國立交通大學 電子物理系 NCTU Electrophysics

Chapter 15

Mechanical Waves

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Outline

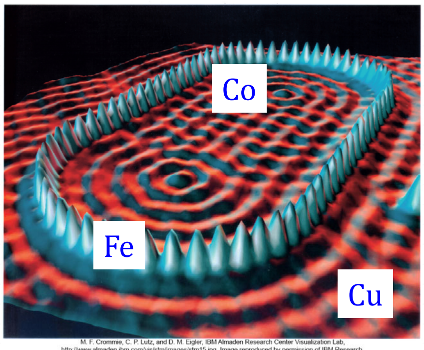
1. Wave - Propagation of a Disturbance
2. Wave function
3. The Speed of Waves on Strings
4. The Linear Wave Equation
5. Rate of Energy Transfer
6. Superposition and Interference
7. Reflection and Transmission
8. Standing Wave

1. Wave – Propagation of a Disturbance

Types of Waves

- **Mechanical waves**
  - Some physical medium is being disturbed.
  - The wave is the propagation of a disturbance through a medium.
- **Electromagnetic waves**
  - No medium required.
  - Examples are light, radio waves, x-rays.
- **Matter waves**
  - **de Broglie waves.** All matter can exhibit wave-like behavior.
- **Gravitational waves**
  - Gravitational waves are ripples in the curvature of spacetime that propagate as a wave.

Quantum corral



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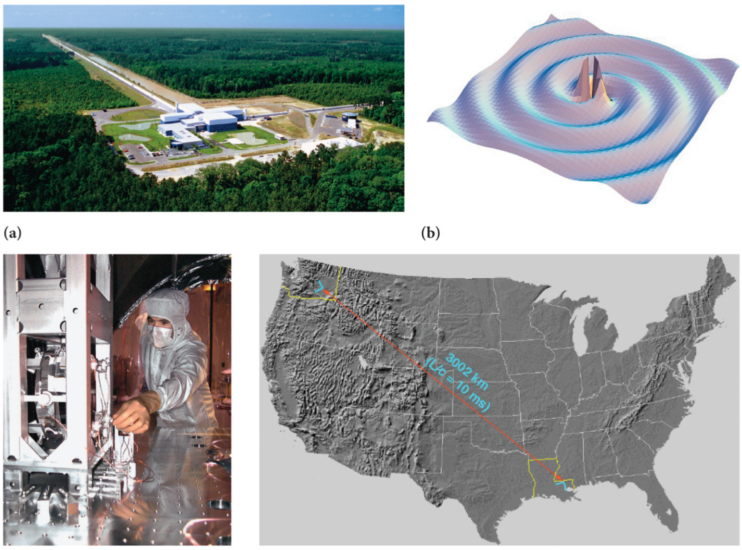

(a) (b) (c) (d)

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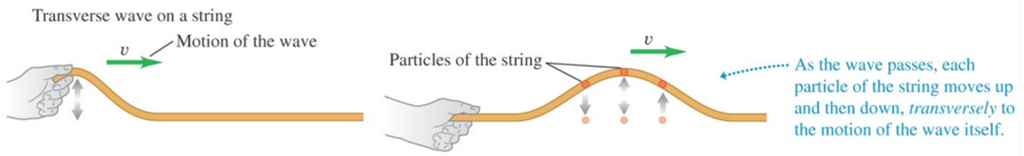
Wave properties

- All waves carry energy.
- In wave motion, energy is transferred over a distance.
- Matter is not transferred over a distance.

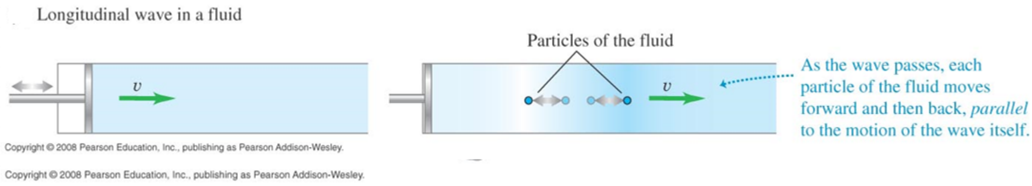
Transverse Wave

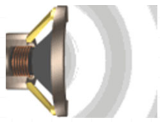
- A traveling wave or pulse that causes the elements of the disturbed medium to move perpendicular to the direction of propagation is called a transverse wave.



Longitudinal Wave

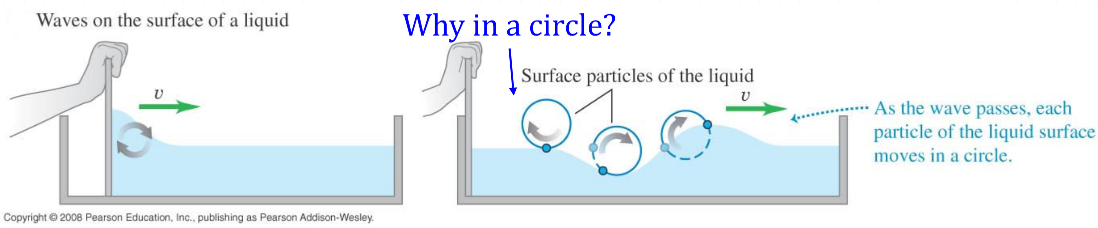
- A traveling wave or pulse that causes the elements of the disturbed medium to move parallel to the direction of propagation is called a longitudinal wave.

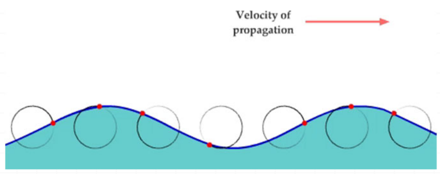


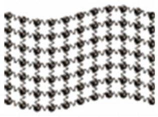


Complex Waves

- Some waves exhibit a combination of transverse and longitudinal waves.
- Surface water waves are an example.

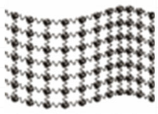




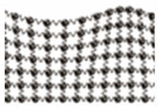


Transverse

Longitudinal



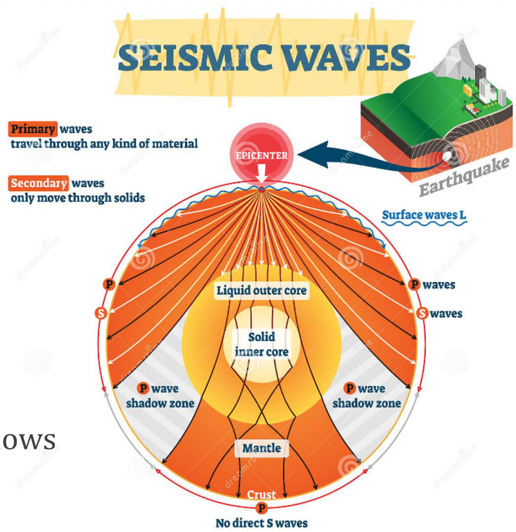
mixed



Water wave

Earthquake Waves

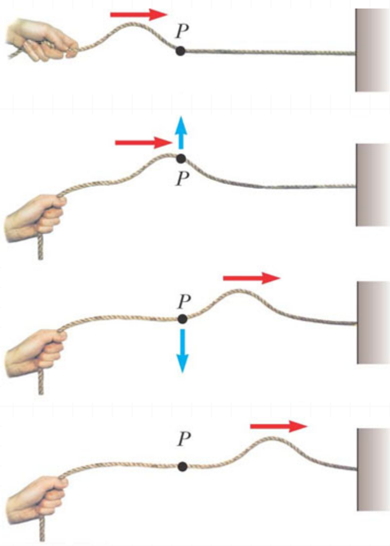
- P waves
  - "P" stands for primary
  - Fastest, at 7 – 8 km / s
  - Longitudinal
- S waves
  - "S" stands for secondary
  - Slower, at 4 – 5 km/s
  - Transverse



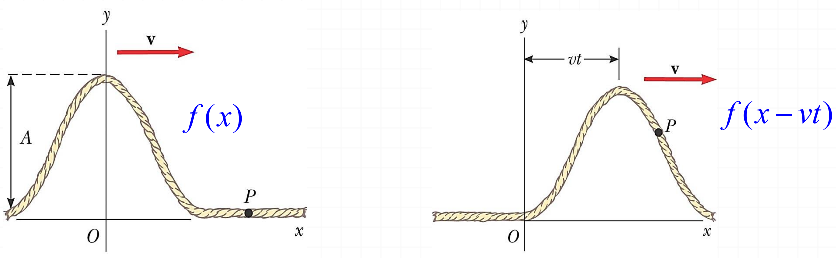
- A seismograph records the waves and allows determination of information about the earthquake's place of origin.

2. Wave function

Pulse on a Rope



Traveling Pulse



(a) Pulse at $t = 0$ (b) Pulse at time t

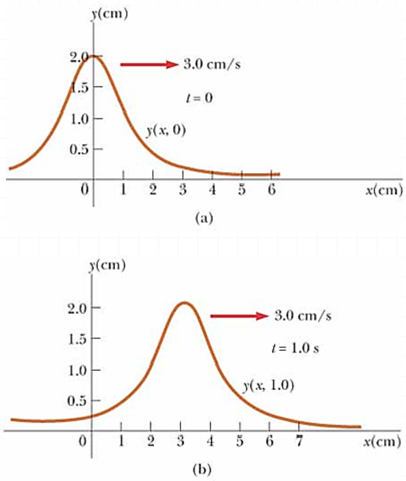
$y\,(x, t{=}0) = f\,(x)$ $y\,(x, t) = f\,(x - vt) = y(x - vt, 0)$

Wave function $y(x, t)$

- For a pulse traveling to the right ($+x$)
  - **$y\,(x, t) = f\,(x - vt)$**
- For a pulse traveling to the left ($-x$)
  - **$y\,(x, t) = f\,(x + vt)$**

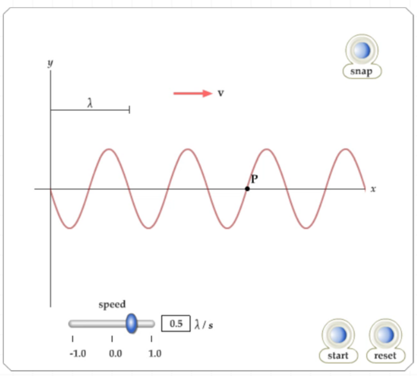
Ex.

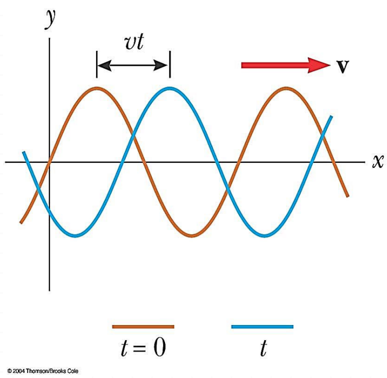
$$y(x, t) = \frac{2}{(x - 3.0t)^2 + 1}$$



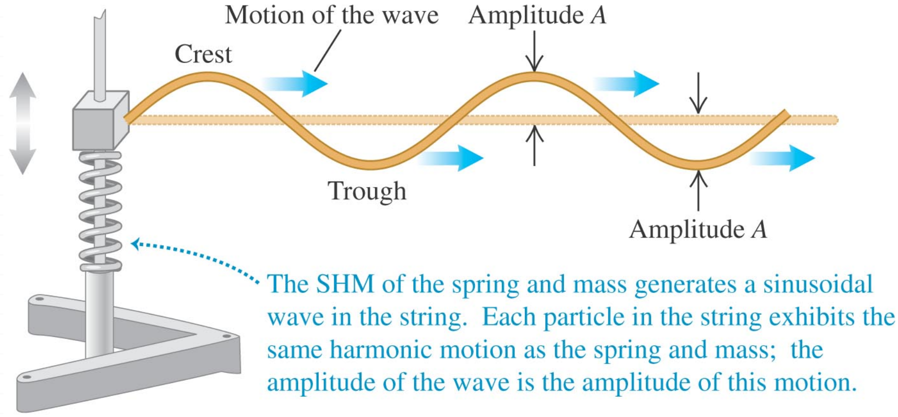
Sinusoidal Waves

- Sinusoidal wave is the simplest example of a periodic continuous wave.
  - Any periodic wave can be represented as a combination of sinusoidal waves.





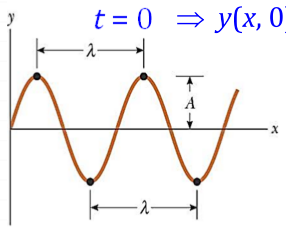
$y(t) = A\cos(\omega t + \phi)$



Sinusoidal wave, $y(x,t) = ?$

By $y\,(x, t) = y(x - vt, 0)$

$t = 0 \;\Rightarrow\; y(x, 0) \qquad y(x,0) = ? \quad A\cos(kx + \phi)$

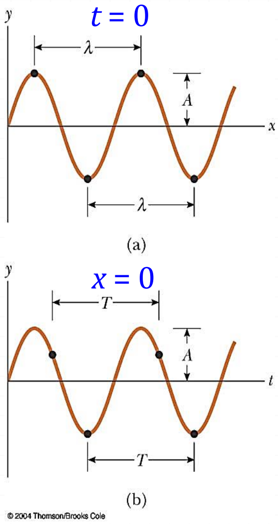


$$\begin{cases} y(x=0,0) = A\cos(0+\phi) = 0 \;\Rightarrow\; \phi = \dfrac{\pi}{2} \\ y(x=\dfrac{\lambda}{2},0) = A\cos(k\dfrac{\lambda}{2}+\phi) = 0 \;\Rightarrow\; k = \dfrac{2\pi}{\lambda} \end{cases}$$

$$\Rightarrow y(x,0) = A\cos(\frac{2\pi}{\lambda}x + \frac{\pi}{2})$$

So $y(x,t) = y(x-vt,0) = A\cos\left[\dfrac{2\pi}{\lambda}(x-vt) + \dfrac{\pi}{2}\right]$

$$= A\cos\left[2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right) + \frac{\pi}{2}\right]$$



$$y(x,t) = A\cos\left[2\pi\left(\frac{x}{\lambda}-\frac{t}{T}\right)+\frac{\pi}{2}\right]$$

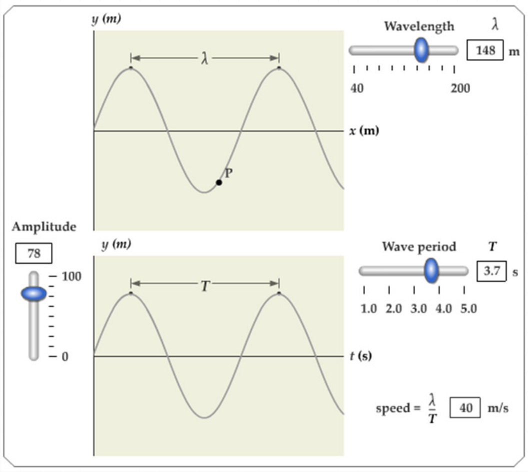
$$\Rightarrow\ y(x,t) = A\cos(kx-\omega t+\frac{\pi}{2})$$

- Amplitude: A
- Wavelength: λ
- Period: T
- Frequency: $f = 1/T$

Wave speed: $v = \frac{\lambda}{T}$

Wave number: $k = \frac{2\pi}{\lambda}$

Angular frequency: $\omega = \frac{2\pi}{T}$



Traveling sinusoidal wave:

$$y(x,t) = \begin{cases} A\cos(kx - \omega t + \phi), & "+x" \text{ direction} \\ A\cos(kx + \omega t + \phi), & "-x" \text{ direction} \end{cases}$$

Wave number: $k = \dfrac{2\pi}{\lambda}$

Angular frequency: $\omega = \dfrac{2\pi}{T}$

Transverse speed: $v_y = \dfrac{\partial y}{\partial t} = \pm\omega A\sin(kx \mp \omega t)$

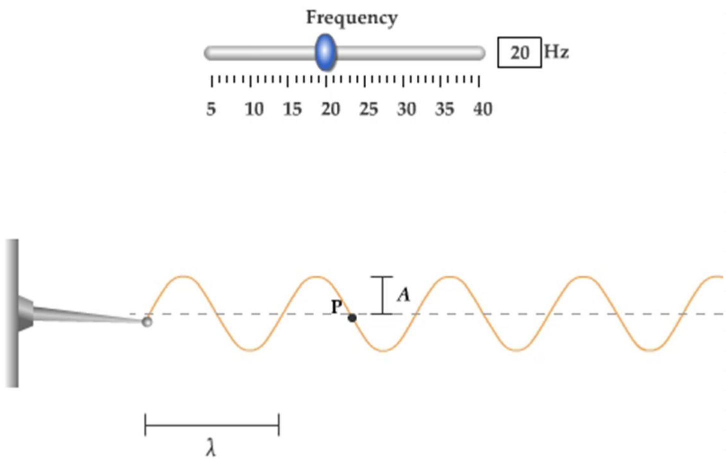
Transverse acceleration: $a_y = \dfrac{\partial v_y}{\partial t} = \mp\omega^2 A\cos(kx \mp \omega t)$

Wave (phase) speed: $v = \dfrac{\omega}{k} = \dfrac{\lambda}{T} = \lambda f$

Wave (group) speed: $v = \dfrac{\partial \omega}{\partial k}$

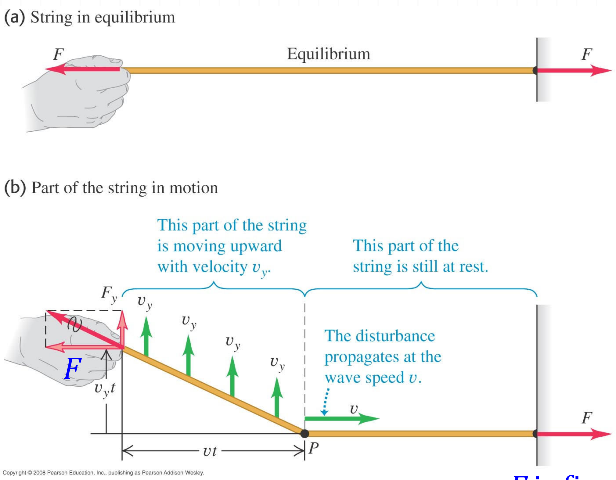
The red square moves with the phase velocity, and

the green circles propagate with the group velocity.



3. The speed of waves on strings

Method#1



$$F_y t = (\Delta m) v_y = (\mu v t) v_y$$

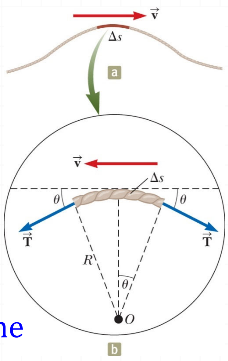
$$\frac{F_y}{F} = \frac{v_y t}{vt} \Rightarrow F_y = \frac{v_y}{v} F$$

$$\Rightarrow v = \sqrt{\frac{F}{\mu}}$$

F is fixed.

Method#2

Rest frame



Moving frame

(along with the wave)

$$F_r = ma = m\frac{v^2}{R}$$

$$\begin{cases} F_r = 2T\sin\theta \approx 2T\theta \\ m = \mu \Delta s = 2\mu R\theta \end{cases}$$

$$\Rightarrow 2T\theta = 2\mu R\theta \frac{v^2}{R}$$

$$\Rightarrow v = \sqrt{\frac{T}{\mu}}$$

$$v = \sqrt{\frac{\text{tension}}{\text{mass/length}}} = \sqrt{\frac{T}{\mu}}$$

p.s., Here, "T" is tension, not period.

The speed of all mechanical waves follows a general form:

$$v = \sqrt{\frac{\text{Restoring force}}{\text{Inertial property}}}$$

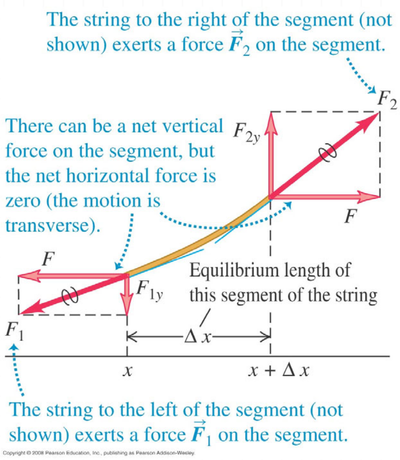
😎 Wave speed $v = \frac{\omega}{k} = \sqrt{\frac{T}{\mu}}$, depends on medium;

Transverse speed $v_{y,\max} = \omega A$, depends on wave source.

$$\Rightarrow \; v_{y,\max} = \omega A = (kA)v$$

Q: A sinusoidal wave of frequency f is traveling along a stretched string. The string is brought to rest, and a second traveling wave of frequency $2f$ is established on the string. The wave speed of the second wave is (a) twice that of the first wave (b) half that of the first wave (c) the same as that of the first wave (d) impossible to determine.

4. The linear wave equation



$$\frac{F_{1y}}{|F|} = -\left(\frac{\partial y}{\partial x}\right)_x , \quad \frac{F_{2y}}{|F|} = \left(\frac{\partial y}{\partial x}\right)_{x+\Delta x}$$

$$F_y = ma_y$$

$$\Rightarrow \quad F_{1y} + F_{2y} = F\left[\left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial y}{\partial x}\right)_x\right] \approx \mu \Delta x \frac{\partial^2 y}{\partial t^2}$$

$$\Rightarrow \quad \frac{\left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial y}{\partial x}\right)_x}{\Delta x} = \frac{\mu}{F}\frac{\partial^2 y}{\partial t^2}$$

$$\Rightarrow \quad \frac{\partial^2 y}{\partial x^2} = \frac{\mu}{F}\frac{\partial^2 y}{\partial t^2}$$

Substitute $y(x,t) = A\sin(kx - \omega t)$ into $\frac{\mu}{T}\left(\frac{\partial^2 y}{\partial t^2}\right) = \frac{\partial^2 y}{\partial x^2}$

$$\Rightarrow \quad -\frac{\mu}{T}\omega^2 \sin(kx - \omega t) = -k^2 \sin(kx - \omega t)$$

$$\Rightarrow \quad k^2 = \frac{\mu}{T}\omega^2$$

$$v = \frac{\omega}{k} \quad \Rightarrow \quad v = \sqrt{\frac{T}{\mu}}$$

So the general linear wave equation is $\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\left(\frac{\partial^2 y}{\partial t^2}\right)$.

In fact,

the general wave function $y(x,t) = y(x \pm vt)$ is the solution of

the linear wave equation $\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\left(\frac{\partial^2 y}{\partial t^2}\right)$.

$$\left[\begin{array}{l} s \equiv x \pm vt \\ \dfrac{\partial y(x \pm vt)}{\partial t} = \dfrac{\partial y(s)}{\partial s}\dfrac{\partial s}{\partial t} = \pm v \dfrac{\partial y(s)}{\partial s} \\ \dfrac{\partial^2 y(x \pm vt)}{\partial t^2} = \pm v \dfrac{\partial}{\partial t}\left[\dfrac{\partial y(s)}{\partial s}\right] = \pm v \dfrac{\partial^2 y(s)}{\partial s^2}\dfrac{\partial s}{\partial t} = v^2 \dfrac{\partial^2 y(s)}{\partial s^2} \\ \Rightarrow \quad \dfrac{\partial^2 y(x \pm vt)}{\partial x^2} = \dfrac{\partial^2 y(s)}{\partial s^2} = \dfrac{1}{v^2}\dfrac{\partial^2 y(x \pm vt)}{\partial t^2} \end{array}\right]$$

"Linear" wave equation: $\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\left(\frac{\partial^2 y}{\partial t^2}\right)$

Why called "linear" ?

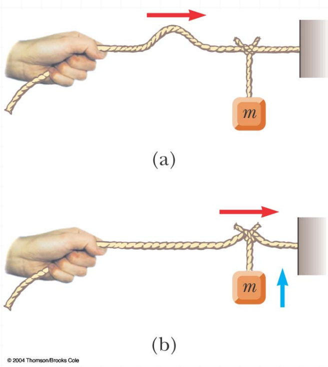
If y_1 and y_2 are solutions, then the linear combination,

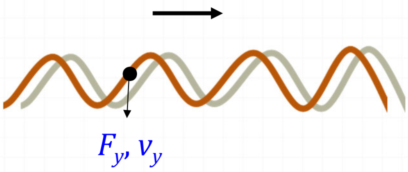
$y = ay_1 + by_2$ is also a solution.

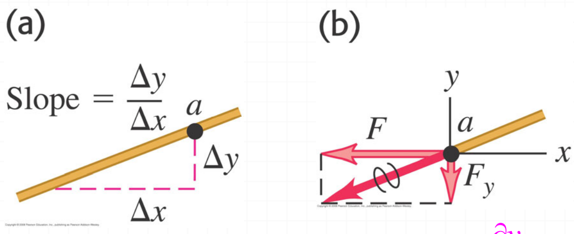
5. Rate of energy transfer

Energy in Waves in a String

- Waves transport energy when they propagate through a medium.







$$F_y = -F\frac{\partial y}{\partial x}$$

Power $P(x,t) = F_y(x,t)v_y(x,t) = \left(-F\frac{\partial y(x,t)}{\partial x}\right)\left(\frac{\partial y(x,t)}{\partial t}\right)$

$$y(x,t) = A\cos(kx-\omega t)$$

$$\frac{\partial y}{\partial x} = -Ak\sin(kx-\omega t)$$

$$\frac{\partial y}{\partial t} = A\omega\sin(kx-\omega t)$$

$$\Rightarrow \quad P(x,t) = F\omega k A^2 \sin^2(kx-\omega t)$$

$$v = \sqrt{\frac{F}{\mu}}, \quad \omega = kv$$

$$\Rightarrow \quad P(x,t) = [(\mu\omega^2 A^2)v]\sin^2(kx-\omega t)$$

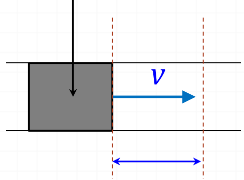
$$\langle P(x,t)\rangle = [(\mu\omega^2 A^2)v]\underbrace{\frac{\int_0^T \sin^2(kx-\omega t)dt}{T}}_{=1/2} = \frac{1}{2}\underbrace{(\mu\omega^2 A^2)}_{=\mu v_{y,\max}^2}v = (\frac{1}{2}\mu v_{y,\max}^2)v$$

$$\underbrace{\text{Energy density}}_{\text{Energy per length}}\ E_\ell = \frac{1}{2}\mu v_{y,\max}^2 = \frac{1}{2}\mu\omega^2 A^2$$

$$\text{Power: } P_\lambda = \frac{\Delta E}{\Delta t} = \frac{E_\ell \lambda}{T} = \frac{1}{2}\mu\omega^2 A^2 \frac{\lambda}{T}$$

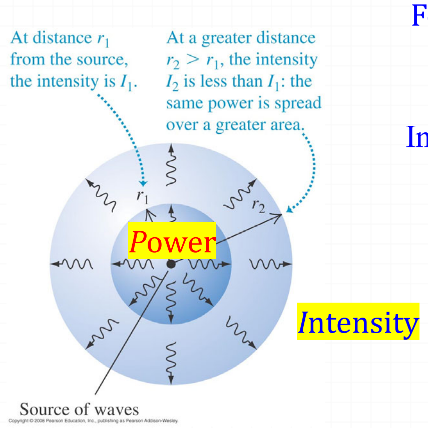
$$= \underbrace{\left(\frac{1}{2}\mu\omega^2 A^2\right)}_{\text{Energy density}} v \quad \propto \omega^2 A^2$$

$$\overbrace{\frac{1}{2}\mu\omega^2 A^2}^{\text{Energy density}}$$



$\lambda = vT$

Wave Intensity



For spherical wave,

$$\text{Intensity } I = \frac{(\text{Power})_{average}}{\text{area}} = \frac{(\text{Power})_{average}}{4\pi r^2} \propto \frac{(\text{Amplitude})^2}{r^2}$$

$$\text{Same power} \Rightarrow \frac{I_1}{I_2} = \frac{r_2^2}{r_1^2}$$ An inverse-square law.

$$\text{Per unit area} \Rightarrow I \propto \text{Power} \propto (\text{Amplitude})^2$$

$$\Rightarrow \frac{(\text{Amplitude})_1}{(\text{Amplitude})_2} = \frac{\sqrt{I_1}}{\sqrt{I_2}} = \frac{r_2}{r_1}$$

For 2D waves, $I \propto \frac{1}{r}$

$$\text{Amplitude} \propto \sqrt{I} \propto \frac{1}{\sqrt{r}}$$

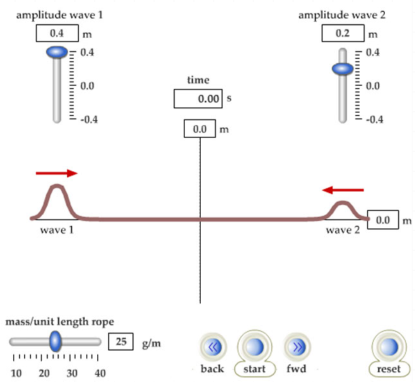
6. Wave Superposition and Interference

Superposition Principle

- If two or more traveling waves are moving through a medium, the resultant value of the wave function at any point is the algebraic sum of the values of the wave functions of the individual waves.
- Waves that obey the superposition principle are linear waves.
  - For mechanical waves, linear waves usually have amplitudes much smaller than their wavelengths.
  - Waves that violate the superposition principle are called non-linear waves, which usually have larger amplitudes.

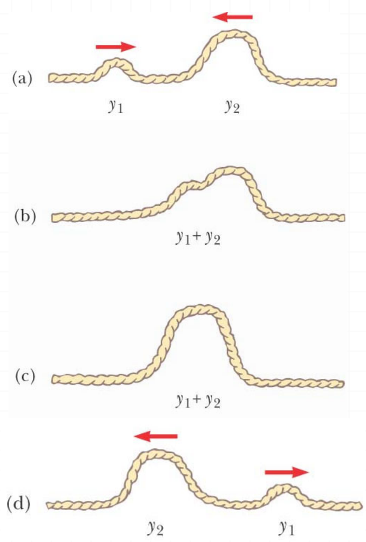
Interference

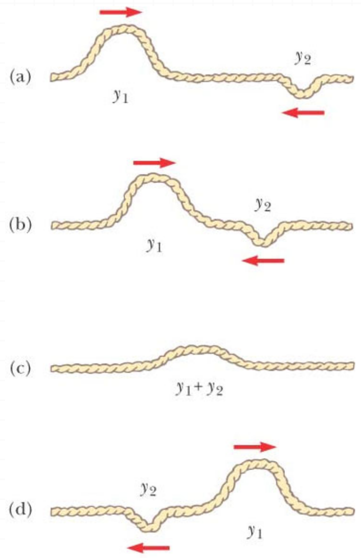
- The combination of separate waves in the same region of space to produce a resultant wave is called interference.



Types of Interference

- Constructive interference occurs when the displacements caused by the two pulses are in the same direction.
  - The amplitude of the resultant pulse is greater than either individual pulse.
- Destructive interference occurs when the displacements caused by the two pulses are in opposite directions.
  - The amplitude of the resultant pulse is less than either individual pulse.





Constructive Destructive

Superposition of Sinusoidal Waves

- Assume two waves are traveling in the same direction, with the same frequency, wavelength and amplitude, and the waves differ in phase:

$y_1(x, t) = A \sin(kx - \omega t)$, $y_2(x, t) = A \sin(kx - \omega t + \phi)$

$$\left[\sin a + \sin b = 2\cos\left(\frac{a-b}{2}\right)\sin\left(\frac{a+b}{2}\right)\right]$$

$\Rightarrow$ $y(x, t) = y_1(x, t) + y_2(x, t) = 2A\cos(\phi/2)\sin(kx - \omega t + \phi/2)$

- Amplitude is a function of ϕ.
- Still a sinusoidal wave.

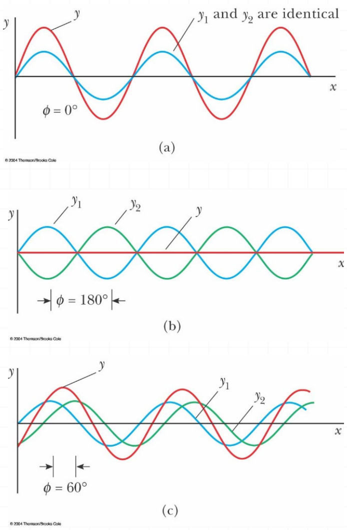
Spatial Interference

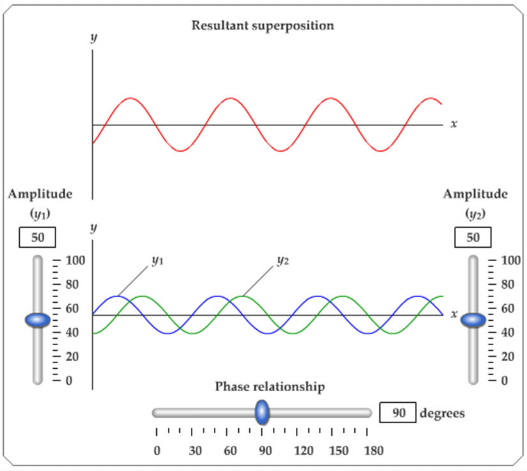
In general,

$y_1(x, t) = A \sin(kx - \omega t)$

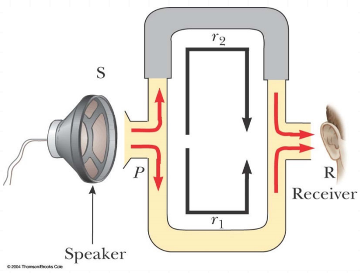
$y_2(x, t) = A \sin(kx - \omega t + \phi)$

- Constructive interference occurs when $\phi = 0$.
- Destructive interference occurs when $\phi = \pi$.
- General interference occurs when $0 < \phi < \pi$.





Interference of Sound Waves

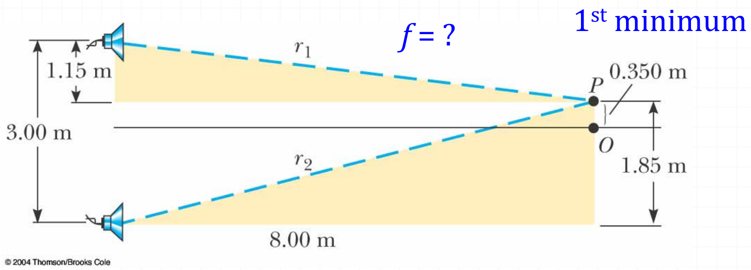


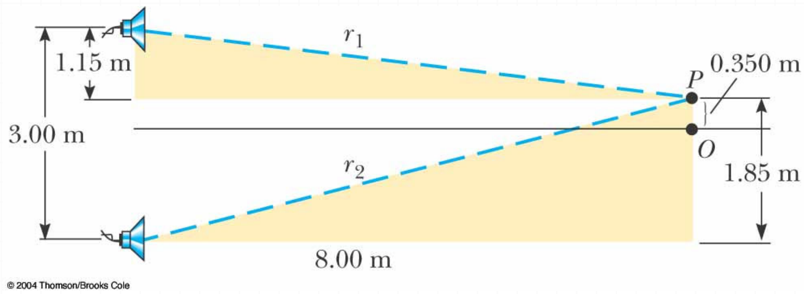
$$y_1(r,t) = A\sin(kr_1 - \omega t)$$
$$y_2(r,t) = A\sin(kr_2 - \omega t)$$
$$= A\sin\left[(kr_1 - \omega t) + k(r_2 - r_1)\right]$$
$$= A\sin\left[(kr_1 - \omega t) + \underbrace{k\Delta r}_{\Delta\phi}\right]$$

$$\begin{cases} \Delta\phi = n(2\pi), & \text{constructive} \\ \Delta\phi = (2n+1)\pi, & \text{destructive} \end{cases} \Rightarrow \begin{cases} k\Delta r = n(2\pi) & \Rightarrow \Delta r = n\lambda \\ k\Delta r = (2n+1)\pi & \Rightarrow \Delta r = (n+\frac{1}{2})\lambda \end{cases}$$

$n = 0,\ \pm 1,\ \pm 2,\ \ldots$

Ex. A pair of speakers placed 3.00 m apart are driven by the same oscillator (Fig.). A listener is originally at point O, which is located 8.00 m from the center of the line connecting the two speakers. The listener then walks to point *P*, which is a perpendicular distance 0.350 m from *O*, before reaching the first minimum in sound intensity. What is the frequency of the oscillator?





$$\Delta r = |r_2 - r_1|$$

1st minimum: $\Delta r = \dfrac{\lambda}{2} \quad \Rightarrow \lambda = 2(r_2 - r_1)$

$$\Rightarrow f = \frac{v}{\lambda} = \frac{v}{2(r_2 - r_1)}$$

😎 What if the speaks were connected 180^0 out of phase?

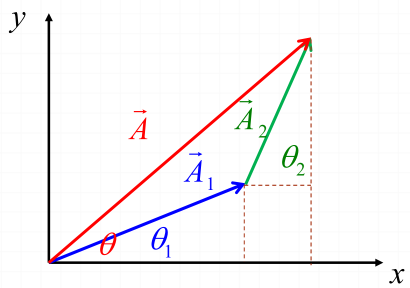
$$y_1(r,t) = A\sin(kr_1 - \omega t)$$
$$y_2(r,t) = A\sin(kr_2 - \omega t + \pi)$$
$$= A\sin\left[(kr_1 - \omega t) + k(r_2 - r_1) + \pi\right]$$
$$= A\sin\left[(kr_1 - \omega t) + \underbrace{k\Delta r + \pi}_{\Delta\phi}\right]$$

$$\begin{cases} \Delta\phi = n(2\pi), & \text{constructive} \\ \Delta\phi = (2n+1)\pi, & \text{destructive} \end{cases} \Rightarrow \begin{cases} k\Delta r = (2n-1)\pi & \Rightarrow \Delta r = (n - \frac{1}{2})\lambda \\ k\Delta r = (2n)\pi & \Rightarrow \Delta r = n\lambda \end{cases}$$

$n = 0,\ \pm 1,\ \pm 2,\ \ldots$

1^{st} minimum: $\Delta r = \lambda \quad \Rightarrow f = \dfrac{v}{\lambda} = \dfrac{v}{\Delta r}$

😎 What if $A_1 \neq A_2$, $y = y_1 + y_2 = ?$



$$\begin{cases} y_1(x,t) = A_1 \sin(\underbrace{kx - \omega t + \phi_1}_{\theta_1}) \\ y_2(x,t) = A_2 \sin(\underbrace{kx - \omega t + \phi_2}_{\theta_2}) \end{cases}$$

$$\vec{A} = (A\cos\theta, A\sin\theta)$$
$$= \vec{A}_1 + \vec{A}_2 = (A_1\cos\theta_1 + A_2\cos\theta_2, A_1\sin\theta_1 + A_2\sin\theta_2)$$

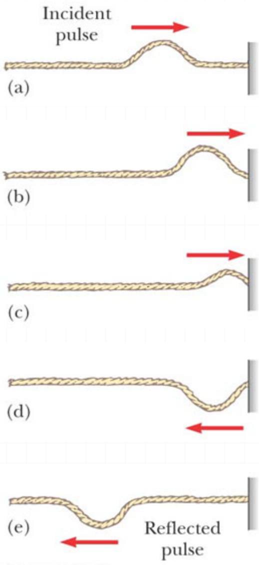
$$\Rightarrow \underbrace{A\sin\theta}_{y} = \underbrace{A_1\sin\theta_1}_{y_1} + \underbrace{A_2\sin\theta_2}_{y_2}$$

$$A = \sqrt{(A_1\cos\theta_1 + A_2\cos\theta_2)^2 + (A_1\sin\theta_1 + A_2\sin\theta_2)^2}$$

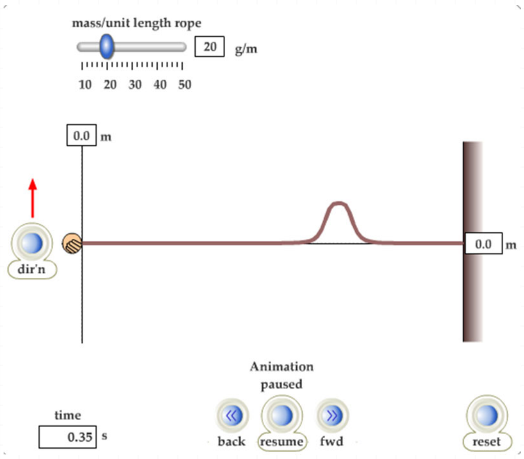
$$\theta = \tan^{-1}\left(\frac{A_1\sin\theta_1 + A_2\sin\theta_2}{A_1\cos\theta_1 + A_2\cos\theta_2}\right)$$

7. Reflection and Transmission

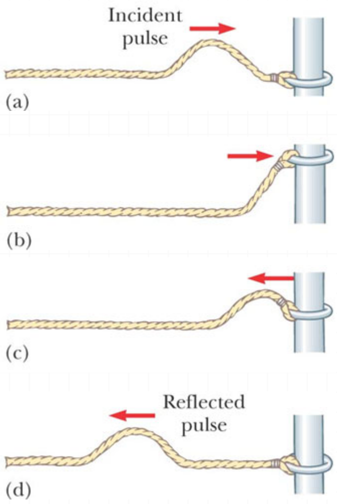
Reflection of a Wave, Fixed End



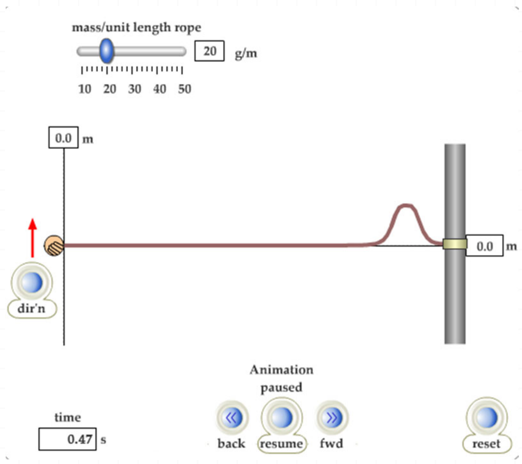
Phase change $\Delta\phi = \pi$.



Reflection of a Wave, Free End

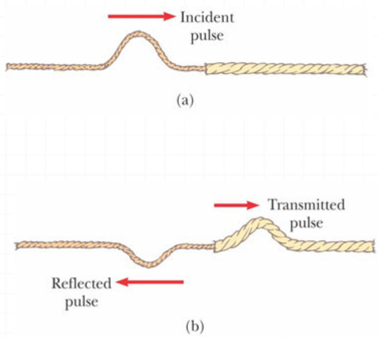


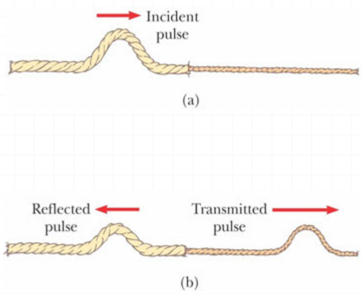
Phase change $\Delta\phi = 0$.



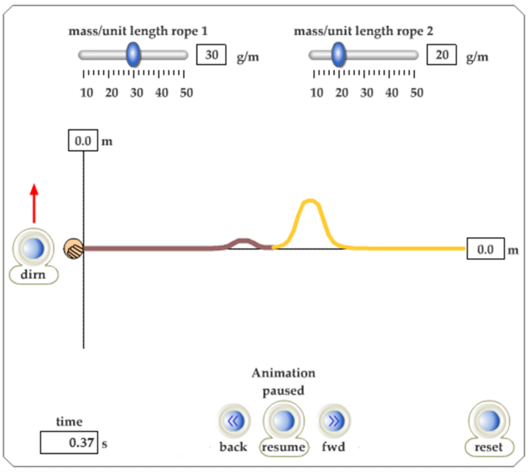
Transmission of a Wave

$$y_I(x,t) = A_I \sin(kx - \omega t)$$



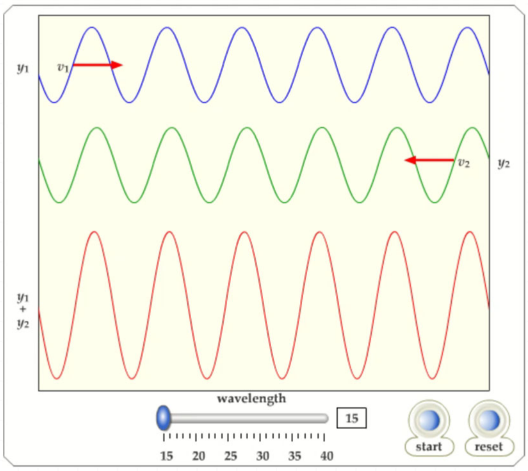


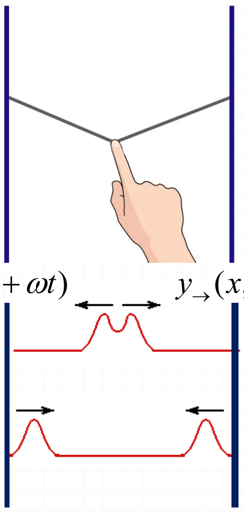
$$y_R(x,t) = A_R \sin(kx + \omega t + \pi) \qquad y_R(x,t) = A_R \sin(kx + \omega t + 0)$$



1. Wave - Propagation of a Disturbance
2. Wave function
3. The Speed of Waves on Strings
4. The Linear Wave Equation
5. Rate of Energy Transfer
6. Superposition and Interference
7. Reflection and Transmission
8. Standing Wave

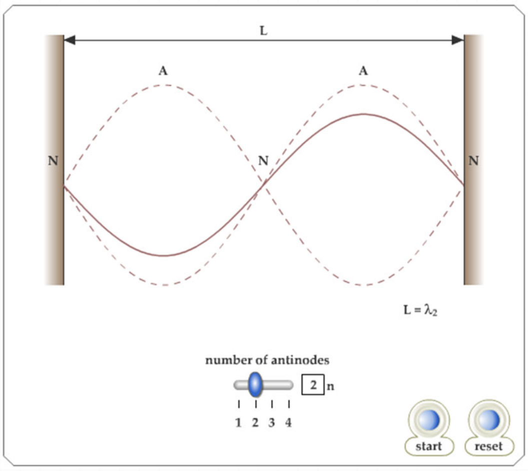
8. Standing wave





$y_{\leftarrow}(x,t) = A\sin(kx + \omega t)$ $y_{\rightarrow}(x,t) = A\sin(kx - \omega t)$

$y_{\rightarrow}(x,t) = A\sin(kx - \omega t + \pi)$ $y_{\leftarrow}(x,t) = A\sin(kx + \omega t + \pi)$



Standing Waves

- Assume two waves with the same amplitude, frequency and wavelength, traveling in opposite directions in a medium.

  $y_1(x, t) = A \sin(kx - \omega t)$

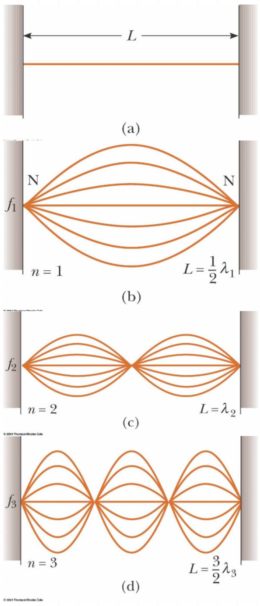
  $y_2(x, t) = A \sin(kx + \omega t)$

  $\Rightarrow y(x, t) = y_1(x, t) + y_2(x, t) = [2A \sin(kx)] \cos \omega t$

  A standing wave, not a traveling wave.

  $\Rightarrow$ Every element in the medium oscillates in simple harmonic motion with the same frequency, ω.

Standing Waves in a String



$$y(x,t) = 2A\sin(kx)\cos\omega t$$

Boundary condition:

$$y(0,t) = y(L,t) = 0$$

$$kL = n\pi, \quad n = \text{integer}$$

$$\Rightarrow \left(\frac{2\pi}{\lambda_n}\right)L = n\pi, \quad n = 1, 2, 3, \ldots$$

$$\Rightarrow \lambda_n = \frac{2L}{n} \quad \text{or} \quad L = n\left(\frac{\lambda_n}{2}\right)$$

Nodes and Antinodes

- A node occurs at a point of zero amplitude.

$$y(x,t) = 2A\sin(kx)\cos\omega t = 0$$

  - These correspond to positions of x where

$$x_n = \frac{n\lambda}{2} \qquad n = 0, 1, \ldots$$

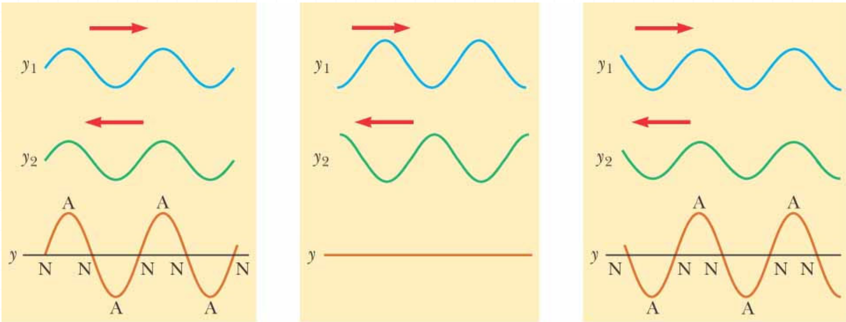
- An antinode occurs at a point of maximum displacement, $2A$.

$$y(x,t) = 2A\sin(kx)\cos\omega t = 2A$$

  - These correspond to positions of x where

$$x_n = \frac{n\lambda}{4} \qquad n = 1, 3, \ldots$$

$$y = y_1 + y_2 = (2A \sin kx) \cos \omega t$$



(a) $t = 0$ (b) $t = T/4$ (c) $t = T/2$

Normal modes

- The wavelengths of the normal modes for a string of length L fixed at both ends are

$$\lambda_n = 2L / n, \quad n = 1, 2, 3, \ldots$$

  - n is the n^{th} normal mode of oscillation.
- The natural frequencies are

$$f_n = n\frac{v}{2L} = \frac{n}{2L}\sqrt{\frac{T}{\mu}}$$

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Quantization

- This situation in which only certain frequencies of oscillation are allowed is called quantization.
- Quantization is a common occurrence when waves are subject to boundary conditions.

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Harmonic Series

- The fundamental frequency corresponds to $n = 1$.
  - It is the lowest frequency, f_1.
- The frequencies of the remaining natural modes are integer multiples of the fundamental frequency.
  - $f_n = nf_1$.
  - Musicians call f_2, f_3, and so on overtones.
  - These normal modes are called *harmonics*.

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Musical Note of a String

- The musical note is defined by its fundamental frequency.
- The frequency of the string can be changed by changing either its length or its tension.

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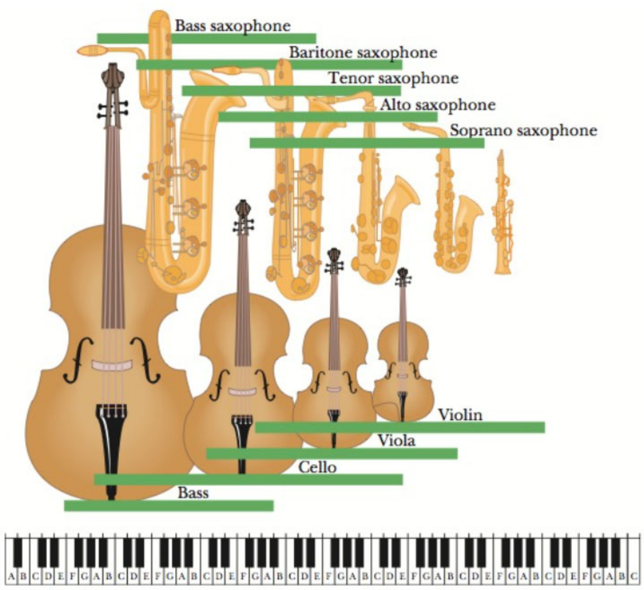
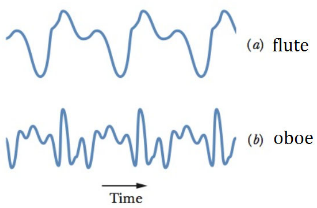
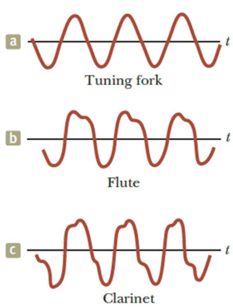
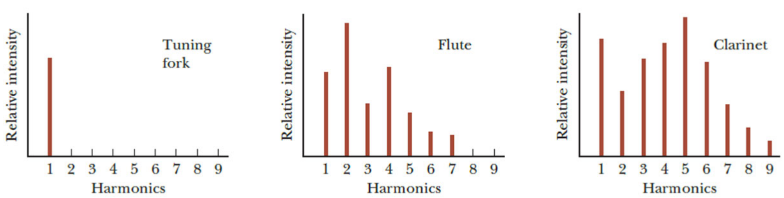




Complex Standing Waves

The standing wave could be a combination of superposition of many modes.

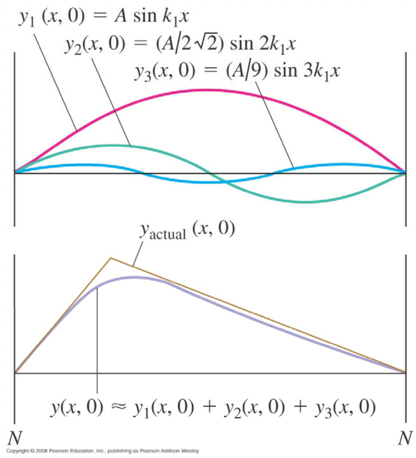




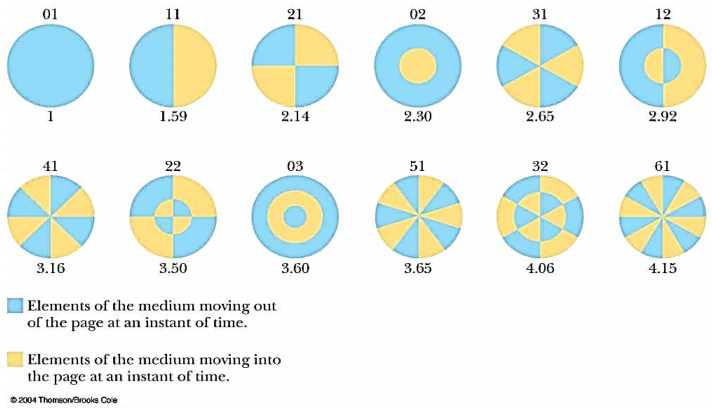
Fourier series:

$$y(t) = \sum_{n} (a_n \cos 2\pi f_n t + b_n \sin 2\pi f_n)$$

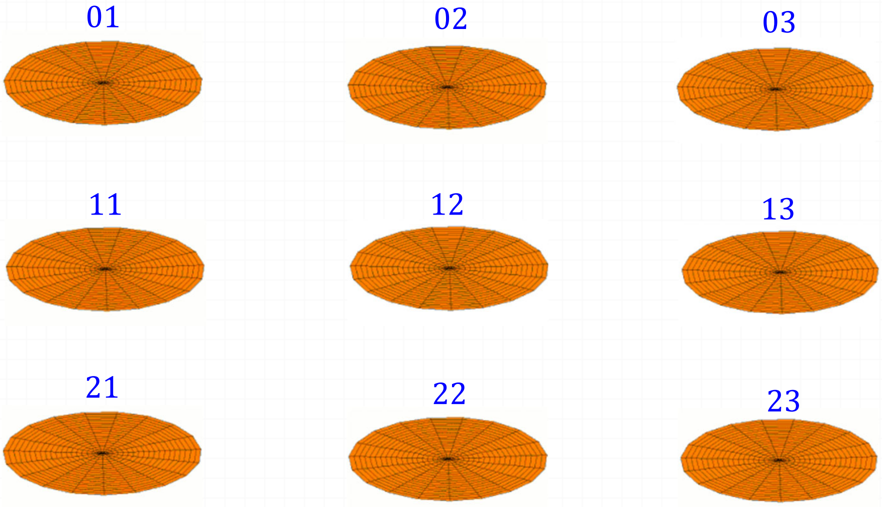
The harmonic content depends on how the string is initially set into motion.



Standing Waves in 2-dimensional Membranes



- The resulting sound is not harmonic because the standing waves have frequencies that are not related by integer multiples.
- The fundamental frequency contains one nodal curve.



 https://en.wikipedia.org/wiki/Vibrations_of_a_circular_membrane

Chladni plates

 https://en.wikipedia.org/wiki/Ernst_Chladni

Standing waves in a 3-dimensional sphere