University of Electronics Science and Technology of China

Title: A paper on A Relationship among FT,DTFT,DFT and Z-Transform.

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INTRODUCTION

The Fourier series and Fourier transforms are mathematical correlations between the time and frequency domains. The Fourier representations are extremely important to engineering and science because they can take a time-domain function and convert it to the corresponding frequency-domain representation. The time-domain functions are not limited to a certain type of time domain signal, or waveform, but may represent a wide variety of natural and man-made signals such as seismic vibrations, mechanical vibrations, electrical signals, and even stock fluctuations. Fourier analysis can be an extremely useful tool in certain situations. Often times, converting a signal into another domain or transforming the signal can make the mathematics easier. Also, if a signal is passed through a linear time-invariant system (LTI), then the frequency components themselves are not shifted, but altered only in magnitude and phase. This means that the effect of the LTI system on the spectral content of the signal is much easier to predict than the effect on the signal's time domain The Fourier series and Fourier transform take a continuous, time-domain function and represent it as a weighted sum of complex sinusoids. The Concept discovered after Fourier's work that the signals did not have to be continuous in the time domain but could instead be discrete data points. This led to the inception of the discrete-time Fourier series and the discrete-time Fourier transform

Four different Fourier representations exist:
1. Fourier series (FS)
2. Fourier transform (FT)
3. Discrete-time Fourier transform (DTFT)
4.Discrete Fourier Series or Discrete Fourier Transform (DFT)

The factor that determines whether to use a series or a transform is periodicity in the time-domain signal. For a periodic signal in the time domain, the series is used. For a non-periodic signal in the time domain, the transform is used. The Fourier series is used to convert a continuous and periodic time-domain signal into the frequency domain. The resulting frequency domain representation from performing the Fourier series is discrete and non-periodic.
The Fourier transform is used to convert a continuous and non-periodic time domain signal into the frequency domain and the resulting frequency domain representation from performing the Fourier transform is continuous and non-periodic. The discrete-time Fourier series(DFT) is used to convert a discrete and periodic time domain into the frequency domain and the resulting frequency domain representation from performing the discrete Fourier series(DFT) is discrete and periodic. The discrete-time Fourier transform is used to convert a discrete and non-periodic time-domain signal into the frequency domain and the resulting frequency domain representation from performing the discrete Fourier transform is continuous and periodic.

The Z-transform is a generalization of the discrete-time Fourier transform (DTFT). The DTFT can be found by evaluating the Z-transform X() at $z = e^{j\omega}$ (where ω is the normalized frequency) or, in other words, evaluated on the unit circle. In order to determine the frequency response of the system the Z-transform must be evaluated on the unit circle, meaning that the system's region of convergence must contain the unit circle. Otherwise, the DTFT of the system does not exist. Because of the convergence condition, in many cases, the DTFT of a sequence may not exist As a result, it is not possible to make use of such frequency-domain characterization in these cases z-transform may exist for many sequences for which the DTFT does not exist.

Here the different transforms- **Fourier Transform(FT), Discrete Time Fourier Transforms(DTFT)**, **Discrete Fourier Transforms(DFT)** and the **Z-Transform** below mentioned transforms and relationships among them will be discussed briefly:

Fourier Transform(FT)

The Fourier Transform of signal $g(t)$ is denoted F $[g(t)]$ and is defined as

$$G(\omega) = \mathcal{F}[g(t)] = \int_{-\infty}^{\infty} g(t)e^{-j\omega t}dt, \qquad \text{(A)}$$

and the IFT of $G(\omega)$ is denoted F $^{-1}[G(\omega)]$ is defined as

$$g(t) = \mathcal{F}^{-1}[G(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)e^{j\omega t}d\omega \qquad \text{(B)}$$

We say that $g(t)$ and $G(\omega)$ for a FT pair, or $g(t) \Leftrightarrow G(\omega)$.
Notice that the exponent term in (A) has a negative sign but no negative sign exists in the exponent in (B). Also, notice that the integration in (A) is in terms of t and it is in terms of ω in (B).
Also notice that $G(\omega)$ in general is a complex signal that has both a magnitude $|G(\omega)|$ and a phase $\theta_G(\omega)$, or

$$G(\omega) = |G(\omega)| \cdot e^{j\theta_G(\omega)}.$$

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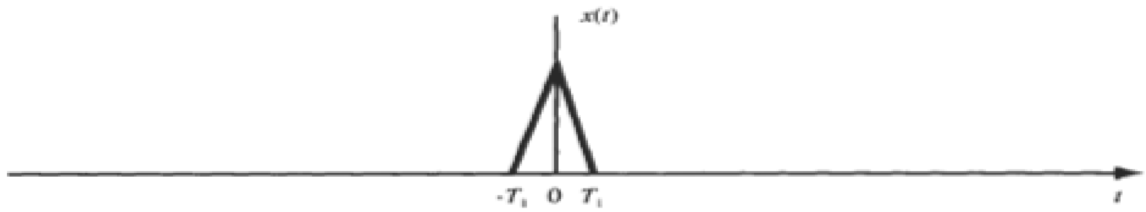
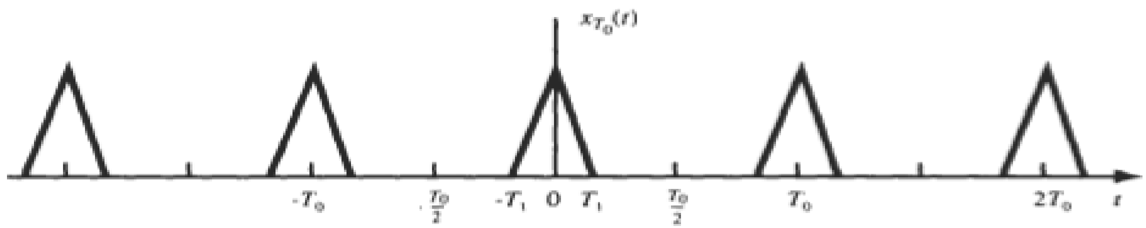
Fourier Transform is used to analyse non periodic signals also we see how we can also use it for periodic signals , also Fourier Transform is another method for representing signals and system in the frequency domain.

Let $x(t)$ be a non periodic signal of finite duration, i.e.,

$$x(t) = 0 \qquad |t| > T_1$$



(a)



Let us form a periodic signal by extending $x(t)$ to $x_{T_0}(t)$ as,

$$\lim_{T_0 \to \infty} x_{T_0}(t) = x(t) \; , \qquad \text{[i.e., the period is infinity]}$$

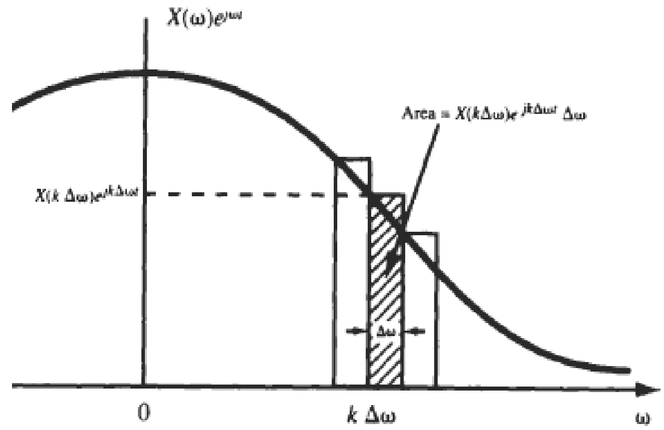
Then,

$$x_{T_0}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T_0}$$

$$c_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x_{T_0}(t) e^{-jk\omega_0 t} dt \qquad \text{.................eq.1}$$

Or,

$$c_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) e^{-jk\omega_0 t} dt = \frac{1}{T_0} \int_{-\infty}^{\infty} x(t) e^{-jk\omega_0 t} dt$$



Let us now define $X(\omega)$ as, $X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$

Thus, $\qquad c_k = \dfrac{1}{T_0} X(k\omega_0)$.

Substituting this in eq.1 we get,

$$x_{T_0}(t) = \sum_{k=-\infty}^{\infty} \frac{X(k\omega_0)}{T_0} e^{jk\omega_0 t} = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \frac{X(k\omega_0)}{T_0} e^{jk\omega_0 t} \omega_0$$

As $T_0 \to \infty$, $\omega_0 \to 0$. Let us assume $\omega_0 = \Delta\omega$.

Thus, $\quad \displaystyle\lim_{T_0 \to \infty} x_{T_0}(t) = \lim_{\Delta\omega \to 0} \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(k\Delta\omega) e^{jk\omega_0 t} \Delta\omega = x(t)$

Or, $x(t) = \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$ eq.2

$x(t)$ in eq.2 is called the Fourier Integral. Thus a finite duration signal is represented by Fourier integral instead of Fourier series.

The function $X(\omega)$ is called the Fourier transform of $x(t)$.

Symbolically these two pairs are represented as,

$$X(\omega) = F\{x(t)\} = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

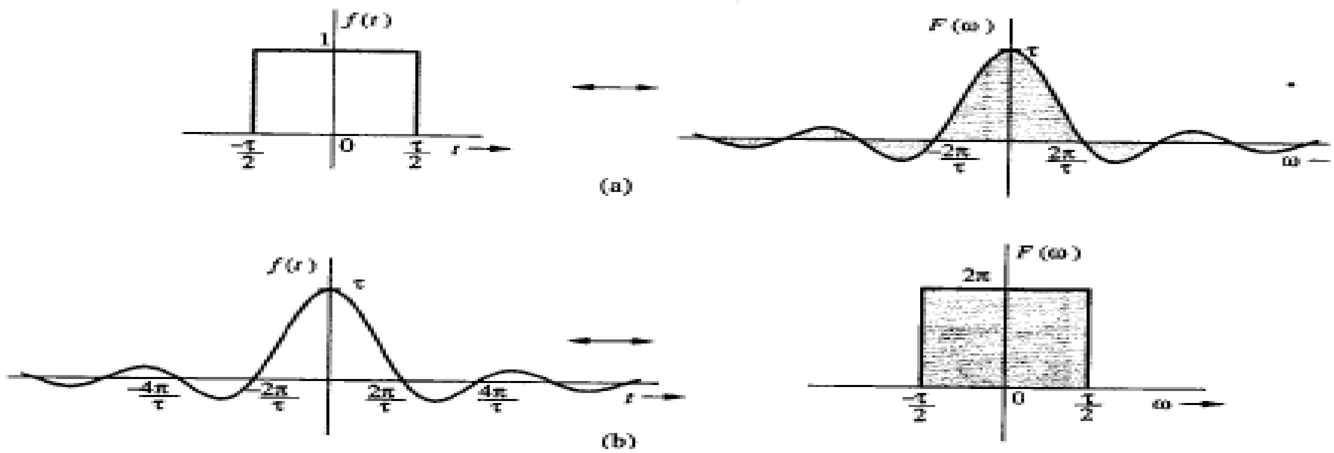
And

$$x(t) = F^{-1}\{X(\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

Alternatively, $x(t) \xleftrightarrow{F.T.} X(\omega)$.

Properties of the Fourier Transform

Symmetry Property:



If $\quad f(t) \longleftrightarrow F(\omega)$

Then $\quad F(t) \longleftrightarrow 2\pi\, f(-\omega)$

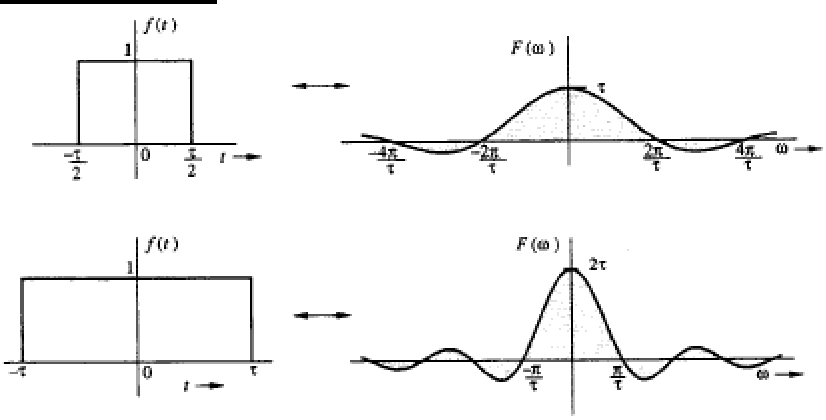
Linearity Property:

If $\quad f_1(t) \longleftrightarrow F_1(\omega)$

And $\quad f_2(t) \longleftrightarrow F_2(\omega)$

Then $\quad [a*f_1(t) + b*f_1(t)] \longleftrightarrow [a*F_1(\omega) + b*F_2(\omega)]$

Scaling Property:



If $\quad f(t) \longleftrightarrow F(\omega)$

Then for a real

$$f(a*t) \longleftrightarrow \frac{1}{|a|} F\left(\frac{\omega}{a}\right)$$

Frequency Shifting:

If $f(t) \leftrightarrow F(\omega)$

Then $f(t) * \varepsilon^{j\omega_0 t} \leftrightarrow F(\omega - \omega_0)$

Time Shifting:

If $f(t) \leftrightarrow F(\omega)$

Then $f(t - t_0) \leftrightarrow F(\omega_0) * \varepsilon^{-j\omega t_0}$

Time Differentiation and Integration

If $f(t) \leftrightarrow F(\omega)$

Then $\frac{d}{dt}[f(t)] \leftrightarrow (j\omega)F(\omega)$

And $\int_{-\infty}^{t} f(\tau)d\tau \leftrightarrow \frac{1}{j\omega}F(\omega)$

Frequency Differentiation

If $f(t) \leftrightarrow F(\omega)$

Then $(-jt)^n f(t) \leftrightarrow \frac{d^n}{dt^n}F(\omega)$

The Convolution Theorem:

Definition: the convolution of two functions $f_1(t)$ *and* $f_2(t)$ is defined as:

$$f_1(t) \otimes f_2(t) \equiv \int_{-\infty}^{\infty} f_1(\tau) * f_2(t - \tau)d\tau = \int_{-\infty}^{\infty} f_2(\tau) * f_1(t - \tau)d\tau$$

Time Convolution

If $f_1(t) \leftrightarrow F_1(\omega)$

And $f_2(t) \leftrightarrow F_2(\omega)$

Then $\Im\{f_1(t) \otimes f_2(t)\} \leftrightarrow F_1(\omega) * F_2(\omega)$

Frequency Convolution:

If $f_1(t) \leftrightarrow F_1(\omega)$

And $f_2(t) \leftrightarrow F_2(\omega)$

Then $f_1(t) * f_2(t) \leftrightarrow \frac{1}{2\pi}F_1(\omega) \otimes F_2(\omega)$

Discrete Time Fourier Transform (DTFT):

The Discrete Time Fourier Transform (DTFT) can be viewed as the limiting form of the DFT when its length N is allowed to approach infinity:

$$X(\tilde{\omega}) \triangleq \sum_{n=-\infty}^{\infty} x(n)e^{-j\tilde{\omega}n}$$

where $\tilde{\omega} \triangleq \omega T \in [-\pi, \pi)$ denotes the continuous normalized radian frequency variable, and x[n] is the signal amplitude at sample number n.

The inverse DTFT is

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\tilde{\omega})e^{j\tilde{\omega}n}d\tilde{\omega}$$

since x[n] can be recovered uniquely from its DTFT, they form a Fourier pair x[n] $\Longleftrightarrow$ X(ω)

Convergence of DTFT

$\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$ For the DTFT to exist, the given series must converge .

$$X_M(\omega) = \sum_{n=-M}^{M} x[n]e^{-j\omega n}$$

That is, the partial sum must converge to a limit $X(\omega)$ as $M \Rightarrow \infty$.

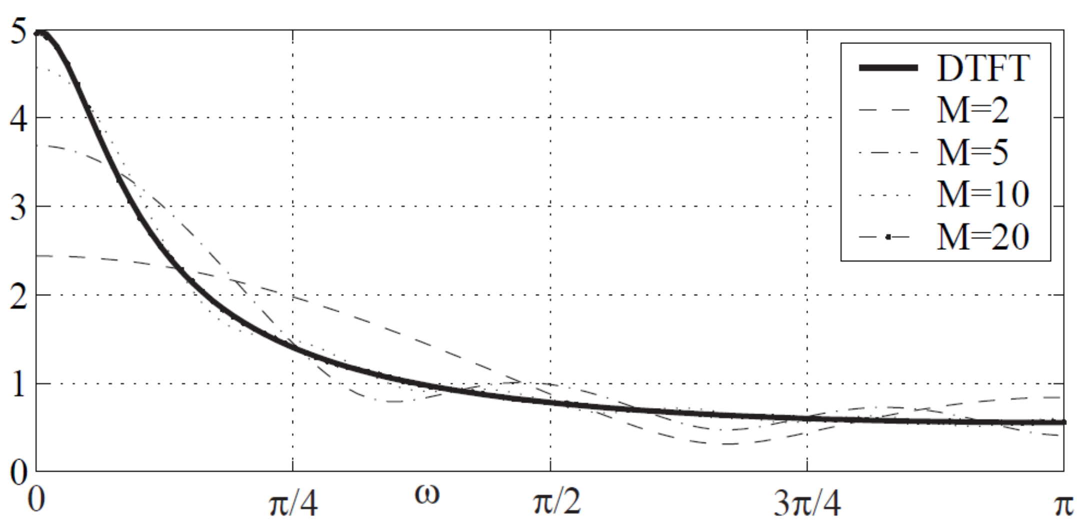


Fig: Illustration of uniform convergence for an exponential sequence.

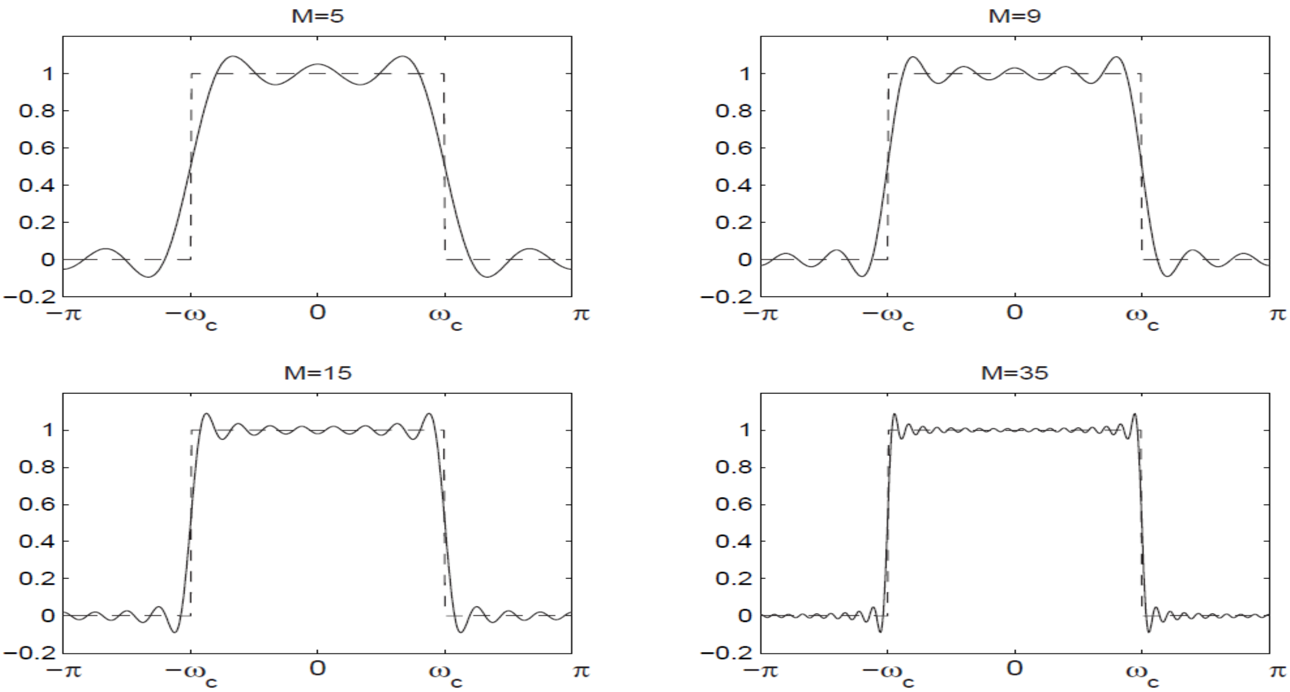


Fig: Illustration of the Gibb's phenomenon

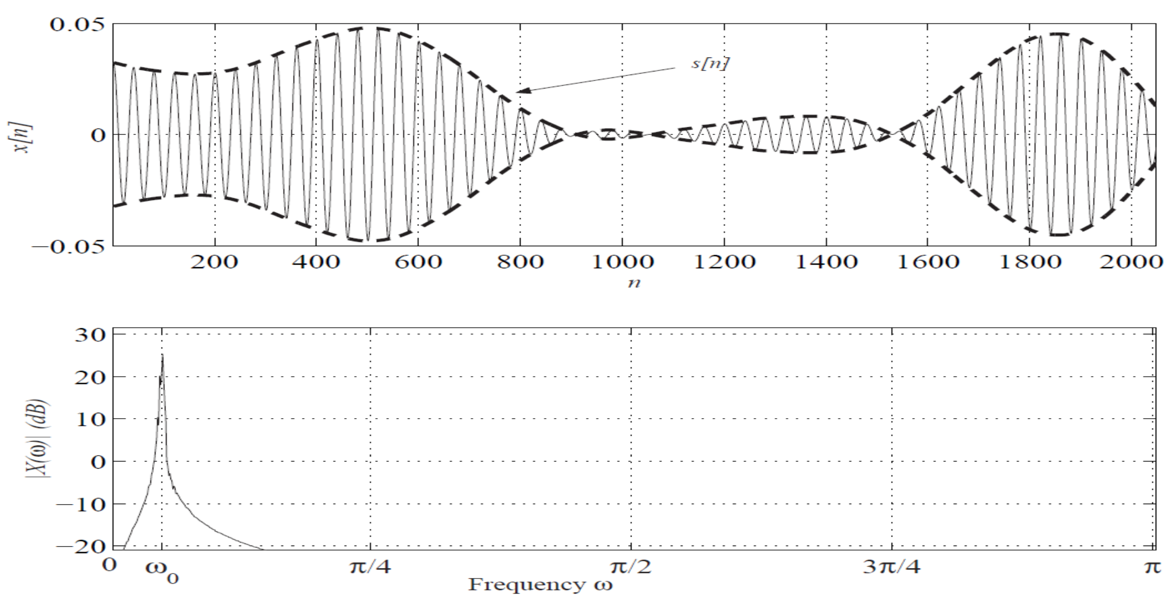


Fig: An example of a narrowband signal as defined by $x[n] = s[n]\cos(w_0 n)$.
Top: thesignal $x[n]$ is shown, together with the dashed envelope $s[n]$ (Samples are connected to ease viewing). Bottom: the magnitude of the DTFT of $x[n]$

Discrete-Time Fourier Transform Properties

Linearity	$a\,x[n]+b\,v[n]\leftrightarrow a\,X(\Omega)+bV(\Omega)$
Right/left time shift	$x(n-q)\leftrightarrow X(\Omega)e^{-jq\Omega},\quad q$ any integer
Time reversal	$x[-n]\leftrightarrow X(-\Omega)=\overline{X(\Omega)}$
Multiplication by *n*	$n\,x[n]\leftrightarrow j\dfrac{d\,X(\Omega)}{d\Omega}$
Multiplication by $e^{jn\Omega_0}$	$x[n]\,e^{jn\Omega_0}\leftrightarrow X(\Omega-\Omega_0),\quad \Omega_0$ real
Multiplication by $\sin\left(n\Omega_0\right)$	$x[n]\sin\left(\Omega_0\,n\right)\leftrightarrow \frac{j}{2}\left[X(\Omega+\Omega_0)-X(\Omega-\Omega_0)\right]$
Multiplication by $\cos\left(n\Omega_0\right)$	$x[n]\cos\left(\Omega_0\,n\right)\leftrightarrow \frac{1}{2}\left[X(\Omega+\Omega_0)+X(\Omega-\Omega_0)\right]$
Convolution in time-domain	$x[n]*v[n]\leftrightarrow X(\Omega)V(\Omega)$
Multiplication in time-domain	$x[n]v[n]\leftrightarrow \dfrac{1}{2\pi}\int_{t_0}^{t_0+2\pi}X(\Omega-\lambda)V(\lambda)\,d\lambda,\quad$ any 2π interval
Summation	$\sum_{i=0}^{n}x[i]\leftrightarrow \dfrac{1}{1-e^{-j\Omega}}X(\Omega)+\sum_{n=-\infty}^{\infty}\pi\,X(2\pi n)\,\delta(\Omega-2\pi n)$
Parseval's theorem	$\sum_{n=-\infty}^{\infty}x[n]v[n]=\frac{1}{2\pi}\int_{t_0}^{t_0+2\pi}\overline{X(\Omega)}\,V(\Omega)\,d\Omega,\quad$ any 2π interval
Special case of Parseval's theorem	$\sum_{n=-\infty}^{\infty}x^2[n]=\frac{1}{2\pi}\int_{t_0}^{t_0+2\pi}\left\lvert X(\Omega)\right\rvert^2\,d\Omega,\quad$ any 2π interval
Relationship to inverse CTFT	If $x[n]\leftrightarrow X(\Omega)$ and $\gamma(t)\leftrightarrow X(\omega)\,p_{2\pi}(\omega)$, then $x[n]=\gamma(t)\big\rvert_{t=n}=\gamma(n)$

DTFT of a periodic unit sample train

We have two expressions for a periodic unit sample train,

$$s[n]=\sum_{m=-\infty}^{\infty}\delta[n-mN]=\frac{1}{N}\sum_{m=0}^{N-1}e^{-j2\pi\frac{m}{N}}$$

The DTFT of the first expression is

$$\tilde{S}[\varphi] = \sum_{m=-\infty}^{\infty} e^{-j2\pi\varphi mN}$$

To obtain the DTFT of the second expression recall that

$$e^{j2\pi\frac{mn}{N}} \overset{F}{\Leftrightarrow} \sum_{k=-\infty}^{\infty} \delta(\varphi - \frac{m}{N} - k)$$ so that,

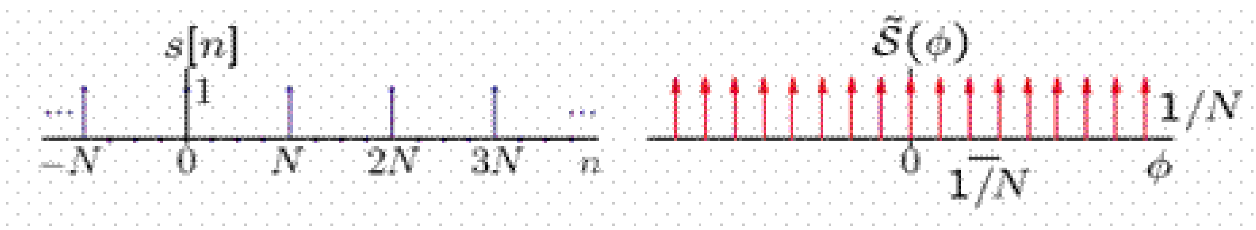
$$\sum_{m=0}^{N-1} e^{j2\pi\frac{mn}{N}} \overset{F}{\Leftrightarrow} \sum_{m=0}^{N-1}\sum_{k=-\infty}^{\infty} \delta(\varphi - \frac{m}{N} - k) = \sum_{m=-\infty}^{\infty} (\varphi - \frac{m}{N})$$

The periodic unit sample train

$$s[n] = \sum_{m=-\infty}^{\infty} \delta[n - mN] = \frac{1}{N}\sum_{m=0}^{N-1} e^{j2\pi\frac{mn}{N}}$$ has the Fourier transform

$$\tilde{S}[\varphi] = \sum_{m=-\infty}^{\infty} e^{-j2\pi\varphi mM} = \frac{1}{N}\sum_{m=-\infty}^{\infty} \delta(\varphi - \frac{m}{N})$$

Therefore, the DT Fourier transform of a periodic unit sample train in time is a periodic unit impulse train in frequency.



DTFT of an arbitrary periodic sequence

We can form an arbitrary periodic sequence x[n] by convolving an aperiodic sequence xN[n] that represents one period of the periodic sequence with a periodic unit sample sequence s[n]. Recall

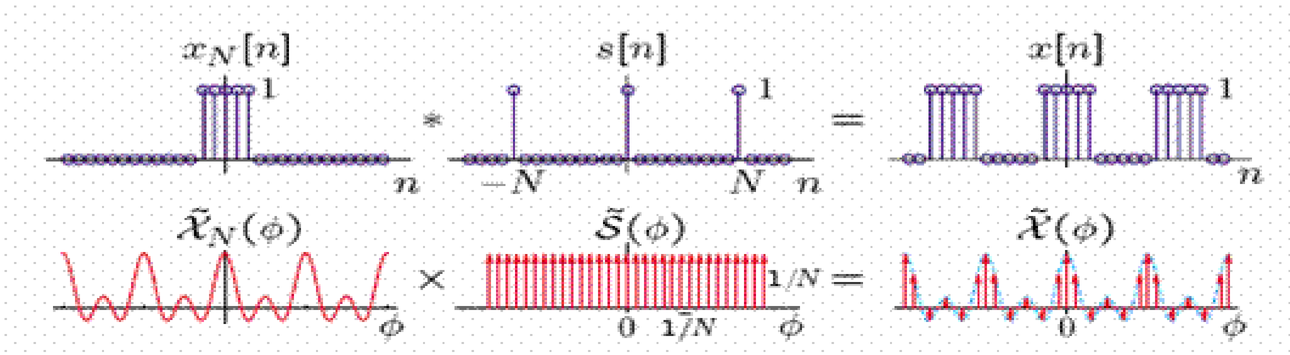
$$s[n] = \sum_{m=-\infty}^{\infty} \delta[n - mN]$$ and

$$x[n] = x_N[n] * s[n] = x_N[n] * \sum_{m=-\infty}^{\infty} \delta[n - mN] = \sum_{m=-\infty}^{\infty} x_N[n - mN]$$

Therefore,

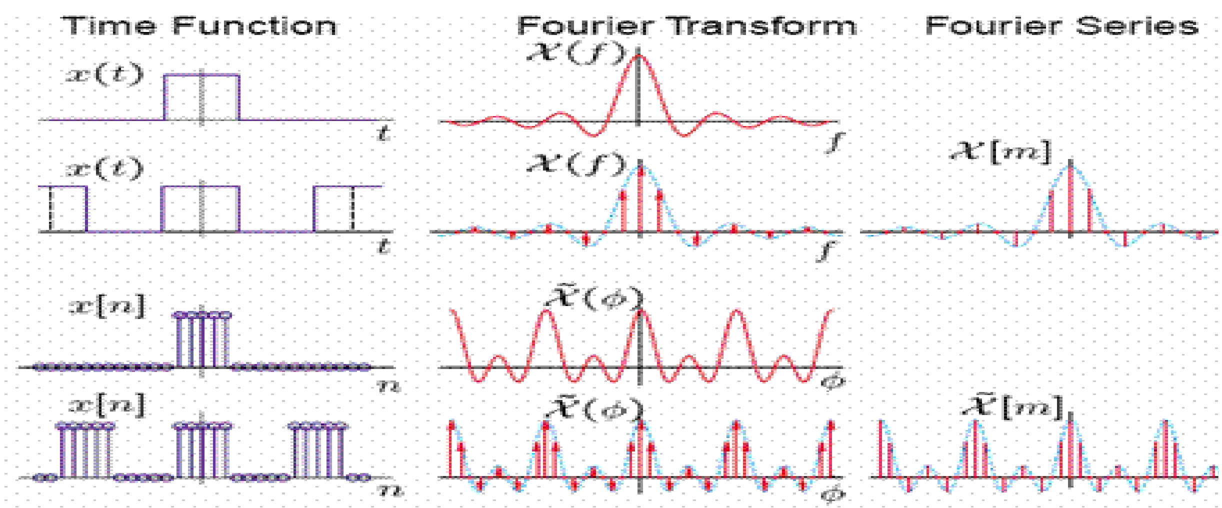
$$\tilde{X}(\varphi) = \tilde{X}_N(\varphi) \times \tilde{S}(\varphi) = \tilde{X}_N(\varphi) \times \frac{1}{N}\sum_{m=-\infty}^{\infty} \delta(\varphi - \frac{m}{N})$$

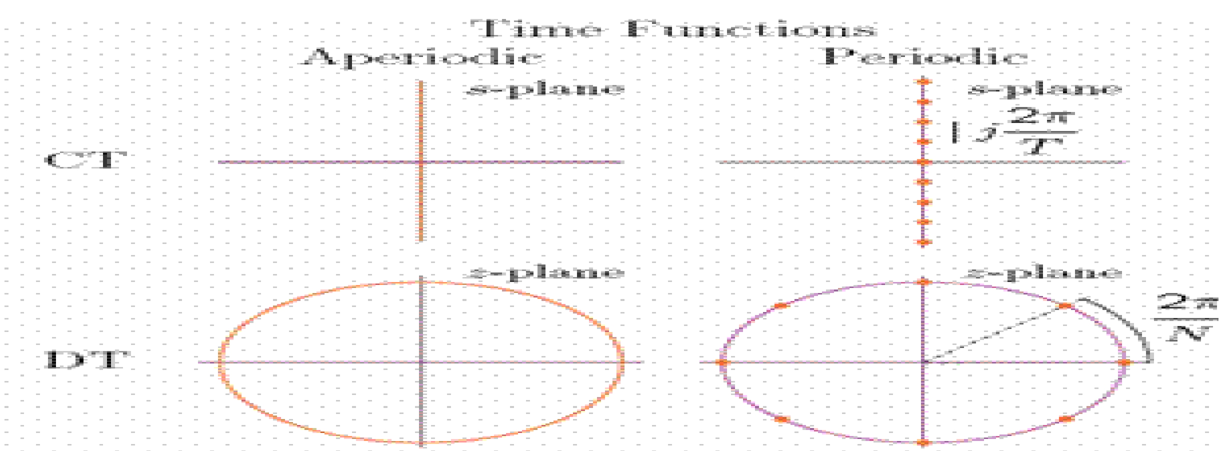
$$= \sum_{m=-\infty}^{\infty} \frac{\tilde{X}_N(m/N)}{N} \delta(\varphi - \frac{m}{N})$$



The DTFT of a periodic sequence consists of scaled impulses at multiples of the fundamental frequency.

Summary of frequency domain representations of signals





Discrete Fourier Transform(DFT)

The discrete time Fourier transform (DTFT) of a sequence {x[n]} is:

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-jn\Omega} \quad \text{where } \Omega = \omega / f_s = \omega T \text{ radians/sample}$$

If {x[n]} is obtained by sampling a suitably band limited signal $x_a(t)$, whose analogue Fourier transform is $X_a(j\omega)$, then

$$X(e^{j\Omega}) = (1/T)\ X_a(j\omega) \quad \text{for } -\pi < \Omega = \omega T < \pi$$

The DTFT provides a convenient means of computing $X_a(j\omega)$ by a summation rather than an integral. However, there are two practical difficulties:

(i) the range of summation is infinite,

(ii) $X(e^{j\Omega})$ is a continuous function of Ω.

The first difficulty enforces 'windowing' or restricting the sequence {x[n]} to a finite block of non-zero samples; say for n in the range 0 to N-1. The resulting windowed sequence:

$$\{\ldots, 0, \underline{x[0]}, x[1], \ldots, x[N-1], 0, \ldots\}$$

may be represented by the finite length sequence:

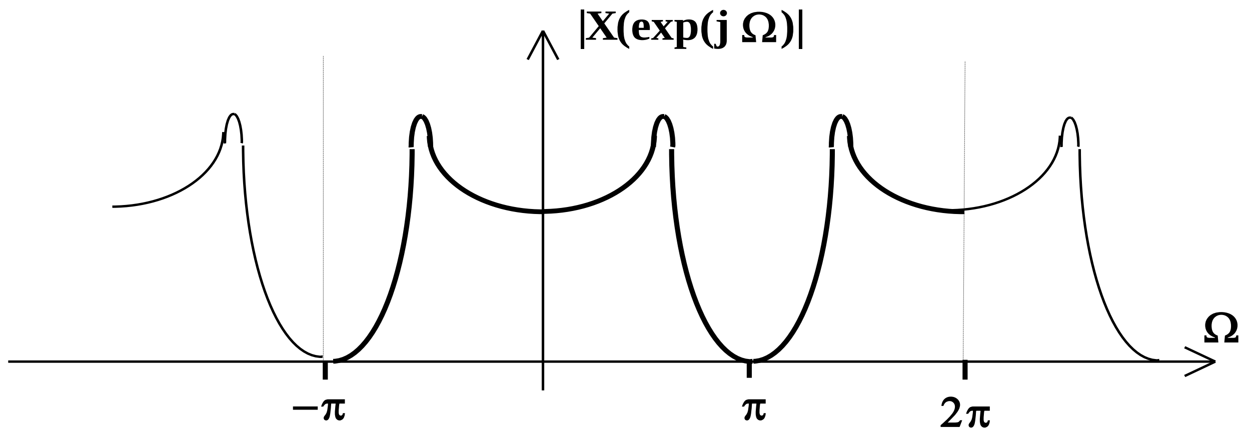
$$\{\underline{x[0]}, x[1], \ldots, x[N-1]\} \quad \text{denoted by } \{x[n]\}_{0,N-1}.$$

The second difficulty means that 'frequency-domain sampling' must be used.

It is normal, because of the inverse DTFT formula, to consider $X(e^{j\Omega})$ for Ω in the range $-\pi$ to π. For real signals, it would be sufficient to consider values of $X(e^{j\Omega})$ only in the range $0 \le \Omega \le \pi$.

There are applications where we need to apply the DTFT to complex signals, and in such cases we then need to know $X(e^{j\Omega})$ for Ω in the range $-\pi$ to π. However, because the spectrum $X(e^{j\Omega})$ repeats at frequency intervals of 2π as illustrated below, equivalent information is obtained if we evaluate $X(e^{j\Omega})$ for Ω in the range 0 to 2π instead of $-\pi$ to π. This is simply because

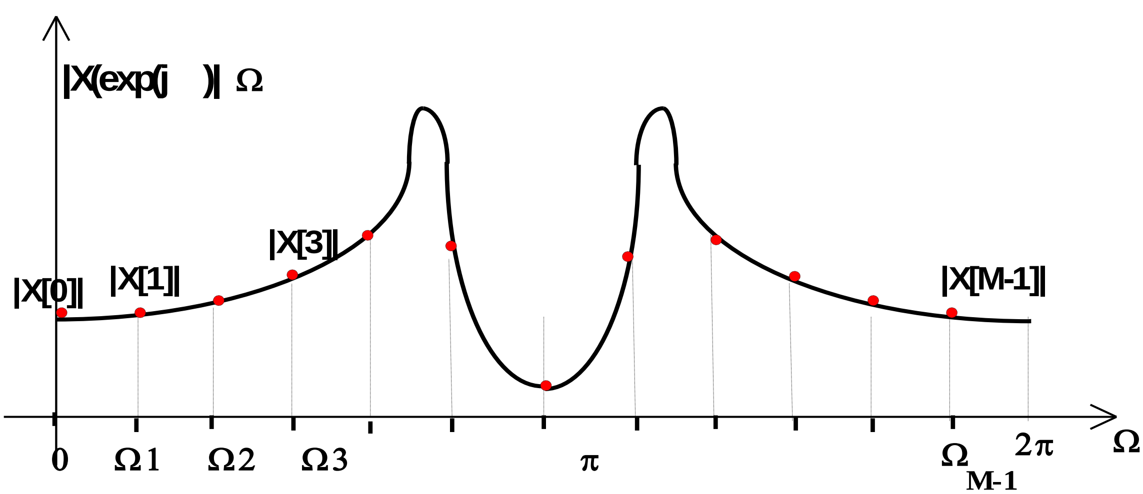
$X(e^{j\Omega}) = X(e^{j[\Omega+2\pi]})$ so the spectrum from $-\pi$ to 0 is repeated for Ω in the range π to 2π.



Taking M equally spaced frequency-domain samples in the range $0 \le \Omega \le 2\pi$ produces the finite sequence of complex numbers:

$$\left\{X(e^{j\Omega_k})\right\}_{O,M-1} = \{ X(e^{j\Omega_0}), \quad X(e^{j\Omega_1}), \ldots\ldots\ldots\ldots, X(e^{j\Omega_{M-1}})\}$$

where $\Omega_k = 2\pi k / M$ for $k = 0, 1, \ldots, M-1$.



The imposition of windowing and frequency-domain sampling on the DTFT produces the following equation:

$$X(e^{j\Omega_k}) = \sum_{n=0}^{N-1} x[n]\, e^{-j\Omega_k n} \quad \text{where } \Omega_k = 2\pi k / M$$

which is usefully evaluated for k = 0, 1, 2, ..., M-1, so that Ω_k goes from 0 to 2π. The larger M, the easier it is to draw smooth and accurate spectral graphs. However, it is often important to evaluate just sufficient samples to obtain an unambiguous spectral representation of a signal quickly. It may be shown that just N frequency-domain samples are sufficient for this purpose since, in this case, an inverse transform exists to recover the original time-domain signal from the N samples of its DTFT spectrum. With fewer than N frequency-domain samples we cannot be guaranteed a way of getting back the original N time-domain samples. Computing more than N frequency-domain samples has advantages for drawing graphs, as mentioned earlier, but this would be at the cost of slowing down the computation, which for some applications is undesirable. When M = N, the complex sequence defined above equation becomes the Discrete Fourier Transform (DFT) of the sequence $\{x[n]\}_{0,N-1}$.

Introducing the notation: $X(e^{j\Omega_k}) = X[k]$, the DFT may be defined as the transformation:

$$\{x[n]\}_{O,N-1} \rightarrow \{X[k]\}_{O,N-1}$$

with

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\Omega_k n} \quad \text{where } \Omega_k = 2\pi k / N \text{ for k } = 0, 1, 2, \ldots\ldots, \text{N - 1}$$

It is normal to consider $\{x[n]\}_{O,N-1}$ as a complex sequence, though in practice the imaginary parts of the sample values are often set to zero.

The inverse DFT:

$$\{X[k]\}_{O,N-1} \rightarrow \{x[n]\}_{O,N-1},$$

may be performed using the formula:

$$x[n] = (1/N) \sum_{k=0}^{N-1} X[k] e^{jn\omega_k} \text{ for } n = 0, 1, 2, \ldots\ldots, N\text{ - 1} \quad \text{with } \Omega_k = 2\pi k / N$$

Important Properties of DFT

Linearity: Let $\{x_0, x_1, ...x_{N-1}\}$ and $\{y_0, y_1, ...y_{N-1}\}$ be two sets of discrete samples with corresponding DFT's given by $X(m)$ and $Y(m)$. Then DFT of sample set $\{x_0 + y_0, x_1 + y_1, ..., x_{N-1} + y_{N-1}\}$ is given by $X(m) + Y(m)$

Periodicity: We have evaluated DFT at $m = 0,1,........, N-1$. Thereafter, $(m \geq N)$ it shows periodicity. For example

$$X(m) = X(N+m) = X(2N+m) = X(-N+m) = X(-2N+m) = X(kN+m)$$

i.e. $X(m) = X(kN+m)$, Where k is an integer

DFT symmetry: If the samples x_n are real, then extracting in frequency domain $X(0)........X(N-1)$ seems counter intuitive; because, from N bits of information in one domain (time), we are deriving 2N bits of information in frequency domain. This suggests that there is some redundancy in computation of $X(0).............X(N-1)$. As per DFT symmetry property, following relationship holds.

$$X(N-m) = X^*(m) \qquad m = 0,1,....., N-1$$

If the samples x_n are real; then they contain at most N bits of information. On the other hand, $X(m)$ is a complex number and hence contains 2 bits of information. Thus, from sequence $\{x_0, x_1, ..., x_{N-1}\}$, if we derive $\{X(0), X(1), ..., X(N-1)\}$, it implies that from N-bit of information, we are deriving $2N$ bits of information. This is counter intuitive. We should expect some relationship in the sequence $\{X(0), X(1), ..., X(N-1)\}$

Thus, we conclude that

$$|X(N-m)| = |X(m)| \qquad \text{[Symmetry]}$$

and $\angle X(N-m) = -\angle X(m)$ [Anti-symmetry]

Thus, a typical DFT magnitude and phase plots appear as follows. $\angle X(m)$

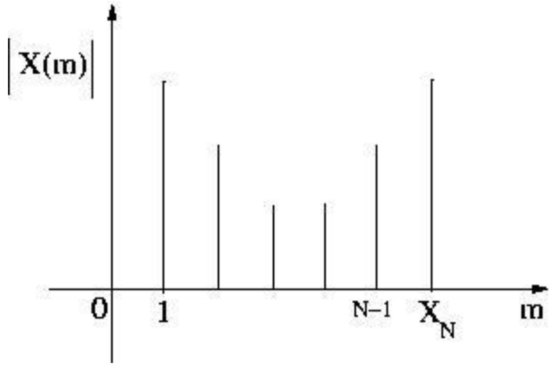


Fig (3)

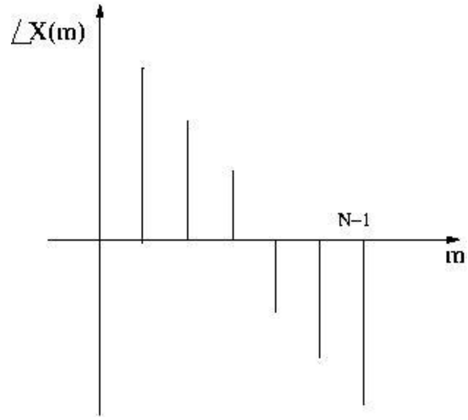


Fig (4)

DFT phase shifting:-

DFT shifting property states that, for a periodic sequence with periodicity N i.e. $x(m) = x(m + lN)$, l an integer, an offset in sequence manifests itself as a phase shift in the frequency domain. In other words, if we decide to sample x(n) starting at n equal to some integer K, as opposed to n=0, the DFT of those time shifted samples. $X_{shifted}(m)$

$$X_{\text{shifted}}(m) = e^{\frac{j2\pi Km}{N}} X(m)$$

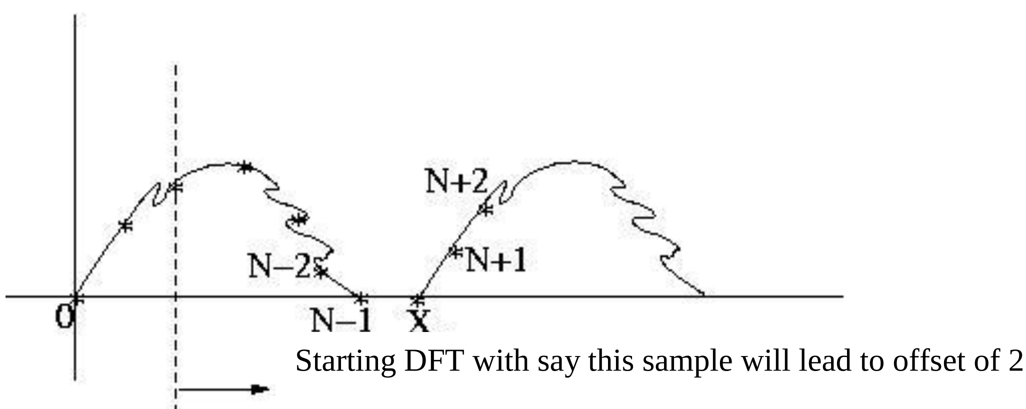


Fig (5)

$$X_{shifted}(m) = \sum_{n=k}^{k+n-1} x(k) e^{\frac{j}{N}}$$

Relationship between Discrete Fourier Transform and Fourier Transform

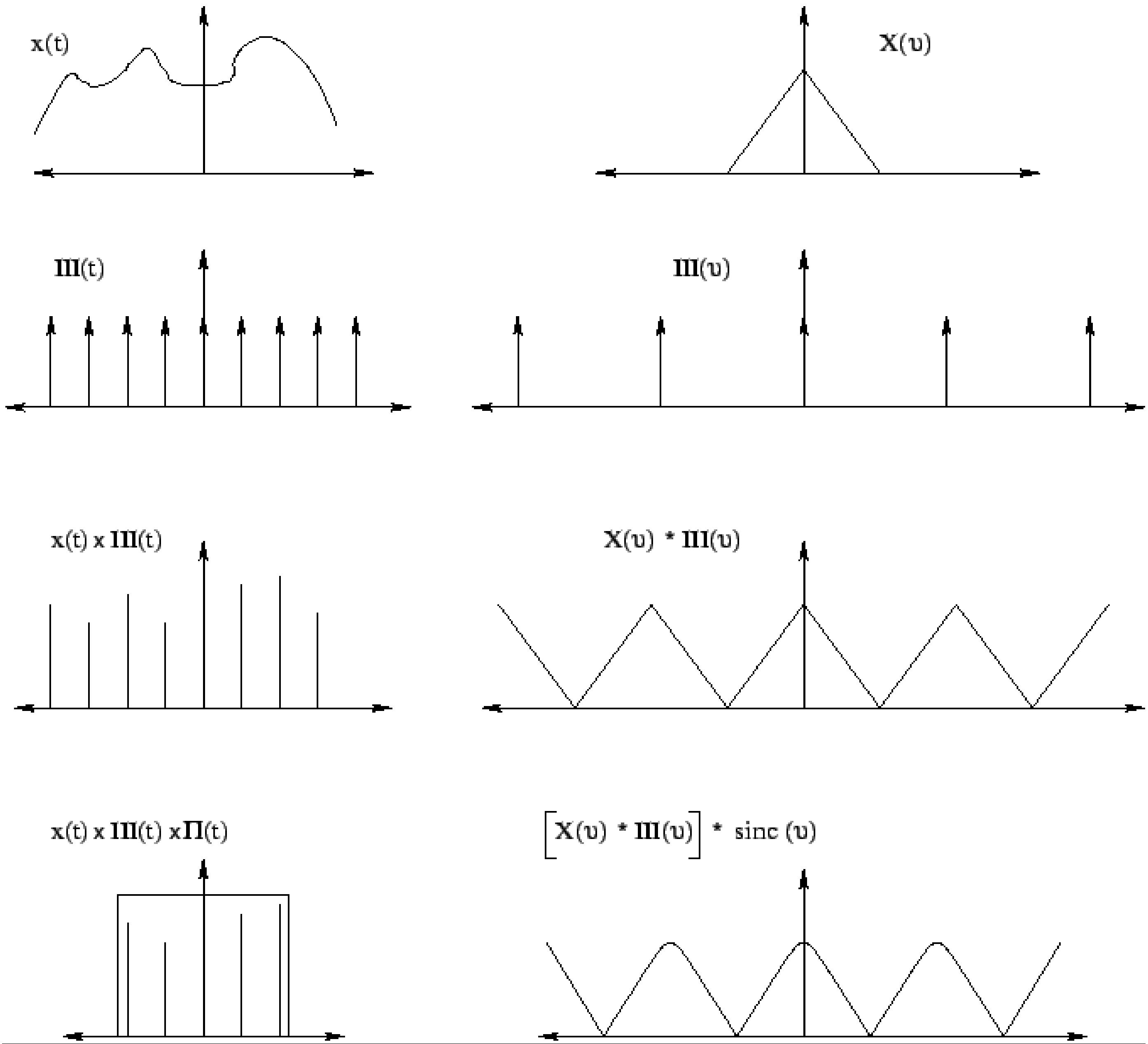


Figure : The relation between the continuous Fourier transform and the discrete Fourier transform. The panels on the left show the time domain signal and those on the right show the corresponding spectra.

The Fourier Transform (FT) of a signal s(t) is defined as

$$S(w) = \int_{-\infty}^{+\infty} s(t) e^{-j\omega t} \mathbf{d}t$$

Discrete Fourier Transform (DFT) is an operation to evaluate the FT of the sampled signal s[n] ($\equiv s(n\frac{1}{f_s}$)) with a finite number of samples (say N). It is defined as

$$S(k) = \sum_{n=0}^{N-1} s(n) e^{-j2\pi nk/N}; \quad 0 \leq k \leq N-1$$

The relationship between FT and DFT and some properties of DFT are discussed here. Consider a time series s[n], which is obtained by sampling a continuous band limited signal s(t) at a rate f_s. The sampling function is a train of delta function III(t). The length of the series is restricted to N samples by multiplying with a rectangular window function II(t) . The modification of the signal s(t)due to these operations and the corresponding changes in the spectrum are shown in Fig. The spectral modifications can be understood from the properties of Fourier transforms. The FT of the time series can now be written as a summation (assuming N is even)

$$\begin{aligned} S(\omega) &= \int_{-\infty}^{+\infty} s(t) \sum_{n=-N/2}^{N/2-1} \delta(t - \frac{n}{f_s}) e^{-j\omega t} \mathbf{d}t \\ &= \sum_{n=-N/2}^{N/2-1} s(\frac{n}{f_s}) e^{-\frac{j\omega n}{f_s}} \end{aligned}$$

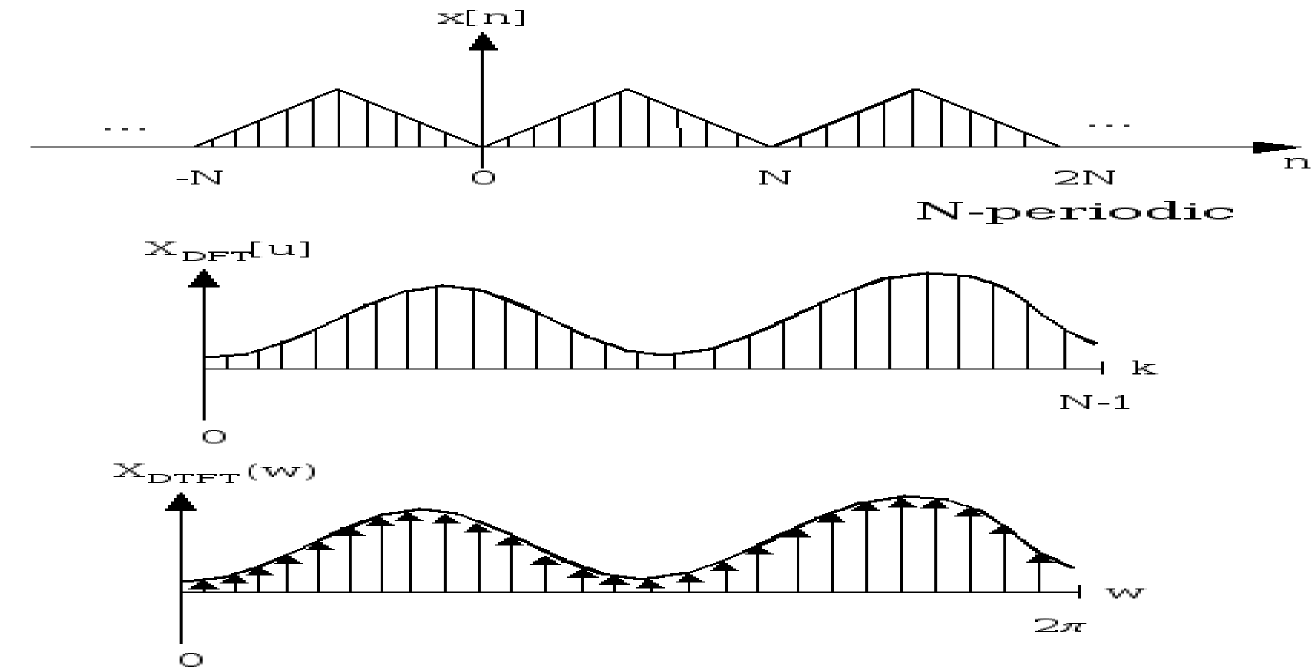
What remains is to quantize the frequency variable. For this the frequency domain is sampled such that there is no aliasing in the time domain. This is satisfied if $\Delta\omega$ = $2\pi f_s/N$. Thus we have,

$$S(k\Delta\omega) = \sum_{n=-N/2}^{N/2-1} s(\frac{n}{f_s}) e^{-\frac{jk\Delta\omega n}{f_s}}$$

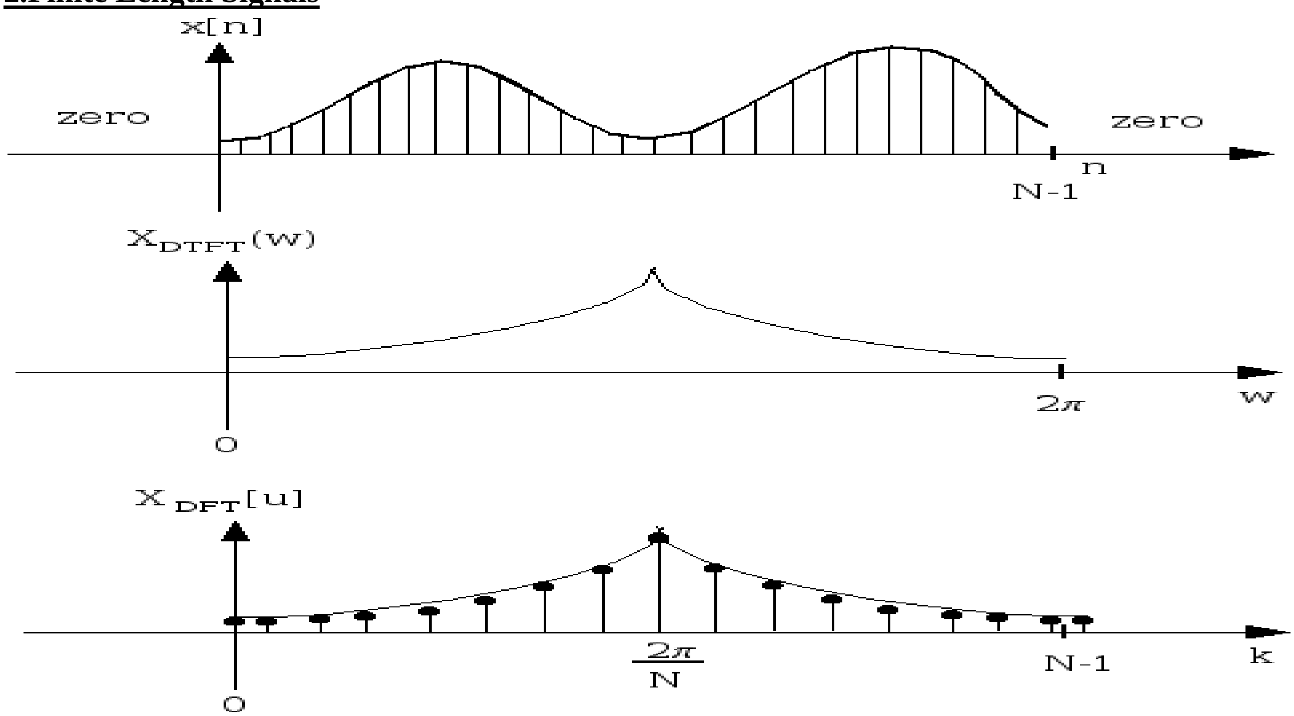
Using the relation $\Delta\omega / f_s = 2\pi / N$ and writing the variables as discrete indices we get the DFT equation. The cyclic nature of DFT (see below) allows n and k to range from 0 to (N-1) instead of (-N/2) to ((N/2)-1)).

Relationship Between DFT and DTFT

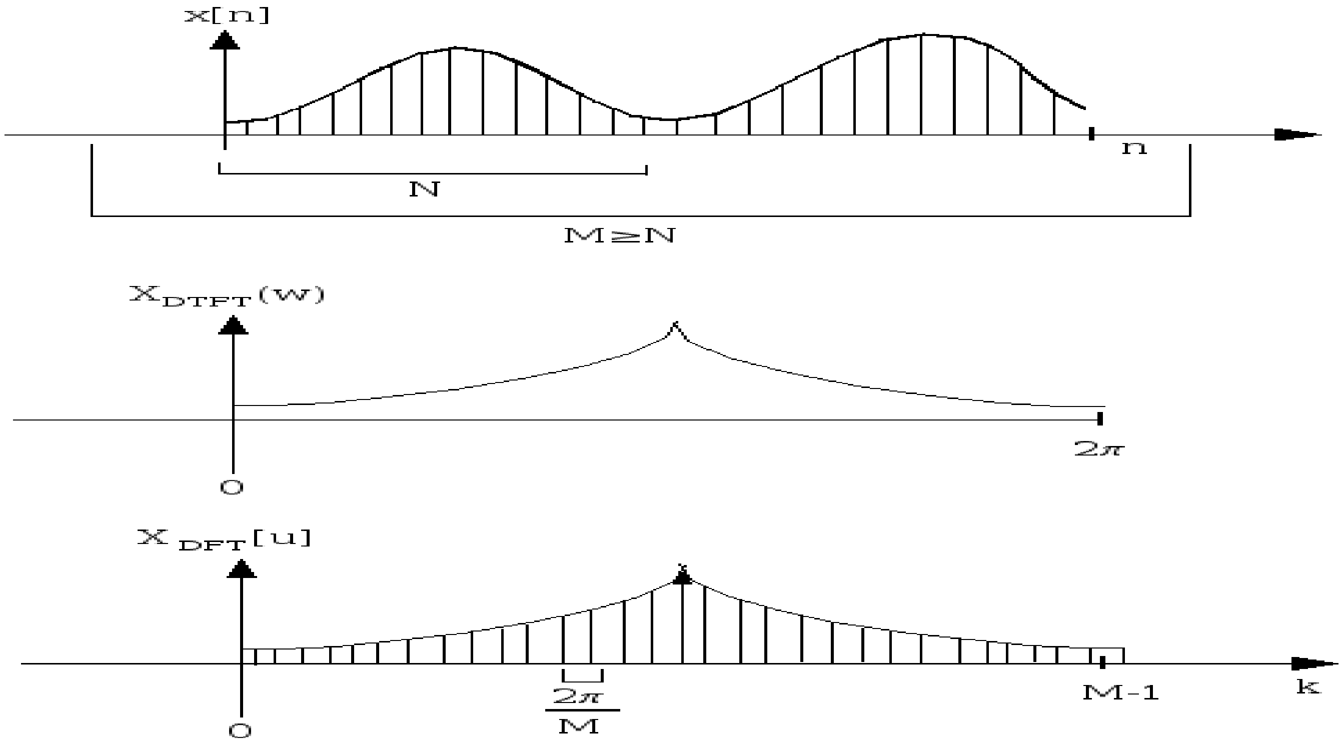
1.Periodic Signals Case



2.Finite Length Signals



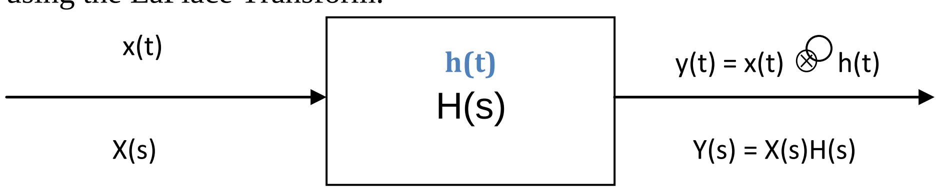
3.Extension(zero padded DFT)



The Z-Transform

The DTFT provides a frequency-domain representation of discrete-time signals and LTI discrete-time systems Because of the convergence condition, in many cases, the DTFT of a sequence may not exist As a result, it is not possible to make use of such frequency-domain characterization in these cases z-transform may exist for many sequences for which the DTFT does not exist Moreover, use of z-transform techniques permits simple algebraic manipulations.

A linear system can be represented in the complex frequency domain (s-domain where s = σ + jω) using the LaPlace Transform.



Where the direct transform is:

$$L\{x(t)\} = X(s) = \int_{t=0}^{\infty} x(t)\varepsilon^{-st}dt$$

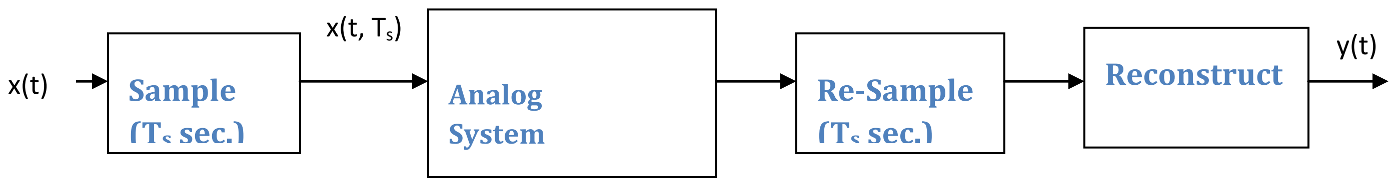
And x(t) is assumed zero for $t \leq 0$

The Inversion integral is a contour integral in the complex plane (seldom used, tables are used instead)

$$L^{-1}\{X(s)\} = x(t) = \frac{1}{2\pi j}\int_{s=\sigma-j\infty}^{\sigma+j\infty} X(s)\varepsilon^{st}ds$$

Where σ is chosen such that the contour integral converges.

If we now assume that x(t) is ideally sampled as in:



Where:

$$x_n = x(n*T_s) = x(t)\big|_{t=n*T_s}$$

and

$$y_n = y(n*T_s) = y(t)\big|_{t=n*T_s}$$

Analyzing this equivalent system using standard analog tools will establish the z-Transform.

Substituting the Sampled version of x(t) into the definition of the LaPlace Transform we get

$$L\{x(t,T_s)\} = X_T(s) = \int_{t=0}^{\infty} x(t,T_s)\varepsilon^{-st}dt$$

But

$$x(t,T_s) = \sum_{n=0}^{\infty} x(t) * p(t - n*T_s)$$

(For x(t) = 0 when t < 0)

Therefore

$$X_T(s) = \int_{t=0}^{\infty}\left[\sum_{n=0}^{\infty} x(n*T_s) * \delta(t - n*T_s)\right]\varepsilon^{-st}dt$$

Now interchanging the order of integration and summation and using the sifting property of δ-functions

$$X_T(s) = \sum_{n=0}^{\infty} x(n*T_s)\int_{t=0}^{\infty}\delta(t - n*T_s)\varepsilon^{-st}dt$$

$$X_T(s) = \sum_{n=0}^{\infty} x(n * T_s)\varepsilon^{-nT_s s}$$

(We are assuming that the first sample occurs at t = 0+)

if we now adjust our nomenclature by letting:

$z = \varepsilon^{sT}$, $x(n*Ts) = x_n$, and $X(z) = X_T(s)\big|_{z=\varepsilon^{sT}}$

$$\boxed{X(z) = \sum_{n=0}^{\infty} x_n z^{-n}}$$

Which is the direct z-transform (one-sided; it assumes $x_n = 0$ for n < 0).

The inversion integral is:

$$x_n = \frac{1}{2\pi j}\oint_c X(z) z^{n-1} dz$$

(This is a contour integral in the complex z-plane)

(The use of this integral can be avoided as tables can be used to invert the transform.)

To prove that these form a transform pair we can substitute one into the other.

$$x_k = \frac{1}{2\pi j}\oint_c \left[\sum_{n=0}^{\infty} x_n z^{-n}\right] z^{k-1} dz$$

Now interchanging the order of summation and integration (valid if the contour followed stays in the region of convergence):

$$x_k = \frac{1}{2\pi j}\sum_{n=0}^{\infty} x_n \oint_c z^{k-n-1} dz$$

If “C” encloses the origin (that’s where the pole is), the Cauchy Integral theorem says:

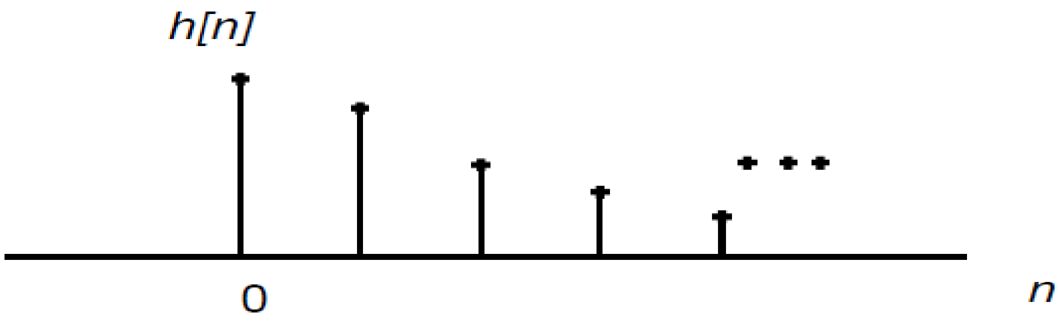
$$\oint_c z^{k-n-1} dz = \begin{matrix} o & for\, n \neq k \\ 2\pi j & for\, n = k \end{matrix}$$

And we get $x_k = x_k$

Poles, Zeros, Regions of Convergence

Consider the causal sequence : $h_1[n] = a^n u[n]$

This impulse response might be used to model signals which decay over time, like echoes.

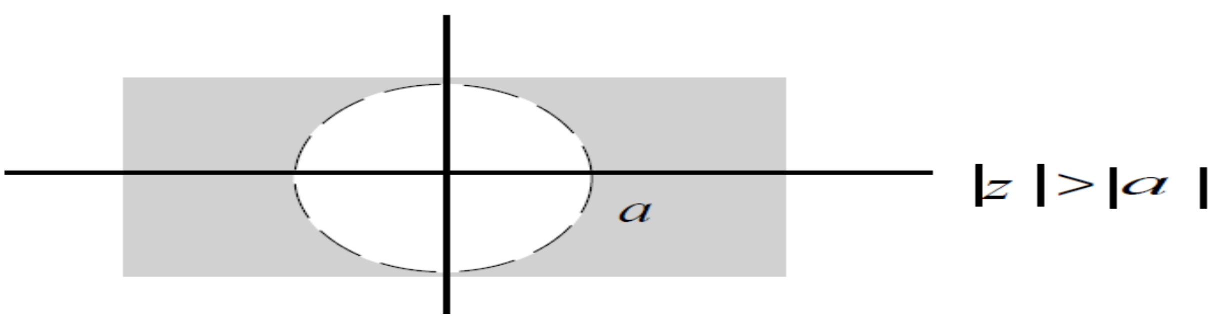


Consider its *Z*-transform: $H_1(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n}$

$$= \sum_{n=0}^{\infty} a^n z^{-n} \quad = \sum_{n=0}^{\infty} (a z^{-1})^n$$

The region in the Z-plane where the transform exists is called the region of convergence (ROC), e.g., $|z| > |a|$ for this example. Even though we can evaluate $\frac{1}{1-az^{-1}}$ for $|z| \leq |a|$ usually inappropriate to do so since this expression for the Z-transform is only defined within its ROC.

Let's draw a picture of this. First we need to think about the poles and zeros of $H_1(z)$.



Poles are the values of z for which the Z-transform is 1 and zeros are the values of z for which the *Z*-transform is 0. Every *Z*-transform has the same number of poles as it has of zeros. It is good practice to count them up when you think you've found them all. For the example above there is one pole and one zero, and the ROC is outside the pole.

$$H_1(z) = \frac{1}{1 - az^{-1}}, \quad |z| > |a|. \quad \begin{array}{l} \text{pole: } z = a \\ \text{zero: } z = 0 \end{array}$$

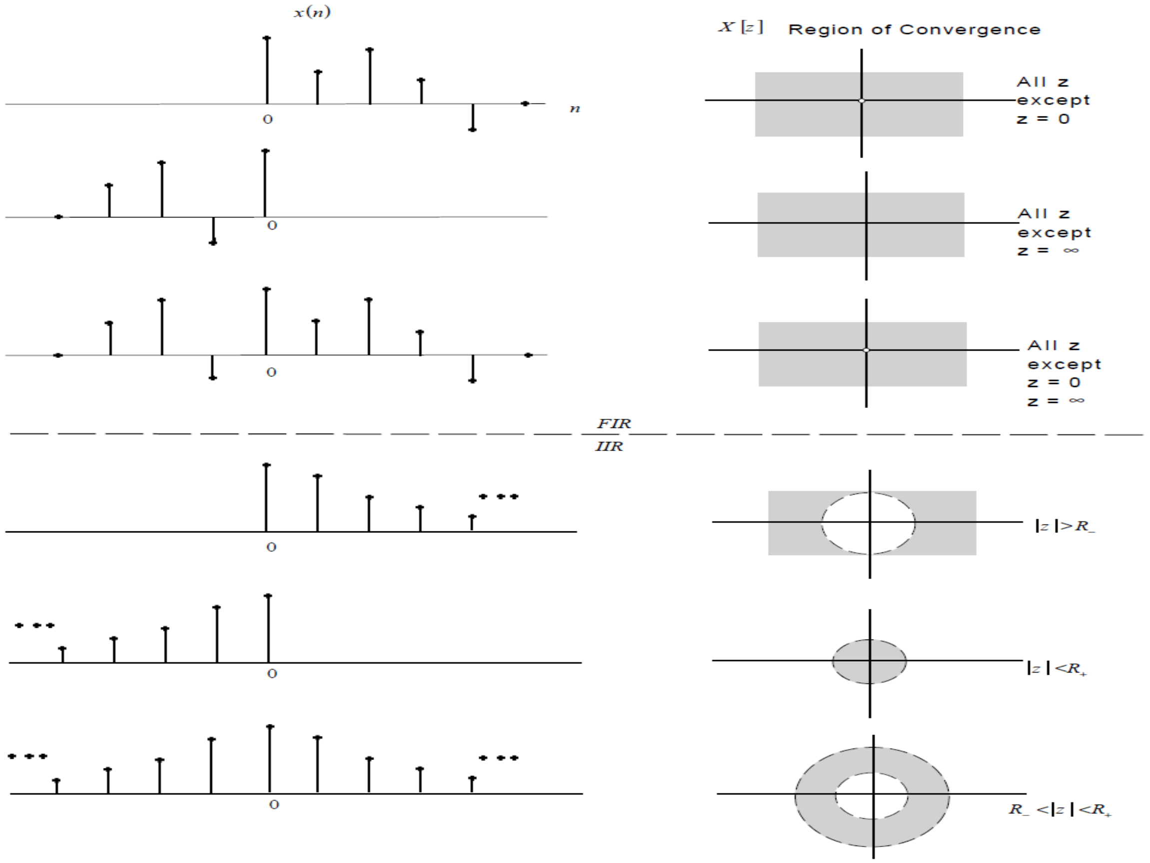


Fig: Example ROC's for finite and infinite length sequences.

Properties of the z-Transform:

For the following $Z\{f[n]\} = \sum_{n=0}^{n=\infty} f[n]z^{-n} = F(z)$

$$Z\{g_n\} = \sum_{n=0}^{n=\infty} g_n z^{-n} = G(z)$$

Linearity:

$Z\{af_n + bg_n\} = aF(z) + bG(z)$. and ROC is $R_f \cap R_g$

which follows from definition of *z*-transform.

Time Shifting

If we have $f[n] \Leftrightarrow F(z)$ then $f[n-n_0] \Leftrightarrow z^{-n_0}F(z)$ The ROC of $Y(z)$ is the same as $F(z)$ except that there are possible pole additions or deletions at $z = 0$ or $z = \infty$.

Multiplication by an Exponential Sequence

Let $y[n] = z_0^n f[n]$ then $Y(z) = X\left(\dfrac{z}{z_0}\right)$

The consequence is pole and zero locations are scaled by z_0. If the ROC of $FX(z)$ is $r_R < |z| < r_L$, then the ROC of $Y(z)$ is

$r_R < |z/z_0| < r_L$, *i.e.*, $|z_0|r_R < |z| < |z_0|r\mathrm{L}$

Differentiation of *X*(*z*)

If we have $f[n] \Leftrightarrow F(z)$ then $nf[n] \xleftrightarrow{Z} -z\dfrac{dF(z)}{z}$ and ROC = R_f

Conjugation of a Complex Sequence

If we have $f[n] \Leftrightarrow F(z)$ then $f^*[n] \xleftrightarrow{Z} F^*(z^*)$ and ROC = R_f

Time Reversal

If we have $f[n] \Leftrightarrow F(z)$ then $f^*[-n] \xleftrightarrow{Z} F^*(1/z^*)$

Let $y[n] = f^*[-n]$, then

$Y(z) = \sum_{n=-\infty}^{\infty} f^*[-n] z^{-n} = \left(\sum_{n=-\infty}^{\infty} f[-n][z^*]^{-n}\right)^* = \left(\sum_{k=-\infty}^{\infty} f[k](1/z^*)^{-k}\right)^* = F^*(1/z^*)$ If the ROC of $F(z)$ is $r_R < |z| < r_L$, then the ROC of $Y(z)$ is

$$r_R < \left|1/z^*\right| < r_L \qquad i.e., \qquad \frac{1}{r_R} > |z| > \frac{1}{r_L}$$

When the time reversal is without conjugation, it is easy to show

$$f[-n] \xleftrightarrow{Z} F(1/z) \quad \text{and ROC is } \frac{1}{r_R} > |z| > \frac{1}{r_L}$$

A Short Table of z-Transforms

f(t) (sampled)	F(z)	Region of Convergence
U(t)	$\frac{z}{z-1}$	$\|z\| > 1$
δ_{n-k}	z^{-k}	$\|z\| > 1$
T	$\frac{Tz}{(z-1)^2}$	$\|z\| > 1$
t^2	$\frac{T^2 z(z+1)}{(z-1)^3}$	$\|z\| > 1$
ε^{at}	$\frac{z}{z-\varepsilon^{aT}}$	$\|z\| > \varepsilon^{at}$
$\sin(\beta t)$	$\frac{z * \sin(\beta T)}{z^2 - 2z * \cos(\beta T) + 1}$	$\|z\| > 1$
$\cos(\beta t)$	$\frac{z * [z - \cos(\beta T)]}{z^2 - 2z * \cos(\beta T) + 1}$	$\|z\| > 1$

Relationship between Z-Transform and DFT

Let us consider a sequence x(n) having z-transform with ROC that includes the

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \qquad \ldots(1)$$

unit circle. If X(z) is sampled at the N equally spaced points on the unit circle. If X(z) is sampled at N equally spaced points on the unit circle.

$$Z_k = e^{j\frac{2\pi k}{N}},\ k = 0, 1, 2, 3 \ldots.\ N-1.$$

We obtain

$$X(k) = X(z)\Big|_{z=e^{\frac{j2\pi k}{N}}}\ ;\ k = 0,1\ , \ldots\ldots\ldots\ldots, N-1$$

$$= \sum_{n=-\infty}^{\infty} x(n)\ e^{\frac{-j2\pi nk}{N}} \qquad \ldots(2)$$

Expression is (2) identical to the Fourier transform X(w) evaluated at the N. equally spaced. Frequencies

$$w_k = \frac{2\pi k}{N},\ k = 0,1,\ \ldots\ldots\ N-1.$$

If the sequence x(n) has a finite duration of length N or less, the sequence can berecovered from its N-point DFT. Hence its Z-transform is uniquely determined by its N-point DFI'. Consequently, X(z) can be expressed as a function of the DFT {X(k)} as follows

$$X(z) = \sum_{n=0}^{N-1} x(n)\, z^{-n}$$

$$X(z) = \sum_{n=0}^{N-1}\left[\frac{1}{N}\sum_{k=0}^{N-1} X(k) e^{\frac{j2\pi kn}{N}}\right] z^{-n} = \frac{1}{N}\sum_{k=0}^{N-1} X(k) \sum_{n=0}^{N-1}\left(e^{\frac{j2\pi k}{N}} z^{-1}\right)^n$$

$$= \frac{1-z^{-N}}{N} \sum_{k=0}^{N-1} \frac{X(k)}{1-e^{\frac{j2\pi k}{N}} z^{-1}} \qquad \ldots(3)$$

When evaluated on the unit circle (3) yields the Fourier transform of the finite duration sequence in terms of its DFT in the form:

$$X(w) = \frac{1-e^{-jwN}}{N} \sum_{k=0}^{N-1} \frac{X(k)}{1-e^{-j\left(w-\frac{2\pi k}{N}\right)}}$$

This expression for Fourier transform is a polynomial interpolation formula for X(w) expressed in terms of the, values {x(k)) of the polynomial at a set of equally spaced discrete frequencies

$$\omega_k = \frac{2\pi k}{N},\ k = 0,1,\ \ldots\ldots\ldots\ldots N-1.$$

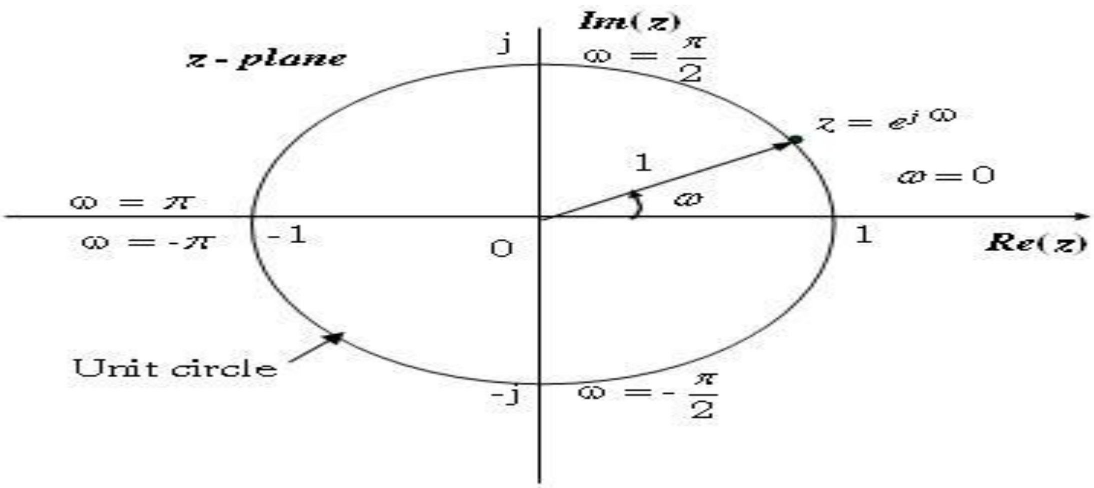


Fig**: Along the unit circle, the z-transform is just the Fourier transform**

CONCLUSION

TRANSFORM	*TIME-DOMAIN* *(Analysis)*	*FREQUENCY-DOMAIN* *(Synthesis)*
Fourier Series (FS)	$x(t)$ continuous periodic $X_n = \frac{1}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}} x(t)e^{-j2\pi nt/T}dt$	X_n discrete aperiodic $x(t) = \sum_{n=-\infty}^{+\infty} X_n e^{j2\pi nt/T}$
Fourier Transform (FT)	$x(t)$ continuous aperiodic $X(\Omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\Omega t}dt$ (or in f where $\Omega = 2\pi f$)	$X(\Omega)$ continuous aperiodic $x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\Omega)e^{j\Omega t}d\Omega$
Discrete-Time Fourier Transform ($DTFT$)	$x[n]$ discrete aperiodic $X(\omega) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$	$X(\omega)$ continuous periodic $x[n] = \frac{1}{2\pi}\int_{-\pi}^{+\pi} X(\omega)e^{j\omega n}d\omega$
Discrete Fourier Transform (DFT)	$x[n]$ discrete periodic $X[k] = \sum_{n=0}^{N-1} x[n]W_N^{kn}$ (where $W_N^{kn} = e^{-j2\pi kn/N}$)	$X[k]$ discrete periodic $x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]W_N^{-kn}$

The table shown above briefly explains the relations among the different transforms and The Z-transform is a generalization of the discrete-time Fourier transform (DTFT). The DTFT can be found by evaluating the Z-transform X(z) at z= $e^{j\omega}$ or, evaluated on the unit circle.