University of Electronic Science and Technology of China

School of Automation and Engineering

Subject: Digital Signal Processing

Technical Report

Title: The relationship among FT, DTFT, DFT and z-transform

Student Name: Mutaz Elradi Saadalla Saeed

Student ID : 201414070113

Supervisor : Assoc Prof Jing-Ran Lin

September 2014

Introduction

Four Fourier Transforms family commonly used for signal analysis and system design in modern telecommunications, radar, and image processing system, A time domain signal can be either continuous or discrete, Also it's very importance in Digital Signal Processing , this report discus , Fourier Transformation , Discrete-time Fourier transform (DTFT) , Discrete Fourier Transform (DFT) and Z transformation and relationship among four family.

1. Fourier Transformation

The Fourier transform represents a generalization of Fourier series , Fourier transform extends the Fourier series expansion for periodic signal to aperiodic signals. In fact, both periodic and aperiodic signals have Fourier transforms (via the impulse function in spectrum .

1.1 Derivation of Fourier transform from Fourier series :

$$X(\omega) = \int_{t:-\infty}^{\infty} x(t)e^{-j\omega t}\ dt \text{ , Fourier transform}$$

: $X(\omega)$ represents the project of $x(t)$ on the orthogonal bases $e^{j\omega t}$

$$x(t) = \frac{1}{2\pi}\int_{\omega:-\infty}^{\infty} X(\omega)e^{+j\omega t}\ d\omega \text{ , Inverse Fourier transform}$$

:$x(t)$ is written as the continuous sum (integral) of orthogonal wave forms $e^{j\omega t}$, $-\infty \leq \omega \leq \infty$

Or equivalently (with$\boldsymbol{\omega = 2\pi f}$)

$$X(f) = \int_{t:-\infty}^{\infty} x(t)e^{-j2\pi f t}\ dt \ ,$$

$$x(t) = \int_{f:-\infty}^{\infty} X(f)e^{+j2\pi f t}\ df \ ,$$

1.2 Properties of Fourier Transform

- Linearity (superposition thm)

$$\mathrm{a_1x_1 + a_2x_2\ (t) \leftrightarrow a_1X_1(j\omega) + a_2X_2(j\omega)}$$

- Time-delay ! linear phase shift; magnitude spectrum remains unchanged

$$x(t - t_0) \leftrightarrow X(\omega)e^{-j\omega t_0}$$

- Time-compression/expansion Spectral expansion/compression

$$x(at) \leftrightarrow \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

- Symmetry:

$$\boldsymbol{(a) x^*(t) \Leftrightarrow X^*(-\omega)}$$

- Differentiation: $\frac{d^n}{dt^n} x(t) \leftrightarrow (j\omega)^n X(\omega)$, which is useful in Preemphasis / Deemphasis of an FM system.

2. Discrete-time Fourier transform (DTFT)

Definition:

The DTFT is a transformation that maps DT signal x[n] into a complex-valued function of the real variable, namely

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \quad , \ \omega \in \mathbb{R} \qquad (2.1)$$

Remarks:

- In general, X (ω) 2 C
- X (ω + 2π) = X (ω)) ω 2 [iπ; π] is sufficient
- X (ω) is called the spectrum of x[n]:

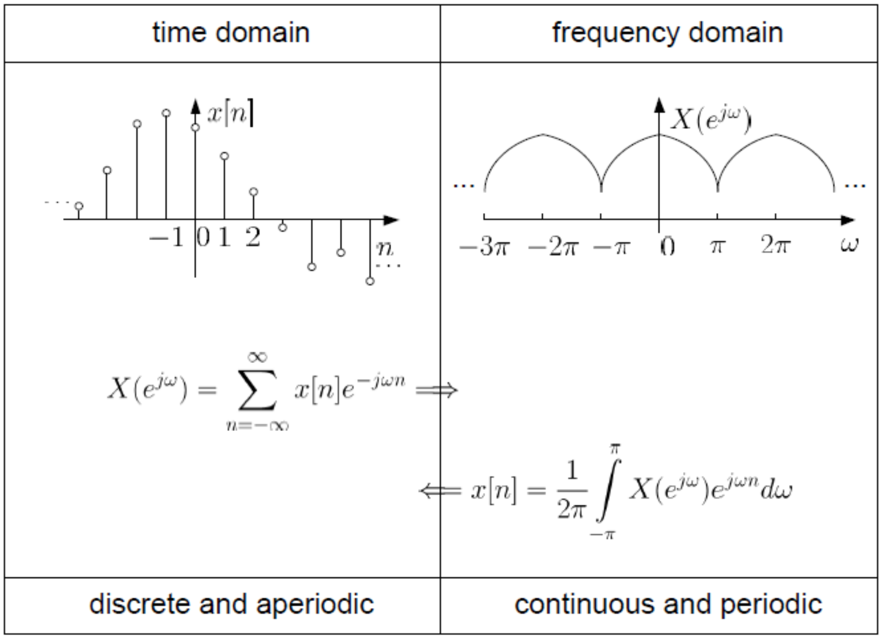
$$X(\omega) = |X(\omega)|e^{jX(\omega)} \Rightarrow \begin{cases} |X(\omega)| = magnitude\ spectrum \\ X(\omega) = phase\ spectrum \end{cases} \qquad (2.2)$$

Example:

The DTFT of the unit sample sequence $\delta[n]$ is given by:

$$\Delta(\omega) = \sum_{n=-\infty}^{\infty} \delta[n]e^{-j\omega n} = \delta[0] = 1$$

The DTFT $X(e^{j\omega})$ of a sequence x[n] is a continuous function of ω.It is also a periodic function of ω with a period 2π. Inverse Discrete-Time Fourier Transform: $x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$



2.1 Inverse DTFT:

Let X (ω) be the DTFT of DT signal x[n]. Then

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}\ d\omega\ ,\ n \in \mathbb{Z} \qquad (3.3)$$

Proof: First note that

$$\int_{-\pi}^{\pi} e^{j\omega n}\ d\omega = 2\pi\delta[n].\ \text{Then we have}$$

$$\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}d\omega = \int_{-\pi}^{\pi}\{\textstyle\sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}\}\, e^{j\omega n}d\omega$$

$$= \sum_{k=-\infty}^{\infty} x[k]\int_{-\pi}^{\pi} e^{j\omega\,(n-k)}\, d\omega$$

$$= 2\pi \textstyle\sum_{k=-\infty}^{\infty} x[k]\delta[n-k] = 2\pi x[n] \qquad (2.4)$$

Remarks:

- In (3.3), x[n] is expressed as a weighted sum of complex exponential signals $e^{j\omega n}$, $\omega \in [-\pi, \pi]$ With weight X (ω)
- Accordingly, the DTFT X (ω) describes the frequency content of x[n]
- Since the DT signal x[n] can be recovered uniquely from its DTFT X (ω), we say that x[n] together with X (ω) form a DTFT pair, and write:

$$x[n] \overset{f}{\leftrightarrow} X(\omega) \qquad (2.5)$$

- DTFT equation (3.1) is called analysis relation; while inverse DTFT equation (3.4) is called synthesis relation.

2.2 Convergence of the DTFT:

Introduction:

For the DTFT to exist, the series $\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$ must converge

- That is, the partial sum

$$\boldsymbol{X_M(\omega) = \sum_{n=-M}^{M} x[n]e^{-j\omega n}} \quad \textbf{(2.6)}$$

must converge to a limit X (ω) as $M \to \infty$.

- Below, we discuss the convergence of XM(ω) for three different signal classes of practical interest, namely:

  - absolutely summable signals
  - energy signals
  - power signals

- In each case, we state without proof the main theoretical results and illustrate the theory with corresponding examples of DTFTs.

Absolutely summable signals:

- Recall: x[n] is said to be absolutely summable (sometimes denoted as L1) if

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty \qquad (2.7)$$

In this case, X (ω) always exists because:

$$\left|\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}\right| \leq \left|\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}\right| = \sum_{n=-\infty}^{\infty} |x[n]| < \infty \qquad (2.8)$$

- Possible to show that XM(ω) converges uniformly to X (ω), that is:

For all ε > 0, can find Mε such that |X (ω) - MX_M(ω)| < ε for all M > Mε and for all ω ∈ ℝ.

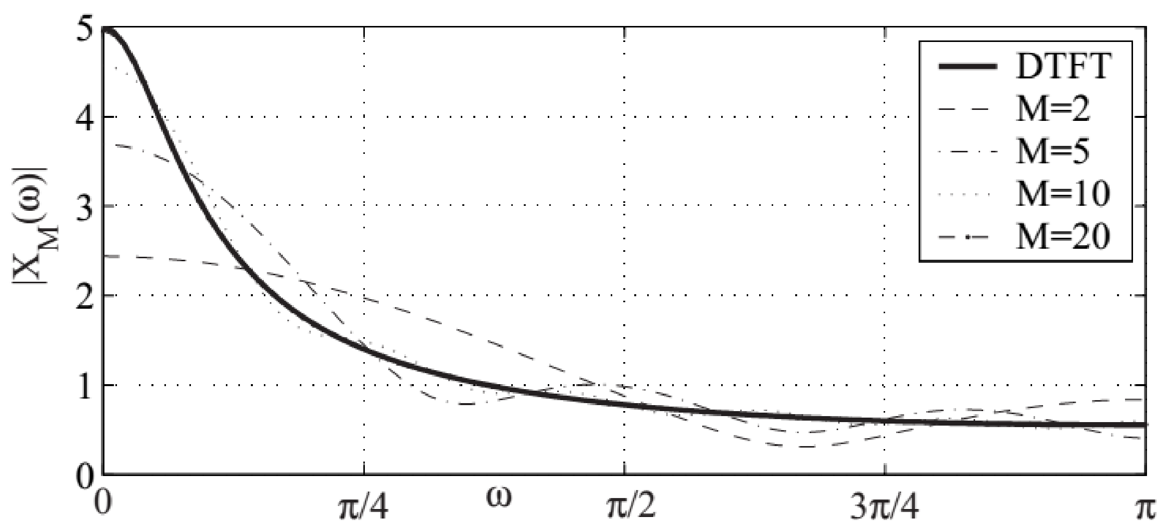
- X (ω) is continuous and $d^pX(\omega)/\ d\omega^p$ exists and continuous for all p ≥1

Example:

- The exponential sequence $x[n] = \ a^n u[n]\ with\ |a| < 1$ is absolutely summable , it's DTFT is easily computed to be

$$X(\omega) = \sum_{n=0}^{\infty} (ae^{j\omega})^n = 1/(1 - ae^{-j\omega}$$

Figure (1) illustrates the uniform convergence of $X_M(\omega)$ as defined in (1) to X (ω) in the special case a = 0:8. As M increases, the whole $X_M(\omega)$ curve tends to X (ω) at every



frequency

figure (1) illustration of uniform convergence for an exponential sequence

Power signals

- Recall: x[n] is a power signal if

$$p_x \triangleq \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2 < \infty \qquad (2.9)$$

- If x[n] has infinite energy but finite power, X_M(ω) may still converge to a generalized function X (ω):
- The expression of X (ω) typically contains continuous delta functions in the variable ω.
- Most power signals do not have a DTFT even in this sense. The exceptions include the following useful DT signals:

  - Periodic signals
  - Unit step

2. 3 Properties of the DTFT

i. Linearity:

$$ax[n] + by[n] \overset{f}{\leftrightarrow} aX(\omega) + bY(\omega)$$

ii. Time shift (very important):

$$x[n-d] \overset{f}{\leftrightarrow} e^{-j\omega d}X(\omega)$$

iii. Frequency modulation:

$$e^{j\omega_0 n}x[n] \overset{f}{\leftrightarrow} X(\omega - \omega_0)$$

iv. Differentiation:

$$nx[n] \overset{f}{\leftrightarrow} \frac{dX(\omega)}{d\omega}$$

v. Convolution:

$$x[n] * y[n] \overset{f}{\leftrightarrow} X(\omega)Y(\omega)$$

vi. Multiplication:

$$x[n]y[n] \overset{f}{\leftrightarrow} \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\emptyset)Y(\omega - \emptyset)d\emptyset$$

3. The Discrete Fourier Transform (DFT)

Introduction:

The DTFT has proven to be a valuable tool for the theoretical analysis of signals and systems. However, if one looks at its definition, i.e.:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \ , \omega \ \in [-\pi, \pi] \qquad (3.1)$$

From a computational viewpoint, it becomes clear that it suffers several drawbacks indeed; its numerical evaluation poses the following difficulties:

- The summation over n is infinite;
- The variable ω is continuous.

In many situations of interest, it is either not possible, or not necessary, to implement the infinite summation $\sum_{n=-\infty}^{\infty}$ in (2.1)

- Only the signal samples x[n] from n = 0 to n = N - 1 are available;
- The signal is known to be zero outside this range; or
- The signal is periodic with period N.

In all these cases, we would like to analyze the frequency content of signal x[n] based only on the finite set of samples x[0],x[1], … ,x[N-1]. We would also like a frequency domain representation of these samples in which the frequency variable only takes on a finite set of values, say ω_k for k = 0,1,…, N-1, in order to better match the way a processor will be able to compute the frequency representation.

The discrete Fourier transform (DFT) fulfills these needs. It can be seen as an approximation to the DTFT.

3.1 The DFT and its inverse

Definition:

The N-point DFT is a transformation that maps DT signal samples {x[0],…, x[N-1]} into a periodic sequence X [k], defined by

$$X[k] = DFT_N\{x[n]\} \triangleq \sum_{n=0}^{N-1} x[n]e^{-\frac{j\pi kn}{N}}, k \ \in \ \mathbb{Z} \qquad (3.2)$$

Remarks:

- Only the samples x[0],…, x[N-1] are used in the computation.
- The N-point DFT is periodic, with period N:

$$X[k+N] = X[k] \qquad (3.3)$$

Thus, it is sufficient to specify X [k] for k = 0,1,…,N-1 only.

- The DFT X [k] may be viewed as an approximation to the DTFT $X(\omega)$ at frequency $\omega_k = 2\pi k/N$
- The ”D” in DFT stands for discrete frequency (i.e. ω_k)
- Other common notations:

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j\omega_k n} \quad where\ \omega_k \triangleq 2\pi k/N \qquad (3.4)$$

$$= \sum_{n=0}^{N-1} x[n]W_N^{kn} \quad where\ W_N \triangleq e^{-j2\pi/N} \qquad (3.5)$$

Example:

Consider

$$x[n] = \begin{cases} 1 & n = 0, \\ 0 & n = 1, \dots, N-1 \end{cases}$$

We have

$$X[n] = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N} = 1\ , all\ k \in \mathbb{Z}$$

3.2 Inverse DFT (IDFT):

The N-point IDFT of the samples X [0],…,X [N-1] is defined as the periodic sequence

$$x\tilde{}\{n] = IDFT_N\{X[k]\} \triangleq \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j2\pi kn/N}\ , n \in \mathbb{Z} \qquad (3.6)$$

Remarks:

- In general, x˜ [n] 6 = x[n] for all $n \in \mathbb{Z}$.
- Only the samples X [0],…,X [N-1] are used in the computation.
- The N-point IDFT is periodic, with period N:

3-3 Relationship between the DFT and the DTFT

Introduction

The DFT may be viewed as a finite approximation to the DTFT:

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j\omega_k n} \approx X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

at frequency $\omega = \omega_k = 2\pi k/N$ As pointed out earlier, in general, an arbitrary signal x[n], $n \in \mathbb{Z}$, cannot be recovered entirely from its N-point DFT X [k]; k ∈ {0,…,N-1}. Thus, it should not be possible to recover the DTFT exactly from the DFT. However, in the following two special cases:

- finite length signals
- N-periodic signals,

x[n] can be completely recovered from its N-point DFT. In these two cases, the DFT is not merely an approximation to the DTFT: the DTFT can be evaluated exactly, at any given frequency $\omega \in [-\pi, \pi]$ if the N-point DFT X [k] is known.

3.3 Properties of the DFT

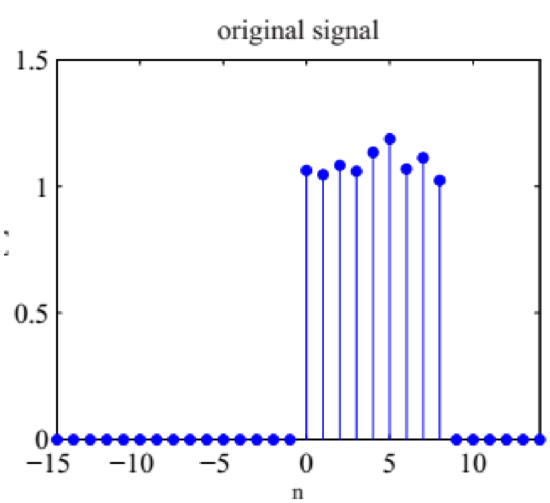
Notations:

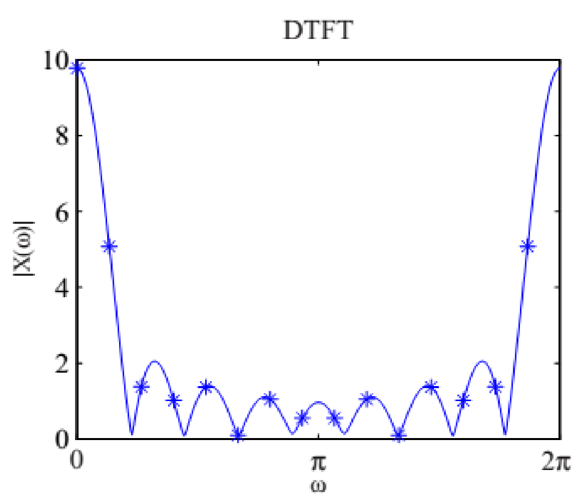
In this section, we assume that the signals x[n] and y[n] are defined over $0 \leq n \leq N-1$. Unless explicitly stated, we make no special assumption about the signal values outside this range. We denote the N-point DFT of x[n] and y[n] by X [k] and Y [k]:

$$x[n] \overset{DFT_N}{\longleftrightarrow} X[k]$$

$$y[n] \overset{DFT_N}{\longleftrightarrow} Y[k]$$

We view X [k] and Y [k] as N-periodic sequences, defined for all k 2 Z $k \in \mathbb{Z}$





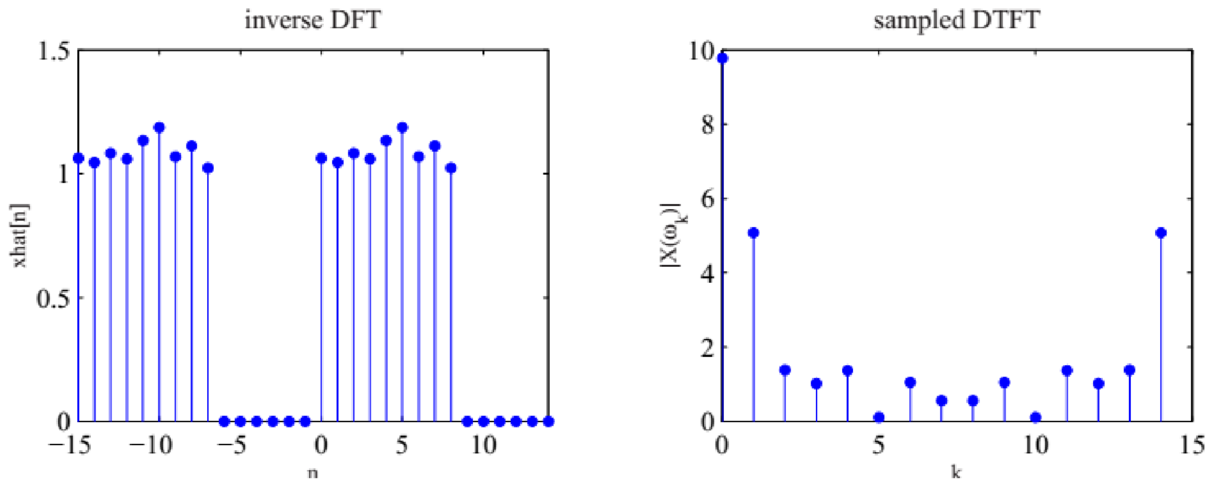
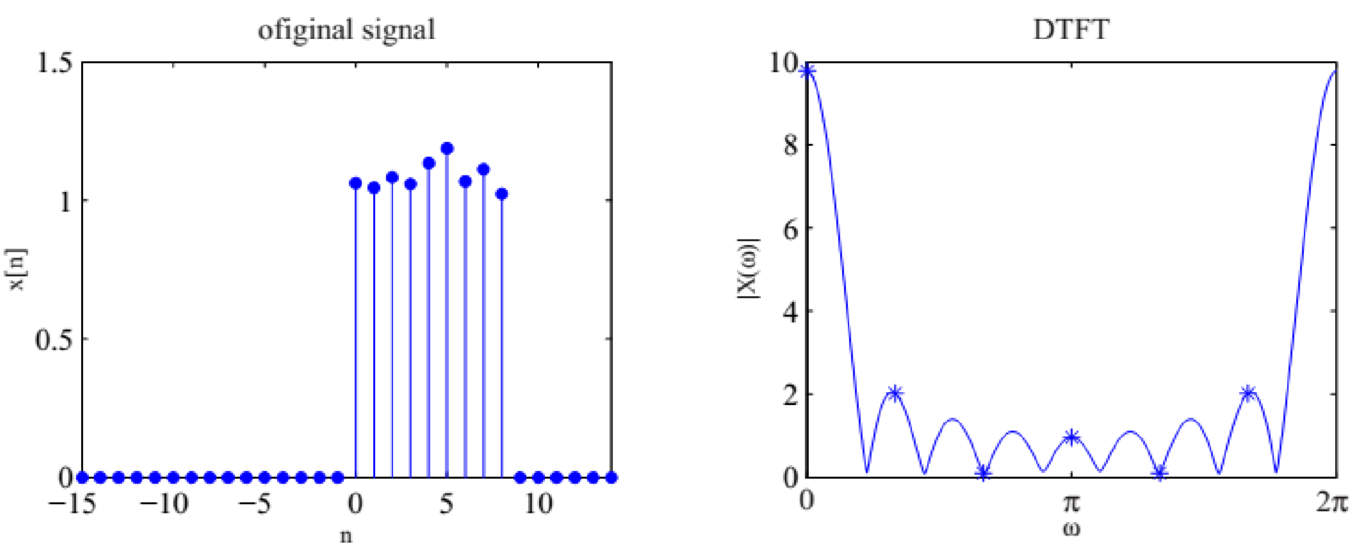


Figure (3.1)

Fig. (3.1) Illustration of reconstruction of a signal from samples of its DTFT. Top left: the original signal x[n] is time limited. Top right, the original DTFT X(ω), from which 15 samples are taken. Bottom right: the equivalent impulse spectrum corresponds by IDFT to a 15-periodic sequence x˜[n] shown on the bottom left. Since the original sequence is zero outside $0 \le n \le 14$, there is no overlap between replicas of the original signal in time.



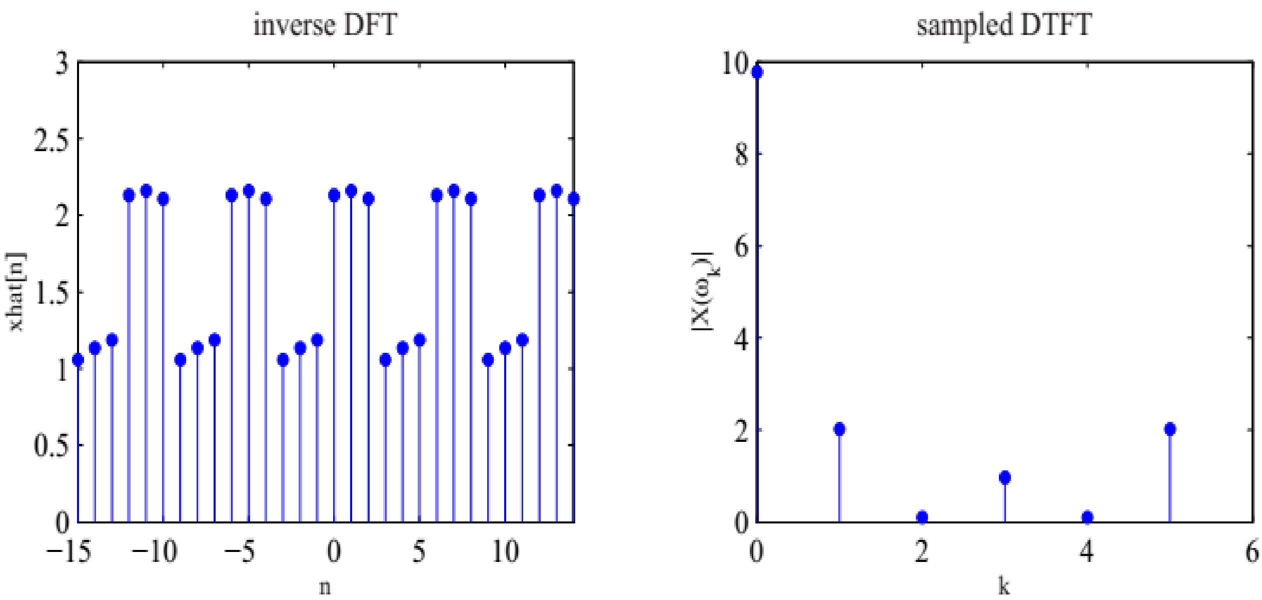


Figure (3.2)

Fig.(3.2) Illustration of reconstruction of a signal from samples of its DTFT. Top left: the original signal x[n] is time limited. Top right, the original DTFT X(ω), from which 6 samples are taken. Bottom right: the equivalent impulse spectrum corresponds by IDFT to a 6-periodic sequence x˜[n] shown on the bottom left. Since the original sequence is not zero outside $0 \leq n \leq 5$ there is overlap between replicas of the original signal in time, and thus aliasing.

4. The z-transform (ZT)

Motivation:

- While very useful, the DTFT has a limited range of applicability.
- For example, the DTFT of a simple signal like $x[n] = 2^n u[n]$ does not exist.
- One may view the ZT as a generalization of the DTFT that is applicable to a larger class of signals.
- The ZT is the discrete-time equivalent of the Laplace transform for continuous-time signals.

4.1 The ZT

Definition:

The ZT is a transformation that maps DT signal x[n] into a function of the complex variable z, defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \qquad (4.1)$$

The domain of X (z) is the set of all $z \epsilon \mathbb{C}$ such that the series converges absolutely, that is:

$$Dom(X) = \{ z \in \mathbb{C} : \sum_n |x[n] z|^{-n} < \infty \qquad (4.2)$$

Remarks:

- The domain of X (z) is called the region of convergence (ROC).
- The ROC only depends on |z|: if z ∈ ROC, so is $ze^{j\emptyset}$ for any angle $\emptyset$.
- Within the ROC, X (z) is an analytic function of complex variable z. (That is, X (z) is smooth, derivative exists, etc.)
- Both X (z) and the ROC are needed when specifying a ZT.

Example:

- Consider $x[n] = 2^n u[n]$. we have

$$X(z) = \sum_{n=\infty}^{\infty} 2^n z^{-n} = \frac{1}{1-2z^{-1}}$$

Where the series converges provides$|2z^{-1}| < 1$. According, ROC: |z| > 2.

4.2 Connection with DTFT:

- The ZT is more general than the DTFT. Let $\boldsymbol{z = e^{j\omega}}$ **so that**

$$ZT\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]\, r^{-n} e^{-j\omega n} = DTFT\{x[n] r^{-n}\} \qquad (4.3)$$

With ZT, possibility of adjusting r so that series converges.

- Consider previous example:

$- x[n] = 2^n u[n]$ does not have a DTFT (Note : $\sum_n |x[n]| = \infty$)

$$- x[n]\ has\ a\ ZT\ for\ |z| = r > 2$$

- If $z = e^{j\omega} \in ROC$,

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = DTFT\{x[n]\} \tag{4.4}$$

- In the sequel the DTFT is either denoted $X(e^{j\omega})$, or simply by $X(\omega)$ when there is no ambiguity

4.3 Inverse ZT:

Let X (z), with associated ROC denoted R_x , be the ZT of DT signal x[n]. Then

$$x[n] = \frac{1}{2\pi j} \oint_C X(z)Z^{n-1}\, dZ, n \in \mathbb{Z} \tag{4.5}$$

Where C is any closed, simple contour around z = 0 within R_x.

Remarks:

- In analogy with the inverse DTFT, signals Z^n , with relative weight X (z).
- In practice, we do not use (4.5) explicitly to compute the inverse ZT (more on this later); the use of (4.5) is limited mostly to theoretical considerations.
- Since DT signal x[n] can be recovered uniquely from its ZT X (z) (and associated ROC R_x), we say that x[n] together with X (z) and R_x form a ZT pair, and write:

$$x[n] \overset{Z}{\leftrightarrow} X(z), z \in R_x$$

4.4 Properties of the ZT

Introductory remarks:

- Notations for ZT pairs:

$$x[n] \overset{Z}{\leftrightarrow} X(z), z \in R_x$$

$$y[n] \overset{Z}{\leftrightarrow} Y(z), z \in R_y$$

R_x and R_y respectively denote the ROC of X (z) and Y (z)

- When stating a property, we must also specify the corresponding ROC.
- In some cases, the true ROC may be larger than the one indicated.

Basic symmetries:

$$x[-n] \overset{Z}{\leftrightarrow} X(z^{-1}), z^{-1} \in R_x$$

$$x^*[n] \overset{Z}{\leftrightarrow} X^*(z^*), z \in R_x$$

Linearity:

$$ax[n]by[n] \overset{Z}{\leftrightarrow} a\,X(Z) + \,bY(z), z \in \,R_x \,\cap\, R_y$$

Time shift (very important):

$$x[n-d] \overset{Z}{\leftrightarrow} \,z^{-d}, X(z)\; z \in \,R_x$$

Exponential modulation:

$$a^n x[n] \overset{Z}{\leftrightarrow} \,X\left(\frac{z}{a}\right), \frac{z}{a} \in \,R_x$$

Differentiation:

$$nx[n] \overset{Z}{\leftrightarrow} -\frac{zdX(z)}{dz}, z \in \,R_x$$

Convolution:

$$x[n] * y[n] \overset{Z}{\leftrightarrow} \,X(z)Y(z), z \in \,R_x \,\cap\, R_y$$

Initial value:

For x[n] causal (i.e x[n] $=0$) for n< 0), we have

$$\lim_{z\to\infty} \boldsymbol{X}(\boldsymbol{z}) = \boldsymbol{x}[\boldsymbol{0}]$$

Conclusion

All members of the Fourier transform family (DFT, DTFT, Fourier Transform & Fourier Series) and Z Transformation can be carried out with either real numbers or complex numbers. Since digital signal processing (DSP) is very important domain. It has many applications, such as audio signal processing, audio compression, digital image processing, Radar. In this discussion we have presented some concepts and important properties of FT, DTFT, DFT, and Z-transform, including showing the relationships between them and also inverse.

References

[1] Analog Devices Inc., Norwood, MA. TIgerSHARC DSP Microcomputer Preliminary Technical data, 2002.

[2] Ashok Bindra. Novel architectures pack multiple dsp cores on-chip. Electronic Design, December 2001.

[3] Yu Hen Hu, editor. Programmable Digital Signal Processors: Architecture, Programming and Applications. Marcel Dekker, 2002.

[4] Emmanuel C. Ifeachor and Barrie W. Jervis. Digital Signal Processing, a practical approach. PrenticeHall, New Jersey, 2nd edition edition, 2002.

[5] Phil Lapsley, Jeff Bier, Amit Shoham, and Edward A. Lee. DSP Processor Funcamentals: Architectures and Features. IEEE Press, 1997.

[6] Ravi A. Managuli and Yongmin Kim. VLIW processor architectures and algorithm mappings for DSP applications. In Hu [3], chapter 2.

[7] James H. McClellan, Ronald W. Schafer, and Mark A. Yoder. DSP First: a multimedia approach. Prentice-Hall, 1998.

[8] Sanjit K. Mitra. Digital Signal Processing, a computer-based approach. McGraw-Hill, 2nd edition edition, 2001.

[9] Motorola, Inc., Denver, CO. DSP56300 Family Manual, 2000.

[10] Allan V. Oppehneim, Ronald W. Schfafer, and J. R. Buck. Discrete-Time Signal Processing. PrenticeHall, New Jersey, 2nd edition edition, 1999.

[11] John G. Proakis and Dimitris G. Manolakis. Digital Signal Processing, Principles, algorithms and applications. Prentice-Hall, 3rd edition edition, 1996.

[12] Russell Tessier and Wayne Burleson. Reconfigurable computing and digital signal processing: Past, present and future. In Hu [3], chapter 4.

[13] Texas Instruments, Houston, TX. TMS320C6204 Data Sheet, 2000