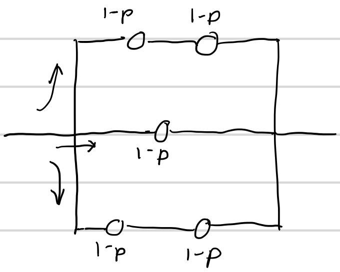
mmellinger66@gmail.com

UC Irvine Statistics 131a - Summer 2013

2:55 HW Problem #74 : Pipes



Each unit fails independently with probability p

What's the probability the system works? i.e. the water gets through?

P(fails to operate) = P(top, middle, bottom all closed)

by independence: = P(top closed) * P(middle closed) * P(bottom closed)

3:40

4:40 $= \left(1-(1-p)^2\right)^2 p$

6:25 More formally: A = {top closed}, B = {middle closed}, C = {bottom closed}

$$P(A \cap B \cap C) = P(A)\,P(B)\,P(C)$$

8:15 Random Variables (R.V.S)

Example 1: Given sample space Ω = {H,T}, define a function $X : \Omega \to \mathbb{R}$ $X(H) = 1$ (win \$1), $X(T) = -1$ (lose \$1)

This kind of function from a sample space to a real line is called a Random Variable.

Joe Doob (Chance Variables) and William Feller (Random Variables) argued over the naming of these functions. They tossed a coin and Feller won.

10:55 Example 2: Ω = {HH...H, HH...T, HH...HTH, ...} = {all sequences of length n with components H or T}

Could replace H and T with S and F:

H → S - a success
T → F - a failure

12:43

Defn: $X : \Omega \to \mathbb{R}$ by $X(\omega) = \#\, H\text{'s in } \omega$ for $\omega \in \Omega$

13:38 Generally, a random variable is a function on a sample space with real values: $X : \Omega \to \mathbb{R}$

14:20 We identify a random variable through its statistical properties by means of the function.

15:20 $F(x) = P(X \le x)$ — The distribution function of X.

There is a probability on the sample space. We look at the probability of all outcomes for which the random variable has a value less than or equal to x.

15:39 $= (= P(\{w \in \Omega : X((w) \le x\}))$ — longer/explicit way of saying the same thing

(1) $0 \le F(x) \le 1$ 0 ≤ F(x) ≤ 1

16:57 If $x < y$ then $\{X \le x\} \subseteq \{X \le y\}$

18:15 (2) $F(x) = P(X \le x) \le P(X \le y) = F(y)$ — i.e. F is a non-decreasing function (or increasing but not strictly increasing)

F(x) = P(X ≤ x) ≤ P(X ≤ y) = F(y) — i.e. F is a non-decreasing function (or increasing but not strictly increasing)

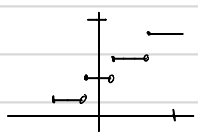
(3) F is right-continuous F is right-continuous

19:30 $\lim_{x \to -\infty} F(x) = \lim_{x \to -\infty} P(X \le x) = 0$ $\lim_{x \to \infty} F(x) = \lim_{x \to \infty} P(X \le x) = 1$

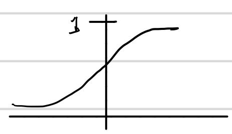
21:58 - Different random variables have different distribution functions
- We identity random variables by their distribution functions

22:40 Two major classes of random variables (r.v.s):
* Class 1: Discrete
* Class 2: Continuous

24:25 X is a discrete random variable if its distribution function has jumps interrupted by flat spots



25:05 X is a continuous random variable if its distribution function is continuous



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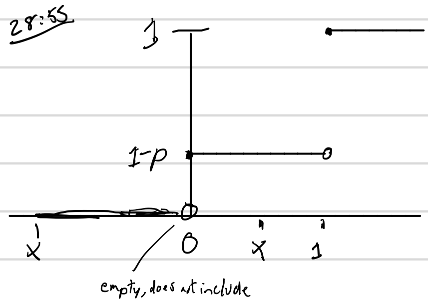
Examples of Discrete R.V.S.

Ex 1: Bernoulli Random Variable

PMF $\begin{cases} P(X = 0) = 1 - p \text{ — failure} \\ P(X = 1) = p \text{ — success} \end{cases}$

Probability Mass Function - PMF

28:55



F(x) = P(X ≤ x)

31:50

Ex 2: Binomial Random Variable

In this example, n independent Bernoulli trials are performed. i.e. an experiment with probability p of success, 1-p of failure and X = # successes in n trials.

$X \in \{0, 1, \ldots, n\}$

$$p(k) = P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, \ldots, n$$

$$\underset{1}{\underline{S}} \ \underset{2}{\underline{S}} \ \ldots \ \underset{k}{\underline{S}} \ \underset{k+1}{\underline{F}} \ \ldots \ \underset{n}{\underline{F}}$$

Such an X is called Bi(n,p)

38:05

Probability Mass Function (PMF)

How can we get from probability mass function (PMF) to the cumulative distribution function (CDF)?

$$F(x) = P(X \leq x) = \sum_{y \leq x} p(y), \quad p(y) = P(X=y)$$

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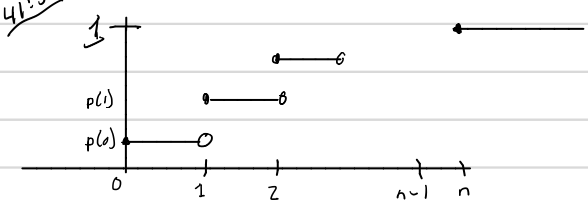
$$F(x) = \sum_{y \leq x} p(y)$$

- The relationship between CDF and a PMF
- If you know the PMF, you can get the CDF by this summation

41:30



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Ex 3: Geometric Random Variable

Perform independent Bernoulli trials until a success occurs
- Let X = # of trial on which first success occurred

X could take on any value in {1,2,3,...}

$$P(X=k) = (1-p)^{k-1} p, \quad k=1,2,3,\ldots$$

$$\underset{1}{F} \ \underset{2}{F} \ \underset{3}{F} \ldots \underset{k-1}{F} \ \underset{k}{S}$$

Geometric Series

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

$r<1 \Rightarrow$ finite

47:15

$$P(X<\infty) = \sum_{k<\infty} p(k) = \sum_{k=1}^{\infty} (1-p)^{k-p} p$$

48:10

$$= p \sum_{j=0}^{\infty} (1-p)^j \quad k-1=j \quad = p \frac{1}{1-(1-p)} = 1$$

50:04

Negative Binomial Random Variable

Perform n independent Bernoulli trials until r successes occur

Let X = # of trials on which rth success occurs

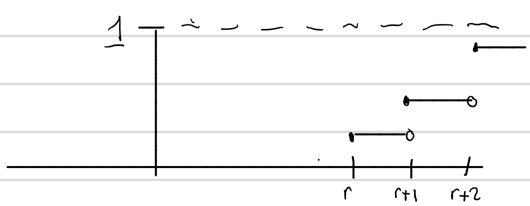
X could take on any value in {r, r+1,r+2,...} = k

$$\text{PMF:} \quad p(k) = P(X=k) = \binom{k-1}{r-1} p^r (1-p)^{k-r}$$

$$\underset{1}{\_\_} \ \underset{2}{\_\_} \ \_\_ \ldots \underset{k-1}{\_\_} \ \underset{k}{S}$$ — last success

54:12

$$\text{CMF: } F(x) = \sum_{k \leq x} p(k)$$



Never reaches 1

55:55

Poisson Random Variable

$$\text{PMF: } p(k) = P(X=k) = \frac{\lambda^x}{k!} e^{-\lambda}, \quad \lambda > 0 \quad k=0,1,2,\ldots$$

Sum of PMF (always 1):

$$1 \stackrel{?}{=} \sum_{k=0}^{\infty} p(k) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{-\lambda} e^{\lambda} = 1$$

59:57

Examine relationship of Binomial R.V. to the Poisson

Let X_n be $Bi(n, p_n)$ with $\lim_{n\to\infty} n p_n = \lambda \in (0,\infty)$

$$P(X_n > 0) = (1-p_n)^n = \left(1 - \frac{np_n}{n}\right)^n \approx \left(1-\frac{\lambda}{n}\right)^n$$

by L'Hopital $e^{-\lambda}$

From before $P(X_n = k) = \binom{n}{k} p_n^k (1-p_n)^{n-k}$

Assume $p_n = \frac{\lambda}{n}$

1:03:24

Let's consider $\frac{P(X_n = k+1)}{P(X_n = k)} = \frac{\binom{n}{k+1} p_n^{k+1} (1-p_n)^{n-k-1}}{\binom{n}{k} p_n^k (1-p_n)^{n-k}}$

1:08:12

1st part

$$\frac{\binom{n}{k+1}}{\binom{n}{k}} = \frac{\cancel{n!}}{(k+1)!(n-k-1)!} \cdot \frac{k!(n-k)!}{\cancel{n!}} = \frac{1}{k+1}(n-k)$$

2nd part

$$\frac{p_n^{k+1}(1-p_n)^{n-k-1}}{p_n^k (1-p_n)^{n-k}} = p_n(1-p_n)^{-1}$$

Put the two together: $\frac{P(X_n = k+1)}{P(X_n = k)} = \frac{1}{k+1} \cdot ((n-k)p_n)(1-p_n)^{-1}$ $n \to \infty$

2nd part converges to λ

$$(n-k)p_n = \left(\frac{n-k}{n}\right) n p_n \to \lambda \longrightarrow \frac{\lambda}{k+1}$$

1:12:10

So far we know: $\lim_{n\to\infty} \frac{p_n(k+1)}{p_n(k)} = \frac{\lambda}{k+1}$ $\quad \lim_{\to\infty} p_n(0) = e^{-\lambda}$

where $p_n(k) = P(X_n = k)$

1:13:26

What is $\lim_{n\to\infty} p_n(1) \overset{?}{=} \lim_{n\to\infty} \frac{p_n(1)}{p_n(0)} p_n(0) = \lambda e^{-\lambda}$

$$\lim_{n\to\infty} p_n(2) \overset{?}{=} \lim_{n\to\infty} \frac{p_n(2)}{p_n(1)} p_n(1) = \frac{\lambda}{2}\lambda e^{-\lambda} = \frac{\lambda^2}{2} e^{-\lambda} = \frac{\lambda^2}{2!} e^{-\lambda}$$

$$\lim_{n\to\infty} p_n(3) \overset{?}{=} \lim_{n\to\infty} \frac{p_n(3)}{p_n(2)} p_n(2) = \frac{\lambda}{3}\frac{\lambda^2}{2!} e^{-\lambda} = \frac{\lambda^3}{3!} e^{-\lambda}$$

General case:

$$\frac{\lambda^k}{k!} e^{-\lambda}$$

The PMF of Poisson

1:15:52

So, large number of Bernoulli trials with very small probability of success is almost like a Poisson r.v.

This is the Law of Rare Events

1:19:25

Continuous Random Variables

Remember “continuous“ means the CDF is a continuous function of X $F(x) = P(X \leq x)$

If F has a derivative F’= f then (mostly true) the integral of its derivative:

$$F(x) = \int_{-\infty}^{\infty} f(y)\,dy$$

Little f is called the density of X. — Probability Density Function (PDF)

We usually identify continuous random variables by the density.

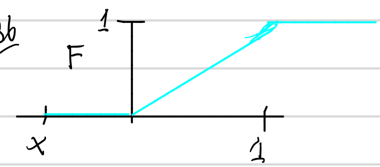
1:22:00

Example: The Uniform on [0,1] r.v. U([0,1]) has density:

$$f(x) \begin{cases} 1, & \text{if success} \\ 0, & \text{otherwise} \end{cases} \qquad F(x) = P(X \leq x) = \int_{-\infty}^{x} f(y)\,dy$$

Integral of 1 from 0 to x

1:23:36



1:25:25

If $0 < a < b$ what is $P(a < X \leq b)$?

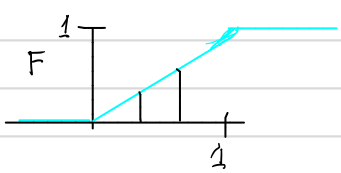
A   A ∪ B   B

$$A \cap B \neq \emptyset \Rightarrow P(A \cup B) = P(A) + P(B)$$

$P(a < X \leq b) = P(X \leq b) - P(X \leq a)$

$= \int_a^b f(x)\,dx$ —— Just integrate the density between the two numbers

1:29:05



$$\boxed{P(a < X \leq b) = \int_a^b f(x)\,dx}$$

Normal Random Variable

1:30:35

Example 2: $X \sim N(\mu, \sigma^2)$ $N(\mu, \sigma^2)$ = normal, mean μ, variance σ^2

Its density is given by: $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$, $-\infty < x < \infty$

1:32:15

If $X \sim N(0,1)$ then $f(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}$, $-\infty < x < \infty$

1:32:35

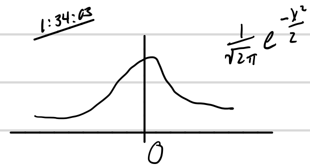
$1 = P(-\infty < X < \infty) = \int_{-\infty}^{\infty} f(x)\,dx$

For any continuous random variable with density little f the probability X is between -∞ and ∞

Is it 1? $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-x^2}{2}}\,dx \overset{?}{=} 1$

Observe it's an even function

1:34:05



So, $= \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{\frac{-x^2}{2}}\,dx$

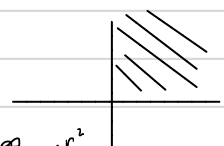
No way to integrate

Square this gives us: $\frac{2}{\pi}\left(\int_0^{\infty} e^{\frac{-x^2}{2}}\,dx \int_0^{\infty} e^{\frac{-y^2}{2}}\,dy \overset{?}{=} 1\right.$

1:36:20

$\left(\int_0^{\infty} e^{\frac{-x^2}{2}}\,dx\right)^2 = \int_0^{\infty} e^{\frac{-x^2}{2}}\,dx \int_0^{\infty} e^{\frac{-y^2}{2}}\,dy = \int_0^{\infty}\int_0^{\infty} e^{-\frac{x^2+y^2}{2}}\,dx\,dy$

1:37:15 Let's change to polar coordinates



we're in 1st quadrent on graph

$= \int_0^{\pi/2}\int_0^{\infty} e^{\frac{-r^2}{2}}\, r\,dr\,d\theta = \frac{\pi}{2}\int_0^{\infty} r e^{\frac{-r^2}{2}}\,dr$ Let $u = \frac{r^2}{2}$, $du = r\,dr$

$= \frac{\pi}{2}\int_0^{\infty} e^{-u}\,du = \frac{\pi}{2}\left[-e^{-u}\Big|_0^{\infty}\right] = \frac{\pi}{2}$

1:39:35 Does this integrate to 1? Yes!

1:40:10 If X is $Bi(n, \frac{1}{2})$

Like tossing a coin n times and counting the # heads

$P\left(\frac{X}{\sqrt{n}} \le b\right) \longrightarrow \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{b} e^{\frac{-x^2}{2}}\,dx$

When n is large, a binomial random variable with those parameters is essentially a normal random variable.

That's the Central Limit Theorem