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MASTERS PROGRAMME

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DIGITAL SIGNAL PROCESSING (DSP)

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4. Talk about filter specifications and how to design a digital filter based on the given specifications.

To begin with, a digital filter is simply a discrete-time, discrete-amplitude convolver. Basic Fourier transform theory states that the linear convolution of two sequences in the domain is the same as multiplication of two corresponding spectral sequences in the frequency domain. Filtering is in essence the multiplication of the signal spectrum by the frequency domain impulse response of the filter. For an ideal lowpass filter the band part of the signal spectrum is multiplied by one and the stopband part of the signal by zero.

The determination of a realizable transfer function G(z) approximating a given frequency response specification is an important step in the development of a digital filter. If an IIR filter is desired, G(z) should be a stable real rational function. Digital filter design is the process of deriving the transfer function G(z). Usually, either the magnitude and / or the phase (delay) response is specified for the design of digital filter for most applications. In some situations the unit sample response or the step response may be specified. In most practical applications, the problem of interest is the development of a realizable approximation of a given magnitude response specification.

In designing a digital filter, there are four basic types of ideal filters with magnitude responses as indicated below.

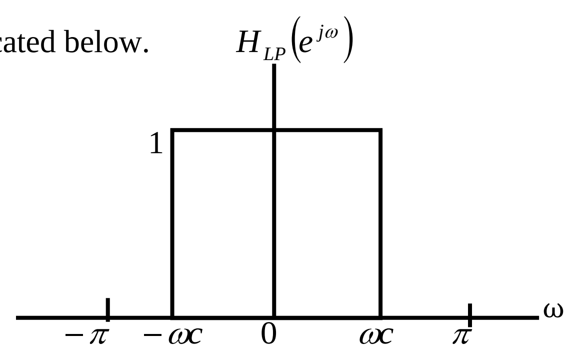


Figure 1(a) filter

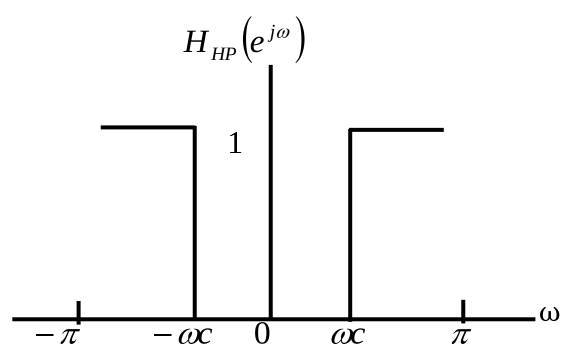
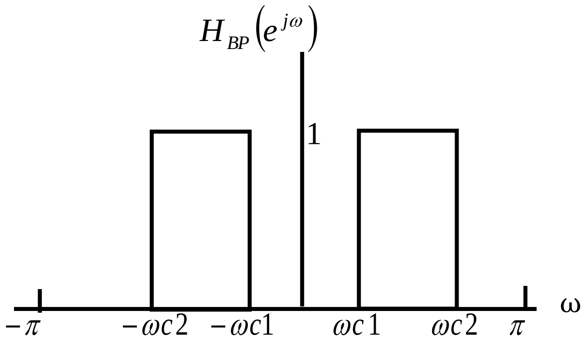


Figure 1(b) filter



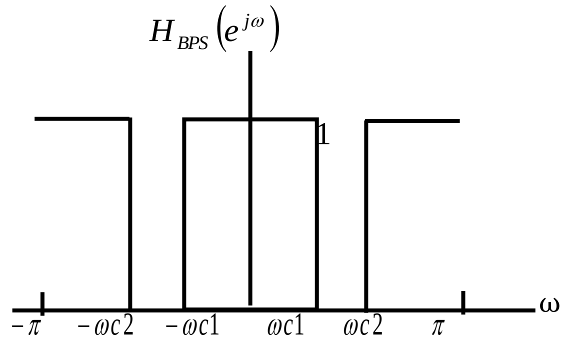


Figure 1(c) filter

Figure 1(d) filter

As the impulse response corresponding to each of these ideal filters is noncausal and of infinite length, these filters are not realizable. In practice, the magnitude response specifications of a digital filter in the passband and in the stopband are given with some acceptable tolerances. In addition, a transition band is specified between the passband and stopband [1].

For example, the magnitude response $|G(e^{j\omega})$ of a digital lowpass filter may be given as indicated below.

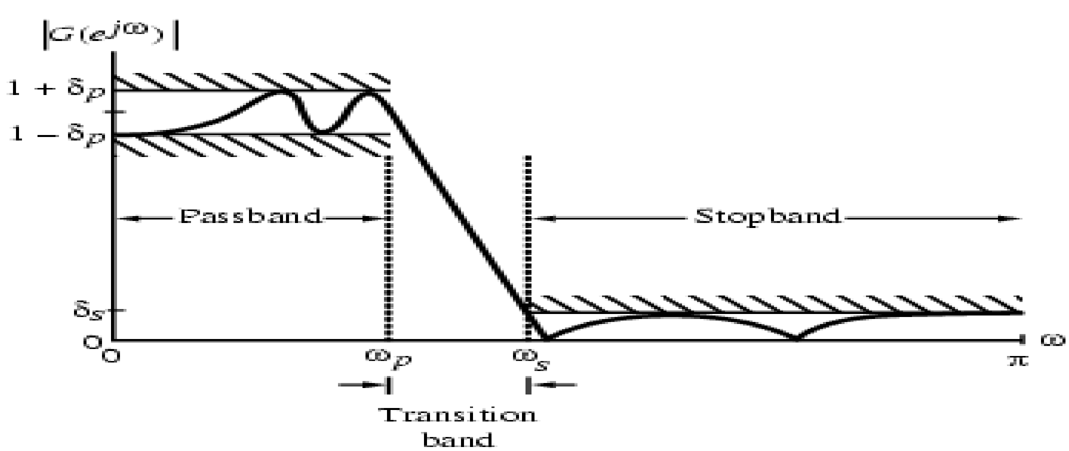


Figure 2: digital Lowpass filter

As indicated in the figure above, in the passband, defined by $0 \le \omega \le \omega_p$, it is required that $\left|G\left(e^{j\omega}\right)\right| \cong 1$ with an error $\pm\delta_p$ i.e $1-\delta_p \le \left|G\left(e^{j\omega}\right)\right| \le 1+\delta_p, |\omega| \le \omega_p$. In the stopband, defined by $\omega_s \le \omega \le \pi$, it required that $\left|G\left(e^{j\omega}\right)\right| \cong 0$ with an error $\delta_{s,}$ i.e $\left|G\left(e^{j\omega}\right)\right| \le \delta_{s,}$ $\omega_s \le |\omega| \le \pi$

ω_p- passband edge frequency

ω_s - stopband edge frequency

δ_p - peak ripple value in the passband

δ_s - peak ripple value in the stopband

Since $\left|G\left(e^{j\omega}\right)\right|$ is a periodic function of ω, and $\left|G\left(e^{j\omega}\right)\right|$ of a real-coefficient digital filter is an even function of ω. As a result, filter specifications are given only for the frequency range $0 \le |\omega| \le \pi$.

Specifications are often given in terms of loss function $G(\omega) = -20\log_{10}\left|G\left(e^{j\omega}\right)\right|$ in dB. Peak passband ripple α_p = - $20\log_{10}(1-\delta_p)$dB. Minimum stopband attenuation α_p = - $20\log_{10}(\delta_s)$ dB. Magnitude specifications may alternately be given in a normalized form as indicated below.

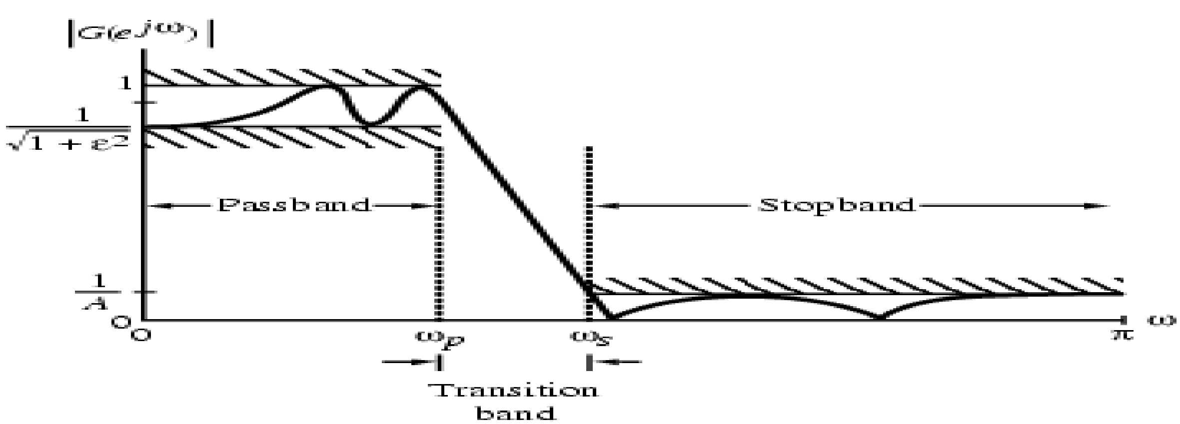


Figure 3: Magnitude response in filters

In this case, the maximum value of the magnitude in the passband is assumed to be unity.

$1/\sqrt{1+\varepsilon^2}$ – Maximum passband deviation, given by the minimum value of the magnitude in the passband . $\frac{1}{A}$ – Maximum stopband magnitude.

For the normalized specification, maximum value of the gain function or the minimum value of the loss function is 0 dB. Maximum passband attenuation - $\alpha_{\max} = 20\log_{10}\left(\sqrt{1+\varepsilon^2}\right)dB$. For δ_p <<1, it can be shown that $\alpha_{\max} \cong -20\log_{10}(1-2\delta_p)dB$

In practice, passband edge frequency F_p and stopband edge frequency F_s are specified in Hz. For digital filter design, normalized bandedge frequencies need to be computed from specifications in Hz using

$$\omega_p = \frac{\Omega_p}{F_T} = \frac{2\pi F_p}{F_T} = 2\pi F_p T \qquad \text{Eq. 1} \quad , \quad \omega_s = \frac{\Omega_s}{F_T} = \frac{2\pi F_s}{F_T} = 2\pi F_s T \qquad \text{Eq.2}$$

For example, let F_p=7 kHz, F_s = 3 kHz, and F_T = 25 kHz then $\omega_p = \frac{2\pi(7x10^3)}{25x10^3} = 0.56\pi$

$$\omega_s = \frac{2\pi(3x10^3)}{25x10^3} = 0.24\pi$$

The transfer function H(z) meeting the frequency response specifications should be a causal transfer function. For IIR digital filter design, the IIR transfer function is a real rational function of Z^{-1}:

$$H(z)=\frac{P_0 + P_1 z^{-1} + P_2 z^{-2} + ... + P_M z^{-M}}{d_0 + d_1 z^{-1} + d_2 z^{-2} + ... + d_N z^{-N}}. \qquad \text{Eq. 3}$$

H(z) must be a stable transfer function and must be of lowest order N for reduced computational complexity. For FIR digital filter design, the FIR transfer function is a polynomial in z^{-1} with real

coefficients: $H(z) = \sum_{n=0}^{N} h[n] z^{-n}$ Eq. 4

For reduced computational complexity, degree N of H(z) must be as small as possible. If linear phase is desired, the filter coefficient must satisfy the constraint: $h[n] = \pm h[N-n]$ Eq. 5

Advantages in using an FIR filter are that, it can be designed with the exact linear phase, filter structure always stable with quantized coefficients. However, some of the disadvantages include, the order of an FIR filter in most cases, is considerably higher than the order of an equivalent IIR filter meeting the same specifications, and FIR filters has thus higher computational complexity [1].

Basic Approach to IIR filter design

- Convert the digital filter specifications into an analog prototype lowpass filter specifications.
- Determine the analog lowpass filter transfer function $H_a(s)$
- Transform $H_a(s)$ in to the desired digital transfer function G(z).

Reasons for the application of the above approaches

- Analog approximation techniques are highly advanced
- They usually yield closed-form solutions
- Extension tables are available for analog filter design.
- Many applications require digital simulation of analog systems.

An analog transfer function to be denoted as $H_a(S) = \frac{P_a(S)}{D_a(S)}$ Eq. 6 where the substrate "a" specifically indicates the analog domain.

A digital transfer function derived from $H_a(S)$ shall be denoted as $G(z) = \frac{P(z)}{D(z)}$. Eq. 7

The Basic idea behind the conversion of $H_a(S)$ into G(z) is to apply a mapping from the s-domain to the z-domain so that essential properties of the analog frequency response are preserved. Thus mapping function should be such that, imaginary ($j\Omega$) axis in the s-plane be mapped onto the unit circle of the z-plane, A stable analog transfer function be mapped into a stable digital transfer function [1].

FIR filter design is based on a direct approximation of the specified magnitude response, with the often added requirement that the phase be linear.

The design of an FIR filter order N may be accomplished by finding either the length-(N+1) impulse response samples $\{h[n]\}$ or the (N+1) samples of its frequency response $H(e^{j\omega})$.

Three commonly used approaches to FIR filter design include;

- Windowed Fourier series approach
- Frequency sampling approach
- Computer-based optimization methods.

IIR Digital Filter Design through Bilinear Transformation Method.

Bilinear transformation – $s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$. Eq. 8 This transformation maps a single point in the s-plane to a unique point in the z-plane and vice-versa. The relationship between G(z) and $H_a(s)$ is then given by $G(z)=H_a(s)\Big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$. Eq. 9

The digital filter design consists of three (3) steps:

- Develop the specifications of $H_a(s)$ by applying the inverse bilinear transformation to specifications of G(z).
- Design $H_a(s)$
- Determine G(z) by applying bilinear transformation to $H_a(s)$.

As a result, the parameter *T* has no effect on G(z) and *T* =2 is chosen for convenience.

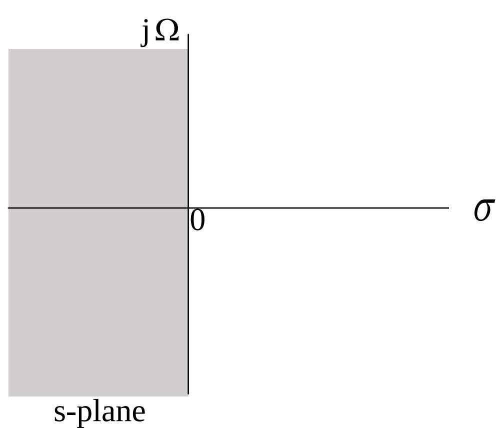
Inverse bilinear transformation for T=2 is $z = \frac{1+s}{1-s}$ Eq. 10

For $s = \sigma_0 + j\Omega_0$ Eq. 11

$$Z = \frac{(1+\sigma_0) + j\Omega_0}{(1-\sigma_0) - j\Omega_0} \Rightarrow |z|^2 = \frac{(1+\sigma_0)^2 + \Omega_0^2}{(1-\sigma_0)^2 + \Omega_0^2} \qquad \text{Eq. 12}$$

$$\text{Thus,}\quad \begin{array}{l} \sigma_0 = \quad \rightarrow |z| = 1 \\ \sigma_0 < 0 \rightarrow |z| < 1 \\ \sigma_0 > 0 \rightarrow |z| > 1 \end{array}$$

Mapping of s-plane into the z-plane



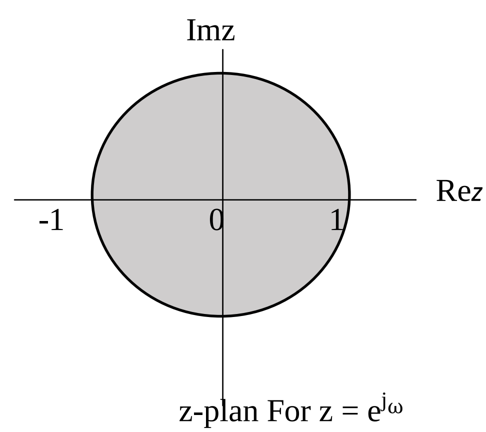


Figure 4. S-plane

Figure 5. Z-plane

with T = 2 we have $j\Omega = \frac{1-e^{-j\omega}}{1+e^{-j\omega}} = \tan(\omega/2)$ *Eq.13*

$or\ \Omega = \tan(\omega/2)$

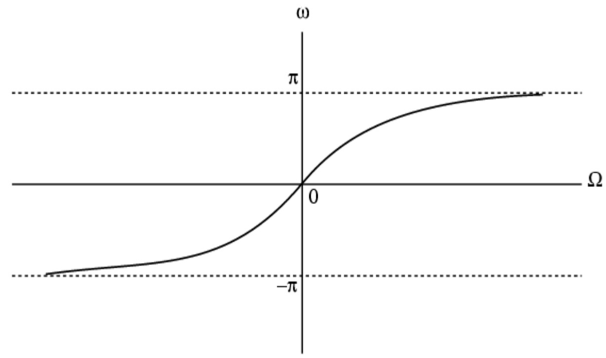


Figure 6. for bilinear transformation

In Bilinear transformation, mapping is highly nonlinear and a complete negative imaginary axis in the s-plane from $\Omega = -\infty to\, \Omega = 0$ is mapped into the lower half of the unit circle in the z-plane from z=-1 to z=1.

A complete positive imaginary axis in the s-plane from Ω=0 to $\Omega = \infty$ is mapped into the upper half of the unit circle in the z-plane from z=1 to z=-1.

Nonlinear mapping introduces a distortion in the frequency axis called frequency warping. The effect of warping is shown below.

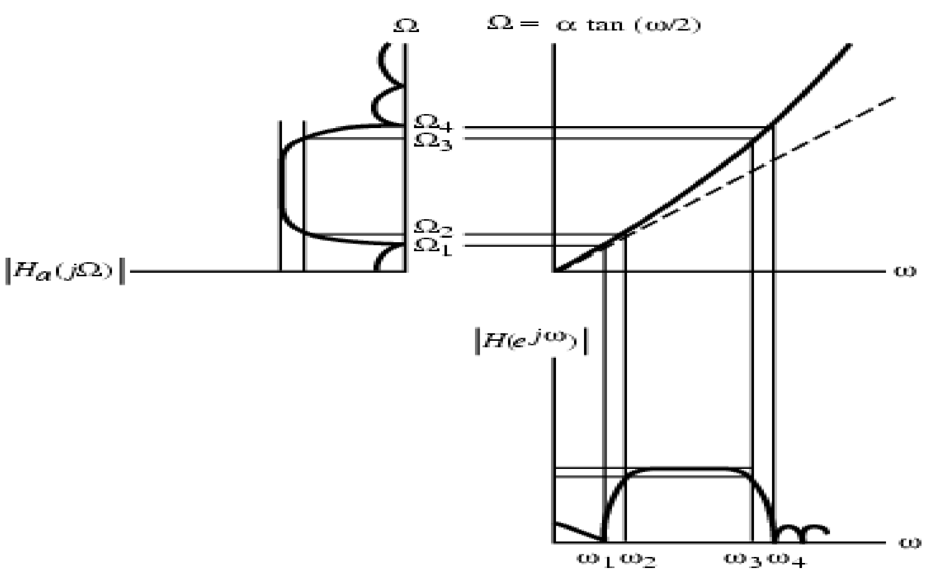


Figure 7. Effect of warping

The steps in the design of a digital filter include;

- Prewarp (ω_p, ω_s) to find their analog equivalents (Ω_p, Ω_s)
- Design the analog filter $H_a(s)$
- Design the digital filter G(z) by applying bilinear transformation to $H_a(s)$
- Transformation can be used only to design digital filters with prescribed magnitude response with piecewise constant values.
- Transformation does not preserve phase response of analog filter

IIR Digital Design using Bilinear Transformation.

For instance, consider $H_a(s) = \dfrac{\Omega_c}{s+\Omega}$ Eq. 14

Applying bilinear transformation to the above we get the transfer function of a first-order digital lowpass Butterworth filter in this form; $G(z) = H_a(s)\Big|_{s=\frac{1-z^{-1}}{1+z^{-1}}} = \dfrac{\Omega_{c(1+z^{-1})}}{(1-z^{-1})+\Omega_c(1+z^{-1})}$ Eq. 15

Rearranging terms we get $G(z) = \dfrac{1-\alpha}{2}.\dfrac{1+z^{-1}}{1-\alpha z^{-1}}$ Eq. 16

Where $\alpha = \dfrac{1-\Omega_c}{1+\Omega_c} = \dfrac{1-\tan(\omega_{c/2})}{1+\tan(\omega_c/2)}$ Eq. 17

Example 2. Consider the second-order analog notch transfer function,

$H_a(s) = \dfrac{s^2+\Omega_0^2}{s^2+Bs+\Omega_0^2}$ for which $\begin{array}{l}|H_a(j\Omega_0)| = 0 \\ |H_a(j0)| = |H_a(j\infty)| = 1\end{array}$ Eq. 18

Ω_0 is called the notch frequency.

If $|H_a(j\Omega_2)| = |H_a(j\Omega_1)| = 1/\sqrt{2}$ *then* $B = \Omega_{2-\Omega_1}$ is the 3-dB notch bandwidth.

$$\text{Then } G(z) = H(s)\Big|_{s=\frac{1-z^{-1}}{1+z^{-1}}} = \frac{\left(1+\Omega_0^2\right) - 2\left(1-\Omega_0^2\right)z^{-1} + \left(1+\Omega_0^2\right)z^{-2}}{\left(1+\Omega_0^2+B\right) - 2\left(1-\Omega_0^2\right)z^{-1} + \left(1+\Omega_0^2-B\right)z^{-2}} \qquad \text{Eq. 19}$$

$$= \frac{1+\alpha}{2} \cdot \frac{1-2\beta z^{-1} + z^{-2}}{1-2\beta(1+\alpha)z^{-1} + \alpha z^{-2}}$$

Where

$$\alpha = \frac{1+\Omega_0^2-B}{1+\Omega_0^2+B} = \frac{1-\tan(Bw/2)}{1+\tan(Bw/2)}$$

$$\beta = \frac{1-\Omega_0^2}{1+\Omega_0^2} = \cos\omega_o \qquad \text{Eq. 20}$$

In another example, design a 2^{nd}-order digital notch filter operating at a sampling rate of 400Hz with a notch frequency at 60Hz, 3-dB notch bandwidth of 6 Hz. Thus $\omega_o = 2\pi(60/400) = 0.3\pi$, $B_w = 2\pi\,(6/400) = 0.03\pi$.

From the above values, we get α = 0.90993 and β = 0.587785.

$$\text{Thus } G(z) = \frac{0.954965 - 1.1226287z^{-2} + 0.954965z^{-2}}{1 - 1.1226287z^{-1} + 0.90993z^{-2}}$$

The gain and phase responses are shown below

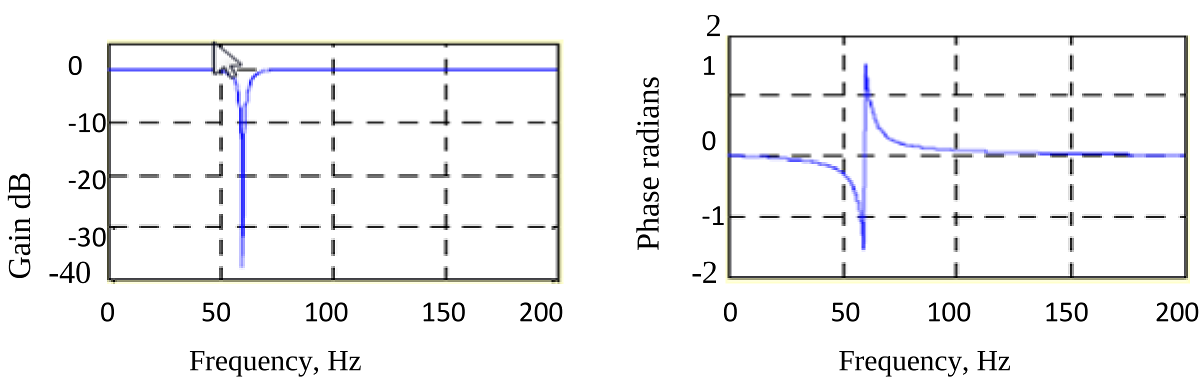


Figure 8. Gain and phase response

The design of a lowpass Butterworth digital filter with

$\omega_p = 0.25\pi,\ \omega = 0.55\pi,\ \alpha_p \le 0.5dB,\ and\ \alpha_s \ge 15\ dB$

Thus $\varepsilon^2 = 0.1220185$ $A^2 = 31.622777$

If $\left|G\left(e^{j0}\right)\right| = 1 \textit{this implies } 20\log_{10}\left|G\left(e^{j0.25\pi}\right)\right| \ge -0.5$
$20\log_{10}\left|G\left(e^{j0.25\pi}\right)\right| \le -15$

In prewarping we get

$$\Omega_s = \tan(\omega_p/2) = \tan(0.25\pi/2) = 0.4142136$$
$$\Omega_s = \tan(\omega_s/2) = \tan(0.55\pi/2) = 1.1708496$$

The inverse transition ratio is $\frac{1}{k} = \frac{\Omega_s}{\Omega_p} = 2.8266809$

The inverse discrimination ratio is $\frac{1}{k} = \frac{\sqrt{A^2 - 1}}{\varepsilon} = 15.841979$

Thus $N = \frac{\log_{10}(1/k_1)}{\log_{10}(1/k)} = 2.6586997$

We choose $N = 3$. To determine Ω_c we use $\left|H_a\left(j\Omega_p\right)\right|^2 = \frac{1}{1 + (\Omega_p/\Omega_c)^{2N}} = \frac{1}{1+\varepsilon^2}$ Eq. 21

We then get $\Omega_c = 1.419915(\Omega_p) = 0.588148$.

3rd-order lowpass Butterworth transfer function for $\Omega_c = 1$ is H $H_{an}(s) = \frac{1}{(s+1)(s^2+s+1)}$

Eq. 22

Denormalizing to get $\Omega_c = 0.588148$ we arrive at $H_a(s) = H_{an}\left(\frac{s}{0.588148}\right)$

Applying bilinear transformation to $H_a(s)$ we obtain the desired digital transfer function; G(z) =

$$H_a(s)\Big|_{s=\frac{1-z^{-1}}{1+z^{-1}}} \qquad \text{Eq. 23}$$

The magnitude and gain responses of G(z) are shown below;

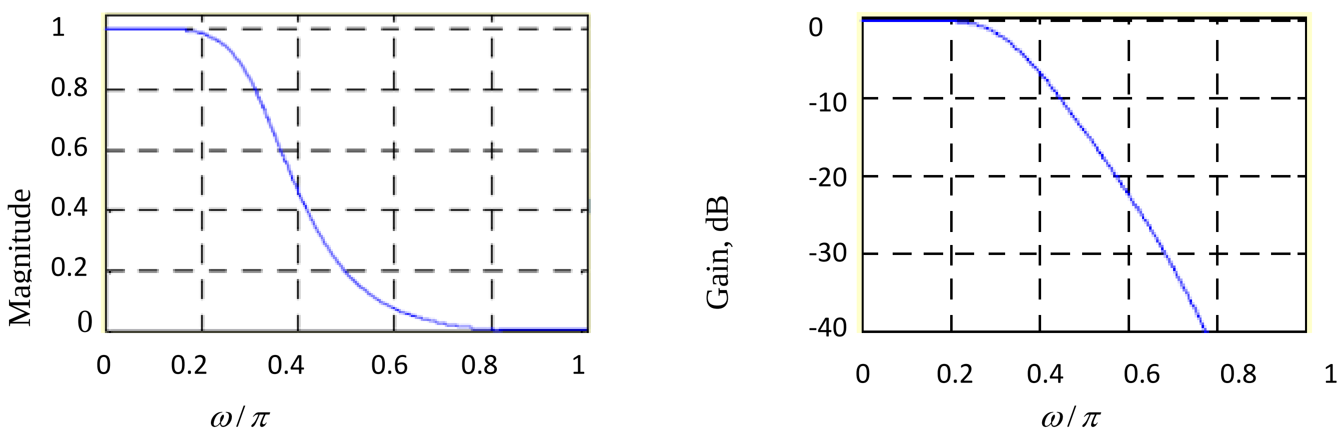


Figure 9 . Magnitude and Gain response

Designing IIR Highpass, Bandpass, and Bandstop Digital Filters.

First approach:

1. Prewarp digital frequency specifications of desired digital filter $G_D(z)$ to arrive at frequency specifications of analog filter $H_D(s)$ of same type.
2. Convert frequency specifications of $H_D(s)$ into that of prototype analog lowpass filter $H_{LP}(s)$.
3. Design analog lowpass filter $H_{LP}(s)$.
4. Convert $H_{LP}(s)$ into $H_D(s)$ using inverse frequency transformation used in step 2.
5. Design desired digital filter $G_D(z)$ by applying bilinear transformation to $H_{LP}(s)$.

Second approach [1].

1. Prewarp digital frequency specifications of desired digital filter $G_D(s)$ to arrive at frequency specifications of analog filter $H_D(s)$ of same type.
2. Convert frequency specifications of $H_D(s)$ into that of prototype analog lowpass filter $H_{LP}(s)$.
3. Design analog lowpass filter $H_{LP}(s)$.
4. Convert $H_{LP}(s)$ into an IIR digital transfer function $G_{LP}(z)$ using bilinear transformation.
5. Transform $G_{LP}(z)$ into the desired digital transfer function $G_D(z)$.

We then illustrate the first approach,

- Design of a type 1 Chebyshew IIR digital highpass filter
- Specifications: F_p = 700hz, F_s= 500Hz, α_p=1dB, α_s = 32dB, F_T=2 kHz.
- Normalized angular bandedge frequencies.

$$\omega_p = \frac{2\pi F_p}{F_T} = \frac{2\pi x 700}{2000} = 0.7\pi$$
$$\omega_s = \frac{2\pi F_s}{F_T} = \frac{2\pi x 500}{2000} = 0.5\pi$$

Eq. 24 and 25 respectively

Prewarping these frequencies we get $\Omega_p = \tan(\omega_p / 2) = 1.9626105$
$\Omega_s = \tan(\omega_s / 2) = 1.0$

For the prototype analog lowpass filter choose $\Omega_p = 1$ using $\Omega = -\frac{\Omega_p \hat{\Omega}_p}{\hat{\Omega}}$ we get

$\Omega_p = 1.962105$

Analog lowpass filter specifications: Ω_p = 1, Ω_s =1.926105, α_p=1dB, α_s =32dB

The design of an elliptic IIR digital bandstop filter. Its specifications include; ω_{s1} = 0.45, ω_{s2} = 0.65, $\omega p1$ = 0.3π, ω_{p2}=0.75, α_p=1dB, α_s= 40dB [1].

Prewarping we get $\hat{\Omega}_{s1}$ = 0.8540806, $\hat{\Omega}$ = 1.6318517, $\hat{\Omega}$ = 0.5095254, $\hat{\Omega}$ = 2.4142136. The width of the stopband is

$$\hat{\Omega}_0^2 = \hat{\Omega}_{s2}\hat{\Omega}_{s1} = 1.393733$$
$$\hat{\Omega}_{p2}\Omega_{p1} = 1.230103 \neq \hat{\Omega}_o^2$$

We therefore modify $\hat{\Omega}_{p1}$ so that $\hat{\Omega}_{p1}$ and $\hat{\Omega}_{p2}$ exhibit geometric symmetry with respect to $\hat{\Omega}_o^2$.

Then we set $\hat{\Omega}_{p1}$ = 0.577303 and for the prototype analog lowpass filter we choose $\hat{\Omega}_s$ = 1 using $\Omega = \Omega_s \dfrac{\hat{\Omega}B_w}{\hat{\Omega}_o^2 - \hat{\Omega}^2}$ we get $\Omega_p = \dfrac{0.5095254x0.777771}{1.393733 - 0.3332787} = 0.4234126$ [1].

Conclusion.

It is very lucid from the above explanations that in the design of a Digital Filter, a number of factors or design specifications such as ω_p- passband edge frequency, ω_s - stopband edge frequency, δ_p - peak ripple value in the passband and δs - peak ripple value in the stopband are very essential and should be given the necessary attention.

Meanwhile, in practice, passband edge frequency F_p and stopband edge frequency F_s are specified in Hz. For digital filter design, normalized bandedge frequencies need to be computed from specifications and both the FIR and IIR digital Filters must be considered when designing digital Filters.

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