Lemma 3.2. *Let $k \in \mathbb{N}_0$, $s \in 2\mathbb{N}-1$ and $M > 0$. For every $\epsilon > 0$, there exists a shallow tanh neural network $\psi_{s,\epsilon} : [-M, M] \to \mathbb{R}^s$ of width $\frac{3(s+1)}{2}$ such that*

$$\max_{p \le s} \left\| f_p - (\psi_{s,\epsilon})_p \right\|_{W^{k,\infty}} \le \epsilon. \tag{26}$$

Furthermore, the weights scale as $O\left(\epsilon^{-s/2}(\sqrt{M}(s+2))^{3s(s+3)/2}\right)$ for small ϵ and large s.

將偶數次方表示成奇數次相減一偶數次

$$y^{2n} = \frac{1}{2\alpha(2n+1)}\left((y+\alpha)^{2n+1} - (y-\alpha)^{2n+1} - 2\sum_{k=0}^{n-1}\binom{2n+1}{2k}\alpha^{2(n-k)+1}y^{2k}\right).$$

轉為神經網路

$$(\psi_{s,\epsilon}(y))_{2n} = \frac{1}{2\alpha(2n+1)}\left(\hat{f}_{2n+1,h}(y+\alpha) - \hat{f}_{2n+1,h}(y-\alpha) - 2\sum_{k=0}^{n-1}\binom{2n+1}{2k}\alpha^{2(n-k)+1}(\psi_{s,\epsilon}(y))_{2k}\right)$$

定義 $E_p = \| f_p - (\psi_{s,\epsilon})_p \|_W$

we want to prove.

$$E_p \le E_p^* := \frac{2^{p/2}(1+\alpha)^{(p^2+p)/2}}{\alpha^{p/2}} \cdot \epsilon.$$

Use 數學歸納法.

For $p=2$,

$$E_2 \le \frac{1}{6\alpha} \cdot 2\varepsilon \le E_2^*$$

Then we assume for all $k<n$

that $E_{2k} \le E_{2k}^*$

Then we got

$$E_{2n} \le \frac{1}{2\alpha(2n+1)}\left(E_{2n+1} + E_{2n+1} + 2\sum_{k=1}^{n-1}\binom{2n+1}{2k}\alpha^{2(n-k)+1}E_{2k}\right).$$

since $\max_{p \text{ odd}} E_p \le \varepsilon$

and $E_{2k} \le E_{2k}^* \le E_{2(n-1)}^*$

and $\varepsilon \le E_{2(n-1)}^*$

we got

$$\begin{aligned} E_{2n} &\le \frac{1}{\alpha(2n+1)}\left(\max_{\substack{p\le s,\\ p \text{ odd}}} E_p + \sum_{k=1}^{n-1}\binom{2n+1}{2k}\alpha^{2(n-k)+1}E_{2(n-1)}^*\right) \\ &\le \frac{1}{\alpha}\left(E_{2(n-1)}^* + (1+\alpha)^{2n+1}E_{2(n-1)}^*\right) \\ &\le \frac{2}{\alpha}(1+\alpha)^{2n+1}E_{2(n-1)}^*. \end{aligned}$$

Recalling the definition of $E_{2(n-1)}^*$, we obtain

$$E_{2n} \le \frac{2}{\alpha}(1+\alpha)^{2n+1}E_{2(n-1)}^* \le \left(\frac{2}{\alpha}(1+\alpha)^{2n+1}\right)^n \cdot \epsilon = E_{2n}^*.$$

證明完畢.

$\therefore \max_{p \le s} \| f_p - (\psi_{s,\epsilon}) \| \le E_p \le \varepsilon$

#.