REPORT ON

Discuss the relationship among FT, DTFT, DFT, and z-transform.

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INTRODUCTION

The **Fourier transform** Very commonly it transforms a mathematical function of time, $f(t)$ into a new function, sometimes denoted by $\hat{f}$ or F, whose argument is frequency with units of cycles or radians per second. The new function is then known as the **Fourier transform** and/or the frequency spectrum of the function f . The DTFT frequency-domain representation is always a periodic function. Since one period of the function contains all of the unique information, it is sometimes convenient to say that the DTFT is a transform to a "finite" frequency-domain (the length of one period), rather than to the entire real line. The **discrete Fourier transform** (**DFT**) converts a finite list of equally-spaced samples of a function into the list of coefficients of a finite combination of complex sinusoids, ordered by their frequencies, that has those same sample values. It can be said to convert the sampled function from its original domain (oftentimes or position along a line) to the frequency domain. The **Z-transform** converts a discrete time-domain signal, which is a sequence of real or complex numbers, into a complex frequency-domain representation. It can be considered as a discrete-time equivalent of the Laplace transform.

ABSTRACT

The basic definitions of Fourier Transfer, DTFT, DFT and Z-Transform are given which is further represented mathematically by giving their condition and relation between them is shown mathematically and diagrammatically which is further clarified with few examples.

Fourier Transform: - Fourier transform, a powerful mathematical tool for the analysis of non-periodic functions, the process of decomposing a function into simpler pieces is often called Fourier analysis; The decomposition process itself is called a Fourier transform. The motivation for the Fourier transform comes from the study of Fourier series. In the study of Fourier series, complicated but periodic functions are written as the sum of simple waves mathematically represented by sines and cosines.

$$F_n(x) = a_0 + \sum_{k=1}^{k=n}\left(\left(a_k cos(kx) + b_k sin(kx)\right)\right)$$

The constants a_0, a_k, b_k are coefficients of $F_n(x)$.

Fourier series for Continuous-time Periodic Signals: - If $x(t)$ be the periodic signal with fundamental period $T_p = \frac{1}{F_0}$ then Fourier series is represented by the following Synthesis equation,

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi F_0 t} \tag{1}$$

Dirichlet conditions guarantee the synthesis of above equation

1. The signal $x(t)$ has finite number of discontinuities in any periods.
2. The signal $x(t)$ contains a finite number of maxima and minima during any period.
3. The signal $x(t)$is absolutely integrable in any period, that is,

$$\int_{T_p} |x(t)|\, dt < \infty$$

Where coefficient is specified by Analysis equation

$$c_k = \frac{1}{T_p} \int_{T_p} x(t)\, e^{j2\pi k F_0 t} dt \qquad (2)$$

Since the c_k is complex conjugate we rewrite above equation (1) in form of

$$x(t) = a_0 + \sum_{k=1}^{\infty} (a_k cos2\pi k F_0 t - b_k sin2\pi k F_0 t) \qquad (3)$$

Where

$$a_0 = c_0$$

$$a_k = 2|c_k| cos\theta_k$$

$$b_k = 2|c_k| sin\theta_k$$

We can illustrate Fourier series and its transform by taking the example of square wave

We have

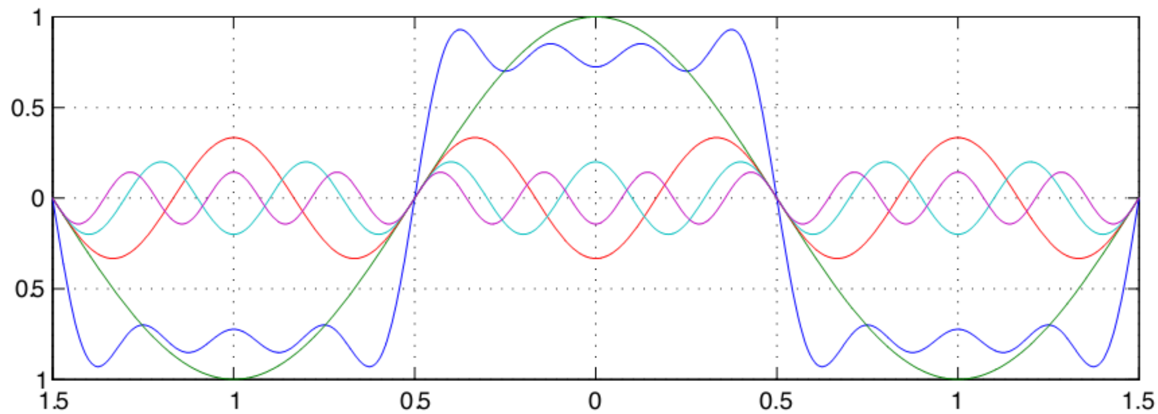
$$x(t) = a_0 + \sum_{k=1}^{m} (a_k cos2\pi k F_0 t + \theta_k)$$

for square wave

θ_k=0, and $a_{k=} \begin{cases} (-1)^{\frac{k-1}{2}} \frac{1}{k} & k = 1,3,5 \ldots\ldots \\ 0 \quad otherwise & \end{cases}$

then

$$x(t) = cos\left(\frac{2\pi}{T} t\right) - \frac{1}{3} cos\left(\frac{2\pi}{T} 3t\right) + \frac{1}{5} cos\left(\frac{2\pi}{T} 5t\right) - \ \ldots \ldots .. \qquad (4)$$



Fourier domain

$x(t)$ is equivalently described by its Fourier series parameters:

$$a_k = (-1)^{\frac{k-1}{2}} \frac{1}{k} \qquad k = 1,3,5 \ldots$$

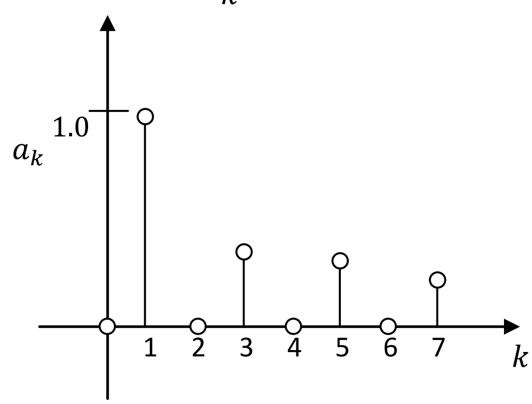


Fig: - Showing magnitude of a_k versus k

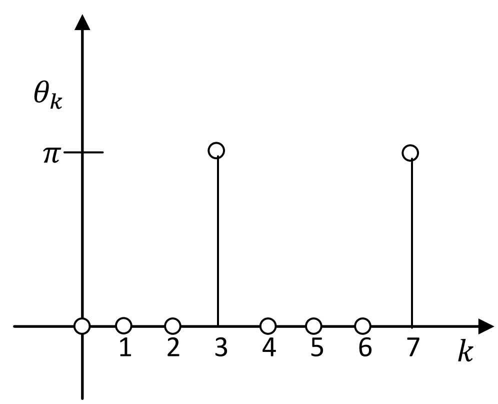


Fig: - Showing magnitude of θ_k versus k

Fourier analysis

We know from the analysis equation

$$c_k = \frac{1}{T_p}\int_{T_p} x(t)\, e^{j2\pi k F_0 t} dt$$

Consider

$$x(t) = cos\left(l\frac{2\pi}{T}t\right)$$

so

$$c_k = 0 \text{ except } k = \pm l$$

$$c_k = \frac{1}{T_p}\left(\int x(t)cos\frac{2\pi kt}{T_p}dt - j\int x(t)sin\frac{2\pi kt}{T_p}dt\right)$$

but, $\int x(t)sin\frac{2\pi kt}{T_p}dt = 0$

$$c_k = \frac{1}{T_p}\left(\int x(t)cos\frac{2\pi kt}{T_p}dt\right)$$

if k, l are positive integers

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(kt).\cos(lt)\,dt = \begin{cases}1 & k = \mp l\\ 0 & otherwise\end{cases}$$

$$c_k = \frac{1}{4\pi}\int_{-\pi}^{\pi}(\cos(k+l)t + \cos(k-l)t)\,dt$$

$$c_k = \frac{1}{4\pi}\left[\frac{\sin(k+l)t}{(k+l)} + \frac{\sin(k-l)t}{(k-l)}\right]_{-\pi}^{\pi}$$

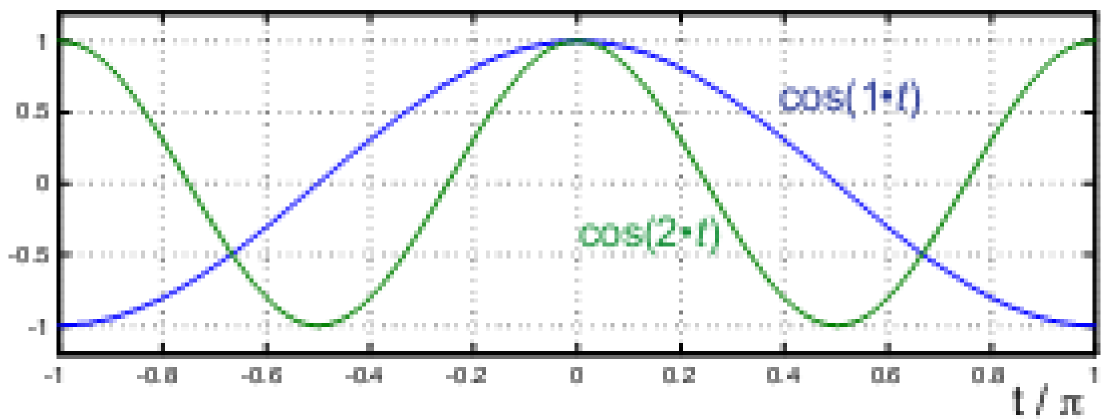


Fig: - Showing Fourier analysis of above example

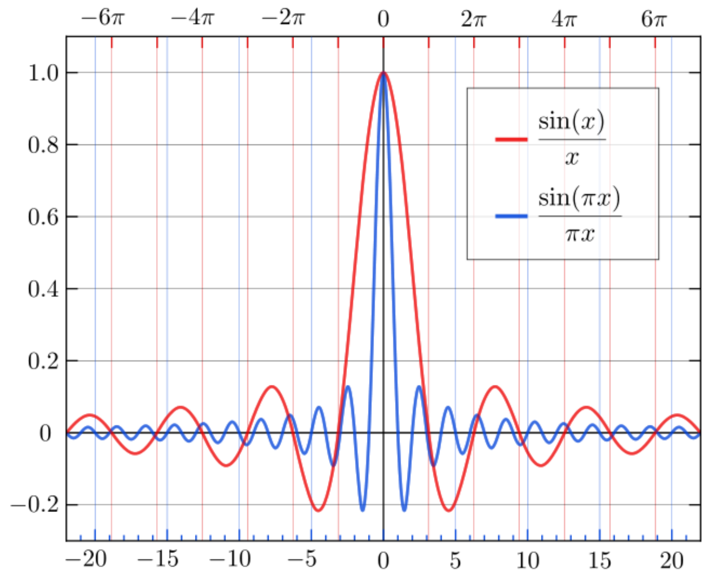


Fig: - showing the picture of sinc function

Fourier series for Continuous-time Aperiodic Signals: -When the period of a signal becomes infinite, the signal becomes Aperiodic and its spectrum becomes continuous that is spectrum of aperiodic signal will be envelope of the line spectrum. The Fourier transform pair of aperiodic signal is given by the following equation.

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\Omega)\, e^{j\Omega t} d\Omega$$

$$\mathrm{X}(\Omega) = \int_{-\infty}^{\infty} x(\mathrm{t})\, e^{-j\Omega t} d\mathrm{t}$$

Summary of the development of the Fourier Transform from the Fourier series of aperiodic and periodic signal.

1. $x(t)$ Aperiodic
   - Construct periodic signal for which one period is $x(t)$.
   - $x_p(t)$ has a Fourier series
   - as period of $x_p(t)$ increase
   - $x_p(t) \rightarrow x(t)$ and Fourier series of
   - $x_p(t) \rightarrow$ Fourier Transform of $x(t)$.
2. $x_p(t)$ Periodic, $x(t)$represents one period
   - Fourier series coefficients of $x_p(t)$

     = $(\frac{1}{T_p})$ times samples of Fourier transform of $x(t)$.

3. $x_p(t)$ periodic

- Fourier transform of $x(t)$defined as impulse train:

  $X_p(\omega) \triangleq \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$

Summary example illustrating some of the relationships between the Fourier series and Fourier transform.

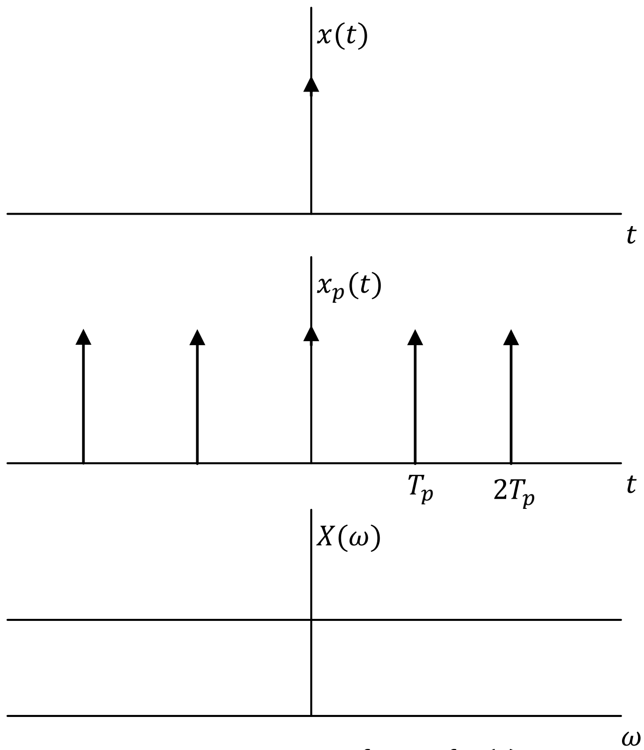


Fig: - Fourier Transform of $x(t)$

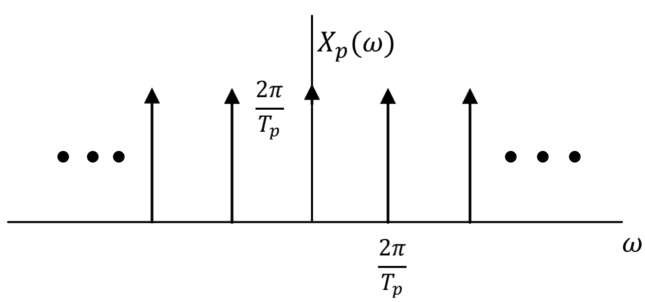


Fig: - Fourier Transform of $x_p(t)$

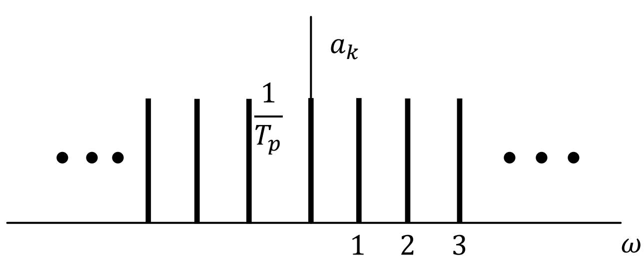


Fig: - Fourier series coefficient of $x_p(t)$

	Time	Frequency
(Continuous time Periodic signal)CTPS	Continuous periodic $x(t)$	Discrete infinitive c_k
(Continuous time Aperiodic signal)CTAS	Continuous infinitive $x(t)$	Continuous infinitive $X(\Omega)$

Discrete Time Fourier Transfer (DTFT)

The discrete-time Fourier transform (DTFT) is one of the specific forms of Fourier analysis. As such, it transforms one function into another, which is called the *frequency domain* representation, or simply the "DTFT", of the original function (which is often a function in the time-domain). The DTFT requires an input function that is *discrete*. Such inputs are often created by digitally sampling a continuous function, like a person's voice.

The DTFT frequency-domain representation is always a periodic function. Since one period of the function contains all of the unique information, it is sometimes convenient to say that the DTFT is a transform to a "finite" frequency-domain (the length of one period), rather than to the entire real line.

Fourier transfer for discrete sequence

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

- Summation(not integral)
- Discrete (normalized)
  Frequency variable ω
- Argument is$e^{j\omega}$,not $j\omega$

Inverse DTFT (IDTFT)

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X\left(e^{j\omega}\right) e^{j\omega n} d\omega$$

- Continuous, periodic $X\left(e^{j\omega}\right)$
  Discrete, infinite $x[n]$
- IDTFT is actually forward Fourier Series
  (except for sign of ω)

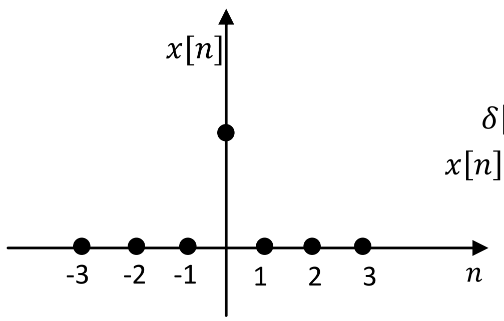
Illustration of DTFTs taking simple example

(1) $x[n] = \delta[n]$

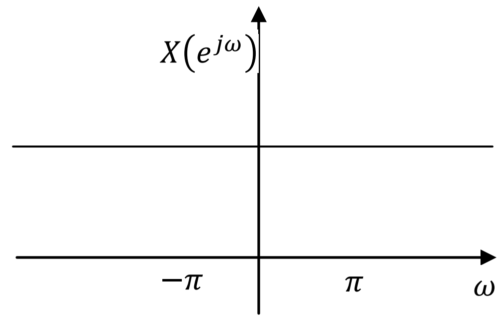
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$X(e^{j\omega}) = e^{-j\omega 0}$$

$$X(e^{j\omega}) = 1 \text{ for all } \omega$$



$$\delta[n] \leftrightarrow 1$$
$$x[n] \leftrightarrow X(e^{j\omega})$$



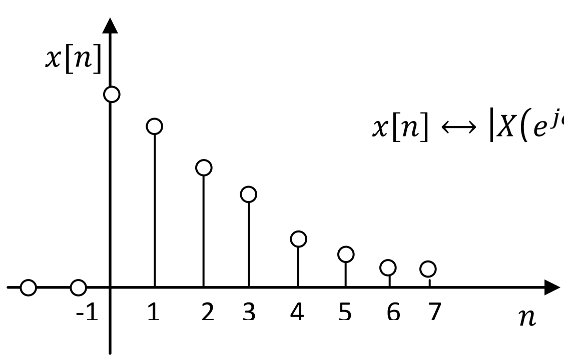
(2) $x[n] = \delta[n]\alpha^n u[n], \quad |\alpha| < 1$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

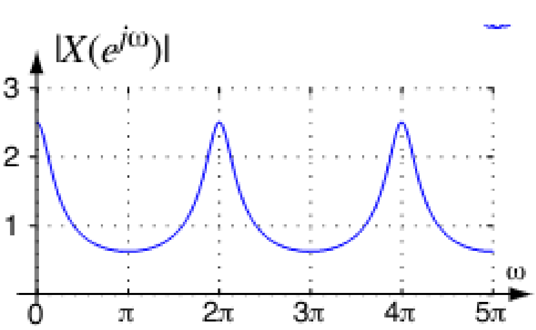
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \alpha^n u[n]e^{-j\omega n}$$

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n}$$

$$X(e^{j\omega}) = \frac{1}{1 - \alpha e^{-j\omega}}$$



$$x[n] \leftrightarrow |X(e^{j\omega})|$$



Properties of DTFT

Property	Time Domain	Frequency Domain
Linearity	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(e^{j\omega}) + a_2X_2(e^{j\omega})$
Time shifting	$x(n-k)$	$e^{-j\omega k}\, X(e^{j\omega})$
Time reversal	$x(-n)$	$X(e^{-j\omega})$
Convolution	$x_1(n) * x_2(n)$	$X_1(e^{j\omega})X_2(e^{j\omega})$
Correlation	$r_{x_1x_2}(l) = x_1(l) * x_2(-l)$	$s_{x_1x_2}\left(e^{j\omega}\right) = X_1\left(e^{j\omega}\right) * X_2(e^{-j\omega})$
Conjugation	$x^*(n)$	$X(e^{-j\omega})$
Frequency Shifting	$e^{j\omega_0 n}x(n)$	$X(e^{j(\omega-\omega_0)})$
Modulation	$x(n)cos\omega_0 n$	$\frac{1}{2}X(e^{j(\omega+\omega_0)}) + \frac{1}{2}X(e^{j(\omega-\omega_0)})$
Multiplication	$x_1(n)x_2(n)$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\lambda)X_2(e^{j(\omega-\lambda)})\, d\lambda$
Differentiation	$nx(n)$	$j\frac{dX(e^{j\omega})}{d\omega}$
Perseval’s theorem	$\sum_{n=-\infty}^{\infty} x_1(n) * x_2{}^*(n)$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X_1\left(e^{j\omega}\right) * X_2(e^{-j\omega})\, d\omega$

Discrete Fourier Transfer (DFT)

the discrete Fourier transform (DFT) converts a finite list of equally-spaced samples of a function into the list of coefficients of a finite combination of complex sinusoids, ordered by their frequencies, that has those same sample values. It can be said to convert the sampled function from its original domain to the frequency domain.

The input samples are complex numbers (in practice, usually real numbers), and the output coefficients are complex too. The frequencies of the output sinusoids are integer multiples of a fundamental frequency, whose corresponding period is the length of the sampling interval. The combination of sinusoids obtained through the DFT is therefore periodic with that same period. The DFT differs from the discrete-time Fourier transform (DTFT) in that its input and output sequences are both finite; it is therefore said to be the Fourier analysis of finite-domain (or periodic) discrete-time functions.

- A finite or periodic sequence has only N unique values,$x[n]$ for $0 \leq n < N$
- Spectrum is completely defined by N distinct frequency samples
- Divide 0 ... .2π into N equal steps, $\{w_k\} = 2\pi k/N$

DFT and IDFT

- Uniform sampling of DTFT spectrum

$$X[k] = X(e^{-j\omega})\Big|_{\omega=\frac{2\pi k}{N}} = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi k}{N}n}$$

DFT

$$X[k] = \sum_{n=0}^{N-1} x[n]W_N^{kn}$$

Where $W_N \quad = e^{-j\frac{2\pi}{N}}$ that is $1/N^{th}$ of a revolution

IDFT

$$x[n] = \frac{1}{N}\sum_{n=0}^{N-1} X[k]W_N^{-nk}$$

Check

$$x[n] = \frac{1}{N}\sum\nolimits_k \left(\sum_l x[l]W_N^{-nl}\right)W_N^{-nk}$$

$$x[n] = \frac{1}{N}\sum_{n=0}^{N-1} x[l]\sum_{n=0}^{N-1} W_N^{k(l-n)}$$

$$x[n] = x[n] \qquad 0 \leq n < N$$

Illustration of DFT with some examples

1. Finite impulse

$$x[n] = \begin{cases} 1 & n = 0 \\ 0 & n = 1 \ldots . N-1 \end{cases}$$

$$X[k] = \sum_{n=0}^{N-1} x[n]W_N^{kn} = W_N^0 = 1$$

2. Periodic sinusoid

$$x[n] = cos\left(\frac{2\pi rn}{N}\right)$$

$$x[n] = \frac{1}{N}(W_N^{-rn} + W_N^{rn})$$

$$X[k] = \frac{1}{2}\sum_{n=0}^{N-1}(W_N^{-rn} + W_N^{rn})W_N^{rn}$$

$$X[k] = \begin{cases} \frac{N}{2} & k = r, k = N - r \\ 0 & otherwise \end{cases}$$

DFT in matrix form

$$X[k] = \sum_{n=0}^{N-1} X[k].W_N^{kn}$$

$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ \vdots \\ . \\ X[N-1] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & . & . & 1 \\ 1 & W_N^1 & W_N^2 & . & . & W_N^{(N-1)} \\ 1 & W_N^2 & W_N^4 & . & . & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ . & . & . & . & . & . \\ 1 & W_N^{(N-1)} & W_N^{2(N-1)} & . & . & W_N^{(N-1)^2} \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ . \\ x[N-1] \end{bmatrix}$$

$$X = D_N.x$$

IDFT in matrix form

$$X = D_N.x$$

$$x = {D_N}^{-1}.X$$

$${D_N}^{-1} = \frac{1}{N}\begin{bmatrix} 1 & 1 & 1 & . & . & 1 \\ 1 & W_N^{-1} & W_N^{-2} & . & . & W_N^{-(N-1)} \\ 1 & W_N^{-2} & W_N^{-4} & . & . & W_N^{-2(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ . & . & . & . & . & . \\ 1 & W_N^{-(N-1)} & W_N^{-2(N-1)} & . & . & W_N^{-(N-1)^2} \end{bmatrix}$$

$$\begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ . \\ x[N-1] \end{bmatrix} = \frac{1}{N}\begin{bmatrix} 1 & 1 & 1 & . & . & 1 \\ 1 & W_N^{-1} & W_N^{-2} & . & . & W_N^{-(N-1)} \\ 1 & W_N^{-2} & W_N^{-4} & . & . & W_N^{-2(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ . & . & . & . & . & . \\ 1 & W_N^{-(N-1)} & W_N^{-2(N-1)} & . & . & W_N^{-(N-1)^2} \end{bmatrix}\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ \vdots \\ . \\ X[N-1] \end{bmatrix}$$

$$x = D_N{}^{-1}.X$$

DFT and DTFT

DTFT

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

- Continuous frequencyω
- Infinite $x[n], -\infty < n < \infty$

DFT

- Discrete frequency $k = N\omega/2\pi$
- Finite $x[n], 0 \leq n < N$
- DFT samples DTFT at discrete frequency
  $X[k] = X(e^{-j\omega})\big|_{\omega=\frac{2\pi k}{N}}$

DTFT from DFT

N-point DFT completely specifies the continuous DTFT of the finite sequence

$$X\left[e^{j\omega}\right] = \sum_{n=0}^{N-1}\left(\frac{1}{N}\sum_{k=0}^{N-1} X[k]W_N^{-kn}\right)e^{-j\omega n}$$

$$X\left[e^{j\omega}\right] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\sum_{k=0}^{N-1} e^{-j(\omega-\frac{2\pi k}{N})n}$$

$$X\left[e^{j\omega}\right] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\sum_{k=0}^{N-1} e^{-j\frac{(N-1)}{2}.\Delta\omega_k}$$

Where $\Delta\omega_k = \omega - \frac{2\pi k}{N}$

$$X[e^{j\omega}] = \frac{1}{N}\sum_{k=0}^{N-1} X[k].\frac{sinN\,\dfrac{\Delta\omega_k}{2}}{sin\,\dfrac{\Delta\omega_k}{2}}\, e^{-j\frac{(N-1)}{2}.\Delta\omega_k}$$

Where $\frac{sinN\frac{\Delta\omega_k}{2}}{sin\frac{\Delta\omega_k}{2}}$ =periodic sinc

Periodic sinc

$$\sum_{k=0}^{N-1} e^{-j.\Delta\omega_k n} = \frac{1-e^{-jN\Delta\omega_k}}{1-e^{-j\Delta\omega_k}}$$

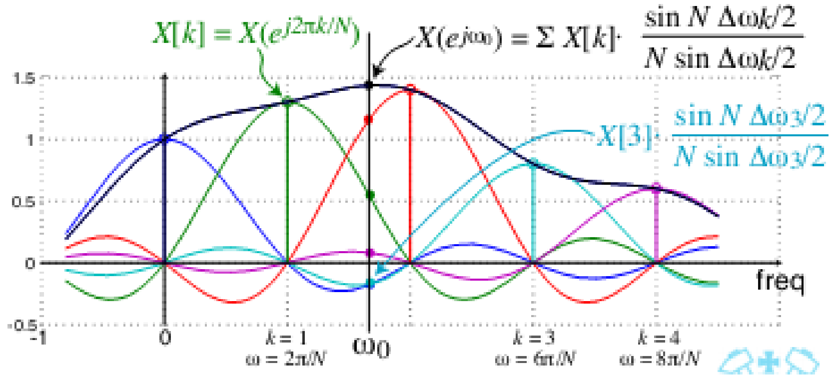
$$\sum_{k=0}^{N-1} e^{-j.\Delta\omega_k n} = \frac{e^{-jN\Delta\omega_k}/2}{e^{-j\Delta\omega_k}/2}.\frac{e^{jN\Delta\omega_k}/2-e^{-jN\Delta\omega_k}/2}{e^{j\Delta\omega_k}/2-e^{-j\Delta\omega_k}/2}$$

$$\sum_{k=0}^{N-1} e^{-j.\Delta\omega_k n} = e^{-j\frac{(N-1)}{2}.\Delta\omega_k}\frac{sinN\,\dfrac{\Delta\omega_k}{2}}{sin\,\dfrac{\Delta\omega_k}{2}}$$

$$\sum_{k=0}^{N-1} e^{-j.\Delta\omega_k n} = \begin{cases} N \;\; when \;\; \Delta\omega_k = 0; \; -N \;\; when \;\; \dfrac{\Delta\omega_k}{2} = \pi \\ 0 \;\; when \;\; \dfrac{\Delta\omega_k}{2} = r.\dfrac{\pi}{N}, r = \pm 1, \pm 2, \ldots\ldots.. \end{cases}$$

other values are in between

DFT to DTFT interpolation by periodic sinc



DFT from DTFT

$$\underset{-A \le n < B}{x[n]} \xrightarrow{DTFT} X[e^{j\omega}] \xrightarrow{sample} X[k] \xrightarrow{IDFT} \underset{0 \le n < N}{\tilde{x}[n]}$$

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1}\left(\sum_{l=-\infty}^{\infty} x[l]W_N^{kl}\right)W_N^{-kn}$$

$$\tilde{x}[n] = \sum_{l=-\infty}^{\infty} x[l]\left(\frac{1}{N}\sum_{k=0}^{N-1} W_N^{k(l-n)}\right)$$

Where $\frac{1}{N}\sum_{k=0}^{N-1} W_N^{k(l-n)} = \begin{cases} 1 & \text{for } n-l = rN, r \in I \\ 0 & otherwise \end{cases}$

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n-rN]$$

All values shifted by exact multiples of N points to lie in $0 \le n < N$.

DFT properties

1. Circular convolution
$$\sum_{m=0}^{N-1} g[m]h[\langle n-m\rangle_N] \leftrightarrow G[k]H[k]$$
2. Modulation
$$g[n].h[n] \leftrightarrow \frac{1}{N}\sum_{m=0}^{N-1} G[m]H[\langle k-m\rangle_N]$$
3. Duality
$G[n] \leftrightarrow N.g[\langle -k\rangle_N]$
4. Perseval
$$\sum_{n=0}^{N-1}|x[n]|^2 = \frac{1}{N}\sum_{m=0}^{N-1}|X[k]|^2$$

Z-Transform

The z-transform is very powerful tool for the study of discrete-time signal and system, thus z-transform of a discrete-time signal $x(n)$ is defined as the power series,

$$X(z) = Z\{x(n)\}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \qquad (6)$$

Where n is integer and z is in general complex variable, since the z-transform is an infinitive power series it exists only for that value of z for which series converges. The region of convergence (ROC) of $X(z)$ is the set of all values of z for which $X(z)$ attains the finite value. Thus any time we cite a z-transform we should also indicate its ROC.

$$ROC = \{z: |\sum_{n=-\infty}^{\infty} x(n) z^{-n}| < \infty\} \tag{7}$$

The stability of a system can also be determined by knowing the ROC alone. If the ROC contains the unit circle ($|z| = 1$) then the system is stable. If you need a causal system then the ROC must contain infinity and the system function will be a right-sided sequence. If you need an anticausal system then the ROC must contain the origin and the system function will be a left-sided sequence. If you need both, stability and causality, all the poles of the system function must be inside the unit circle.

Properties of z-transform

Property	**Time Domain**	**z-Domain**	**ROC**
Linearity	$a_1 x_1(n) + a_2 x_2(n)$	$a_1 X_1(z) + a_2 X_2(z)$	At least intersection of ROC_1 and ROC_2
Time shifting	$x(n-k)$	$z^{-k} X(z)$	That of $X(z)$ except for $z = 0$ if $k > 0$ and $z = \infty$ if $k < 0$
Scaling	$a^n x(n)$	$X(a^{-1} z)$	$\|a\| r_1 < \|z\| < \|a\| r_2$
Time reversal	$x(-n)$	$X(z^{-1})$	$\frac{1}{r_2} < \|z\| < \frac{1}{r_1}$
Differentiation	$n x(n)$	$-z \frac{dX(z)}{dz}$	$r_2 < \|z\| < r_1$
Convolution	$x_1(n) * x_2(n)$	$X_1(z) X_2(z)$	At least intersection of ROC_1 and ROC_2

Correlation	$r_{x_1x_2}(l) = \sum_{n=-\infty}^{\infty} x_1(n)x_2(n-l)$	$R_{x_1x_2}(z) = X_1(z)X_2(z^{-1})$	At least intersection of ROC of $X_1(z)$ and $X_2(z^{-1})$
Multiplication	$x_1(n)x_2(n)$	$\frac{1}{2\pi j}\oint X_1(v)\, X_1\left(\frac{z}{v}\right) v^{-1}dv$	$r_{1l}r_{2l} < \lvert z\rvert < r_{1u}r_{2u}$
Initial value theorem	If $x(n)$ is causal	$x(0) = \lim_{z\to\infty} X(z)$	
Parseval's relation	$\sum_{n=-\infty}^{\infty} x_1(n)x_2{}^*(n)$	$\frac{1}{2\pi j}\oint X_1(v)\, X_1{}^*\left(\frac{z}{v^*}\right) v^{-1}$	

Relation between Z-transform and Fourier Transfer

The z-transform of a discrete-time signal $x(n)$ is defined as the power series

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \qquad ROC: r_2 < |z| < r_1$$

Let $z = re^{j\omega}$

Where $r = |z|$ and $\omega = \measuredangle z$, thus within the region of convergence of $X(z)$ we get.

$$X(z)|_{z=re^{j\omega}} = \sum_{n=-\infty}^{\infty}[x(n)r^{-n}]e^{-j\omega n} \tag{8}$$

If $X(z)$ converges for $|z| = 1$, then

$$X(z)|_{z=re^{j\omega}} \equiv X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

Therefore, the Fourier transform can be viewed as the z-transform of the sequence evaluated on the unit circle. If the unit circle is not contained in the region of convergence of $X(z)$ the Fourier transform $X(\omega)$ does not exist.

We should note that the existence of the z-transform requires that the sequence $[x(n)r^{-n}]$be absolutely summable for some value of r, that is,

$$\sum_{n=-\infty}^{\infty}|x(n)r^{-n}| < \infty \tag{9}$$

There are sequences; however that do not satisfy the requirement in (9) for example, the sequence

$$x(n) = \frac{sin\omega_c n}{\pi n} \quad -\infty < n < \infty$$

This sequence does not have a z-transform, since it has a finite energy thus its Fourier transfer converges to the discontinuous function. In conclusion the existence of z-transfer requires that (9) be satisfied for some region in the z-plane. If this region contains the unit circle, the Fourier transform exists. The existence of the Fourier transform which is defined for finite energy signals does not necessarily ensure the existence of the z-transfer.

Relationship to s-plane and z-plane

The Z-transform is a useful approximation for converting continuous time filters into discrete time filters (represented in z space), and vice versa. We can use the following substitutions in $H(s)$ or $H(z)$.

$$s = \frac{2}{T}\frac{(z-1)}{(z+1)}$$

Now from Laplace to z

$$z = \frac{2+sT}{2-sT}$$

or from z to Laplace. Through the z-transformation, the complex s-plane is mapped to the complex z-plane (of the z-transform).while this mapping is necessarily nonlinear, it is useful in that it maps the entire $j\Omega$ axis of the s-plane into the unit circle in the z-plane.As such, the Fourier transform (which is the Laplace transform evaluated on the $j\Omega$ axis) becomes the discrete-time Fourier transform. This assumes that the Fourier transform exists that is the $j\Omega$ axis is in the region of convergence of the Laplace transform.

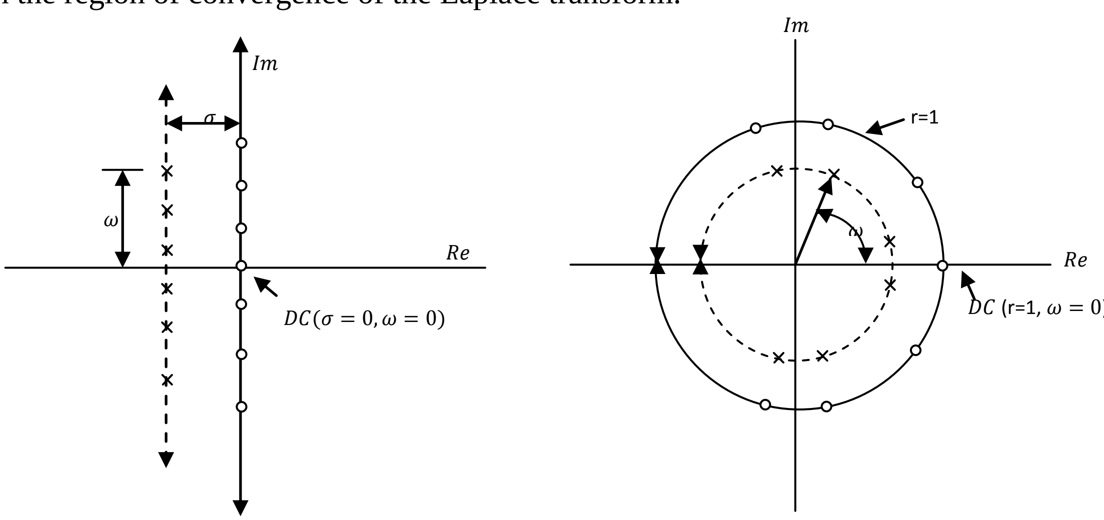


Fig: - Relationship between the s-plane and the z-plane

CONCLUSION

The relation between Fourier Transfer, DTFT, DFT and Z-Transform are drawn on the basis of their definitions and with some examples, their subtle relation may be shown by simulation.

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