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UC Irvine Statistics 131a - Summer 2013

Random Variables (R.V.S)

– missing first few minutes of video

If Ω is a sample space, there is a function: $X: \Omega \to \mathbb{R}$ there is a probability P on the sample space then

$$F(x) = P(X \le x)$$

is called the [cumulative] distribution function of X.

$\{\omega \in \Omega : X(\omega) \le x\}$ is an event

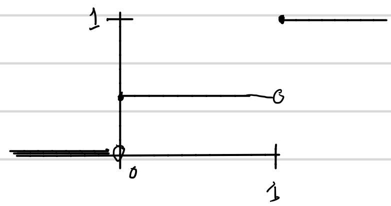
$F(x) = P(\{\omega \in \Omega : X(\omega) \le x\})$

Examples

Bernoulli R.V.S

$\Omega = \{S, F\}$ $X(S) = 1$, $X(F) = 0$

0:06 $$P(X \le x) = \begin{cases} 0, & \text{if } x < 0 \\ 1-p, & \text{if } 0 \le x < 1 \\ 1, & \text{if } 1 \le x \end{cases}$$



$P(X=0) = 1-p$ Jump at 0
$P(X=1) = p$ Jump at 1

2:40 For function $F(x) = P(X \le x)$, if $y < x$ then $F(x) - F(y) = P(y < X \le x)$

5:35 Bernoulli random variables led to a bunch of other r.v.s. Almost all the discrete r.v.s studied in the class are from Bernoulli.

6:00 The probability mass function (PMF) of X is

$p(0) = P(X=0) = 1-p$
$p(1) = P(X=1) = p$

6:57 When you don't assign the values to a PMF, assume they're zero

7:35 Bernoulli Led to Binomial with parameters n and p: Bi(n,p)

You perform n Bernoulli trials

$\Omega = \{SS\ldots S, SS\ldots SF, SS\ldots SFS, \ldots, FF\ldots F\}$, $X(\omega) = \#$ S's in ω

Probability of any outcome: $P(\omega) = p^{\#\text{S's in } \omega}(1-p)^{\#\text{F's in } \omega}$

PMF: $P(X=k) = p^k(1-p)^{n-k}\binom{n}{k}$

10:55 Bernoulli Led to Geometric

Perform independent Bernoulli trials. X = # of trial on which first S occurs.

PMF: $P(X=k) = (1-p)^{k-1}p$, $k = 1, 2, \ldots$

12:25 Distribution function: $F(k) = P(X \le k) = \sum_{j=1}^{k} (1-p)^{j-1} p$

We get the distribution function from probability mass function by adding.

13:20

Bernoulli Led to Poisson

Perform n independent Bernoulli trials with probability: $p = \frac{\lambda}{n}$ If X is the number of successes

$$\text{PMF}: P(X=k) \approx \frac{\lambda^k}{k!} e^{-\lambda}$$

15:02

Bernoulli Led to Normal Distribution

Perform n independent Bernoulli trials with probability: $p = \frac{1}{2}$ then if X = # successes

$$P\left(\frac{X}{\sqrt{n}} \leq n\right) \approx \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2}\,dy$$

17:23

X geometric r.v with parameter p

Given conditional probability:

18:38 $P(\underbrace{X > n+k-1}_{A} \mid \underbrace{X > n-1}_{B})$ $\quad P(A|B) = \frac{P(A \cap B)}{P(B)}$

means ∩ – both events occur

$$= \frac{P(X > n+k-1, X > n-1)}{P(X > n-1)}$$ We can remove the 2nd term 21:34

$$\boxed{\sum_{j=0}^{\infty} r^j = \frac{1}{1-r} \quad \text{as long as } |r| < 1}$$ 25:40

22:10 A distribution function has this form: $P(X \leq x)$

How to compute $P(X > x) = ?$

$1 - F(x)$ from $P(A^c) = 1 - P(A)$

Remember Geometric distribution is

$$F(k) = P(X \leq k) = \sum_{j=1}^{k} (1-p)^{j-1} p$$

Now we can use this

23:25 $P(X > k) = 1 - F(k) = \underbrace{\sum_{j=1}^{\infty} (1-p)^{j-1} p}_{\text{this is } 1} - \sum_{j=0}^{k} (1-p)^{j-1} p = \sum_{j=k+1}^{\infty} (1-p)^{j-1} p = p \sum_{j=k+1}^{\infty} (1-p)^{j-1} = p(1-p)^k \sum_{j=0}^{\infty} (1-p)^j$

factor out $(1-p)^k$

$$= p(1-p)^k \cdot \frac{1}{1-(1-p)} = (1-p)^k$$

Now we can evaluate ratio

29:15 $$\frac{P(X > n+k-1, \cancel{X > n-1})}{P(X > n-1)} \qquad P(X > k) = (1-p)^k \qquad = \frac{(1-p)^{n+k-1}}{(1-p)^{n-1}} = (1-p)^k$$

Same as starting from the beginning. You are not due!

Coin or die has no memory

31:50

Continuous Random Variables

X is continuous if $P(X \leq x) = F(x)$ is a continuous function of X

32:35 These functions can have flat spots but no jumps.

33:05

Exponential R.V. with parameter λ

If $\lambda > 0$, X is called an exponential r.v. with parameter λ if

$$F(x) = P(X \le x) = \begin{cases} 1 - e^{-\lambda x}, & \text{if } x \ge 0 \\ 0, & \text{if } x < 0 \text{ — can't have negative values} \end{cases}$$

If $x > 0$, $P(X > x) = 1 - e^{-\lambda x} = e^{-\lambda x}$ — dies off exponentially fast as x goes to infinity

35:30

Derivative of F(x):

$$f(x) = F'(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & x < 0 \end{cases}$$

not differentiable at 0

— the density of X

36:30

take difference of distribution functions. OR integrate the density from y to x

$$P(y < X \le x) = F(x) - F(y) = \int_y^x f(t)\,dt \text{ if } F' = f$$

used to model waiting times

e.g Waiting for bus
- Car in front of you to move after light changes.
- Smoke detector.
- Light bulbs to burn out.

38:15

Has same memoryless property as Geometric random variable: $P(X > s + t \mid X > t)\ P(X > s)$, s, t > 0

Let's verify this equation is true. Use definition of Conditional Probability

s,t > 0 means s+t >, therefore second event x > t has no affect, so ignore it

$$= \frac{P(X > s+t, X > t)}{P(X > t)} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda t}} = e^{-\lambda s} = P(X > s)$$

41:55

Gamma Random Variable

The gamma function is $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\,dx$, $\alpha > 0$

$u = x^{\alpha-1}$, $du = (\alpha-1)x^{\alpha-2}\,dx$

$dv = e^{-x}\,dx$, $v = -e^{-x}$

Can integrate by parts

43:48

$$\Gamma(\alpha) = -x^{\alpha-1} e^{-x} \Big|_0^\infty + (\alpha-1)\int_0^\infty x^{\alpha-1} e^{-x}\,dx$$

$$= (\alpha-1)\int_0^\infty x^{(\alpha-1)-1} e^{-x}\,dx = (\alpha - 1)\,\Gamma(\alpha-1)$$

47:30

$$\lambda^\alpha \int_0^\infty x^{\alpha-1} e^{-\lambda x}\,dx \stackrel{?}{=} \lambda^{-1}\lambda^{1-\alpha}\int_0^\infty y^{\alpha-1} e^{-y}\,dy$$

$y = \lambda x$, $dy = \lambda\,dx$

$$= \lambda^{-\alpha}\,\Gamma(\alpha)$$

$$\left.\begin{array}{l} \Gamma(\alpha) = (\alpha - 1)\,\Gamma(\alpha-1) \\ \Gamma(1) = (1) \end{array}\right\} \Rightarrow \Gamma(n) = (n-1)!$$

That says the following is the density:

$$f(x) = \begin{cases} \dfrac{\lambda^2}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, & \text{if } x > 0 \\ 0, & \text{if } x < 0 \end{cases}$$

Used to model successive waiting time. e.g. 5 lightbulbs burnout.

X is a Γ(α,λ) random variable if its density is ?

52:30

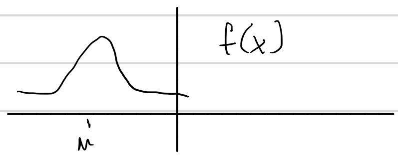
X is a normal random variable with mean μ and variance σ^2 if

Distribution function $P(X \le x) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{(y-\mu)^2}{2\sigma^2}} dy$

The density of X is $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-\mu)^2}{2\sigma^2}}, \ \forall x$

The famous bell-shaped curve

symmetric with respect to μ



56:55

Functions of Random Variables

If X is a random variable with distribution function F, and g is a strictly increasing function, compute the distribution function of g(X). Using strictly to make function one-to-one (simpler that way)

58:40

A function of a random variable is a random variable

59:00

The distribution function of g(X) is $P(g(X) \le x)$

Let's compute in terms of capital F: $F(x) = P(X \le x)$

1:00:10

If g is increasing, its inverse is increasing

$$P(g(X) \le x) = P(X \le g^{-1}(x)) = F(g^{-1}(x))$$

1:01:10

Example Suppose X is uniformly distributed on [0,1] and

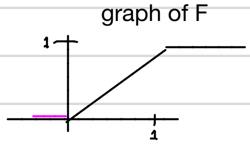
$g(x) = -\frac{1}{\lambda}\ln(1-x)$ —an increasing function

What is the distribution of g(X)? i.e. What is distribution of $P(g(X) \le x)$?

The distribution of X is

$$F(x) = \begin{cases} 0, & \text{if } x \le 0 \\ x, & \text{if } 0 < x < 1 \\ 1, & \text{if } 1 \le x \end{cases}$$

graph of F



If g is strictly decreasing P

1:04:10

Let's try to compute the distribution function of g(X)

If g is strictly decreasing: $P(g(X) \le x) = P(X\ g(x)) = 1 - F(g\ (x))$

$x > 0,\ P\left(\frac{-1}{\lambda}\ln(1-X) \le x\right) = P(X \le g^{-1}(x)) = P(X \le 1 - e^{-\lambda x}) = 1 - e^{-\lambda x}$

1:07:20

Find $g^{-1}(x)$: $g(x) = -\frac{1}{\lambda}\ln(1-x) = y$

$= \ln(1-x) = -\lambda y$

$1 - x = e^{-\lambda y}$

$g^{-1}(y) = 1 - e^{-\lambda y} = x$

1:10:05

Example Suppose $X \sim N(\mu, \sigma^2)$ — normal, mean μ, variance σ^2

Y = aX + b — What's the distribution of Y? i.e. P(Y ≤ x)?

$$P(Y \le x) = P(aX + b \le x)$$
$$= P(aX \le x - b) \quad \text{Let's assume} > 0$$
$$= P\left(X \le \frac{x-b}{a}\right)$$

So, distribution function of Y

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\frac{x-b}{a}} e^{\frac{-(y-\mu)^2}{2\sigma^2}} dy$$

1:14:00

Take the derivative of the distribution function to get the density of Y:

$$f_Y(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-\left(\frac{x-b}{a} - \mu\right)^2}{2\sigma^2}} \frac{1}{a}$$

$$\boxed{\frac{d}{dx} \int_{-\infty}^{h(x)} f(y)\,dy = f(h(x))\,h'(x)}$$

1:16:15

Simplify

$$\frac{1}{\sqrt{2\pi a^2 \sigma^2}} \exp\left\{-\frac{(x - (b - a\mu))^2}{2\sigma^2 a^2}\right\}$$

This is a $N(b + a\mu, \sigma^2 a^2)$

1:19:27

$$N(\mu, \sigma^2) \xrightarrow{?} N(0,1)$$

$$a = \frac{1}{\sigma}, \quad b = \frac{\mu}{\sigma}$$

If $X \sim N(0,1)$ then $\sigma X + \mu \sim N(\mu, \sigma^2)$

1:27:10

Chapter 3: Jointly Distributed Random Variables