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201424020116

SCHOOL

SCHOOL OF COMMUNICATION AND INFORMATION ENGINEERING

MASTERS PROGRAMME

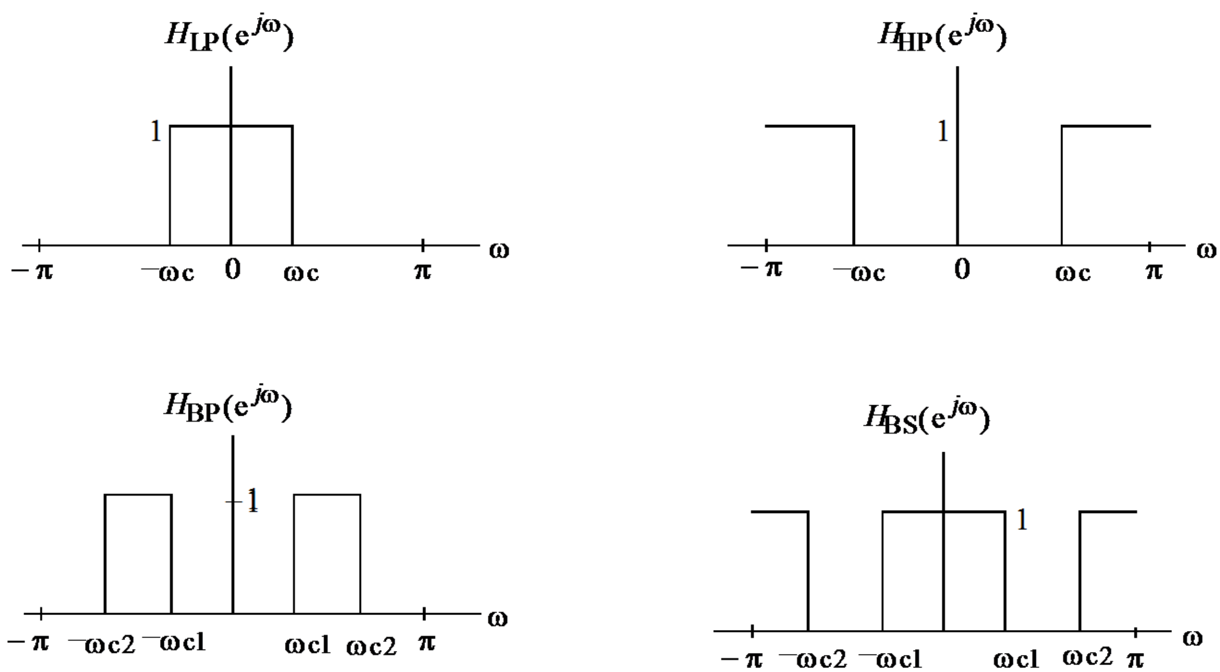
DIGITAL SIGNAL PROCESSING (DSP)

COURSE TITLED

2014, DECEMBER 18

Digital Filter Specifications

- Only the magnitude approximation problem
- Four basic types of ideal filters with magnitude responses as shown below (Piecewise flat)



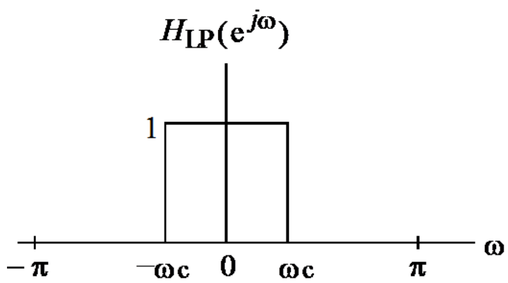
These filters are unealisable because (one of the following is sufficient)

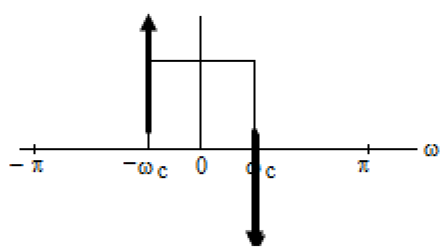
their impulse responses infinitely long non-causal

Their amplitude responses cannot be equal to a constant over a band of frequencies

Another perspective that provides some understanding can be obtained by looking at the ideal amplitude squared.

Consider the ideal LP response squared (same as actual LP response)





- The realisable squared amplitude response transfer function (and its differential) is continuous in ω
- Such functions

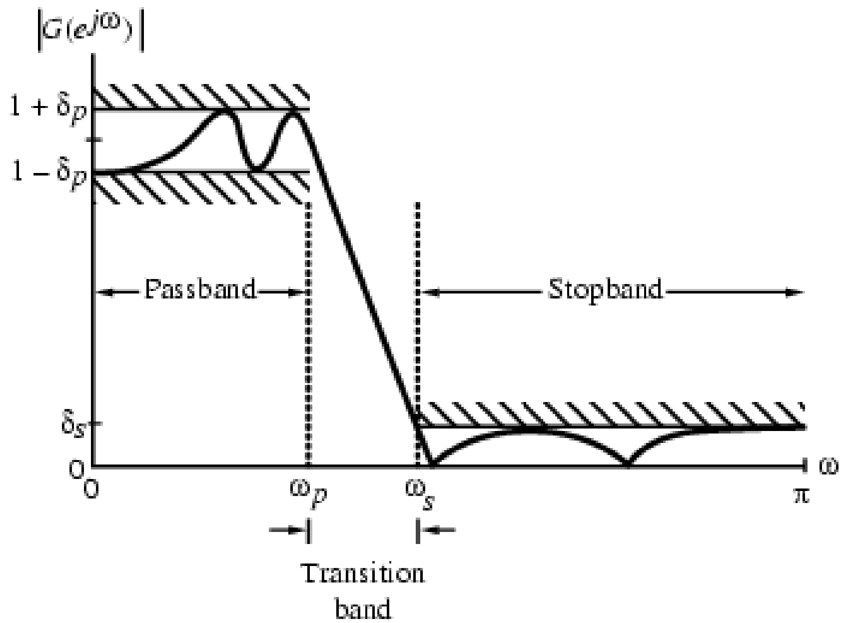
if IIR can be infinite at point but around that point cannot be zero.

if FIR cannot be infinite anywhere.

- Hence previous defferential of ideal response is unrealisable

- A realisable response would effectively need to have an approximation of the *delta functions* in the differential
- This is a necessary condition

- For example the magnitude response of a digital lowpass filter may be given as indicated below



- In the **passband** $0 \le \omega \le \omega_p$ we require that $|G(e^{j\omega})| \cong 1$ with a deviation $\pm\delta_p$

$$1-\delta_p \le |G(e^{j\omega})| \le 1+\delta_p, \quad |\omega| \le \omega_p$$

In the **stopband** $\omega_s \le \omega \le \pi$ we require that $|G(e^{j\omega})| \cong 0$ with a deviation δ_s

$$|G(e^{j\omega})| \le \delta_s, \quad \omega_s \le |\omega| \le \pi$$

Filter specification parameters

- ω_p - **passband edge frequency**
- ω_s - **stopband edge frequency**
- δ_p - **peak ripple value** in the passband
- δ_s - **peak ripple value** in the stopband
- Practical specifications are often given in terms of **loss function (in dB)**

$$\mathrm{G}(\omega) = -20\log_{10}|G(e^{j\omega})|$$

- **Peak passband ripple**

$$\alpha_p = -20\log_{10}(1-\delta_p) \quad \text{dB}$$

- **Minimum stopband attenuation**

$$\alpha_s = -20\log_{10}(\delta_s) \quad \text{dB}$$

- In practice, passband edge frequency F_p and stopband edge frequency F_s are specified in Hz

- For digital filter design, normalized bandedge frequencies need to be computed from specifications in Hz using

$$\omega_p = \frac{\Omega_p}{F_T} = \frac{2\pi F_p}{F_T} = 2\pi F_p T$$

$$\omega_s = \frac{\Omega_s}{F_T} = \frac{2\pi F_s}{F_T} = 2\pi F_s T$$

<u>Example</u> - Let $F_p = 7$ kHz, $F_s = 3$ kHz, and $F_T = 25$ kHz

Then

$$\omega_p = \frac{2\pi(7\times 10^3)}{25\times 10^3} = 0.56\pi$$

$$\omega_s = \frac{2\pi(3\times 10^3)}{25\times 10^3} = 0.24\pi$$

Selection of Filter Type

- The transfer function *H*(*z*) meeting the specifications must be a causal transfer function

- For IIR real digital filter the transfer function is a real rational function of z^{-1}

$$H(z) = \frac{p_0 + p_1 z^{-1} + p_2 z^{-2} + \cdots + p_M z^{-M}}{d_0 + d_1 z^{-1} + d_2 z^{-2} + \cdots + d_N z^{-N}}$$

- $H(z)$ must be stable and of lowest order N or M for reduced computational complexity
- FIR real digital filter transfer function is a polynomial in z^{-1} (order N) with real coefficients

$$H(z) = \sum_{n=0}^{N} h[n] z^{-n}$$

- For reduced computational complexity, degree N of $H(z)$ must be as small as possible
- If a linear phase is desired then we must have:

$$h[n] = \pm h[N-n]$$

- Advantages in using an FIR filter -

  (1) Can be designed with exact linear phase

  (2) Filter structure always stable with quantised coefficients

- Disadvantages in using an FIR filter - Order of an FIR filter is considerably higher than that of an equivalent IIR filter meeting the same specifications; this leads to higher computational complexity for FIR
- **FIR Digital Filter Design**
- Three commonly used approaches to FIR filter design -
- (1) Windowed Fourier series approach
- (2) Frequency sampling approach
- (3) Computer-based optimization methods

<u>Finite Impulse Response Filters</u>

- The transfer function is given by

$$H(z) = \sum_{n=0}^{N-1} h(n).z^{-n}$$

- The length of Impulse Response is N

All poles are at $z = 0$

- Zeros can be placed anywhere on the z-plane

FIR: Linear phase

For phase linearity the FIR transfer function must have zeros outside the unit circle

- **To develop expression for phase response set transfer function (order *n*)**

$$H(z) = h_0 + h_1 z^{-1} + h_2 z^{-2} + \ldots + h_n z^{-n}$$

- In factored form

$$H(z) = K \prod_{i=1}^{n_1} (1 - \alpha_i z^{-1}) . \prod_{i=1}^{n_2} (1 - \beta_i z^{-1})$$

- Where $\left|\alpha_i\right| < 1, \quad \left|\beta_i\right| > 1$ K, is real & zeros occur in conjugates

- Let $H(z) = K N_1(z) N_2(z)$

Where

$$N_1(z) = \prod_{i=1}^{n_1} (1 - \alpha_i z^{-1}) \qquad N_2(z) = \prod_{i=1}^{n_2} (1 - \beta_i z^{-1})$$

Thus

$$\ln(H(z)) = \ln(K) + \sum_{i=1}^{n_1} \ln(1 - \alpha_i z^{-1}) + \sum_{i=1}^{n_2} \ln(1 - \beta_i z^{-1})$$

- Expand in a Laurent Series convergent within the unit circle
- To do so modify the second sum as

$$\sum_{i=1}^{n_2} \ln(1-\overset{s_m^{N_1}}{\beta_i} z^{-1}) = \sum_{i=1}^{n_2} \ln(-\beta_i z^{-1}) + \sum_{i=1}^{n_2} \ln(1-\frac{1}{\beta_i} z)$$

- So that

$$\ln(H(z)) = \ln(\overline{K}) - n_2 \ln(z) + \sum_{i=1}^{n_1} \ln(1-\alpha_i z^{-1}) + \sum_{i=1}^{n_2} \ln(1-\frac{1}{\beta_i} z)$$

- Thus

$$\ln(H(z)) = \ln(\overline{K}) - n_2 \ln(z) + \sum_{m=1}^{\infty} \frac{s_m^{N_1}}{m} z^{-m} + \frac{s_{-m}^{N_2}}{m} z^{m}$$

where

$$s_m^{N_1} = \sum_{i=1}^{n_1} \alpha_i^m \qquad s_{-m}^{N_2} = \sum_{i=1}^{n_1} \beta_i^{-m}$$

$s_m^{N_1}$ are the root moments of the minimum phase component

$s_{-m}^{N_2}$ are the inverse root moments of the maximum phase component

Now on the unit circle we have $z = e^{j\theta}$ and $H(e^{j\theta}) = A(\theta) e^{j\phi(\theta)}$

Fundamental Relationships

$$\ln(H(e^{j\theta})) = \ln(\overline{K}) - jn_2\theta + \sum_{m=1}^{\infty} \frac{s_m^{N_1}}{m} e^{-jm\theta} + \frac{s_{-m}^{N_2}}{m} e^{jm\theta}$$

$$\ln(H(e^{j\theta})) = \ln(A(\theta)e^{j\phi(\theta)}) = \ln(A(\theta)) + j\phi(\theta)$$

hence (note Fourier form)

$$\ln(A(\theta)) = \ln(\overline{K}) + \sum_{m=1}^{\infty} \left(\frac{s_m^{N_1}}{m} + \frac{s_{-m}^{N_2}}{m}\right)\cos m\theta$$

FIR: Linear phase

Thus for linear phase the second term in the fundamental phase relationship must be identically zero for all index values.

Hence

1) the maximum phase factor has zeros which are the inverses of the those of the minimum phase factor

2) the phase response is linear with group delay (normalised) equal to the number of zeros outside the unit circle

It follows that zeros of linear phase FIR trasfer functions not on the circumference of the unit circle occur in the form $\left[\rho_i e^{\pm j\theta_i}\right]^{\pm 1}$

For Linear Phase t.f. (order *N-1*)

$$h(n) = \pm h(N-1-n)$$

so that for *N* even:

$$H(z) = \sum_{n=0}^{N/2-1} h(n).z^{-n} \pm \sum_{n=N/2}^{N-1} h(n).z^{-n}$$

for *N* odd:

$$H(z)=\sum_{n=0}^{\frac{N-1}{2}-1} h(n).\left[z^{-n}\pm z^{-m}\right]+h\left(\frac{N-1}{2}\right)z^{-\left(\frac{N-1}{2}\right)}$$

- I) On $C:|z|=1$ we have for N even, and +ve sign

$$H(e^{j\omega T})=e^{-j\omega T\left(\frac{N-1}{2}\right)}.\sum_{n=0}^{N/2-1} 2h(n).\cos\left(\omega T\left(n-\frac{N-1}{2}\right)\right)$$

- II) While for –ve sign

$$H(e^{j\omega T})=e^{-j\omega T\left(\frac{N-1}{2}\right)}.\sum_{n=0}^{N/2-1} j2h(n).\sin\left(\omega T\left(n-\frac{N-1}{2}\right)\right)$$

- [Note: antisymmetric case adds $\pi/2$ rads to phase, with discontinuity at $\omega=0$]
- III) For N odd with +ve sign

$$H(e^{j\omega T})=e^{-j\omega T\left[\frac{N-1}{2}\right]}\left\{h\left(\frac{N-1}{2}\right)+\sum_{n=0}^{\frac{N-3}{2}} 2h(n).\cos\left[\omega T\left(n-\frac{N-1}{2}\right)\right]\right\}$$

- IV) While with a –ve sign

$$H(e^{j\omega T})=e^{-j\omega T\left[\frac{N-1}{2}\right]}\left\{\sum_{n=0}^{\frac{N-3}{2}} 2j.h(n).\sin\left[\omega T\left(n-\frac{N-1}{2}\right)\right]\right\}$$

- [Notice that for the antisymmetric case to have linear phase we require

$$h\left(\frac{N-1}{2}\right)=0.$$

The phase discontinuity is as for *N* even]

The cases most commonly used in filter design are (I) and (III), for which the amplitude characteristic can be written as a polynomial in $\cos\frac{\omega T}{2}$

<u>Design of FIR filters: Windows</u>

(i) Start with ideal infinite duration $\{h(n)\}$

(ii) Truncate to finite length. (This produces unwanted ripples increasing in height near discontinuity.)

(iii) Modify to $\tilde{h}(n) = h(n).w(n)$

Weight *w*(*n*) is the window

Windows

Commonly used windows

- Rectangular $1-\dfrac{2|n|}{N}$ $|n| < \dfrac{N-1}{2}$
- Bartlett $1+\cos\left(\dfrac{2\pi n}{N}\right)$
- Hann
- Hamming $0.54+0.46\cos\left(\dfrac{2\pi n}{N}\right)$
- Blackman $0.42+0.5\cos\left(\dfrac{2\pi n}{N}\right)+0.08\cos\left(\dfrac{4\pi n}{N}\right)$

Kaiser $J_0\left[\beta\sqrt{1-\left(\dfrac{2n}{N-1}\right)^2}\right]\Big/ J_0(\beta)$

Kaiser window

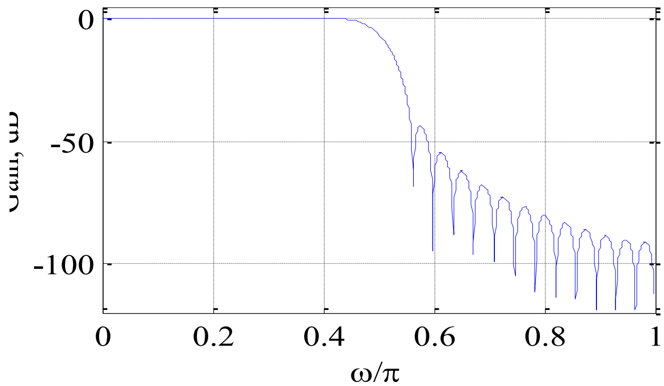
Kaiser window

β	Transition width (Hz)	Min. stop attn dB
2.12	1.5/N	30
4.54	2.9/N	50
6.76	4.3/N	70
8.96	5.7/N	90

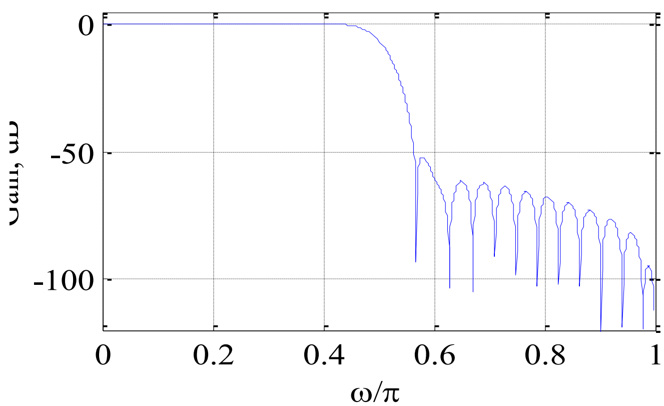
Example

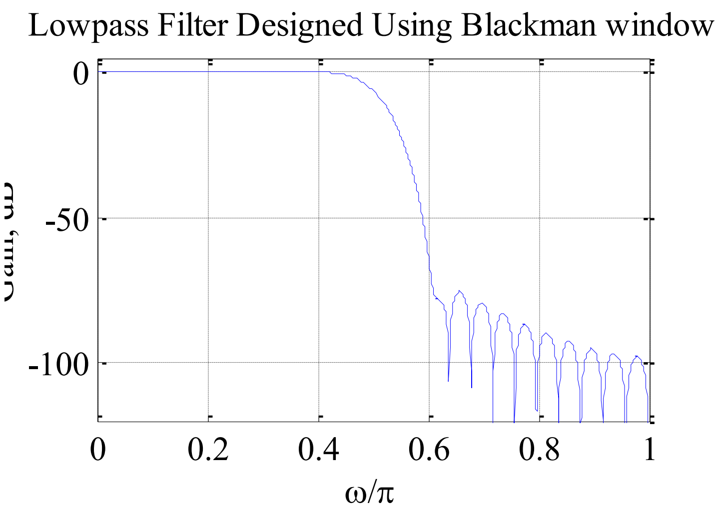
Lowpass filter of length 51 and $\omega_c = \pi / 2$

Lowpass Filter Designed Using Hann window



Lowpass Filter Designed Using Hamming window





Frequency Sampling Method

- In this approach we are given $H(k)$ and need to find $H(z)$
- This is an interpolation problem and the solution is given in the DFT part of the course

$$H(z)=\frac{1}{N}\sum_{k=0}^{N-1}H(k).\frac{1-z^{-N}}{1-e^{j\frac{2\pi}{N}k}.z^{-1}}$$

It has similar problems to the windowing approach

Linear-Phase FIR Filter Design by Optimisation

Amplitude response for all 4 types of linear-phase FIR filters can be expressed as

$$\breve{H}(\omega)=Q(\omega)A(\omega)$$

Where

$$Q(\omega) = \begin{cases} 1, & \text{for Type1} \\ \cos(\omega/2), & \text{for Type2} \\ \sin(\omega), & \text{for Type3} \\ \sin(\omega/2), & \text{for Type4} \end{cases}$$

Modified form of weighted error function

$$\begin{aligned} \mathrm{E}(\omega) &= W(\omega)[Q(\omega)A(\omega) - D(\omega)] \\ &= W(\omega)Q(\omega)[A(\omega) - \tfrac{D(\omega)}{Q(\omega)}] \\ &= \tilde{W}(\omega)[A(\omega) - \tilde{D}(\omega)] \end{aligned}$$

Where $\tilde{W}(\omega) = W(\omega)Q(\omega)$

$\tilde{D}(\omega) = D(\omega)/Q(\omega)$

- Optimisation Problem - Determine $\tilde{a}[k]$ which minimise the peak absolute value of

$$\mathrm{E}(\omega) = \tilde{W}(\omega)[\sum_{k=0}^{L} \tilde{a}[k]\cos(\omega k) - \tilde{D}(\omega)]$$

over the specified frequency bands $\omega \in R$

- After $\tilde{a}[k]$ has been determined, construct the original $A(e^{j\omega})$ and hence $h[n]$

- Solution is obtained via the Alternation Theorem
- The optimal solution has equiripple behaviour consistent with the total number of available parameters.
- Parks and McClellan used the Remez algorithm to develop a procedure for designing linear FIR digital filters.

- Kaiser's Formula:

$$N \cong \frac{-20\log_{10}(\sqrt{\delta_p \delta_s})}{14.6(\omega_s - \omega_p)/2\pi}$$

- ie *N* is inversely proportional to transition band width and not on transition band location

FIR Digital Filter Order Estimation

- Hermann-Rabiner-Chan's Formula:

$$N \cong \frac{D_\infty(\delta_p, \delta_s) - F(\delta_p, \delta_s)[(\omega_s - \omega_p)/2\pi]^2}{(\omega_s - \omega_p)/2\pi}$$

Where

$$D_\infty(\delta_p, \delta_s) = [a_1(\log_{10}\delta_p)^2 + a_2(\log_{10}\delta_p) + a_3]\log_{10}\delta_s \quad + [a_4(\log_{10}\delta_p)^2 + a_5(\log_{10}\delta_p) + a_6]$$

$$F(\delta_p, \delta_s) = b_1 + b_2[\log_{10}\delta_p - \log_{10}\delta_s]$$

With $a_1 = 0.005309,\ a_2 = 0.07114,\ a_3 = -0.4761$

$a_4 = 0.00266,\ a_5 = 0.5941,\ a_6 = 0.4278$

$b_1 = 11.01217,\quad b_2 = 0.51244$

Formula valid for $\delta_p \geq \delta_s$

For $\delta_p < \delta_s$ formula to be used is obtained by interchanging δ_p and δ_s

Both formulae provide only an estimate of the required filter order *N*

If specifications are not met, increase filter order until they are met

Fred Harris' guide: $N \cong \dfrac{A}{20(\omega_s - \omega_p)/2\pi}$

where *A* is the attenuation in dB

Then add about 10% to it

Conclusion

With recursive IIR filters, we can generally achieve a desired frequency response characteristic with a filter of lower order than for a non-recursive filter (especially if elliptic designs are used). A recursive filter has both poles and zeroes which can be selected by the designer, hence there are more free parameters than for a non-recursive filter of the same order (only zeroes can be varied). However, when the poles of an IIR filter are close to the unit circle, they need to be specified very accurately (typically 3 to 6 decimal places) if instability is to be avoided.