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Report on Digital Signal Processing

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Based on the definition of delta function discuss the relationship and the difference between δ(t) and δ[n]

Introduction

The signals are the sources of the measurement information. Generally, the systems that are dealt with in practical situations evolve in time with continuity. Due to this property, they can be mathematically represented by functions of the independent variable *time* that belongs to the set of the *real numbers*. For this reason, these functions are generally referred to as belonging to the *continuous time* domain, and the signals are called *continuous-time signals*. On the other hand, situations exist where the signals do not evolve with continuity in their appertaining domain. This is the typical case of the signals that represent quantities in the quantum mechanics, and it is also the more simple case of the periodic signals when they are represented in the frequency domain. The Fourier theory shows that, in this case, the signal is defined, in the frequency domain, only for discrete values of the independent variable frequency. When the independent variable is time and the signal is defined only for discrete values of the independent variable, the signal is defined as belonging to the *discrete time* domain and is synthetically called a *discrete-time signal*. When the independent variable takes only discrete values, it sweeps over its axis by *quanta*; therefore it can be represented only by *integer numbers* that represent actually the serial number of the quantum. For this reason, the independent variable of the mathematical object that represents a discrete-time signal belongs to the set of the integer numbers; the mathematical object is called a *sequence*. An example of discrete-time signal is provided by a signal obtained by sampling a continuous-time signal with a constant sampling period. Usually, most discrete-time signals are obtained by sampling continuous-time signals. Anyway, for the sake of generality, a discrete-time signal can be seen as generated by a process defined in the discrete time domain. For this reason, the discrete-time signals will be analyzed by their own, without referring to their possible origin in a sampling operation. In this way the properties of the discrete-time signals can be fully perceived, and the mathematical tools can be defined that are required to provide an answer to the fundamental question of the digital signal processing theory how and with which changes the information associated to a continuous-time signal is transferred to a discrete-time signal by sampling the continuous-time signal.

Continuous-time impulse function

1. Introduction

In our discussion of the unit step function $u(t)$ we saw that it was an idealized model of a quantity that goes from 0 to 1 very quickly. In the idealization we assumed it jumped directly from 0 to 1 in no time.

In this note we will have an idealized model of a large input that acts over a short time. We will call this model the *delta function* or *Dirac delta function* or *unit impulse*.

After constructing the delta function we will look at its properties. The first is that it is not really a function. This won't bother us, we will simply call it a *generalized function*. The reason it won't bother us is that the delta function is useful and easy to work with. Inside integrals or as input to differential equations we will see that it is much simpler than almost any other function.

2. Delta Function as Idealized Input

Suppose that radioactive material is dumped in a container. The equation governing the amount of material in the tank is.

$$X' + kx = q(t),$$

where, $x(t)$ is the amount of radioactive material (in kg), k is the decay rate of the material (in 1/year), and $q(t)$ is the rate at which material is being added to the dump (in kg/year).

The input $q(t)$ is in units of mass/time, say kg/year. So, the total amount dumped into the container from time 0 to time t is

$$Q(t) = \int_0^t q(u)\, du.$$

Equivalently

$$\dot{Q}(t) = q(t).$$

To keep things simple we will assume that $q(t)$ is only nonzero for a short amount of time and that the total amount of radioactive material dumped over that period is 1 kg. Here are the graphs of two possibilities for $q(t)$ and $Q(t)$.

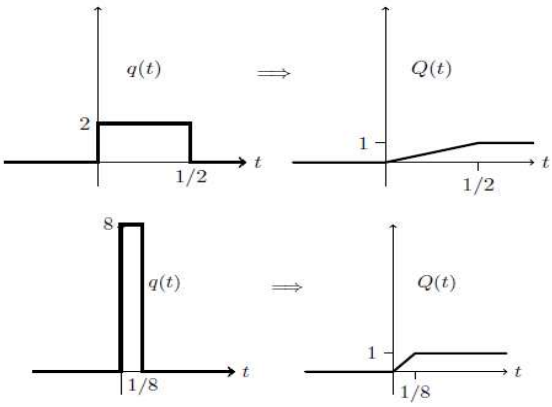


Figure 1: two possible graphs of $q(t)$ and $Q(t)$, both with total input = 1.

It is easy to see that each of the boxes on the left side of Figure 1 has total area equal to 1. Thus, the graphs for $Q(t)$ rise linearly to 1 and then stay equal to 1 thereafter. In other words, the total amount dumped in each case is 1.

Now let $qh(t)$ be a box of width h and height $1/h$. As $h\to 0$, the width of the box becomes 0, the graph looks more and more like a spike, yet it still has area 1 (see Figure 2).

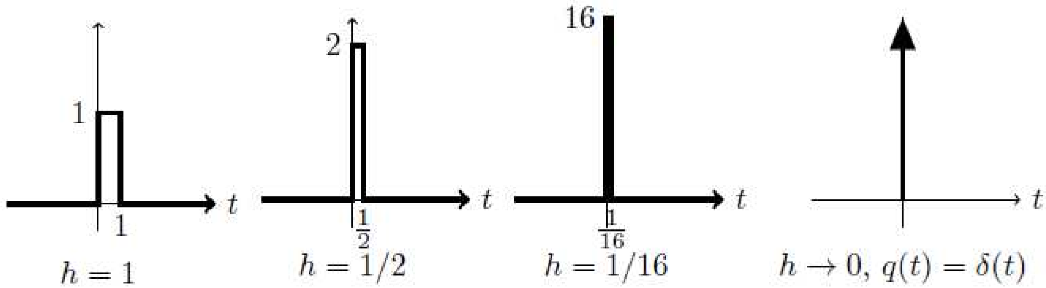


Figure 2: Box functions $q_h(t)$ becoming the delta function as $h \to 0$.

We define the **delta function** to be the formal limit

$$\delta(t) = \lim_{h\to 0} qh(t).$$

Graphically $\delta(t)$ is represented as a spike or harpoon at $t = 0$. It is an infinitely tall spike of infinitesimal width enclosing a total area of 1 (see figure 2, rightmost graph). As an input function $\delta(t)$ represents the ideal case where 1 unit of material is dumped in all at once at time $t = 0$.

Properties of $\delta(t)$

We list the properties of $\delta(t)$ below.

1. From the previous section we have

$$\delta(t) = \begin{cases} 0 & \text{if } t \neq 0, \\ \infty & \text{if } t = 0. \end{cases}$$

The graph is represented as a spike at $t = 0$. (See figure 2)

2. Because $\delta(t)$ is the limit of graphs of area 1, the area under its graph is 1. More precisely:

$$\int_c^d \delta(t)\, dt = \begin{cases} 1 & \text{if } c < 0 < d \\ 0 & \text{otherwise} \end{cases}$$

3. For any continuous function $f(t)$ we have

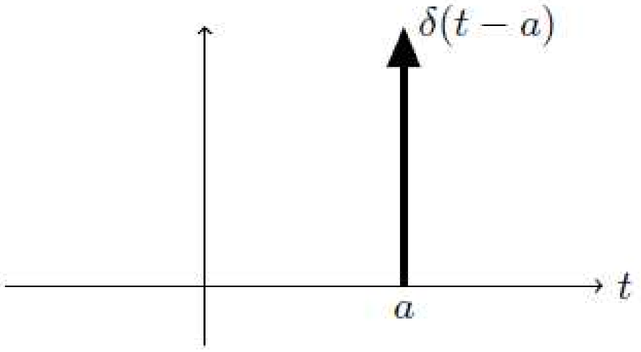
$$f(t)\delta(t) = f(0)\delta(t) \quad \text{and} \quad \int_c^d f(t)\delta(t)\, dt = \begin{cases} f(0) & \text{if } c < 0 < d \\ 0 & \text{otherwise} \end{cases}$$

   The first statement follows because $\delta(t)$ is 0 everywhere except at $t = 0$. The second follows from the first and property (2).

4. We can place the delta function over any value of t: $\delta(t - a)$ is 0 everywhere but at $t = a$. Its total area remains 1. Its graph is now a spike shifted to be over $t = a$; and we have

$$f(t)\delta(t-a) = f(a)\delta(t-a).$$

$$\int_c^d f(t)\delta(t-a)\, dt = \begin{cases} f(a) & \text{if } c < a < d \\ 0 & \text{otherwise} \end{cases}$$



5. δ (t) = u'(t), where u(t) is the unit step function. Because u(t) has a jump at 0, δ(t) is not a derivative in the usual sense, but is called a *generalized derivative*. This is explained below.

6. We defined δ(t) as a limit of a sequence of box functions, all with unit area and which, in the limit, become a infinite spike over t = 0. Box functions are simple, but not special. Any sequence of functions with these properties has δ(t) as its limit.

7. In practical terms, you should think of δ(t) as any function of unit area, concentrated very near t = 0.

8. δ (t) is not really a function. We call it a **generalized function**.

9. In arriving at these properties we have skipped over some important technical details in the analysis. Generally property (3) is taken to be the formal definition of δ(t), from which the other properties follow.

Examples of integration

Properties (3) and (2) show that $\delta(t)$ is very easy to integrate, as the following examples show:

Example 1. $\int_{-5}^{5} 7e^{t^2}\cos(t)\delta(t)\,dt = 7$. All we had to do was evaluate the integrand at $t = 0$.

Example 2. $\int_{-5}^{5} 7e^{t^2}\cos(t)\delta(t-2)\,dt = 7e^4\cos(2)$. All we had to do was evaluate the integrand at $t = 2$.

Example 3. $\int_{-5}^{1} 7e^{t^2}\cos(t)\delta(t-2)\,dt = 0$. Since $t = 2$ is not in the interval of integration the integrand is 0 on the entire interval.

The value $t = 0^-$ represents the 'left-side' of 0 and $t = 0^+$ is the 'right-side'. So, 0 is in the interval $[0^-, \infty)$ and not in $[0^+, \infty)$. Thus

$$\int_{0^-}^{\infty} \delta(t)\,dt = 1 \quad \text{and} \quad \int_{0^+}^{\infty} \delta(t)\,dt = 0.$$

In fact, since all the area under the graph is concentrated at 0, we can even write

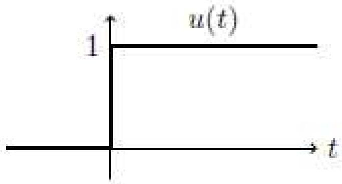
.

$$\int_{0^-}^{0^+} \delta(t)\,dt = 1.$$

5. Generalized Derivatives

Our goal in this section is to explain property (5). A look at the graph of the unit step function $u(t)$ shows that it has slope 0 everywhere except

at $t = 0$ and that its slope is ∞ at $t = 0$.



That is, its derivative is

$$u'(t) = \begin{cases} 0 & \text{if } t \neq 0 \\ \infty & \text{if } t = 0. \end{cases}$$

Since $u(t)$ has a jump of 1 at $t = 0$ this derivative matches properties (1) and (2) of $\delta(t)$ and we conclude that $u'(t) = \delta(t)$.

Now this derivative does not exist in the calculus sense. The function $u(t)$ is not even defined at 0. So we call this derivative a **generalized derivative**.

We can also explain property (5) by looking at the anti-derivative of $\delta(t)$. Let

$$f(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau.$$

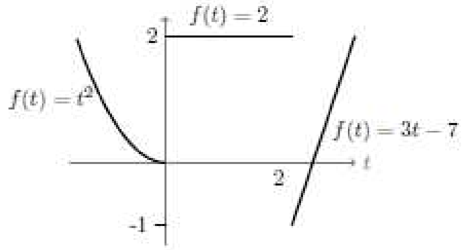
The fundamental theorem of calculus leads us to say that $f'(t) = \delta(t)$. (Again, this is only in a generalized sense since technically the fundamental theorem of calculus requires the integrand to be continuous.) Property (3) makes it easy to compute

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t > 0. \end{cases}$$

That is, $f(t) = u(t)$, so $u(t)$ is the antiderivative of $\delta(t)$.

In general, a jump discontinuity contributes a delta function to the generalized derivative.

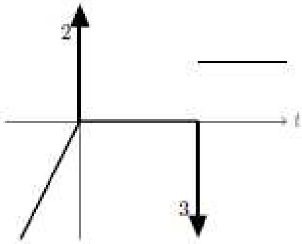
Example 4. Suppose $f(t)$ has the following graph.



The formula for each piece of the graph is indicated. For the smooth parts of the graph the derivative is just the usual one. Each jump discontinuity adds a delta function scaled by the size of the jump to $f'(t)$.

$$f'(t) = 2\delta(t) - 3\delta(t-2) + \begin{cases} 2t & \text{if } t < 0 \\ 0 & \text{if } 0 < t < 2 \\ 3 & \text{if } 2 < t \end{cases}$$

In the graph for $f'(t)$ we represent the delta functions as spikes with the magnitude written next to the spike. The sign is indicated by the direction of the spike. The rest of the $f'(t)$ is plotted normally.



We say $f'(t)$ is a generalized function. In a **generalized function** will mean a sum of a regular function and a linear combination of delta functions. (In the wider world of mathematics there are other generalized functions.)

If we want to refer to the different parts of a generalized function we will call the delta function pieces the **singular part** and the remainder will be called the **regular part**. If the singular part contains a multiple of $\delta(t-a)$ we will say the function *contains* $\delta(t-a)$.

Example. Consider $f(t) = u(t) + \delta(t) + e^{-t} + 3\delta(t-2)$. The regular part of f is $u(t) + e^{-t}$. The singular part is $\delta(t) + 3\delta(t-2)$. The function *contains* $\delta(t)$ and $\delta(t-2)$. It does not contain $\delta(t-1)$.

Important: In this unit, whenever a discontinuous function is differentiated we will mean the generalized derivative.

Discrete-time impulse function

The counterpart of the continuous-time step function is the discrete-time *unit step*, denoted by $u[n]$ and defined by

$$u[n] = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}$$

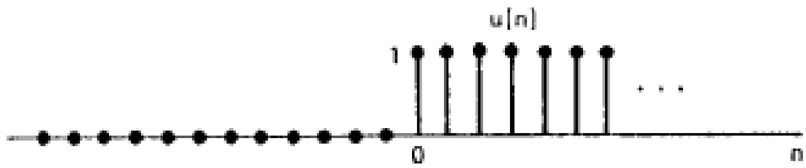


Figure 2 Unit step sequence.

very important continuous-time signal is the unit impulse. In discrete time we define the *unit impulse* (or *unit sample*) as

$$\delta[n] = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases}$$

Throughout the book we will refer to $\delta[n]$ interchangeably as the unit sample or unit impulse. Note that unlike its continuous-time counterpart, there are no analytical difficulties in defining $\delta[n]$.

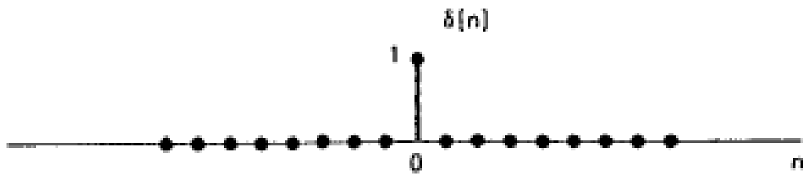


Figure 2.2 Unit sample (impulse).

The discrete-time unit sample possesses many properties that closely parallel the characteristics of the continuous-time unit impulse. For example, since $\delta[n]$ is nonzero (and equal to 1) only for $n = 0$, it is immediately seen that

$$x[n]\,\delta[n] = x[0]\,\delta[n]$$

Relationship and the difference between δ(t) and δ[n].

In discrete time the unit step is a well-defined sequence, whereas in continuous time there is the mathematical complication of a discontinuity at the origin. **A** similar distinction applies to the unit impulse. In discrete time the unit impulse is simply a sequence that is zero except at $n = 0$, where it is unity. in continuous time, it is somewhat badly behaved mathematically, being of infinite height and zero width but having finite area. The unit step and unit impulse are closely related. In discrete time the unit impulse is the first difference of the unit step, and the unit step is the running sum of the unit impulse. Correspondingly, in continuous time the unit impulse is the derivative of the unit step, and the unit step is the running integral of the impulse. the fact that it is a first difference and a running sum that relate the step and the impulse in discrete time and a derivative and running integral that relate them in continuous time should not be misinterpreted to mean that a first difference is a good "representation" of a derivative or that a running sum is a good "representation" of a running integral.

Rather, for this particular situation those operations play corresponding roles in continuous time and in discrete time. As indicated above, there are a variety of mathematical difficulties with the continuous-time unit step and unit impulse. This topic is treated formally mathematically through the use of what are referred to as generalized functions, which is a level of formalism well beyond what we require for our purposes. The important aspect of these functions, in particular of the impulse, is not what its value is at each instant of time but how it behaves under integration.

In their most general form, systems are hard to deal with analytically because they have no particular properties to exploit. In other words, general systems are simply too general. We define, discuss, and illustrate a number of system properties that we will find useful to refer to and exploit as the lectures proceed, among them memory, inevitability, causality, stability, time invariance, and linearity. The last two, linearity and time invariance, become particularly significant from this point on. Somewhat amazingly, as we'll see, simply knowing that a system is linear and time-invariant affords us an incredibly powerful array of tools for analyzing and representing it. While not all systems have these properties, many do, and those that do are often easiest to understand and implement. Consequent**ly,** both continuous-time and discrete-time systems that are linear and time invariant become extremely significant in system design, implementation, and analysis in a broad array of applications.