

$$\frac{dv/dt}{v - 49} = -\frac{1}{5}. \tag{23}$$

By integrating both sides, we obtain

$$\ln|v(t) - 49| = -\frac{t}{5} + C, \tag{24}$$

and then the general solution of equation (21) is

$$v(t) = 49 + ce^{-t/5}, \tag{25}$$

where the constant c is arbitrary. To determine the particular value of c that corresponds to the initial condition (22), we substitute $t = 0$ and $v = 0$ into equation (25), with the result that $c = -49$. Then the solution of the initial value problem (21), (22) is

$$v(t) = 49\left(1 - e^{-t/5}\right). \tag{26}$$

Equation (26) gives the velocity of the falling object at any positive time after being dropped—until it hits the ground, of course.

Graphs of the solution (25) for several values of c are shown in Figure 1.2.2, with the solution (26) shown by the green curve. It is evident that, regardless of the initial velocity of the object, all solutions tend to approach the equilibrium solution $v(t) = 49$. This confirms the conclusions we reached in Section 1.1 on the basis of the direction fields in Figures 1.1.2 and 1.1.3.

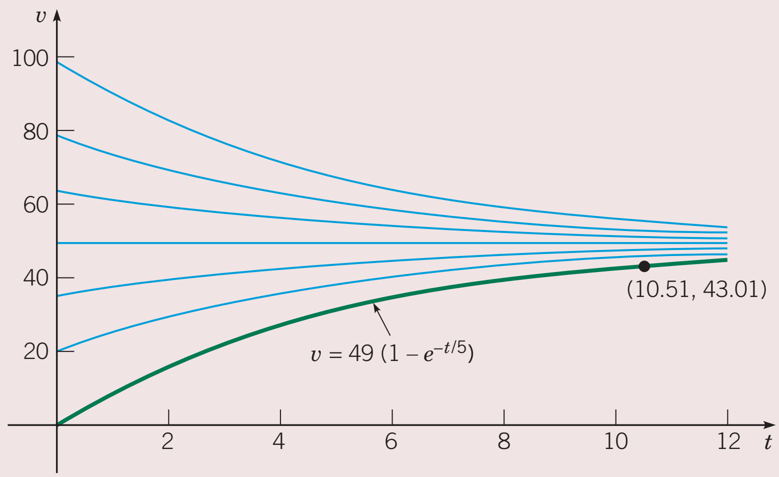

FIGURE 1.2.2 Graphs of the solution (25), $v = 49 + ce^{-t/5}$, for several values of c. The green curve corresponds to the initial condition $v(0) = 0$. The point $(10.51, 43.01)$ shows the velocity when the object hits the ground.

To find the velocity of the object when it hits the ground, we need to know the time at which impact occurs. In other words, we need to determine how long it takes the object to fall 300 m. To do this, we note that the distance x the object has fallen is related to its velocity v by the differential equation $v = dx/dt$, or

$$\frac{dx}{dt} = 49\left(1 - e^{-t/5}\right). \tag{27}$$

Consequently, by integrating both sides of equation (27) with respect to t, we have

$$x = 49t + 245e^{-t/5} + k, \tag{28}$$

where k is an arbitrary constant of integration. The object starts to fall when $t = 0$, so we know that $x = 0$ when $t = 0$. From equation (28) it follows that $k = -245$, so the distance the object has fallen at time t is given by

$$x = 49t + 245e^{-t/5} - 245. \tag{29}$$

Let T be the time at which the object hits the ground; then $x = 300$ when $t = T$. By substituting these values in equation (29), we obtain the equation

$$49T + 245e^{-T/5} - 245 = 300. \tag{30}$$

The value of T satisfying equation (30) can be approximated by a numerical process[4] using a calculator or other computational tool, with the result that $T \cong 10.51$ s. At this time, the corresponding velocity v_T is found from equation (26) to be $v_T \cong 43.01$ m/s. The point $(10.51, 43.01)$ is also shown in Figure 1.2.2.

[4] A computer algebra system provides this capability; many calculators also have built-in routines for solving such equations.

Further Remarks on Mathematical Modeling. Up to this point we have related our discussion of differential equations to mathematical models of a falling object and of a hypothetical relation between field mice and owls. The derivation of these models may have been plausible, and possibly even convincing, but you should remember that the ultimate test of any mathematical model is whether its predictions agree with observations or experimental results. We have no actual observations or experimental results to use for comparison purposes here, but there are several sources of possible discrepancies.

In the case of the falling object, the underlying physical principle (Newton's laws of motion) is well established and widely applicable. However, the assumption that the drag force is proportional to the velocity is less certain. Even if this assumption is correct, the determination of the drag coefficient γ by direct measurement presents difficulties. Indeed, sometimes one finds the drag coefficient indirectly — for example, by measuring the time of fall from a given height and then calculating the value of γ that predicts this observed time.

The model of the field mouse population is subject to various uncertainties. The determination of the growth rate r and the predation rate k depends on observations of actual populations, which may be subject to considerable variation. The assumption that r and k are constants may also be questionable. For example, a constant predation rate becomes harder to sustain as the field mouse population becomes smaller. Further, the model predicts that a population above the equilibrium value will grow exponentially larger and larger. This seems at variance with the behavior of actual populations; see the further discussion of population dynamics in Section 2.5.

If the differences between actual observations and a mathematical model's predictions are too great, then you need to consider refining the model, making more careful observations, or perhaps both. There is almost always a tradeoff between accuracy and simplicity. Both are desirable, but a gain in one usually involves a loss in the other. However, even if a mathematical model is incomplete or somewhat inaccurate, it may nevertheless be useful in explaining qualitative features of the problem under investigation. It may also give satisfactory results under some circumstances but not others. Thus you should always use good judgment and common sense in constructing mathematical models and in using their predictions.

Historical Background, Part II: Euler, Lagrange, and Laplace. The greatest mathematician of the eighteenth century, Leonhard Euler (1707–1783), grew up near Basel, Switzerland and was a student of Johann Bernoulli. He followed his friend Daniel Bernoulli to St. Petersburg in 1727. For the remainder of his life he was associated with the St. Petersburg Academy (1727–1741 and 1766–1783) and the Berlin Academy (1741–1766). Losing sight in his right eye in 1738, and in his left eye in 1766, did not stop Euler from being one of the most prolific mathematicians of all time. In addition to publishing more than 500 books and papers during his life, an additional 400 have appeared posthumously.

Of particular interest here is Euler's formulation of problems in mechanics in mathematical language and his development of methods of solving these mathematical problems. Lagrange said of Euler's work in mechanics, "The first great work in which analysis is applied to the science of movement." Among other things, Euler identified the condition for exactness of first-order differential equations (Section 2.6) in 1734–1735, developed the theory of integrating factors (Section 2.6) in the same paper, and gave the general solution of homogeneous linear differential equations with constant coefficients (Sections 3.1, 3.3, 3.4, and 4.2) in 1743. He extended the latter results to nonhomogeneous differential equations in 1750–1751. Beginning about 1750, Euler made frequent use of power series (Chapter 5) in solving differential equations. He also proposed a numerical procedure (Sections 2.7 and 8.1) in 1768–1769, made important contributions in partial differential equations, and gave the first systematic treatment of the calculus of variations.

Joseph-Louis Lagrange (1736–1813) became professor of mathematics in his native Turin, Italy, at the age of 19. He succeeded Euler in the chair of mathematics at the Berlin Academy in 1766 and moved on to the Paris Academy in 1787. He is most famous for his monumental work *Mécanique analytique*, published in 1788, an elegant and comprehensive treatise of Newtonian mechanics. With respect to elementary differential equations, Lagrange showed in 1762–1765 that the general solution of a homogeneous nth order linear differential equation is a linear combination of n independent solutions (Sections 3.2 and 4.1). Later, in 1774–1775, he offered a complete development of the method of variation of parameters (Sections 3.6 and 4.4). Lagrange is also known for fundamental work in partial differential equations and the calculus of variations.

Pierre-Simon de Laplace (1749–1827) lived in Normandy, France, as a boy but arrived in Paris in 1768 and quickly made his mark in scientific circles, winning election to the Académie des Sciences in 1773. He was preeminent in the field of celestial mechanics; his greatest work, *Traité de mécanique céleste*, was published in five volumes between 1799 and 1825. Laplace's equation is fundamental in many branches of mathematical physics, and Laplace studied it extensively in connection with gravitational attraction. The Laplace transform (Chapter 6) is also named for him, although its usefulness in solving differential equations was not recognized until much later.

By the end of the eighteenth century many elementary methods of solving ordinary differential equations had been discovered. In the nineteenth century interest turned more toward the investigation of theoretical questions of existence and uniqueness and to the development of less elementary methods such as those based on power series expansions (see Chapter 5). These methods find their natural setting in the complex plane. Consequently, they benefitted from, and to some extent stimulated, the more or less simultaneous development of the theory of complex analytic functions. Partial differential equations also began to be studied intensively, as their crucial role in mathematical physics became clear. In this connection a number of functions, arising as solutions of certain ordinary differential equations, occurred repeatedly and were studied exhaustively. Known collectively as higher transcendental functions, many of them are associated with the names of mathematicians, including Bessel (Section 5.7), Legendre (Section 5.3), Hermite (Section 5.2), Chebyshev (Section 5.3), Hankel, and many others.

Problems

N **1.** Solve each of the following initial value problems and plot the solutions for several values of y_0. Then describe in a few words how the solutions resemble, and differ from, each other.

a. $dy/dt = -y + 3, \quad y(0) = y_0$
b. $dy/dt = -2y + 3, \quad y(0) = y_0$
c. $dy/dt = -2y + 6, \quad y(0) = y_0$

G **2.** Follow the instructions for Problem 1 for the following initial value problems:

a. $dy/dt = y - 3, \quad y(0) = y_0$
b. $dy/dt = 2y - 3, \quad y(0) = y_0$
c. $dy/dt = 2y - 6, \quad y(0) = y_0$

3. **Undetermined Coefficients.** Here is an alternative way to solve the equation

$$\frac{dy}{dt} = ay - b. \tag{31}$$

a. Solve the simpler equation

$$\frac{dy}{dt} = ay. \tag{32}$$

Call the solution $y_1(t)$.

b. Observe that the only difference between equations (31) and (32) is the constant $-b$ in equation (31). Therefore, it may seem reasonable to assume that the solutions of these two equations also differ only by a constant. Test this assumption by trying to find a constant k such that $y = y_1(t) + k$ is a solution of equation (31).

c. Compare your solution from part b with the solution given in the text in equation (17).

Note: This method can also be used in some cases in which the constant b is replaced by a function $g(t)$. It depends on whether you can guess the general form that the solution is likely to take. This method is described in detail in Section 3.5 in connection with second-order equations.

4. Use the method of Problem 3 to solve the equation

$$\frac{dy}{dt} = -ay + b.$$

5. Consider the differential equation

$$\frac{dy}{dt} = -ay + b,$$

where both a and b are positive numbers.

a. Find the general solution of the differential equation.
G **b.** Sketch the solution for several different initial conditions.
c. Describe how the solutions change under each of the following conditions:

i. a increases.
ii. b increases.
iii. Both a and b increase, but the ratio b/a remains the same.

6. Consider the differential equation $dy/dt = ay - b$.

a. Find the equilibrium solution y_e.
b. Let $Y(t) = y - y_e$; thus $Y(t)$ is the deviation from the equilibrium solution. Find the differential equation satisfied by $Y(t)$.

7. The field mouse population in Example 1 satisfies the differential equation

$$\frac{dp}{dt} = 0.5p - 450.$$

a. Find the time at which the population becomes extinct if $p(0) = 800$.
b. Find the time of extinction if $p(0) = p_0$, where $0 < p_0 < 900$.
c. Find the initial population p_0 if the population is to become extinct in 1 year.

8. Consider a population p of field mice that grows at a rate proportional to the current population, so that $dp/dt = rp$.

a. Find the rate constant r if the population doubles in 20 days.
b. Find r if the population doubles in N days.

9. Consider the falling object of mass 10 kg in Example 2, but assume now that the drag force is proportional to the square of the velocity.

a. If the limiting velocity is 49 m/s (the same as in Example 2), show that the equation of motion can be written as

$$dv/dt = [(49)^2 - v^2]/245.$$

b. If $v(0) = 0$, find an expression for $v(t)$ at any time.

G **c.** Plot your solution from part b and the solution (26) from Example 2 on the same axes.

d. Based on your plots in part c, compare the effect of a quadratic drag force with that of a linear drag force.

e. Find the distance $x(t)$ that the object falls in time t.

N **f.** Find the time T it takes the object to fall 250 m.

10. Radium-226 has a half-life of 1620 years. Find the time period during which a given amount of this material is reduced by one-third.

11. A radioactive material, such as the isotope thorium-234, disintegrates at a rate proportional to the amount currently present. If $Q(t)$ is the amount present at time t, then $dQ/dt = -rQ$, where $r > 0$ is the decay rate.

a. If 100 mg of thorium-234 decays to 82.04 mg in 1 week, determine the decay rate r.

b. Find an expression for the amount of thorium-234 present at any time t.

c. Find the time required for the thorium-234 to decay to one-half its original amount.

12. Suppose that a building loses heat in accordance with Newton's law of cooling and that the rate constant k has the value 0.15 h^{-1}. Assume that the interior temperature is 20°C when the heating system fails. If the external temperature is −10°C, how long will it take for the interior temperature to fall to 0°C?

13. Your swimming pool containing 60,000 L of water has been contaminated by 5 kg of a nontoxic dye that leaves a swimmer's skin an unattractive green. The pool's filtering system can take water from the pool, remove the dye, and return the water to the pool at a flow rate of 200 L/min.

a. Write down the initial value problem for the filtering process; let $q(t)$ be the amount of dye in the pool at any time t.

b. Solve the problem in part a.

c. You have invited several dozen friends to a pool party that is scheduled to begin in 4 h. You have also determined that the effect of the dye is imperceptible if its concentration is less than 0.02 g/L. Is your filtering system capable of reducing the dye concentration to this level within 4 h?

d. Find the time T at which the concentration of dye first reaches the value 0.02 g/L.

e. Find the flow rate that is sufficient to achieve the concentration 0.02 g/L within 4 h.

14. Consider an electric circuit containing a capacitor, resistor, and battery; see Figure 1.2.3. The charge $Q(t)$ on the capacitor satisfies the equation[5]

$$R\frac{dQ}{dt} + \frac{Q}{C} = V,$$

where R is the resistance, C is the capacitance, and V is the constant voltage supplied by the battery.

G **a.** If $Q(0) = 0$, find $Q(t)$ at any time t, and sketch the graph of Q versus t.

b. Find the limiting value Q_L that $Q(t)$ approaches after a long time.

G **c.** Suppose that $Q(t_1) = Q_L$ and that at time $t = t_1$ the battery is removed and the circuit is closed again. Find $Q(t)$ for $t > t_1$ and sketch its graph.

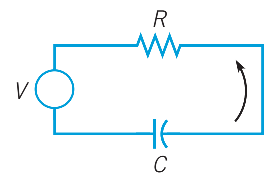

FIGURE 1.2.3 The electric circuit of Problem 14.

[5]This equation results from Kirchhoff's laws, which are discussed in Section 3.7.

2.3 Modeling with First-Order Differential Equations

Differential equations are of interest to nonmathematicians primarily because of the possibility of using them to investigate a wide variety of problems in the physical, biological, and social sciences. One reason for this is that mathematical models and their solutions lead to equations relating the variables and parameters in the problem. These equations often enable you to make predictions about how the natural process will behave in various circumstances. It is often easy to vary parameters in the mathematical model over wide ranges, whereas this may be very time-consuming or expensive, if not impossible, in an experimental setting. Nevertheless, mathematical modeling and experiment or observation are both critically important and have somewhat complementary roles in scientific investigations. Mathematical models are validated by comparison of their predictions with experimental results. On the other hand, mathematical analyses may suggest the most promising directions to explore experimentally, and they may indicate fairly precisely what experimental data will be most helpful.

In Sections 1.1 and 1.2 we formulated and investigated a few simple mathematical models. We begin by recapitulating and expanding on some of the conclusions reached in those sections. Regardless of the specific field of application, there are three identifiable steps that are always present in the process of mathematical modeling.

Step 1: Construction of the Model. In this step the physical situation is translated into mathematical terms, often using the steps listed at the end of Section 1.1. Perhaps most critical at this stage is to state clearly the physical principle(s) that are believed to govern the process. For example, it has been observed that in some circumstances heat passes from a warmer to a cooler body at a rate proportional to the temperature difference, that objects move about in accordance with Newton's laws of motion, and that isolated insect populations grow at a rate proportional to the current population. Each of these statements involves a rate of change (derivative) and consequently, when expressed mathematically, leads to a differential equation. The differential equation is a mathematical model of the process.

It is important to realize that the mathematical equations are almost always only an approximate description of the actual process. For example, bodies moving at speeds comparable to the speed of light are not governed by Newton's laws, insect populations do not grow indefinitely as stated because of eventual lack of food or space, and heat transfer is affected by factors other than the temperature difference. Thus you should always be aware of the limitations of the model so that you will use it only when it is reasonable to believe that it is accurate. Alternatively, you can adopt the point of view that the mathematical equations exactly describe the operation of a simplified physical model, which has been constructed (or conceived of) so as to embody the most important features of the actual process. Sometimes, the process of mathematical modeling involves the conceptual replacement of a discrete process by a continuous one. For instance, the number of members in an insect population changes by discrete amounts; however, if the population is large, it seems reasonable to consider it as a continuous variable and even to speak of its derivative.

Step 2: Analysis of the Model. Once the problem has been formulated mathematically, you are often faced with the problem of solving one or more differential equations or, failing that, of finding out as much as possible about the properties of the solution. It may happen that this mathematical problem is quite difficult, and if so, further approximations may be indicated at this stage to make the problem mathematically tractable. For example, a nonlinear equation may be approximated by a linear one, or a slowly varying coefficient may be replaced by a constant. Naturally, any such approximations must also be examined from the physical point of view to make sure that the simplified mathematical problem still reflects the essential features of the physical process under investigation. At the same time, an intimate knowledge of the physics of the problem may suggest reasonable mathematical approximations that will make the mathematical problem more amenable to analysis. This interplay of understanding of physical phenomena and knowledge of mathematical techniques and their limitations is characteristic of applied mathematics at its best, and it is indispensable in successfully constructing useful mathematical models of intricate physical processes.

Step 3: Comparison with Experiment or Observation. Finally, having obtained the solution (or at least some information about it), you must interpret this information in the context in which the problem arose. In particular, you should always check that the mathematical solution appears physically reasonable. If possible, calculate the values of the solution at selected points and compare them with experimentally observed values. Or ask whether the behavior of the solution after a long time is consistent with observations. Or examine the solutions corresponding to certain special values of parameters in the problem. Of course, the fact that the mathematical solution appears to be reasonable does not guarantee that it is correct. However, if the predictions of the mathematical model are seriously inconsistent with observations of the physical system it purports to describe, this suggests that errors have been made in solving the mathematical problem, that the mathematical model itself needs refinement, or that observations must be made with greater care.

The examples in this section are typical of applications in which first-order differential equations arise.

EXAMPLE 1 Mixing

At time $t = 0$ a tank contains Q_0 lb of salt dissolved in 100 gal of water; see Figure 2.3.1. Assume that water containing $\frac{1}{4}$ lb of salt per gallon is entering the tank at a rate of r gal/min and that the well-stirred mixture is draining from the tank at the same rate. Set up the initial value problem that describes this flow process. Find the amount of salt $Q(t)$ in the tank at any time, and also find the limiting amount Q_L that is present after a very long time. If $r = 3$ and $Q_0 = 2Q_L$, find the time T after which the salt level is within 2% of Q_L. Also find the flow rate that is required if the value of T is not to exceed 45 min.

Solution:

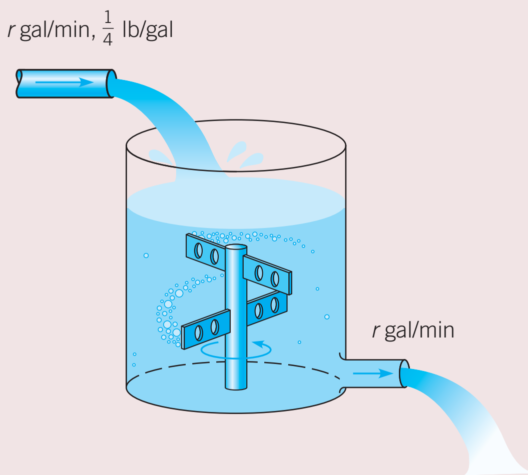

FIGURE 2.3.1 The water tank in Example 1.

We assume that salt is neither created nor destroyed in the tank. Therefore, variations in the amount of salt are due solely to the flows in and out of the tank. More precisely, the rate of change of salt in the tank, dQ/dt, is equal to the rate at which salt is flowing in minus the rate at which it is flowing out. In symbols,

$$\frac{dQ}{dt} = \text{rate in} - \text{rate out}. \tag{1}$$

The rate at which salt enters the tank is the concentration $\frac{1}{4}$ lb/gal times the flow rate r gal/min, or $r/4$ lb/min. To find the rate at which salt leaves the tank, we need to multiply the concentration of salt in the tank by the rate of outflow, r gal/min. Since the rates of flow in and out are equal, the volume of water in the tank remains constant at 100 gal, and since the mixture is "well-stirred," the concentration throughout the tank is the same, namely, $Q(t)/100$ lb/gal. Therefore, the rate at which salt leaves the tank is $rQ(t)/100$ lb/min. Thus the differential equation governing this process is

$$\frac{dQ}{dt} = \frac{r}{4} - \frac{rQ}{100}. \tag{2}$$

The initial condition is

$$Q(0) = Q_0. \tag{3}$$

Upon thinking about the problem physically, we might anticipate that eventually the mixture originally in the tank will be essentially replaced by the mixture flowing in, whose concentration is $\frac{1}{4}$ lb/gal. Consequently, we might expect that ultimately the amount of salt in the tank would be very close to 25 lb. We can also find the limiting amount $Q_L = 25$ by setting dQ/dt equal to zero in equation (2) and solving the resulting algebraic equation for Q.

To solve the initial value problem (2), (3) analytically, note that equation (2) is linear. (It is also separable, see Problem 23 in Section 2.2.) Rewriting the differential equation (2) in the standard form for a linear differential equation, we have

$$\frac{dQ}{dt} + \frac{rQ}{100} = \frac{r}{4}. \tag{4}$$

Thus the integrating factor is $e^{rt/100}$ and the general solution is

$$Q(t) = 25 + ce^{-rt/100}, \tag{5}$$

where c is an arbitrary constant. To satisfy the initial condition (3), we must choose $c = Q_0 - 25$. Therefore, the solution of the initial value problem (2), (3) is

$$Q(t) = 25 + (Q_0 - 25)e^{-rt/100}, \tag{6}$$

or

$$Q(t) = 25(1 - e^{-rt/100}) + Q_0 e^{-rt/100}. \tag{7}$$

From either form of the solution, (6) or (7), you can see that $Q(t) \to 25$ (lb) as $t \to \infty$, so the limiting value Q_L is 25, confirming our physical intuition.

Further, $Q(t)$ approaches the limit more rapidly as r increases. In interpreting the solution (7), note that the second term on the right-hand side is the portion of the original salt that remains at time t, while the first term gives the amount of salt in the tank as a consequence of the flow processes. Plots of the solution for $r = 3$ and for several values of Q_0 are shown in Figure 2.3.2.

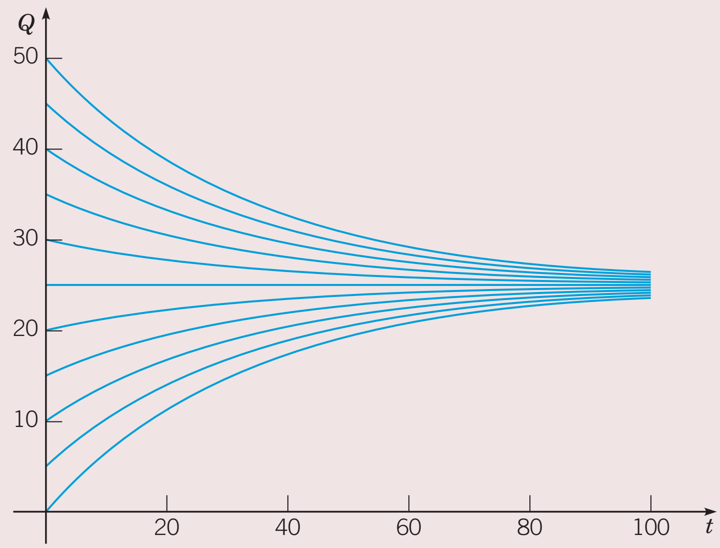

FIGURE 2.3.2 Solutions of the initial value problem (2): $dQ/dt = r/4 - rQ/100$, $Q(0) = Q_0$ for $r = 3$ and several values of Q_0.

Now suppose that $r = 3$ and $Q_0 = 2Q_L = 50$; then equation (6) becomes

$$Q(t) = 25 + 25e^{-0.03t}. \tag{8}$$

Since 2% of 25 is 0.5, we wish to find the time T at which $Q(t)$ has the value 25.5. Substituting $t = T$ and $Q = 25.5$ in equation (8) and solving for T, we obtain

$$T = \frac{\ln(50)}{0.03} \cong 130.4 \text{ (min)}. \tag{9}$$

To determine r so that $T = 45$, return to equation (6), set $t = 45$, $Q_0 = 50$, $Q(t) = 25.5$, and solve for r. The result is

$$r = \frac{100}{45} \ln 50 \cong 8.69 \text{ gal/min}. \tag{10}$$

Since this example is hypothetical, the validity of the model is not in question. If the flow rates are as stated, and if the concentration of salt in the tank is uniform, then the differential equation (1) is an accurate description of the flow process. Although this particular example has no special significance, models of this kind are often used in problems involving a pollutant in a lake, or a drug in an organ of the body, for example, rather than a tank of salt water. In such cases the flow rates may not be easy to determine or may vary with time. Similarly, the concentration may be far from uniform in some cases. Finally, the rates of inflow and outflow may be different, which means that the variation of the amount of liquid in the problem must also be taken into account.

EXAMPLE 2 Compound Interest

Suppose that a sum of money, S_0, is deposited in a bank or money fund that pays interest at an annual rate r. The value $S(t)$ of the investment at any time t depends on the frequency with which interest is compounded as well as on the interest rate. Financial institutions have various policies concerning compounding: some compound monthly, some weekly, and some even daily. Assume that compounding takes place *continuously*. Set up an initial value problem that describes the growth of the investment.

Solution:

The rate of change of the value of the investment is dS/dt, and this quantity is equal to the rate at which interest accrues, which is the interest rate r times the current value of the investment $S(t)$. Thus

$$\frac{dS}{dt} = rS \tag{11}$$

is the differential equation that governs the process. If we let t denote the time, in years, since the original deposit, the corresponding initial condition is

$$S(0) = S_0. \tag{12}$$

Then the solution of the initial value problem (8) gives the balance $S(t)$ in the account at any time t. This initial value problem is readily solved, since the differential equation (11) is both linear and separable. Consequently, by solving equations (11) and (12), we find that

$$S(t) = S_0 e^{rt}. \tag{13}$$

Thus a bank account with continuously compounding interest grows exponentially.

The model in Example 2 is easily extended to situations involving deposits or withdrawals in addition to the accrual of interest, dividends, or annual capital gains. If we assume that the deposits or withdrawals take place at a constant rate k, then equation (11) is replaced by

$$\frac{dS}{dt} = rS + k,$$

or, in standard form,

$$\frac{dS}{dt} - rS = k, \tag{14}$$

where k is positive for deposits and negative for withdrawals.

Equation (14) is linear with the integrating factor e^{-rt}, so its general solution is

$$S(t) = ce^{rt} - \frac{k}{r},$$

where c is an arbitrary constant. To satisfy the initial condition (12), we must choose $c = S_0 + k/r$. Thus the solution of the initial value problem (10), (8) is

$$S(t) = S_0 e^{rt} + \frac{k}{r}(e^{rt} - 1). \tag{15}$$

The first term in expression (15) is the part of $S(t)$ that is due to the return accumulated on the initial amount S_0, and the second term is the part that is due to the deposit or withdrawal rate k.

The advantage of stating the problem in this general way without specific values for S_0, r, or k lies in the generality of the resulting formula (15) for $S(t)$. With this formula we can readily compare the results of different investment programs or different rates of return.

For instance, suppose that one opens an individual retirement account (IRA) at age 25 and makes annual investments of \$2000 thereafter in a continuous manner. Assuming a rate of return of 8%, what will be the balance in the IRA at age 65? We have $S_0 = 0$, $r = 0.08$, and $k = \$2000$, and we wish to determine $S(40)$. From equation (15) we have

$$S(40) = 25{,}000(e^{3.2} - 1) = \$588{,}313. \tag{16}$$

It is interesting to note that the total amount invested is \$80,000, so the remaining amount of \$508,313 results from the accumulated return on the investment. The balance after 40 years is also fairly sensitive to the assumed rate. For instance, $S(40) = \$508{,}948$ if $r = 0.075$ and $S(40) = \$681{,}508$ if $r = 0.085$.

Let us now examine the assumptions that have gone into the model. First, we have assumed that the return is compounded continuously and that additional capital is invested continuously. Neither of these is true in an actual financial situation. We have also assumed that the return rate r is constant for the entire period involved, whereas in fact it is likely to fluctuate considerably. Although we cannot reliably predict future rates, we can use solution (15) to determine the approximate effect of different rate projections. It is also possible to consider r and k in equation (14) to be functions of t rather than constants; in that case, of course, the solution may be much more complicated than equation (15).

The initial value problem (10), (8) and the solution (15) can also be used to analyze a number of other financial situations, including annuities, mortgages, and automobile loans.

Let us now compare the results from the model with continuously compounded interest (and no other deposits or withdrawals) with the corresponding situation in which compounding occurs at finite time intervals. If interest is compounded once a year, then after t years

$$S(t) = S_0(1 + r)^t.$$

If interest is compounded twice a year, then at the end of 6 months the value of the investment is $S_0(1 + (r/2))$, and at the end of 1 year it is $S_0(1 + r/2)^2$. Thus, after t years, we have

$$S(t) = S_0\left(1 + \frac{r}{2}\right)^{2t}.$$

In general, if interest is compounded m times per year, then

$$S(t) = S_0\left(1 + \frac{r}{m}\right)^{mt}. \tag{17}$$

The relation between formulas (13) and (17) is clarified if we recall from calculus that

$$\lim_{m\to\infty} S_0\left(1 + \frac{r}{m}\right)^{mt} = S_0 e^{rt}.$$

The same model applies equally well to more general investments in which dividends and perhaps capital gains can also accumulate, as well as interest. In recognition of this fact, we will from now on refer to r as the rate of return.

Table 2.3.1 shows the effect of changing the frequency of compounding for a return rate r of 8%. The second and third columns are calculated from equation (17) for quarterly and daily compounding, respectively, and the fourth column is calculated from equation (13) for continuous compounding. The results show that the frequency of compounding is not particularly important in most cases. For example, during a 10-year period the difference between quarterly and continuous compounding is \$17.50 per \$1000 invested, or less than \$2/year. The difference would be somewhat greater for higher rates of return and less for lower rates. From the first row in the table, we see that for the return rate $r = 8\%$, the annual yield for quarterly compounding is 8.24% and for daily or continuous compounding it is 8.33%.

TABLE 2.3.1 Growth of Capital at a Return Rate $r = 8\%$ for Several Modes of Compounding

	$S(t)/S(t_0)$ From Equation (17)		$S(t)/S(t_0)$
Years	$m = 4$	$m = 365$	From Equation (13)
1	1.0824	1.0833	1.0833
2	1.1717	1.1735	1.1735
5	1.4859	1.4918	1.4918
10	2.2080	2.2253	2.2255
20	4.8754	4.9522	4.9530
30	10.7652	11.0203	11.0232
40	23.7699	24.5239	24.5325

EXAMPLE 3 Chemicals in a Pond

Consider a pond that initially contains 10 million gallons of fresh water. Water containing an undesirable chemical flows into the pond at the rate of 5 million gallons per year, and the mixture in the pond flows out at the same rate. The concentration $\gamma(t)$ of chemical in the incoming water varies periodically with time according to the expression $\gamma(t) = 2 + \sin(2t)$ g/gal. Construct a mathematical model of this flow process and determine the amount of chemical in the pond at any time. Plot the solution and describe in words the effect of the variation in the incoming concentration.

Solution:

Since the incoming and outgoing flows of water are the same, the amount of water in the pond remains constant at 10^7 gal. Let us denote time by t, measured in years, and the chemical by $Q(t)$, measured in grams. This example is similar to Example 1, and the same inflow/outflow principle applies. Thus

$$\frac{dQ}{dt} = \text{rate in} - \text{rate out},$$

where "rate in" and "rate out" refer to the rates at which the chemical flows into and out of the pond, respectively. The rate at which the chemical flows in is given by

$$\text{rate in} = (5 \times 10^6)\ \text{gal/yr}\ (2 + \sin(2t))\ \text{g/gal}. \tag{18}$$

The concentration of chemical in the pond is $Q(t)/10^7$ g/gal, so the rate of flow out is

$$\text{rate out} = (5 \times 10^6)\ \text{gal/year}\ (Q(t)/10^7)\ \text{g/gal} = Q(t)/2\ \text{g/yr}. \tag{19}$$

Thus we obtain the differential equation

$$\frac{dQ}{dt} = (5 \times 10^6)(2 + \sin(2t)) - \frac{Q(t)}{2}, \tag{20}$$

where each term has the units of g/yr.

To make the coefficients more manageable, it is convenient to introduce a new dependent variable defined by $q(t) = Q(t)/10^6$, or $Q(t) = 10^6\, q(t)$. This means that $q(t)$ is measured in millions of grams, or megagrams (metric tons). If we make this substitution in equation (20), then each term contains the factor 10^6, which can be canceled. If we also transpose the term involving $q(t)$ to the left-hand side of the equation, we finally have

$$\frac{dq}{dt} + \frac{1}{2}q = 10 + 5\sin(2t). \tag{21}$$

Originally, there is no chemical in the pond, so the initial condition is

$$q(0) = 0. \tag{22}$$

Equation (21) is linear, and although the right-hand side is a function of time, the coefficient of q is a constant. Thus the integrating factor is $e^{t/2}$. Multiplying equation (21) by this factor and integrating the resulting equation, we obtain the general solution

$$q(t) = 20 - \frac{40}{17}\cos(2t) + \frac{10}{17}\sin(2t) + ce^{-t/2}. \tag{23}$$

The initial condition (22) requires that $c = -300/17$, so the solution of the initial value problem (17), (18) is

$$q(t) = 20 - \frac{40}{17}\cos(2t) + \frac{10}{17}\sin(2t) - \frac{300}{17}e^{-t/2}. \tag{24}$$

A plot of the solution (24) is shown in Figure 2.3.3, along with the line $q = 20$ (shown in black). The exponential term in the solution is important for small t, but it diminishes rapidly as t increases. Later, the solution consists of an oscillation, due to the $\sin(2t)$ and $\cos(2t)$ terms, about the constant level $q = 20$. Note that if the $\sin(2t)$ term were not present in equation (21), then $q = 20$ would be the equilibrium solution of that equation.

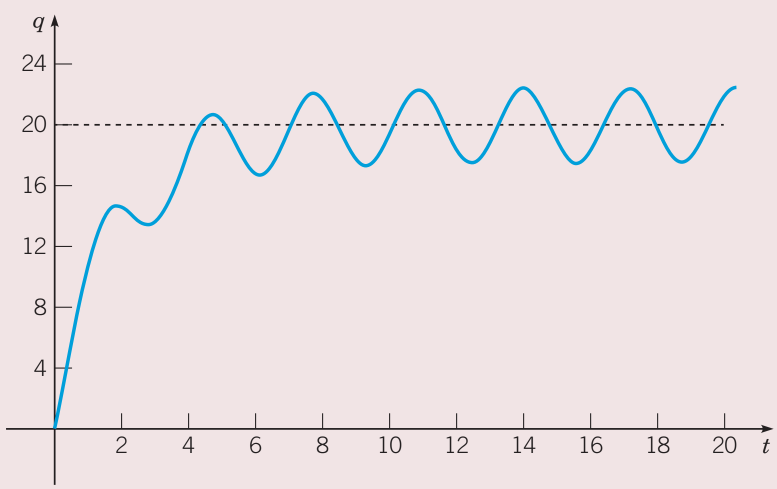

FIGURE 2.3.3 Solution of the initial value problem (17), (18): $dq/dt + q/2 = 10 + 5\sin(2t)$, $q(0) = 0$.

Let us now consider the adequacy of the mathematical model itself for this problem. The model rests on several assumptions that have not yet been stated explicitly. In the first place, the amount of water in the pond is controlled entirely by the rates of flow in and out — none is lost by evaporation or by seepage into the ground, and none is gained by rainfall. The same is true of the chemical; it flows into and out of the pond, but none is absorbed by fish or other organisms living in the pond. In addition, we assume that the concentration of chemical in the pond is uniform throughout the pond. Whether the results obtained from the model are accurate depends strongly on the validity of these simplifying assumptions.

EXAMPLE 4 Escape Velocity

A body of constant mass m is projected away from the earth in a direction perpendicular to the earth's surface with an initial velocity v_0. Assuming that there is no air resistance, but taking into account the variation of the earth's gravitational field with distance, find an expression for the velocity during the ensuing motion. Also find the initial velocity that is required to lift the body to a given maximum altitude A_{max} above the surface of the earth, and find the least initial velocity for which the body will not return to the earth; the latter is the **escape velocity**.

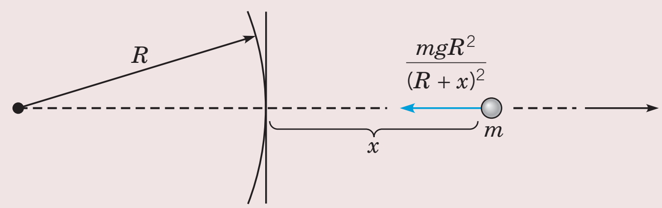

FIGURE 2.3.4 A body in the earth's gravitational field is pulled towards the center of the earth.

Solution:

Let the positive x-axis point away from the center of the earth along the line of motion with $x = 0$ lying on the earth's surface; see Figure 2.3.4. The figure is drawn horizontally to remind you that gravity is directed toward the center of the earth, which is not necessarily downward from a perspective away from the earth's surface. The gravitational force acting on the body (that is, its weight) is inversely proportional to the square of the distance from the center of the earth and is given by $w(x) = -k/(x+R)^2$, where k is a constant, R is the radius of the earth, and the minus sign signifies that $w(x)$ is directed in the negative x direction. We know that on the earth's surface $w(0)$ is given by $-mg$, where g is the acceleration due to gravity at sea level. Therefore, $k = mgR^2$ and

$$w(x) = -\frac{mgR^2}{(R+x)^2}. \tag{25}$$

Since there are no other forces acting on the body, the equation of motion is

$$m\frac{dv}{dt} = -\frac{mgR^2}{(R+x)^2}, \tag{26}$$

and the initial condition is

$$v(0) = v_0. \tag{27}$$

Unfortunately, equation (26) involves too many variables since it depends on t, x, and v. To remedy this situation, we can eliminate t from equation (26) by thinking of x, rather than t, as the independent variable. Then we can express dv/dt in terms of dv/dx by using the chain rule; hence

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx},$$

and equation (26) is replaced by

$$v\frac{dv}{dx} = -\frac{gR^2}{(R+x)^2}. \tag{28}$$

Equation (28) is separable but not linear, so by separating the variables and integrating, we obtain

$$\frac{v^2}{2} = \frac{gR^2}{R+x} + c. \tag{29}$$

Since $x = 0$ when $t = 0$, the initial condition (27) at $t = 0$ can be replaced by the condition that $v = v_0$ when $x = 0$. Hence $c = (v_0^2/2) - gR$ and

$$v = \pm\sqrt{v_0^2 - 2gR + \frac{2gR^2}{R+x}}. \tag{30}$$

Note that equation (30) gives the velocity as a function of altitude rather than as a function of time. The plus sign must be chosen if the body is rising, and the minus sign must be chosen if it is falling back to earth.

To determine the maximum altitude A_{max} that the body reaches, we set $v = 0$ and $x = A_{max}$ in equation (30) and then solve for A_{max}, obtaining

$$A_{max} = \frac{v_0^2 R}{2gR - v_0^2}. \tag{31}$$

Solving equation (31) for v_0, we find the initial velocity required to lift the body to the altitude A_{max}, namely,

$$v_0 = \sqrt{2gR\frac{A_{max}}{R + A_{max}}}. \tag{32}$$

The escape velocity v_e is then found by letting $A_{max} \to \infty$. Consequently,

$$v_e = \sqrt{2gR}. \tag{33}$$

The numerical value of v_e is approximately 6.9 mi/s, or 11.1 km/s.

The preceding calculation of the escape velocity neglects the effect of air resistance, so the actual escape velocity (including the effect of air resistance) is somewhat higher. On the other hand, the effective escape velocity can be significantly reduced if the body is transported a considerable distance above sea level before being launched. Both gravitational and frictional forces are thereby reduced; air resistance, in particular, diminishes quite rapidly with increasing altitude. You should keep in mind also that it may well be impractical to impart too large an initial velocity instantaneously; space vehicles, for instance, receive their initial acceleration during a period of a few minutes.

Problems

1. Consider a tank used in certain hydrodynamic experiments. After one experiment the tank contains 150 L of a dye solution with a concentration of 1 g/L. To prepare for the next experiment, the tank is to be rinsed with fresh water flowing in at a rate of 2 L/min, the well-stirred solution flowing out at the same rate. Find the time that will elapse before the concentration of dye in the tank reaches 1% of its original value.

2. A tank contains 100 L of water and 50 dkg of salt. Water containing a salt concentration of $\frac{1}{4}(1 + \frac{1}{2}\sin t)$ dkg/L flows into the tank at a rate of 2 L/min, and the mixture in the tank flows out at the same rate.

a. Find the amount of salt in the tank at any time.

G **b.** Plot the solution for a time period long enough so that you see the ultimate behavior of the graph.

c. The long-time behavior of the solution is an oscillation about a certain constant level. What is this level? What is the amplitude of the oscillation?

3. A tank initially contains 180 L of pure water. A mixture containing a concentration of γ g/L of salt enters the tank at a rate of 3 L/min, and the well-stirred mixture leaves the tank at the same rate. Find an expression in terms of γ for the amount of salt in the tank at any time t. Also find the limiting amount of salt in the tank as $t \to \infty$.

4. Suppose that a tank containing a certain liquid has an outlet near the bottom. Let $h(t)$ be the height of the liquid surface above the outlet at time t. Torricelli's[2] principle states that the outflow velocity v at the outlet is equal to the velocity of a particle falling freely (with no drag) from the height h.

a. Show that $v = \sqrt{2gh}$, where g is the acceleration due to gravity.

b. By equating the rate of outflow to the rate of change of liquid in the tank, show that $h(t)$ satisfies the equation

$$A(h)\frac{dh}{dt} = -\alpha a\sqrt{2gh}, \tag{34}$$

where $A(h)$ is the area of the cross section of the tank at height h and a is the area of the outlet. The constant α is a contraction coefficient that accounts for the observed fact that the cross section of the (smooth) outflow stream is smaller than a. The value of α for water is about 0.6.

c. Consider a water tank in the form of a right circular cylinder that is 4 m high above the outlet. The radius of the tank is 2 m, and the radius of the circular outlet is 0.1 m. If the tank is initially full of water, determine how long it takes to drain the tank down to the level of the outlet.

5. Suppose that a sum S_0 is invested at an annual rate of return r compounded continuously.

a. Find the time T required for the original sum to double in value as a function of r.

b. Determine T if $r = 8\%$.

c. Find the return rate that must be achieved if the initial investment is to double in 8 years.

6. A young person with no initial capital invests k dollars per year at an annual rate of return r. Assume that investments are made continuously and that the return is compounded continuously.

a. Determine the sum $S(t)$ accumulated at any time t.

[2] Evangelista Torricelli (1608–1647), successor to Galileo as court mathematician in Florence, published this result in 1644. In addition to this work in fluid dynamics, he is also known for constructing the first mercury barometer and for making important contributions to geometry.

b. If $r = 6\%$, determine k so that \$1 million will be available for retirement in 40 years.

c. If $k =$ \$2000/year, determine the return rate r that must be obtained to have \$1 million available in 40 years.

7. A certain college graduate borrows \$9000 to buy a car. The lender charges interest at an annual rate of 8%. Assuming that interest is compounded continuously and that the borrower makes payments continuously at a constant annual rate k, determine the payment rate k that is required to pay off the loan in 3 years. Also determine how much interest is paid during the 3-year period.

8. A home buyer wishes to borrow \$200,000 at an interest rate of 6% to finance the purchase. Assume that interest is compounded continuously and that payments are also made continuously.

a. Determine the monthly payment that is required to pay off the loan in 20 years; in 30 years.

b. Determine the total interest paid during the term of the loan in each of the cases in part a.

9. An important tool in archeological research is radiocarbon dating, developed by the American chemist Willard F. Libby.[3] This is a means of determining the age of certain wood and plant remains, and hence of animal or human bones or artifacts found buried at the same levels. Radiocarbon dating is based on the fact that some wood or plant remains contain residual amounts of carbon-14, a radioactive isotope of carbon. This isotope is accumulated during the lifetime of the plant and begins to decay at its death. Since the half-life of carbon-14 is long (approximately 5730 years[4]), measurable amounts of carbon-14 remain after many thousands of years. If even a tiny fraction of the original amount of carbon-14 is still present, then by appropriate laboratory measurements the *proportion* of the original amount of carbon-14 that remains can be accurately determined. In other words, if $Q(t)$ is the amount of carbon-14 at time t and Q_0 is the original amount, then the ratio $Q(t)/Q_0$ can be determined, as long as this quantity is not too small. Present measurement techniques permit the use of this method for time periods of 50,000 years or more.

a. Assuming that Q satisfies the differential equation $Q' = -rQ$, determine the decay constant r for carbon-14.

b. Find an expression for $Q(t)$ at any time t, if $Q(0) = Q_0$.

c. Suppose that certain remains are discovered in which the current residual amount of carbon-14 is 25% of the original amount. Determine the age of these remains.

10. Newton's law of cooling states that the temperature of an object changes at a rate proportional to the difference between its temperature and that of its surroundings. Suppose that the temperature of a cup of coffee obeys Newton's law of cooling. If the coffee has a temperature of 90°C when freshly poured, and 1 min later has cooled to 85°C in a room at 20°C, determine when the coffee reaches a temperature of 65°C.

N **11.** Suppose that a certain population satisfies the initial value problem

$$dy/dt = r(t)y - k, \quad y(0) = y_0,$$

where the growth rate $r(t)$ is given by $r(t) = (1 + \sin t)/5$, and k represents the rate of predation.

G **a.** Suppose that $k = 1/5$. Plot y versus t for several values of y_0 between 1/2 and 1.

b. Estimate the critical initial population y_c below which the population will become extinct.

c. Choose other values of k and find the corresponding y_c for each one.

G **d.** Use the data you have found in parts b and c to plot y_c versus k.

N **12.** Suppose that a certain population has a growth rate that varies with time and that this population satisfies the differential equation

$$\frac{dy}{dt} = (0.5 + \sin t)\frac{y}{5}.$$

a. If $y(0) = 1$, find (or estimate) the time τ at which the population has doubled. Choose other initial conditions and determine whether the doubling time τ depends on the initial population.

b. Suppose that the growth rate is replaced by its average value 1/10. Determine the doubling time τ in this case.

c. Suppose that the term $\sin t$ in the differential equation is replaced by $\sin 2\pi t$; that is, the variation in the growth rate has a substantially higher frequency. What effect does this have on the doubling time τ?

d. Plot the solutions obtained in parts a, b, and c on a single set of axes.

N **13.** Consider an insulated box (a building, perhaps) with internal temperature $u(t)$. According to Newton's law of cooling, u satisfies the differential equation

$$\frac{du}{dt} = -k[u - T(t)], \tag{35}$$

where $T(t)$ is the ambient (external) temperature. Suppose that $T(t)$ varies sinusoidally; for example, assume that $T(t) = T_0 + T_1 \cos(\omega t)$.

a. Solve equation (35) and express $u(t)$ in terms of t, k, T_0, T_1, and ω. Observe that part of your solution approaches zero as t becomes large; this is called the transient part. The remainder of the solution is called the steady state; denote it by $S(t)$.

G **b.** Suppose that t is measured in hours and that $\omega = \pi/12$, corresponding to a period of 24 h for $T(t)$. Further, let $T_0 = 15$°C, $T_1 = -10$°C, and $k = 0.2$/h. Draw graphs of $S(t)$ and $T(t)$ versus t on the same axes. From your graph estimate the amplitude R of the oscillatory part of $S(t)$. Also estimate the time lag τ between corresponding maxima of $T(t)$ and $S(t)$.

c. Let k, T_0, T_1, and ω now be unspecified. Write the oscillatory part of $S(t)$ in the form $R\cos[\omega(t - \tau)]$. Use trigonometric identities to find expressions for R and τ. Let T_1 and ω have the values given in part b, and plot graphs of R and τ versus k.

14. Heat transfer from a body to its surroundings by radiation, based on the Stefan–Boltzmann[5] law, is described by the differential equation

$$\frac{du}{dt} = -\alpha(u^4 - T^4), \tag{36}$$

where $u(t)$ is the absolute temperature of the body at time t, T is the absolute temperature of the surroundings, and α is a constant depending on the physical parameters of the body. However, if u is much larger than T, then solutions of equation (36) are well approximated by solutions of the simpler equation

$$\frac{du}{dt} = -\alpha u^4. \tag{37}$$

Suppose that a body with initial temperature 2000 K is surrounded by a medium with temperature 300 K and that $\alpha = 2.0 \times 10^{-12}\ \mathrm{K^{-3}/s}$.

a. Determine the temperature of the body at any time by solving equation (37).

G **b.** Plot the graph of u versus t.

[3] Willard F. Libby (1908–1980) was born in rural Colorado and received his education at the University of California at Berkeley. He developed the method of radiocarbon dating beginning in 1947 while he was at the University of Chicago. For this work he was awarded the Nobel Prize in Chemistry in 1960.

[4] *McGraw-Hill Encyclopedia of Science and Technology* (8th ed.) (New York: McGraw-Hill, 1997), Vol. 5, p. 48.

[5] Jozef Stefan (1835–1893), professor of physics at Vienna, stated the radiation law on empirical grounds in 1879. His student Ludwig Boltzmann (1844–1906) derived it theoretically from the principles of thermodynamics in 1884. Boltzmann is best known for his pioneering work in statistical mechanics.

N **c.** Find the time τ at which $u(\tau) = 600$—that is, twice the ambient temperature. Up to this time the error in using equation (37) to approximate the solutions of equation (36) is no more than 1%.

15. A skydiver weighing 90 kg (including equipment) falls vertically downward from an altitude of 1500 m and opens the parachute after 10 s of free fall. Assume that the force of air resistance, which is directed opposite to the velocity, is of magnitude $11|v|$ when the parachute is closed and is of magnitude $180|v|$ when the parachute is open, where the velocity v is measured in m/s.

a. Find the speed of the skydiver when the parachute opens.
b. Find the distance fallen before the parachute opens.
c. What is the limiting velocity v_L after the parachute opens?
d. Determine how long the skydiver is in the air after the parachute opens.
G **e.** Plot the graph of velocity versus time from the beginning of the fall until the skydiver reaches the ground.

16. A rocket sled having an initial speed of 240 km/h is slowed by a channel of water. Assume that during the braking process, the acceleration a is given by $a(v) = -\mu v^2$, where v is the velocity and μ is a constant.

a. As in Example 4 in the text, use the relation $dv/dt = v(dv/dx)$ to write the equation of motion in terms of v and x.
b. If it requires a distance of 600 m to slow the sled to 24 km/h, determine the value of μ.
c. Find the time τ required to slow the sled to 24 km/h.

N **17.** A ball with mass 0.15 kg is thrown upward with initial velocity 20 m/s from the roof of a building 30 m high. Neglect air resistance.

a. Find the maximum height above the ground that the ball reaches.
b. Assuming that the ball misses the building on the way down, find the time that it hits the ground.
G **c.** Plot the graphs of velocity and position versus time.

N **18.** Assume that the conditions are as in Problem 17 except that there is a force due to air resistance of magnitude $|v|/30$ directed opposite to the velocity, where the velocity v is measured in m/s.

a. Find the maximum height above the ground that the ball reaches.
b. Find the time that the ball hits the ground.
G **c.** Plot the graphs of velocity and position versus time. Compare these graphs with the corresponding ones in Problem 17.

19. Assume that the conditions are as in Problem 17 except that there is a force due to air resistance of magnitude $v^2/1325$ directed opposite to the velocity, where the velocity v is measured in m/s.

a. Find the maximum height above the ground that the ball reaches.
b. Find the time that the ball hits the ground.
G **c.** Plot the graphs of velocity and position versus time. Compare these graphs with the corresponding ones in Problems 17 and 18.

20. A body of constant mass m is projected vertically upward with an initial velocity v_0 in a medium offering a resistance $k|v|$, where k is a constant. Neglect changes in the gravitational force.

a. Find the maximum height x_m attained by the body and the time t_m at which this maximum height is reached.
b. Show that if $kv_0/mg < 1$, then t_m and x_m can be expressed as

$$t_m = \frac{v_0}{g}\left[1 - \frac{1}{2}\frac{kv_0}{mg} + \frac{1}{3}\left(\frac{kv_0}{mg}\right)^2 - \cdots\right],$$

$$x_m = \frac{v_0^2}{2g}\left[1 - \frac{2}{3}\frac{kv_0}{mg} + \frac{1}{2}\left(\frac{kv_0}{mg}\right)^2 - \cdots\right].$$

c. Show that the quantity kv_0/mg is dimensionless.

21. A body of mass m is projected vertically upward with an initial velocity v_0 in a medium offering a resistance $k|v|$, where k is a constant. Assume that the gravitational attraction of the earth is constant.

a. Find the velocity $v(t)$ of the body at any time.
b. Use the result of part a to calculate the limit of $v(t)$ as $k \to 0$—that is, as the resistance approaches zero. Does this result agree with the velocity of a mass m projected upward with an initial velocity v_0 in a vacuum?
c. Use the result of part a to calculate the limit of $v(t)$ as $m \to 0$—that is, as the mass approaches zero.

22. A body falling in a relatively dense fluid, oil for example, is acted on by three forces (see Figure 2.3.5): a resistive force R, a buoyant force B, and its weight w due to gravity. The buoyant force is equal to the weight of the fluid displaced by the object. For a slowly moving spherical body of radius a, the resistive force is given by Stokes's law, $R = 6\pi\mu a|v|$, where v is the velocity of the body, and μ is the coefficient of viscosity of the surrounding fluid.[6]

a. Find the limiting velocity of a solid sphere of radius a and density ρ falling freely in a medium of density ρ' and coefficient of viscosity μ.
b. In 1910 R. A. Millikan[7] studied the motion of tiny droplets of oil falling in an electric field. A field of strength E exerts a force Ee on a droplet with charge e. Assume that E has been adjusted so the droplet is held stationary ($v = 0$) and that w and B are as given above. Find an expression for e. Millikan repeated this experiment many times, and from the data that he gathered he was able to deduce the charge on an electron.

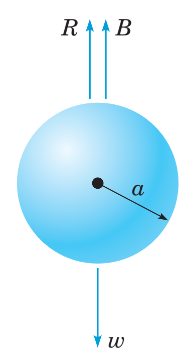

FIGURE 2.3.5 A body falling in a dense fluid (See problem 22).

23. A mass of 0.25 kg is dropped from rest in a medium offering a resistance of $0.2\,|v|$, where v is measured in m/s.

a. If the mass is dropped from a height of 25 m, find its velocity when it hits the ground.
b. If the mass is to attain a velocity of no more than 10 m/s, find the maximum height from which it can be dropped.

[6] Sir George Gabriel Stokes (1819–1903) was born in Ireland but spent most of his life at Cambridge University, first as a student and later as a professor. Stokes was one of the foremost applied mathematicians of the nineteenth century, best known for his work in fluid dynamics and the wave theory of light. The basic equations of fluid mechanics (the Navier–Stokes equations) are named partly in his honor, and one of the fundamental theorems of vector calculus bears his name. He was also one of the pioneers in the use of divergent (asymptotic) series.

[7] Robert A. Millikan (1868–1953) was educated at Oberlin College and Columbia University. Later he was a professor at the University of Chicago and California Institute of Technology. His determination of the charge on an electron was published in 1910. For this work, and for other studies of the photoelectric effect, he was awarded the Nobel Prize for Physics in 1923.

c. Suppose that the resistive force is $k|v|$, where v is measured in m/s and k is a constant. If the mass is dropped from a height of 25 m and must hit the ground with a velocity of no more than 10 m/s, determine the coefficient of resistance k that is required.

24. Let $v(t)$ and $w(t)$ be the horizontal and vertical components, respectively, of the velocity of a batted (or thrown) baseball. In the absence of air resistance, v and w satisfy the equations

$$\frac{dv}{dt} = 0, \quad \frac{dw}{dt} = -g.$$

a. Show that

$$v = u\cos A, \quad w = -gt + u\sin A,$$

where u is the initial speed of the ball and A is its initial angle of elevation.

b. Let $x(t)$ and $y(t)$ be the horizontal and vertical coordinates, respectively, of the ball at time t. If $x(0) = 0$ and $y(0) = h$, find $x(t)$ and $y(t)$ at any time t.

G **c.** Let $g = 32$ ft/s^2, $u = 125$ ft/s, and $h = 3$ ft. Plot the trajectory of the ball for several values of the angle A; that is, plot $x(t)$ and $y(t)$ parametrically.

d. Suppose the outfield wall is at a distance L and has height H. Find a relation between u and A that must be satisfied if the ball is to clear the wall.

e. Suppose that $L = 350$ ft and $H = 10$ ft. Using the relation in part d, find (or estimate from a plot) the range of values of A that correspond to an initial velocity of $u = 110$ ft/s.

f. For $L = 350$ and $H = 10$, find the minimum initial velocity u and the corresponding optimal angle A for which the ball will clear the wall.

N **25.** A more realistic model (than that in Problem 24) of a baseball in flight includes the effect of air resistance. In this case the equations of motion are

$$\frac{dv}{dt} = -rv, \quad \frac{dw}{dt} = -g - rw,$$

where r is the coefficient of resistance.

a. Determine $v(t)$ and $w(t)$ in terms of initial speed u and initial angle of elevation A.

b. Find $x(t)$ and $y(t)$ if $x(0) = 0$ and $y(0) = h$.

G **c.** Plot the trajectory of the ball for $r = 1/5$, $u = 125$, $h = 3$, and for several values of A. How to the trajectories differ from those in Problem 24 with $r = 0$?

d. Assuming that $r = 1/5$ and $h = 3$, find the minimum initial velocity u and the optimal angle A for which the ball will clear a wall that is 350 ft distant and 10 ft high. Compare this result with that in Problem 24f.

26. Suppose that a rocket is launched straight up from the surface of the earth with initial velocity $v_0 = \sqrt{2gR}$, where R is the radius of the earth. Neglect air resistance.

a. Find an expression for the velocity v in terms of the distance x from the surface of the earth.

b. Find the time required for the rocket to go 384,440 km (the approximate distance from the earth to the moon). Assume that $R = 6400$ km.