University of Electronic Science and Technology Of China

Digital signal processing

Discussion of the relationship among FT, DTFT, DFT, and Z-transform

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Introduction

Signal processing is an area of systems engineering, electrical engineering and applied mathematics that deals with operations on or analysis of signals, in either discrete or continuous time. Signals of interest can include sound, images, time-varying measurement values and sensor data, for example biological data such as electrocardiograms, control system signals, telecommunication transmission signals, and many others. Signals are analog or digital electrical representations of time-varying or spatial-varying physical quantities. In the context of signal processing, arbitrary binary data streams and on-off signalling are not considered as signals, but only analog and digital signals that are representations of analog physical quantities.

FT:

The Fourier transform is a mathematical operation that decomposes a function into its constituent frequencies, known as its frequency spectrum. For instance, the transform of a musical chord made up of pure notes (without overtones) is a mathematical representation of the amplitudes and phases of the individual notes that make it up. The composite waveform depends on time, and therefore is called the time domain representation. The frequency spectrum is a function of frequency and is called the frequency domain representation. Each value of the function is a complex number (called complex amplitude) that encodes both a magnitude and phase component. The term "Fourier transform" refers to both the transform operation and to the complex-valued function it produces.

In the case of a periodic function, like the musical chord, the Fourier transform can be simplified to the calculation of a discrete set of complex amplitudes, called Fourier series coefficients. Also, when a time-domain function is sampled to facilitate storage and/or computer-processing, it is still possible to recreate a version of the original Fourier transform according to the Poisson summation formula, also known as discrete-time Fourier transform.

FFT IS:

- ✓ It is an algorithm which calculates the DFT and the IDFT efficiently.
- ✓ It has a very small complexity of calculations compared to the direct Calculations of the DFT or the IDFT.
- ✓ There are 2 possible algorithms :

- The decimation in time algorithm, and
- The decimation in frequency algorithm.

- ✓ The decimation in time algorithm needs the arrangement of the input time sequence {x (n)}.

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- ✓ The decimation in frequency algorithm needs the arrangement of the output frequency sequence {X (k)}.

Relationship between FFT and DFT

An FFT computes the DFT and produces exactly the same result as evaluating the DFT definition directly; the only difference is that an FFT is much faster. (In the presence of round-off error, many FFT algorithms are also much more accurate than evaluating the DFT definition directly, as discussed below.)

Let x_0, ..., x_{N-1} be complex numbers. The DFT is defined by the formula

$$X_k = \sum_{n=0}^{N-1} x_n e^{-i2\pi k \frac{n}{N}} \qquad k = 0, \dots, N-1.$$

Evaluating this definition directly requires $O(N^2)$ operations: there are N outputs X_k, and each output requires a sum of N terms. An FFT is any method to compute the same results in O(N log N) operations. More precisely, all known FFT algorithms require Θ(N log N) operations (technically, O only denotes an upper bound), although there is no known proof that better complexity is impossible.

To illustrate the savings of an FFT, consider the count of complex multiplications and additions. Evaluating the DFT's sums directly involves N^2 complex multiplications and N(N − 1) complex additions [of which O(N) operations can be saved by eliminating trivial operations such as multiplications by 1]. The well-known radix-2 Cooley–Tukey algorithm *, for N a power of 2, can compute the same result with only (N/2) log2 N complex multiplies (again, ignoring simplifications of multiplications by 1 and similar) and N log2N complex additions.

DTFT and IDTFT:

The expressions for the DTFT $X(e^{j\omega})$ and the IDTFT $x(n)$ are

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} x(n)e^{-j\omega n}$$

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

The terms frequency response of a discrete-time signal and the discrete-time Fourier transform (DTFT) are synonymous and will be used interchangeably. This is also known as the frequency spectrum; its magnitude response and phase response are generally known as the magnitude spectrum and phase spectrum, respectively. The terms discrete-time signal, discrete-time sequence, discrete-time function, and discrete-time series synonymously.

DTFT Properties:

1. Time-Shifting Property

If $x(n)$ has a DTFT $X(e^{j\omega})$, then $x(n-k)$ has a DTFT equal to $e^{-j\omega k}X(e^{j\omega})$, where k is an integer.

Prove DTFT of

$$x(n-k) = \sum_{n=-\infty}^{\infty} x(n-k)e^{-j\omega n} = e^{-j\omega k}\sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} = e^{-j\omega k}X(e^{j\omega}).$$

So we denote this property by

$$x(n-k) \Leftrightarrow e^{-j\omega k}X(e^{j\omega})$$

- This is a divide and conquer algorithm that recursively breaks down a DFT of any composite size $N = N_1N_2$ into many smaller DFTs of sizes N_1 and N_2

2. Frequency-Shifting Property

If *x (n)* ⇔ *X* ($e^{j\omega}$), then $e^{j\omega_0 n}x(n) \Leftrightarrow X(e^{j(\omega-\omega_0)})$

Prove:

$$\sum_{n=-\infty}^{\infty} x(n)e^{j\omega_0 n}e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x(n)e^{-j(\omega-\omega_0)n} = X(e^{j(\omega-\omega_0)})$$

3. Time Reversal Property

Consider *x (n)* = $a^n u(n)$. It's DTFT $X(e^{j\omega}) = \sum_{n=0}^{\infty} a^n e^{-j\omega n}$

Next, to find the DTFT of *x (−n)*, if we replace *n* by −*n*, we would write the DTFT of *x (−n)* as $\sum_{n=0}^{-\infty} a^{-n} e^{j\omega n}$ (but that is wrong), as illustrated by the following example:

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} a^n e^{-j\omega n} = 1 + ae^{-j\omega} + a^2 e^{-j2\omega} + a^3 e^{-j3\omega} + \cdots$$

But the correct expression for the DTFT of *x (−n)* is of the form $1 + ae^{j\omega} + a^2e^{j2\omega} + a^3e^{j3\omega} + \cdots$

So the compact form for this series is $\sum_{n=-\infty}^{0} a^{-n} e^{-j\omega n}$.

With this clarification, we now prove the property that if *x(n)* ⇔ *X*($e^{j\omega}$) then

Proof: DTFT of $x(-n) = \sum_{n=-\infty}^{\infty} x(-n)e^{-j\omega n}$. We substitute $(-n) = m$, and we get $\sum_{n=-\infty}^{\infty} x(-n)e^{-j\omega n} = \sum_{m=-\infty}^{\infty} x(m)e^{j\omega m} = \sum_{m=-\infty}^{\infty} x(m)e^{-j(-\omega)m} = X(e^{-j\omega})$.

Example

Consider *x (n)* = *δ (n)*. Then, from the definition for DTFT, we see that *δ (n)* ⇔ *X* ($e^{j\omega}$) = 1 *for all ω*.

From the time-shifting property, we get

$$\delta(n-k) \Leftrightarrow e^{-j\omega k}$$

Example

We consider $x(n) = \delta(n+k) + \delta(n-k)$. Its DTFT is given by $X(e^{j\omega}) = e^{j\omega k} + e^{-j\omega k} = 2\cos(\omega k)$.

4. Differentiation Property

To prove that $nx(n) \Leftrightarrow j\,[dX(e^{j\omega})]/d\omega$, we start with $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$ and differentiate both sides to get and multiplying both sides by j, we get

$$[dX(e^{j\omega})]/d\omega = \sum_{n=-\infty}^{\infty} x(n)(-jn)e^{-j\omega n}$$

$$j[dX(e^{j\omega})]/d\omega = \sum_{n=-\infty}^{\infty} nx(n)e^{-j\omega n}.$$

The proof is similar to that used in Chapter 2 to prove that the z transform of $nx(n)u(n)$ is $-z\,[dX(z)]/dz$.

5. Multiplication Property

When two discrete -time sequences are multiplied, for example, $x(n)h(n) = y(n)$, the DTFT of $y(n)$ is the convolution of $X(e^{j\omega})$ and $H(e^{j\omega})$ that is carried out in the frequency domain as an integral over one full period. Choosing the period $[-\pi \;\; \pi]$ in the convolution integral, symbolically denoted by $X(e^{j\omega})*H(e^{j\omega}) = Y(e^{j\omega})$, we have the property

$$x(n)h(n) \Leftrightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\varsigma})H(e^{j(\omega-\varsigma)})\,d\varsigma$$

6. Conjugation Property

Here we assume the samples $x(n)$ to be complex-valued and in the general form $x(n) = (ae^{j\theta})^n$ and the complex conjugate to be $x^*(n) = (ae^{-j\theta})^n$. Let us find their DTFT:

$$x(n) \Leftrightarrow X_1(e^{j\omega}) = \sum_{n=-\infty}^{\infty} (ae^{j\theta})^n \, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} (a)^n e^{j\theta n} \, e^{-j\omega n}$$

$$x^*(n) \Leftrightarrow X_2(e^{j\omega}) = \sum_{n=-\infty}^{\infty} (ae^{-j\theta})^n \, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} (a)^n e^{-j\theta n} \, e^{-j\omega n}$$

TABLE 3.1 Properties of Discrete-Time Fourier Transform

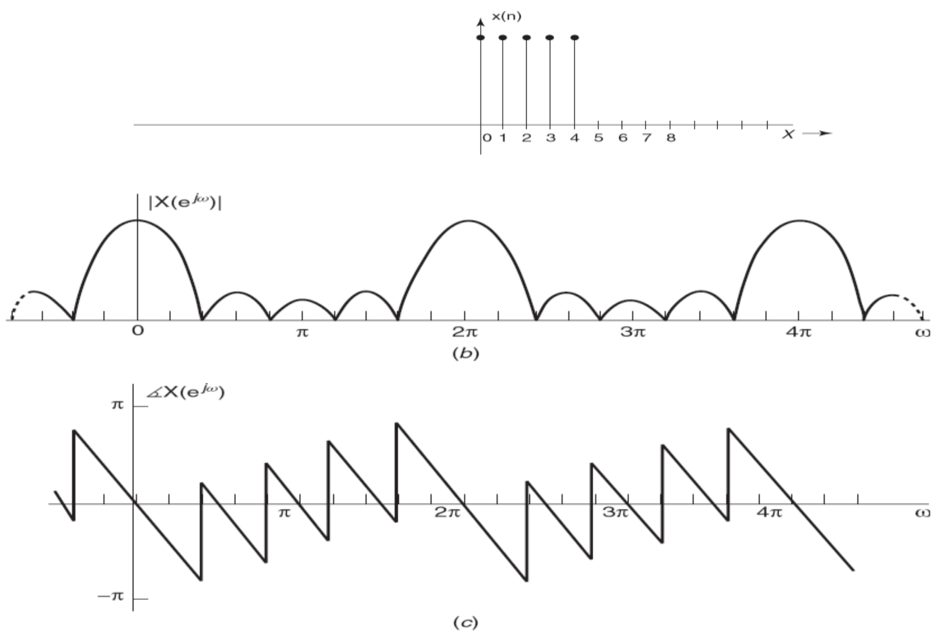
Property	Time Domain $x(n), x_1(n), x_2(n)$	Frequency Domain $X(e^{j\omega}), X_1(e^{j\omega}), X_2(e^{j\omega})$
Linearity	$ax_1(n) + bx_2(n)$	$aX_1(e^{j\omega}) + bX_2(e^{j\omega})$
Convolution	$x_1(n) * x_2(n)$	$X_1(e^{j\omega})X_2(e^{j\omega})$
Time shifting	$x(n-k)$	$e^{-j\omega k}X(e^{j\omega})$
Frequency shifting	$e^{j\omega_0 n}x(n)$	$X(e^{j(\omega-\omega_0)})$
Time reversal	$x(-n)$	$X(e^{-j\omega})$
Multiplication	$x_1(n)x_2(n)$	$(1/2\pi)\int_{-\pi}^{\pi} X_1(e^{j\zeta})X_2(e^{j(\omega-\zeta)})d\zeta$
Differentiation	$nx(n)$	$j[dX(e^{j\omega})]/d\omega$
Conjugation	$x^*(n)$	$X^*(e^{-j\omega})$
Even part of $x(n)$	$x_e(n) = \frac{1}{2}[x(n) + x(-n)]$	$\mathrm{Re}\{X(e^{j\omega})\}$
Odd part of $x(n)$	$x_0(n) = \frac{1}{2}[x(n) - x(-n)]$	$j\mathrm{Im}\{X(e^{j\omega})\}$
Symmetry		$\mathrm{Re}\{X(e^{j\omega})\} = \mathrm{Re}\{X(e^{-j\omega})\}$
		$\mathrm{Im}\{X(e^{j\omega})\} = -\mathrm{Im}\{X(e^{-j\omega})\}$
		$\lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert$
		$\mathrm{Ang}\ X(e^{j\omega}) = -\mathrm{Ang}\ X(e^{-j\omega})$

The DTFT-IDTFT pair for a discrete-time function given by

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} x(n)e^{-j\omega n}$$

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

Let us consider one more example of a discrete-time function *x(n)* and its DTFT $X(e^{j\omega})$. Figure below shows a nonperiodic discrete-time function *x(n)*



ɡure 3.28 A nonperiodic signal; (b) its magnitude response; (c) its phase response.

Relationship between continuous-time FT and DFT

The goal of spectrum analysis is often to determine the frequency content of an analog (continuous-time) signal; very often, as in most modern spectrum analyzers, this is actually accomplished by sampling the analog signal, windowing (truncating) the data, and computing and plotting the magnitude of its DFT. It is thus essential to relate the DFT frequency samples back to the original analog frequency.

Discrete-Time Fourier DFT

The discrete Fourier transform is the discrete version of the Fourier transform. This does not mean that it is a discretization of the continuous transform result, as is sometimes mistaken, but that it is a Fourier-type transform applied to discrete data. The DFT is sufficiently different from the

FT, however, to warrant careful study, especially since it (or a varient of it) is usually the primary spectral method used to analyze new data.

The DFT is actually the Z- transform computed on the unit circle of the complex plane. In this case, z = exp(jw), which is a point on the unit circle at an angle of w radians. The DFT of a real function is usually computed as the Z transform of points on the upper half of the circle (for w = 0 to π), while the DFT of a complex function is usually computed for points on the entire circle. There is really only one copy of the spectrum, but it can be obtained over and over again by computing values of W past –π or π (or outside any other 2 pi range). As W goes around and around the unit circle, the same value of the Z transform is returned as the value of the DFT for different values of W. If the signal is sufficiently band-limited, its spectrum will decline to zero before W reaches -pi or pi, and thus the "copies" of the spectrum will not overlap. However, if the spectrum extends beyond -pi or pi, it can overlap the spectrum coming from the other direction, causing alaising and distortion.Consider a discrete-time a periodic signal $x(n)$ that is finite in length equal to N samples and generate a periodic sequence with a period N so that it satisfies the condition $x_p(n + KN) = x_p(n)$, where K is any integer. The complex Fourier series representation of this periodic signal contains the sum of the discrete-time complex exponentials $e^{jk\omega 0n}$, where $\omega 0 = 2\pi/N$ is its fundamental frequency and $\omega_0 k$ is its kth harmonic. The Fourier series is a weighted sum of the fundamental and higher harmonics in the form $x_p(n) = \sum_{k=0} X_p(k)e^{j\omega_0 kn}$

where $Xp(k)$ is the coefficient of the kth harmonic in the Fourier series. These coefficients are complex-valued in general and hence have a magnitude and a phase: $Xp(k) = |Xp(k)| \angle Xp(k)$

The exponentials with frequencies that are separated by integer multiples of 2π are the same

$$e^{j(\omega_0 \pm 2\pi)k} = e^{j\omega_0 k} e^{j\pm 2\pi k} = e^{j\omega_0 k}$$

the coefficient of the complex Fourier series $Xp(k) = Xp(k \pm N)$. Its complex Fourier series form

$$x_p(n) = \sum_{k=0}^{N-1} X_p(k) e^{j(2\pi/N)kn}$$

To find these coefficients, let us multiply both sides by $e^{-jm\omega 0k}$ and sum over n from $n = 0$ to $(N - 1)$:

$$\sum_{n=0}^{N-1} x_p(n) e^{-jm\omega_0 k} = \sum_{n=0}^{N-1} \sum_{k=0}^{N-1} X_p(k) e^{j(2\pi/N)kn} e^{-jm\omega_0 k}$$

By interchanging the order of summation on the right side, we get

$$\sum_{k=0}^{N-1} X_p(k) \left[\sum_{n=0}^{N-1} e^{j(2\pi/N)k(n-m)} \right]$$

$[\sum_{n=0}^{N-1} e^{j(2\pi/N)k(n-m)}]$ is equal to N when $n = m$ and zero for all values of $n \neq m$.

When $n = m$, the summation reduces to $[\sum_{n=0}^{N-1} e^{j0}] = N$

and when $n \neq m$, we apply $$\sum_{n=-N}^{N} r^n = \frac{r^{N+1} - r^{-N}}{r - 1}; \quad r \neq 1$$

and find that the summation yields zero. Hence there is only one nonzero term $Xp(k)N$. The final result is

$$X_p(k) = \frac{1}{N} \sum_{n=0}^{N-1} x_p(n) e^{-jn\omega_0 k}$$

Now we notice that

$$
\begin{aligned}
\frac{1}{N}\sum_{n=0}^{N-1} x_p(n)e^{-jn\omega_0 k} &= \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j(2\pi/N)nk} \\
&= \left(\frac{1}{N}\right) X_p(e^{j\omega})\big|_{\omega_k=(2\pi/N)k} = X_p(k)
\end{aligned}
$$

The DTFT of the finite length sequence *x(n)* is evaluated at the discrete frequency $\omega k = (2\pi/N)k$, (which is the *k*th sample when the frequency range $[0, 2\pi]$ is divided into *N* equally spaced points) and dividing by *N*, we get the value of the Fourier series coefficient *Xp(k)*. The expression

$$
x_p(n) = \sum_{k=0}^{N-1} X_p(k)e^{j(2\pi/N)kn}
$$

is known as the *discrete-time Fourier series* (DTFS) representation for the discrete-time, periodic function *xp(n)*. Because both *xp(n)* and *Xp(k)* are periodic, with period *N*, we observe that the two expressions above are valid for $-\infty < n < \infty$ and $-\infty < k < \infty$, respectively. Note that some authors abbreviate DTFS to DFS. To simplify the notation, let us denote $e^{-j(2\pi/N)n}$ by W_N so that

$$
X_p(k) = \frac{1}{N}\sum_{n=0}^{N-1} x_p(n)e^{-jn\omega_0 k}
$$

are rewritten in compact form for the DTFS-IDTFS pair as

$$
\begin{aligned}
x_p(n) &= \sum_{k=0}^{N-1} X_p(k)W^{-kn}, \qquad -\infty < n < \infty \\
X_p(k) &= \frac{1}{N}\sum_{n=0}^{N-1} x_p(n)W^{kn}, \qquad -\infty < k < \infty
\end{aligned}
$$

Figure 3.29a shows an example of the periodic discrete-time function xp(n) constructed from Figure 3.28a while Figures 3.29b,c show the samples of the magnitude NXp(k) and phase Xp(k) of the DTFT at the discrete frequencies $\omega k = k(2\pi/N)$, $k = 0, 1, 2, \ldots, (N - 1)$.

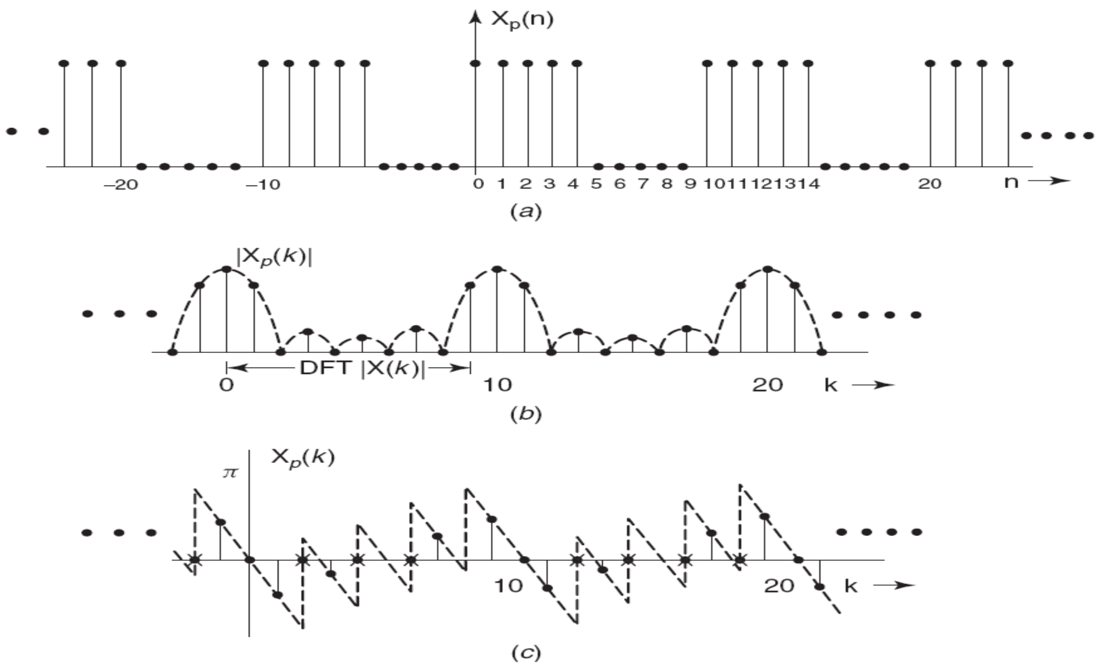


Figure 3.29 A periodic signal; (b) its magnitude response; (c) its phase response.

Considering the DFT as a subset of the ZT, the circular nature of W implies that there should be an infinite number of copies of the spectrum. However, this prediction conflicts with what has just been said about the finite nature of the discretizing process.

Reconstruction of DTFT from DFT

Consider the DTFT of *x(n)* and substitute *x(n)* by the formula

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j(2\pi/N)kn} = \frac{1}{N}\sum_{k=0}^{N-1} X(k)W^{-kn}$$

for finding the IDFT of *X(k)* as explained below:

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x(n)e^{-j\omega n} = \sum_{n=0}^{N-1} \left[\frac{1}{N} \sum_{k=0}^{N-1} X(k)e^{j(2\pi kn/N)} \right] e^{-j\omega n}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X(k) \sum_{n=0}^{N-1} e^{j(2\pi kn/N)} e^{-j\omega n}$$

$$\sum_{n=0}^{N-1} e^{j(2\pi kn/N)} e^{-j\omega n} = \frac{1 - e^{-j(\omega N - 2\pi k)}}{1 - e^{-j[\omega - (2\pi k/N)]}}$$

$$= \frac{e^{-j[(\omega N - 2\pi k)/2]}}{e^{-j[(\omega N - 2\pi k)/2N]}} \cdot \frac{\sin\left[\frac{\omega N - 2\pi k}{2}\right]}{\sin\left[\frac{\omega N - 2\pi k}{2N}\right]}$$

$$= \frac{\sin\left[\frac{\omega N - 2\pi k}{2}\right]}{\sin\left[\frac{\omega N - 2\pi k}{2N}\right]} e^{-j[\omega - (2\pi k/N)][(N-1)/2]}$$

Substituting the last expression in above equation we obtain the final result to reconstruct the DTFT $X(e^{j\omega})$, from only the finite number of the DFT samples $X(k)$, as given below:

$$X(e^{j\omega}) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) \frac{\sin\left[\frac{\omega N - 2\pi k}{2}\right]}{\sin\left[\frac{\omega N - 2\pi k}{2N}\right]} e^{-j[\omega - (2\pi k/N)][(N-1)/2]}$$

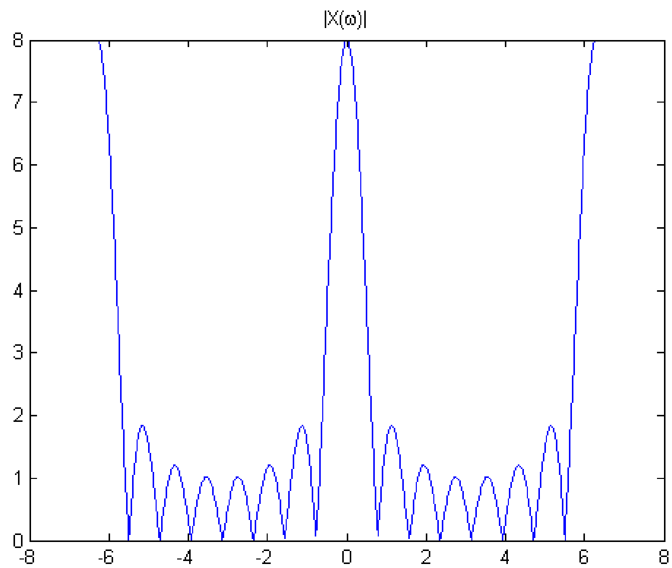
Relationship between DFT and DTFT

To start looking at the relationship between the discrete Fourier transform (DFT) and the discrete-time Fourier transform (DTFT). Let's look at a simple rectangular pulse, $x[n] = 1$ for $0 \leq n < M$. The DTFT of $x[n]$ is:

$$X(\omega) = \frac{sin(\omega M/2)}{sin(\omega/2)} e^{-j\omega(M-1)/2}$$

Let's plot $|X(\omega)|$ for $M = 8$ over a couple of periods:

```
M = 8;

w = linspace(-2*pi, 2*pi, 800);

X_dtft = (sin(w*M/2) ./ sin(w/2)) .* exp(-1j * w * (M-1) / 2);

plot(w, abs(X_dtft))

title('|X(\omega)|')
```



It turns out that, under certain conditions, the DFT is just equally-spaced samples of the DTFT. Suppose $X_P[k]$ is the P-point DFT of $x[n]$. If $x[n]$ is nonzero only over the finite domain $0 \leq n < M$, then $X_P[k]$ equals $X(\omega)$ at equally spaced intervals of ω :

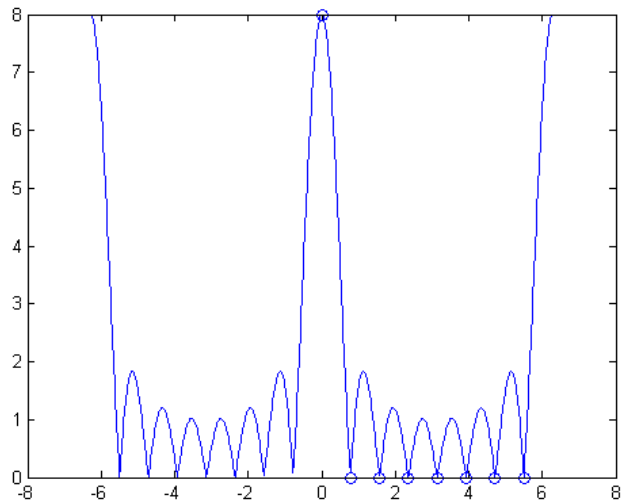
$$X_P[k] = X(2\pi k/P),\ k = 0, \dots, P-1$$

The MATLAB function FFT computes the DFT. Here's the 8-point DFT of our 8-point rectangular pulse:

```
x = ones(1, M);

X = fft(x)

X =

     8     0     0     0     0     0     0     0
```

One 8 and a bunch of zeros?? That doesn't seem anything like the DTFT plot above. But when you superimpose the output of fft in the right places on the DTFT plot, it all becomes clear.

```
P = 8;

w_k = (0:P-1) * (2*pi/P);

X = fft(x);

plot(w, abs(X_dtft))

hold on

plot(w_k, abs(X), 'o')

hold off
```



Now you can see that the seven zeros in the output of FFT correspond to the seven places (in each period) where the DTFT equals zero.

Properties of DTFS and DFT

TABLE 3.3 Properties of DTFs Coefficients

Property	Periodic Signal $x(n)$ and $y(n)$ with Period N	DTFS Coefficients $X_p(k)$ $X_p(k)$ and $Y_k(k)$
Linearity	$ax(n) + by(n)$	$aX_p(k) + bY_p(k)$
Periodic convolution	$x(n) \circledast y(n)$	$X_p(k)Y_p(k)$
Time shifting	$x(n-k)$	$e^{-j(2\pi/N)km}X_p(k) = W_N^{km}X_p(k)$
Frequency shifting	$e^{j(2\pi/N)mn}x(n) = W^{-mn}x(n)$	$X_p(k-m)$
Time reversal	$x(-n)$	$X_p(-k)$
Multiplication	$x(n)y(n)$	$\frac{1}{N}\sum_{m=0}^{N-1} X_p(m)Y_p(k-m)$
Conjugation	$x^*(n)$	$X_p^*(-k)$
Even part of $x(n)$	$x_e(n) = \frac{1}{2}\left[x(n) + x^*(-n)\right]$	$\text{Re}\left[X_p(k)\right]$
Odd part of $x(n)$	$x_0(n) = \frac{1}{2}\left[x(n) - x^*(-n)\right]$	$j\ \text{Im}\left[X_p(k)\right]$
Symmetry	$x(n)$ is a real sequence	$X_p(k) = X_p^*(-k)$ $X_p(k) = X_p^*(-k)$ $\text{Re}X_p(k) = \text{Re}X_p(-k)$ $\text{Im}X_p(k) = -\text{Im}X_p(-k)$ $\lvert X_p(k)\rvert = \lvert X_p(-k)\rvert$ $\measuredangle X_p(k) = -\measuredangle X_p(-k)$

TABLE 3.4 Properties of DFT Coefficients

Property	Signal $x(n)$ and $y(n)$ of Length N	DFT Coefficients $X(k)$ and $Y(k)$ of Length N
Linearity	$ax(n) + by(n)$	$aX(k) + bY(k)$
Convolution	$\sum_{m=0}^{N-1} x(m)y((n-m))_N$	$X(k)Y(k)$
Time shifting	$x(n-m)_N$	$e^{-j(2\pi/N)km} X(k) = W_N^{km} X(k)$
Frequency shifting	$e^{j(2\pi/N)mn} x(n) = W^{-mn} x(n)$	$X((k-m))_N$
Multiplication	$x(n)y(n)$	$\frac{1}{N}\sum_{m=0}^{N-1} X(m)Y((k-m))_N$
Conjugation	$x^*(n)$	$X^*((-k))_N$
Even part of $x(n)$	$x_e(n) = \frac{1}{2}\left[x(n) + x^*((-n))_N\right]$	$\mathrm{Re}X(k)$
Odd part of $x(n)$	$x_0(n) = \frac{1}{2}\left[x(n) - x^*((-n))_N\right]$	$j\ \mathrm{Im}X(k)$
		$X(k) = X^*((-k))_N$
		$\mathrm{Re}X(k) = \mathrm{Re}X((-k))_N$
		$\mathrm{Im}X(k) = -\mathrm{Im}X((-k))_N$
		$\lvert X(k)\rvert = \lvert X((-k))_N\rvert$
		$\measuredangle X(k) = -\measuredangle X_p((-k))_N$

Z- Transform

The *z* transform of a sequence *x (n)* is simply defined as:

$$\mathcal{Z}[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$z = e^{sT} = e^{j\omega T}$$

And the inverse *z* transform defined as:

$$\mathcal{Z}^{-1}[X(z)] = x(n) = \frac{1}{2\pi j}\int_C X(z) z^{n-1} dz$$

Why do we need to study Z transform??

–This is an important tool in the analysis of signals and linear time invariant systems

•What are the differences between Fourier transform, Laplace transform and Z transform?

–Fourier transform (FT) –the major and widely used. But cannot be used for all systems. Sometimes FT does not exist for some particular reasons. FT types are Discrete Fourier Transform (DFT), Fast Fourier Transform (FFT), Discrete Time Fourier Transform (DTFT)
–Laplace transform (LT) –used to simplify continuous systems, e.g., RCL circuits, controls, etc. LT applies to a wider class of signals compared to FT.
–Z transform (ZT) –used to simplify discrete time systems, e.g., digital signal processing, digital filter design, etc.
•The advantages of Z transform
–Simplify the analysis of the response of an LTI systems to various signals
–Provide a means of characterizing an LTI systems and its response to various signals, by its pole-zeros locations

The Z-transform is a generalization of the discrete-time Fourier transform (DTFT). The DTFT can be found by evaluating the Z-transform $X(z)$ at $z = e^{j\omega}$ or, in other words, evaluated on the unit circle. In order to determine the frequency response of the system the Z-transform must be evaluated on the unit circle, meaning that the system's region of convergence must contain the unit circle. Otherwise, the DTFT of the system does not exist.

The relation between the Z, Laplace and Fourier transform is illustrated by the above equation. It shows that the Fourier Transform of a sampled signal can be obtained from the Z Transform of the signal by replacing the variable z with e^{jwT}.

This procedure is equivalent to restricting the value of z to the unit circle in the z plane. Since the signal is discrete and the spectrum is continuous, the resulting transform is referred to as the Discrete Time Frequency Transform (DTFT).

The DTFT of a signal is usually found by finding the Z transform and making the above substitution.

We express the discrete sequence as a function of the continuous variable *t*, which allows us to treat signal processing mathematically. The product is denoted as:

$$x^*(t) = \sum_{n=0}^{\infty} x(t)\delta(t-nT)$$
$$= \sum_{n=0}^{\infty} x(nT)\delta(t-nT)$$

This expression has a Laplace transform denoted as

$$X^*(s) = \sum_{n=0}^{\infty} x(nT)e^{-snT}$$

Now we use a frequency transformation $e^{sT} = z$, (where z is a complex variable), and substituting it in expression above we get

$$X^*(s)\big|_{e^{sT}=z} = \sum_{n=0}^{\infty} x(nT)z^{-n}$$

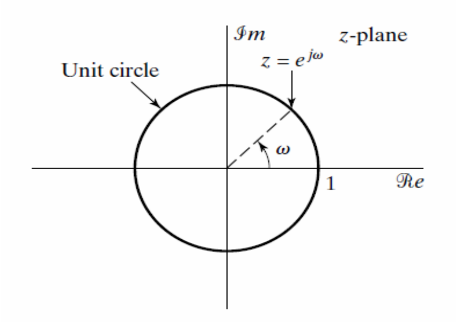
<u>The relationship between ZT and FT</u>

The following Eq.(1) and (2) are FT and ZT, respectively.

$X(e^{j\omega}) = Xn=\sum x[n]e^{-j\omega}$ (1)
$X(z) = Xn=\sum\ x[n]z^{-n}$ (2)

Replacing Z with e^{jw} , ZT will become FT

When ZT has unity magnitude (and FT exists), it means that ZT = FT.



ZT is possible to converge even if FT does not due to the sequence is multiplied by real exponential (r^{-n})

Z-transform properties

1. **Differentiation**

If *X(z)* is the *z* transform of *x(n)u(n)*, -*z*[*dX(z)*]/*dz* is the *z* transform of *nx(n)u(n)*. We denote this

$$nx(n)u(n) \Longleftrightarrow -z\frac{dX(z)}{dz}$$

Proof: $X(z) = \sum_{n=0}^{\infty} x(n)z^{-n}$. Differentiating both sides with respect to z, we get

$$\frac{dX(z)}{dz} = \sum_{n=0}^{\infty} x(n)\left[-nz^{-n-1}\right] = -z^{-1}\sum_{n=0}^{\infty} nx(n)z^{-n}$$

$$-z\frac{dX(z)}{dz} = \sum_{n=0}^{\infty} nx(n)z^{-n} = \mathcal{Z}[nx(n)u(n)]$$

. Delay

Let the z transform of x(n)u(n) be X(z) = $X(z) = \sum_{n=0}^{\infty} x(n)z^{-n}$ = $x(0) + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + \cdots$. Then the z transform of x(n - 1)u(n - 1) is $z^{-1}X(z) + x(-1)$:

By repeated application of this property, we derive

$x(n-2)u(n-2) \Leftrightarrow z^{-2}X(z) + z^{-1}x(-1) + x(-2)$

$x(n-3)u(n-3) \Leftrightarrow z^{-3}X(z) + z^{-2}x(-1) + z^{-1}x(-2) + x(-3)$

and

$x(n-m)u(n-m) \Leftrightarrow z^{-m}X(z) + z^{-m+1}x(-1) + z^{-m+2}x(-2)$

$+ \cdots + x(-m)$

$$x(n-m)u(n-m) \Longleftrightarrow z^{-m}X(z) + \sum_{n=0}^{m-1} x(n-m)z^{-n}$$

If the initial conditions are zero, we have the simpler relationship: $x(n - m)u(n - m) \Leftrightarrow z^{-m}X(z)$

TABLE 2.3 Properties of z Transforms

Operation	$x(n)u(n)$	$X(z)$
Addition	$x_1(n) + x_2(n)$	$X_1(z) + X_2(z)$
Scalar multiplication	$Kx(n)$	$KX(z)$
Delay	$x(n-1)u(n-1)$	$z^{-1}X(z) + x(-1)$
	$x(n-2)u(n-2)$	$z^{-2}X(z) + z^{-1}x(-1) + x(-2)$
	$x(n-3)u(n-3)$	$z^{-3}X(z) + z^{-2}x(-1) + z^{-1}x(-2) + x(-3)$
	$x(n-m)u(n-m)$	$z^{-m}X(z) + \sum_{n=0}^{m-1} x(n-m)z^{-n}$
Time reversal	$x(-n)u(-n)$	$X(z^{-1})$
Multiplication by n	$nx(n)$	$-z\dfrac{dX(z)}{dz}$
Multiplication by r^n	$r^n x(n)$	$X(r^{-1}z)$
Time convolution	$x_1(n) * x_2(n)$	$X_1(z)X_2(z)$
Modulation	$x_1(n)x_2(n)$	$(1/2\pi j)\int_C X_1(z)X_2\left(\dfrac{z}{u}\right)u^{-1}du$
Initial value	$x(0)$	$\lim_{z\to\infty} X(z)$
Final value	$\lim_{N\to\infty} x(N)$	$\lim_{z\to 1}(z-1)X(z)$, when poles of $(z-1)X(z)$ are inside the unit circle

Conclusion

A signal in the time domain can be converted to its counterpart in the frequency domain by means of Fourier Transform (FT). The signal must be sampled at discrete time by an A/D converter before it can be analyzed by a computer. Discrete Fourier Transform (DFT) can be used to convert the discrete signal (discrete in time) in the time domain to its counterpart (discrete in frequency) in the frequency domain. DFT can be computed efficiently in practice using a Fast Fourier Transform (FFT) algorithm, which is generally N/log (N)-1 times faster than DFT, where N is called DFT or FFT size, which is the number of data points used in the computation.