CHAPTER 3

Motion in Two or Three Dimensions

楊 本 立 副教授

Outline

1. Position, Velocity, and Acceleration Vectors
2. Two-dimensional motion with constant acceleration
3. Projectile motion
4. Uniform circular motion
5. Relative motion

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1. Position, Velocity, and Acceleration Vector

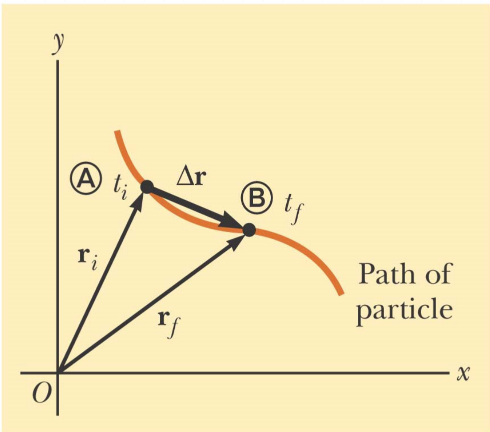
Position, Velocity, and Acceleration Vectors

- Position vector:

$$\vec{r}$$

- Displacement:

$$\Delta\vec{r} = \vec{r}_f - \vec{r}_i$$



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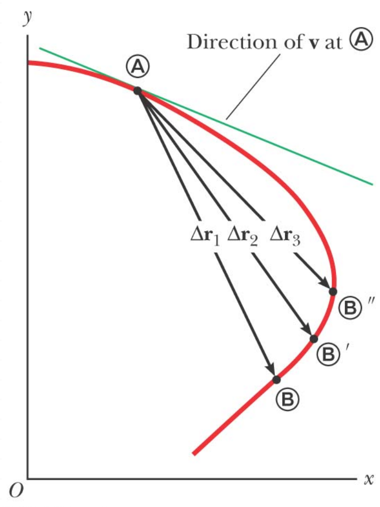
- Average velocity:

$$\bar{\vec{v}} \equiv \frac{\Delta \vec{r}}{\Delta t}, \quad \text{indepdent of path}$$

- Instantaneous velocity

$$\vec{v} \equiv \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}$$

- Instantaneous speed = $|\vec{v}|$



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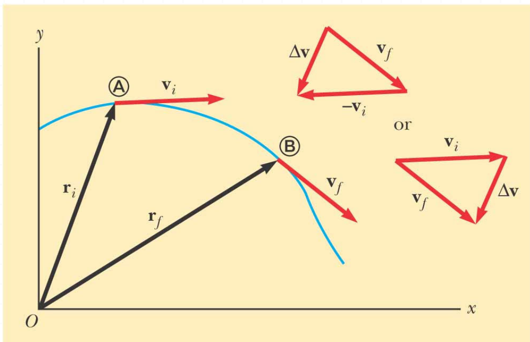
- Average acceleration:

$$\bar{\vec{a}} \equiv \frac{\vec{v}_f - \vec{v}_i}{t_f - t_i} = \frac{\Delta \vec{v}}{\Delta t}$$

- Instantaneous acceleration:

$$\vec{a} \equiv \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt}$$



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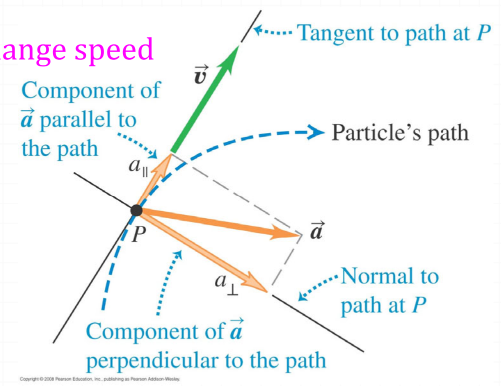
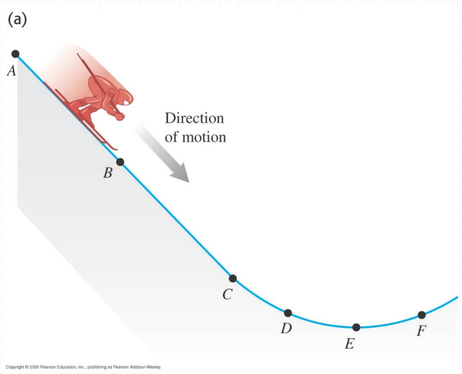
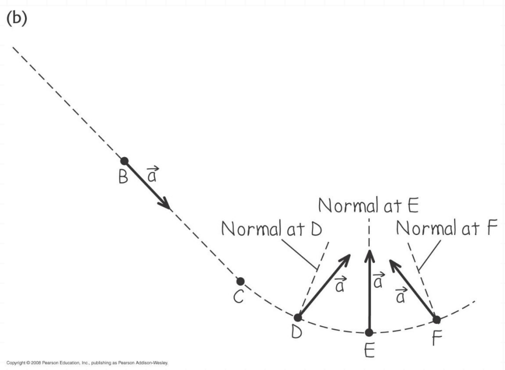
😎 Any particle following a curved path is accelerating!

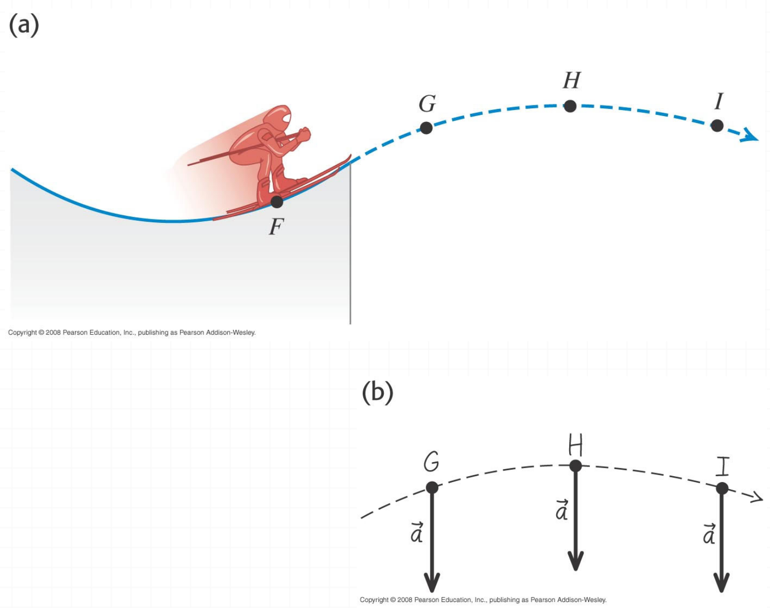
$a_{\parallel}$: change speed



$a_{\perp}$: change direction







1. Position, Velocity, and Acceleration Vectors
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2. Two-dimensional motion with constant acceleration

Two-dimensional motion with constant acceleration

$$\frac{d\vec{v}}{dt}=\vec{a} \quad \Rightarrow \quad \frac{dv_x}{dt}\hat{i}+\frac{dv_y}{dt}\hat{j}=a_x\hat{i}+a_y\hat{j}$$

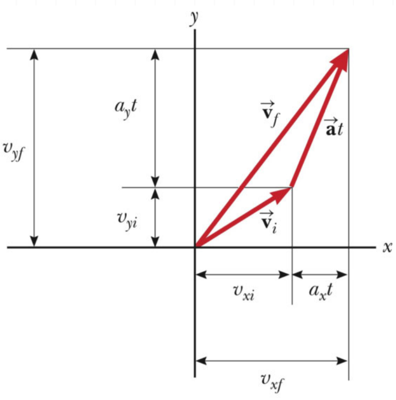
$$\Rightarrow \begin{cases} \dfrac{dv_x}{dt}=a_x \\ \dfrac{dv_y}{dt}=a_y \end{cases} \quad \Rightarrow \quad \begin{cases} v_x(t)=v_x(0)+a_xt \\ v_y(t)=v_y(0)+a_yt \end{cases}$$

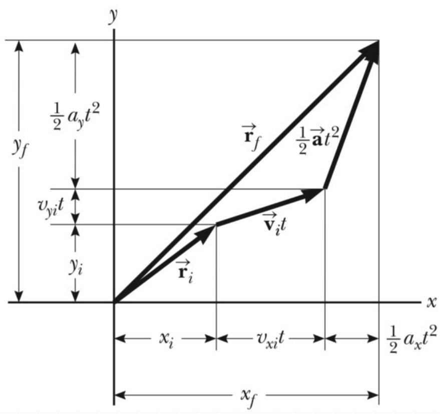
The motion in the *x* and *y* directions is independent of each other.

$$\Rightarrow \quad \vec{v}(t)=\vec{v}(0)+\vec{a}t$$

(1) $\vec{v}_f=\vec{v}_i+\vec{a}t$

(2) $\vec{r}_f=\vec{r}_i+\vec{v}_it+\dfrac{1}{2}\vec{a}t^2$





Note:

$$(3)\ \vec{r}_f = \vec{r}_i + \langle \vec{v} \rangle t$$

$$(4)\ v_f^2 = v_i^2 + 2\vec{a}\cdot(\vec{r}_f - \vec{r}_i)$$

Proof:

$$\begin{aligned} v_f^2 &= \vec{v}_f \cdot \vec{v}_f = (\vec{v}_i + \vec{a}t)\cdot(\vec{v}_i + \vec{a}t) \\ &= v_i^2 + 2\vec{v}_i \cdot \vec{a}t + \vec{a}t \cdot \vec{a}t \\ &= v_i^2 + 2\vec{a}\cdot\left(\vec{v}_i t + \frac{1}{2}\vec{a}t^2\right) \\ &= v_i^2 + 2\vec{a}\cdot(\vec{r}_f - \vec{r}_i) \end{aligned}$$

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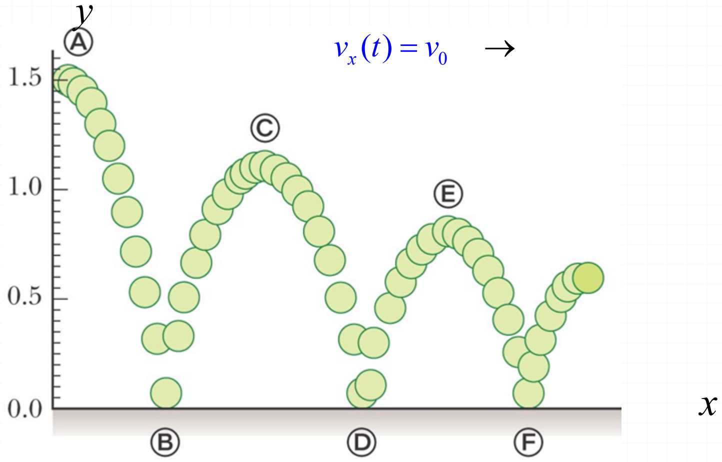
Q: Plot $y = y(t) = ?$

$v_y = v_y(t) = ?$

$a_y = a_y(t) = ?$



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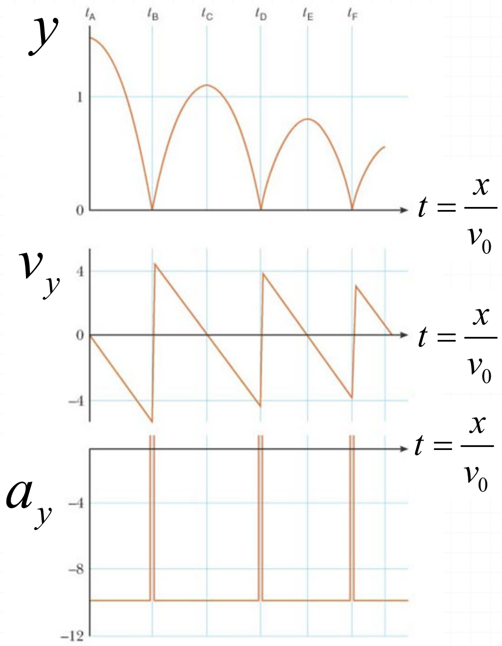

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3. Projectile Motion

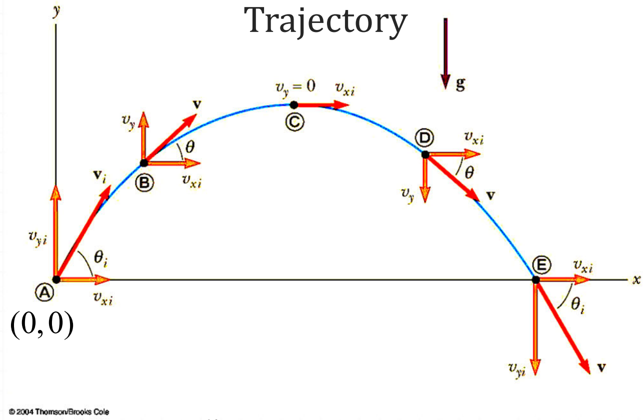
Projectile Motion

- An object in projectile motion will follow a parabolic path.
- Reference frame chosen y is vertical with upward positive.
- Acceleration components:

  $a_y = -g$ and $a_x = 0$
- Initial velocity components:

  $v_{xi} = v_i \cos\theta$ and $v_{yi} = v_i \sin\theta$

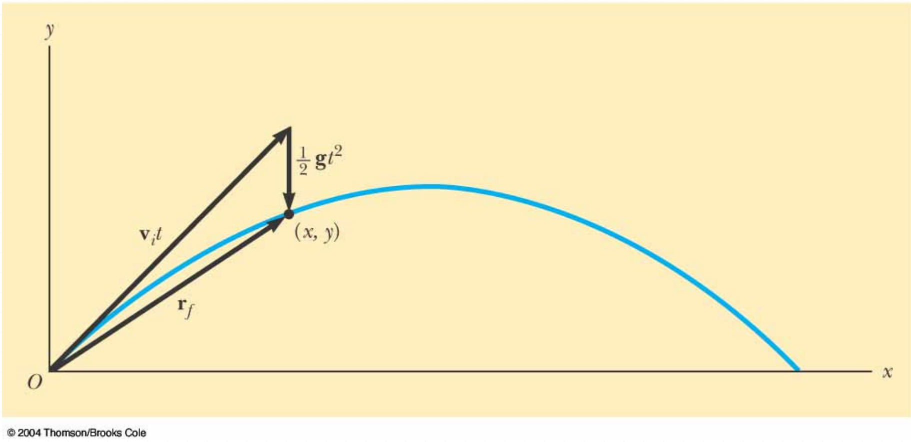
Trajectory



$$\begin{cases} x_f = v_{xi}t = v_i\cos\theta_i t \Rightarrow t = \dfrac{x_f}{v_i\cos\theta_i} \\ y_f = v_{yi}t + \dfrac{1}{2}a_y t^2 = (v_i\sin\theta_i)t - \dfrac{1}{2}gt^2 \end{cases}$$

$$\Rightarrow y_f = \tan\theta_i x_f - \left(\frac{g}{2v_i^2\cos^2\theta_i}\right)x_f^2$$

“Parabola”



$$\vec{r}_f = \vec{v}_i t + \frac{1}{2}\vec{g}t^2$$

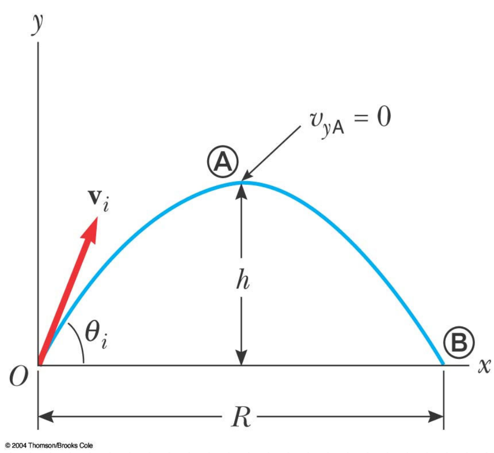
Range and Maximum Height of a Projectile

- The range, *R*, is the horizontal distance of the projectile.

$$R = \frac{v_i^2 \sin 2\theta_i}{g}$$

- The maximum height the projectile reaches is *h*

$$h = \frac{v_i^2 \sin^2 \theta_i}{2g}$$



Proof: Time to max height:

$$v_{yf} = v_{yi} + a_y t \Rightarrow 0 = v_i \sin\theta_i - g t_0$$

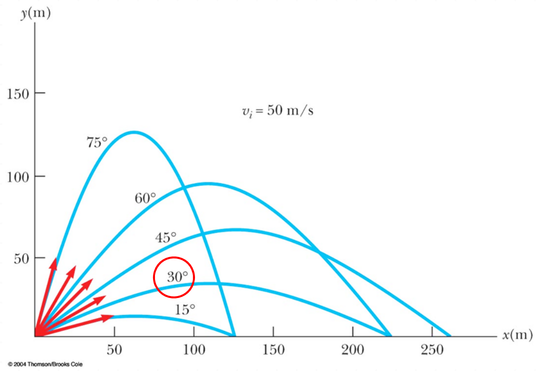
$$\Rightarrow t_0 = v_i \sin\theta_i / g$$

$$h = v_{yi}t - \frac{1}{2}gt^2 = v_i \sin\theta_i \frac{v_i \sin\theta_i}{g} - \frac{1}{2}g\left(\frac{v_i \sin\theta_i}{g}\right)^2$$

$$\Rightarrow h = \frac{v_i^2 \sin^2\theta_i}{2g}$$

$$R = v_{xi}t = v_{xi}(2t_0) = v_i \cos\theta_i (2\frac{v_i \sin\theta_i}{g})$$

$$\Rightarrow R = \frac{2v_i^2 \sin\theta_i \cos\theta_i}{g} = \frac{v_i^2 \sin 2\theta_i}{g}$$



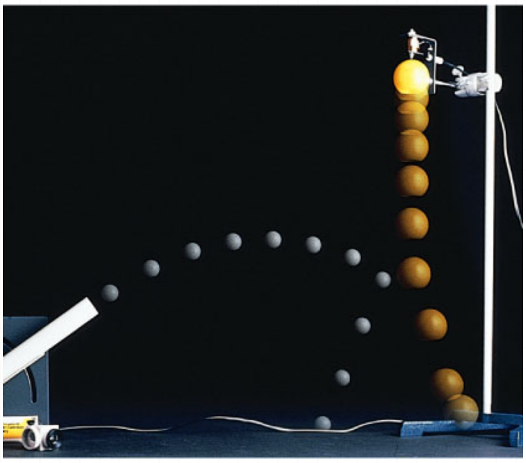
- The maximum range occurs at $\theta_i = 45^o$.
  - What if there is air resistance ($F_r \propto v$)? $\theta_i < 45^o$ [Am. J. Phys. 72, 1404 (2004)]
- Complementary angles will produce the same range, but flight time and max height are different!
- In this figure, which angle has the shortest flight time? 15°

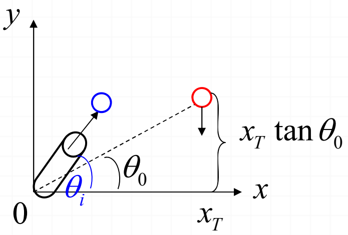
😎 A projectile is thrown from the origin O.

It moves in such a way that its distance from O is always increasing.

Find the maximum angle above the horizontal with which the projectile could have been thrown.

Ex. At which angle can the ball hit the target?





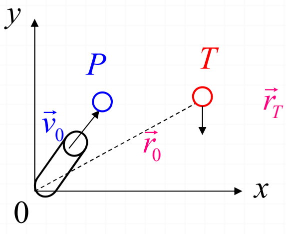
Target: $y_T = x_T \tan\theta_0 - \frac{1}{2} g t^2$

Projectile: $y_P = \underbrace{x_P \tan\theta_i}_{\text{The expected } y_P \text{ if } g=0.} - \frac{1}{2} g t^2$

When $(x_P, y_P) = (x_T, y_T)$, $\Rightarrow \theta_i = \theta_0$

Or in terms of vector,

$\vec{r}_P = \vec{v}_0 t + \frac{1}{2}\vec{g}t^2$

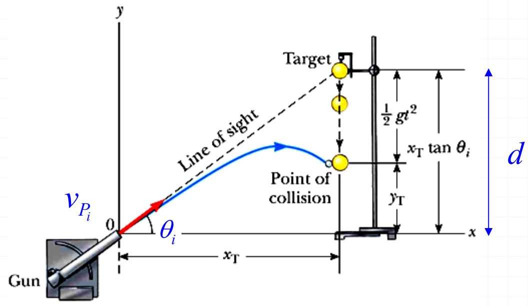


$\vec{r}_T = \vec{r}_0 + \frac{1}{2}\vec{g}t^2$

$\vec{r}_P = \vec{r}_T \quad \Rightarrow \quad \vec{v}_0 = \vec{r}_0 / t$

$\Rightarrow \quad \vec{v}_0 \parallel \vec{r}_0$

😎 What is the requirement for the target to be hit before it reaches ground?



$$y_T > 0 \quad \Rightarrow \quad d - \frac{1}{2}gt^2 > 0$$

$$t = \frac{x_T}{v_{Pi}\cos\theta_i} = \frac{d/\tan\theta_i}{v_{Pi}\cos\theta_i} = \frac{d}{v_{Pi}\sin\theta_i}$$

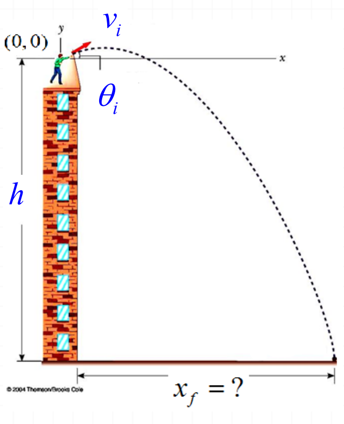
$$\Rightarrow \; v_{Pi}\sin\theta_i \geq \sqrt{\frac{gd}{2}}$$

Q: When the bullet is at P, where will be the monkey?

Projectile Motion – Problem Solving Hints

- Select a coordinate system.
- $v \rightarrow v_x, v_y$
- Analyze the horizontal motion using constant velocity techniques
- Analyze the vertical motion using constant acceleration techniques
- Remember that both directions share the same time.

Ex.



1. Flight time t = ?
2. x_f = ?
3. v_f = ?

$$1.\quad y_f = y_i + v_{yi}t + \frac{1}{2}a_y t^2$$

$$\Rightarrow \quad -h = 0 + v_i \sin\theta_i t - \frac{1}{2}gt^2$$

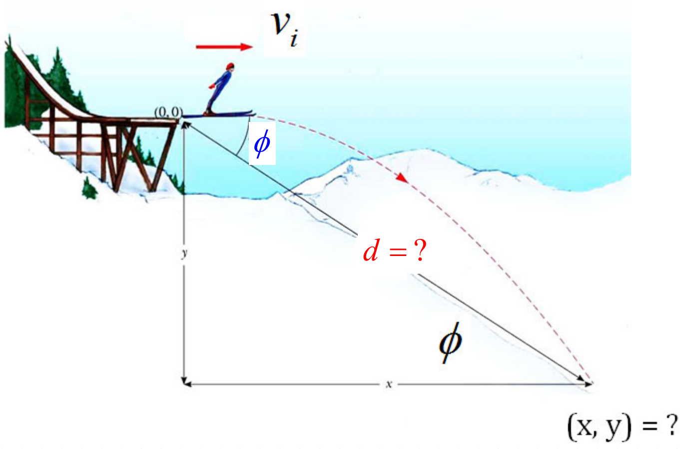
$$\Rightarrow \quad t = \cdots$$

$$2.\quad x_f = v_{xi}t = \cdots$$

3.

$$\begin{cases} v_{xf} = v_{xi} = v_i \cos\theta_i \\ v_{yf} = v_{yi} + a_y t = v_i \sin\theta_i - gt \end{cases} \Rightarrow \quad \vec{v}_f = v_{xf}\hat{i} + v_{yf}\hat{j}$$

Ex.

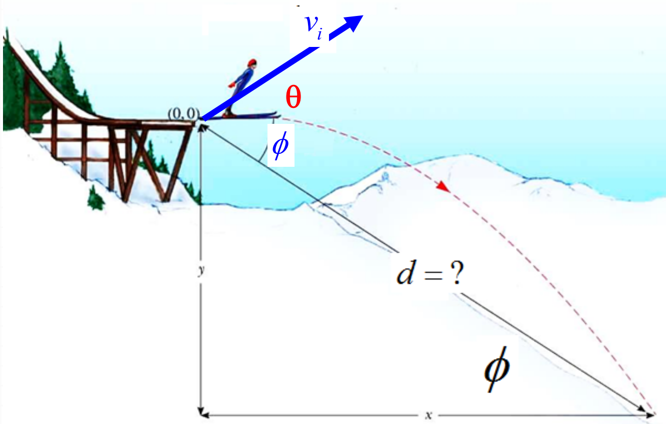


$$\begin{cases} x_f = v_{xi}t = v_i t \\ y_f = v_{yi}t + \frac{1}{2}a_y t^2 = -\frac{1}{2}gt^2 \end{cases}$$

$$\tan\phi = \frac{y_f}{x_f} = \frac{-\frac{1}{2}gt}{v_i} \quad \Rightarrow \; t = \frac{-2v_i \tan\phi}{g}$$

$$\Rightarrow \; (x_f, y_f) = \cdots$$

😎 Which θ makes the distance d maximum?



Hint: To express $d = d(\theta)$= ?

$$\begin{cases} x_f = v_i \cos\theta t = d\cos\phi & \Rightarrow t = \dfrac{d\cos\phi}{v_i\cos\theta} \\ y_f = (v_i \sin\theta) t - \dfrac{1}{2} g t^2 = -d\sin\phi \end{cases}$$

$$\Rightarrow d = d(\theta) = \frac{v_i^2}{g\cos^2\phi}\left(\sin 2\theta\cos\phi + 2\cos^2\theta\sin\phi\right)$$

$$\text{Max } d \Rightarrow \frac{d}{d\theta}\left[d(\theta)\right] = 0 \quad \Rightarrow \theta = \frac{\pi}{4} - \frac{\phi}{2}$$

Note:

$$\phi = 0 \Rightarrow \theta = 45°$$

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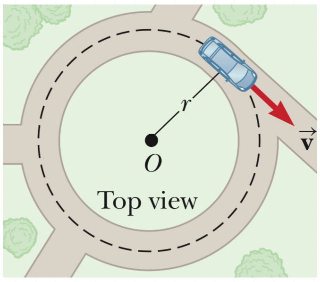
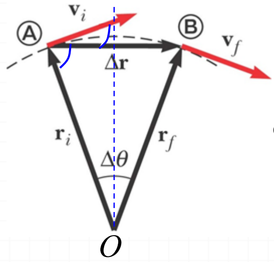
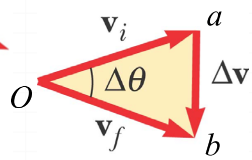
4. Uniform Circular Motion

Uniform Circular Motion

- *Uniform circular motion* occurs when an object moves in a *circular* path with a *constant speed.*
- The velocity vector is always *tangent* to the path of the object.
- An acceleration exists since the *direction* of the motion is changing, and the acceleration is perpendicular to the path.



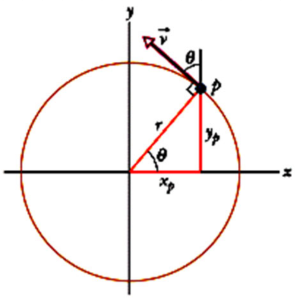


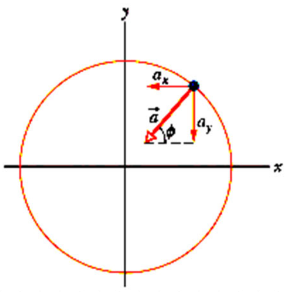


$$|\vec{r}_i| = |\vec{r}_f| = r \qquad |\vec{v}_i| = |\vec{v}_f| = v$$

$$\angle OAB \sim \angle Oab \quad \Rightarrow \frac{|\Delta\vec{v}|}{v} = \frac{|\Delta\vec{r}|}{r}$$

$$|\vec{a}| = \lim_{\Delta t \to 0} \frac{|\Delta\vec{v}|}{\Delta t} = \lim_{\Delta t \to 0} \frac{v}{r}\left(\frac{|\Delta\vec{r}|}{\Delta t}\right) = \frac{v^2}{r}$$





$$\vec{v} = v_x\hat{i} + v_y\hat{j} = (-v\sin\theta)\hat{i} + (v\cos\theta)\hat{j}$$

$$= (-v\frac{y_p}{r})\hat{i} + (v\frac{x_p}{r})\hat{j}$$

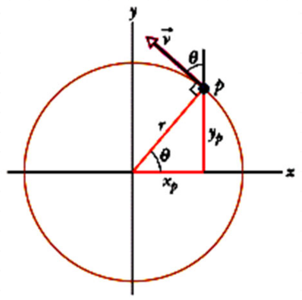
$$\vec{a} = \frac{d\vec{v}}{dt} = (-\frac{v}{r}\frac{dy_p}{dt})\hat{i} + (\frac{v}{r}\frac{dx_p}{dt})\hat{j}$$

$$= (-\frac{v}{r}v_y)\hat{i} + (\frac{v}{r}v_x)\hat{j}$$

$$= (-\frac{v}{r}v\cos\theta)\hat{i} + [\frac{v}{r}(-v\sin\theta)]\hat{j}$$

$$= (-\frac{v^2}{r}\cos\theta)\hat{i} + (-\frac{v^2}{r}\sin\theta)\hat{j}$$

$$\Rightarrow |\vec{a}| = \frac{v^2}{r}$$



$$\vec{r}(t) = r\cos(\omega t)\hat{i} + r\sin(\omega t)\hat{j}$$

$$\vec{v}(t) = \frac{d\vec{r}(t)}{dt} = -r\omega\sin(\omega t)\hat{i} + r\omega\cos(\omega t)\hat{j}$$

$$\vec{a}(t) = \frac{d\vec{v}(t)}{dt} = -r\omega^2\cos(\omega t)\hat{i} - r\omega^2\sin(\omega t)\hat{j}$$

$$\Rightarrow \vec{a}(t) = -r\omega^2\hat{r}$$

Centripetal Acceleration

- The acceleration always points toward the center of the circle of motion, perpendicular to the path of the motion, is called the *centripetal acceleration*.
- The magnitude of the centripetal acceleration vector is given by

$$a_C = \frac{v^2}{r}$$

- The direction of the centripetal acceleration vector is always changing, to stay directed toward the center of the circle of motion.
- This concept also applies to nonuniform circular motion or any curved path, where r is the radius of the curvature.

Period

- The *period, T*, is the time required for one complete revolution.
- The speed of the particle would be the circumference of the circle of motion divided by the period.
- Therefore, the period is

$$T \equiv \frac{2\pi r}{v}$$

Q: What is the centripetal acceleration of the Earth as it moves around the Sun?

$$a_c = \frac{v^2}{r} = \frac{\left(\frac{2\pi r}{T}\right)^2}{r} = 5.93\times10^{-3}\,\mathrm{m/s^2} \ll 9.8\mathrm{m/s^2}$$

Q: What is the centripetal acceleration of a point on the surface of the Earth at the equator caused by the Earth's spinning about its axis?

$$a_c = \frac{v^2}{r} = \frac{\left(\frac{2\pi r}{T}\right)^2}{r} = 3.37\times10^{-2}\,\mathrm{m/s^2} \ll 9.8\mathrm{m/s^2}$$

Centripetal Acceleration: $a_c = \underbrace{\frac{v^2}{r}}_{\substack{\text{Valid for any} \\ \text{radius of curvature}}} = \underbrace{r\omega^2}_{\substack{\text{Only for} \\ \text{uniform circular motion}}}$

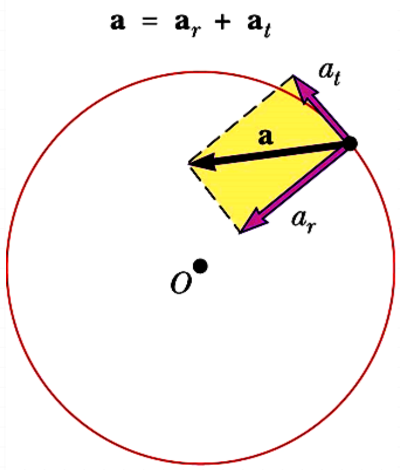
Period: $T \equiv \frac{2\pi r}{v}$

Angular speed: $\omega \equiv \frac{2\pi}{T} = 2\pi\left(\frac{v}{2\pi r}\right) = \frac{v}{r}$

$\Rightarrow \quad v = r\omega$

Tangential and Radial Acceleration

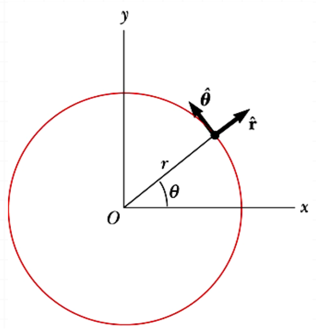
- The tangential acceleration causes the change in the *speed* of the particle.
- The radial acceleration comes from a change in the *direction* of the velocity vector.

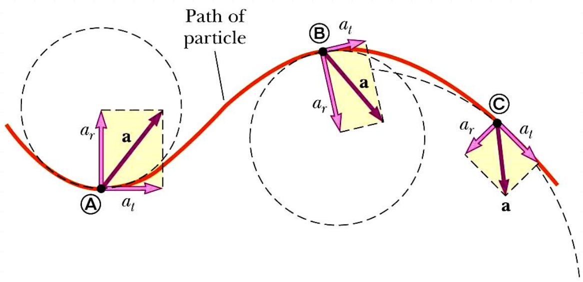


- The total acceleration is

$$\vec r(t) = r\hat r \quad \Rightarrow \quad \vec a(t) = \frac{d^2\vec r(t)}{dt^2}$$

$$\vec a = \vec a_r + \vec a_t = \underbrace{\left(-\frac{v^2}{r}\right)\hat r}_{\vec v \text{ changes direction.}} + \underbrace{\left(\frac{d|\vec v|}{dt}\right)\hat\theta}_{\vec v \text{ changes magnitude}}$$





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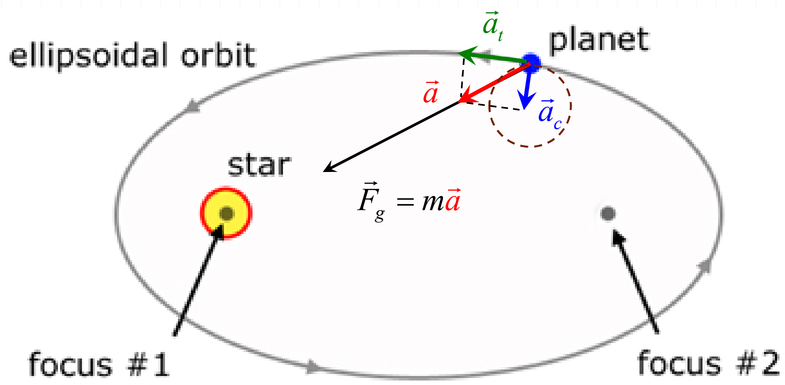
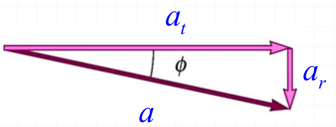


Ex.

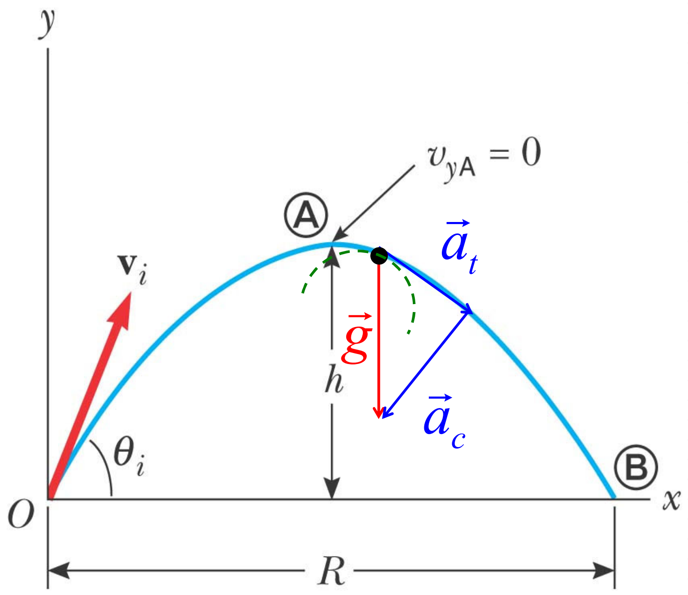
Total acceleration =?



$$a = \sqrt{a_r^2 + a_t^2} = \sqrt{(\frac{v^2}{r})^2 + a_t^2}$$

$$\theta = \tan^{-1}\frac{|a_r|}{|a_t|}$$

Ex.



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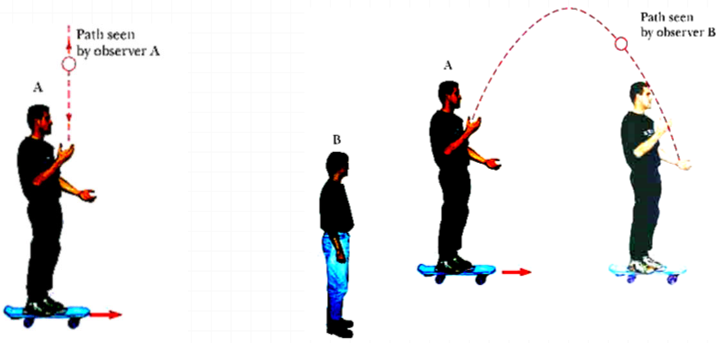
5. Relative Motion

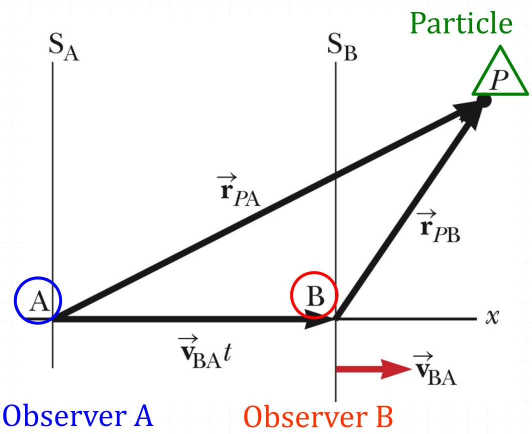
Relative Motion

- Two observers moving relative to each other generally do NOT agree on the outcome of an experiment.
- For example, observers A and B below see different paths for the ball.



台灣歷史博物館 (台南) – 時光列車

[https://www.youtube.com/watch?v=0vYPPRcm0kg]



$$\boxed{\vec{r}_{PA} = \vec{r}_{PB} + \vec{r}_{BA}}$$

$$\vec{r}_{PA} = \vec{r}_{PB} + \vec{v}_{BA}t$$

$$t_A = t_B = t$$

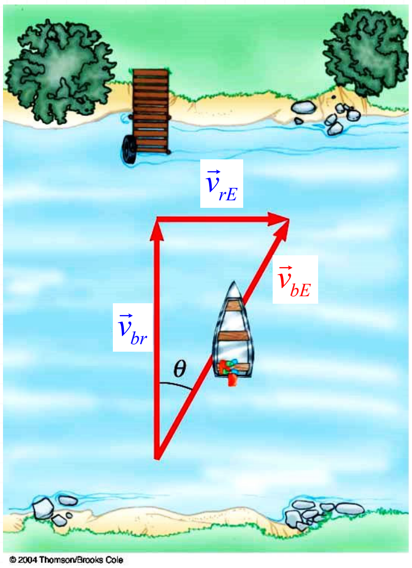
$$\boxed{\Rightarrow \quad \vec{v}_{PA} = \vec{v}_{PB} + \vec{v}_{BA}}$$

p.s., $\vec{v}_{PA} = (-)\vec{v}_{AP}$

In general, the relative velocity of observer *B* to observer *A* can be written as

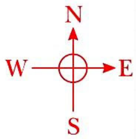
$$\boxed{\vec{v}_{BA} = \vec{v}_{BP} - \vec{v}_{AP}}$$

Ex.



Assume $\vec{v}_{br}$, $\vec{v}_{rE}$ are known.
The boat heads north,
what is $\vec{v}_{bE} = ?$

(**b**oat, **r**iver, **E**arth)



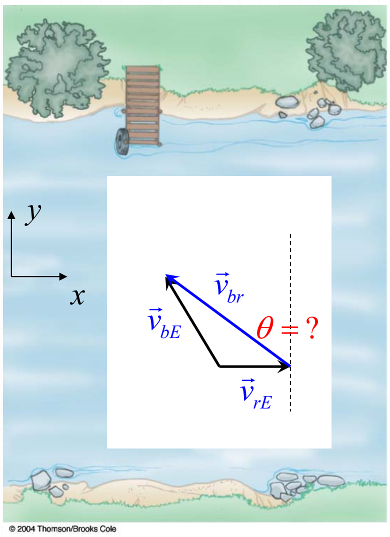
The boat heads north

$\Rightarrow \vec{v}_{br}$ is known.

$$\vec{v}_{br} = \vec{v}_{bE} - \vec{v}_{rE}$$

$$\Rightarrow \vec{v}_{bE} = \vec{v}_{br} + \vec{v}_{rE}$$

Ex.



Which θ will be so that the boat lands exactly opposite the starting point?

$$v_{bE,x} = 0$$

$$\vec{v}_{bE} = \vec{v}_{br} + \vec{v}_{rE}$$

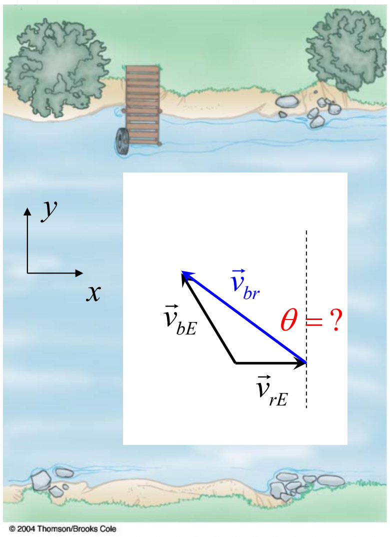
$$= (-v_{br}\sin\theta)\hat{i} + (v_{br}\cos\theta)\hat{j} + v_{rE}\hat{i}$$

$$= (v_{rE} - v_{br}\sin\theta)\hat{i} + v_{br}\cos\theta\,\hat{j}$$

$$v_{bE,x} = 0 \quad \Rightarrow v_{rE} - v_{br}\sin\theta = 0$$

$$\Rightarrow \theta = \sin^{-1}\frac{v_{rE}}{v_{br}}$$

Ex.



If $|v_{br}|$ is the same, which θ arrives fastest?

$$\vec{v}_{bE} = \vec{v}_{br} + \vec{v}_{rE}$$

$$= (-v_{br}\sin\theta)\hat{i} + (v_{br}\cos\theta)\hat{j} + v_{rE}\hat{i}$$

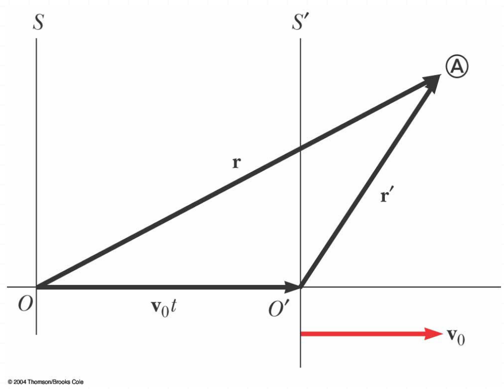
$$= (v_{rE} - v_{br}\sin\theta)\hat{i} + v_{br}\cos\theta\,\hat{j}$$

Fastest $\rightarrow$ $v_{bE,y} = v_{br}\cos\theta$ is max,

$$\Rightarrow \theta = 0^0$$

Relative Velocity and Acceleration

$t' = t$ $\quad$ $\vec{r}\,' = \vec{r} - \vec{v}_0 t$ $\quad$ (1)



$$\frac{d\vec{r}\,'}{dt} = \frac{d\vec{r}}{dt} - \vec{v}_0$$

$\vec{v}\,' = \vec{v} - \vec{v}_0$ $\quad$ (2)

Galilean transformation

(Only valid for $|\vec{v}|,\ |\vec{v}_0| \ll c$)

$$\frac{d\vec{v}\,'}{dt} = \frac{d\vec{v}}{dt} - \frac{d\vec{v}_0}{dt}$$

If $\vec{v}_0$ is constant, then $\vec{a}\,' = \vec{a}$.

Q: A passenger on a train that is moving with constant velocity drops a spoon. What is the acceleration of the spoon relative to (a) the train and (b) the Earth?