Energy and Power

- Signal energy:

$$E_g = \int_{-\infty}^{+\infty} |g(t)|^2 dt$$

  - Energy signal:

$$if \quad E_g < \infty$$

- Signal Power:

$$P_g = \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |g(t)|^2 dt$$

  - Power signal:

$$if \quad P_g < \infty$$

Multiplication, Correlation and Convolution

- Signal Convolution

$$y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

  convolution in time <-> multiplication in frequency

- Signal Correlation

$$r_{xy}(\tau) = \int_{-\infty}^{+\infty} x(t)y^*(t-\tau)dt, r(t) = x(t) * y^*(-t)$$

  correlation in time <-> conjugate multiplication

Signal Space Representation

- Vector Space of Functions

$$Inner \quad Product :< f, g >= \int f(x)g(x)dx$$

$$Norm : ||f|| = \sqrt{< f, f >} = \sqrt{\int f(x)^2 dx}$$

$$Orthonormal \quad functions :< \phi_k(x), \phi_l(x) >= \int \phi_k(x)\phi_l(x)dx = \delta_{kl}$$

$$Arbitrary \quad function \quad f(x) \quad in \quad orthonormal \quad basis : f(x) = \sum_k < f(x), \phi_k(x) > \phi_k(x)$$

- For ***energy signals***

$$< x(t), y(t) >= \int_{-\infty}^{+\infty} x(t)y^*(t)dt$$

- For ***power signals***

$$< x(t), y(t) >= \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)y^*(t)dt$$

- For ***periodic signals***

$$< x(t), y(t) >= \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)y^*(t)dt$$

Fourier Series and Fourier Transform

- ***Fourier Series***: every ***periodic*** signal with fundamental period can be decomposed as

$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}, \quad C_k =< x(t), e^{jk\omega_0 t} >= \frac{1}{T_0} \int_{T_0} x(t)(e^{jk\omega_0 t})^* dt = \frac{1}{T_0} \int_{T_0} x(t)e^{-jk\omega_0 t} dt$$

  - Parseval's Theorem:

$$||x(t)|| =< x(t), x(t) >= \frac{1}{T_0} \int_{T_0} x(t)x^*(t)dt = \sum_{k=-\infty}^{\infty} |C_k|^2$$

$$\Rightarrow \frac{1}{T_0} \int_{T_0} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |C_k|^2$$

- ***Fourier Transform:***

$$F(\omega) = \int_{-\infty}^{+\infty} f(t)e^{-j\omega t} dt, \quad f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega)e^{j\omega t} d\omega$$

  - ***Properties***

$$Linearity: \quad af_1(t) + bf_2(t) \overset{F}{\leftrightarrow} aF_1(\omega) + bF_2(\omega)$$

$$Scaling \quad in \quad Time: \quad f(at) \overset{F}{\leftrightarrow} \frac{1}{|a|} F(\frac{\omega}{a})$$

$$Scaling \quad in \quad Frequency: \quad \frac{1}{|a|} f(\frac{t}{a}) \overset{F}{\leftrightarrow} F(a\omega)$$

$$Shifting \quad in \quad Time: \quad f(t - t_0) \overset{F}{\leftrightarrow} F(\omega)e^{-j\omega t_0}$$

$$Shifting \quad in \quad Frequency: \quad f(t)e^{j\omega_0 t} \overset{F}{\leftrightarrow} F(\omega - \omega_0)$$

$$Convolution: f_1(t) * f_2(t) \overset{F}{\leftrightarrow} F_1(\omega)F_2(\omega)$$

$$Multiplication: \quad f_1(t)f_2(t) \overset{F}{\leftrightarrow} \frac{1}{2\pi} F_1(\omega) * F_2(\omega)$$

$$Differentiation \quad in \quad time: \frac{df(t)}{dt} \overset{F}{\leftrightarrow} j\omega F(\omega), \frac{d^n f(t)}{dt^n} \overset{F}{\leftrightarrow} (j\omega)^n F(\omega)$$

$$Frequency \quad Differentiation: \frac{dF(\omega)}{d\omega} \overset{F}{\leftrightarrow} (-jt)f(t), \frac{dF^n(\omega)}{d\omega^n} \overset{F}{\leftrightarrow} (-jt)^n f(t)$$

$$Integration \quad in \quad Time: \int_{-\infty}^{t} f(\tau)d\tau \overset{F}{\leftrightarrow} \frac{1}{j\omega}F(\omega) + \pi F(0)\delta(\omega)$$

$$differentiation \quad in \quad time(frequency) = multiplying \quad j\omega(-jt) \quad in \quad frequency(time)$$

$$integration \quad in \quad time = dividing \quad j\omega \quad in \quad frequency$$