**4.21**

**Intro**

In this exercise we will study 1D decision boundaries in a simple scenario where the class conditional densities are Gaussians and we have uninformative priors. Since the Gaussian has a bell shape expressed as a quadratic expression inside an exponential, we will eventually solve a quadratic inequality to find the decision boundaries (i.e. $ax^2+bx+c \leq 0$). Gaussians have two parameters $\mu$ and $\sigma$ that define them. If our Gaussians differ in both parameters, we can show that the Gaussians will intersect in two points. This means that one curve is bigger inside the interval defined by these two points and the other curve is bigger in the two remaining intervals. If only the variance of two Gaussians is equal, the quadratic function degenerates into an affine function and the Gaussians will only touch at one point. If only the mean of the Gaussians is equal, the Gaussians will touch in symmetric points with respect to the mean. Obviously, if the two parameters are equal, the curves will be equal everywhere.

**Solution**

a)

In the first case $\mu_1 = 0$, $\sigma_1^2 = 1$, $\mu_2 = 1$, $\sigma_2^2 = 10^6$. In the following development log refers to the natural logarithm.

$$\begin{aligned}
&p(x|\mu_1, \sigma_1) \geq p(x|\mu_2, \sigma_2) \\
&\frac{1}{\sqrt{2\pi}\sigma_1} exp\left(-\frac{1}{2}\frac{(x-\mu_1)^2}{\sigma_1^2}\right) \geq \frac{1}{\sqrt{2\pi}\sigma_2} exp\left(-\frac{1}{2}\frac{(x-\mu_2)^2}{\sigma_2^2}\right) \\
&\log \sigma_2 - \frac{1}{2}\frac{(x-\mu_1)^2}{\sigma_1^2} \geq \log \sigma_1 - \frac{1}{2}\frac{(x-\mu_2)^2}{\sigma_2^2} \\
&3 \log 10 - \frac{1}{2}\frac{(x-0)^2}{1} \geq \log 1 - \frac{1}{2}\frac{(x-1)^2}{10^6} \\
&(10^6 - 1)x^2 + 2x - (1 + 6 \times 10^6 \log 10) \leq 0
\end{aligned} \tag{1}$$

With the inequality at hand, we must find the roots of its left side.

$$\begin{aligned}
x &= \frac{-2 \pm \sqrt{4 + 4(10^6-1)(1 + 6 \times 10^6 \log 10)}}{2(10^6 - 1)} \\
x_1 &= -3.72 \\
x_2 &= 3.72
\end{aligned} \tag{2}$$

Since $a = (10^6 - 1) > 0$, the inequality is valid between the two roots. To put more directly: $R_1 = \{x \in \mathbb{R} : -3.72 \leq x \leq 3.72\}$.

You can check the graphical comparison between the two Gaussians in the notebook that was pushed together with this solution to the repository.

b)

Now, we will assume $\sigma_2 = 1$, which means that the two Gaussians have the same shape and only differ by a horizontal translation controlled by the mean $\mu$. Remembering Equation 1, we have:

$$
\begin{aligned}
&p(x|\mu_1, \sigma_1) \geq p(x|\mu_2, \sigma_2) \\
&\log \sigma_2 - \frac{1}{2}\frac{(x-\mu_1)^2}{\sigma_1^2} \geq \log \sigma_1 - \frac{1}{2}\frac{(x-\mu_2)^2}{\sigma_2^2} \\
&(x-\mu_1)^2 \leq (x-\mu_2)^2 \\
&x^2 \leq x^2 - 2x + 1 \\
&x \leq \frac{1}{2}
\end{aligned}
\tag{3}
$$

As we can see, when the variance of both Gaussians are equal, our quadratic inequality degenerates into a affine function. As a result, our new decision boundary is $R_1 = \{x \in \mathbb{R} : x \leq \frac{1}{2}\}$

**Conclusion**

In this exercise we study decision boundaries when comparing Gaussian distributions. In the first exercise, the two parameters of the Gaussian were distinct and as a result the Gaussians touched in two points. The the second exercise, the variance of both curves were the same, transforming the original quadratic expression into an affine function. Both results were expect based on our discussion in the introduction.