CSC 665: Artificial Intelligence

Constraint Satisfaction Problems

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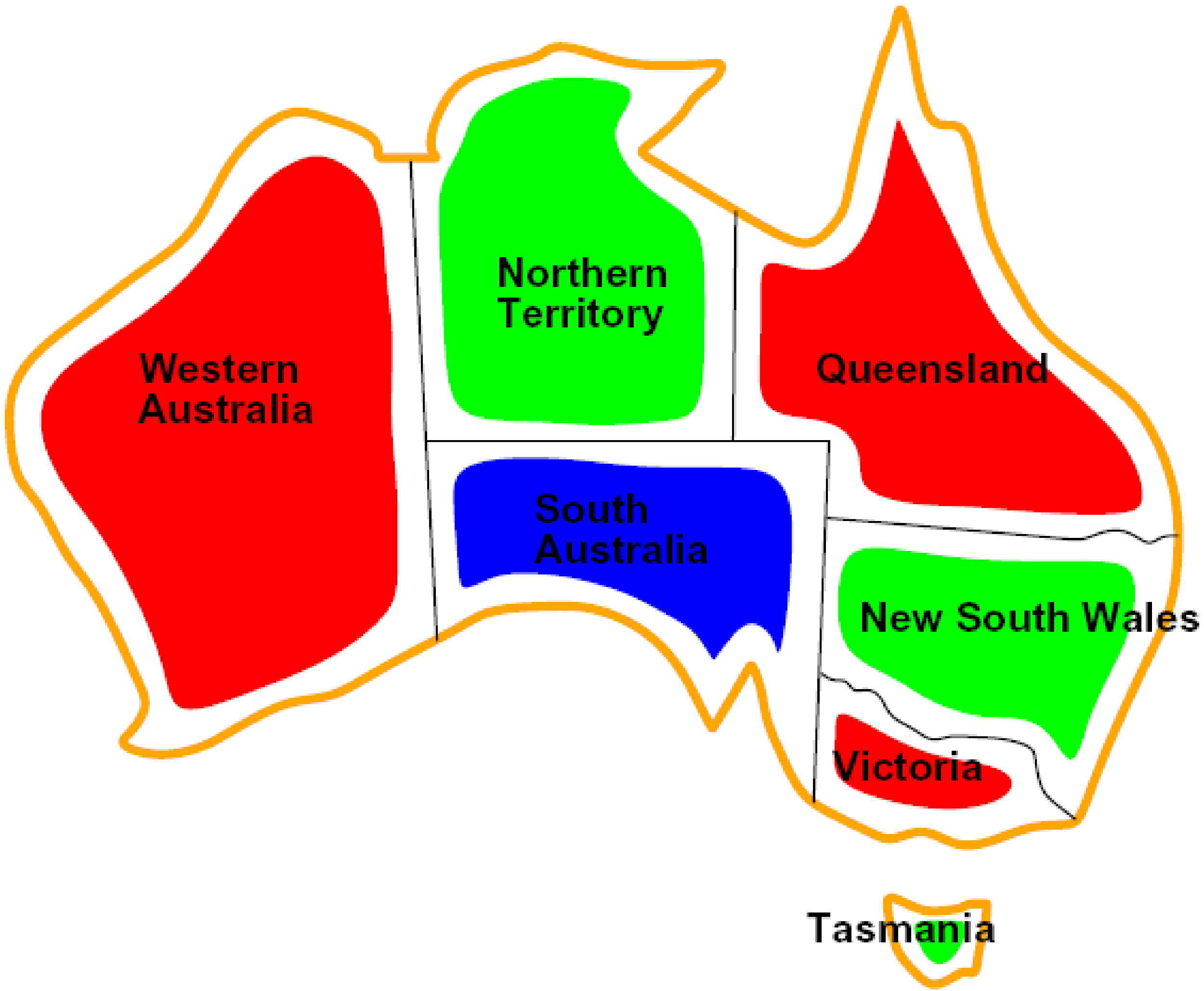
Constraint Satisfaction Problems

- Constraint satisfaction problems (CSPs):
  - A special subset of search problems
  - State is defined by a set of variables X_i and each variable has a domain D_i
  - Goal test is a set of constraints specifying allowable combinations of values for subsets of variables

CSP Examples



Example: Map Coloring

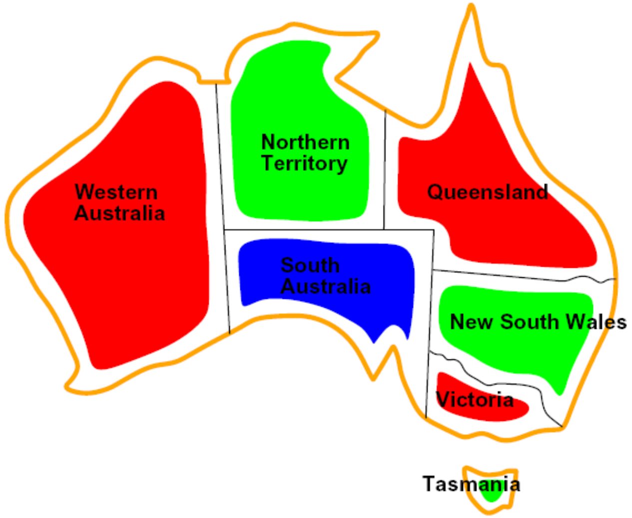
- Variables: WA, NT, Q, NSW, V, SA, T
- Domains: $D = \{\text{red}, \text{green}, \text{blue}\}$
- Constraints: adjacent regions must have different colors

  Implicit: $WA \neq NT$

  Explicit: $(WA, NT) \in \{(\text{red}, \text{green}), (\text{red}, \text{blue}), \ldots\}$

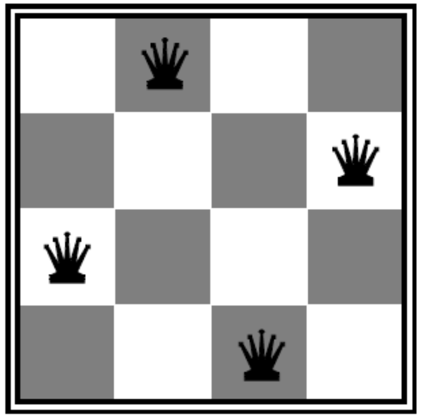
- Solutions are assignments satisfying all constraints, e.g.:

  {WA=red, NT=green, Q=red, NSW=green,
  V=red, SA=blue, T=green}



Example: N-Queens

- Formulation 1:
  - Variables: X_{ij}
  - Domains: $\{0, 1\}$
  - Constraints



$$\forall i, j, k \;\; (X_{ij}, X_{ik}) \in \{(0,0), (0,1), (1,0)\}$$
$$\forall i, j, k \;\; (X_{ij}, X_{kj}) \in \{(0,0), (0,1), (1,0)\}$$
$$\forall i, j, k \;\; (X_{ij}, X_{i+k,j+k}) \in \{(0,0), (0,1), (1,0)\}$$
$$\forall i, j, k \;\; (X_{ij}, X_{i+k,j-k}) \in \{(0,0), (0,1), (1,0)\}$$

$$\sum_{i,j} X_{ij} = N$$

Example: N-Queens

- Formulation 2:
  - Variables: Q_k (Rows)
  - Domains: $\{1, 2, 3, \ldots N\}$

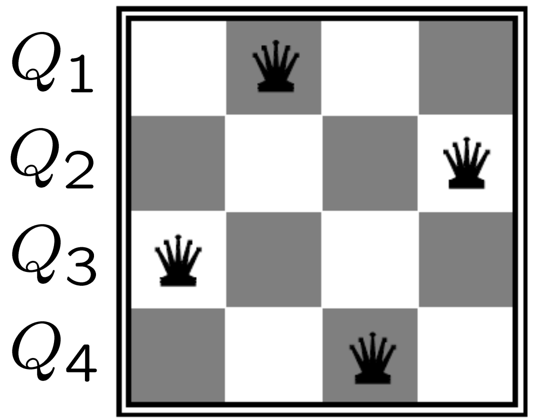
    Queen position at each row

  - Constraints:

Implicit: $\forall i, j$ non-threatening(Q_i, Q_j)

Explicit: $(Q_1, Q_2) \in \{(1,3), (1,4), \ldots\}$

$\cdots$



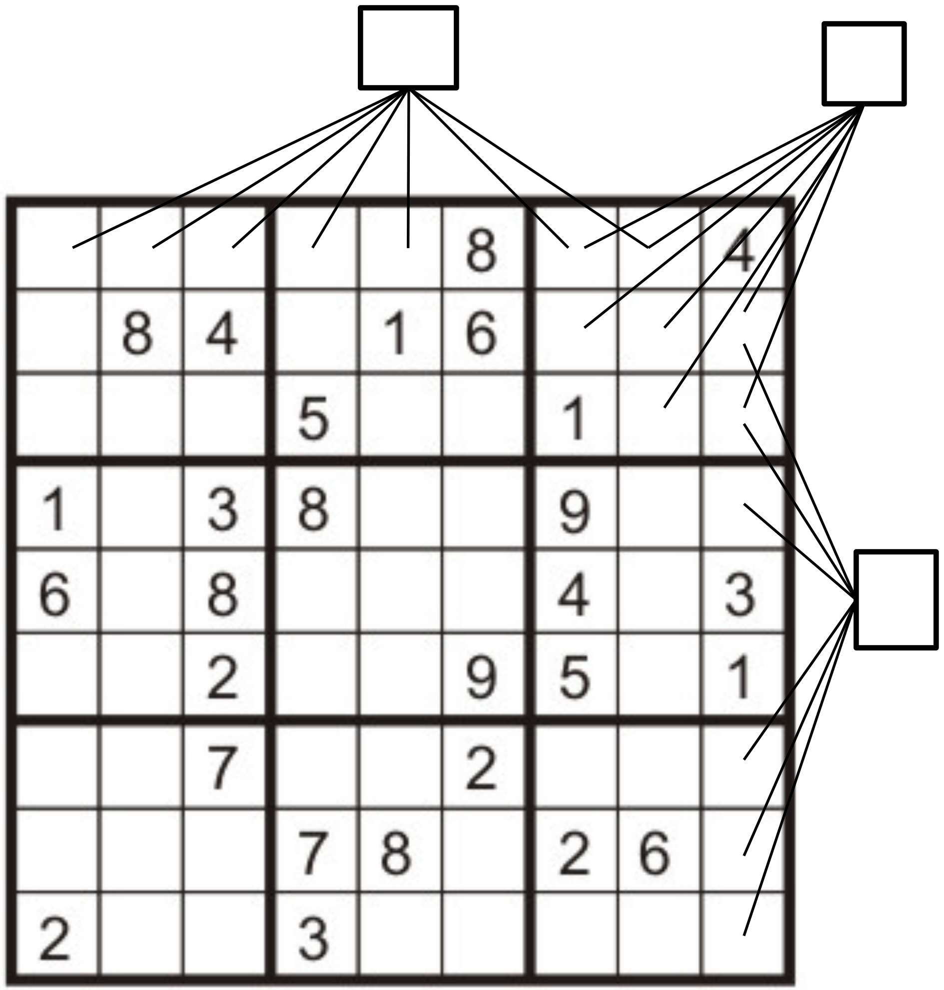
Example: Sudoku

- Variables:
  - Each (open) square
- Domains:
  - {1,2,…,9}
- Constraints:

  9-way alldiff for each column

  9-way alldiff for each row

  9-way alldiff for each region



Constraint Satisfaction Problems: Definition

Definition:
A Constraint Satisfaction Problem (CSP) consists of:

- a set of variables V
- a domain dom(V) for each variable $V \in V$
- a set of constraints C

Simple example:

- $V = \{V_1\}$
  - $dom(V_1) = \{1,2,3,4\}$
- $C = \{C_1, C_2\}$
  - $C_1: V_1 \neq 2$
  - $C_2: V_1 > 1$

Another example:

- $V = \{V_1, V_2\}$
  - $dom(V_1) = \{1,2,3\}$
  - $dom(V_2) = \{1,2\}$
- $C = \{C_1, C_2, C_3\}$
  - $C_1: V_2 \neq 2$
  - $C_2: V_1 + V_2 < 5$
  - $C_3: V_1 > V_2$

Models of a CSP

Definition:
A model of a CSP is an assignment of values to all of its variables that satisfies all of its constraints.

Simple example:

- $V = \{V_1\}$
  - $dom(V_1) = \{1,2,3,4\}$
- $C = \{C_1,C_2\}$
  - $C_1: V_1 \neq 2$
  - $C_2: V_1 > 1$

All models for this CSP:

$\{V_1 = 3\}$

$\{V_1 = 4\}$

Models of a CSP

- $\mathcal{V} = \{V_1, V_2\}$
  - $dom(V_1) = \{1,2,3\}$
  - $dom(V_2) = \{1,2\}$
- $\mathcal{C} = \{C_1, C_2, C_3\}$
  - C_1: $V_2 \neq 2$
  - C_2: $V_1 + V_2 < 5$
  - C_3: $V_1 > V_2$

Which are models for this CSP?

$\{V_1=1, V_2=1\}$

$\{V_1=2, V_2=1\}$ ✓

$\{V_1=3, V_2=1\}$ ✓

$\{V_1=3, V_2=2\}$

Varieties of Variables

- Discrete Variables
  - Finite domains
    - Size d means $O(d^n)$ complete assignments
  - Infinite domains (integers, strings, etc.)

- Continuous variables
  - E.g., start/end times for Hubble Telescope observations

Varieties of Constraints

- Varieties of Constraints
  - Unary constraints involve a single variable (equivalent to reducing domains), e.g.:

$$SA \neq green$$

  - Binary constraints involve pairs of variables, e.g.:

$$SA \neq WA$$

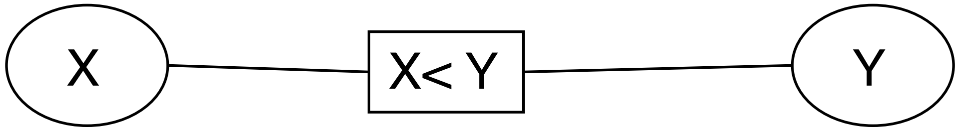
  - Higher-order constraints involve 3 or more variables

Constraint Network: Definition

Definition: A constraint network is defined by a graph, with

- one node for every variable (drawn as circle)
- one node for every constraint (drawn as rectangle)
- Edges/arcs running between variable nodes and constraint nodes whenever a given variable is involved in a given constraint.

- Two variables X and Y
- One constraint: X<Y



Constraint Network: Definition

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- one node for every variable (drawn as circle)
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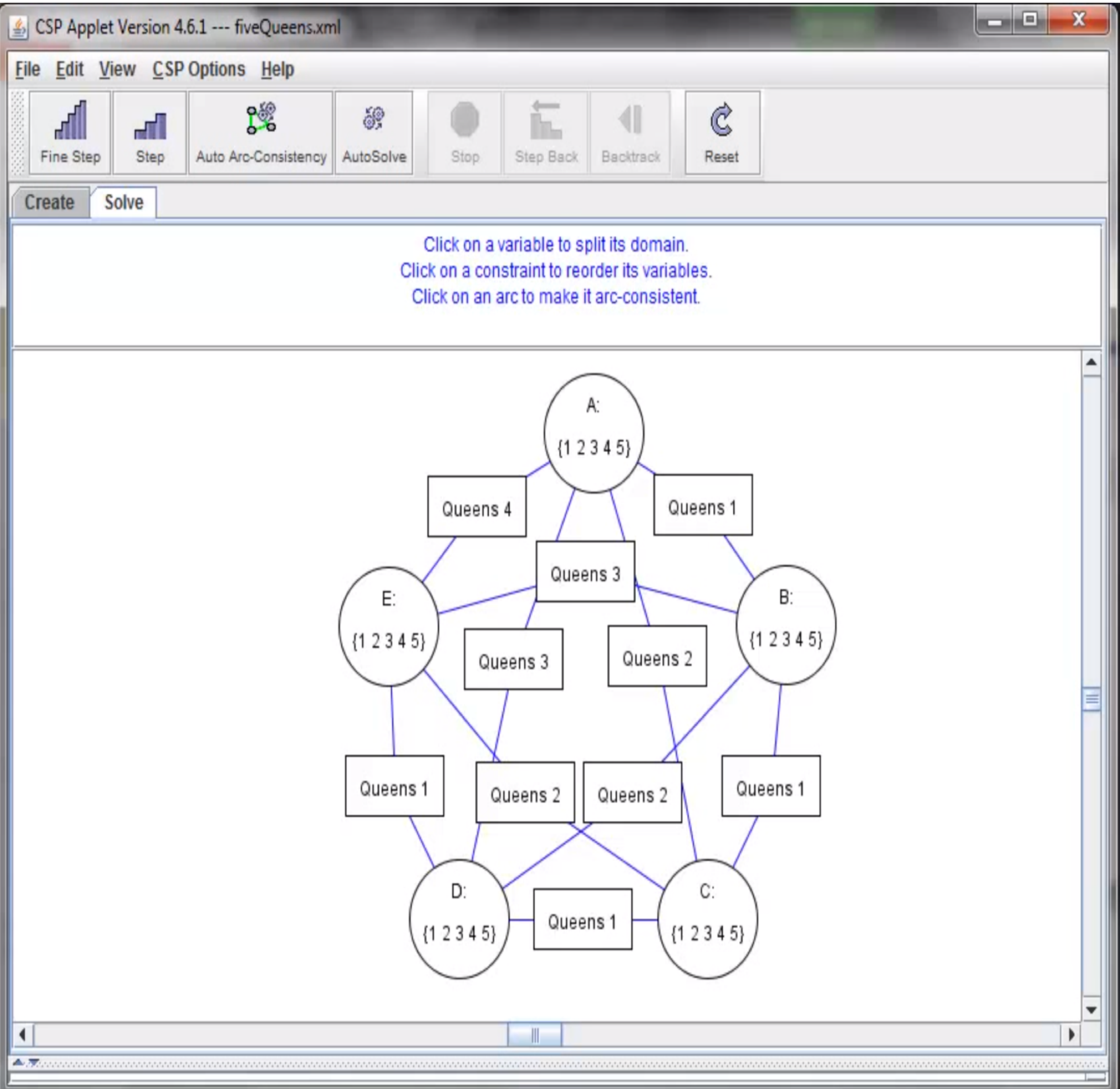
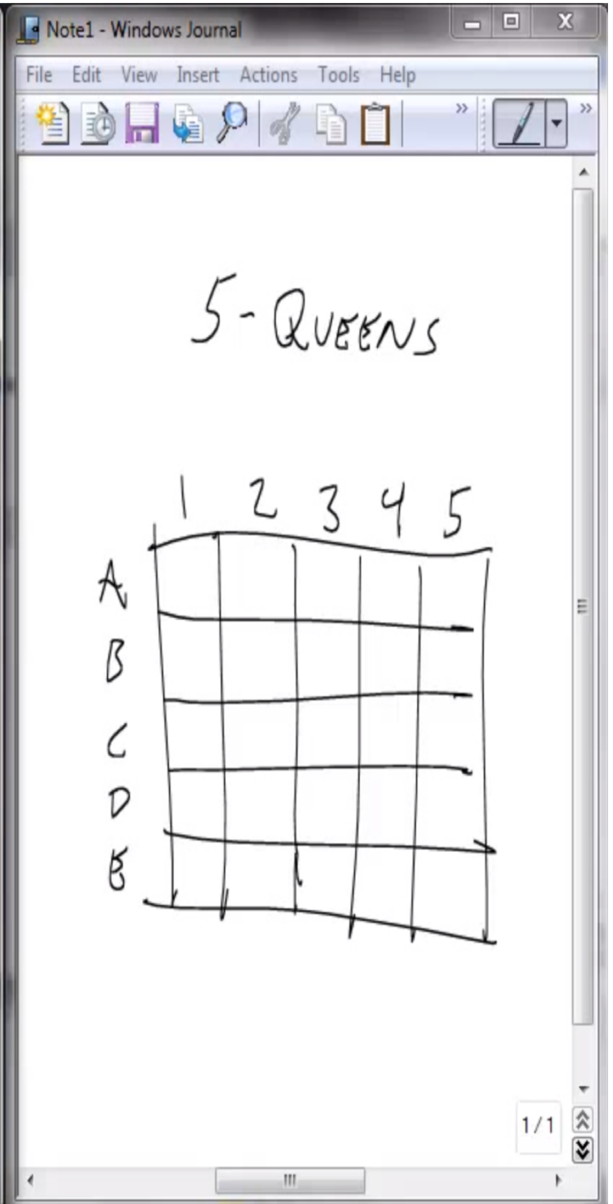
- Edges/arcs running between variable nodes and constraint nodes whenever a given variable is involved in a given constraint.

- Whiteboard example: 3 Variables A,B,C
  - 3 Constraints: A<B, B<C, A+3=C
  - 6 edges/arcs in the constraint network:
    - ⟨A,A<B⟩ , ⟨B,A<B⟩
    - ⟨B,B<C⟩ , ⟨C,B<C⟩
    - ⟨A, A+3=C⟩ , ⟨C,A+3=C⟩

AIspace CSP Demos





http://www.aispace.org/

Real-World CSPs

- Assignment problems: e.g., who teaches what class
- Timetabling problems: e.g., which class is offered when and where?
- Hardware configuration
- Transportation scheduling
- Factory scheduling
- Circuit layout
- Fault diagnosis
- … lots more!

- Many real-world problems involve real-valued variables…

Solving CSPs

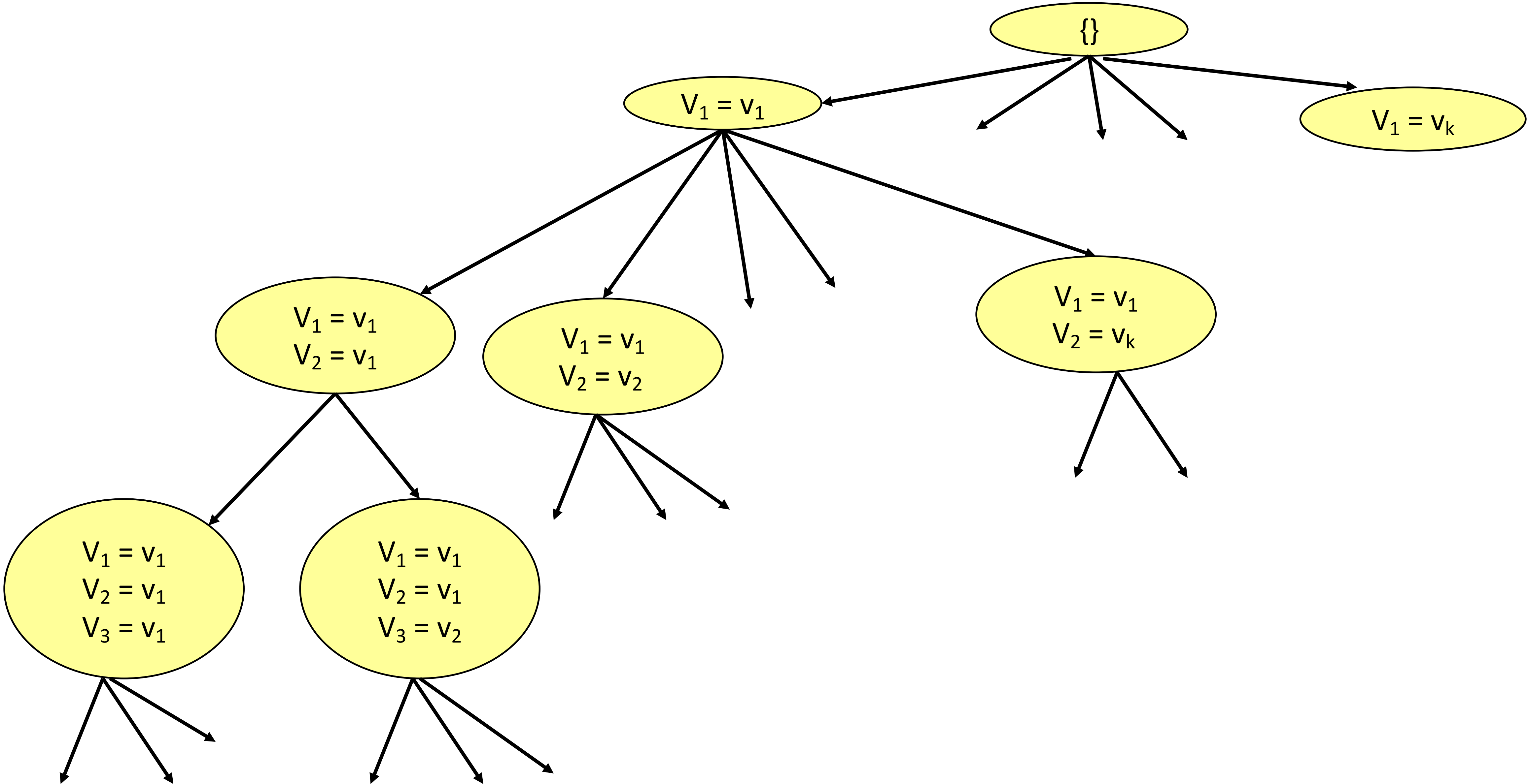
Standard Search Formulation

- Standard search formulation of CSPs

- States defined by the values assigned so far (partial assignments)
  - Initial state: the empty assignment, {}
  - Successor function: assign a value to an unassigned variable
  - Goal test: the current assignment is complete and satisfies all constraints

- We'll start with the straightforward, naïve approach, then improve it

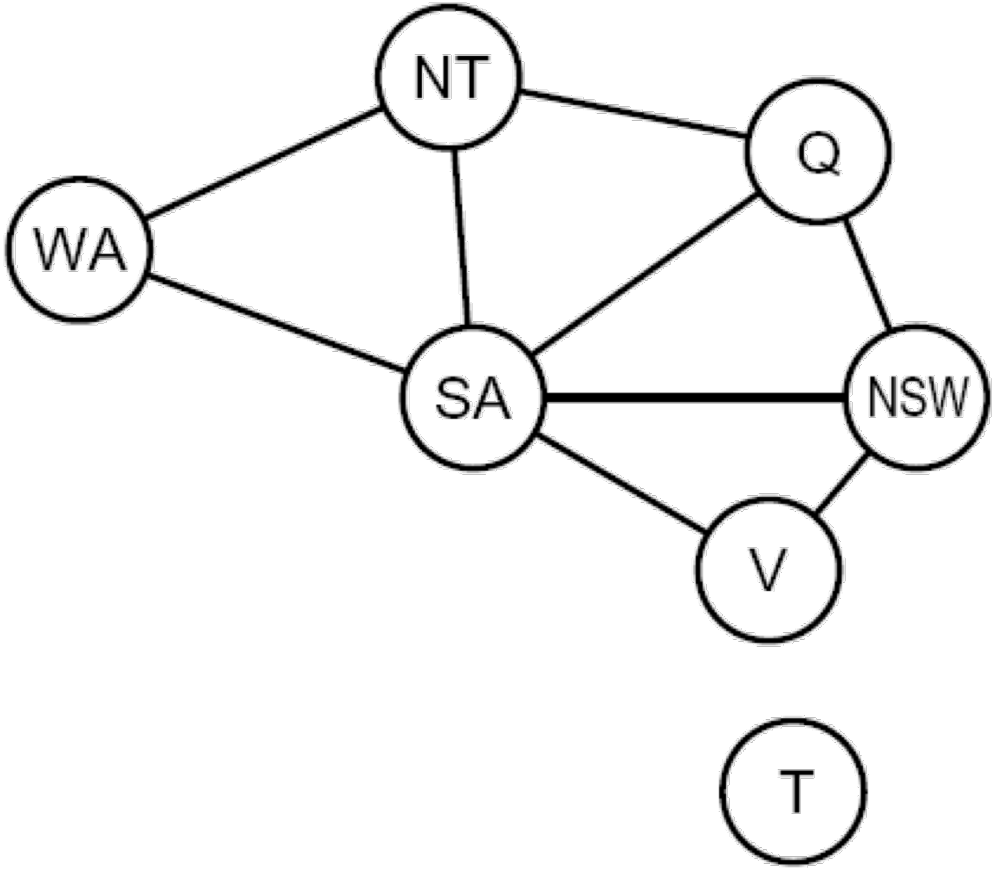
Standard Search Formulation: Example



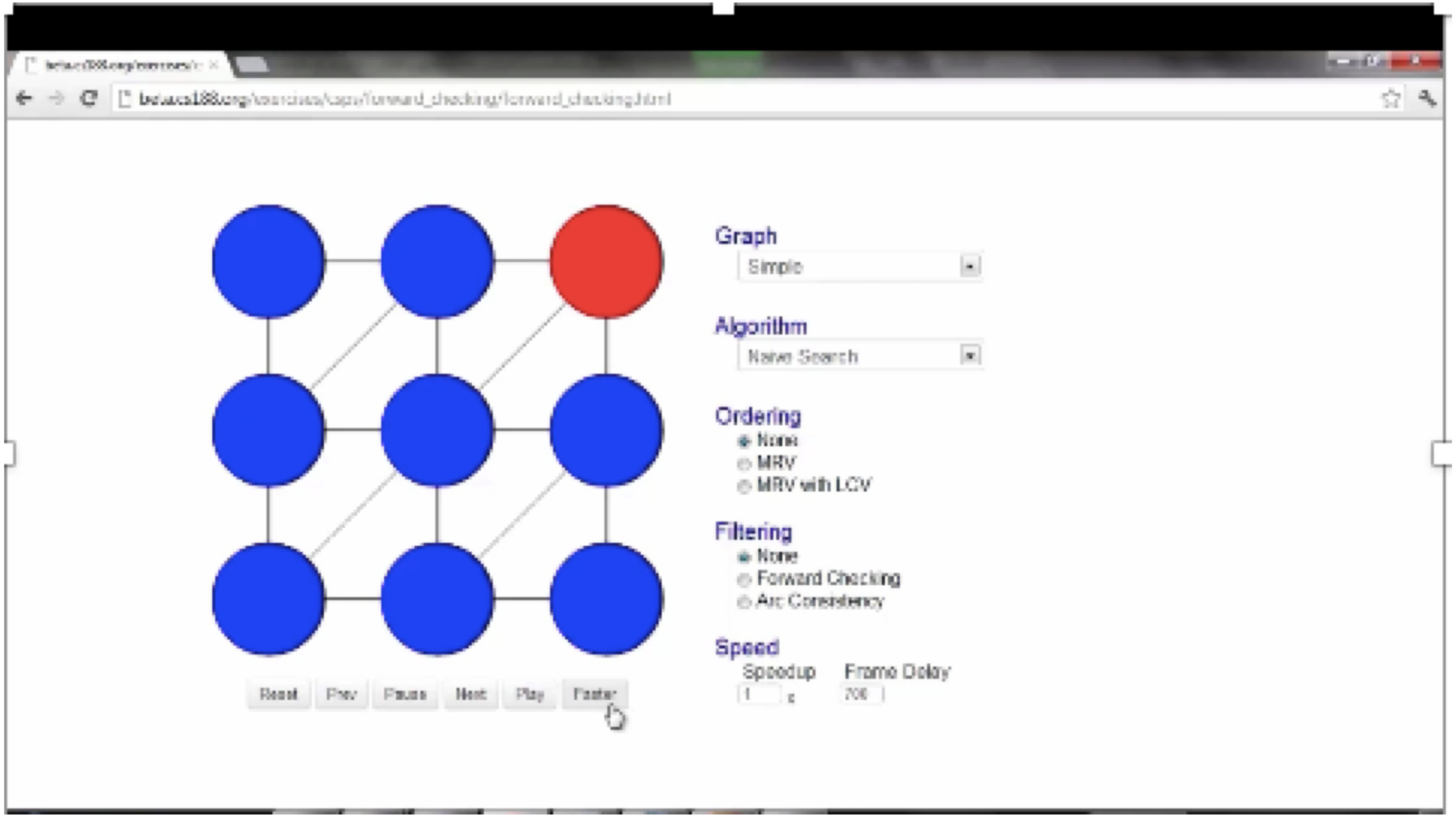
Search Methods

- What would BFS do?

- What would DFS do?



Video of Demo Coloring -- DFS



Backtracking Search

Backtracking Search

- Backtracking search is the basic uninformed algorithm for solving CSPs

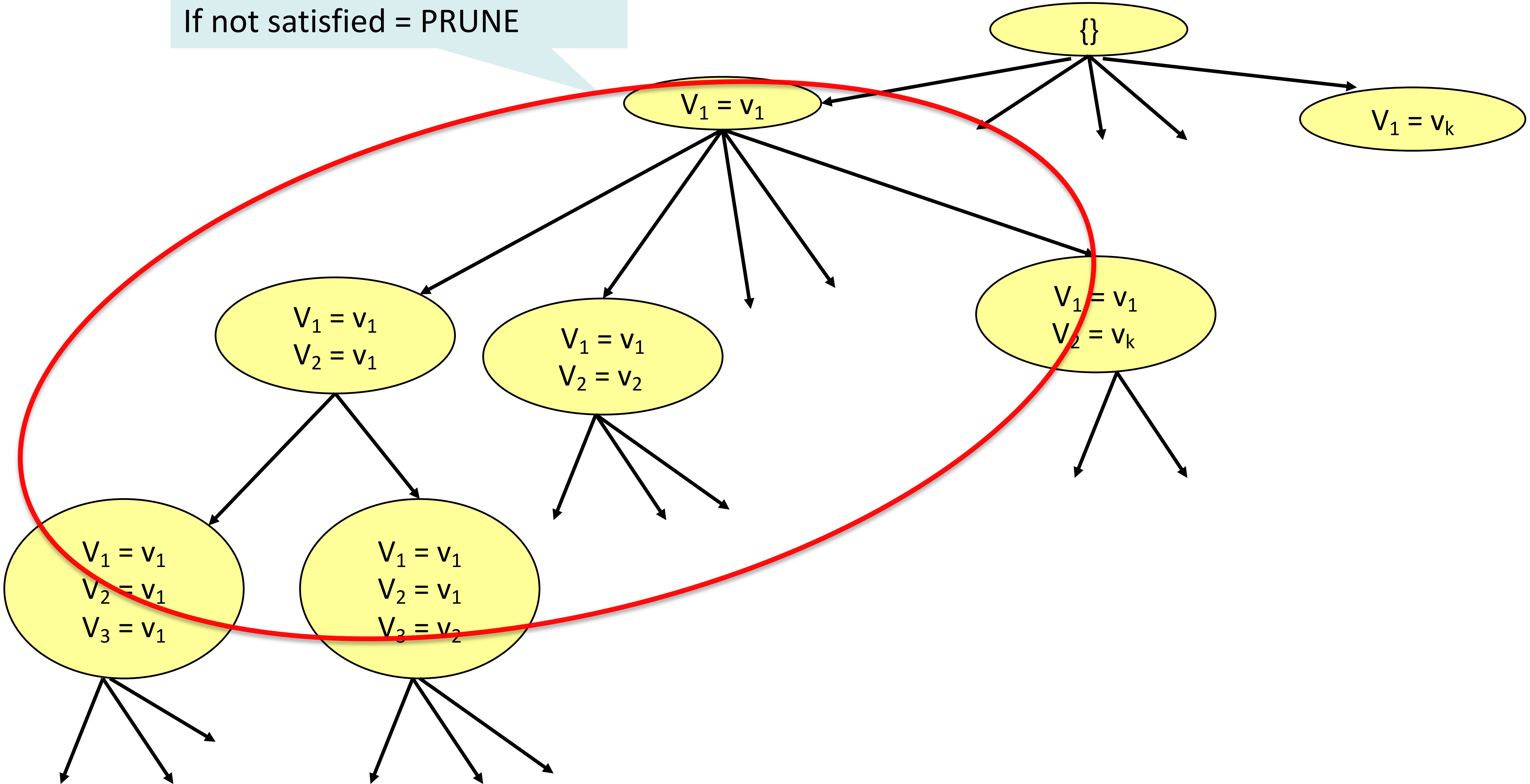
- Idea: Check constraints as you go
  - I.e. consider only values which do not conflict previous assignments
  - Might have to do some computation to check the constraints
  - "Incremental goal test"

- Depth-first search with this improvement is called *backtracking search*

- Can solve n-queens for $n \approx 25$

Backtracking Example

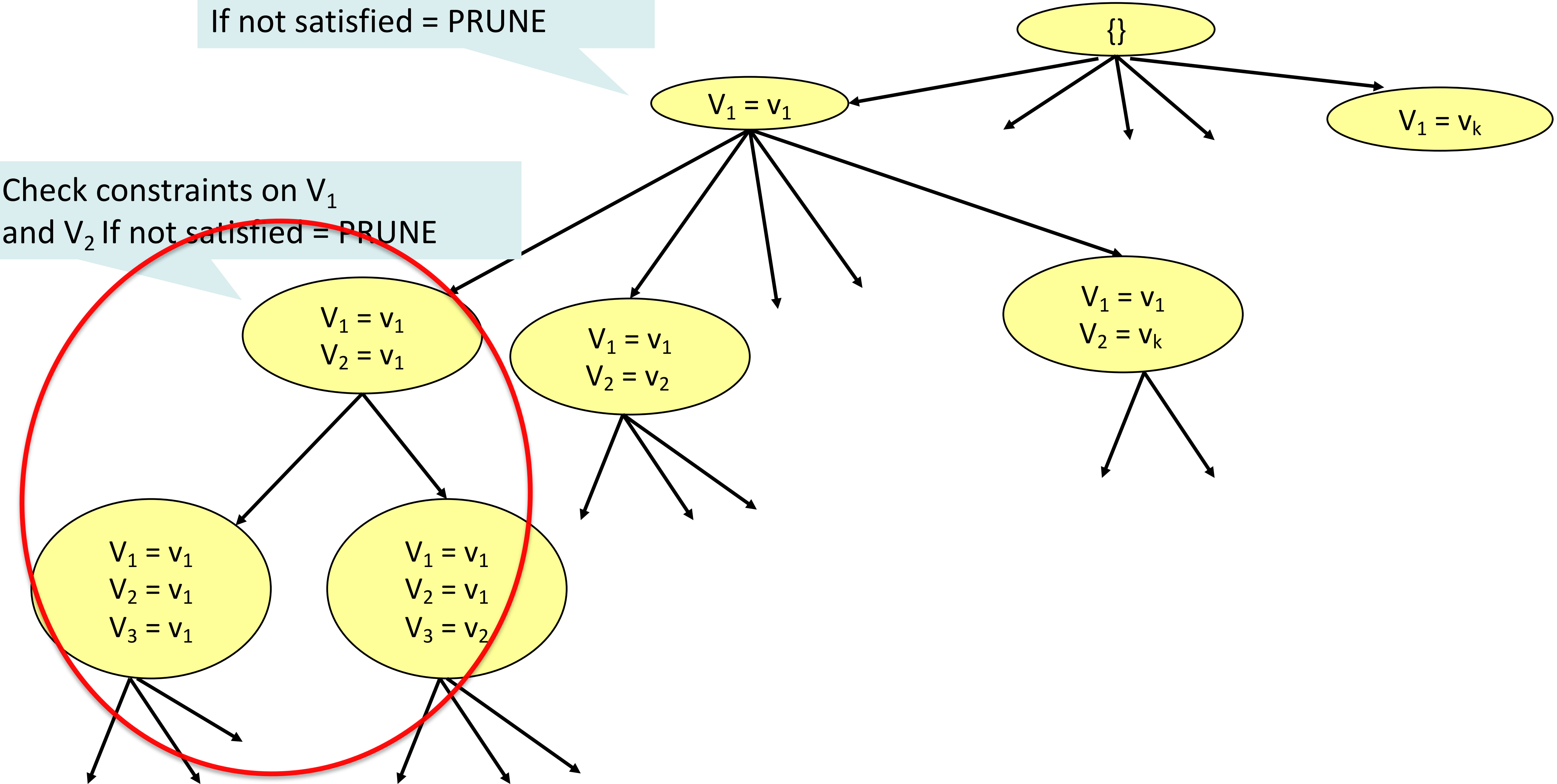
Check unary constraints on V_1
If not satisfied = PRUNE



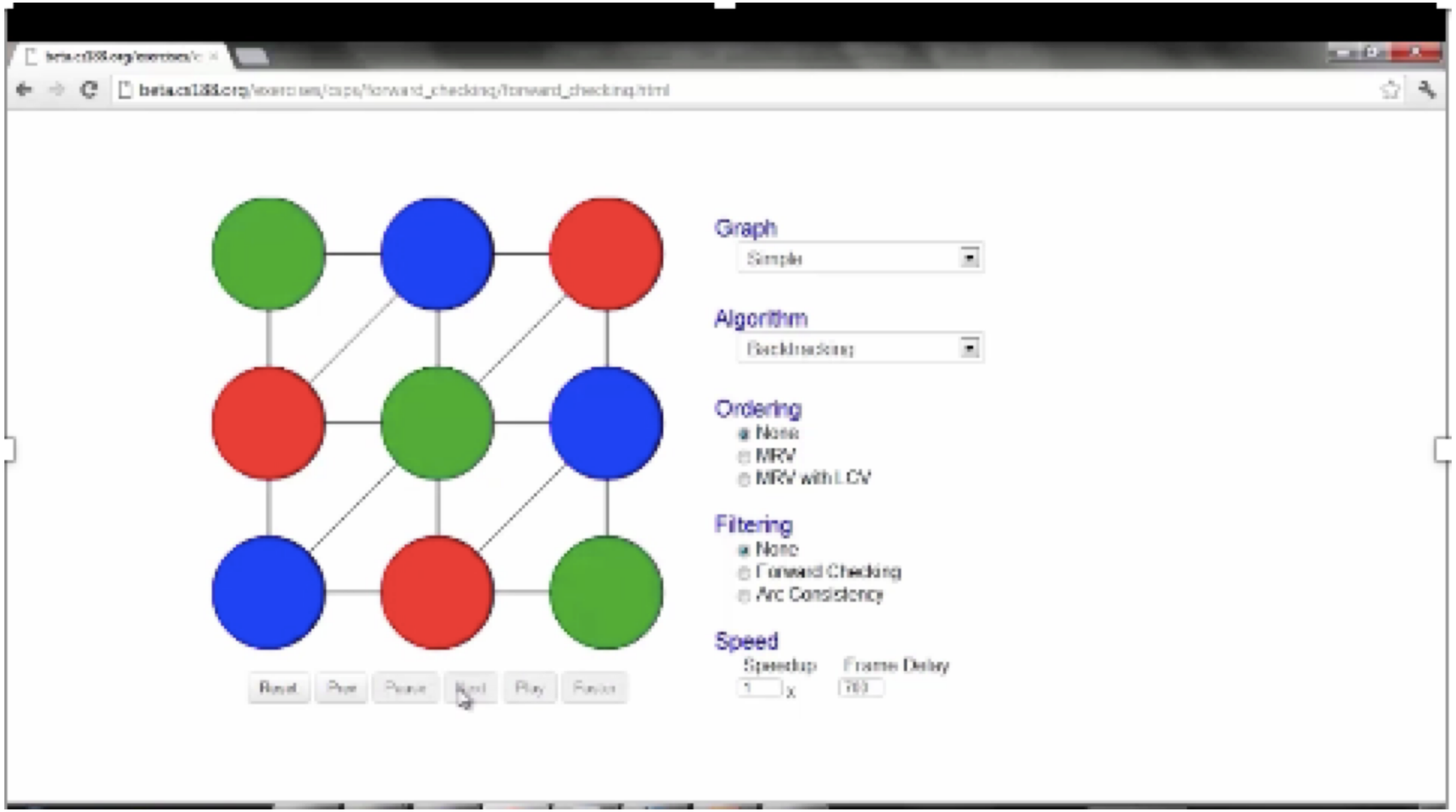
Backtracking Example

Check unary constraints on V_1
If not satisfied = PRUNE

Check constraints on V_1
and V_2 If not satisfied = PRUNE



Video of Demo Coloring – Backtracking



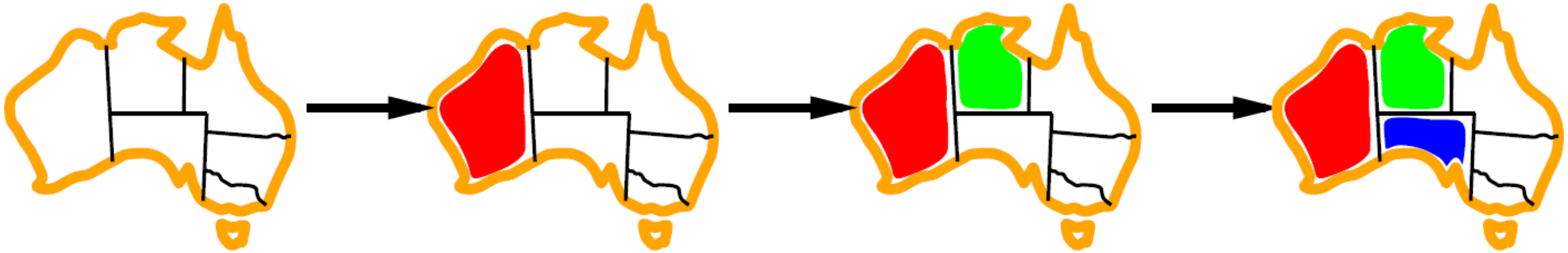
Improving Backtracking

- General-purpose ideas give huge gains in speed
- Ordering:
  - Which variable should be assigned next?
  - In what order should its values be tried?
- Filtering: Can we detect inevitable failure early?

Ordering

Ordering: Minimum Remaining Values

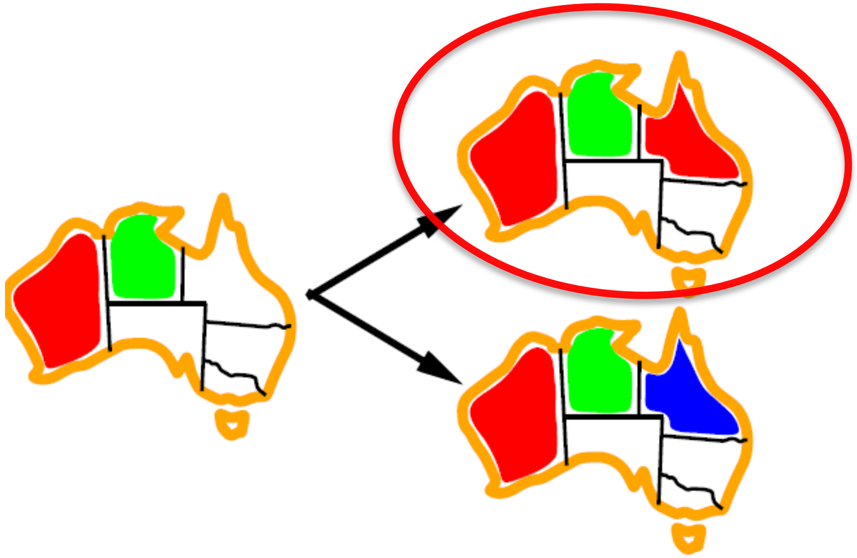
- Variable Ordering: Minimum Remaining Values (MRV):
  - Choose the variable with the fewest legal values left in its domain



- Why min rather than max? **Fail fast variable**
- Also called “most constrained variable”
- “Fail-fast” ordering

Ordering: Least Constraining Value

- Value Ordering: Least Constraining Value
  - Given a choice of variable, choose the *least constraining value*
  - I.e., the one that rules out the fewest values in the remaining variables
  - Note that it may take some computation to determine this!
- Why least rather than most?
  - Tries to avoid failure by assigning values that leave maximal flexibility for the remaining variables.
- Combining these ordering ideas makes 1000 queens feasible



Filtering

Can we detect inevitable failure early?

Node Consistency

- Prune the domains as much as possible before searching for a solution.

Definition: A variable is Node Consistent if no value of its domain is ruled impossible by any unary constraints.

- Example: $dom(V_2) = \{1, 2, 3, 4\}$. $V_2 \neq 2$
- Variable V_2 is not node consistent.

- It is node consistent once we remove 2 from its domain.

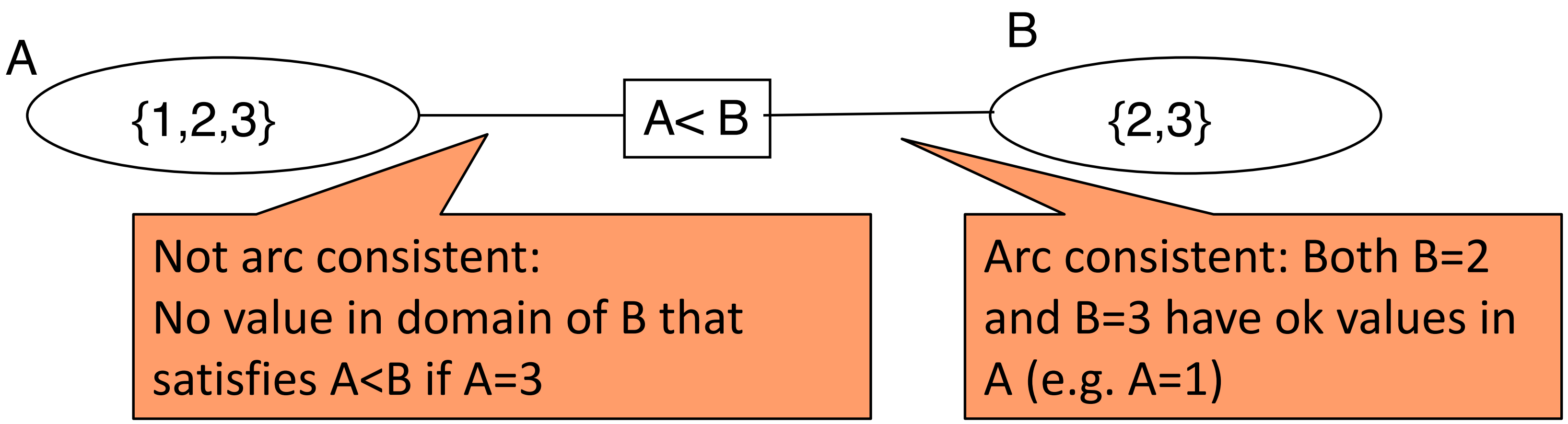
- Trivial for unary constraints. Trickier for k-ary ones.

Arc Consistency

Definitions:

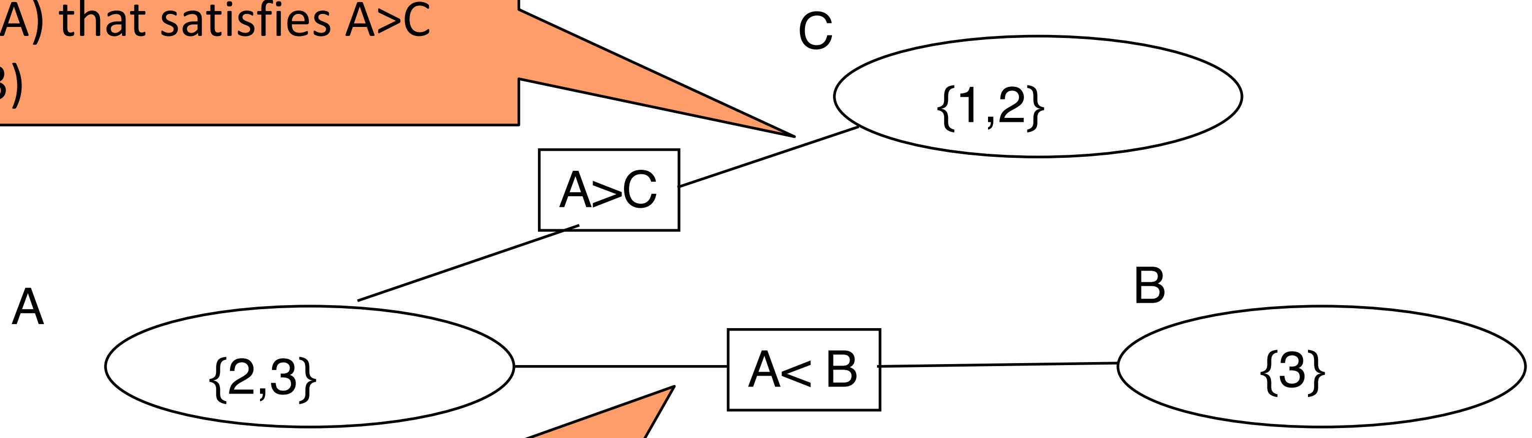
An arc <X, c(X,Y)> is arc consistent if for each value x in dom(X) there is some value y in dom(Y) such that c(X,Y) is satisfied.

A network is arc consistent if all its arcs are arc consistent.



Arc Consistency

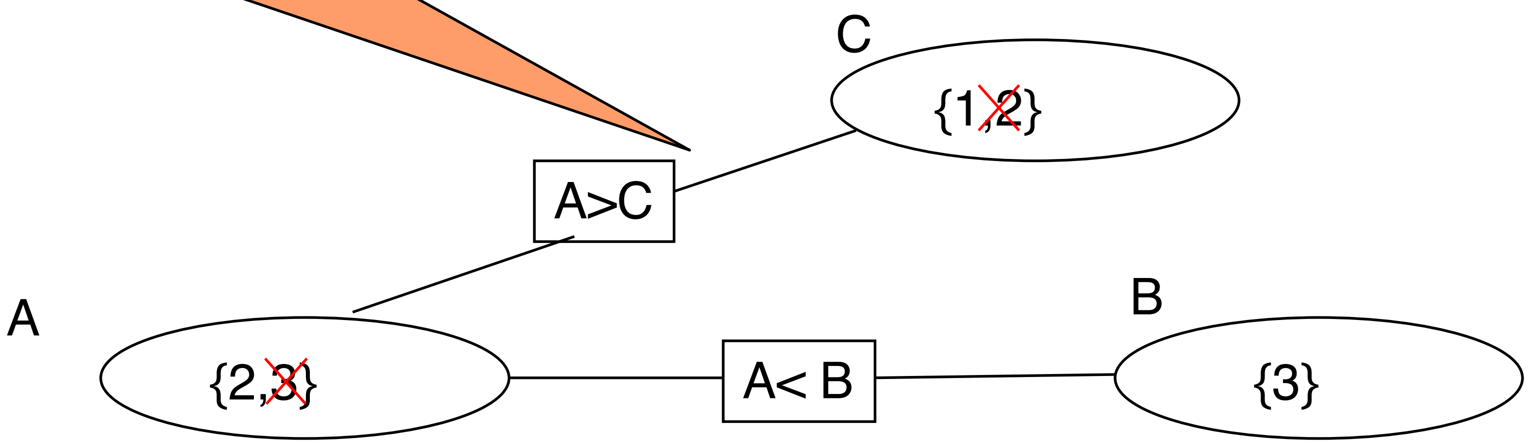
Arc consistent:
For each value in dom(C), there is one in dom(A) that satisfies A>C (namely A=3)



Not arc consistent:
No value in domain of B that satisfies A<B if A=3

Arc Consistency

Not arc consistent anymore:
For C=2, there is no value in dom(A) that satisfies A>C



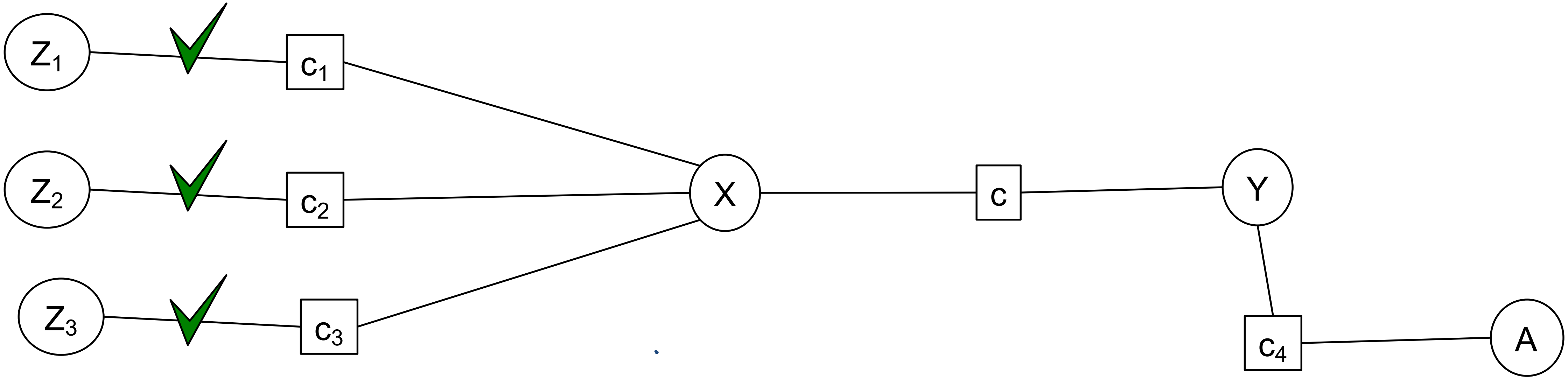
Arc Consistency: High Level Strategy

- Consider the arcs in turn, making each arc consistent
  - Reconsider arcs that could be made inconsistent again by this pruning of the domains
- Eventually reach a ‘fixed point’: all arcs consistent
- Run ‘Simple Problem 1’ in AIspace for an example:

- Arc Consistency can be run as a preprocessor or after each assignment.

Arc Consistency Quiz

- When we reduce the domain of a variable X to make an arc ⟨X,c⟩ arc consistent, which arcs do we need to reconsider?



- You do not need to reconsider other arcs
  - If arc ⟨Y,c⟩ was arc consistent before, it will still be arc consistent
  - If an arc ⟨X,c'⟩ was arc consistent before, it will still be arc consistent
  - Nothing changes for arcs of constraints not involving X

Arc Consistency Algorithm: Interpreting Outcomes

- Three possible outcomes
  (when all arcs are arc consistent):
  - Each domain has a single value
    - We have a (unique) solution.
  - At least one domain is empty
    - No solution! All values are ruled out for this variable.
  - Some domains have more than one value
    - There may be a solution, multiple ones, or none
    - Need to solve this new CSP (usually simpler) problem:
      same constraints, domains have been reduced

Arc Consistency Quiz

Can we have an arc consistent network with non-empty domains that has no solution?

YES ✓ NO

- Example: vars A, B, C with domain {1, 2} and constraints $A = B$, $B = C$, $A \neq C$
- Or see AIspace CSP applet "Simple Problem 2"

Domain Splitting (or Case Analysis)

- Arc consistency ends: Some domains have more than one value $\rightarrow$ may or may not have a solution
  A. Apply Backtracking Search or
  B. Split the problem in a number of disjoint cases:

     CSP with dom(X) = $\{x_1, x_2, x_3, x_4\}$ becomes

     CSP_1 with dom(X) = $\{x_1, x_2\}$ and
     CSP_2 with dom(X) = $\{x_3, x_4\}$

- Solution to CSP is the union of solutions to CSP_i

Searching by Domain Splitting

CSP, apply AC

If domains with multiple values
Split on one

CSP$_1$, apply AC

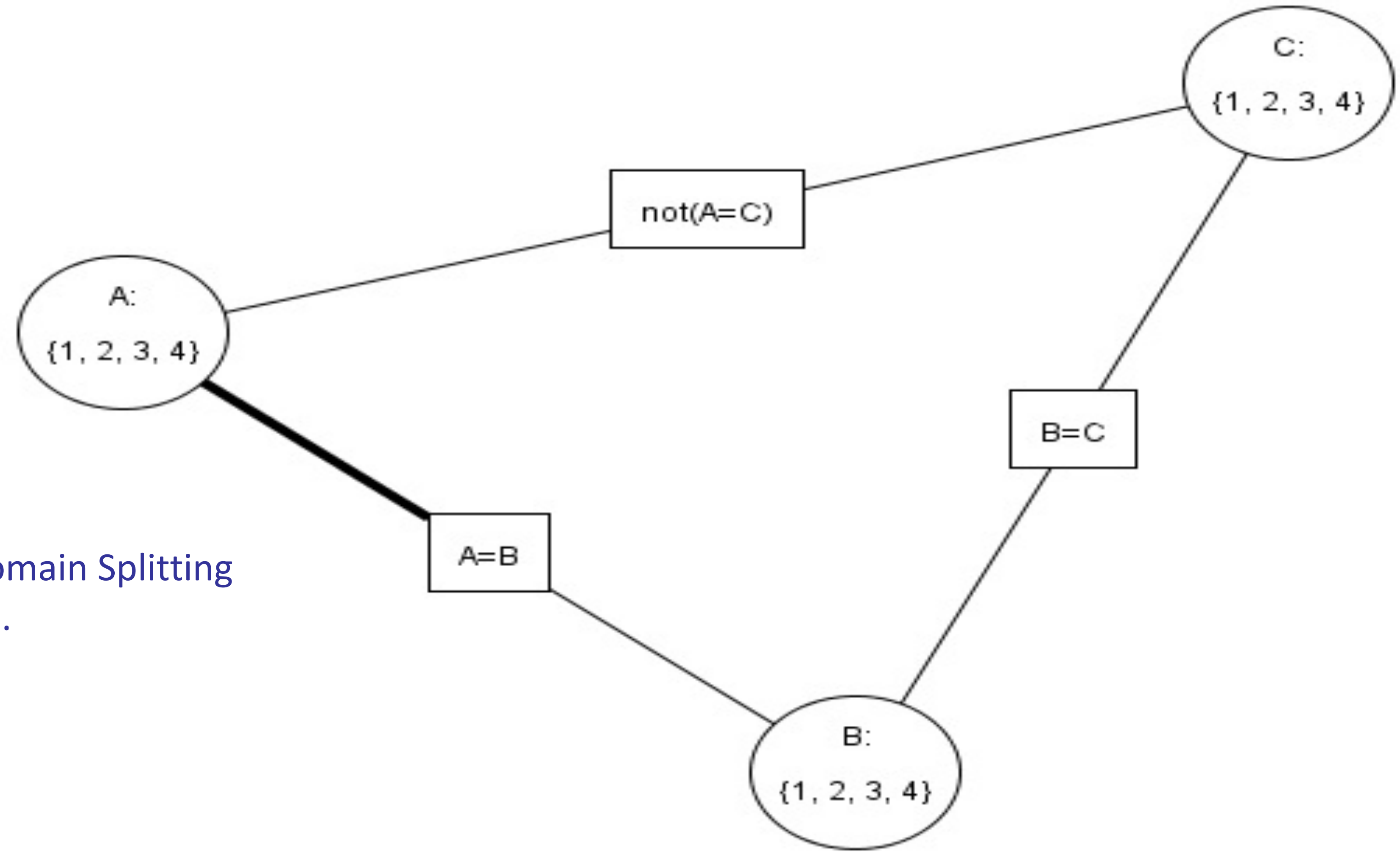
If domains with multiple values,
Split on one

CSP$_2$, apply AC

If domains with multiple values,
Split on one

Example: ‘Simple Problem 2’ in AIspace

- 3 variables: A, B, C
- Domains: all {1,2,3,4}
- A=B, B=C, A≠C



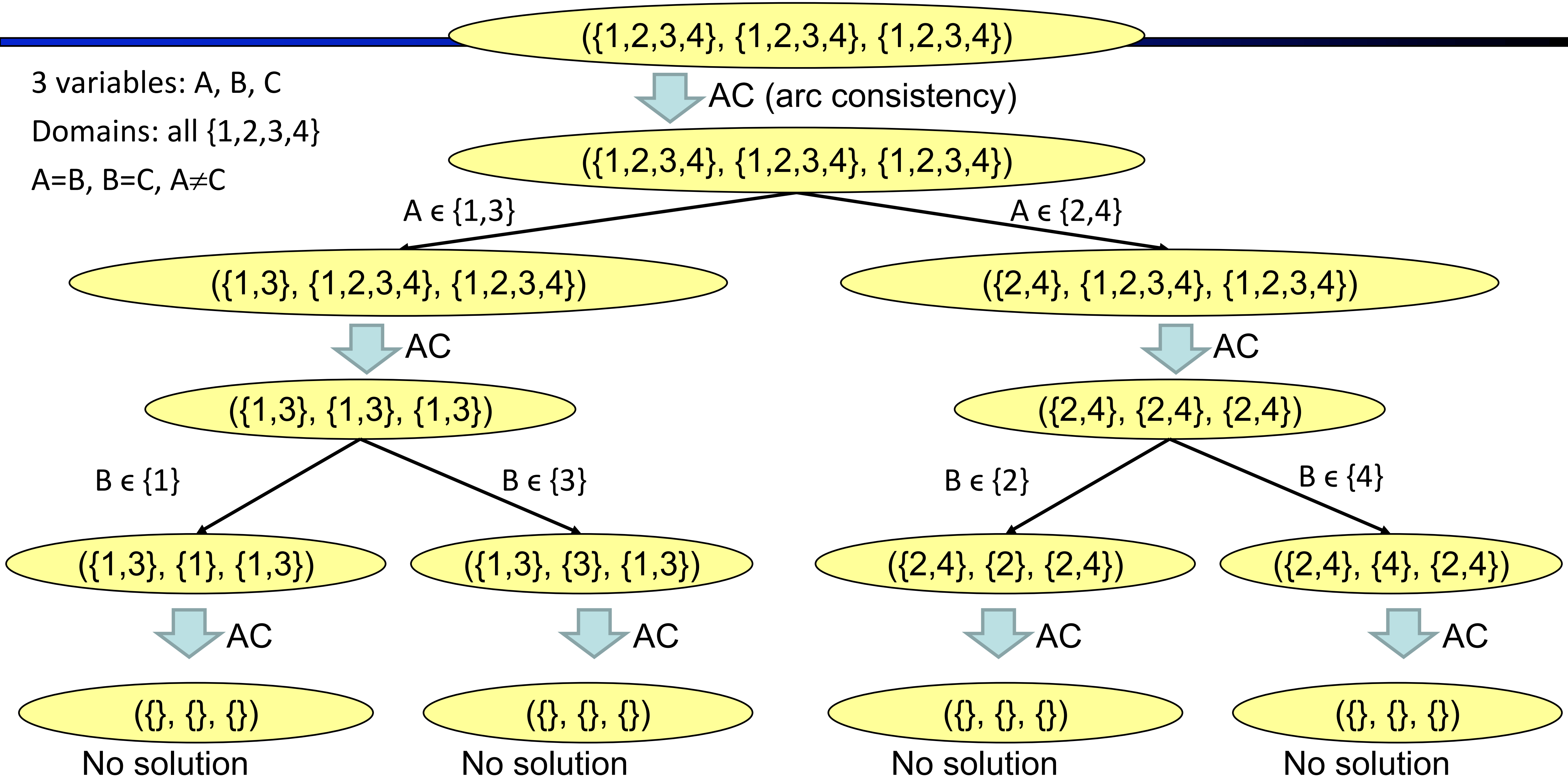
- Trace Arc Consistency + Domain Splitting for this network in AIspace.

Example: ‘Simple Problem 2’ in AIspace

3 variables: A, B, C

Domains: all {1,2,3,4}

A=B, B=C, A≠C

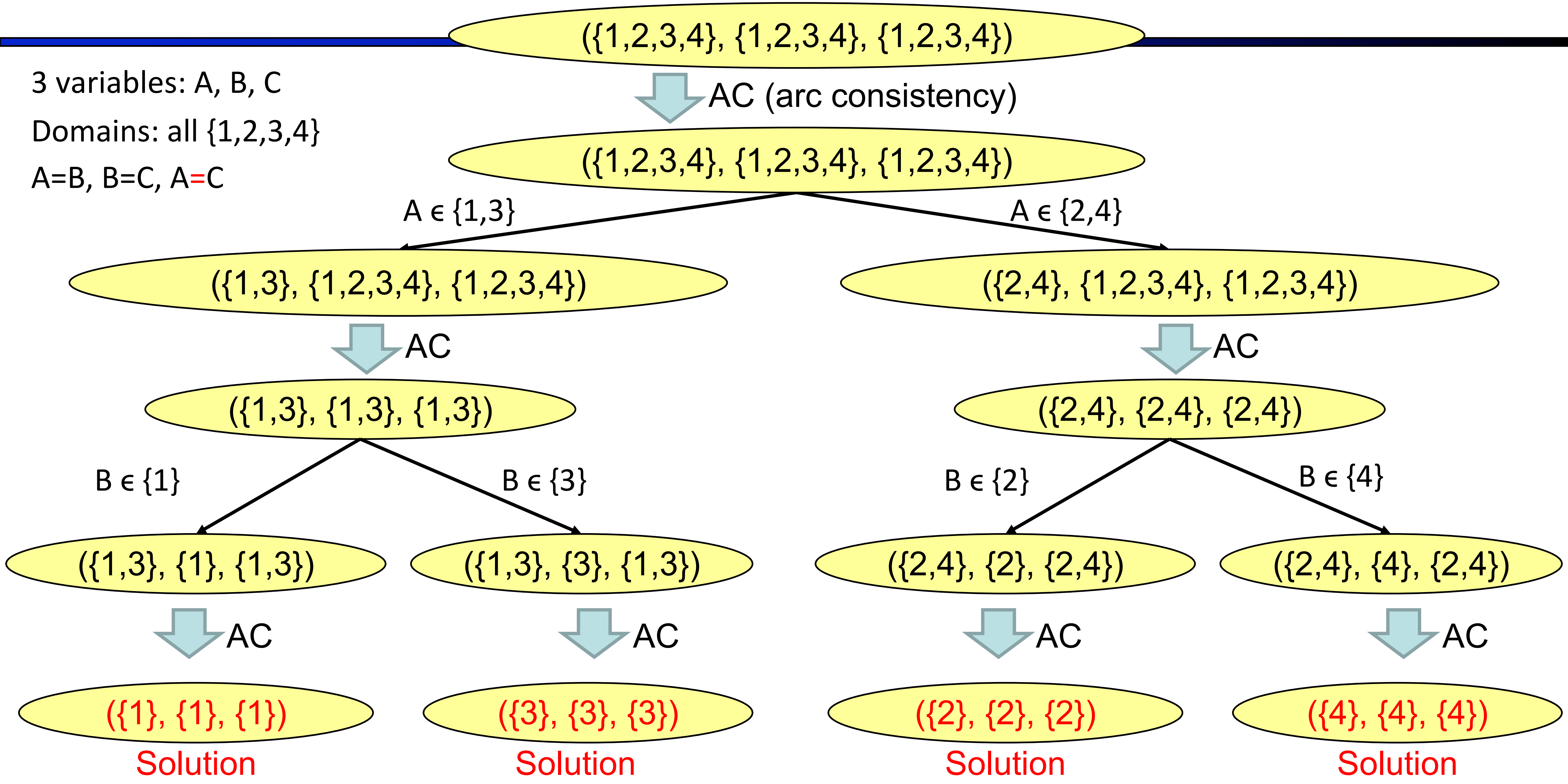


Example: ‘Simple Problem 2’ in AIspace

3 variables: A, B, C

Domains: all {1,2,3,4}

A=B, B=C, A=C



A quick note on local search for CSPs

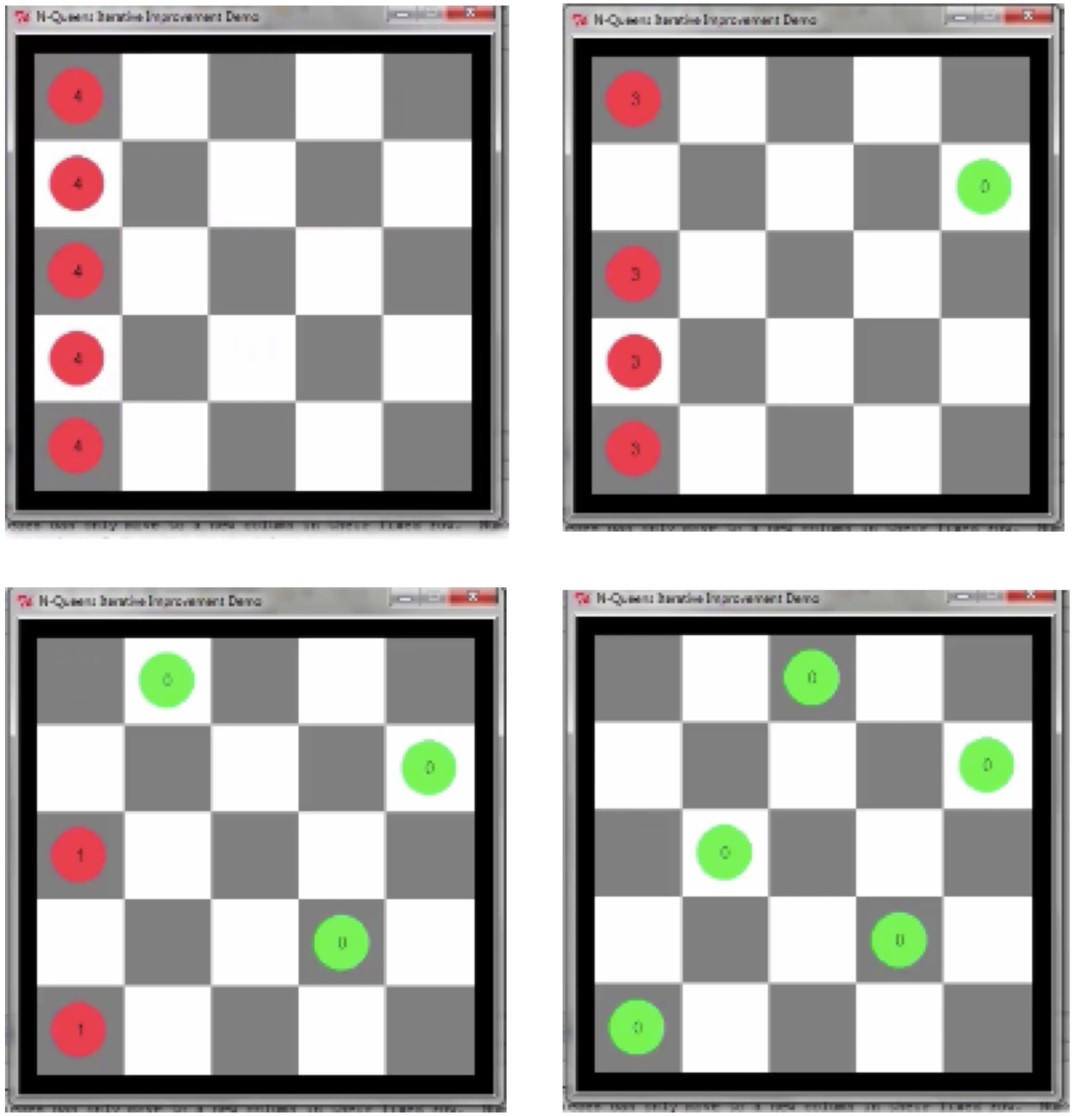
Greedy Descent with Min-Conflict Heuristic

- Algorithm: While not solved,
  - Variable selection: ***randomly select*** any conflicted variable
  - Value selection: min-conflicts heuristic:
    - Choose a value that violates the fewest constraints
    - Break ties randomly

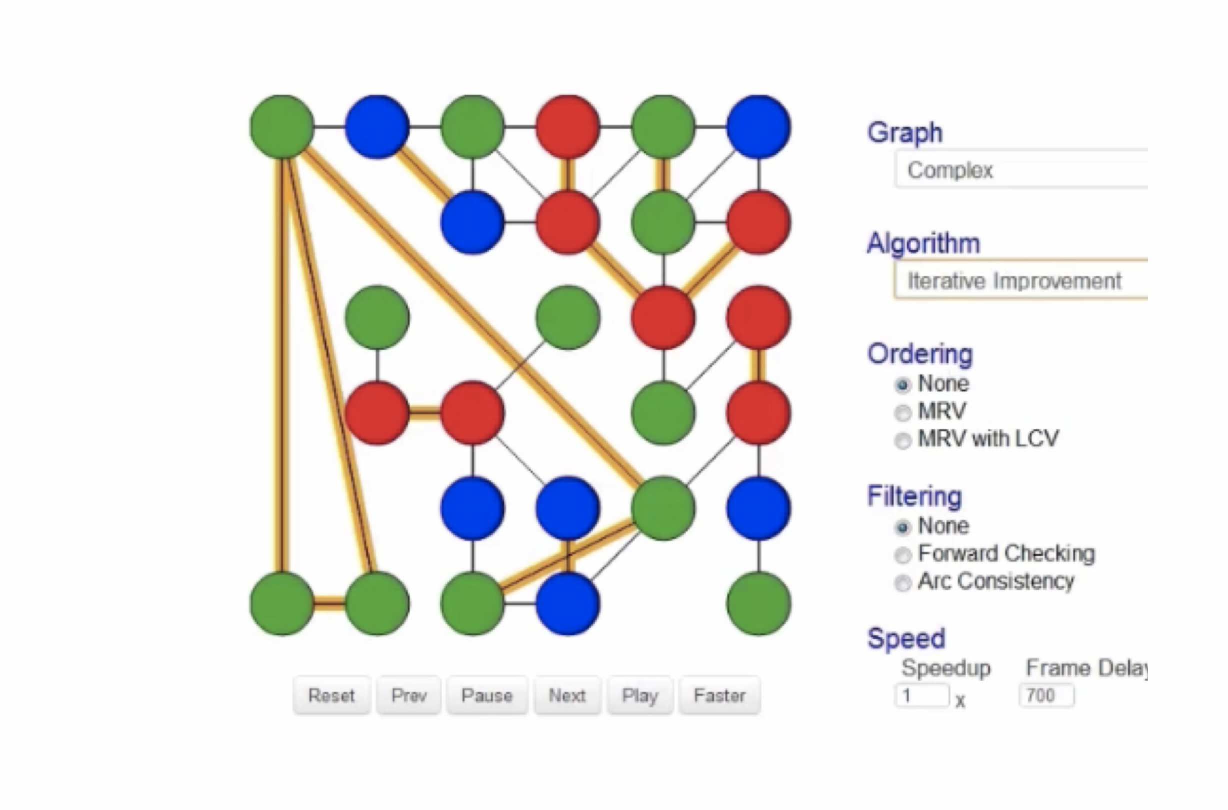
- One of the best local search techniques for CSP solving

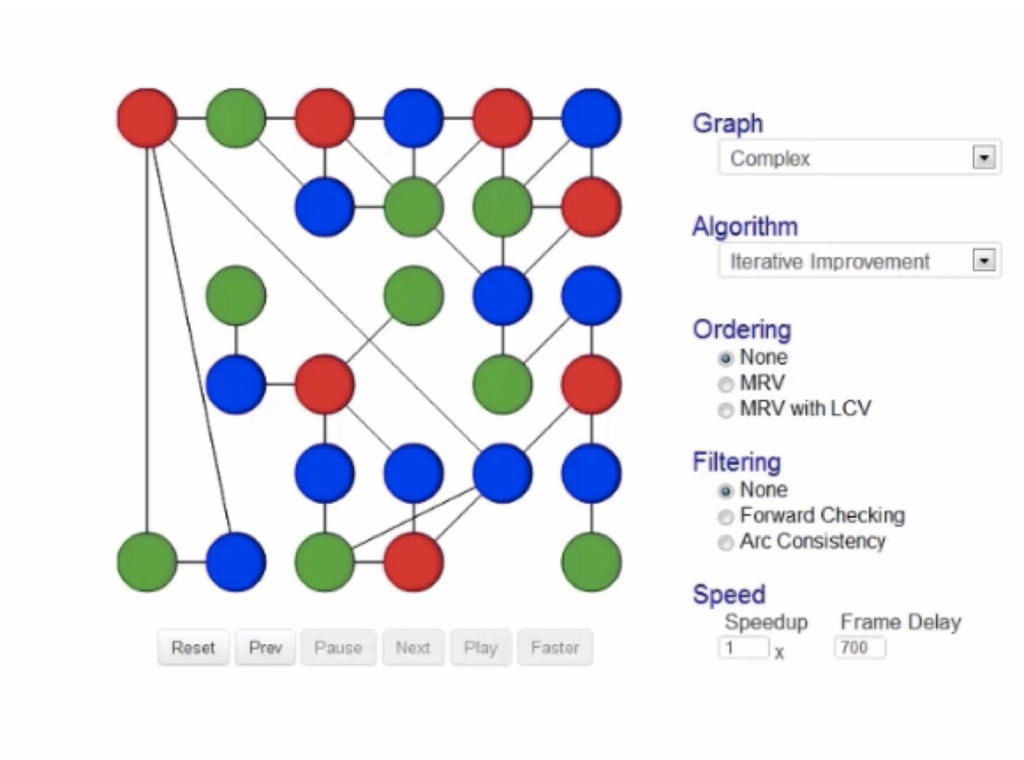
- Given random initial state, can solve n-queens in almost constant time for arbitrary n with high probability (e.g., n = 10,000,000)!

Demo Min-Conflict in N-Queens



Demo Min-Conflict – Coloring





Constraint Optimization Problems

- Constraint Satisfaction Problems
  - Hard constraints: need to satisfy all of them
  - All models are equally good
- Constraint Optimization Problems
  - Hard constraints: need to satisfy all of them
  - Soft constraints: need to satisfy them as well as possible **lower priority**
  - Can have weighted constraints
    - Minimize h(n) = sum of weights of constraints unsatisfied in n
    - Hard constraints have a very large weight
    - Some soft constraints can be more important than other soft constraints $\rightarrow$ larger weight
  - All local search methods work just as well for constraint optimization
    - all they need is an evaluation function h

Example for Constraint Optimization Problem

Exam scheduling

- Hard constraints:
  - Cannot have an exam in too small a room
  - Cannot have multiple exams in the same room in the same time slot
  - …
- Soft constraints
  - Students should not have multiple exams on the same day
  - It would be nice if students had their exams spread out
  - …

Local Search Generality: Dynamically Changing CSPs

- The problem may change over time
  - Particularly important in scheduling
  - E.g., schedule for airline:
    - Thousands of flights and thousands of personnel assignments
    - A storm can render the schedule infeasible

- Goal: Repair the schedule with minimum number of changes
  - Often easy for Local Search starting from the current schedule
  - Other techniques usually:
    - Require more time
    - Might find solutions requiring many more changes

Summary: CSPs

- CSPs are a special kind of search problem:
  - States are partial assignments
  - Goal test defined by constraints
- Basic solution: backtracking search
- Speed-ups:
  - Ordering
  - Filtering
- Greedy descent with min-conflict heuristic is often effective in practice

Reading

- Read Chapters 6, 6.1, 6.2, 6.2.1, 6.2.2, 6.2.6, 6.3, 6.3.1, 6.4 in the AIMA textbook
- Read Chapters 4.6 in the P&M textbook (Domain Splitting)