University of Toronto at Scarborough
**CSCC73 - Algorithm Design and Analysis**

# Assignment #7 Linear Programming

Due: March 26, 2023 at 11.59pm This exercise is worth 5% of your final grade.

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1. (15 marks) Your favourite natural food company sells two types of trail mix each of which are made from a blend of dried fruits and nuts. Trail mix $A$ contains $1lb$ of dried fruits and $1.5lbs$ of nuts and retails for \$7. A package of trail mix $B$ contains $2lbs$ of dried fruit and $1lb$ of nuts and retails for \$6. Dried fruits when bought in bulk cost $\$1/lb$ and bulk nuts cost $\$2/lb$. The packaging for trail mix $A$ is a nice metal tin and costs $\$1.40$ to package whereas type $B$ tail mix is packaged in a resealable bag and costs $\$0.60$ to for the packaging. A total of 240,000lbs of dried fruits and 180,000lbs of nuts are available each month. Due to the nature of the packaging, the bottleneck in the production is for type $A$ in that the factory can only produce 110,000 tins of trail mix $A$ per month.

   (a) Formulate the problem as a linear program in two variables where the objective function maximizes profit.
   Soln.

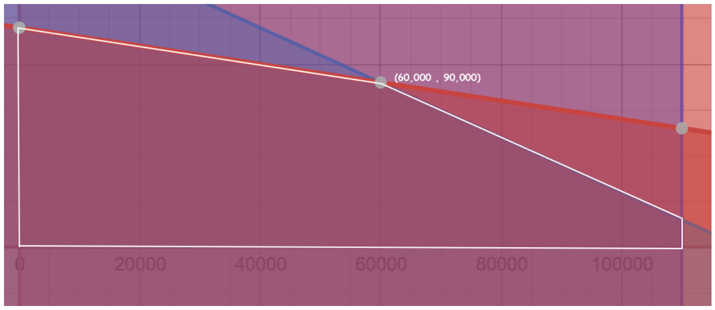
   First note that trail mix A costs $\$1 + 2x1.5 + 1.40$ to make and retails for \$7 so profit is $\$1.60$ per container. Trail mix B costs $\$2x1 + 2 + 0.6$ to make and retails for \$6 so makes a profit of $\$1.40$ per bag.

   $$\max 1.6x_1 + 1.4x_2$$

   subject to:

   $$x_1 + 2x_2 \leq 240000$$
   $$1.5x_1 + x_2 \leq 180000$$
   $$x_1 \leq 110000$$
   $$x_1, x_2 \geq 0$$

   (b) Graph the feasible region, give the coordinates of the vertices and state the vertex maximizing the profit and the value of the maximum profit.




(c) Confirm the maximizing vertex by applying the Simplex method to the problem.
The profit is maximized when $x_1 = 60000$ and $x_2 = 90000$ and has value \$222000.

2. (0 marks - will not be graded, just for practice, does not need to be handed in) Consider the following LP problem:

$$\max 18x_1 + 12.5x_2$$

subject to

$$\begin{aligned} &x_1 + x2 \leq 20 \;\; (1) \\ &x_1 \leq 12 \;\; (2) \\ &x_2 \leq 16 \;\; (3) \\ &x_1, x_2 \geq 0 \;\; (4) \end{aligned}$$

(a) Solve the program using the SIMPLEX method.
Note that $(0, 0)$ is a feasible starting vertex with obj. value 0.

If we now move to tighten constraint 2, we get $x_1 = 12$ and $x_2 = 0$, for vertex $(12, 0)$ and objective value 216.

Let $y_1 = 12 - x_1$ and $y_2 = x_2$. Then $(y_1, y_2) = (0, 0)$ is a feasible solution. The objective function becomes $\max 18(12 - x_1) + 12.5y_2 = \max 216 - 18y_1 + 12.5y_2$. Constraints become:

$$\begin{aligned} &-y_1 + y_2 \leq 8 \;\; (1) \\ &y_1 \geq 0 \;\; (2) \\ &y_2 \leq 16 \;\; (3) \\ &y_1 \leq 12(4) \\ &y_2 \geq 0 \;\; (4) \end{aligned}$$

Notice that the next feasible vertex is when we increase $y_2$ and make constraint (1) tight. This gives us:
$y_2 \leq 8$, so let $z_2 = 8 - y_2$ and $z_1 = y_1$. Then the obj. fun. becomes:
$\max 216 - 18z_1 + 12.5(8 - z_2)$ or $\max 316 - 18z_1 - 12.5z_2$
The max value is 316 and we can do no better. Substituting back in for $y_1$ and $y_2$ we get $z_2 = 0 = 8 - y_2$ or $y_2 = 8 = x_2$ and $0 = z_1 = y_1 = 12 - x_1$ so $x_1 = 12$. The optimal vertex point is $(12, 8)$.

(b) Write the dual LP and show that your solution in part (a) is optimal.

To get the dual we want to take a linear combination of the constraints that is the same as the optimization function.

$y_1(x_1 + x_2) \leq 20y_1$
$y_2(x_1) \leq 12y_2$
$y_3(x_2) \leq 16y_3$

$x_1(y_1 + y_2) + x_2(y_1 + y_3) \leq 20y_1 + 12y_2 + 16y_3$

Which gives
$\min 20y_1 + 12y_2 + 16y_3$ 66 + 250 = 312 subject to:

$$\begin{aligned} &y_1 + y_2 \geq 18 \ \ (1) \\ &y_1 + y_3 \geq 12.5 \ \ (2) \\ &y_1, y_2, y_3 \geq 0 \ \ (3) \end{aligned}$$

Notice that if we let $y_1 = 12.5$, $y_3 = 0$ and $y_2 = 5.5$ then our objective function value is $20(12.5) + 12(5.5) + 16(0) = 316$. Since the dual and primal have the same objective values this must be the optimum value.

3. The standard form of an LP problem is to solve for vector $x \in R^n$ defined by: `minimize` $c^t x$ subject to $x \geq 0$ and $Ax \geq b$ where $A$ is an $m \times n$ matrix and $c \in R^n$ and $b \in R^m$ are vectors.

When we consider an ILP (integer linear program), the added constraint that $x$ is an integer valued vector is added. Consider the following objective function and set of constraints.

$$Minimize : h(x_1) + h(x_2) + \ldots + h(x_n)$$

where

$$h(x_i) = \begin{cases} c_i x_i + f_i & x_i > 0 \\ 0 & x_i = 0 \end{cases}$$

subject to:

$$\begin{aligned} x_1 + x_2 + \ldots + x_n &\geq d \\ x_i &\geq 0,\ 1 \leq i \leq n \\ x_i &\leq B_i,\ 1 \leq i \leq n \\ x_i \in \mathbb{Z} & \qquad 1 \leq i \leq n \end{aligned}$$

Notice that the objective function is *not* a linear equation. Introduce new variables and constraints to convert this optimization problem into an ILP problem. Hint: Try using indicator variables for your new variables.

SAMPLE SOLUTION:

*We can correct the objective function by adding $n$ new variables $z_1, \ldots, z_n$. Each $z_i$ will be an indicator variable for $x_i$ in that $z_i = 1$ if $x_i > 0$ and $z_i = 0$ if $x_i = 0$. Thus our new objective function is:*

$$c_1x_1 + f_1z_1 + c_2x_2 + f_2z_2 + \cdots + c_nx_n + f_nz_n$$

*We need to integrate the $z_i$ into the constraints so that the original meaning remains unchanged, but the requirements of $z_i$ are met. Notice that if we change*

$$x_i \geq 0, 1 \leq i \leq n$$

to

$$x_i - z_i \geq 0, 1 \leq i \leq n$$

*then if $x_i$ is 0, the constraint requires that $z_i$ be 0. Also, notice that since $x_i$ is integer, if $x_i > 0$, then $x_i \geq 1$ so $x_i - 1 \geq 0$ is still satisfied (so $z_i = 1$).*
*Secondly, if we take the next constraint,*

$$x_i \leq B_i, 1 \leq i \leq n$$

*We can insert $z_i$ as follows:*

$$x_i \leq z_iB_i \Leftrightarrow x_i - B_iz_i \leq 0$$

*This forces $z_i$ to be 1 if $x_i > 0$ but it does not interfere with the requirement that $x_i \leq B_i$ as when $x_i \geq 0$ $z_iB_i = B_i$. Finally we need to add the last constraint,*

$$z_i \in \{0, 1\}.$$