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Title :

The relationship among FT, DTFT, DFT, and z-transform

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The relationship among FT, DTFT, DFT, and z-transform

Introduction:

Fourier transforms of time signals is commonly used in signal processing and system design in modern application such as Telecommunication, Radar, speech processing and image processing systems.

Classical Fourier methods such as Fourier transform and Fourier series are used for continuous signal $x(t)$ where t within an interval $-\infty < t < \infty$

Discrete Fourier methods using to represent a discrete time sequence $x[n]$ defined only for the values of *n* where *n* is an integer within the interval $-\infty < n < \infty$. LTI discrete time systems represented in the frequency domain because of many considerations such as reducing calculation complexity and design consideration.

The following discussion focuses in the discrete time (DT) Fourier methods with the particular emphasis, the relationships among them as well as z-transform.

The discrete time Fourier transform

The discrete-time Fourier transform (DTFT) or Fourier transform of a discrete time sequence $x[n]$ in term of a complex exponential sequence $e^{-j\omega n}$. The DTFT $X\left(e^{j\omega}\right)$ is defined as the following

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}, \qquad (1)$$

and its inverse given by

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X\left(e^{j\omega}\right) e^{j\omega n} d\omega \qquad (2)$$

DTFT has many properties such as linearity, time-shifting, Frequency-shifting convolution, modulation and Parseval's relation.

Discrete Fourier Transform

DTFT may not be practical for analysing $x[n]$ because $x\left(e^{j\omega}\right)$ is a function of the continuous frequency variable ω and we cannot use a digital computer to calculate a continuum of functional values.

Discrete Fourier Transform (DFT) is a frequency analysis tool for periodic infinite-duration discrete-time signals which is practical because it is discrete in the frequency domain as well as discrete in time domain. DFT is obtained by sampling the continuous frequency domain of the DTFT at N points uniformly spaced around the unit circle in the z-plane. DFT is derived as follows.

Let $x[n]$ be a finite length sequence equal N such that $x[n]=0$ outside $0 \le n \le N-1$. The DFT obtained by sampling $X\left(e^{j\omega}\right)$ on the frequency domain within the interval $-\pi \le \omega \le \pi$ at $\omega_k = 2\pi k/N,\ \ 0 \le k \le N-1$

$$\begin{aligned} X[k] &= \sum_{n=0}^{N} x[n] W_N^{kn}, \qquad 0 \le k \le N-1 \\ where \quad W_N^{kn} &= e^{-j2\pi kn/N} \end{aligned} \tag{3}$$

and the inverse DFT given by

$$x[n] = \frac{1}{N}\sum_{n=0}^{N} X[k] W_N^{-kn}, \qquad 0 \le n \le N-1 \tag{4}$$

The DFT treats the N samples of $x[n]$ as though they are one period of a periodic sequence. DFT has properties namely linearity, circular time-shifting, circular frequency-shifting, circular convolution and modulation property. The duality and symmetry relation also hold for DFT.

Fast Fourier Transform

Fast Fourier Transform (FFT) is a fast algorithm for DFT and inverse DFT computation. Recall equation (3)

$$X[k] = \sum_{n=0}^{N} x[n] W_N^{kn}, \qquad 0 \le k \le N-1 \tag{5}$$

Each $X[K]$ involves and N complex multiplications and $(N-1)$ additions, respectively. Computing all DFT coefficients requires N^2 complex multiplications and $N(N-1)$ complex additions. Assuming that $N = 2^v$, the corresponding computational requirements for FFT are $0.5N\log_2(N)$ complex multiplications and $N\log_2(N)$ complex additions

Basically, FFT makes use of two ideas in its development:

- Decompose the DFT computation of a sequence into successively smaller DFTs.
- Utilize two properties of $W_N^{kn} = e^{-j2\pi kn/N}$,

Complex conjugate symmetry property

$$W_N^{k(N-n)} = W_N^{-kn} = \left(W_N^{kn}\right)^* \qquad (6)$$

And periodicity of n and k

$$W_N^{k(N+n)} = W_N^{n(N+k)} = W_N^{kn} \qquad (7)$$

The most basic FFT algorithms used in applications are: decimation-in-time (D-I-T) algorithms and decimation-in-frequency (D-I-F) algorithms. The basic idea of them is to decompose the N-point sequence $x[n], X[k]$ into a set of smaller subsequences. The Radix-2 algorithm also is widely used in practice.

Other algorithm namely are: bit-reversed algorithms, normally ordered algorithms, mixed-radix algorithms (for block lengths that are not powers of 2) and prime factorization algorithms.

FFT is often used to perform spectral analysis on signals that sampled and recorded as part of lab experiments or in any kinds of data acquisition systems. If the data is not periodic the windowing method used to minimize frequency domain distortion. Many windows used in DSP application such as rectangular , Hamming, Hanning , Bartlett… etc. . Improving spectral analysis may achieved by increasing the length of the FFT by taking more samples from the observation interval or augmenting the original data set with zeroes this process namely called zero padding. It allows better view of the spectrum of signals by giving more frequency domain points.Comparing FFT with DTFT we see that

$$X[k] = X\left(e^{j\omega}\right)\Big|_{\omega=\frac{2\pi k}{N}} \qquad (8)$$

Z-transform

The DTFT provide frequency domain representation of a discrete-time sequences and linear time invariant systems LTI. But with convergence condition the DTFT may not exist. A Generalized DTFT leads to z-transform. Z-transform is defined by

$$X(Z) = X\left(e^{j\omega}\right)\Big|_{Z=e^{j\omega}} = \sum_{n=0}^{N} x[n]\, z^{-n} \qquad (9)$$

In general the inverse z-transform given by the following expression

$$x[n] = \frac{1}{2\pi j} \oint_C X(z)\, z^{n-1} dz , \qquad (10)$$

Where C in ROC of $X(z)$

Combining equation(8) with equation (9) z-transform is related to FFT as:

$$X[k] = X(z)\big|_{Z=e^{j2\pi k/N}} = X\left(e^{j2\pi k/N}\right) \tag{11}$$

Where z is a complex variable The z-transform is equal to FFT evaluated at N equally spaced points in the unit circle., z plane can be represented by taking the magnitude of $|z| = 1$, there are a condition on the convergence for a given sequence, the set of R of values of z for which its z-transform converge it well known as a region of convergence. Figure (1) shows the relationship among FFT, DTFT and z-transform.

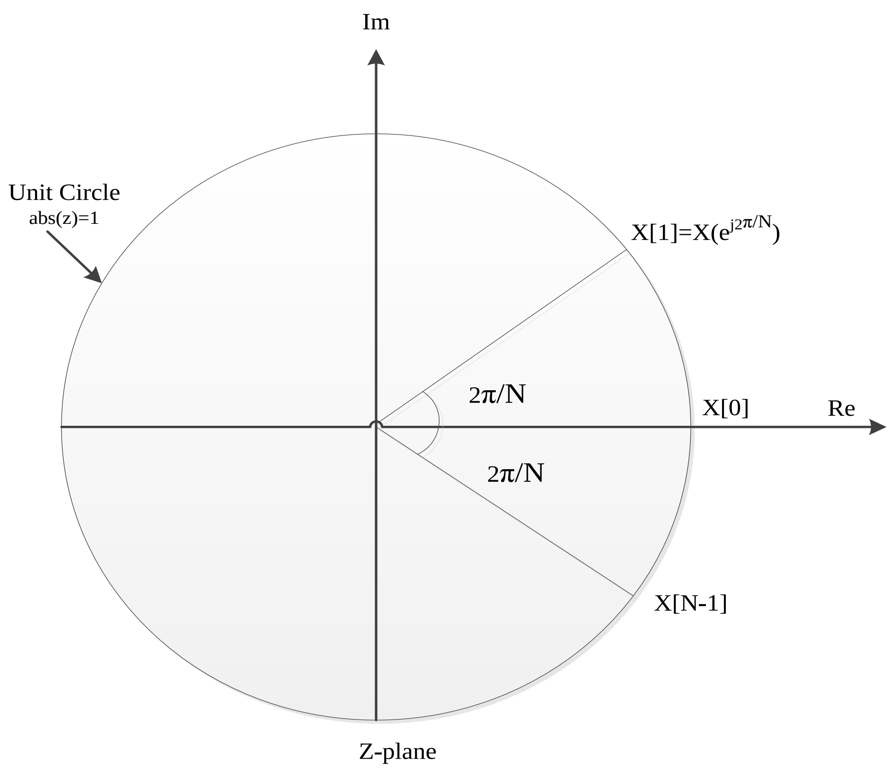


Figure (1) the relationship among FFT, DTFT and z-transform

In the following simple example is done to for plotting DFT and DTFT. In the figure(2-a) is the discrete time sequence $x[n] = [1 1 1 0 0 0]$ and figure (2-b) its DFT. Figure (2-c) shows the DTFT of discrete time sequence and it is clear that the representation of magnitude response and phase response in the frequency domain are continuous.

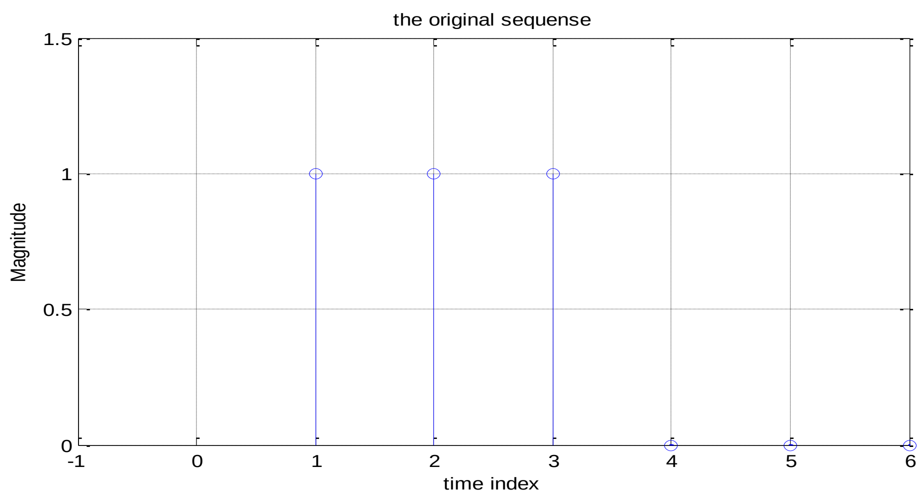
$$x[n] = \delta[n] + \delta[n-1] + \delta[n-2]$$
$$X\left(e^{j\omega}\right) = 1 + e^{-j\omega} + e^{-j2\omega}$$

$$X\left(e^{j\omega}\right)=e^{j\omega}\left\{1+e^{j\omega}+e^{-j\omega}\right\}=e^{j\omega}\left\{1+2\cos(\omega)\right\}$$

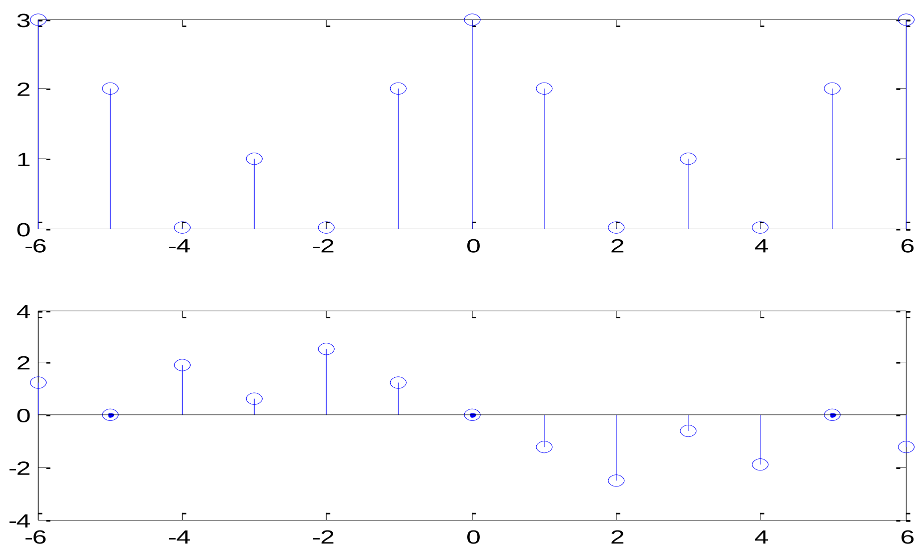
$1+2\cos(\omega)$ is the magnitude response and the exponential factor is the phase response of the DTFT.

By applying Z-transform for our example $X[z]$ expression will be as the following

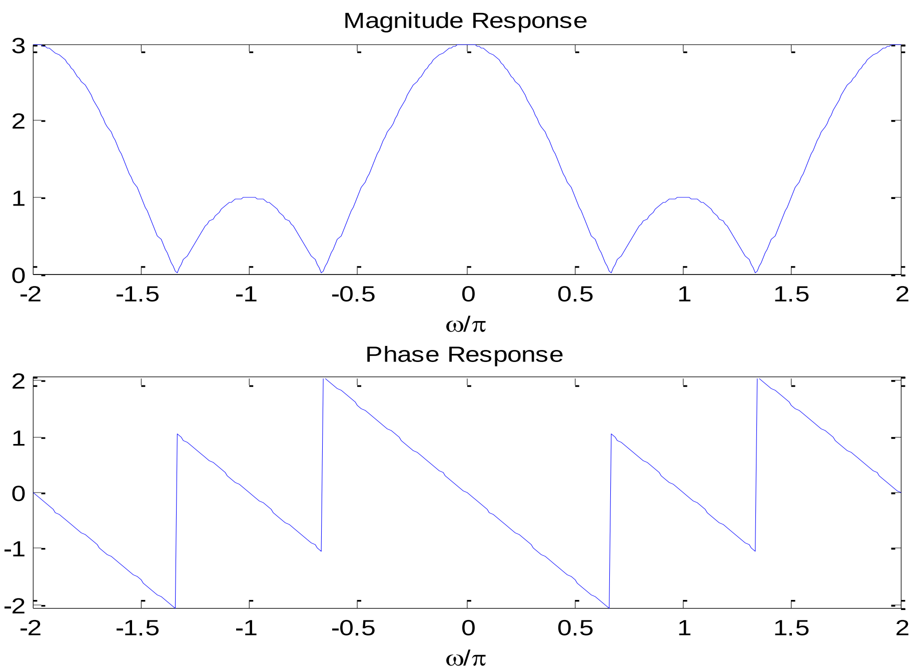
$X[z]=1+z^{-1}+z^{-2}$, the system has two poles at -0.5 ± 0.866 in z-plane, figure(3) shows the pole-zero plot of the z-transform of our example.



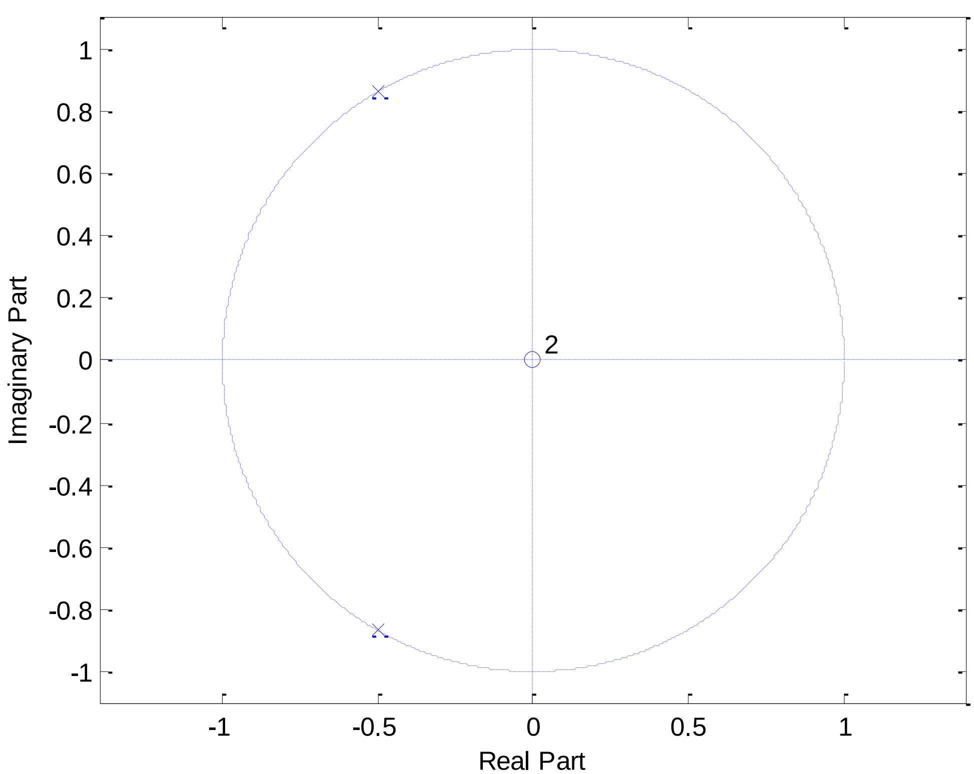
Figure(2-a) the original length-N time sequence



Figure(2-b) the DFT of $x[n]$ (plot for 2 periods)



Figure(2-c) the DTFT of $x\left[n\right]$ (plot for 2 periods)



Figure(3) the pole-zero plot of the z-transform of the given example

Conclusion

In this short report, DTFT, DFT, FFT and Z-transform and their applications in the digital signal processing field was presented. The relationship among them have been considered. Z-transform is often used for design and analysis digital filters. DTFT and Z-transform are applicable for any discrete signals. DFT and FFT can be applied to finite length sequences.FFT used for faster form DFT.