Computer Science C73 | November 8, 2023
Scarborough Campus | University of Toronto

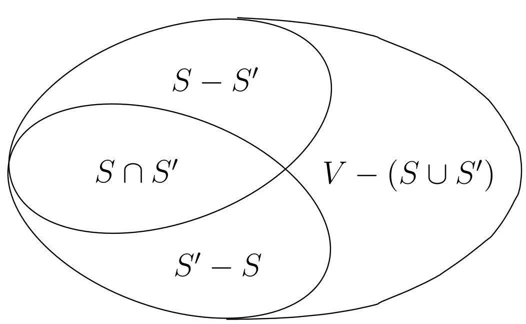
Homework Assignment #7
Due: November 22, 2023, by 11:59 pm

• ***You must submit your assignment through the Crowdmark system.*** *You will receive by email an invitation through which you can submit your work in the form of separate PDF docucments with your answers to each question of the assignment. To work with a partner, you and your partner must form a group on Crowdmark. Crowdmark does not enforce a limit on the size of groups.* ***The course policy that limits the size of each group to at most two remains in effect:*** *submissions by groups of more than two persons will not be graded.*
• *It is your responsibility to ensure that the PDF files you submit are legible. To this end, I encourage you to learn and use the LaTex typesetting system, which is designed to produce high-quality documents that contain mathematical notation. You are not required to produce the PDF files you submit using LaTex; you may produce it any way you wish, as long as the resulting document is legible.*
• *By virtue of submitting this assignment you (and your partner, if you have one) acknowledge that you are aware of the policy on homework collaboration for this course.*[a]
• *For any question, you may use data structures and algorithms previously described in class, or in prerequisites of this course, without describing them. You may also use any result that we covered in class, or is in the assigned sections of the official course textbooks, by referring to it.*
• *Unless we explicitly state otherwise, you may describe algorithms in high-level pseudocode or point-form English, whichever leads to a clearer and simpler description. Do not provide executable code.*
• *Unless we explicitly state otherwise, you should justify your answers. Your paper will be graded based on* ***the correctness and efficiency of your answers, and the clarity, precision, and conciseness of your presentation****.*

[a] "In each homework assignment you may collaborate with at most one other student who is currently taking CSCC73. If you collaborate with another student on an assignment, you and your partner must submit only one copy of your solution, with both of your names. The solution will be graded in the usual way and both partners will receive the same mark. Collaboration involving more than two students is not allowed. **For help with your homework you may consult only the instructor, TAs, your homework partner (if you have one), your textbook, and your class notes. You may not consult any other source.**"

Question 1. (15 marks) Let $\mathcal{F} = (G, s, t, c)$ be a flow network and (S, T), (S', T') be ***minimum*** cuts of $\mathcal{F}$. Prove that $(S \cap S', T \cup T')$ and $(S \cup S', T \cap T')$ are also minimum cuts of $\mathcal{F}$.

Hint: Prove both facts together. In thinking about this problem you may find the following diagram useful:



Question 2. (15 marks) Given a flow network in which all edges have capacity ***one*** and a positive integer k, we want to remove k edges from the flow network so as to reduce as much as possible the maximum flow in the resulting flow network.

Give a polynomial-time algorithm that does this. That is, your algorithm takes as input (a) a flow network $\mathcal{F} = (G, s, t, c)$, where $G = (V, E)$ and $c(e) = 1$ for each $e \in E$, and (b) a positive integer $k \leq |E|$, and produces as output a set $E' \subseteq E$ such that (i) $|E'| = k$, and (ii) the maximum flow in the flow graph (G', s, t, c') is as small as possible, where $G' = (V, E - E')$ and $c'(e) = 1$ for each $e \in E - E'$.

Prove that your algorithm is correct and analyze its running time.

Question 3. (25 marks) Let $\mathcal{F} = (G, s, t, c)$ be a flow network with integer capacities, P be an $s \to t$ simple path of G, and ℓ be the length of P. (Recall that the length of a path is the number of ***edges*** on it.) Let $\mathcal{F}^+ = (G, s, t, c^+)$ be the flow graph obtained from $\mathcal{F}$ by increasing the capacity of every edge on P by one unit. That is, for every edge e of G,

$$c^+(e) = \begin{cases} c(e) + 1, & \text{if } e \text{ is on } P \\ c(e), & \text{otherwise.} \end{cases}$$

Let V be the value of a maximum flow of $\mathcal{F}$ and V^+ be the value of a maximum flow of $\mathcal{F}^+$. In this question we explore the effect of increasing the capacities of the edges on P: What are the possible values of $V^+ - V$?

a. Prove that $V^+ \geq V + 1$. That is, the increase of the capacities of the edges on P ***necessarily*** increases the value of the maximum flow by at least 1.

b. Prove that $\lceil \ell/2 \rceil$ is a ***tight*** bound on $V^+ - V$. That is, prove that (a) for ***any*** flow network $\mathcal{F}$, $V^+ \leq V + \lceil \ell/2 \rceil$, and (b) there is ***some*** flow network $\mathcal{F}$ such that $V^+ = V + \lceil \ell/2 \rceil$.

Question to think about (but not to submit with your answer): For each $d \in [1..\lceil \ell/2 \rceil]$, can you design a flow network whose maximum flow value increases by exactly d if you increase by 1 the capacity of every edge on some $s \to t$ path?