mmellinger66@gmail.com



UC Irvine Statistics 131a - Summer 2013

## Estimating Population Size

n = # members of a coyote population
10 coyotes are captured, tagged, and released.

Some respectable time later, 20 coyotes are captured.

A = {4 of the 20 captured coyotes are tagged]

Problem: From the information, give an estimate of n.

$$P(A) = \frac{\binom{10}{4}\binom{n-10}{16}}{\binom{n}{20}} = f(n)$$

Find n that makes P(A) the largest. The most probable thing is probably what happened.

Look at ratio $\frac{f(n+1)}{f(n)} > 1$ When the ratio stops being > 1, that is our estimator

$$\frac{f(n+2)}{f(n+1)} < 1$$

3:40

$$\frac{f(n+1)}{f(n)} = \frac{\binom{n-9}{16}\binom{n}{20}}{\binom{n+1}{20}\binom{n-10}{16}} = \frac{\frac{(n-9)!}{\cancel{16!}\,(n-25)!}\;\frac{n!}{(n-20)!\,\cancel{20!}}}{\frac{(n-10)!}{\cancel{16!}\,(n-26)!}\;\frac{(n+1)!}{(n-19)!\,\cancel{20!}}}$$

$$= \frac{(n-9)!\,n!\,(n-26)!\,(n-19)!}{(n-10)!\,(n+1)!\,(n-25)!\,(n-20)!} = \frac{(n-9)(n-19)}{(n+1)(n-25)}$$

Find last (n+1) for which this is > 1

OR

First n for which this is < 1

$$(n-9)(n-19) > (n+1)(n-25)$$
$$n^2 - 28n + 171 > n^2 - 24n - 25$$
$$196 > 4n$$
$$49 > n \Rightarrow \underset{n+1}{48+1} = \boxed{49}$$

10:50

estimated population size
After 49, probability starts going down

12:00

## Multinomial Theorem

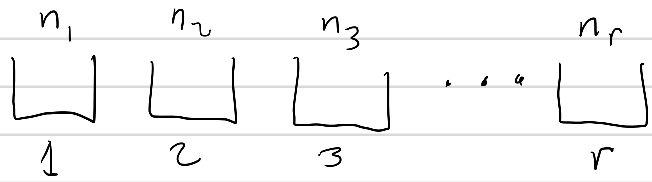
You want to store n objects in r closets, or boxes with $n_1$ objects in closet 1, $n_2$ in closet 2, ... $n_r$ in closet r where

$$\sum_{i=1}^{r} n_i = n$$

How many ways are there to do this?

use mult. princ.

15:15




last one = 1

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-\cdots-n_{r-1}}{n_r}$$

16:00 Simplify this idea

$$= \frac{n!}{(n-n_1)!\,n_1!}\cdot\frac{(n-n_1)!}{(n-n_1-n_2)!\,n_2!}\cdot\frac{(n-n_1-n_2)!}{(n-n_1-n_2-n_3)!\,n_3!}\cdot\ldots\cdot 1$$

$$= \frac{n!}{n_1!\,n_2!\,n_3!\cdots n_r!} = \binom{n}{n_1\, n_2 \cdots n_r}$$

— The multinomial coefficient

18:00 **The Multinomial Theorem**

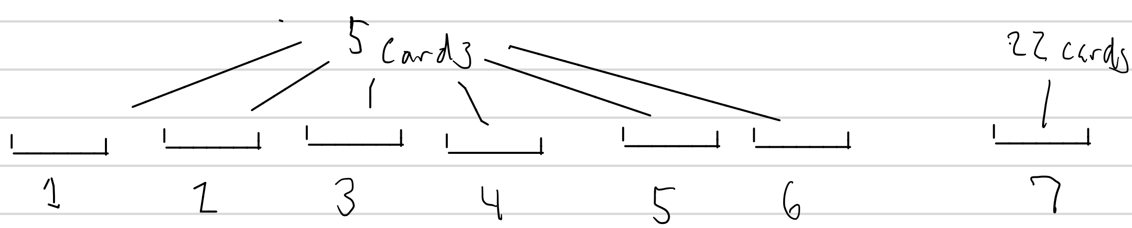
$$(x_1+x_2+\cdots+x_r)^n = \sum_{\Sigma n_i = n}\binom{n}{n_1\, n_2\cdots n_r} x_1^{n_1}x_2^{n_2}\cdots x_r^{n_r}$$

19:20

**Poker Example**

How many ways to deal six hands with five cards each from a deck of 52 cards?

20:35




5 cards in first six boxes
The remainder go in box seven: 22

$$\binom{52}{5\,5\,5\,5\,5\,5\,22} = \frac{52!}{(5!)^6\,22!}$$

22:50

Example: # possible Bridge hands?

4 players each with 13 cards

$$\binom{52}{13\ 13\ 13\ 13} = \frac{52!}{(13!)^4}$$

25:23

## Conditional Probability and Independence

If events $A, B \subset \Omega$ and $P(B) > 0$ then the probability of $A$ given $B$ is:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

26:35

Example: You roll a fair die. What is P(A|B) if A = {6 appears} and B = {the outcome is even}?

$$\frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)} = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}$$

29:25 Can rewrite above equation as: $P(A \cap B) = P(A|B)P(B)$

30:45

## Law of a Total Probability

If $B_1, \ldots, B_n$ are mutually disjoint events

$$\bigcup_{i=1}^{n} B_i = \Omega \text{ and } P(B_i) > 0,\ i = 1, 2, \ldots, n$$

Then $P(A) = \sum_{i=1}^{n} P(A|B_i) P(B_i)$

32:50

$$= \sum_{i=1}^{n} P(A \cap B_i) = P\left(\bigcup_{i=1}^{n} (A \cap B_i)\right)$$

$$= P\left(A \cap \left(\bigcup_{i=1}^{n} B_i\right)\right) \text{ by De Morgan's Law}$$

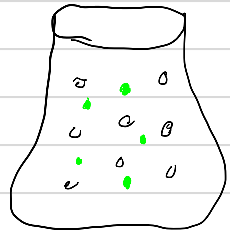
$$= P(A \cap \Omega) = P(A)$$

34:25

## Urn Problem

A gum ball is drawn at random. Its color is noted and replaced together with two more of the same color. Then the step is repeated.

8 white
5 green



37:00

If $G_2 = \{\text{the second gumball drawn is green}\}$ What is $P(G_2)$?

38:20

Condition on what happened the first time. — Law of Total Probability

$G_1 = \{\text{green gumball is drawn first time}\}$
$W_1 = \{\text{white " " " " "}\}$

$G_1 \cup W_1 = \Omega$ The union exhausts all possibilities

$G_1 \cap W_1 = \emptyset$

40:15

$P(G_1) = \frac{5}{13} > 0, \quad P(W_1) = \frac{8}{13} > 0$

$$P(G_2) = P(G_2 \mid G_1)P(G_1) + P(G_2 \mid W_1)P(W_1) = \frac{7}{15}\cdot\frac{5}{13} + \underbrace{\frac{5}{15}\cdot\frac{8}{13}}$$

15 balls on 2nd draw

43:20

If $G_3 = \{\text{the third gumball drawn is green}\}$ What is $P(G_3)$?

shorthand

$G_1 \cap G_2$ - $G_1 G_2$
$G_1 \cap W_2$ - $G_1 W_2$
$W_1 \cap G_2$ - $W_1 G_2$
$W_1 \cap W_2$ - $W_1 W_2$

All disjoint. Union of all possibilities = Ω

All probabilities are > 0. Therefore we can use the Law of Total Probability

46:20

$$P(G_3) = P(G_3 \mid G_2 G_1)P(G_2 G_1) + P(G_3 \mid G_1 W_2)P(G_1 W_2) + P(G_3 \mid W_1 G_2)P(W_1 G_2) + P(G_3 \mid W_1 W_2)P(W_1 W_2)$$

47:35

$$= \frac{9}{17}P(G_2 \mid G_1)P(G_1) + \frac{7}{17}P(W_2 \mid G_1)P(G_1) + \frac{7}{17}P(G_2 \mid W_1)P(W_1) + \frac{5}{17}P(W_2 \mid W_1)P(W_1)$$

$$= \frac{9}{17}\cdot\frac{7}{15}\cdot\frac{5}{13} + \frac{7}{17}\cdot\frac{8}{15}\cdot\frac{5}{13} + \frac{7}{17}\cdot\frac{5}{15}\cdot\frac{8}{13} + \frac{5}{17}\cdot\frac{10}{15}\cdot\frac{8}{13}$$

51:50 **Bayes Rule**

Under the same conditions as the Law of Total Probability

$$P(B_j|A) = \frac{P(A|B_j)P(B_j)}{\sum_{i=1}^{n} P(A|B_i)P(B_i)}$$

53:45 **Example**

Let $A = G_2$, $B_1 = W_1$, $B_2 = G_1$ What's $P(W_1|G_2)$? — What's cause given effect?

55:55

$$P(W_1|G_2) = \frac{P(G_2|W_1)P(W_1)}{P(G_2|W_1)P(W_1) + P(G_2|G_1)P(G_1)} = \frac{\frac{5}{15}\cdot\frac{8}{13}}{\frac{5}{15}\cdot\frac{8}{13} + \frac{7}{15}\cdot\frac{5}{13}}$$

58:05 **Independence**

$A$ & $B$ are called independent if $P(A\cap B) = P(A)P(B)$

If $P(B) > 0$, this is equivalent to $P(A|B) = P(A)$

1:00:50 A = {H second toss}, B = {H first toss}, C = {T on first toss}

$$P(A\cap B) = \phi$$
$$P(A) = P(A\cap B) + P(A\cap C) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$
$$P(B) = \frac{1}{2}$$

$$P(A)P(B) = \frac{1}{4}$$

1:04:35 There's memory in the urn problem so there's not independence.

1:06:00 A dart is thrown repeatedly at a target and the probabílity of a bull's eye is 0.05. If the dart is thrown n times and the trials are independent,

What is P(A) if A = {at least 1 bull's eye occurs}?

1:08:20

$P(A) = 1 - P(A^c)$ — It's easier to calculate the probability of no bull's eye

$= 1 - (.95)^n$ — Probability of failing n times in a row