DIGITAL SIGNAL PROCESSING FINAL REPORT

RELATIONSHIP AMONG FOURIER TRANSFORM (FT), DISCRETE TIME FOURIER TRANSFORM (DTFT), DISCRETE FOURIER TRANSFORM (DTF) AND Z-TRANSFORM

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INTRODUCTION

Fourier transform in general expresses a function of time in terms of amplitude and phase of each frequencies that make it up. Fourier transform is a special case of Fourier series which expresses complicated but periodic functions as the sum of simple waves mathematically represented by sines and cosines.

Fourier Transform has proved to be very useful in many fields of engineering especially digital signal processing. Signals can be analyzed in either or both time and frequency. In quantum mechanics wave solutions can functions of either space or momentum. These things are made possible by the principle of Fourier Transform.

In Digital Signal Processing, when it becomes difficult to work with a signal in the time domain it can quickly be switched into the frequency domain where it might be easier and vice versa. For example, the operation of differentiation in the time domain corresponds to multiplication of frequency domain by a scalar. The Laplace transform is a perfect example of such case. Differential equations are easier solved using Laplace transform.

In time domain analysis of signals, it is not apparent how the input signal is changed by the system.

$$f\,(t) \longrightarrow \boxed{h\,(t)} \longrightarrow \quad y(t)=h(t)*f(t)=\int_{-\infty}^{\infty} f(ɾ)h(t-ɾ)dɾ$$

i.e. y(t) is a convolution of f(t) and h(t) where f(t) is the input signal and h(t) system response.

Consider a discrete signal x (t) which is to be processed by a system h[t]. The output of the system will be y[t] =x (t)*h (t). Assuming x (t) =$e^{-i\omega^{\circ}t}$ then the output response will be as follows: y (n) =$\int_{-\infty}^{\infty} e^{-i(\omega^{\circ}t-ɾ)}$h[ɾ]dɾ = $e^{-i\omega^{\circ}t}\int_{-\infty}^{\infty} e^{-i\omega^{\circ}ɾ}\,h(ɾ)$dɾ. $\int_{-\infty}^{\infty} h[ɾ]\,e^{-i\omega^{\circ}ɾ}$dɾ is denoted by H($e^{-i\omega t}$) and is a complex-valued function of ω, having a magnitude response of |H($e^{-i\omega t}$)| and phase response $\theta(e^{-i\omega t})$. Thus we have the following result:

y (n)= $e^{-i\omega^{\circ}t}$|H($e^{-i\omega t}$)|$e^{-i\theta(e^{-i\omega t})}$ which shows that when the input is a complex exponential function $e^{-i\omega t}$ the magnitude of the output y[n] is |H($e^{-i\omega t}$)| and the phase of the output y(n) is $(\omega^{\circ}+\theta)$. The same can be said when the input signal is sinusoidal i.e. x (t) = Re(A$e^{-i\omega t}$) =$A\cos(\omega t)$, then output y(t) is also a sinusoidal function given by

y (t)=A|H($e^{-i\omega t}$)|$\cos(\omega t+\theta)$. H ($e^{-i\omega t}$) is the frequency response of the system and it apparent how it affects the input signal. Hence signal processing of real life application are carried out in the frequency domain and there is the need to transform them from time to frequency.

FOURIER TRANSFORMS

Signal processing are carried out by computers. Computers however can only work with digital values and since real life signals e.g. heat, speech etc. are all continuous they are first digitized by A/D converter before the processing begins.

The discrete or digital signal are then transformed into the frequency domain using Discrete Time Fourier Transform (DTFT). So then an input discrete signal x[n] is changed into the frequency domain by DTFT (where X ($e^{-i\omega t}$) is the transform version of x[n]) and can be recovered using Inverse Discrete Time Fourier Transform (IDTFT).

$X(e^{i\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-i\omega n}$ --DTFT $\qquad$ $x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{i\omega})e^{i\omega t}\, d\omega$ –IDTFT

In general, X ($e^{-i\omega t}$) is a complex function of the real variable ω and can therefore be written as

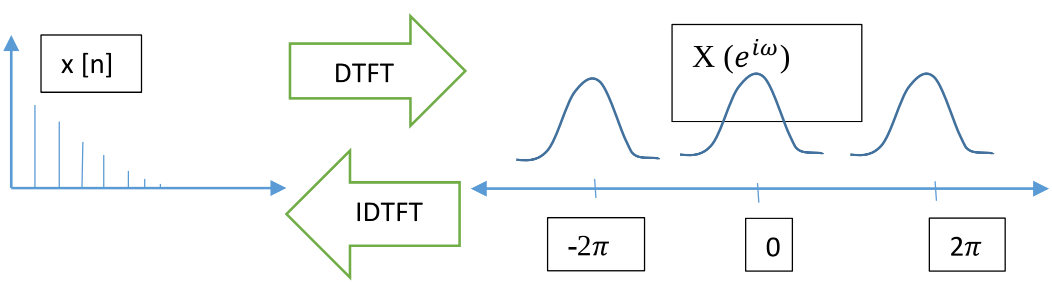
$X(e^{i\omega}) = X_{re}(e^{i\omega}) + X_{im}(e^{\omega t})$. It is also a continuous and a periodic function.

$X(e^{-i(\omega+2\pi k)}) = \sum_{n=-\infty}^{\infty} x[n]e^{-i(\omega+2\pi k)n}$ where k is an integer

$= \sum_{n=-\infty}^{\infty} x[n]e^{-i\omega n}]e^{-i2\pi kn}$

$=\sum_{n=-\infty}^{\infty} x[n]e^{-i\omega n}$

$= X(e^{i\omega})$



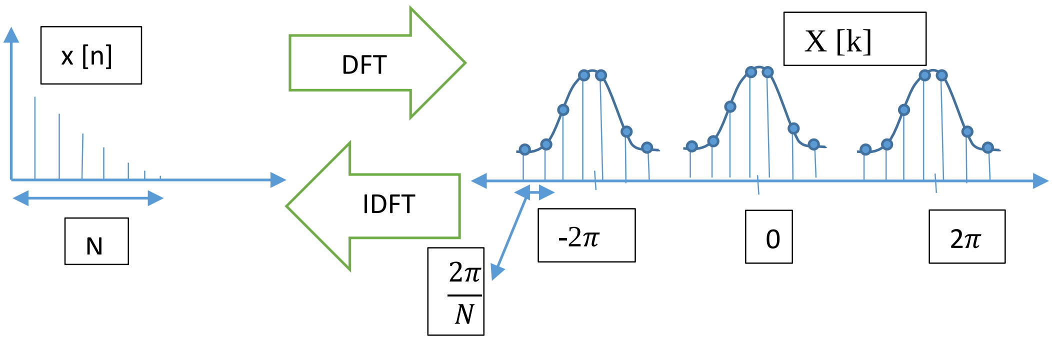
Computers cannot work with DTFT since it is continuous. If however this DTFT could be made discrete just as the original signal x (t) it would be possible for computer to work with them.

X ($e^{i\omega}$) is discretized by sampling uniformly on the ω-axis between 0 and 2π at $\omega_k = \frac{2\pi k}{N}$; $0 \leq k \leq N-1$.

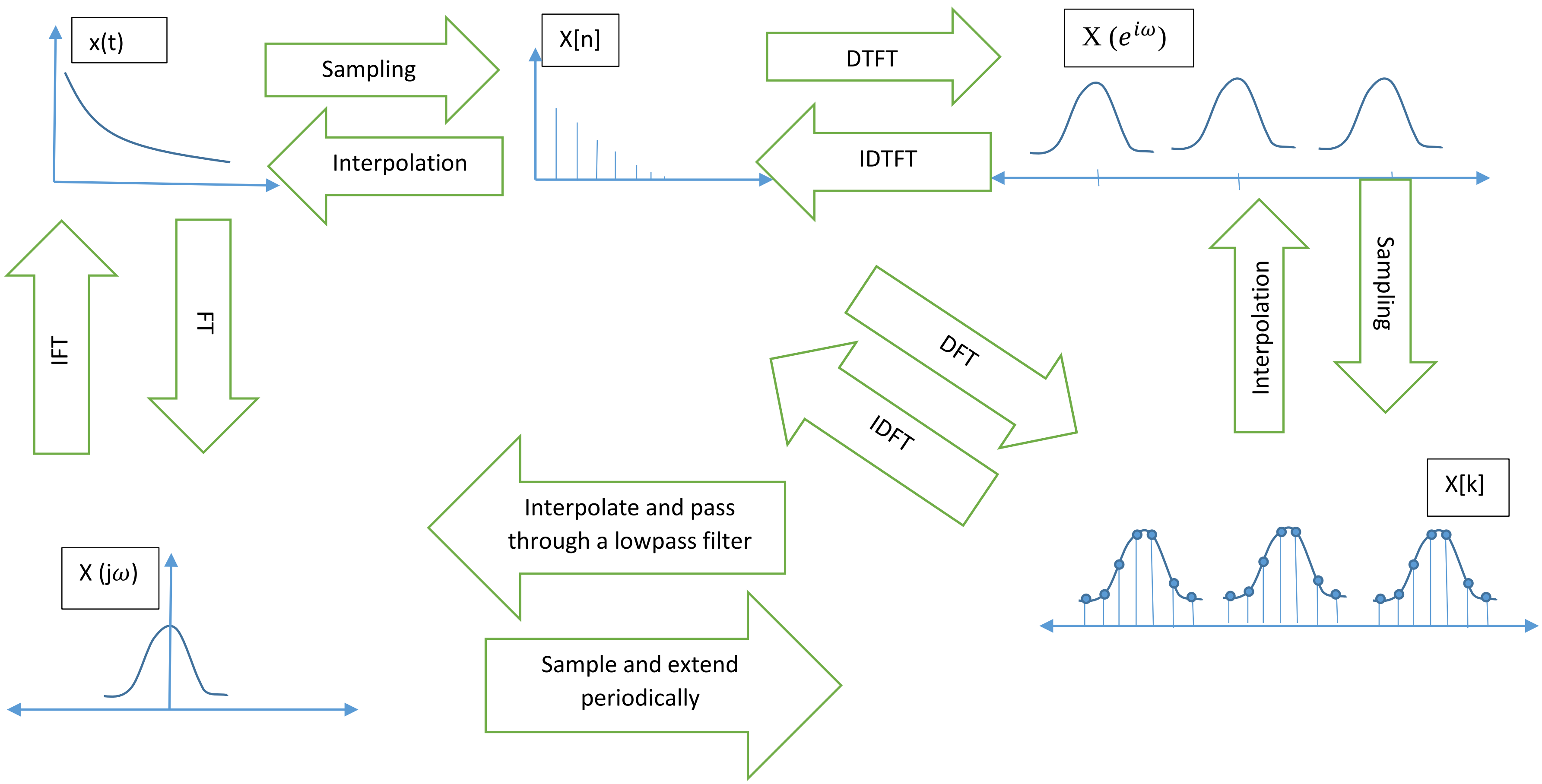
A direct transformation form x [n] to this sampled X ($e^{i\omega}$) is what is called Discrete Fourier Transform (DFT).i.e. X [k] = X $(e^{i\omega})\Big|_{\omega = \frac{2\pi k}{N}}$

And since X ($e^{i\omega}$) = $\sum_{n=-\infty}^{\infty} x[n]e^{-i\omega n}$

X [k] =$\sum_{n=0}^{N-1} x[n]e^{-i\frac{2\pi kn}{N}}$---DFT   x [n] = $\frac{1}{N}\sum_{n=0}^{N-1} X\ [k]e^{i\frac{2\pi kn}{N}}$ ------IDFT



RELATIONSHIP AMONG FOURIER TRANSFORM (FT), DISCRETE TIME FOURIER TRANSFORM (DTFT) AND DISCRETE FOURIER TRANSFOMER (DFT)



- It can be seen from the diagram above that DTFT is a periodic extension of FT in the frequency domain.
- DFT is also a sampled version of FT extended periodically in the frequency domain.

EXISTENCE OF A FOURIER TRANSFORM

Questions are raised about the conditions necessary for a Fourier transform to exist.

F (jω) = $\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$

F (jω) will only exist if = $\int_{-\infty}^{\infty} f(t)dt < \infty$. In other words F (j$\omega$) will only exist for signals which converge to a definite point.

The same conditions is applied to DTFT and since DFT is just a sampled version of DTFT they can both be analyzed the same way.

F ($e^{i\omega}$) = $\sum_{n=-\infty}^{\infty} f[n]e^{-i\omega n}$

F ($e^{i\omega}$) will only be defined if the summation$\sum_{n=-\infty}^{\infty} f[n < \infty]$. This condition is called in the absolute summability of a sequence.

Let $F_k(e^{i\omega}) = \sum_{n=-k}^{k} f[n]e^{-i\omega n}$; where is an integer

$F_k(e^{i\omega})$ will surely be defined because the *f[n]* is finite. Therefore the error criterion for this condition is:

$\lim_{k\to\infty} \left|\text{F (j}\omega) - \text{Fk}(e^{i\omega})\right| \to 0$

However there are sequences which do not satisfy the condition above but as yet have a fourier transform. Absolutely summability of a sequence is therefore NOT A NECESSARY condition for the existence of a Fourier Transform but A SUFFICIENT CONDITION.

A typical example to buttress the above point is the sequence x[n] = $\frac{1}{n}u[n-1]$ is not absolutely summable because $\sum_{1}^{\infty}\frac{1}{n}$ does not converge but $\sum_{1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$. x[n] is square summable and has a Fourier transform. Its error criterion is

$\lim_{k\to\infty} \int_{-\pi}^{\pi} \left|\text{X (j}\omega) - \text{Xk}(e^{i\omega})\right|^2 \to 0$

The sequence is u[n] is neither absolutely summable nor square summable but has a Fourier Transform. This goes on to prove that both conditions are NOT NECESSARY conditions for the existence of a Fourier Transform. However condition 1 is seen as stronger condition according to the Cauchy-Schwarz inequality.

$\int |f(x)g(x)dx|^2 \leq \int |f(x)|^2 dx . \int |g(x)|^2 dx$

For such sequence which do not follow any of the above mentioned conditions the delta function can be used to prove their Fourier Transform eg. U[n],$\propto^n$, $e^{-i\omega n}$, cosωn,sinωn.

Given the sequence $\alpha^n u[n]$, its summation is given by $\sum_0^\infty \propto^n = \frac{1}{1-\alpha}$. This only holds when $|\alpha| < 1$ i.e. $\alpha^n u[n]$ converges given that the magnitude of of α is less than one. α can be viewed as the convergence factor for the sequence.

From the definition of DTFT

X $(e^{i\omega})$ = $\sum_{n=-\infty}^{\infty} x[n]e^{-i\omega n}$

As already stated, X $(e^{i\omega})$ will only be defined if the sequence x[n] converge. If x[n] however does not converge, a convergence factor can be introduced so that X $(e^{i\omega})$ becomes defined

Z-transform applies this principle. It multiplies every sequence by r^{-n}. The Z-transform of x[n] is defined by the equation below.

X (z) = $\sum_{n=-\infty}^{\infty} r^{-n}x[n]e^{-i\omega n}$

$= \sum_{n=-\infty}^{\infty} x[n](re^{i\omega})^{-n}$; z = $re^{i\omega}$

$= \sum_{n=-\infty}^{\infty} x[n]z^{-n}$

It can be seen that when $r = 1$ X (z) = X $(e^{i\omega})$ or X (z) = X $(e^{i\omega})$ when $|z| = 1$

In DTFT, the Region of Convergence (ROC) is only limited to a unit a circle on the z-plane but that is not the same case for the z-transform, r can be as small as zero and as big as infinite. It is therefore prudent to specify the ROC of every Z-transform.