Computer Science C73 November 23, 2022
Scarborough Campus University of Toronto

Homework Assignment #8
Due: MONDAY December 5, 2022, by 11:59 pm (note unusual due day)

• ***You must submit your assignment through the Crowdmark system.*** *You will receive by email an invitation through which you can submit your work in the form of separate PDF docucments with your answers to each question of the assignment. To work with a partner, you and your partner must form a group on Crowdmark. Crowdmark does not enforce a limit on the size of groups.* ***The course policy that limits the size of each group to at most two remains in effect:*** *submissions by groups of more than two persons will not be graded.*
• *It is your responsibility to ensure that the PDF files you submit are legible. To this end, I encourage you to learn and use the LaTex typesetting system, which is designed to produce high-quality documents that contain mathematical notation. You are not required to produce the PDF files you submit using LaTex; you may produce it any way you wish, as long as the resulting document is legible.*
• *By virtue of submitting this assignment you (and your partner, if you have one) acknowledge that you are aware of the policy on homework collaboration for this course.*[a]
• *For any question, you may use data structures and algorithms previously described in class, or in prerequisites of this course, without describing them. You may also use any result that we covered in class, or is in the assigned sections of the official course textbooks, by referring to it.*
• *Unless we explicitly state otherwise, you may describe algorithms in high-level pseudocode or point-form English, whichever leads to a clearer and simpler description.*
• *Unless we explicitly state otherwise, you should justify your answers. Your paper will be graded based on* ***the correctness and efficiency of your answers, and the clarity, precision, and conciseness of your presentation****.*

[a] "In each homework assignment you may collaborate with at most one other student who is currently taking CSCC73. If you collaborate with another student on an assignment, you and your partner must submit only one copy of your solution, with both of your names. The solution will be graded in the usual way and both partners will receive the same mark. Collaboration involving more than two students is not allowed. **For help with your homework you may consult only the instructor, TAs, your homework partner (if you have one), your textbook, and your class notes. You may not consult any other source.**"

Question 1. (20 marks) A group of p neighbours $1, 2, \dots, p$ agree to cooperate on tending a community garden over the next q weekends $1, 2, \dots, q$. Each neighbour i is available to tend the garden on a subset W_i of the next q weekends. (So, for each i, $1 \le i \le p$, $W_i \subseteq \{1, 2, \dots, q\}$.) The garden is far away from the neighbourhood, so the neighbours carpool to drive there. Each neighbour i has a car that can carry up to 5 persons (including him or herself), and is willing to drive at most k_i weekends.

a. We want to find an assignment of neighbours to drive to the garden on each weekend so that:

(1) just enough drivers are assigned each weekend to carry all the neighbours available to garden that weekend;
(2) each of the neighbours assigned to drive on a weekend is available to garden that weekend; and
(3) no neighbour i is assigned to drive more than k_i weekends.

By reduction to the maximum flow problem, give a polynomial-time algorithm that determines if such an assignment is possible; if it is possible, your algorithm also finds an assignment of drivers to weekends that satisfies the above requirements (1)-(3). Your algorithm takes as inputs p; q; W_i and k_i for each i, $1 \le i \le p$. Justify the correctness of your algorithm and analyze its running time.

Hint: From the input we can compute the number n_j of neighbours available to garden on week j, $1 \leq j \leq q$.

b. The neighbours soon realize that gardening becomes problematic if too many people are working at the same time. They decide that in the following season at most 10 people can be gardening each weekend.

We want to find an assignment of neighbours to work on the garden each weekend, some of whom will serve as drivers, that maximizes the total number of neighbours assigned to garden over the q weekends and satisfies all of the following requirements (if such an assignment exists):

(1) at most 10 neighbours are assigned to work on the garden each weeekend;
(2) each of the neighbours assigned to garden on a weekend is available to do so that weekend;
(3) enough of the neighbours assigned to garden on a weekend can serve as drivers to carry all the people assigned to garden that weekend (recall that each driver can drive at most five people, including him or herself);
(4) no neighbour i has to serve as driver more than k_i weekends.

Note that here we are seeking an assignment that tells us who is assigned to garden each weekend, not only who will serve as driver each weekend (as in part (a)).

Formulate this problem as a 0-1 linear program. Explain the role of your program's variables, and how the constraints capture the above requirements.

Question 2. (20 marks) Let $G = (V, E)$ be an undirected graph. Recall that a ***matching*** of G is a subset E' of the edges so that no two edges in E' share an endpoint (i.e., no two edges in E' are incident on the same node). A matching is ***maximal*** if no proper superset of it is a matching. A matching is ***maximum*** if no matching has more edges. Obviously a maximum matching is maximal, but the converse is not true (you may find it instructive to construct a counterexample).

a. Formulate the problem of finding a maximum matching in a graph as a 0-1 linear program.

b. Let E' be a maximal matching of G and V' be the set of endpoints of the edges in E'. Furthermore, let E^* be a maximum matching and V^* be a minimum vertex cover of G. Prove that V' is a vertex cover, that $|V'| \leq 2|V^*|$, and that $|E'| \geq |E^*|/2$.

c. Give linear time (i.e., $O(|V| + |E|)$ approximation algorithms for the vertex cover and maximum matching problems with approximation factors 2 and 1/2, respectively. (**Hint:** Use part (b).)

d. Are the approximation factors tight for your algorithms in part (c)? Justify your answer.

Note: There exist polynomial time algorithms that solve the maximum matching problem exactly, but the fastest known ones are superlinear.

THAT'S IT WITH HOMEWORK, FOLKS!