1 Assignment 1

We have $S(t) = S_0 \exp\left((\mu - \frac{1}{2}\sigma^2)t + \sigma\sqrt{t}Z\right)$, where $Z \sim N(0, 1)$.

1.1 Equation 1

$$\begin{aligned}
\mathbb{E}\left[S(t)\right] &= \mathbb{E}\left[S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma\sqrt{t}Z\right)\right] \\
&= S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t\right) \mathbb{E}\left[\exp\left(\sigma\sqrt{t}Z\right)\right] \\
&= S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t\right) M_Z\left(\sigma\sqrt{t}\right) \\
&= S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t\right) \exp\left(\frac{1}{2}\sigma^2 t\right) \\
&= S_0 \exp(\mu t)
\end{aligned}$$

1.2 Equation 2

$$\begin{aligned}
\mathbb{E}\left[S(t)^2\right] &= \mathbb{E}\left[S_0^2 \exp\left(2\left(\mu - \frac{1}{2}\sigma^2\right)t + 2\sigma\sqrt{t}Z\right)\right] \\
&= S_0^2 \exp\left(\left(2\mu - \sigma^2\right)t\right) M_Z\left(2\sigma\sqrt{t}\right) \\
&= S_0^2 \exp\left(\left(2\mu - \sigma^2\right)t\right) \exp\left(\frac{4}{2}\sigma^2 t\right) \\
&= S_0^2 \exp\left(\left(2\mu + \sigma^2\right)t\right)
\end{aligned}$$

1.3 Equation 3

$$\begin{aligned}
\mathrm{Var}\left(S(t)\right) &= \mathbb{E}\left[S(t)^2\right] - \mathbb{E}\left[S(t)\right]^2 \\
&= S_0^2 \exp\left(\left(2\mu + \sigma^2\right)t\right) - S_0^2 \exp(2\mu t) \\
&= S_0 \exp\left(2\mu t\right)\left(\exp\left(\sigma^2 t\right) - 1\right)
\end{aligned}$$