<u>Title</u>:

The relationship among FT, DTFT, DFT, and z-transform.

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Introduction

The purpose of this paper is to explain the Relationship of FT, DTFT, DFT and Z-transform. The Fourier Transform gives us a unique and powerful way of viewing these waveforms. Fourier transformation is used to decompose time series signals into frequency components each having an amplitude and phase. Using the inverse Fourier transformation the time series signal can be reconstructed from its frequency-domain representation. Fourier transformation is one of the most important concepts in digital signal processing and is not only used for estimating the spectral distribution of a signal in the frequency domain (the power spectrum). Fourier transformation is also the foundation of coherence analysis and certain types of so-called surrogate signals. Finally, the Fourier transformation is implemented in many DSP (Digital Signal Processing) routines because any mathematical operation in the time domain has an equivalent operation in the frequency domain that is often computationally faster. Thus, Fourier transformation is occasionally implemented solely to speed up algorithms. Using the inverse Fourier transformation, the time-domain signal is reconstructed from its frequency domain representation.

In this tutorial we will study the formula for computing the discrete Fourier transform (DFT), DTFT AND Z-TRANSORM order to keep track on the indices in the FT formula (which most people consider complicated and abstract when working with long signals). One “problem” with large time series is that you get as many frequency components in the frequency domain as you have data samples in the time domain. Once you have understood this simple exercise, you can easily invent a slightly more complicated example (longer signal and multiple frequency components) or intuitively accept that the idea also applies to longer signals.

The Fourier Transform

As we have seen, any (sufficiently smooth) function f(t) that is periodic can be built out of sin's and

cos's. We have also seen that complex exponentials may be used in place of sin's and cos's. We shall now use complex exponentials because they lead to less writing and simpler computations, but yet can easily be converted into sin's and cos's. If f(t) has period 2ℓ, its (complex) Fourier series expansion is

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{ik\frac{\pi}{\ell}t} \qquad \text{with} \qquad c_k = \frac{1}{2\ell}\int_{-\ell}^{\ell} f(t)e^{-ik\frac{\pi}{\ell}t}\,dt$$

Not surprisingly, each term $c_k e^{ik\frac{\pi}{\ell}t}$ **in this expansion also has period 2ℓ,**

because $c_k e^{ik\frac{\pi}{\ell}(t+2\ell)} = c_k e^{ik\frac{\pi}{\ell}t}$.

We now develop an expansion for non-periodic functions, by allowing complex exponentials (or equivalently sin's and cos's) of all possible periods, not just 2ℓ, for some fixed ℓ. So, from now on, do not assume that f(t) is periodic.

For simplicity we'll only develop the expansions for functions that are zero for all sufficiently large |t|.

With a little more work, one can show that our conclusions apply to a much broader class of functions.

Let L > 0 be sufficiently large that f(t) = 0 for all |t| ≥ L. We can get a Fourier series expansion for the part of f(t) with −L < t < L by using the periodic extension trick. Define FL(t) to be the unique function determined by the requirements that

$$i)\ F_L(t) = f(t) \quad \text{for} \quad -L < t \leq L$$

$$ii)\ F_L(t) \text{ is periodic of period } 2L$$

Then, for $-L < t < L$,

$$f(t) = F_L(t) = \sum_{k=-\infty}^{\infty} c_k(L) e^{ik\frac{\pi}{L}t} \qquad \text{where} \qquad c_k(L) = \frac{1}{2L} \int_{-L}^{L} f(t) e^{-ik\frac{\pi}{L}t}\, dt$$

If we can somehow take the limit L → ∞, we will get a representation of f that is valid for all t's , not just those in some finite interval −L < t < L. This is exactly what we shall do, by the simple expedient of interpreting the sum in (2) as a Riemann sum approximation to a certain integral. For each integer k define the kth frequency to be

be $\omega_k = k\frac{\pi}{L}$ and denote by $\Delta\omega = \frac{\pi}{L}$ the spacing, $\omega_{k+1} - \omega_k$, between any two successive frequencies. Also define $\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt$. Since $f(t) = 0$ for all $|t| \geq L$

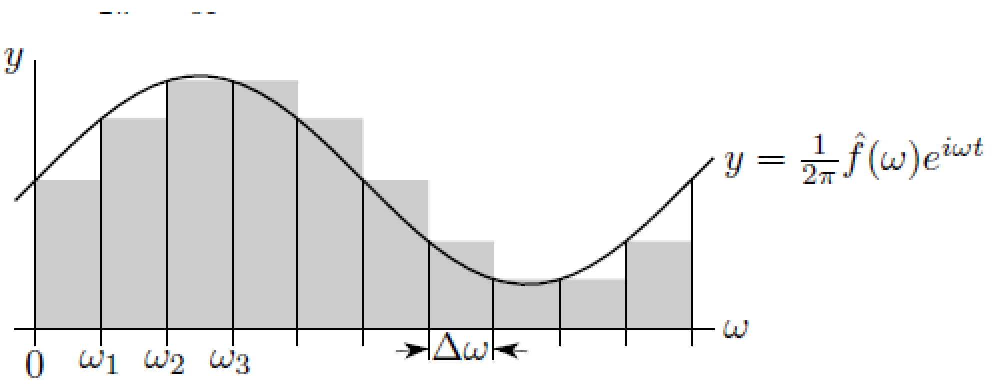
$$c_k(L) = \frac{1}{2L} \int_{-L}^{L} f(t) e^{-ik\frac{\pi}{L}t}\, dt = \frac{1}{2L} \int_{-\infty}^{\infty} f(t) e^{-i(k\frac{\pi}{L})t}\, dt = \frac{1}{2L} \int_{-\infty}^{\infty} f(t) e^{-i\omega_k t}\, dt = \frac{1}{2L} \hat{f}(\omega_k) = \frac{1}{2\pi} \hat{f}(\omega_k)\, \Delta\omega$$

In this notation,

$$f(t) = F_L(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \hat{f}(\omega_k) e^{i\omega_k t} \Delta\omega$$

for any $-L < t < L$. As we let $L \to \infty$, the restriction $-L < t < L$ disappears, and the right hand

side converges exactly to the integral $\frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t}\, d\omega$. To see this, cut the domain of integration into



small slices of width $\Delta\omega$ and approximate, as in the above figure, the area under the part of the graph of $\frac{1}{2\pi}\hat{f}(\omega)e^{i\omega t}$ with ω between ω_k and $\omega_k + \Delta\omega$ by the area of a rectangle of base $\Delta\omega$ and height $\frac{1}{2\pi}\hat{f}(\omega_k)e^{i\omega_k t}$. This approximates the integral $\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{i\omega t}\, d\omega$ by the sum $\frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\hat{f}(\omega_k)e^{i\omega_k t}\Delta\omega$. As $L \to \infty$ the approximation gets better and better so that the sum approaches the integral. So taking the limit of (3) as $L \to \infty$ gives

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{i\omega t}\, d\omega \qquad \text{where} \qquad \hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\, dt$$

The function $\hat{f}$ is called the Fourier transform of f. It is to be thought of as the frequency profile of the signal $f(t)$.

Properties of the Fourier Transform

Linearity

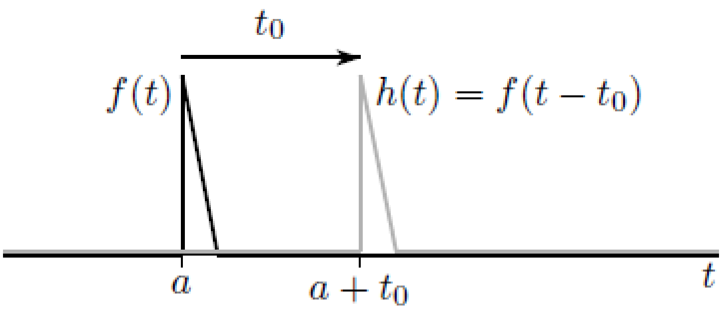
If α and β are any constants and we build a new function h(t) = αf(t) + βg(t) as a linear combination of two old functions f(t) and g(t), then the Fourier transform of h is

$$\hat{h}(\omega) = \int_{-\infty}^{\infty} h(t)e^{-i\omega t}\, dt = \int_{-\infty}^{\infty}[\alpha f(t) + \beta g(t)]e^{-i\omega t}\, dt = \alpha\int_{-\infty}^{\infty} f(t)e^{-i\omega t}\, dt + \beta\int_{-\infty}^{\infty} g(t)e^{-i\omega t}\, dt$$
$$= \alpha\hat{f}(\omega) + \beta\hat{g}(\omega)$$

Time Shifting

Suppose that we build a new function h(t) = f(t − t0) by time shifting a function f(t) by t0. The easy way to check the direction of the shift is to note that if the original signal f(t) has a jump when its argument

t = a, then the new signal h(t) = f(t − t0) has a jump when t − t0 = a, which is at t = a + t0, t0 units to the right of the original jump.



The Fourier transform of h is

$$\hat{h}(\omega) = \int_{-\infty}^{\infty} h(t) e^{-i\omega t}\, dt = \int_{-\infty}^{\infty} f(t - t_0) e^{-i\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} f(t') e^{-i\omega(t'+t_0)}\, dt' \text{ where } t = t' + t_0,\ dt = dt'$$

$$= e^{-i\omega t_0} \int_{-\infty}^{\infty} f(t') e^{-i\omega t'}\, dt' = e^{-i\omega t_0} \hat{f}(\omega)$$

<u>Scaling</u>

If we build a new function $h(t) = f\left(\frac{t}{\alpha}\right)$ **by scaling time by a factor of α > 0, then the Fourier transform of h is**

$$\hat{h}(\omega) = \int_{-\infty}^{\infty} h(t)e^{-i\omega t}\, dt = \int_{-\infty}^{\infty} f\left(\tfrac{t}{\alpha}\right)e^{-i\omega t}\, dt$$

$$= \alpha \int_{-\infty}^{\infty} f(t')e^{-i\omega\alpha t'}\, dt' \text{ where } t = \alpha t',\ dt = \alpha dt'$$

$$= \alpha\hat{f}(\alpha\omega)$$

Introduction to Digital Signal and System Analysis

Z- Domain Analysis:-

The z-transform is defined as

$$\sum_{n=0}^{\infty} x[n]z^{-n} \qquad (1.1)$$

Where z is a complex variable. The transform defined by Eq. (1.1) is a unilateral transform as defined on one side of the axis $0 \le n < \infty$. In the transform, each sample

X[n] is multiplied by the complex variable z^{-n} i.e..

X [0]	**X [1]**	**X [2]**	**X [3]**
z^0	z^{-1}	z^{-2}	z^{-3}

There is advantage in this unilateral transform definition as it can avoid mathematical inconvenience. One can shift the signal of interest to obtain a required origin in its analysis, thus usual y causing no trouble in applications. The inverse z-transform can be found by

X [n] = $\frac{1}{2\pi j} \oint x[z]\, z^{n-1} dz$ **(1.2)**

It involves contour integration, and further discussion is beyond the scope of this basic content. However, an alternative approach is available using partial fractions together with z-transform formulas of basic functions. Table 1.1 lists the basic properties of the z transform and Table 1.2 lists some basic z-transform pairs.

<u>Relationship between z-transform and Fourier transform</u>

Let z = exp(jΩ) , i.e. the complex variable z is only allowed on the unit circle, the z-transform becomes a unilateral Fourier transform

X [Ω] = $\sum_{n=0}^{\infty} x[n] \exp(-j\Omega)$

Obviously, apart from on the unit circle, the complex operator z can be specified into other curves or region, if necessary. Later, it will be shown the unit circle is important boundary on the z-domain

1.3) Z as time shift operator

Multiplying by z implies a time advance and dividing by z, or multiplying by z^{-1} , is to cause a time delay. For the unit

X(z) = $\sum \ \partial[n]\, z^{-n}|_{n=0} = 1$

For the delayed unit impulse,

$$X(z) = \sum_{n=0}^{\infty} \delta\ [n-1] z^{-n} = z^{-n}\Big|_{n=1} = z^{-1}$$

For more general cases of shifting by n_0 samples,

$$X(z) = \sum_{n=0}^{\infty} \delta\ [n-n_0] z^{-n} = z^{-n}\Big|_{n=n_0} = z^{-n_0}$$

For shifted signal x[n], i.e. delayed by n_0 samples, the z-transform

$$\sum_{n=0}^{\infty} x\ [n-n_0]\ u[n-n_0]z^{-n} = X(z)z^{-n_0}$$

1.4 Transfer function

The transfer function describes the input-output relationship, or the transmissibility between input and output, in the z-domain. Applying the z-transform to the output of a system, the relationship between the z-transforms of input and output can be found:

$$Y(z) = \sum_{n=0}^{\infty} y[n]z^{-n} = \sum_{n=0}^{\infty}\ \sum_{r=\infty}^{\infty} x[r]h[n-r]z^{-n}$$

$$= \sum_{r=\infty}^{\infty} x(r)z^{-r} \sum_{n=0}^{\infty}\ h[n-r]z^{-(n-r)}$$

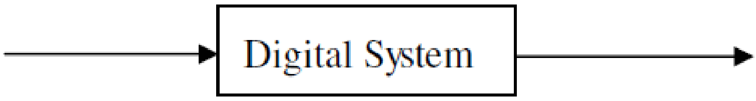
$$= \sum_{r=\infty}^{\infty} x(r)z^{-r} \sum_{m=-r}^{\infty}\ h[m]z^{-m} = X(z)H(z)$$

Therefore,

$$H(z) = \frac{Y(z)}{X(z)}$$

i.e., the transfer function can be obtained from the z-transforms of input and output.

Alternatively, the transfer function *H* (z) can be obtained by applying z-transform directly to the impulse response h[n]. The relationships of input (x [n]and x[z]), output (Y (n) and Y (z)) and the system function (h (n) and h (z)) in the time and z domains are depicted in fig 4.1



	Input		Output
Time domain	$x[n]$	$h[n]$	$y[n] = x[n] * h[n]$
	$\updownarrow$	$\updownarrow$	$\updownarrow$
z-domain	$X(z)$	$H(z)$	$Y(z) = X(z)H(z)$

Input and output relationships in the time domain and z-domain (4.1)

Definition or property	Signal	z-transform
z-transform definition	x[n]	$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$
Inverse z-transform	$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$	$X(z)$
Linearity	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$
Time-shifting property	$x\ [n-n_0]u[n-n_0]$	$X(z)z^{-n_0}$
Convolution	$x[n] * y[n]$	$X(z)Y(z)$

Transform definition and properties.

❖ **Introduction to Digital Signal and System Analysis**

Discrete Fourier Transform

Definition of discrete Fourier transform For a digital signal x[n],

The discrete Fourier transform (DFT) is defined as

X [K]= $\sum_{n=0}^{N-1} x[n]\exp(-j*(\frac{2\pi kn}{N}))$ **1.2**

Where the DFT ***X*****[k] is a discrete periodic function of period N. Therefore one period of distinct values are only taken at** ***k*** **= 0, 1, 2,… .N-1**

Relationship between DFT and DTFT

: Since the DTFT ***X*** **(ω) is a function of the samples** ***x*****[*****n*****], it should be clear that in this case, it is possible to completely reconstruct the DTFT** ***X*** **(ω) from the** ***N*****-point DFT** ***X*****[*****k*****].**

First consider the

frequency□□ω***k*****=2π*****k*****/*****N*****:**

$$X(\omega_k) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega_k n}$$

$$= \sum_{n=0}^{N-1} x[n]e^{-j\omega_k n} = X[k]$$

The general case, i.e.□ω arbitrary, is handled by the following theorem.

<u>Theorem</u>: Let *X*(ω) and *X*[*k*] respectively denote the DTFT and *N*-point DFT of signal *x*[*n*]. If *x*[*n*] = 0

For *n* < 0 and for

***n* ≥0, then:**

$$X(\omega) = \sum_{k=0}^{N-1} X[k] P(\omega - \omega_k)$$

$$P(\omega) \triangleq \frac{1}{N} \sum_{n=0}^{N-1} e^{-j\omega n}$$

$$X(\omega) = \sum_{k=0}^{N-1} X[k] P(\omega - \omega_k)$$

$$P(\omega) \triangleq \frac{1}{N} \sum_{n=0}^{N-1} e^{-j\omega n}$$

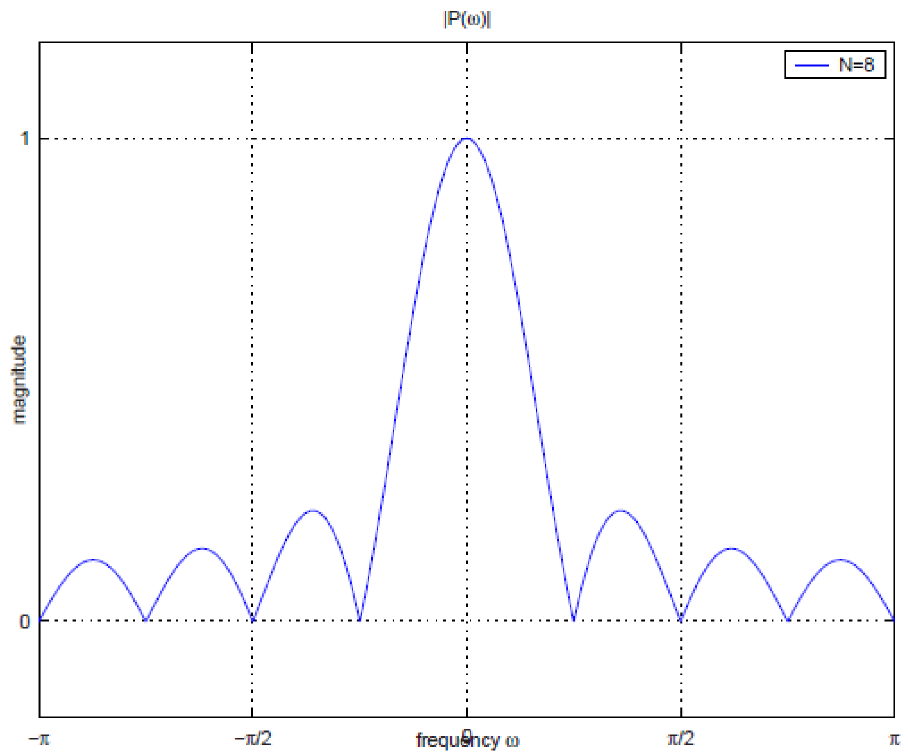
Properties P(ω)

- Periodicity: $P(\omega+2\pi) = P(\omega)$
- If $\omega = 2\pi l$ (l integer), then $e^{-j\omega n} = e^{-j2\pi l n} = 1$ so that $P(\omega) = 1$.
- If $\omega \neq 2\pi l$, then

$$\begin{aligned} P(\omega) &= \frac{1}{N}\frac{1-e^{-j\omega N}}{1-e^{j\omega}} \\ &= \frac{1}{N}e^{-j\omega(N-1)/2}\frac{\sin(\omega N/2)}{\sin(\omega/2)} \end{aligned}$$

- Note that at frequency $\omega_k = 2\pi k/N$:

$$P(\omega_k) = \begin{cases} 1 & k = 0, \\ 0 & k = 1, \ldots, N-1. \end{cases}$$



Magnitude spectrum of the frequency interpolation function $P(\omega)$ for $N = 8$.

Remarks: More generally, suppose that $x[n] = 0$ for $n < 0$ and for $n \geq L$. As long as the DFT size N is

Larger than or equal to L, i.e. $N \geq L$, the results of this section apply. In particular:

- **One can reconstruct x[n] entirely from the DFT samples X[k]**

- **Also, from the theorem: $X[k] = X(\omega_k)$ at $\omega_k = 2\pi k/N$.**

Increasing the value of N beyond the minimum required value, i.e. N = L, is called zero-padding:

•□□x [0]... x [L□– 1]□□□□⇒□□x[0], ..., x[L□– 1], 0, ..., 0□□

• The DFT points obtained give a nicer graph of the underlying DTFT because□$\Delta\omega_k = 2\pi/N$ is smaller.

• However, no new information about the original signal is introduced by increasing N in this way.

Inverse DFT: In this case, $x[n]$ can be recovered entirely from its N-point DFT $X[k]$, $k = 0, 1, ..., N$□-1. Let $\tilde{x}[n]$ denote the IDFT of $X[k]$, as defined in (6.14). For $n = 0, 1, ..., N$□-1, the IDFT theorem yields: $x[n] = \tilde{x}[n]$. Since both $\tilde{x}[n]$ and $x[n]$ are known to be N-periodic, it follows that $x[n]$ $\tilde{x}[n]$ must also be true for $n <$ 0 and for n□$\geq N$. Therefore

$x[n] = \tilde{x}[n]$,□ $\forall n$□$\in \mathbb{Z}$

<u>Relationship between DFT and DTFT</u>: Since the N-periodic signal $x[n]$ can be recovered completely

From its N-point DFT $X[k]$, it should also be possible in theory to reconstruct the DTFT $X(\omega)$ from $X[k]$.

The situation is more complicated here because the DTFT of a periodic signal contains infinite impulses.

The desired relationship is provided by the following theorem.

Theorem: Let $X(\omega)$ and $X[k]$ respectively denote the DTFT and N-point DFT of N-periodic signal $x[n]$. then

$$X(\omega) = \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} X[k]\delta_a(\omega - \omega_k)$$

Where□$\delta_a(\omega)$ denotes an analog delta function centered at□$\omega = 0$.

Remarks on the Discrete Fourier series: In the special case when $x[n]$ is N-

periodic, i.e. $x[n + N] = x[n]$, the DFT admits a Fourier series interpretation. Indeed, the IDFT provides an expansion of $x[n]$ as a sum of Harmonically related complex exponential signals e^{jwnk}

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\omega_k n}, \quad n \in \mathbb{Z}$$

This expansion is called discrete Fourier series (DFS). The DFS coefficients $X[k]$ are identical to the DFT:

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\omega_k n}, \quad n \in \mathbb{Z}$$

Signal reconstruction via DTFT sampling introduce on Let $X(\omega)$ be the DTFT of signal $x[n]$, $n \in Z$, that is:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \quad \omega \in \mathbb{R}.$$

Consider the sampled values of $X(\omega)$ at uniformly spaced frequencies $\omega_k = 2\pi k/N$ for $k = 0, ..., N-1$.

Suppose we compute the IDFT of the samples $X(\omega_k)$:

$$\hat{x}[n] = \text{IDFT}\{X(\omega_k)\} = \frac{1}{N}\sum_{k=0}^{N-1} X(\omega_k)e^{j\omega_k n}$$

<u>PROPARITS OF</u>

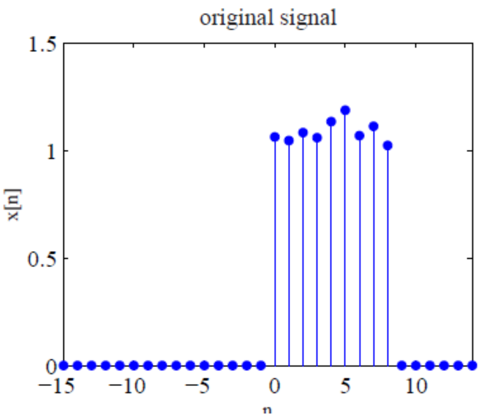
<u>DFT:-</u>

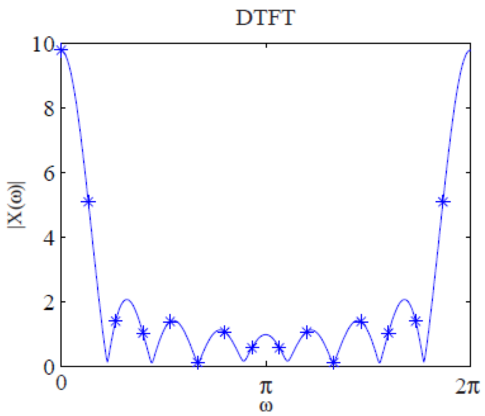
In this section, we assume that the signals $x[n]$ and $y[n]$ are defined over $0 \leq n \leq N-1$. Unless explicitly stated, we make no special assumption about the signal values outside this range. We denote the N-point DFT of $x[n]$ and $y[n]$ by $X[k]$ and $Y[k]$:

$$x[n] \overset{\text{DFT}_N}{\longleftrightarrow} X[k] \qquad x[n] \overset{\text{DFT}_N}{\longleftrightarrow} X[k]$$
$$y[n] \overset{\text{DFT}_N}{\longleftrightarrow} Y[k] \qquad y[n] \overset{\text{DFT}_N}{\longleftrightarrow} Y[k]$$

We view $X[k]$ and $Y[k]$ as N-periodic sequences,

defined for all $k \in Z$





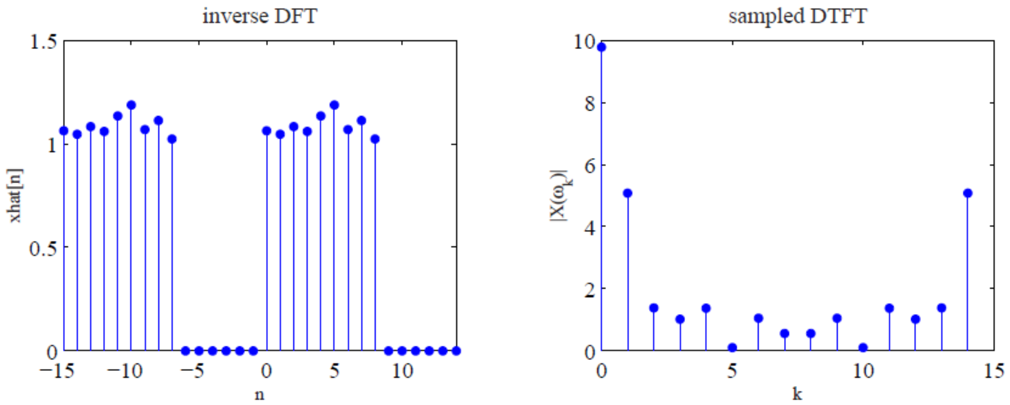


Illustration of reconstruction of a signal from samples of its DTFT. Top left: the original signal $x[n]$ is time limited. Top right, the original DTFT $X(\omega)$, from which 15 samples are taken. Bottom right: the equivalent impulse spectrum corresponds by IDFT to a 15-periodic Sequence $\hat{x}[n]$ shown on the bottom left. Since the original sequence is zero outside $0 \leq n \leq 14$, there is no overlap between replicas of the original signal in time.

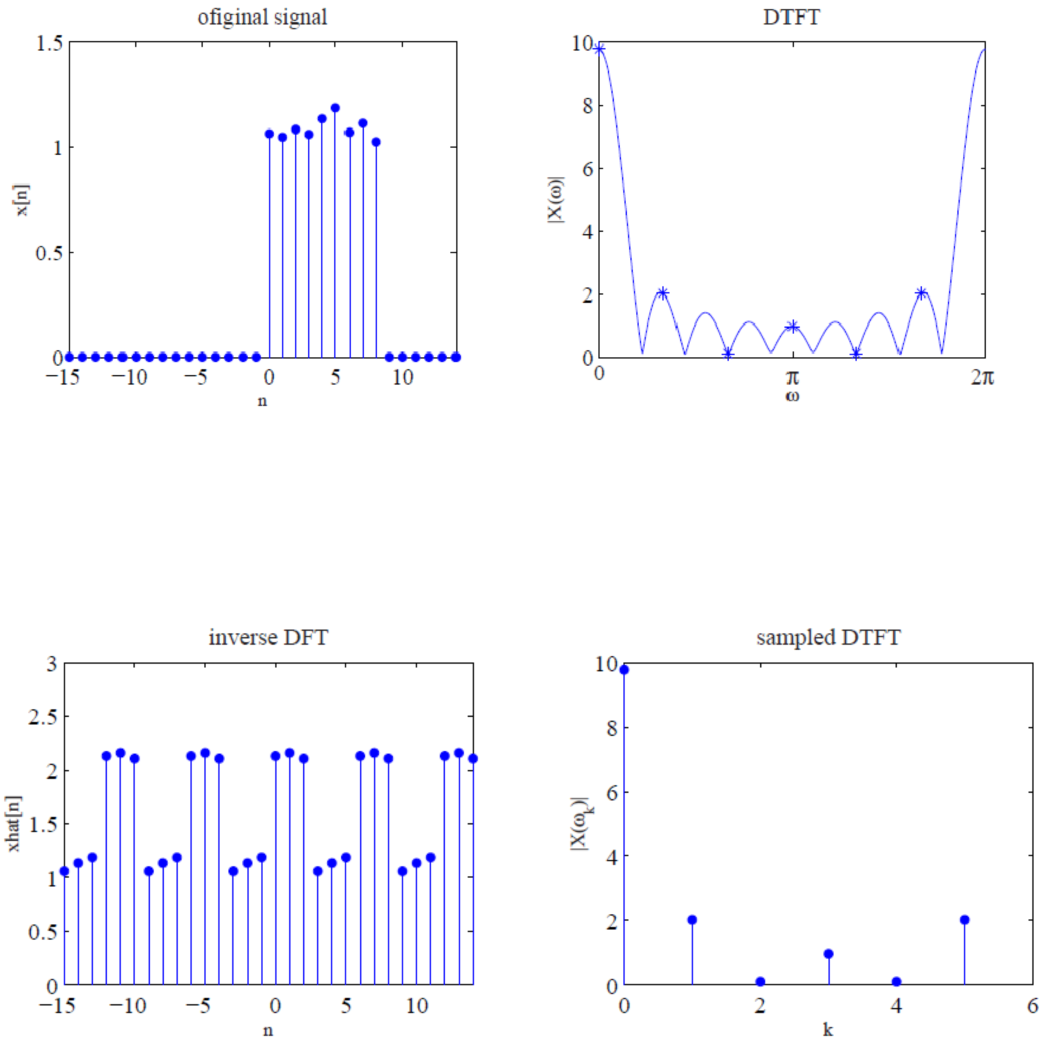


Illustration of reconstruction of a signal from samples of its DTFT. Top left: the original signal $x[n]$ is time limited. Top right, the original DTFT $X(\omega)$, from which 6 samples are taken. Bottom right: the equivalent impulse spectrum corresponds by IDFT to a 6-periodic sequence $\hat{x}[n]$ shown on the bottom left. Since the original sequence is not zero outside $0 \leq n \leq 5$, there is overlap between replicas of the original signal in time, and thus aliasing.

Conclusion:-

• The Fourier Transform provides us with a way to decompose an image into a weighted sum of sin and cos waves. Because of the convolution theorem, we can perform any linear operation by filtering (multiplication) in the frequency domain instead of convolution in spatial domain. The Discrete Time Fourier Tranform is a good way to analyze discrete time signals in the frequency domain in theory, but in application the infinite range of time and frequency in the conversion formulas can make analysis difficult, especially on the computer. The Discrete Fourier Transform is very similar to the DFT, but uses a finite time signal, allowing its formulas to be finite sums which a computer can easily calculate. Signals must be discrete and time-limited (or truncated) for use with the DFT. A discrete periodic signal can be used when only one period of the signal is analyzed. The DFT of a signal will be discrete and have a finite duration.

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