Chapter 11
Equilibrium and Elasticity

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Outline

1. The Conditions for Equilibrium
2. Center of Gravity
3. Solving rigid-body equilibrium problems
4. Elastic Properties of Solids

1. The conditions for equilibrium

Equilibrium

Conditions for equilibrium:

(1) $\sum \vec{F} = 0$, translational equilibrium, and

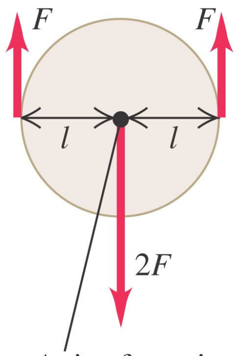
(2) $\sum \vec{\tau} = 0$, rotational equilibrium

(independent of the choice of axis)

Static equilibrium:
the object is at rest and experiences no translational and rotational motion, relative to the observer in an inertial reference frame.

(a) This body is in static equilibrium.

Equilibrium conditions:



First condition satisfied:
Net force = 0, so body at rest has no tendency to start moving as a whole.

Second condition satisfied:
Net torque about the axis = 0, so body at rest has no tendency to start rotating.

Axis of rotation (perpendicular to figure)

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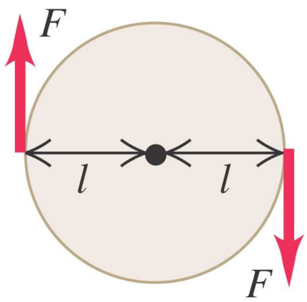
(b) This body has no tendency to accelerate as a whole, but it has a tendency to start rotating.



First condition satisfied:
Net force = 0, so body at rest has no tendency to start moving as a whole.

Second condition NOT satisfied: There is a net clockwise torque about the axis, so body at rest will start rotating clockwise.

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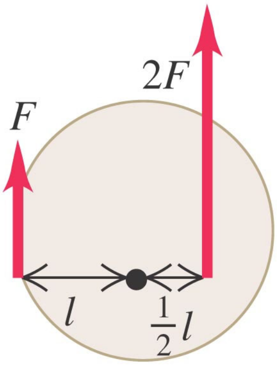
(c) This body has a tendency to accelerate as a whole but no tendency to start rotating.



First condition NOT satisfied: There is a net upward force, so body at rest will start moving upward.

Second condition satisfied: Net torque about the axis = 0 so body at rest has no tendency to start rotating.

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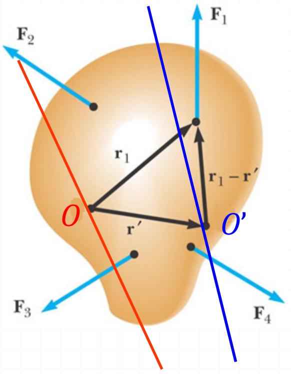
https://www.inc.com/jim-schleckser/seven-secrets-of-successful-people-to-living-a-balanced-life.html

Torque for different axes at O and O':



$$\vec{\tau}_o = \sum_i \vec{r}_i \times \vec{F}_i$$

$$\vec{\tau}_{o'} = \sum_i (\vec{r}_i - \vec{r}\,') \times \vec{F}_i$$

$$= \sum_i \vec{r}_i \times \vec{F}_i - \vec{r}\,' \times \sum_i \vec{F}_i$$

$$\vec{\tau}_o = \vec{\tau}_{o'} + \vec{r}\,' \times \sum_i \vec{F}_i$$

(1) $\sum_i \vec{F}_i = 0$ or $\vec{r}\,' \parallel \sum_i \vec{F}_i \Rightarrow \vec{\tau}_{o'} = \vec{\tau}_o$, indep. of axis.

(2) $\sum_i \vec{F}_i = 0$, $\vec{\tau}_o = 0$ equilibrium $\Rightarrow \vec{\tau}_{o'} = 0$.

1. The Conditions for Equilibrium
2. Center of Gravity
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2. The center of gravity

Center of gravity

3 All the gravitational torques acting on an object equals to a single torque acting through a point, called center of gravity (CG).

😎 In contrast, all the forces acting on an object is equal to a single force acting on the center of mass (CM).

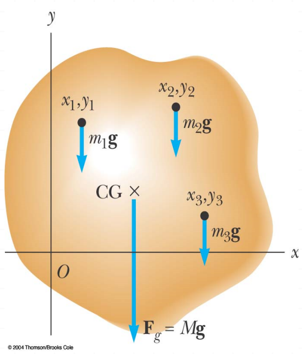
3 If g is uniform over the object,

1. the CG will coincide with CM, and
2. the gravitational torque on an object equals to the torque acting at the CM.

Proof of $x_{CG} = x_{CM}$, only if g is assumed to be a constant.

Torque by gravity,

$$\tau_{\text{total}} = \sum_i m_i g_i x_i$$



If $g_1 = g_2 = \ldots = g_{CM} = g$

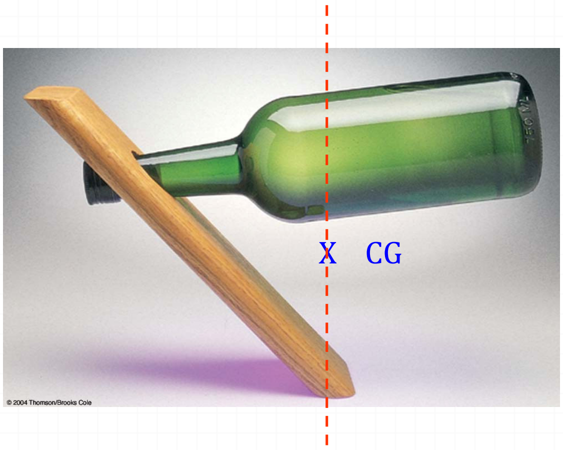
$$\tau_{total} = \sum_i m_i g_i x_i$$

$$= (\sum_i m_i x_i) g = [(\sum_i m_i) x_{CM}] g = \tau_{CM}$$

$$\tau_{CG} = [(\sum_i m_i) g_{CG}] x_{CG} = [(\sum_i m_i) g] x_{CG}$$

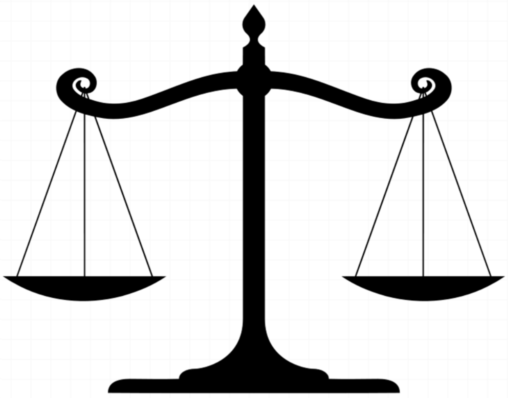
By $\tau_{total} \equiv \tau_{CG} \quad \Rightarrow \quad x_{CG} = x_{CM} = \dfrac{\sum_i m_i x_i}{\sum_i m_i}$

Q: Where is the CG on the horizontal axis?



The system satisfies ΣF=0 & Στ = 0.

$$\vec{\tau}_o = \vec{\tau}_{o'} + \vec{r}\,' \times \sum_i \vec{F}_i \overset{Equalibrium}{\Rightarrow} \vec{\tau}_o = \underbrace{\vec{\tau}_{CM}}_{=0} + \underbrace{\vec{r}_{CM} \times \vec{F}_{CM}}_{=0} = 0$$



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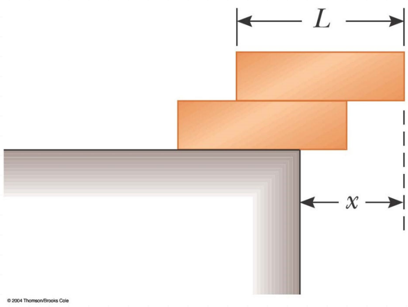
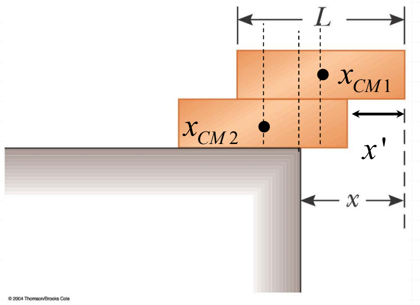
Ex.



What is the maximum *x* without falling?

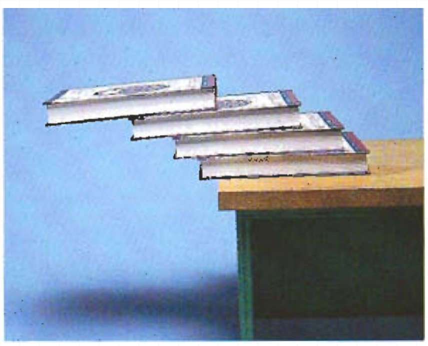


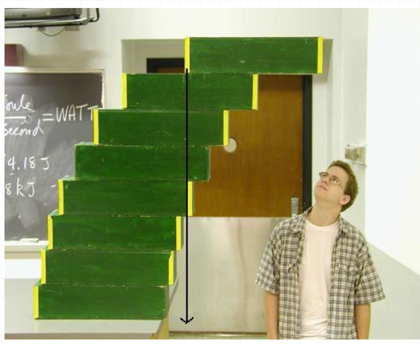
$$x = 0$$

$$x_{CG} = \frac{m_1 x_{CM1} + m_2 x_{CM2}}{m_1 + m_2}$$

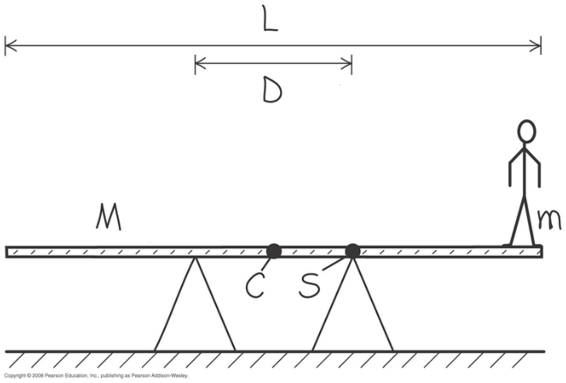
$$= \frac{m(x - L/2) + m[-(L/2 - (x - x'))]}{2m} \le 0 \;\Rightarrow\; x \le \frac{L + x'}{2}$$

But $x' \le L/2$ So $x \le 3L/4$





Ex. What is the maximum mass for the person to keep balanced?



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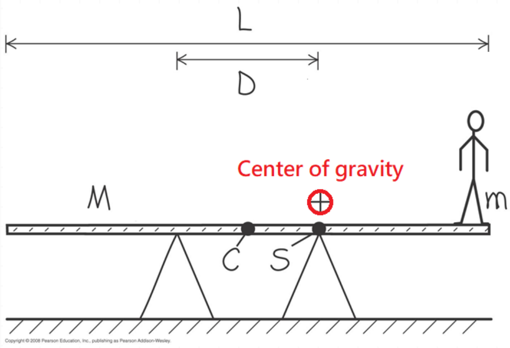
Ans:



Method#1

Set pivot at S:

$$x_{CG} = \frac{mx_m + Mx_M}{M+m} = \frac{m(\frac{1}{2}L - \frac{1}{2}D) + M(-\frac{1}{2}D)}{M+m} \le 0$$

$$\Rightarrow \quad m \le \frac{D}{L-D} M$$

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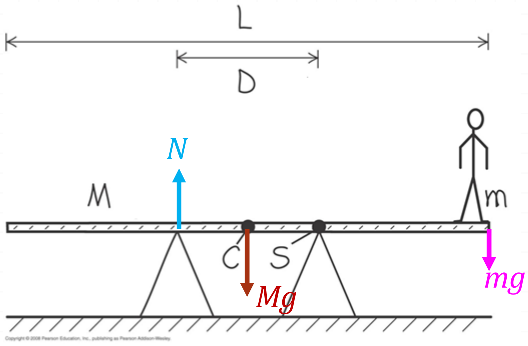
Method#2



Set pivot at S:

$$\sum \tau = 0 \quad \Rightarrow \quad Mg\frac{D}{2} - ND - mg(\frac{L}{2} - \frac{D}{2}) = 0$$

$$N = \frac{Mg\frac{D}{2} - mg(\frac{L}{2} - \frac{D}{2})}{D} \geq 0$$

$$\Rightarrow \quad m \leq \frac{D}{L-D} M$$

3. Solving rigid-body equilibrium problems

Problem-Solving Strategy – Equilibrium Problems

1. Isolate the object being analyzed.
2. Draw a free-body diagram.

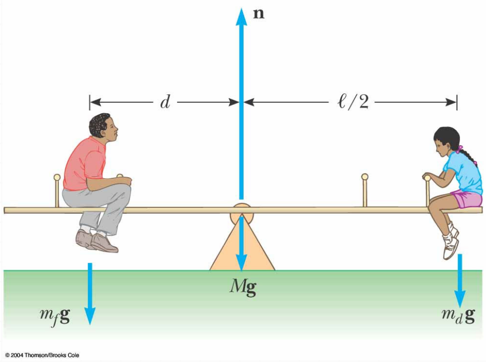
   (or separate free-body diagrams if more than one object)

3. Try to guess the correct direction for each force.
4. Choose a coordinate system and the pivot.

   (the choice of rotational axis is arbitrary)

5. $\Sigma F=0$ & $\Sigma\tau = 0$.

Ex. The seesaw problem



(a) n = ?

(b) To be balanced, what is d ?

Ans:

(a)

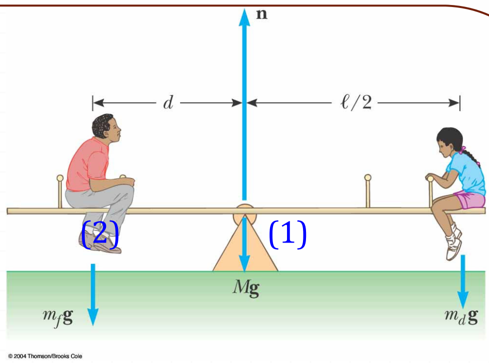
$$n - m_f g - m_d g - Mg = 0$$

$$n = m_f g + m_d g + Mg$$

(b) Set pivot at (1)

$$(m_f g)d - (m_d g)\frac{\ell}{2} = 0$$

$$d = \left(\frac{m_d}{m_f}\right)\frac{\ell}{2}$$



Set pivot at (2)

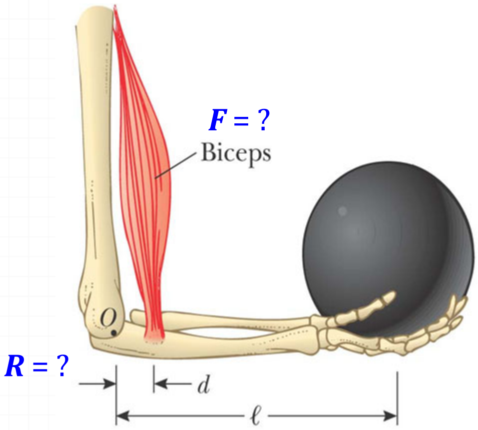
$$nd - Mgd - m_d g(d + \ell/2) = 0$$

$$n = m_f g + m_d g + Mg$$

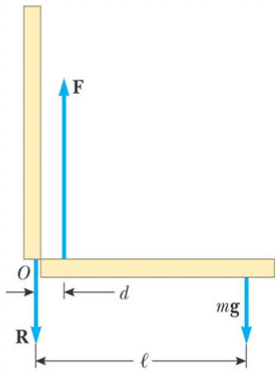
$$\Rightarrow d = \left(\frac{m_d}{m_f}\right)\frac{1}{2}\ell$$

When equilibrium, $\Rightarrow$ Indep. of the choice of axis.

Ex.

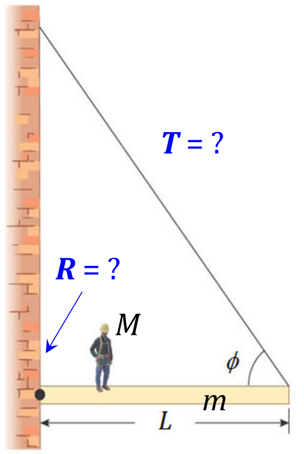


Ans:

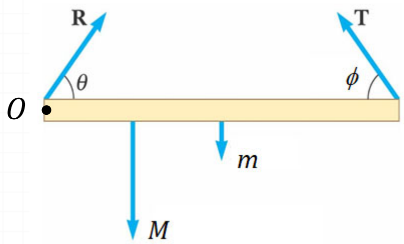


$$\begin{cases} \sum F_y = F - R - mg = 0 \\ \sum \tau_o = Fd - mg\ell = 0 \end{cases} \Rightarrow \begin{cases} R = \dfrac{mg\ell}{d} - mg \\ F = \dfrac{mg\ell}{d} \end{cases}$$

Ex.



Ans:



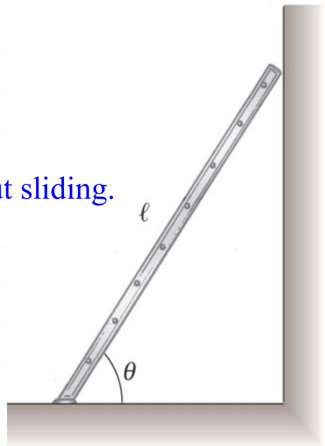
$\Sigma \boldsymbol{F}=0$

$\Sigma \boldsymbol{\tau}=0$

$$\Rightarrow \begin{cases} \sum F_x = R\cos\theta - T\cos\phi = 0 \\ \sum F_y = R\sin\theta + T\sin\phi - M - m = 0 \\ \sum \tau_o = T\sin\phi L - m\dfrac{L}{2} - M\dfrac{L}{4} = 0 \end{cases}$$

Ex.

$\theta_{\min} = ?$ without sliding.



$\mu_s = 0$

$\mu_s \neq 0$

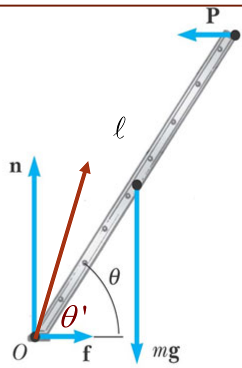
Ans:

$$\sum F_x = 0 \quad \Rightarrow \quad f_s = P$$

$$\sum F_y = 0 \quad \Rightarrow \quad n = mg$$

$$\sum \vec{\tau}_o = 0 \quad \Rightarrow \quad P\ell \sin\theta - mg\frac{\ell}{2}\cos\theta = 0$$

$$\tan\theta = \frac{mg}{2P} = \frac{mg}{2f_s},$$



$f_s \leq \mu_s n = \mu_s mg,\quad f_s = \dfrac{mg}{2\tan\theta}, \quad \Rightarrow \quad \tan\theta \geq \dfrac{1}{2\mu_s} \quad \Rightarrow \tan\theta_{\min} = \dfrac{1}{2\mu_s}$

😎 1. Note that the resultant force can never be along the ladder.

$$\tan\theta' = \frac{n}{f_s} = \frac{mg}{f_s} = 2\tan\theta \quad \Rightarrow \quad \theta' \neq \theta$$

(This can be seen if the pivot is set at the wall contact point.)

😎 2. The problem becomes statically indeterminate if the wall has friction.

Ex.

$\theta_{\min} = ?$ without sliding.

Ans.

$$\sum F_x = 0 = f_s - P$$

$$\sum F_y = 0 = n - (m+M)g$$

$$\sum \vec{\tau}_o = 0 = P\ell \sin\theta - mg\frac{\ell}{2}\cos\theta - Mgd\cos\theta$$

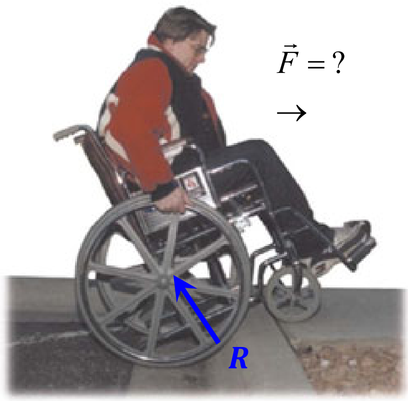
$$\tan\theta = \frac{mg(\ell/2) + Mgd}{P\ell} = \frac{mg(\ell/2) + Mgd}{f_s \ell}$$

$$f_s \le \mu_s n = \mu_s (m+M)g \quad \Rightarrow \quad \tan\theta_{\min} = \frac{m(\ell/2) + Md}{\mu_s \ell (m+M)}$$

$$\Rightarrow \quad \theta_{\min} \text{ depends on "} d \text{" and "} M \text{".}$$

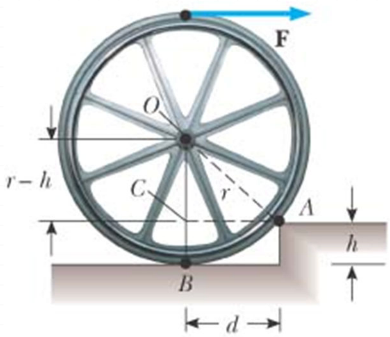
$$\theta_{\min} \nearrow \text{ as } d \nearrow .$$

Ex.

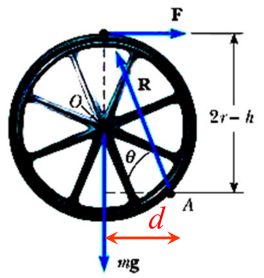


$\vec{F} = ?$

$\rightarrow$



Ans: System = wheel Set the pivot at "A"

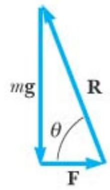


$$d = \sqrt{r^2 - (r-h)^2} = \sqrt{2rh - h^2}$$

$$\sum \tau = mgd - F(2r-h) = 0$$

$$mg\sqrt{2rh - h^2} - F(2r-h) = 0$$

$$\Rightarrow \quad F = \frac{mg\sqrt{2rh-h^2}}{2r-h} < mg$$



$$\sum F_x = 0 = F - R\cos\theta$$
$$\sum F_y = 0 = R\sin\theta - mg$$
$$\Rightarrow \quad \tan\theta = \frac{mg}{F}$$

$$R = \sqrt{(mg)^2 + F^2}$$

Q: 有一人想要測量一個長形物體的重量，他先將一端放在秤上，得到的讀數是W_1，然後再將另一端放在秤上，得到讀數W_2，則此物體的重量是下列何者?

(A) $W_1 + W_2$。

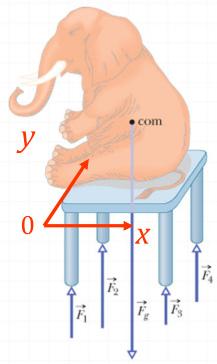
(B) $(W_1 + W_2)/2$。

(C) $W_1 W_2$

(D) $\sqrt{W_1 W_2}$

(E) 無法決定。

Indeterminate structure



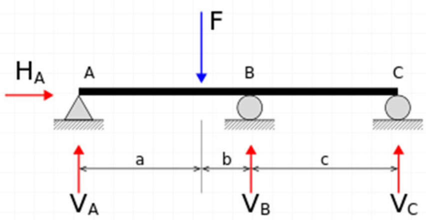
The table is an indeterminate structure. The four forces on the table legs differ from one another in magnitude and cannot be found from the laws of static equilibrium alone.

To solve such indeterminate equilibrium problems, we must supplement equilibrium equations with some knowledge of elasticity of the materials.

$$\left.\begin{array}{l} (1)\ \sum \vec{F} = 0 = \sum F_z = 0 \\ (2)\ \sum \vec{\tau} = 0 = \begin{cases} \sum \tau_x = 0 \\ \sum \tau_y = 0 \end{cases} \end{array}\right\} \Rightarrow \text{3 equations but 4 variables.}$$

$$\left[\sum \vec{r} \times \vec{F} = \sum (r_x \hat{i} + r_y \hat{j} + r_z \hat{k}) \times (F_z \hat{k}) = \sum \tau_x \hat{i} + \sum \tau_y \hat{i} = 0\right]$$

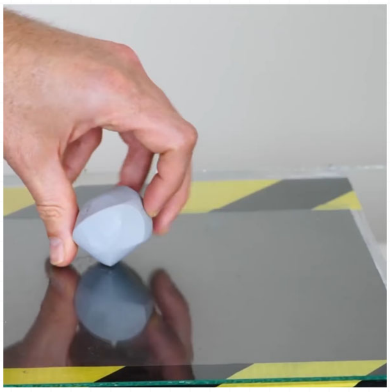
E.g.,



$$\sum F_x = 0 \quad \Rightarrow \quad H_A - F_x = 0$$
$$\sum F_y = 0 \quad \Rightarrow \quad V_A - F_y + V_B + V_C = 0$$
$$\sum \vec{\tau}_A = 0 \quad \Rightarrow \quad F_y a - V_B(a+b) - V_C(a+b+c) = 0$$

4 unknown variables (H_A, V_A, V_B, V_C)

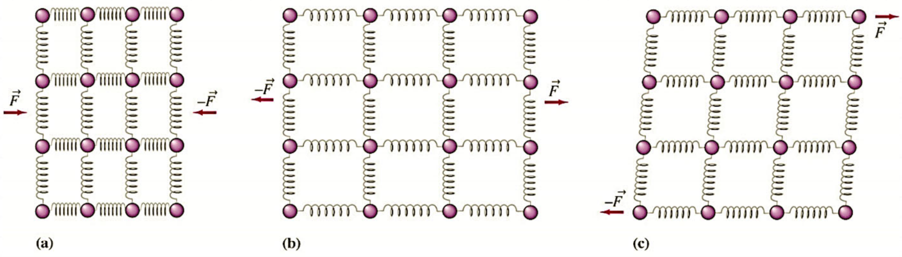
[https://zh.wikipedia.org/wiki/%E9%9D%9C%E4%B8%8D%E5%AE%9A]



https://www.facebook.com/461822001292778/posts/847345822740392/?sfnsn=mo&extid=tooIwDMpAzXQq2DC

4. Elastic properties of solids

Elastic Properties of Solids



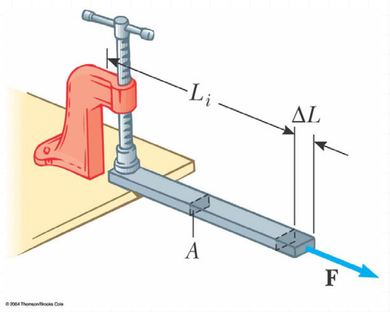
3 Deformation:

- Stress: $\propto F$, causing deformation.
- Strain: the result of stress, a measure of the "degree of deformation".

3 For sufficiently small stress, strain is proportional to stress. The proportional constant is called,

$$\text{Elastic modulus} \equiv \frac{stress}{strain} \quad \text{(Hook's law)}$$

Young's modulus (elasticity in length): measure the resistance of a solid to a change in its length.

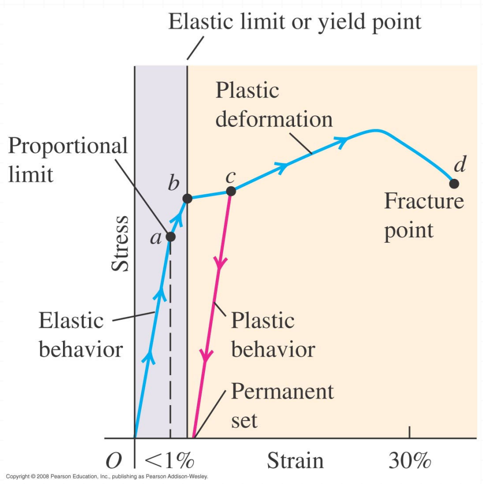


Tensile stress: F/A

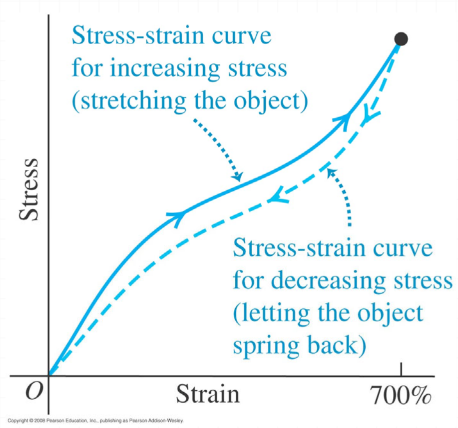
Tensile strain: $\Delta L/L_i$

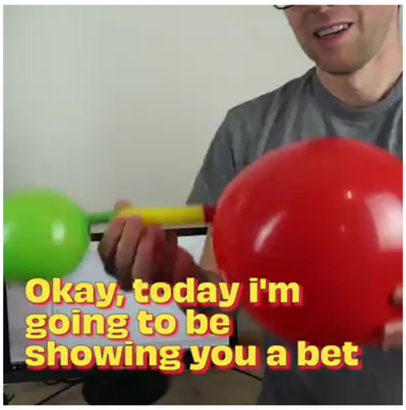
$$Y \equiv \frac{\text{tensile stress}}{\text{tensile strain}} = \frac{F / A}{\Delta L / L_i}$$

unit= $\frac{\text{force}}{\text{area}}$



Elastic hysteresis





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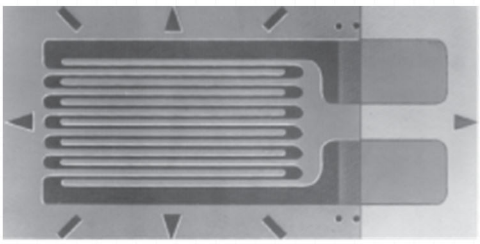
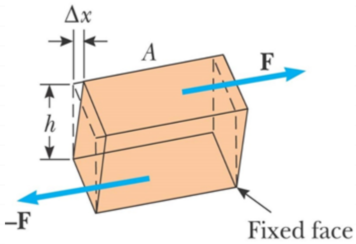
Strain gauge



Courtesy Micro Measurements, a Division of Vishay Precision Group, Raleigh, NC

The gauge is fastened with adhesive to the object whose strain is to be measured; it experiences the same strain as the object. The electrical resistance of the gauge varies with the strain, permitting strains up to 3% to be measured.

Shear modulus (elasticity in shape): measure the resistance to motion of the planes within a solid parallel to each other.

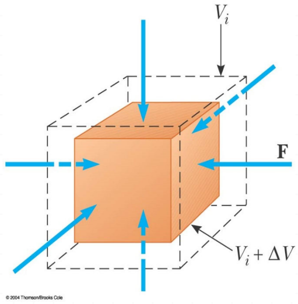


Shear stress: F/A

Shear strain: $\Delta x/h$

$$S \equiv \frac{\text{shear stress}}{\text{shear strain}} = \frac{F/A}{\Delta x/h}$$

Bulk modulus (elasticity in volume): measure the resistance of an object to a change in its volume.



Volume stress: F/A

Volume strain: $\Delta V/V_i$

$$B \equiv \frac{\text{volume stress}}{\text{volume strain}} = \frac{(-)F/A}{\Delta V/V_i}$$

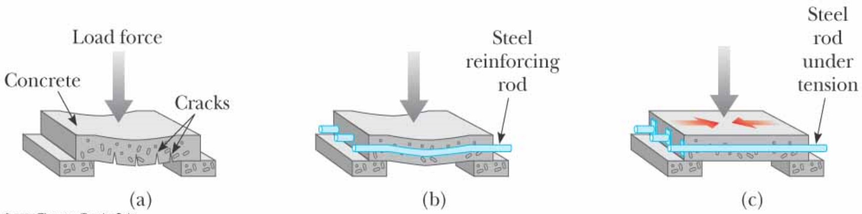
p.s., "–" is to make B "+".

$$\text{Compressibility} \equiv \frac{1}{B}$$

Typical Values for Elastic Moduli

Substance	Young's Modulus (N/m^2)	Shear Modulus (N/m^2)	Bulk Modulus (N/m^2)
Tungsten	35×10^{10}	14×10^{10}	20×10^{10}
Steel	20×10^{10}	8.4×10^{10}	6×10^{10}
Copper	11×10^{10}	4.2×10^{10}	14×10^{10}
Brass	9.1×10^{10}	3.5×10^{10}	6.1×10^{10}
Aluminum	7.0×10^{10}	2.5×10^{10}	7.0×10^{10}
Glass	$6.5–7.8 \times 10^{10}$	$2.6–3.2 \times 10^{10}$	$5.0–5.5 \times 10^{10}$
Quartz	5.6×10^{10}	2.6×10^{10}	2.7×10^{10}
Water	—	—	0.21×10^{10}
Mercury	—	—	2.8×10^{10}

Prestressed Concrete



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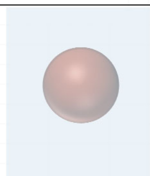
https://fb.watch/gtYq32kjNB/

Ex. A solid brass sphere is initially surrounded by air, and the air pressure exerted on it is $1.0 \times 10^5\ \mathrm{N/m^2}$ (normal atmospheric pressure). The sphere is lowered into the ocean to a depth where the pressure is $2.0 \times 10^7\ \mathrm{N/m^2}$. The volume of the sphere in air is $0.50\ \mathrm{m^3}$. By how much does this volume change once the sphere is submerged?



$$B = -\frac{\Delta P}{\Delta V / V_i}$$

$$\Delta V = -\frac{V_i \Delta P}{B}$$

$$\Delta V = -\frac{(0.50\ \mathrm{m^3})(2.0 \times 10^7\ \mathrm{N/m^2} - 1.0 \times 10^5\ \mathrm{N/m^2})}{6.1 \times 10^{10}\ \mathrm{N/m^2}}$$

$$= \boxed{-1.6 \times 10^{-4}\ \mathrm{m^3}}$$