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Chapter 11.

11.2 (a) a_{2n}, b_n, c_n, d_n are monostone subsequence

(b) The set of subsequential limit For a_n: $S_a = \{-1, 1\}$. For b_n: $S_b = \{0\}$
For c_n: $S_c = \{+\infty\}$. For d_n: $S_d = \{\frac{6}{7}\}$.

(c) $\lim\sup a_n = \sup(S_a) = 1$, $\lim\inf a_n = \inf(S_a) = -1$
$\lim\sup b_n = \sup(S_b) = 0$, $\lim\inf b_n = \inf(S_b) = 0$
$\lim\sup c_n = \sup(S_c) = +\infty$, $\lim\inf c_n = \inf(S_c) = +\infty$
$\lim\sup d_n = \sup(S_d) = \frac{6}{7}$, $\lim\inf d_n = \inf(S) = \frac{6}{7}$.

(d) a_n does not converge. b_n converges to 0.
c_n diverge to $+\infty$. d_n converges to $\frac{6}{7}$.

(e) a_n is bounded. b_n is bounded. c_n is unbounded d_n is bounded.

11.4. (a) w_{2n}, x_{2n}. y_{2n}, z_{4k-2} are monostone subsequence.

(b) The set of subsequential limit For w_n: $S_w = \{-\infty, +\infty\}$. For x_n: $S_x = \{\frac{1}{5}, 5\}$.
For y_n: $S_y = \{0, 2\}$. For z_n: $S_z = \{-\infty, 0, +\infty\}$.

(c) $\lim\sup w_n = \sup(S_w) = +\infty$, $\lim\inf w_n = \inf(S_w) = -\infty$
$\lim\sup x_n = \sup(S_x) = 5$, $\lim\inf x_n = \inf(S_x) = \frac{1}{5}$
$\lim\sup y_n = \sup(S_y) = 2$, $\lim\inf y_n = \inf(S_y) = 0$
$\lim\sup z_n = \sup(S_z) = +\infty$, $\lim\inf z_n = \inf(S_z) = -\infty$

(d) w_n does not converge. x_n does not converge,
y_n does not converge. z_n. does not converge.

(e) w_n is unbounded. x_n is bounded.
y_n is bounded. z_n is unbounded.

11.6 Proof: For sequence s_n. by definition ③, a sequence t is a subsequence of s if and only if $t = s \circ \sigma_1$ for increasing function σ_1 mapping $\mathbb{N}$ to $\mathbb{N}$. Then, $t_k = s_{n_k}$ for $\sigma_1(k) = n_k$.
Also, a sequence u is a subsequence of t if and only if $u = t \circ \sigma_2$ for increasing function σ_2 mapping $\mathbb{N}$ to $\mathbb{N}$. Then, $u_m = t_{k_m}$ for $\sigma_2 = k_m$.
Then, $u = t \circ \sigma_2 = (s \circ \sigma_1) \circ \sigma_2 = s \circ (\sigma_1 \circ \sigma_2)$, and $\sigma_1 \circ \sigma_2$ is also an increasing function mapping $\mathbb{N}$ to $\mathbb{N}$.
Then, by definition, u is a subsequence of s and $u_m = t_{k_m} = s_{n_{k_m}}$. Thus if a sequence $u_m = t_{k_m} = s_{n_{k_m}}$ a subsequence of a subsequence $t_k = s_{n_k}$ of a sequence s_n. then u_n is a subsequence of a given sequence s_n.

11.8. Proof. By def 10.6, $\lim\sup s_n = \lim\sup\{s_n \mid n > N\}$, and $\lim\inf s_n = \lim\inf\{s_n \mid n > N\}$. Then we take $S = \{s_n \mid n > N\}$, then $-S = \{-s_n \mid n > N\}$. By Exercise 5.4. $\inf(S) = -\sup(-S)$.
Thus, $\lim\inf s_n = \lim\inf S = \lim -\sup(-S) = -\lim\sup(-S) = -\lim\sup -s_n$.
Therefore, for every sequence s_n, $\lim\inf s_n = -\lim\sup(-s_n)$.

11.10 (a) There are monotone subsequences: $(1, 1, 1, \cdots)$ converges to 1, $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \cdots)$ converges to $\frac{1}{2}$, $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \cdots)$ converges to $\frac{1}{3}$, $\cdots$, and $(\frac{1}{n}, \frac{1}{n}, \frac{1}{n}, \cdots)$ converges to $\frac{1}{n}$ for $n \in \mathbb{N}^+$. Also, there's a monotone subsequence converges to 0 for large n. So 0 is also in S.
Thus, the set of subsequence limits is $S = \{\frac{1}{n} \mid n \in \mathbb{N}^+\} \cup \{0\}$.

11.10 (b) Since S_n is a sequence in $\mathbb{R}$, and S is nonempty, then $\liminf S_n = \inf S = 0$, and $\limsup S_n = \sup S = 1$.

Chapter 12

12.2 ① When $\lim S_n = 0$. Then, for $\forall \varepsilon > 0$, $\exists N$ st. $\forall n > N$ we have $|S_n - 0| < \varepsilon$. Then, $||S_n| - 0| < \varepsilon$, so $\lim |S_n| = 0$. Since $\lim |S_n|$ is defined on $\mathbb{R}$, then $\liminf |S_n| = \lim |S_n| = \limsup |S_n|$ Thus $\limsup |S_n| = \lim |S_n| = 0$.

② When $\limsup |S_n| = 0$. Since $|S_n| \geq 0$ for $\forall n$, 0 is a lower bound for $|S_n|$. Then, $\inf |S_n| \geq 0$, then for $\forall n > N$, $\liminf |S_n| \geq 0$. ① Also, for non-empty S, $\inf S \leq \sup S$, for $S = \{S_n | n > N\}$ for a fixed N. Then, $\inf S_n \leq \sup S_n$ for $n > N$, $\forall N \in \mathbb{N}$. Then, $\liminf |S_n| \leq \limsup |S_n|$. ② Combining ① and ②, $0 \leq \liminf |S_n| \leq \limsup |S_n| = 0$. Therefore, $\liminf |S_n| = \limsup |S_n| = 0$. By property, $\lim |S_n| = 0$. Based on proof in (a), we then have $\lim S_n = 0$.

Thus, $\lim S_n = 0 \iff \limsup |S_n| = 0$

12.4 Proof: For fixed $N \in \mathbb{N}$ for $\forall n > N$, $s_n \leq \sup\{s_n : n > N\}$ and $t_n \leq \sup\{t_n : n > N\}$, since these are upper bounds of s_n and t_n. Then, for all $n > N$, $s_n + t_n \leq \sup\{s_n : n > N\} + \sup\{t_n : n > N\}$. so $\sup\{s_n : n > N\} + \sup\{t_n : n > N\}$ is an upper bound for $s_n + t_n$, then $\sup\{s_n + t_n : n > N\} \leq \sup\{s_n : n > N\} + \sup\{t_n : n > N\}$. Apply limit on both sides since s_n and t_n are bounded

$\lim(\sup\{s_n + t_n : n > N\}) \leq \lim(\sup\{s_n : n > N\}) + \lim(\sup\{t_n : n > N\})$

Since $s_n + t_n$, s_n, and t_n are sequence in $\mathbb{R}$, by the definition of lim sup, we then have $\lim \sup (s_n + t_n) \leq \lim(\sup s_n) + \lim(\sup t_n)$

12.8 Proof: For fixed $N \in \mathbb{N}$ for $\forall n > N$, $s_n \leq \sup\{s_n : n > N\}$ and $t_n \leq \sup\{t_n : n > N\}$, since these are upper bounds of s_n and t_n. Then, for all $n > N$, since $s_n > 0$, $t_n > 0$, $s_n \cdot t_n \leq \sup\{s_n : n > N\} \cdot \sup\{t_n : n > N\}$. so, $\sup\{s_n : n > N\} \cdot \sup\{t_n : n > N\}$ is an upper bound for $s_n \cdot t_n$, then $\sup\{s_n \cdot t_n : n > N\} \leq \sup\{s_n : n > N\} \cdot \sup\{t_n : n > N\}$. Since s_n and t_n are bounded, apply limit on both sides : $\lim(\sup\{s_n \cdot t_n : n > N\}) \leq \lim(\sup\{s_n : n > N\}) \lim(\sup\{t_n : n > N\})$

Since $s_n \cdot t_n$, s_n, and t_n are sequence in $\mathbb{R}$, by the definition of lim sup, we then have $\lim \sup (s_n \cdot t_n) \leq (\lim(\sup s_n)) \cdot (\lim(\sup t_n))$

12.10 Proof: ① When s_n is bounded. $\exists a > 0$ (a is positive real number) st. $|s_n| < a$ for $\forall n \in \mathbb{N}$. Then, $\sup\{|s_n| : n > N\} \leq a$ for $\forall N > 0$. Then, $\limsup\{|s_n| : n > N\} \leq a$. By definition of lim sup, since $|s_n|$ is sequence on $\mathbb{R}$, we have $\limsup |s_n| < a < +\infty$

② When $\limsup |s_n| < +\infty$. Then, $\limsup |s_n| = \limsup\{|s_n| : n > N\}$ exists finitely. Assume that $\limsup\{|s_n| : n > N\} = A$ for A is a non-negative real number. For $\varepsilon = 1$, we have $N \in \mathbb{N}^+$ st. $\forall n > N \Rightarrow |\sup\{|s_n| : n > N\} - A| < 1 \Rightarrow \sup\{|s_n| : n > N\} - A < 1$ $\Rightarrow \sup\{|s_n| : n > N\} < 1 + A \Rightarrow |s_n| < 1 + A$ for all $n > N$.

Then, $|s_n| \leq \max\{|s_1|, |s_2|, \cdots, |s_{N-1}|, 1 + A\}$ for all $n \in \mathbb{N}$. So s_n is bounded.

Thus, s_n is bounded $\iff \limsup |s_n| < +\infty$.

12.12 (a) Proof: ① For $\limsup \sigma_n \leq \limsup s_n$: Take $n > m > N$, then

$\sigma_n = \frac{1}{n}(s_1 + s_2 + \cdots + s_N + s_{N+1} + \cdots + s_n) = \frac{1}{n}(s_1 + s_2 + \cdots + s_N) + \frac{1}{n}(s_{N+1} + s_{N+2} + \cdots + s_n)$

Since $n > m$, then $\frac{1}{n}(s_1 + s_2 + \cdots + s_N) < \frac{1}{m}(s_1 + s_2 + \cdots + s_N)$, and $\frac{1}{n}(s_{N+1} + s_{N+2} + \cdots + s_n) \leq \sup\{s_n : n > m\}$. Thus $\sigma_n < \frac{1}{m}(s_1 + s_2 + \cdots + s_N) \cdot \sup\{s_n : n > m\}$.

$\Rightarrow \sup\{\sigma_n : n > m\} < \frac{1}{m}(s_1 + s_2 + \cdots + s_N) \cdot \sup\{s_n : n > m\}$. For $m \to +\infty$ for all N,

we have $\lim_{m\to\infty}(\sup\{\sigma_n : n>m\}) \le \lim_{m\to\infty}\frac{1}{m}(s_1+s_2+\cdots+s_m)\cdot \lim_{m\to\infty}\sup\{s_n : n>m\}$.

$\Rightarrow \limsup \sigma_n \le \limsup s_n$.

② For $\liminf s_n \le \liminf \sigma_n$: First, $\liminf s_n = -\limsup(-s_n)$. Given that $\limsup \sigma_n \le \limsup s_n$ can be changed to $\limsup(\sigma_n) \le \limsup(s_n)$ since $-\sigma_n = \frac{1}{n}(-s_1-s_2-\cdots-s_n)$ is still true, we have $-\limsup(-\sigma_n) \ge -\limsup(-s_n)$.

$\Rightarrow \liminf s_n \le \liminf \sigma_n$.

③ For $\liminf \sigma_n \le \limsup \sigma_n$: Take σ_{n_k} as a subsequence of σ_n st. $\lim_{k\to\infty}\sigma_{n_k} = a$. Then, $\liminf_{k\to\infty}\sigma_{n_k} = a = \limsup_{k\to\infty}\sigma_{n_k}$. Since $n_k > k$ for $\forall k$, $\{\sigma_{n_k} : k>N\} \subset \{\sigma_n : k>N\}$ for all $N \in \mathbb{N}$. Thus, $\liminf_{n\to\infty}\sigma_n \le \liminf_{k\to\infty}\sigma_{n_k} = \limsup_{k\to\infty}\sigma_{n_k} \le \limsup_{n\to\infty}\sigma_n$. Then $\liminf \sigma_n \le \limsup \sigma_n$

Thus, $\liminf s_n \le \liminf \sigma_n \le \limsup \sigma_n \le \limsup s_n$

(b) Proof: Since $\lim s_n$ exists, then $\liminf s_n = \lim s_n = \limsup s_n$.

By (a), we then have $\liminf s_n = \liminf \sigma_n = \limsup \sigma_n = \limsup s_n$.

So, since $\liminf \sigma_n = \limsup \sigma_n$, $\lim \sigma_n$ exists and $\lim \sigma_n = \liminf \sigma_n$ $= \liminf s_n = \lim s_n$. Thus, $\lim \sigma_n$ exists and $\lim \sigma_n = \lim s_n$.

(c) When $s_n = (-1)^n + 1 > 0$, $\lim s_n$ doesn't exist since $\limsup s_n = 2 \ne \liminf s_n = 0$.

However, $\sigma_n = \frac{1}{n}[1+(-1)^1+1+(-1)^2+1+(-1)^3+\cdots+1+(-1)^n] = \frac{1}{n}(0+2+0+2+\cdots+1+(-1)^n)$

$= \begin{cases} \frac{n+0}{n}, & \text{when } n \text{ is even} \\ \frac{n+1}{n}, & \text{when } n \text{ is odd.} \end{cases}$ Then, $\lim \sigma_n = 1$, so $\lim \sigma_n$ exists.