Homework Assignment #7
Due: November 23, 2022, by 11:59 pm

• **You must submit your assignment through the Crowdmark system.** *You will receive by email an invitation through which you can submit your work in the form of separate PDF docucments with your answers to each question of the assignment. To work with a partner, you and your partner must form a group on Crowdmark. Crowdmark does not enforce a limit on the size of groups.* **The course policy that limits the size of each group to at most two remains in effect:** *submissions by groups of more than two persons will not be graded.*
• *It is your responsibility to ensure that the PDF files you submit are legible. To this end, I encourage you to learn and use the LaTex typesetting system, which is designed to produce high-quality documents that contain mathematical notation. You are not required to produce the PDF files you submit using LaTex; you may produce it any way you wish, as long as the resulting document is legible.*
• *By virtue of submitting this assignment you (and your partner, if you have one) acknowledge that you are aware of the policy on homework collaboration for this course.[a]*
• *For any question, you may use data structures and algorithms previously described in class, or in prerequisites of this course, without describing them. You may also use any result that we covered in class, or is in the assigned sections of the official course textbooks, by referring to it.*
• *Unless we explicitly state otherwise, you may describe algorithms in high-level pseudocode or point-form English, whichever leads to a clearer and simpler description.*
• *Unless we explicitly state otherwise, you should justify your answers. Your paper will be graded based on* **the correctness and efficiency of your answers, and the clarity, precision, and conciseness of your presentation**.

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[a] "In each homework assignment you may collaborate with at most one other student who is currently taking CSCC73. If you collaborate with another student on an assignment, you and your partner must submit only one copy of your solution, with both of your names. The solution will be graded in the usual way and both partners will receive the same mark. Collaboration involving more than two students is not allowed. **For help with your homework you may consult only the instructor, TAs, your homework partner (if you have one), your textbook, and your class notes. You may not consult any other source.**"

Question 1. (10 marks) For each of the statements below, state whether it is true or false, and justify your answer. A justification by counterexample should include a demonstration that the proposed counterexample possesses all the (non-obvious) properties needed to serve its purpose. In all cases, assume that the edge capacities of the flow graph F are positive integers, and that there is a path from s to t (so the maximum flow of F has positive value).

a. If f is a maximum flow and (S,T) is a minimum cut of flow network $\mathcal{F}$, then every edge that crosses (S,T) is saturated by f. (We say that an edge $e = (u,v)$ crosses a cut (S,T) if $u \in S$ and $v \in T$; we say that e is saturated by f, if $f(e) = c(e)$.)

b. If f is maximum flow in flow network $\mathcal{F}$ and e is an edge that is saturated by f, then e crosses some minimum cut of $\mathcal{F}$.

c. If f is a flow in flow network $\mathcal{F}$ and $\mathcal{V}(f) = 0$, then $f(e) = 0$ for every edge e of $\mathcal{F}$.

d. If (S,T) is a minimum cut of a flow network $\mathcal{F} = (G, s, t, c)$ then (S,T) is a minimum cut of flow network $\mathcal{F}^+ = (G, s, t, c^+)$, which is $\mathcal{F}$ with every edge's capacity increased by one (i.e., for every edge e of $\mathcal{F}$, $c^+(e) = c(e) + 1$.

Question 2. (15 marks) Let $\mathcal{F} = (G, s, t, c)$ be a flow graph with integer capacities and f be a flow in $\mathcal{F}$. A **cycle of flow** f is a cycle of G every edge of which carries some traffic; that is, a path $u_1, u_2, \ldots, u_k$ of G such that $u_1 = u_k$ and $f(u_i, u_{i+1}) > 0$, for each i such that $1 \leq i < k$. A flow f is called **acyclic** if it has no cycle.

a. Suppose we use the version of the Ford-Fulkerson algorithm that always chooses an augmenting path with the fewest edges. Give an example flow network in which this algorithm **necessarily** produces a cyclic flow. Your example should be as simple as you can. You should identify the augmenting paths that the algorithm chooses in your example flow network and the cycle of the maximum flow that it creates.

b. Prove that for any flow network $\mathcal{F}$ and any flow f in $\mathcal{F}$, there is an acyclic flow of $\mathcal{F}$ with the same value as f. You may **not** assume that f is an integral flow.

Question 3. (15 marks) Let $\mathcal{F} = (G, s, t, c)$ be a flow network with integer capacities, f be an **integral** maximum flow in $\mathcal{F}$, and e be an edge of the graph G. Let $\mathcal{F}^+ = (G, s, t, c^+)$ be the flow graph obtained from $\mathcal{F}$ by increasing the capacity of e by one unit, and $\mathcal{F}^- = (G, s, t, c^-)$ be the flow graph obtained from $\mathcal{F}$ by decreasing the capacity of e by one unit. (That is, for every edge $e' \neq e$ of G, $c^+(e') = c^-(e') = c(e')$, $c^+(e) = c(e) + 1$, and $c^-(e) = c(e) - 1$.[†])

a. Prove that the value of a maximum flow in $\mathcal{F}^+$ is $\mathcal{V}(f)$ or $\mathcal{V}(f) + 1$, and the value of a maximum flow in $\mathcal{F}^-$ is $\mathcal{V}(f)$ or $\mathcal{V}(f) - 1$.

b. Give an algorithm that, given $\mathcal{F}$, f, and e, finds a maximum flow in $\mathcal{F}^+$. Your algorithm should be **significantly faster** than recomputing a maximum flow of $\mathcal{F}^+$ from scratch. Justify the correctness of your algorithm and analyze its running time.

c. Do as in part (b), but for $\mathcal{F}^-$. In this case you may find it convenient to assume that G has no cycles. If the correctness of your algorithm depends on this assumption then, for full marks, you must state explicitly where it is needed.

[†]You may assume that $c(e) > 1$, so $c^-(e) > 0$.