CSC 665: Artificial Intelligence

Markov Chains and Hidden Markov Models

Instructor: Pooyan Fazli
San Francisco State University

Notations

Useful notation: $X_{a:b} = X_a, X_{a+1}, \ldots, X_b$

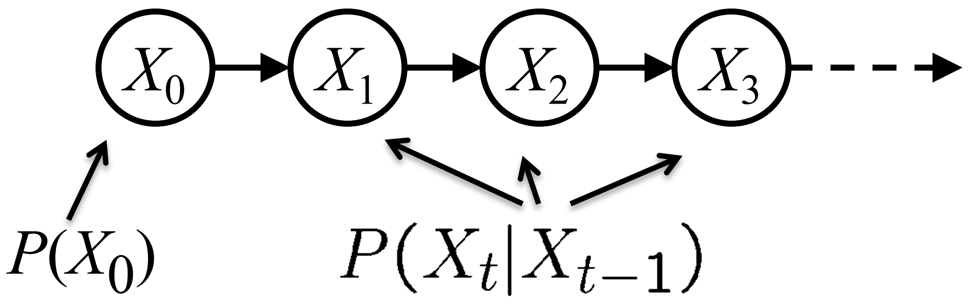
For example: $P(X_{1:2} \mid e_{1:3}) = P(X_1, X_2 \mid e_1, e_2, e_3)$

Reasoning over Time

- Often, we want to reason about a sequence of states
  - Speech recognition
  - Robot localization
  - User attention
  - Medical monitoring
- Need to introduce time into our models

Markov Chain

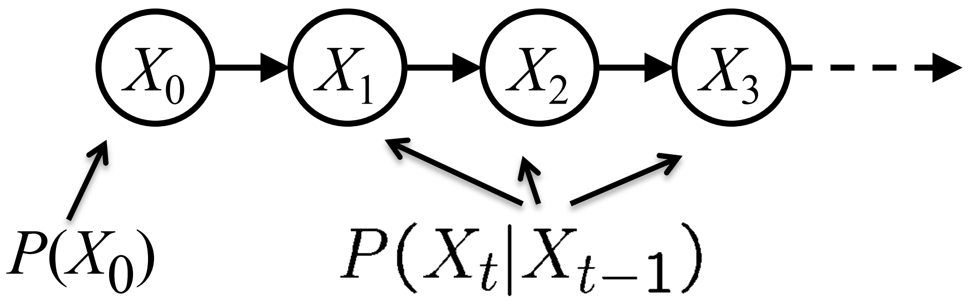
- Value of X at a given time is called the state



- Initial Distribution: $P(X_0)$
- Transition Model: $P(X_t \mid X_{t-1})$ specifies how the state evolves over time

Markov Chain

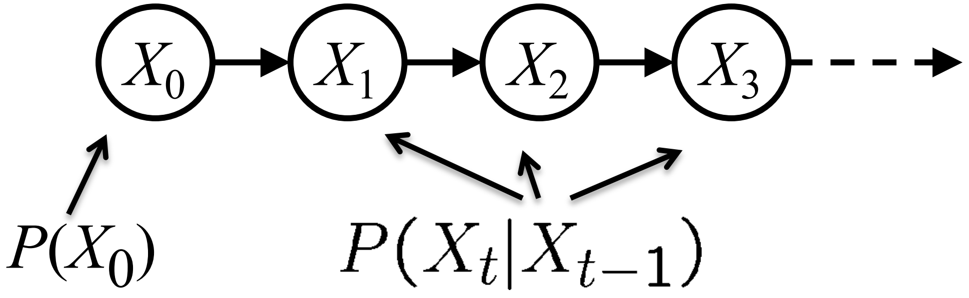
- Value of X at a given time is called the state



- Stationarity assumption: transition probabilities the same at all times

Markov Chain

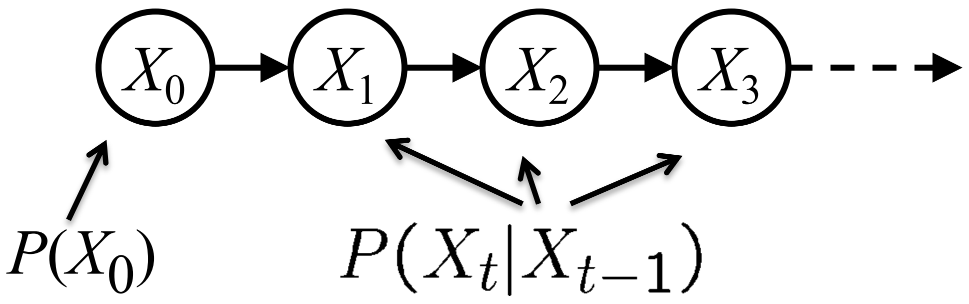
- Value of X at a given time is called the state



- Markov assumption: X_t is independent of $X_0, \ldots, X_{t\text{-}2}$ given $X_{t\text{-}1}$

Markov Chain

- Value of X at a given time is called the state

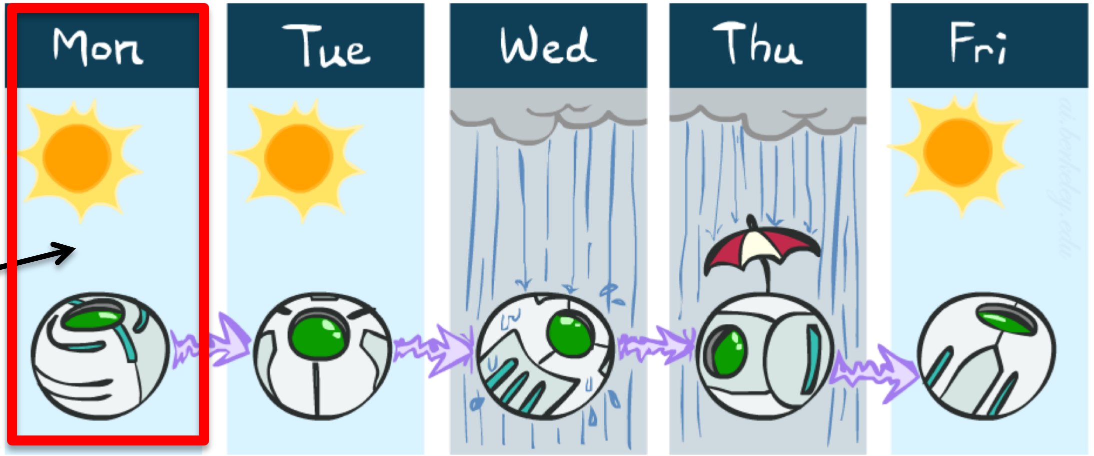


- Joint distribution: $P(X_0, \ldots, X_T) = P(X_0) \prod_t P(X_t \mid X_{t\text{-}1})$

Quiz: are Markov Chains a special case of Bayes nets?

- Yes and no!
- Yes:
  - Directed acyclic graph, joint = product of conditionals
- No:
  - Infinitely many nodes
  - Repetition of transition model not part of standard Bayes Net syntax

Markov Chain: Weather



- States: X = {rain, sun}

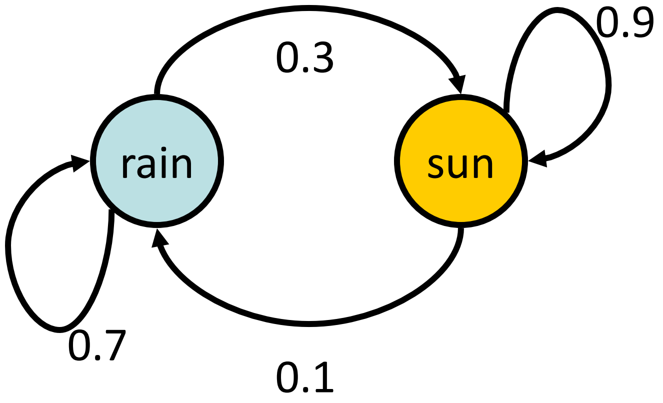
- Initial distribution:

$$P(X_0 = sun) = P(X_0 = rain) = 0.5$$

- $P(X_t \mid X_{t-1})$ or Transition Matrix

X_{t-1}	X_t	$P(X_t \mid X_{t-1})$
sun	sun	0.9
sun	rain	0.1
rain	sun	0.3
rain	rain	0.7

X_{t-1}	$P(X_t \mid X_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7



Markov Chain: Weather

- Initial distribution, Time 0:

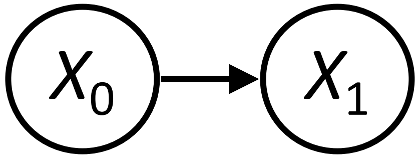
$$P(X_0 = sun) = P(X_0 = rain) = 0.5$$

X_{t-1}	$P(X_t \mid X_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7

- What is the weather like at time 1?

$$
\begin{aligned}
\color{red}{P(X_1 = sun)} &= \textstyle\sum_{x_0} P(X_1{=}sun, X_0{=}x_0) \\
&= P(X_1 = sun \mid X_0 = sun) P(X_0 = sun) + \\
&\quad\; P(X_1 = sun \mid X_0 = rain) P(X_0 = rain) \\
&= \color{green}{0.9}{*}0.5 + \color{green}{0.3}{*}0.5 = 0.6
\end{aligned}
$$

$$
\begin{aligned}
\color{red}{P(X_1 = rain)} &= \textstyle\sum_{x_0} P(X_1{=}rain, X_0{=}x_0) \\
&= P(X_1 = rain \mid X_0 = sun) P(X_0 = sun) + \\
&\quad\; P(X_1 = rain \mid X_0 = rain) P(X_0 = rain) \\
&= \color{green}{0.1}{*}0.5 + \color{green}{0.7}{*}0.5 = 0.4
\end{aligned}
$$



Markov Chain: Weather

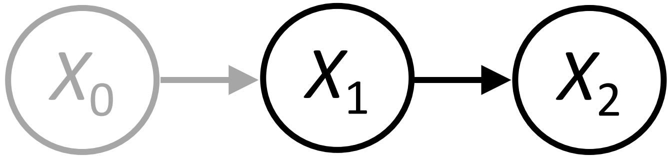
- Time 1:

$$P(X_1 = sun) = 0.6,\ P(X_1 = rain) = 0.4$$

X_{t-1}	$P(X_t \mid X_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7

- What is the weather like at time 2?

$$\begin{aligned} P(X_2 = sun) &= \textstyle\sum_{x_1} P(X_2{=}sun, X_1{=}x_1) \\ &= P(X_2 = sun \mid X_1 = sun)P(X_1 = sun) + \\ &\quad P(X_2 = sun \mid X_1 = rain)P(X_1 = rain) \\ &= 0.9 * 0.6 + 0.3 * 0.4 = 0.66 \end{aligned}$$



$$\begin{aligned} P(X_2 = rain) &= \textstyle\sum_{x_1} P(X_2{=}rain, X_1{=}x_1) \\ &= P(X_2 = rain \mid X_1 = sun)P(X_1 = sun) + \\ &\quad P(X_2 = rain \mid X_1 = rain)P(X_1 = rain) \\ &= 0.1 * 0.6 + 0.7 * 0.4 = 0.34 \end{aligned}$$

Markov Chain: Weather

- Time 2:

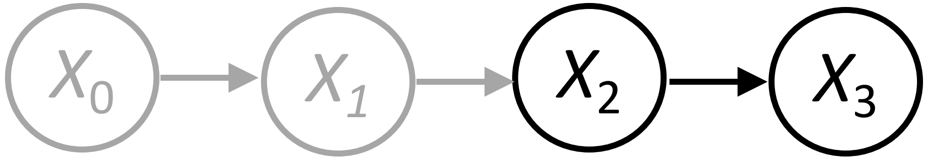
$$P(X_2 = sun) = 0.66,\ P(X_2 = rain) = 0.34$$

X_{t-1}	$P(X_t \mid X_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7

- What is the weather like at time 3?

$$\begin{aligned} P(X_3 = sun) &= \textstyle\sum_{x_2} P(X_3{=}sun, X_2{=}x_2) \\ &= P(X_3 = sun \mid X_2 = sun)P(X_2 = sun) + \\ &\quad P(X_3 = sun \mid X_2 = rain)P(X_2 = rain) \\ &= 0.9 * 0.66 + 0.3 * 0.34 = 0.696 \end{aligned}$$

$$\begin{aligned} P(X_3 = rain) &= \textstyle\sum_{x_2} P(X_3{=}rain, X_2{=}x_2) \\ &= P(X_3 = rain \mid X_2 = sun)P(X_2 = sun) + \\ &\quad P(X_3 = rain \mid X_2 = rain)P(X_2 = rain) \\ &= 0.1 * 0.66 + 0.7 * 0.34 = 0.304 \end{aligned}$$



Forward Algorithm

- Question: What's P(X) on some day t?

$$X_0 \to X_1 \to X_2 \to X_3 \dashrightarrow$$

$$\color{red}{P(X_0)} = Known$$

$$\begin{aligned} P(X_t) &= \textstyle\sum_{x_{t-1}} P(X_t, X_{t-1} = x_{t-1}) \\ &= \textstyle\sum_{x_{t-1}} \underbrace{P(X_t \mid X_{t-1} = x_{t-1})}_{\text{Transition model}} \, \underbrace{P(X_{t-1} = x_{t-1})}_{\text{Probability from previous iteration}} \end{aligned}$$

The same thing in linear algebra

- In matrix-vector form:
  - $P(X_0) = \begin{pmatrix} 0.5 & 0.5 \end{pmatrix}$
  - $P(X_1) = \begin{pmatrix} 0.5 & 0.5 \end{pmatrix} \begin{pmatrix} 0.9 & 0.1 \\ 0.3 & 0.7 \end{pmatrix} = \begin{pmatrix} 0.6 & 0.4 \end{pmatrix}$
  - $P(X_2) = \begin{pmatrix} 0.6 & 0.4 \end{pmatrix} \begin{pmatrix} 0.9 & 0.1 \\ 0.3 & 0.7 \end{pmatrix} = \begin{pmatrix} 0.66 & 0.34 \end{pmatrix}$
  - $P(X_3) = \begin{pmatrix} 0.66 & 0.34 \end{pmatrix} \begin{pmatrix} 0.9 & 0.1 \\ 0.3 & 0.7 \end{pmatrix} = \begin{pmatrix} 0.696 & 0.304 \end{pmatrix}$

$T =$

X_{t-1}	$P(X_t \mid X_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7

- Multiply by T (transition matrix)

Stationary Distributions

- The final distribution is called the Stationary Distribution P_∞ of the chain

- It satisfies $P_\infty = P_{\infty+1} = P_\infty T$

- Solving for P_∞ in the example:

$$\begin{pmatrix} p & 1-p \end{pmatrix} \begin{pmatrix} 0.9 & 0.1 \\ 0.3 & 0.7 \end{pmatrix} = \begin{pmatrix} p & 1-p \end{pmatrix}$$

$$0.9p + 0.3(1-p) = p$$

$$p = 0.75$$

$T =$

X_{t-1}	$P(X_t \mid X_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7

Stationary distribution is (0.75 0.25) regardless of starting distribution

Example Run of Forward Algorithm

- From initial belief of sun

$$\left\langle \begin{matrix} 1.0 \\ 0.0 \end{matrix} \right\rangle \quad \left\langle \begin{matrix} 0.9 \\ 0.1 \end{matrix} \right\rangle \quad \left\langle \begin{matrix} 0.84 \\ 0.16 \end{matrix} \right\rangle \quad \left\langle \begin{matrix} 0.804 \\ 0.196 \end{matrix} \right\rangle \Rightarrow \left\langle \begin{matrix} 0.75 \\ 0.25 \end{matrix} \right\rangle$$

$P(X_0)$ $P(X_1)$ $P(X_2)$ $P(X_3)$ $P(X_\infty)$

- From initial belief of rain

$$\left\langle \begin{matrix} 0.0 \\ 1.0 \end{matrix} \right\rangle \quad \left\langle \begin{matrix} 0.3 \\ 0.7 \end{matrix} \right\rangle \quad \left\langle \begin{matrix} 0.48 \\ 0.52 \end{matrix} \right\rangle \quad \left\langle \begin{matrix} 0.588 \\ 0.412 \end{matrix} \right\rangle \Rightarrow \left\langle \begin{matrix} 0.75 \\ 0.25 \end{matrix} \right\rangle$$

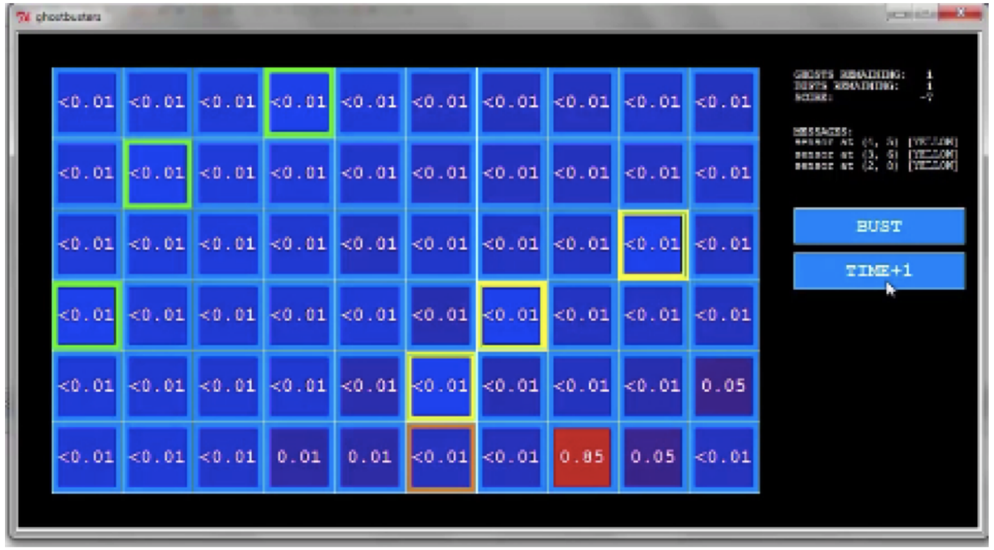
$P(X_0)$ $P(X_1)$ $P(X_2)$ $P(X_3)$ $P(X_\infty)$

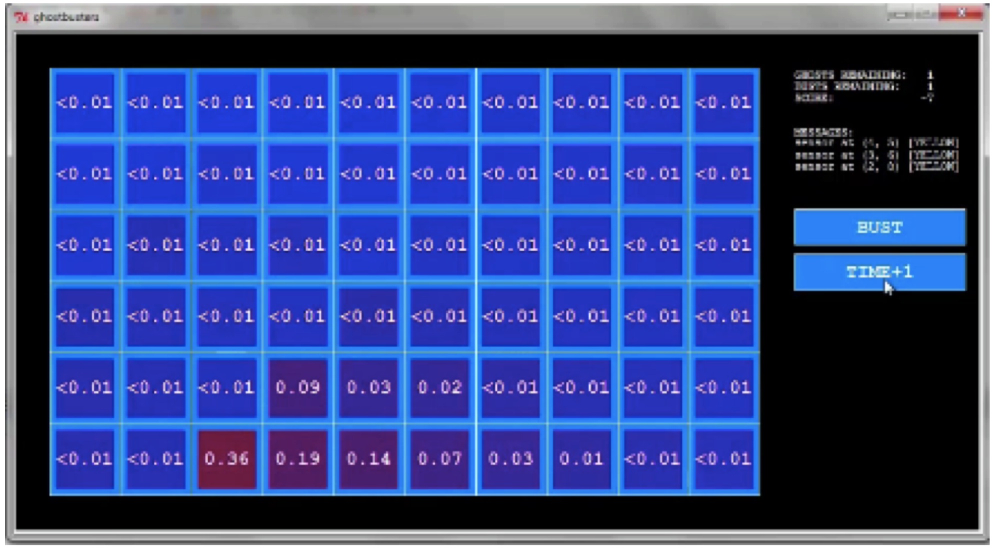
- From yet another initial distribution $P(X_0)$:

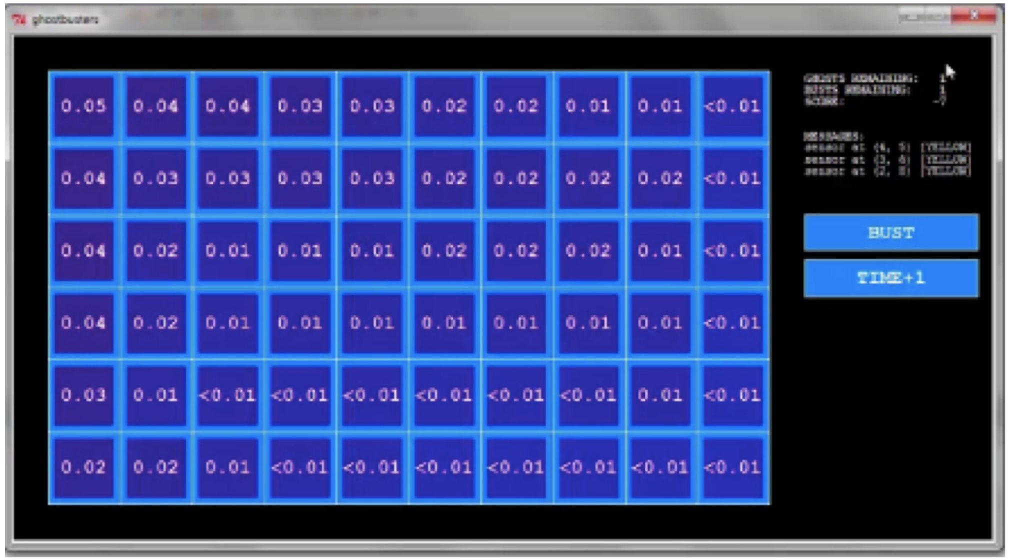
$$\left\langle \begin{matrix} p \\ 1-p \end{matrix} \right\rangle \quad \ldots \Rightarrow \left\langle \begin{matrix} 0.75 \\ 0.25 \end{matrix} \right\rangle$$

$P(X_0)$ $P(X_\infty)$

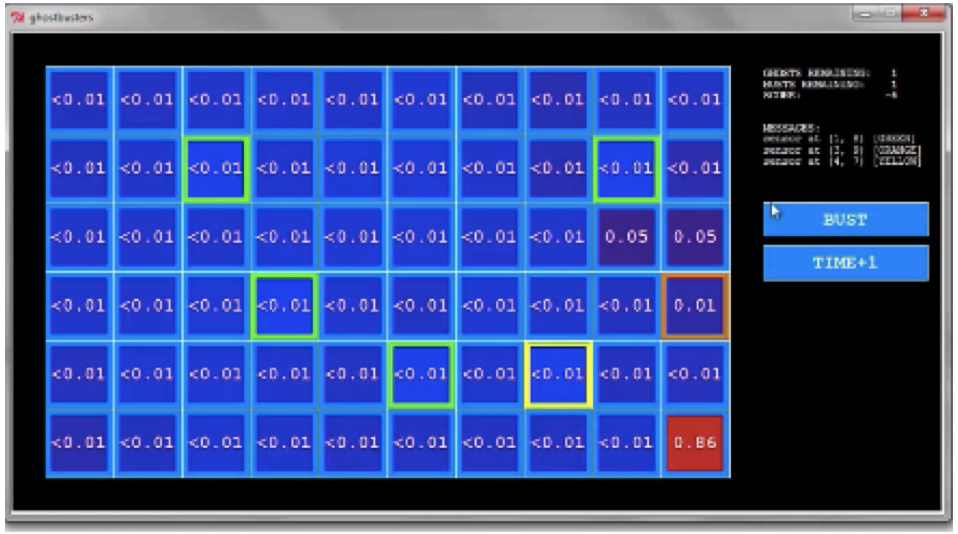
Ghostbusters Circular Dynamics

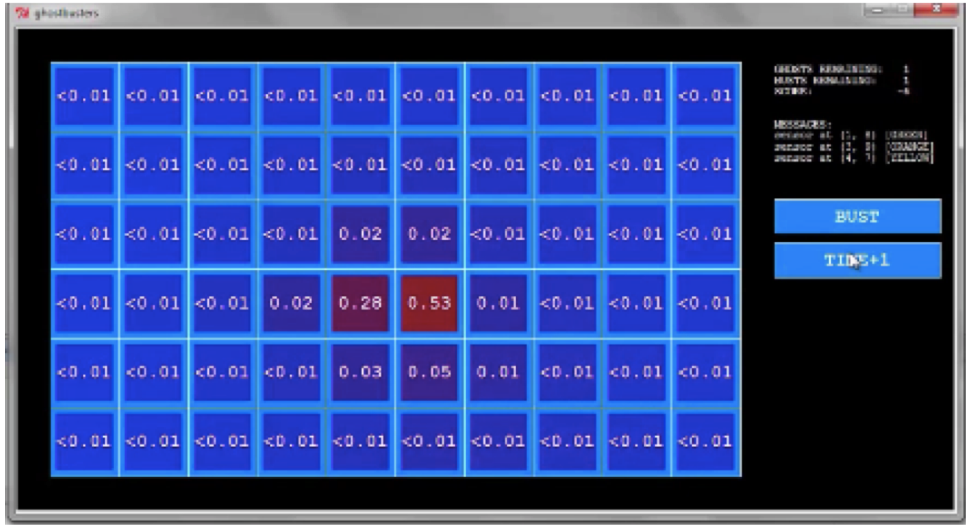


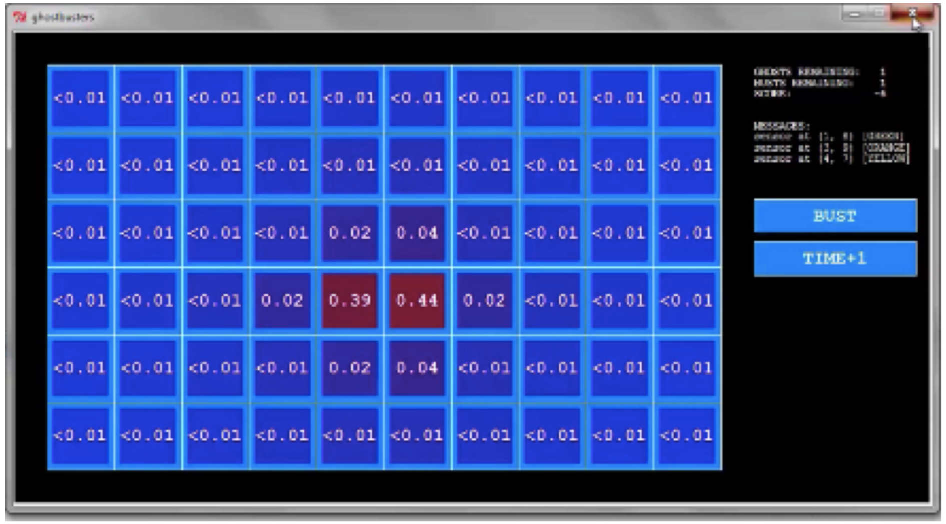




Ghostbusters Whirlpool Dynamics







Web Search: The Web is a Directed Graph

- The nodes or vertices are the web pages.
- The edges are the links coming into the page and going out of the page.

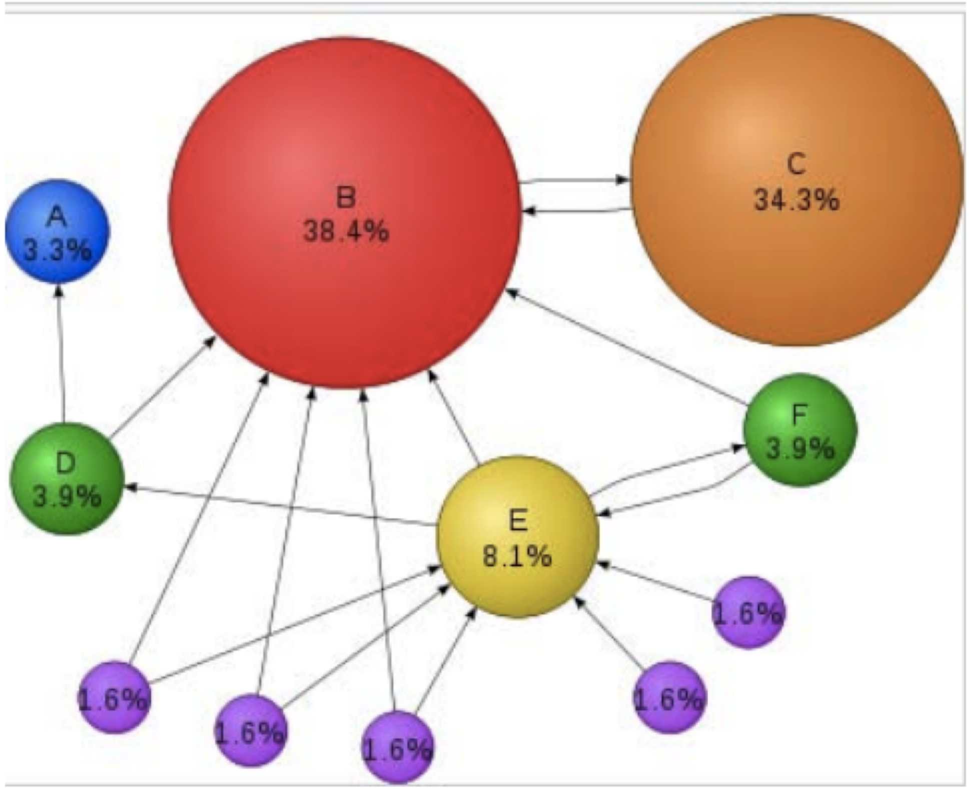
Google PageRank

- Sergey Brin and Larry Page considered web surfing as a Markov Chain

- PageRank: A webpage is important if it is pointed to by other important pages. The algorithm was patented in 2001.

Google PageRank

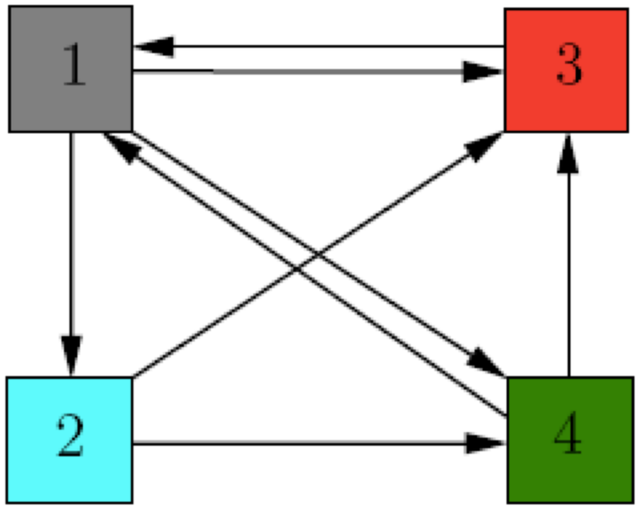
- C has a higher rank than E, even though there are fewer links to C since the one link to C comes from an "important" page.



Google PageRank

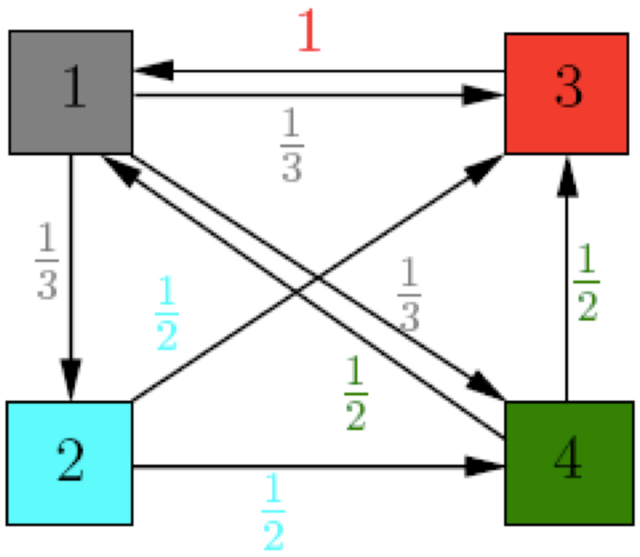
- Markov Chain over a web graph
  - State: page visited at step *t*
  - Initial distribution: uniform over pages
  - Transition Model: Follow a random outlink
- Question: What is the Stationary Distribution over pages?
  - if the process runs forever, what fraction of time does it spend in any given page? PageRank
- For any particular query, Google finds the pages on the Web that match that query and lists those pages in the order of their PageRank.
- Before Google, search engines returned the set of pages containing all your keywords in decreasing rank

Application: Google PageRank



Source: http://www.math.cornell.edu/~mec/Winter2009/RalucaRemus/Lecture3/lecture3.html

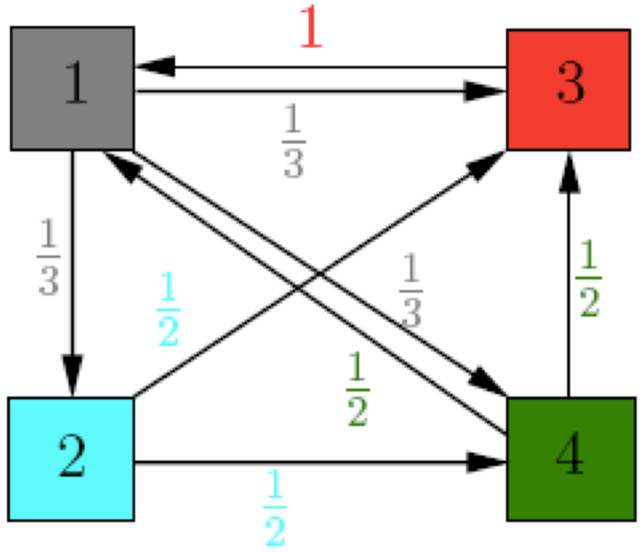
Application: Google PageRank



Application: Google PageRank

Initial Distribution: $v = \begin{bmatrix} 0.25 & 0.25 & 0.25 & 0.25 \end{bmatrix}$

Transition matrix: $A = \begin{bmatrix} 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 1/2 & 1/2 \\ 1 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \end{bmatrix}$



$vA = \begin{bmatrix} 0.37 & 0.08 & 0.33 & 0.20 \end{bmatrix}$

$vA^2 = \begin{bmatrix} 0.43 & 0.12 & 0.27 & 0.16 \end{bmatrix}$

.
.
.

$vA^7 = \begin{bmatrix} 0.38 & 0.12 & 0.29 & 0.19 \end{bmatrix}$

$vA^8 = \begin{bmatrix} 0.38 & 0.12 & 0.29 & 0.19 \end{bmatrix}$

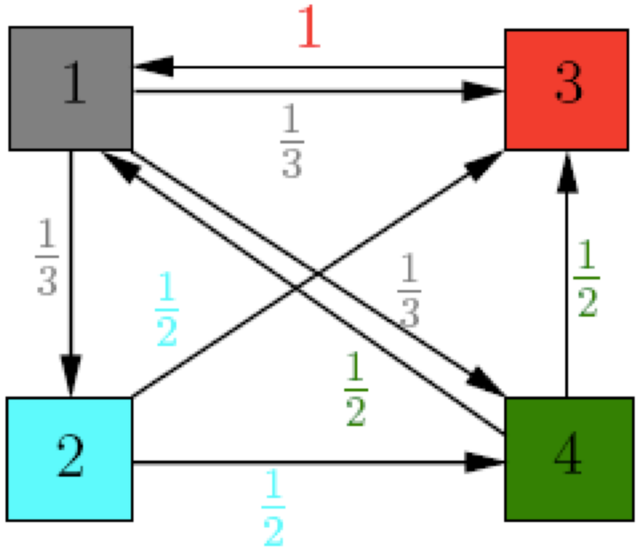
Intuitively, at step 1, one node receives an importance vote from its direct neighbors, at step 2 from the neighbors of its neighbors, and so on.

Rank of the pages

Application: Google PageRank

Initial Distribution: v = $\begin{bmatrix} 0.25 & 0.25 & 0.25 & 0.25 \end{bmatrix}$

Transition matrix: $A = \begin{bmatrix} 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 1/2 & 1/2 \\ 1 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \end{bmatrix}$



$$\begin{bmatrix} v_1 & v_2 & v_3 & v_4 \end{bmatrix} \begin{bmatrix} 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 1/2 & 1/2 \\ 1 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 & v_3 & v_4 \end{bmatrix}$$

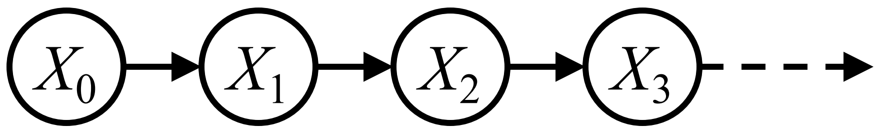
v = $\begin{bmatrix} 0.38 & 0.12 & 0.29 & 0.19 \end{bmatrix}$

Rank of the pages

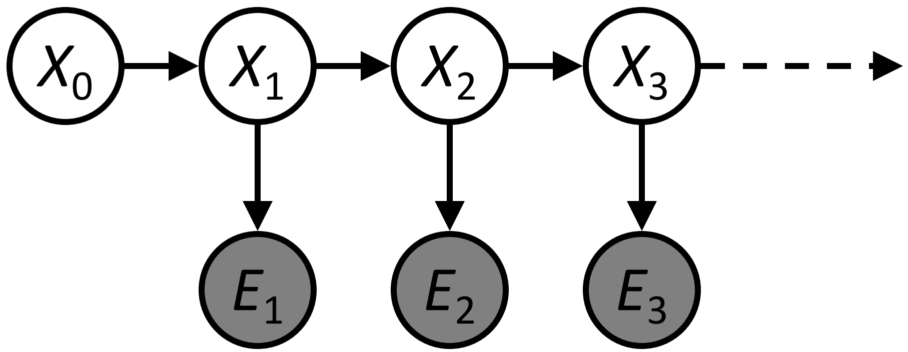
Hidden Markov Models

Hidden Markov Models

- Markov chains not so useful for most agents
  - Need observations to update your beliefs



- Hidden Markov models (HMMs)
  - You observe outputs (effects) at each time step



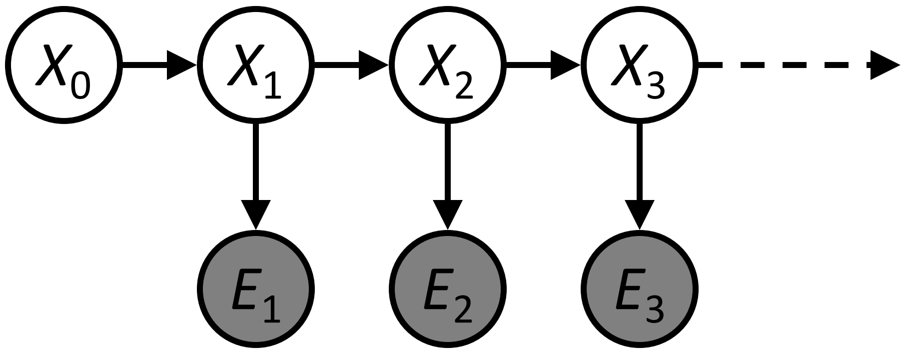
HMM as Probability Model

- Joint distribution for Markov Chain:

$$P(X_0,\ldots,X_T) = P(X_0) \prod_{t=1:T} P(X_t \mid X_{t-1})$$

- Joint distribution for Hidden Markov Model:

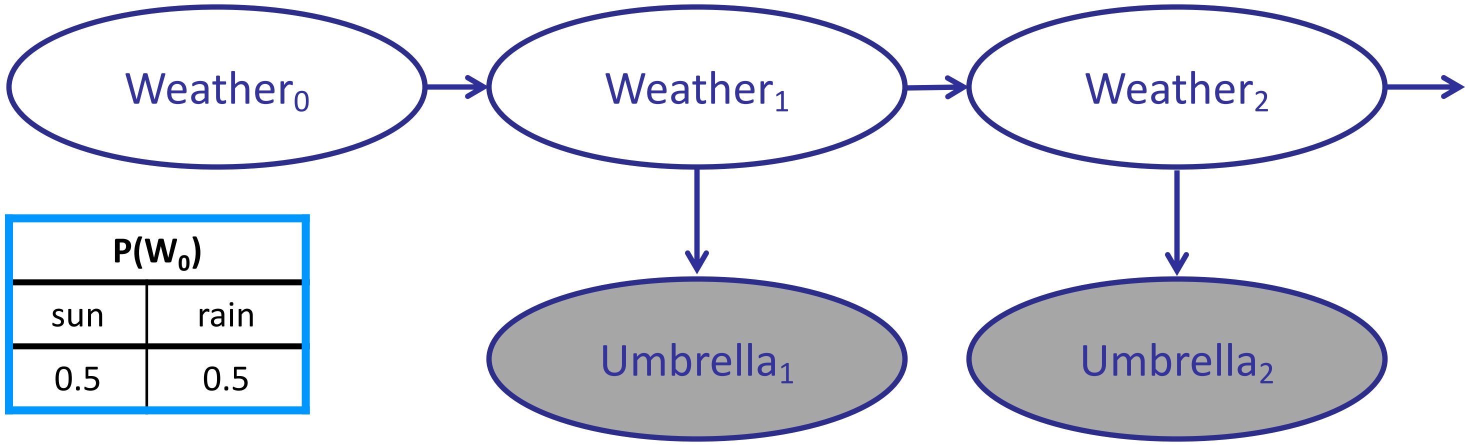
$$P(X_0,X_1,E_1,\ldots,X_T,E_T) = P(X_0) \prod_{t=1:T} P(X_t \mid X_{t-1})\, P(E_t \mid X_t)$$



Example: Weather HMM

- An HMM is defined by:
  - Initial distribution: $P(X_0)$
  - Transition model: $P(X_t \mid X_{t-1})$
  - Sensor model: $P(E_t \mid X_t)$

W_{t-1}	$P(W_t \mid W_{t-1})$	
	sun	rain
sun	0.7	0.3
rain	0.3	0.7

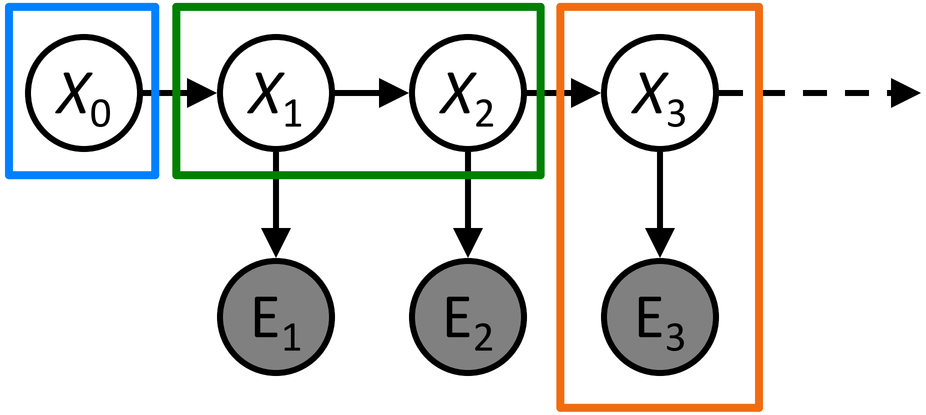


$P(W_0)$	
sun	rain
0.5	0.5

W_t	$P(U_t \mid W_t)$	
	true	false
sun	0.2	0.8
rain	0.9	0.1

Example: Ghostbusters HMM

- State: location of moving ghost
- Observations: Color recorded by ghost sensor at clicked squares
- $P(X_0)$ = uniform
- $P(X_t | X_{t-1})$ = usually move clockwise, but sometimes move randomly or stay in place
- $P(E_t | X_t)$ = same sensor model as before: red means close, green means far away.



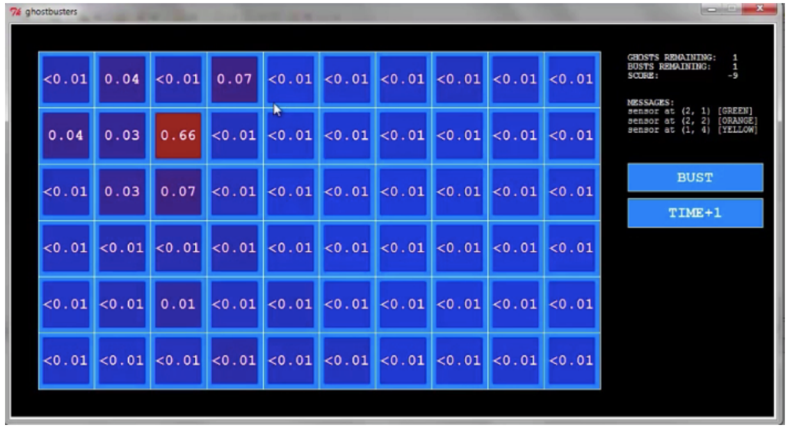
1/9	1/9	1/9
1/9	1/9	1/9
1/9	1/9	1/9

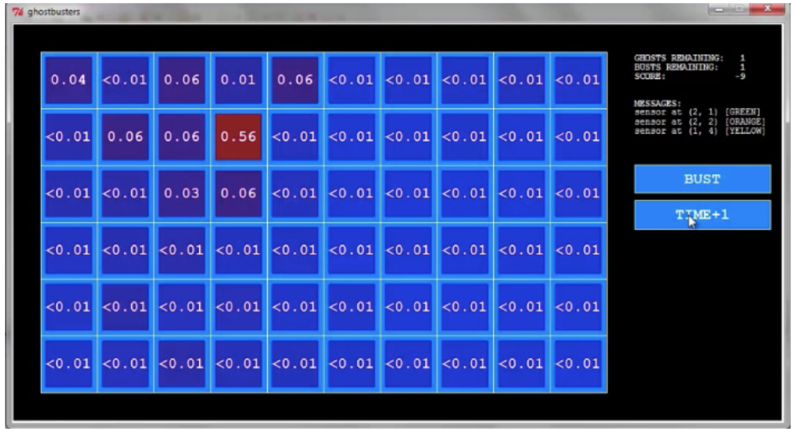
$P(X_0)$

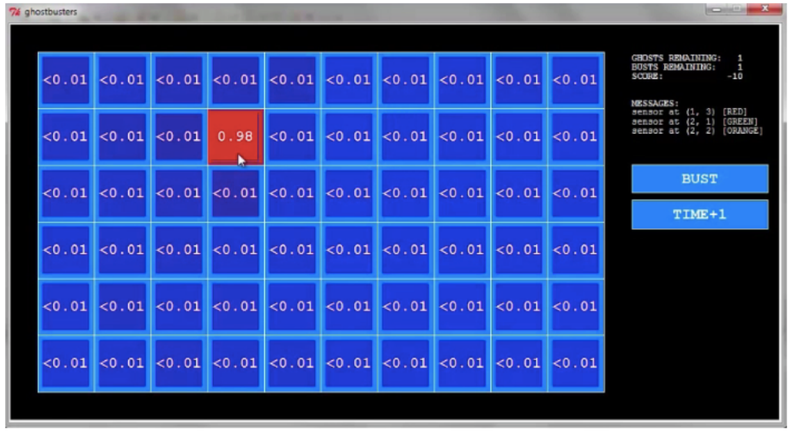
1/6	1/6	1/2
0	1/6	0
0	0	0

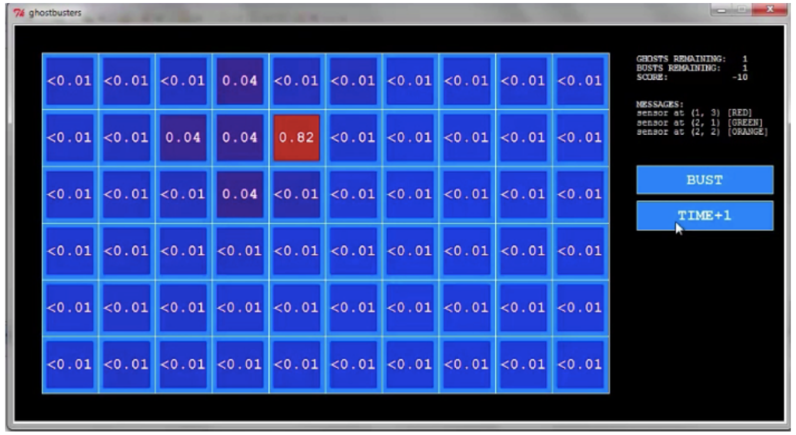
$P(X_1 | X_0=(2,3))$

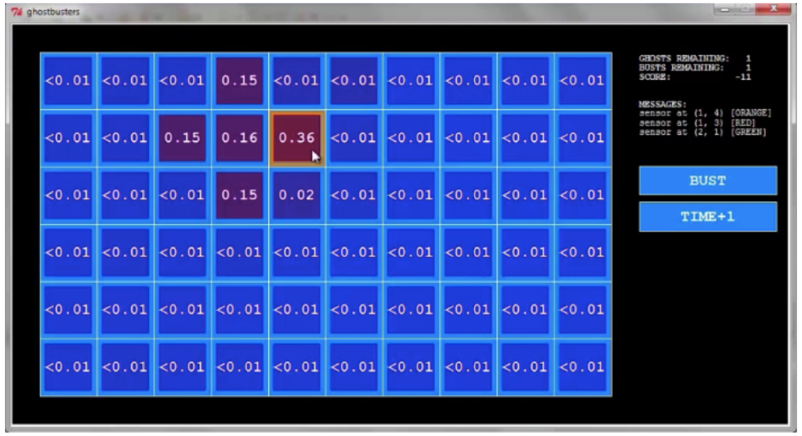
Ghostbusters – Circular Dynamics -- HMM





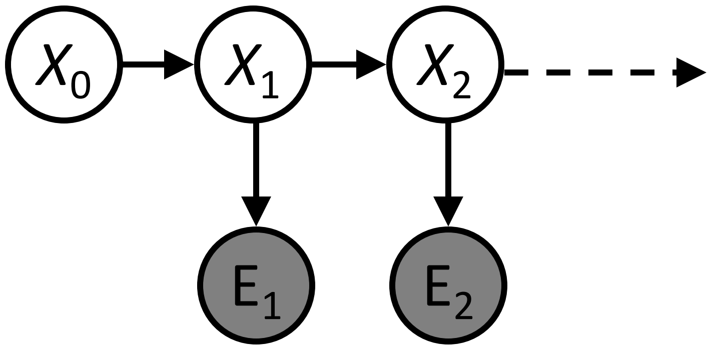






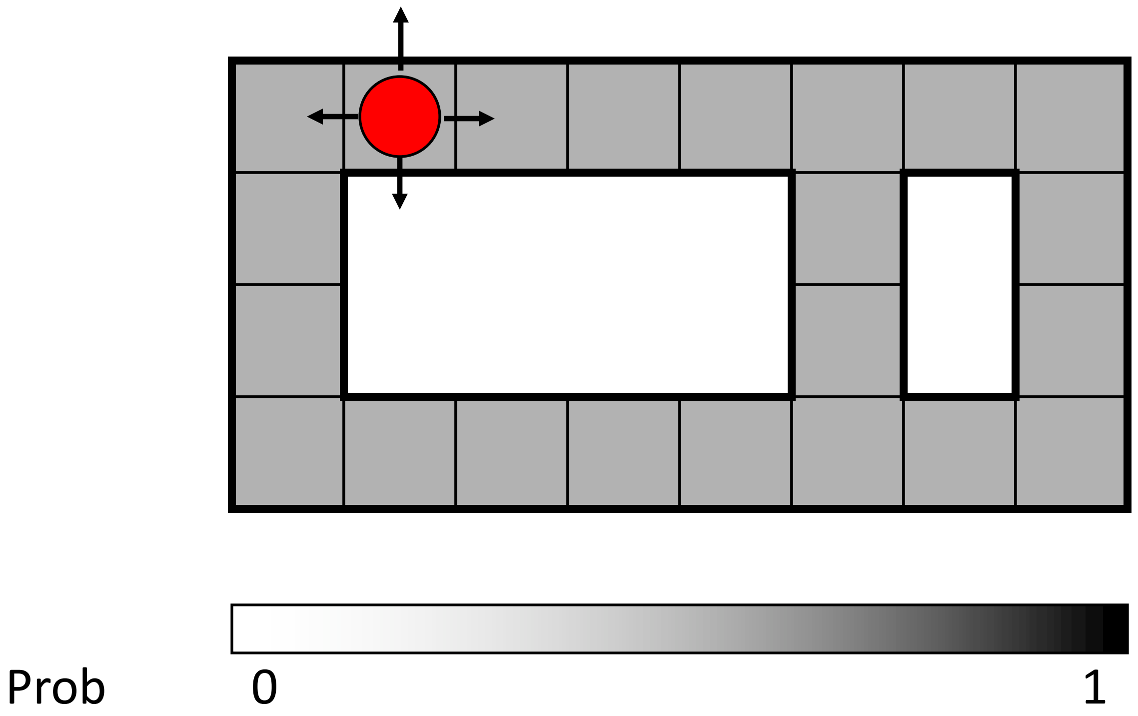
Filtering

- Filtering is the task of tracking the distribution $P(X_t \mid e_1, \ldots, e_t)$ (the belief state) over time
- We start with $P(X_0)$, usually uniform
- As time passes, or we get observations, we predict/update the belief state



Example: Robot Localization

Example from
Michael Pfeiffer

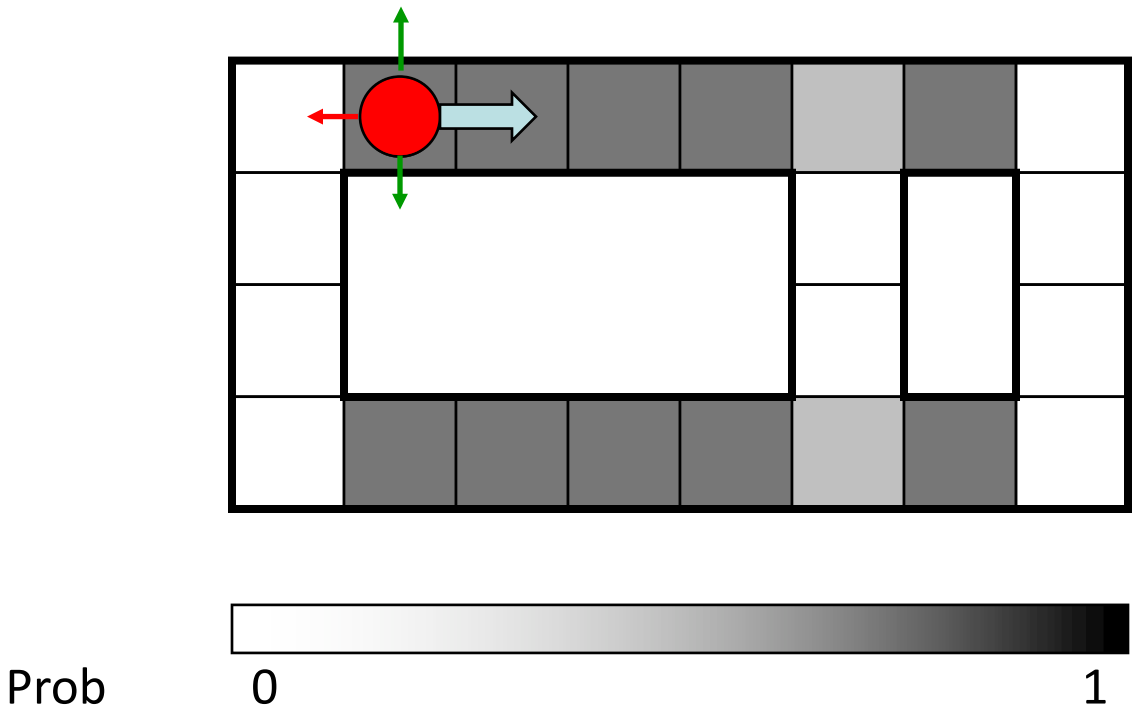


t=0

Sensor model: four bits for wall/no-wall in each direction, never more than 1 mistake

Transition model: action may fail with small prob.

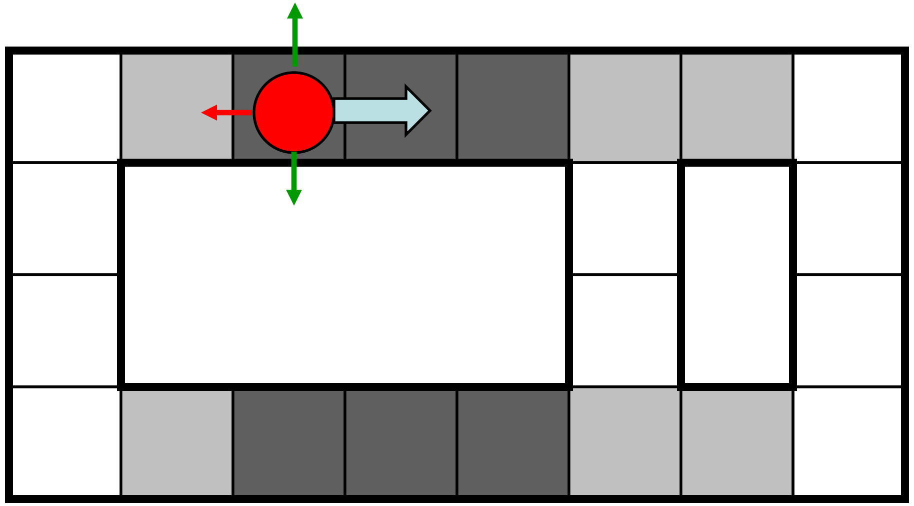
Example: Robot Localization

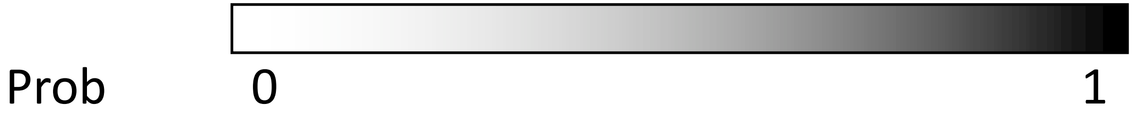


t=1

Lighter grey: was possible to get the reading, but less likely b/c required 1 mistake

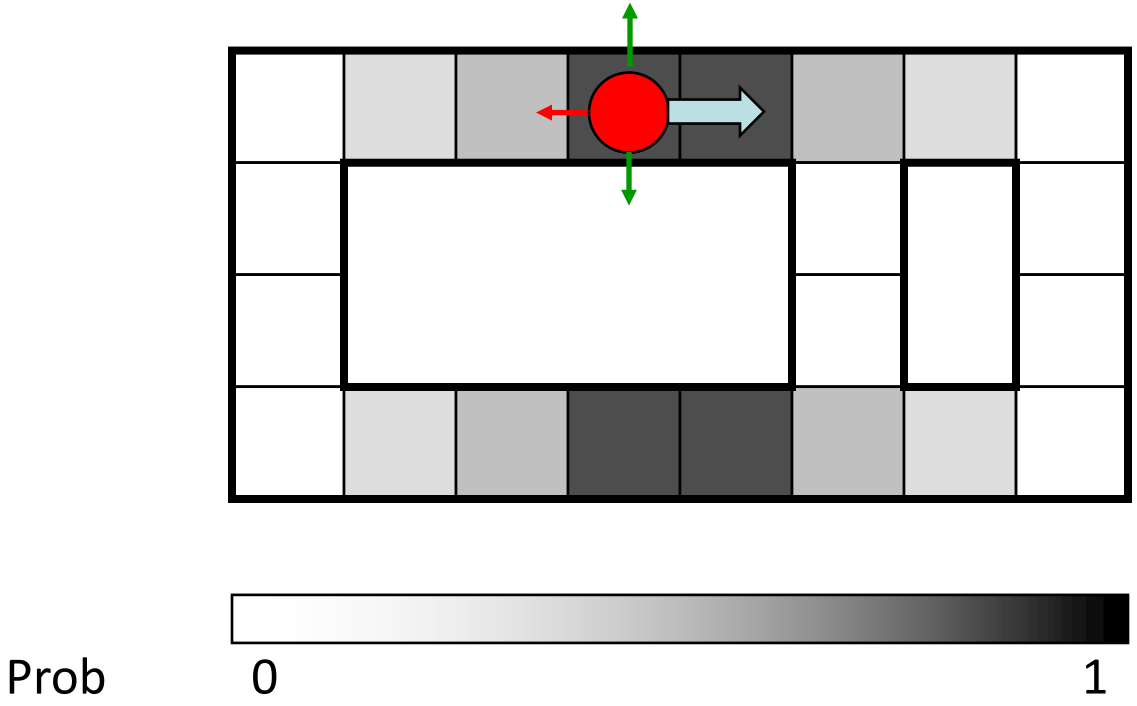
Example: Robot Localization





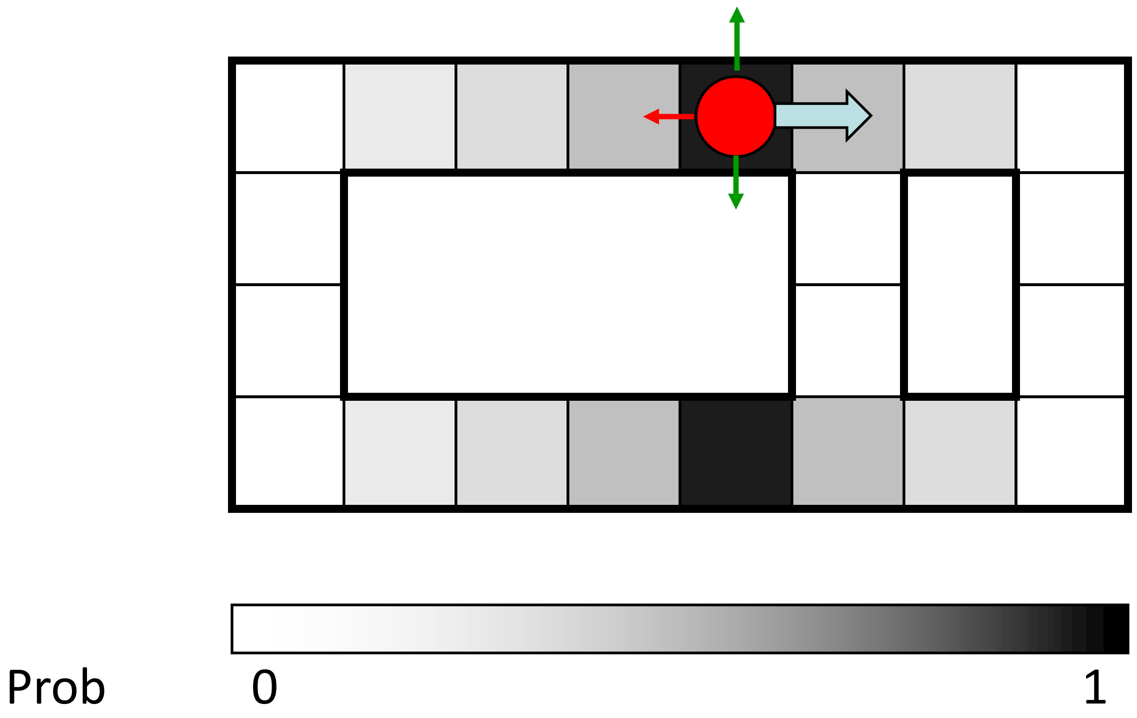
t=2

Example: Robot Localization



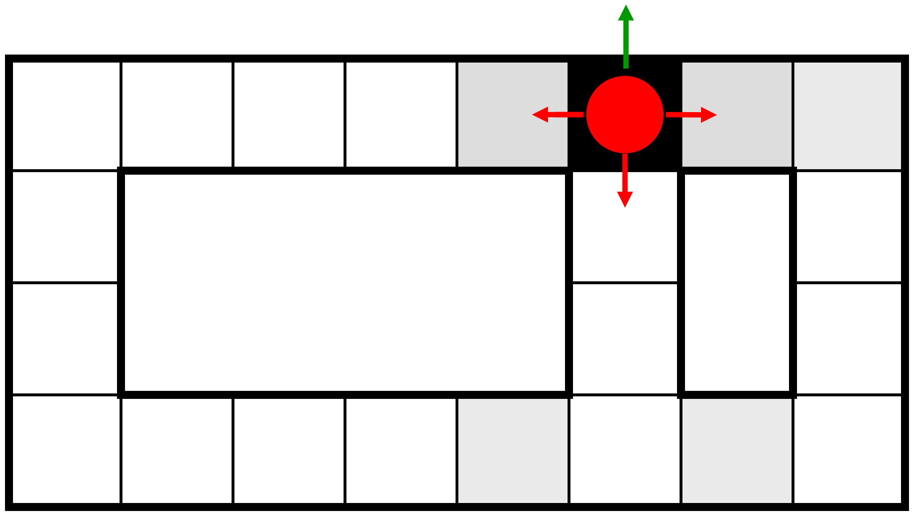
t=3

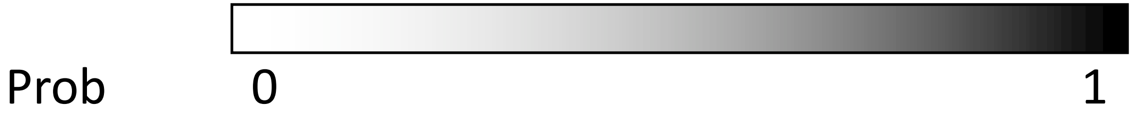
Example: Robot Localization



t=4

Example: Robot Localization



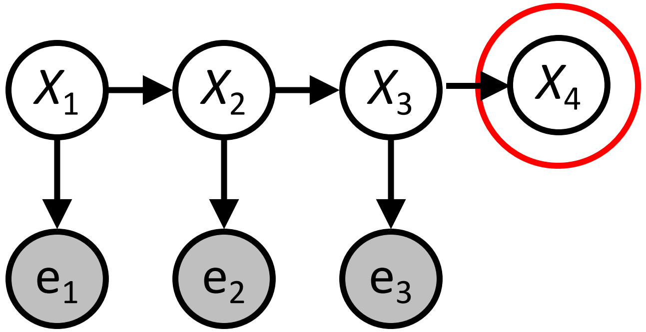


t=5

Passage of Time (Predict)

- Assume we have current belief P(X | evidence to date)

$$P(X_t|e_{1:t})$$



- After one time step passes:

$$P(X_{t+1}|e_{1:t}) = \sum_{x_t} P(X_{t+1}, x_t|e_{1:t})$$

Apply Product Rule

$$= \sum_{x_t} P(X_{t+1}|x_t, e_{1:t}) P(x_t|e_{1:t})$$

Apply Conditional Independence

$$= \sum_{x_t} P(X_{t+1}|x_t) P(x_t|e_{1:t})$$

Example: Passage of Time

- As time passes, uncertainty "accumulates"

(Transition model: ghosts usually go clockwise)

<0.01	<0.01	<0.01	<0.01	<0.01	<0.01
<0.01	<0.01	<0.01	<0.01	<0.01	<0.01
<0.01	<0.01	1.00	<0.01	<0.01	<0.01
<0.01	<0.01	<0.01	<0.01	<0.01	<0.01

T = 1

<0.01	<0.01	<0.01	<0.01	<0.01	<0.01
<0.01	<0.01	0.06	<0.01	<0.01	<0.01
<0.01	0.76	0.06	0.06	<0.01	<0.01
<0.01	<0.01	0.06	<0.01	<0.01	<0.01

T = 2

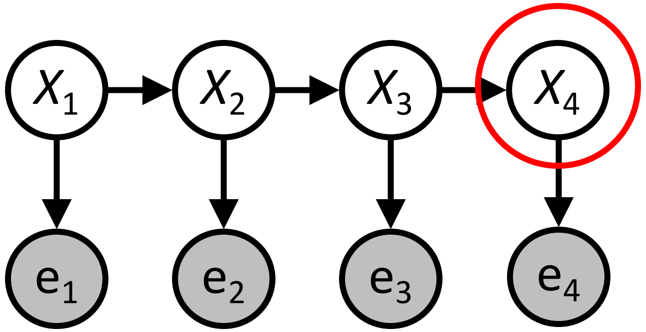
0.05	0.01	0.05	<0.01	<0.01	<0.01
0.02	0.14	0.11	0.35	<0.01	<0.01
0.07	0.03	0.05	<0.01	0.03	<0.01
0.03	0.03	<0.01	<0.01	<0.01	<0.01

T = 5

Observation (Update)

- Assume we have current belief P(X | previous evidence):

$$P(X_{t+1}|e_{1:t})$$



- Then, after evidence comes in:

$$P(X_{t+1}|e_{1:t+1}) = P(X_{t+1}|e_{t+1}, e_{1:t})$$

Conditional Probability

$$= P(X_{t+1}, e_{t+1}|e_{1:t})/P(e_{t+1}|e_{1:t})$$

$$\propto_{X_{t+1}} P(X_{t+1}, e_{t+1}|e_{1:t})$$

Apply Product rule

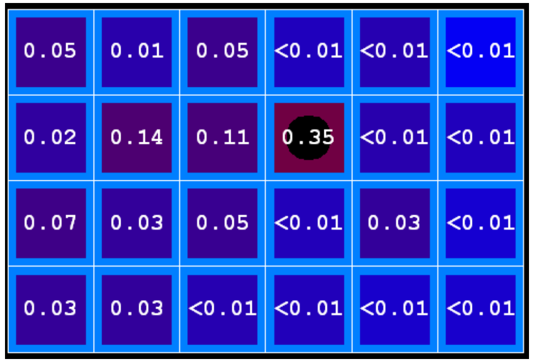
$$= P(e_{t+1}|X_{t+1}, e_{1:t}) \; P(X_{t+1}|e_{1:t})$$

Apply conditional independence

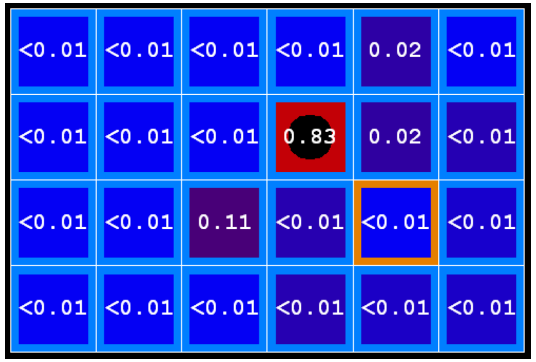
$$= P(e_{t+1}|X_{t+1})P(X_{t+1}|e_{1:t})$$

Example: Observation

- As we get observations, beliefs get reweighted, uncertainty "decreases"



Before observation



After observation

Example: Weather HMM

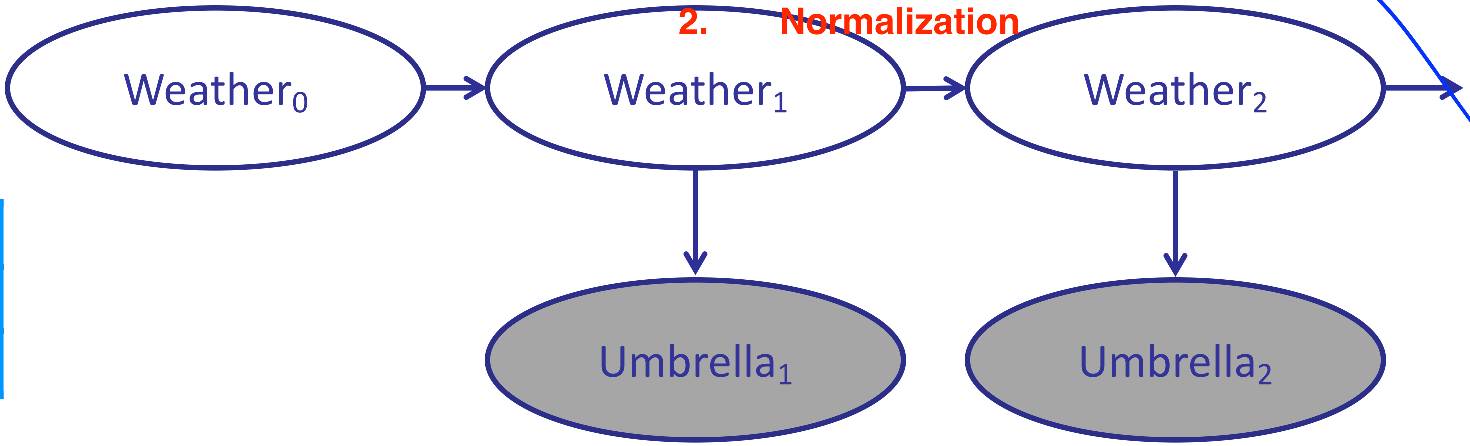

Belief: <P(sun), P(rain)>

$P(W_1) = <0.5\ 0.5>$ $\quad$ $P(W_2|U_1{=}true) = <0.373\ 0.627>$

predict $\quad$ update $\quad$ predict $\quad$ update

$P(W_0) = <0.5\ 0.5>$ $\quad$ $P(W_1|U_1{=}true) = <0.182\ 0.818>$ $\quad$ $P(W_2|U_1{=}true, U_2{=}true) = <0.117\ 0.883>$

1. <0.5*0.2 0.5*0.9>
2. Normalization



$P(W_0)$	
sun	rain
0.5	0.5

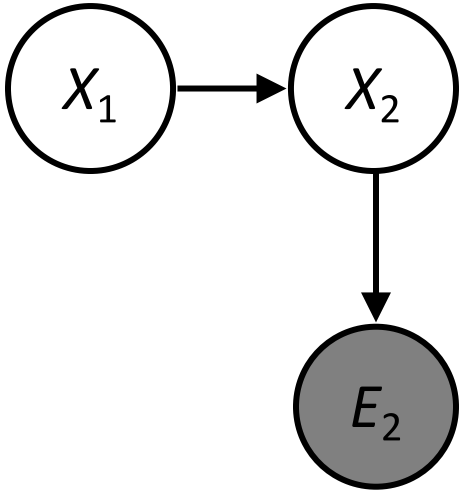
W_{t-1}	$P(W_t\|W_{t-1})$	
	sun	rain
sun	0.7	0.3
rain	0.3	0.7

W_t	$P(U_t\|W_t)$	
	true	false
sun	0.2	0.8
rain	0.9	0.1

Summary

- Every time step, we start with current P(X | evidence)
- We predict for time:

$$P(x_t|e_{1:t-1}) = \sum_{x_{t-1}} P(x_{t-1}|e_{1:t-1}) \cdot P(x_t|x_{t-1})$$



- We update for evidence:

$$P(x_t|e_{1:t}) \propto_X P(x_t|e_{1:t-1}) \cdot P(e_t|x_t)$$

Reading

- Read Sections 15.1-15.3 in the AIMA textbook (Third Edition)