Probability review

Adopted from notes of Andrew W. Moore and Eric Xing from CMU

So far our classifiers are deterministic!

- For a given X, the classifiers we learned so far give a single predicted y value
- In contrast, a probabilistic prediction returns a probability over the output space

  $P(y=0|X)$, $P(y=1|X)$
- We can easily think of situations when this would be very useful!
  - Given $P(y=1|X) = 0.49$, $P(y=-1|X)=0.51$, how would you predict?
  - What if I tell you it is much more costly to miss an positive example than the other way around?

Discrete Random Variables

- A is a Boolean-valued random variable if A denotes an event, and there is some degree of uncertainty as to whether A occurs.
- Examples
- A = The US president in 2023 will be male
- A = You wake up tomorrow with a headache
- A = You have Ebola

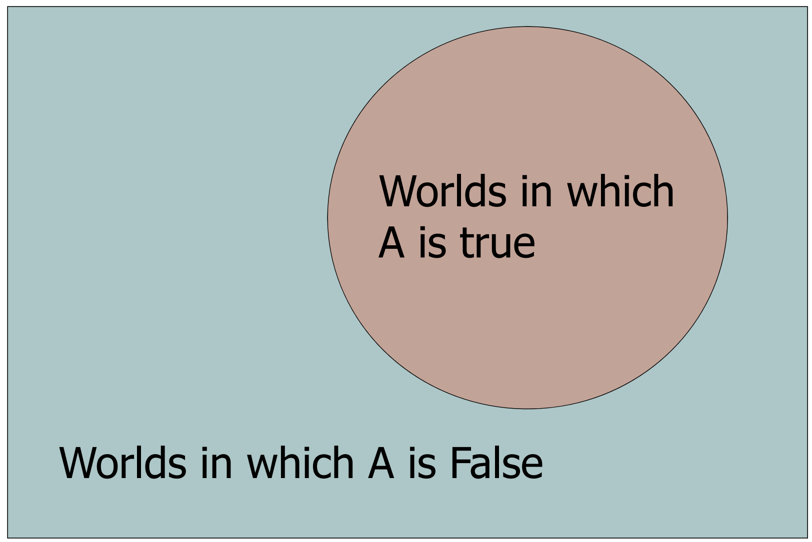
Probabilities

- We write P(A) as "the fraction of possible worlds in which A is true"
- We could at this point spend 2 hours on the philosophy of this.
- But we won't.

Visualizing A

Event space of all possible worlds

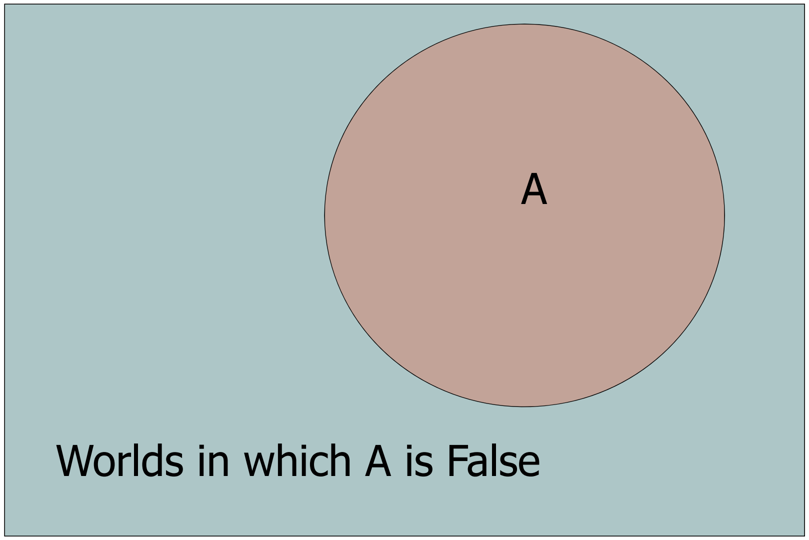
Its area is 1

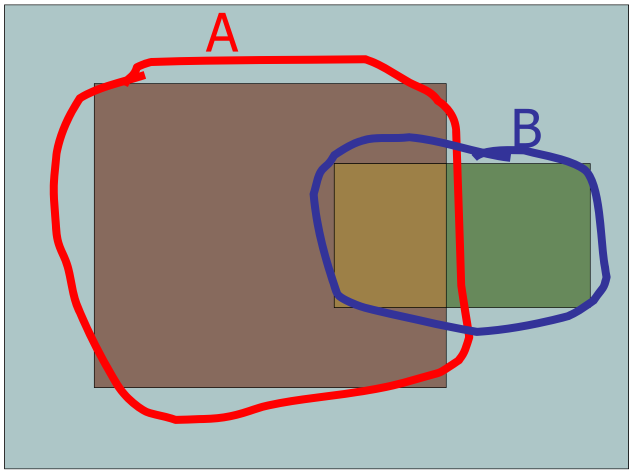


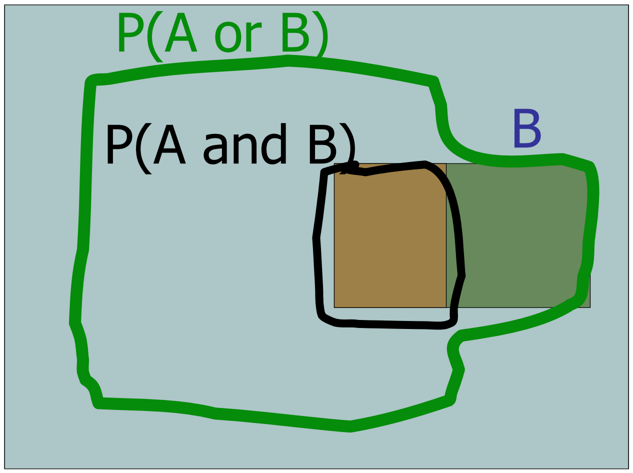
P(A) = Area of reddish oval

Basic axioms

- 0 <= P(A) <= 1
- P(True) = 1
- P(False) = 0
- P(A or B) = P(A) + P(B) - P(A and B)







Simple addition and subtraction

Elementary Probability Theorems

- $P(\sim A) + P(A) = 1$
- $P(B) = P(B \wedge A) + P(B \wedge \sim A)$

Multinomial Random Variables

- Suppose A can take on more than 2 values
- A is a *random variable with arity k* if it can take on exactly one value out of $\{v_1, v_2, \ldots v_k\}$
- Thus…

$$P(A = v_i \wedge A = v_j) = 0 \text{ if } i \neq j$$
$$P(A = v_1 \vee A = v_2 \vee A = v_k) = 1$$

An easy fact about Multinomial Random Variables:

- Using the axioms of probability…

  0 <= P(A) <= 1, P(True) = 1, P(False) = 0

  P(A or B) = P(A) + P(B) - P(A and B)

- And assuming that A obeys…

$$P(A=v_i \wedge A=v_j)=0 \text{ if } i \neq j$$

$$P(A=v_1 \vee A=v_2 \vee A=v_k)=1$$

- It's easy to prove that

$$P(A=v_1 \vee A=v_2 \vee A=v_i)=\sum_{j=1}^{i} P(A=v_j)$$

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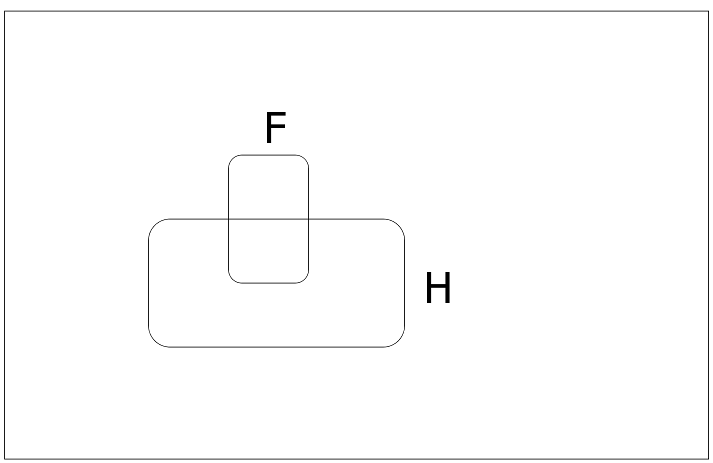
- And thus we can prove

$$P(B) = \sum_{j=1}^{k} P(B \wedge A = v_j)$$

Conditional Probability

- P(A|B) = Fraction of worlds in which B is true that also have A true

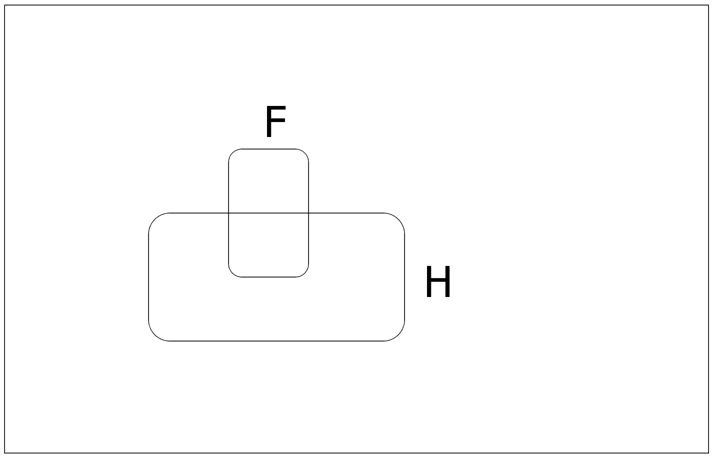


H = "Have a headache"
F = "Coming down with Flu"

P(H) = 1/10
P(F) = 1/40
P(H|F) = 1/2

"Headaches are rare and flu is rarer, but if you're coming down with 'flu there's a 50-50 chance you'll have a headache."

Conditional Probability



H = "Have a headache"
F = "Coming down with Flu"

P(H) = 1/10
P(F) = 1/40
P(H|F) = 1/2

P(H|F) = Fraction of flu-inflicted worlds in which you have a headache

$$= \frac{\text{\#worlds with flu and headache}}{\text{\#worlds with flu}}$$

$$= \frac{\text{Area of "H and F" region}}{\text{Area of "F" region}}$$

$$= \frac{P(H \wedge F)}{P(F)}$$

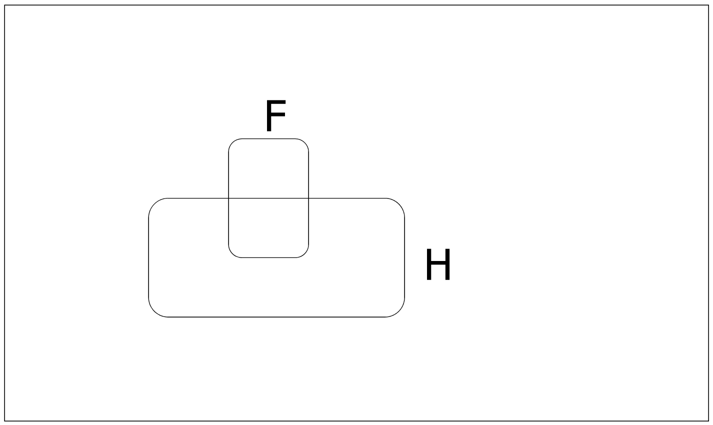
Definition of Conditional Probability

$$P(A|B) = \frac{P(A \wedge B)}{P(B)}$$

Corollary: The Chain Rule

$$P(A \wedge B) = P(A|B)\, P(B)$$

Probabilistic Inference



H = "Have a headache"
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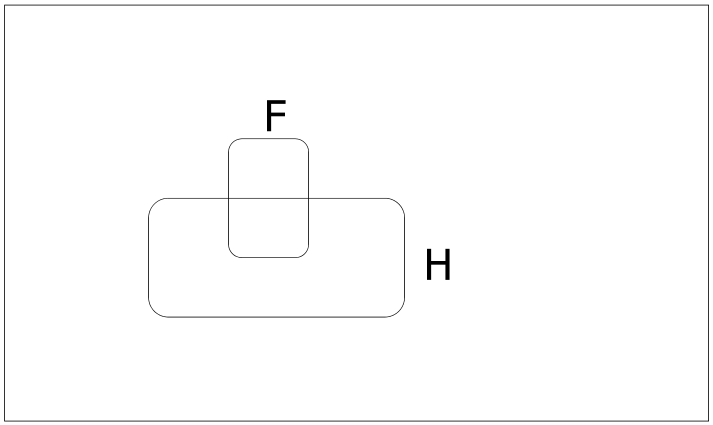
One day you wake up with a headache. You think: "Drat! 50% of flus are associated with headaches so I must have a 50-50 chance of coming down with flu"

Is this reasoning good?

Probabilistic Inference



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P(F ^ H) = …

P(F|H) = …

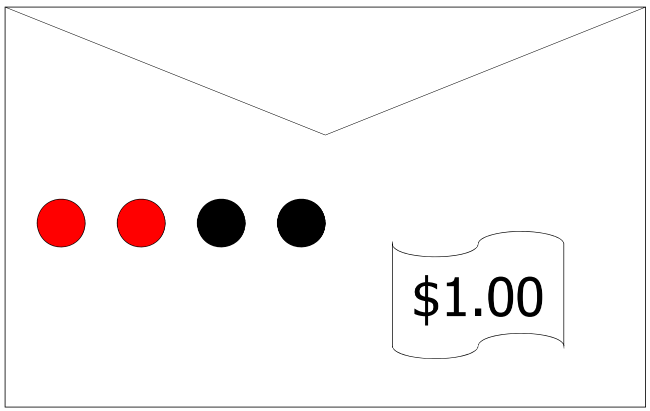
What we just did…

$$P(B|A) = \frac{P(A \wedge B)}{P(A)} = \frac{P(A|B)\,P(B)}{P(A)}$$

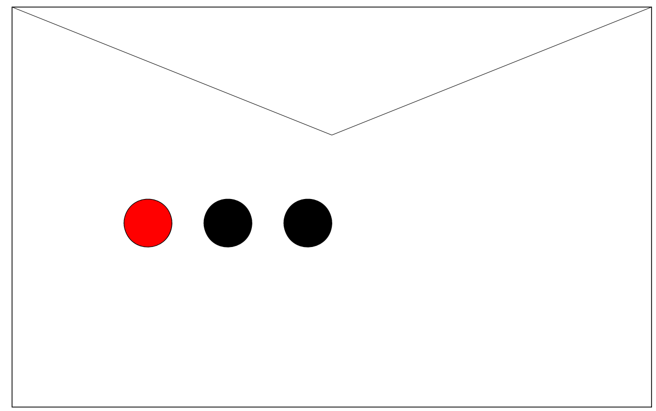
This is Bayes Rule

Bayes, Thomas (1763) An essay towards solving a problem in the doctrine of chances. *Philosophical Transactions of the Royal Society of London,* **53:370-418**

Using Bayes Rule to Gamble



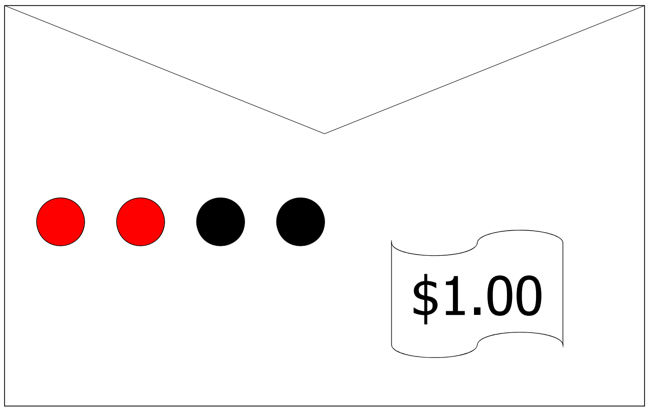
The "Win" envelope has a dollar and four beads in it



The "Lose" envelope has three beads and no money

Trivial question: someone draws an envelope at random and offers to sell it to you. How much should you pay?

Using Bayes Rule to Gamble



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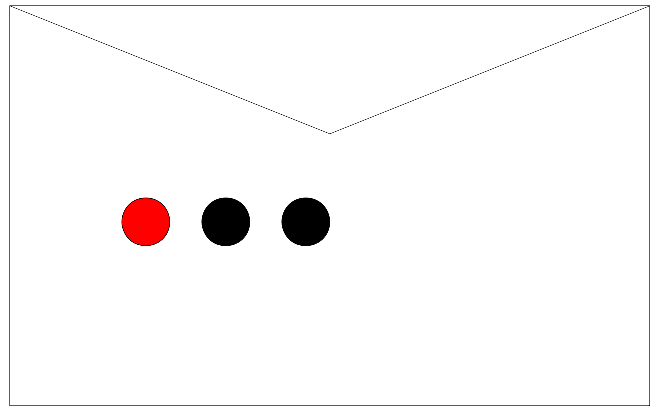


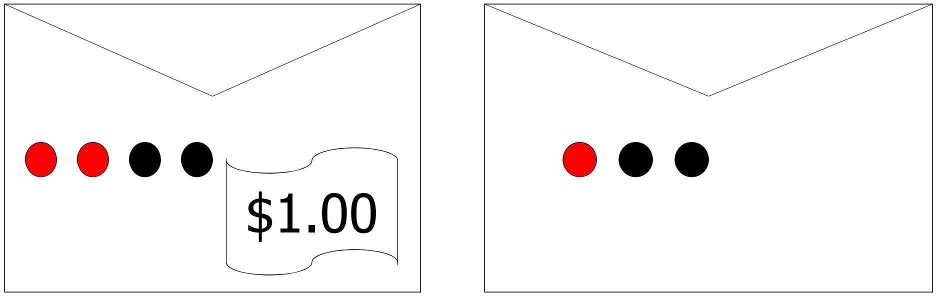
The "Lose" envelope has three beads and no money

Interesting question: before deciding, you are allowed to see one bead drawn from the envelope.

Suppose it's black: How much should you pay?

Suppose it's red: How much should you pay?

Calculation…



More General Forms of Bayes Rule

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|\sim A)P(\sim A)}$$

$$P(A|B \wedge X) = \frac{P(B|A \wedge X)P(A \wedge X)}{P(B \wedge X)}$$

More General Forms of Bayes Rule

$$P(A=v_i|B)=\frac{P(B|A=v_i)P(A=v_i)}{\sum_{k=1}^{n_A}P(B|A=v_k)P(A=v_k)}$$

Useful Easy-to-prove facts

$$P(A \mid B) + P(\neg A \mid B) = 1$$

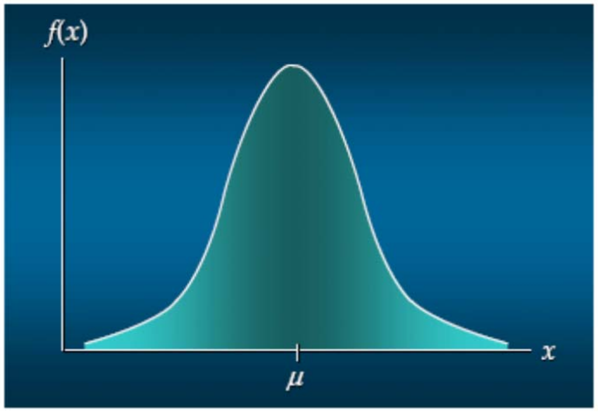
$$\sum_{k=1}^{n_A} P(A = v_k \mid B) = 1$$

Continuous Probability Distribution

- A continuous random variable x can take any value in an interval on the real line
  - X usually corresponds to some real-valued measurements, e.g., today's lowest temperature
  - It is not possible to talk about the probability of a continuous random variable taking an exact value --- $P(x=56.2)=0$
  - Instead we talk about the probability of the random variable taking a value within a given interval $P(x \in [50, 60])$

PDF: probability density function

- The probability of X taking value in a given range [x1, x2] is defined to be the area under the PDF curve between x1 and x2
- We use *f*(x) to represent the PDF of x
- Note:
  - *f*(x) ≥0
  - *f*(x) can be larger than 1
  - $\int_{-\infty}^{\infty} f(x)dx = 1$
  - $P(X \in [x1, x2]) = \int_{x1}^{x2} f(x)dx$



What is the intuitive meaning of *f(x)*?

If $f(x1)=\alpha * a$ and $f(x2)=a$

Then when x is sampled from this distribution, you are α times more likely to see that x is "very close to" x1 than that x is "very close to" x2

Some commonly used distributions

Bernoulli distribution: Ber(p)

$$P(x) = \begin{cases} 1-p & \text{for } x = 0 \\ p & \text{for } x = 1 \end{cases} \quad \Rightarrow \quad P(x) = p^x (1-p)^{1-x}$$

Binomial distribution: Binomial(n , p)

the probability to see x heads out of n flips

$$\boxed{P(x) = \frac{n(n-1)\cdots(n-x+1)}{x!} p^x (1-p)^{n-x}}$$

Multinomial distribution: Multinomial(n , [x_1, x_2, ..., x_k])

The probability to see x_1 ones, x_2 twos, etc, out of n dice rolls

$$\boxed{P([x_1, x_2, \ldots, x_k]) = \frac{n!}{x_1! x_2! \cdots x_k!} \theta_1^{x_1} \theta_2^{x_2} \cdots \theta_k^{x_k}}$$

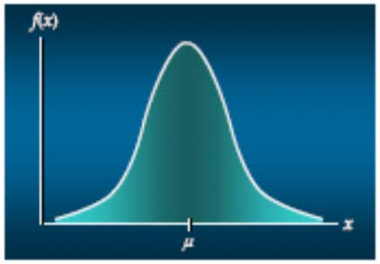
Continuous Distributions

Uniform Probability Density Function

$$f(x) = 1/(b-a) \quad \text{for } a \le x \le b$$
$$= 0 \quad \text{elsewhere}$$

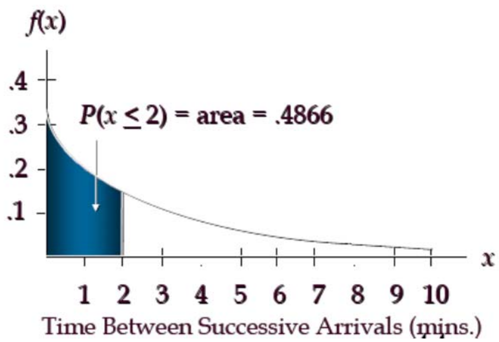
Normal (Gaussian) Probability Density Function

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}$$



Exponential Probability Distribution

$$f(x) = \frac{1}{\mu} e^{-x/\mu}$$



Expected value

Given a random variable x characterized by a distribution $p(x)$, what is the *expected* value of x?

The *expectation* of x.

$$\mathbb{E}[x] = \sum_x p(x)x$$

or

$$\mathbb{E}[x] = \int p(x)x\mathrm{d}x$$

Example

- What is the expected value when rolling one die?

x	p(x)
1	$\frac{1}{6}$
2	$\frac{1}{6}$
3	$\frac{1}{6}$
4	$\frac{1}{6}$
5	$\frac{1}{6}$
6	$\frac{1}{6}$

$$\begin{aligned}\mathbb{E}[x] &= \sum_x p(x)x \\ &= 1\cdot\frac{1}{6}+2\cdot\frac{1}{6}+3\cdot\frac{1}{6}+4\cdot\frac{1}{6}+5\cdot\frac{1}{6}+6\cdot\frac{1}{6} \\ &= \frac{21}{6}=3.5\end{aligned}$$

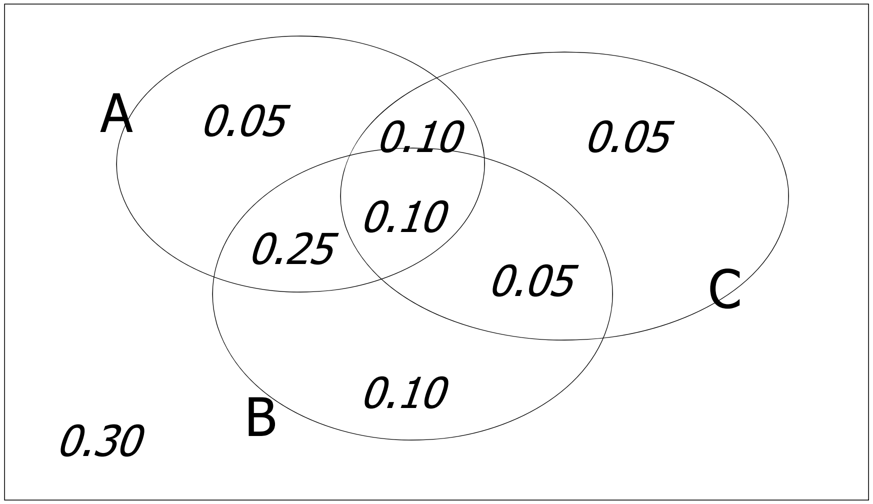
The Joint Distribution

Example: Boolean variables A, B, C

Recipe for making a joint distribution of M variables:

1. Make a truth table listing all combinations of values of your variables (if there are M Boolean variables then the table will have 2^M rows).
2. For each combination of values, say how probable it is.
3. If you subscribe to the axioms of probability, those numbers must sum to 1.

A	**B**	**C**	**Prob**
0	0	0	0.30
0	0	1	0.05
0	1	0	0.10
0	1	1	0.05
1	0	0	0.05
1	0	1	0.10
1	1	0	0.25
1	1	1	0.10



Using the Joint

gender	hours_worked	wealth	
Female	v0:40.5-	poor	0.253122
		rich	0.0245895
	v1:40.5+	poor	0.0421768
		rich	0.0116293
Male	v0:40.5-	poor	0.331313
		rich	0.0971295
	v1:40.5+	poor	0.134106
		rich	0.105933

One you have the JD you can ask for the probability of any logical expression involving your attribute

$$P(E) = \sum_{\text{rows matching } E} P(\text{row})$$

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		rich	0.105933	

P(Poor Male) = 0.4654

$$P(E) = \sum_{\text{rows matching } E} P(\text{row})$$

Inference with the Joint

gender	hours_worked	wealth	
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$$P(E_1 \mid E_2) = \frac{P(E_1 \wedge E_2)}{P(E_2)} = \frac{\sum\limits_{\text{rows matching } E_1 \text{ and } E_2} P(\text{row})}{\sum\limits_{\text{rows matching } E_2} P(\text{row})}$$