國立交通大學 電子物理系 NCTU Electrophysics

Chapter 13

PERIODIC MOTION

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Outline

1. Simple Harmonic Motion (SHM)
2. Energy of the Simple Harmonic Oscillator
3. Simple Harmonic Motion vs Circular Motion
4. The Pendulum
5. The Damped Oscillation and Forced Oscillation
6. Resonance

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1. Simple Harmonic Motion (SHM)

Periodic Motion

- ***Periodic motion*** is motion of an object that regularly repeats.
  - The object returns to a given position after a fixed time interval.

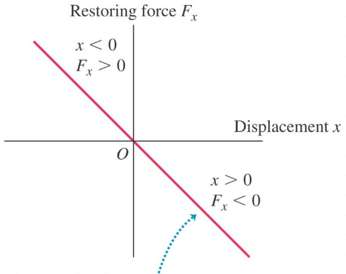
Simple Harmonic Motion

- A special kind of periodic motion occurs in mechanical systems when the force acting on the object is proportional to the position of the object relative to some equilibrium position, i.e.,

$$\vec{F} = -k\vec{x},$$

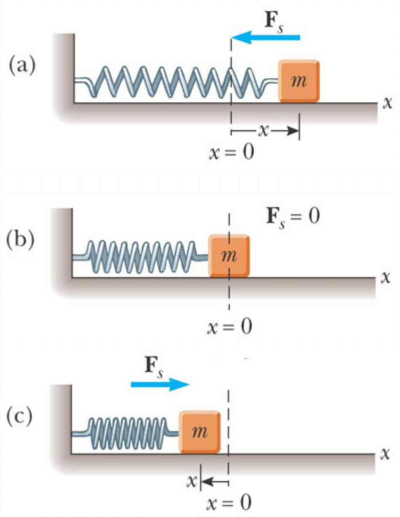
called ***simple harmonic motion***.

It is a conservative force!

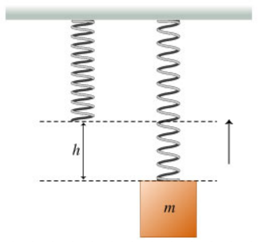


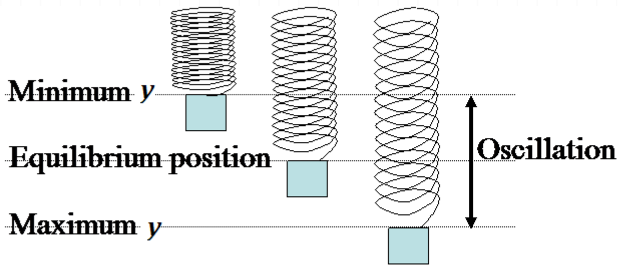
The restoring force exerted by an idealized spring is directly proportional to the displacement (Hooke's law, $F_x = -kx$): the graph of F_x versus x is a straight line.

Motion of a Spring-Mass System



$Hook's$ law: $\vec{F}_s = -k\vec{x}$





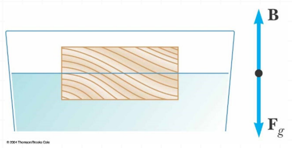
$$\vec{F}_0 = +kh\hat{j} - mg\hat{j} = 0$$

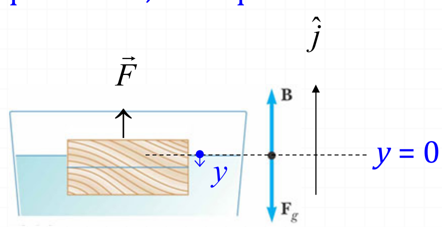
$$\vec{F}_y = +k(h-\Delta y)\hat{j} - mg\hat{j} = -k\Delta y\hat{j}$$

$$\Rightarrow \quad \vec{F} = -k\Delta\vec{y}, \text{ a SHM}$$

Ex.

Initially the block is at y away from the equilibrium, not equilibrium.





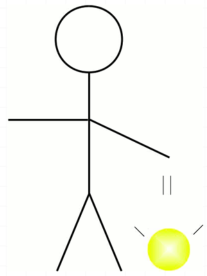
$$\vec{F}_g + \vec{B} = 0$$

$$m\vec{g} + \rho_{\text{fluid}} g V_0 \hat{j} = 0$$

$$\vec{F} = \vec{F}_g + \vec{B}'$$

$$\vec{F} = \underbrace{m\vec{g} + \rho_{\text{fluid}} g V_0 \hat{j}}_{=0} - \rho_{\text{fluid}} g A\vec{y}$$

$$\Rightarrow \quad \vec{F} = -(\rho_{\text{fluid}} g A)\vec{y} \quad \text{a SHM}$$



The bouncing ball $\Rightarrow$ Not a SHM.

😎 Not all periodic motions are simple harmonic!

Kinematics of SHM

$$F = -kx$$

$$\Rightarrow \frac{d^2x}{dt^2} = -\frac{k}{m}x$$ A 2nd-order differential equation

$$\text{Define } \omega^2 = \frac{k}{m}$$

$$\Rightarrow \frac{d^2x}{dt^2} = -\omega^2 x$$

What functions satisfy this equation?

e^{-x}, $\sin x$, $\cos x$

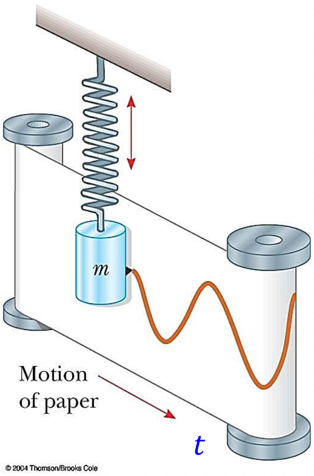
The solution to the differential equation is

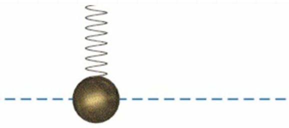
$$x(t) = A\cos(\omega t + \phi)$$

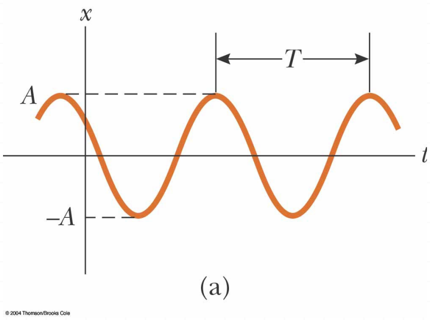
Amplitude   Angular frequency   Phase constant

(1) $\omega = \sqrt{\dfrac{k}{m}}$ (rad/s)

(2) $A > 0$ and ϕ, both depend on initial conditions.







$$\omega = \frac{2\pi}{T} = \sqrt{\frac{k}{m}} \quad \text{(rad/s)}$$

$$\text{Period } T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$

$$f = \frac{1}{T} = \frac{\omega}{2\pi} \quad \text{(Hertz)}$$

😎The frequency and period do not depend on amplitude.

The simple harmonic motion

$$x(t) = A\cos(\omega t + \phi)$$

$$v(t) = \frac{dx}{dt} = -\omega A\sin(\omega t + \phi) \qquad \Rightarrow \quad v_{\max} = \omega A$$

$$a(t) = \frac{dv}{dt} = -\omega^2 A\cos(\omega t + \phi) \qquad \Rightarrow \quad a_{\max} = \omega^2 A$$

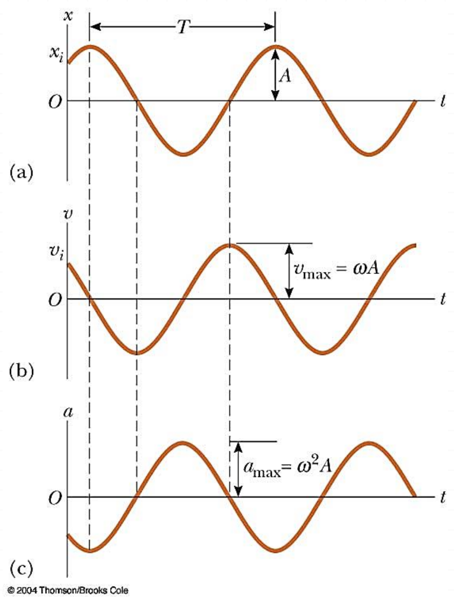
Not a uniform acceleration.

😎 To get the phase constant and amplitude:

$$\frac{v(0)}{x(0)} = \frac{-\omega A \sin\phi}{A\cos\phi} = -\omega\tan\phi \quad \Rightarrow \quad \phi = \tan^{-1}\left[-\frac{v(0)}{\omega x(0)}\right]$$

$$\begin{aligned}[x(0)]^2 &= A\cos^2\phi \\ [v(0)]^2 &= \omega^2 A^2 \sin^2\phi\end{aligned} \quad \Rightarrow \quad A = \sqrt{[x(0)]^2 + \frac{[v(0)]^2}{\omega^2}}$$

- The velocity is 90° out of phase with the displacement and the acceleration is 180° out of phase with the displacement .



$$x(t) = A\cos(\omega t + \phi)$$

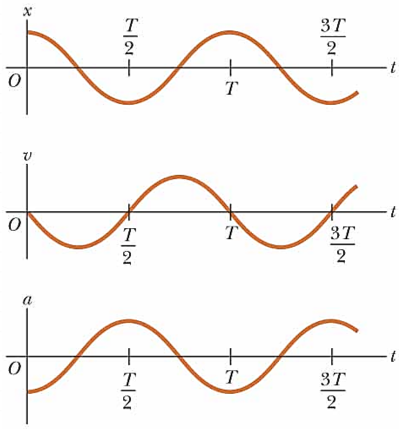
- Initial conditions at $t = 0$ are

  $x(0) = A_0$

  $v(0) = 0$

$$\begin{cases} x(0) = A\cos(\omega\cdot 0 + \phi) = A_0 \\ v(0) = -\omega A\sin(\omega\cdot 0 + \phi) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \phi = 0 \\ A = A_0 \end{cases}$$



- Initial conditions at $t = 0$ are

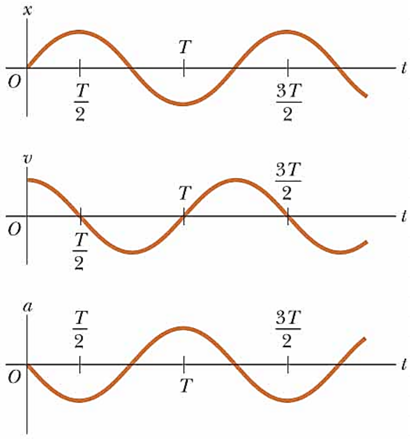
  $x(0)=0$

  $v(0) = v_i$

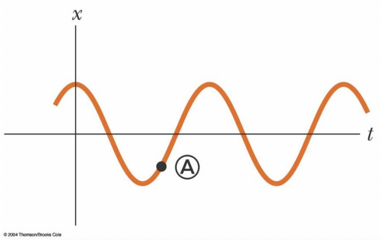
$$\begin{cases} x(0) = A\cos(\omega\cdot 0 + \phi) = 0 \\ v(0) = -\omega A\sin(\omega\cdot 0 + \phi) = v_i \end{cases}$$

$$\Rightarrow \begin{cases} \phi = (-)\dfrac{\pi}{2} \\ A = \dfrac{v_i}{\omega} \end{cases}$$

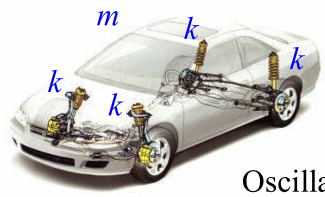
p.s., $\begin{cases} \phi = +\dfrac{\pi}{2} \\ A = (-)\dfrac{v_i}{\omega} \end{cases}$ not allowed. $\because$ amplitude must > 0.



Q: Velocity at A: positive or negative?



Ex.



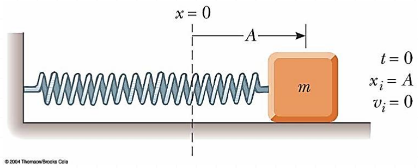
Oscillating frequency $f = ?$

$$F_{total} = \sum(-kx) = -\left(\sum k\right)x = -k_{\text{eff}}\,x$$

$$k_{\text{eff}} = \sum k = 4k$$

$$\Rightarrow \quad f = \frac{1}{2\pi}\sqrt{\frac{k_{eff}}{m}}$$

Ex. A box-spring system



a. The period T = ?

b. The max. speed of the block = ?

c. The max. acceleration of the block =?

d. $x(t)$, $v(t)$, $a(t)$ = ?

Ans:

(a) $$\omega = \sqrt{\frac{k}{m}} \quad \Rightarrow \quad T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$

(b) $$v_{\max} = \omega A$$

(c) $$a_{\max} = \omega^2 A$$

(d) $$x(0) = A\cos\phi = A \quad \Rightarrow \quad \phi = 0$$

$$x(t) = A\cos\omega t$$
$$v(t) = -\omega A\sin\omega t$$
$$a = -\omega^2 A\cos\omega t$$

2. Energy of the Simple Harmonic Motion

Energy of the Simple Harmonic Oscillator

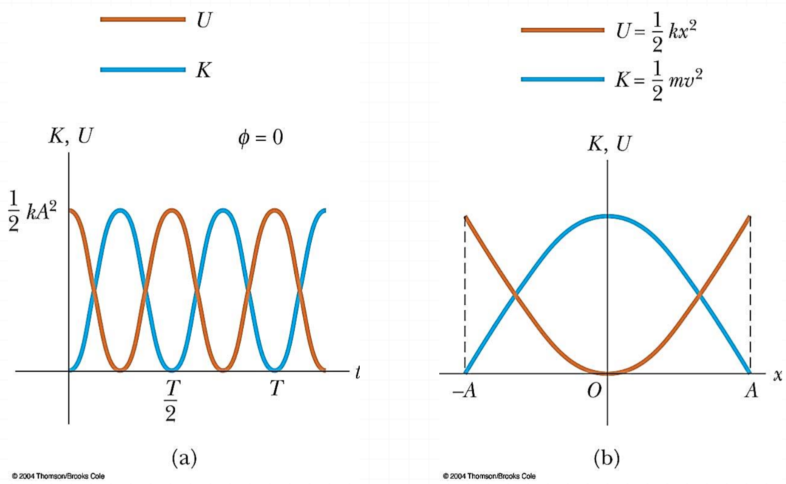
$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t + \phi)$$

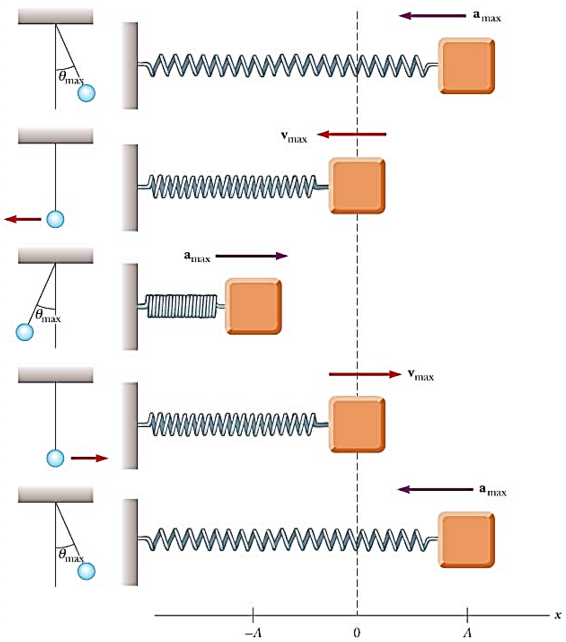
$$U = \frac{1}{2}kx^2 = \frac{1}{2}kA^2 \cos^2(\omega t + \phi)$$

$$\Rightarrow \quad E = K + U = \frac{1}{2}kA^2$$

The total mechanical energy of the SHM is proportional to A^2.

$$\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2 \quad \Rightarrow \quad v(x) = \pm\sqrt{\frac{k}{m}(A^2 - x^2)} = \pm\omega\sqrt{A^2 - x^2}$$

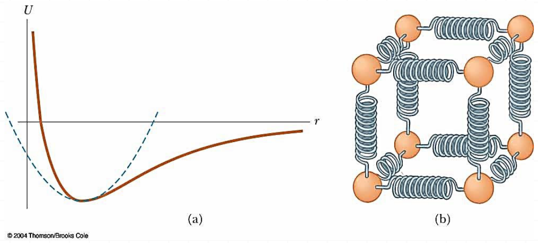


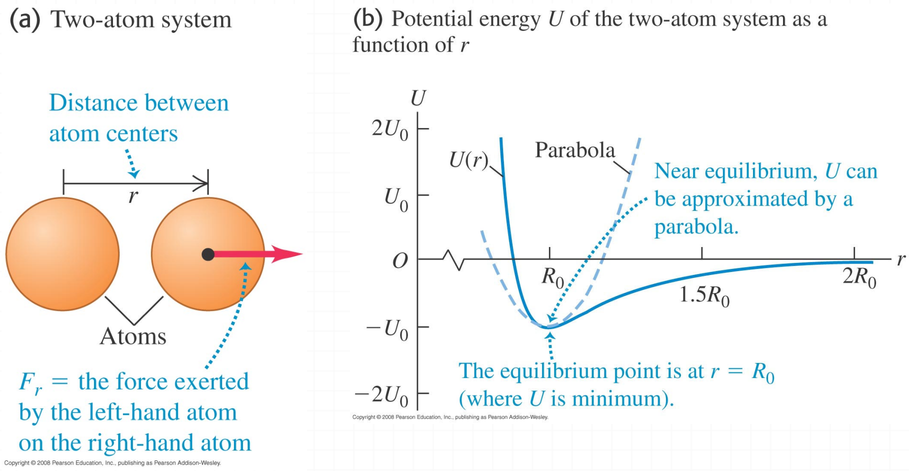


t	x	v	a	K	U
0	A	0	$-\omega^2 A$	0	$\frac{1}{2}kA^2$
$T/4$	0	$-\omega A$	0	$\frac{1}{2}kA^2$	0
$T/2$	$-A$	0	$\omega^2 A$	0	$\frac{1}{2}kA^2$
$3T/4$	0	ωA	0	$\frac{1}{2}kA^2$	0
T	A	0	$-\omega^2 A$	0	$\frac{1}{2}kA^2$

Molecular Model of SHM

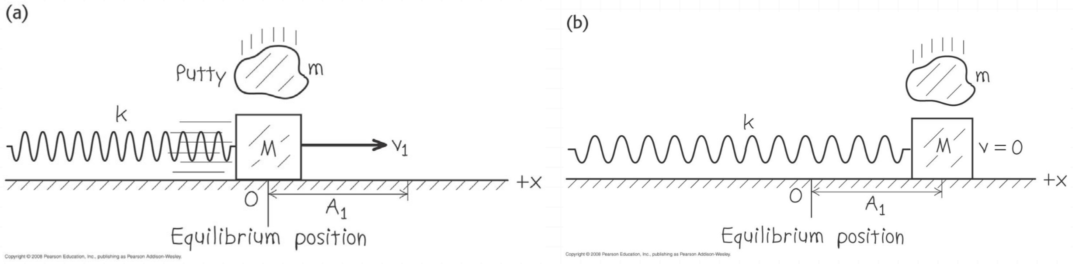
- If the atoms in the molecule do not move too far, the force between them can be modeled as if there were springs between the atoms.
- The potential energy acts similar to that of the SHM oscillator.





😎 Why does the volume of solids usually increase as the temperature increases?

Ex.



Find the new amplitude and period.

Ans:

(a) Collision ⇒ Momentum conservation

At $x = 0$,

$$\frac{1}{2}(M+m)v_{2x}^2 + \frac{1}{2}k\underbrace{x}_{=0}^2 = \frac{1}{2}kA_2^2 \quad \Rightarrow \quad A_2 = \sqrt{\frac{(M+m)}{k}}v_{2x}$$

$$\text{Final:} \quad Mv_{1x} = (M+m)v_{2x} \quad \Rightarrow \quad v_{2x} = \frac{M}{M+m}v_{1x} \qquad \Rightarrow \quad A_2 = \sqrt{\frac{M}{M+m}}A_1$$

$$\text{Initial:} \quad \frac{1}{2}Mv_{1x}^2 + \frac{1}{2}k\underbrace{x}_{=0}^2 = \frac{1}{2}kA_1^2 \quad \Rightarrow \quad v_{1x} = \sqrt{\frac{k}{M}}A_1$$

$$T_1 = 2\pi\sqrt{\frac{M}{k}} \quad \Rightarrow \quad T_2 = 2\pi\sqrt{\frac{M+m}{k}}$$

(b) At $x = A_1$,

$$\frac{1}{2}(M+m)\underbrace{v_{2x}^2}_{=0} + \frac{1}{2}k\underbrace{x}_{=A_1}{}^2 = \frac{1}{2}kA_2^2 \quad \Rightarrow \quad A_2 = \sqrt{A_1^2 + \frac{M+m}{k}v_{2x}^2}$$

Final: $Mv_{1x} = (M+m)v_{2x} \quad \Rightarrow \quad v_{2x} = \dfrac{M}{M+m}v_{1x} \quad \Rightarrow \quad A_2 = A_1$

Initial: $\dfrac{1}{2}Mv_{1x}^2 + \dfrac{1}{2}k\underbrace{x}_{=A_1}{}^2 = \dfrac{1}{2}kA_1^2 \quad \Rightarrow \quad v_{1x} = 0$

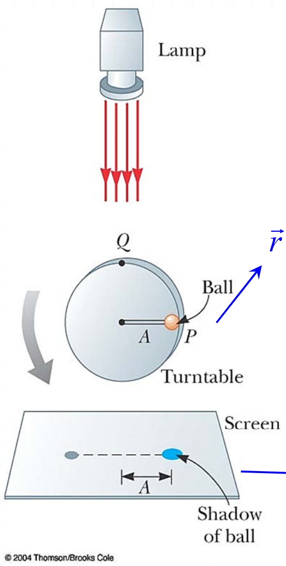
$$T_1 = 2\pi\sqrt{\frac{M}{k}} \quad \Rightarrow \quad T_2 = 2\pi\sqrt{\frac{M+m}{k}}$$

3. Simple Harmonic Motion vs. Circular Motion

SHM and Circular Motion

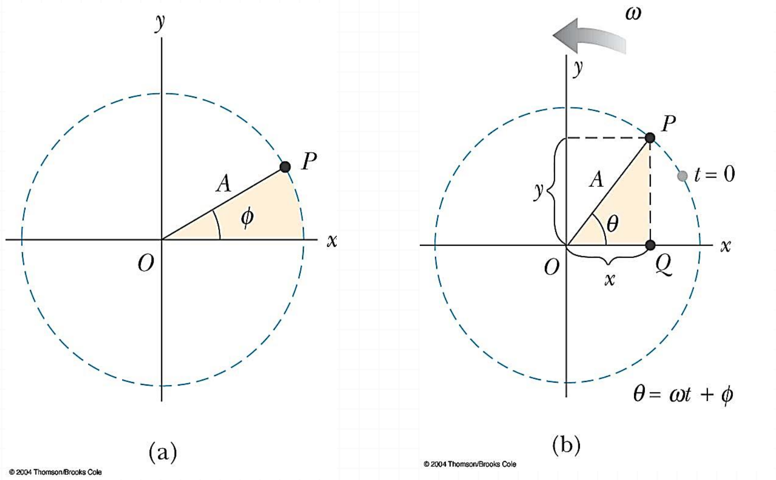
- **Simple Harmonic Motion** along a straight line can be represented by the projection of uniform circular motion along the diameter of a reference circle.
- **Uniform circular motion** can be considered a combination of two perpendicular simple harmonic motions.

$$\begin{aligned}\vec{r}(r,\theta) &= A\cos(\omega t+\phi)\hat{i} + A\sin(\omega t+\phi)\hat{j} \\ &= A\cos(\omega t+\phi)\hat{i} + A\cos(\omega t+\phi+90^0)\hat{j}\end{aligned}$$



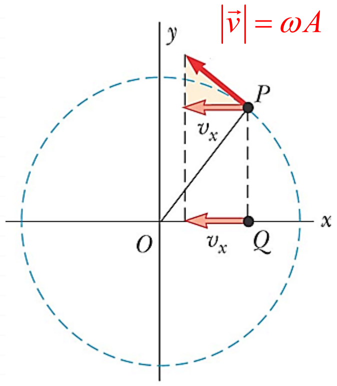
$$\vec{r}(r,\theta) = A\cos(\omega t+\phi)\hat{i} + A\sin(\omega t+\phi)\hat{j}$$

$$\vec{x}(t) = A\cos(\omega t+\phi)\hat{i}$$

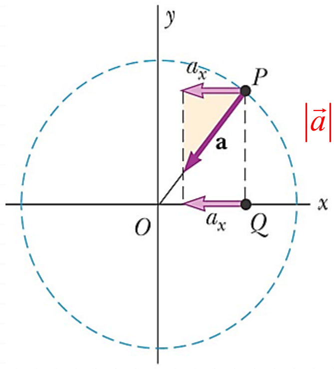


$$x(t) = A\cos\theta = A\cos(\omega t + \phi)$$

$$|\vec{v}| = \omega A$$

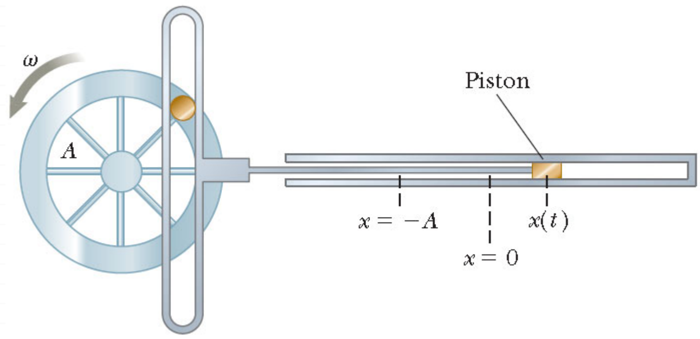


$$v_x(t) = -\underbrace{\omega A}_{v_{x,\ \max}} \sin(\omega t + \phi)$$

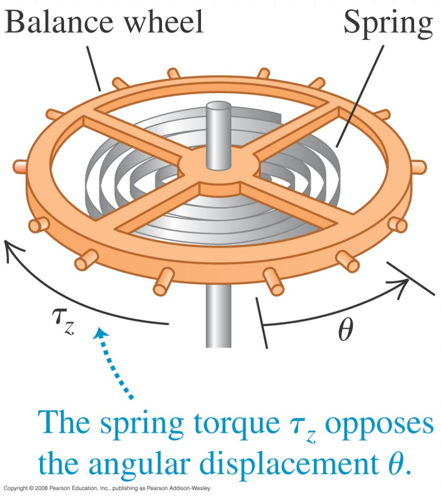


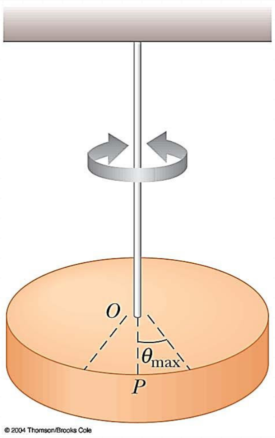
$$|\vec{a}| = \omega^2 A$$

$$a_x(t) = -\underbrace{\omega^2 A}_{a_{x,\,\max}}\cos(\omega t + \phi)$$



Torsional Pendulum





$\tau = -\kappa\theta$ κ: torsion constant

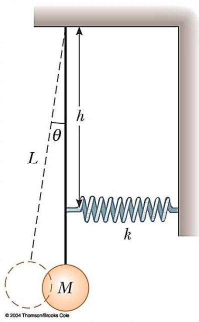
$$\Rightarrow \quad -\kappa\theta = I\frac{d^2\theta}{dt^2}$$

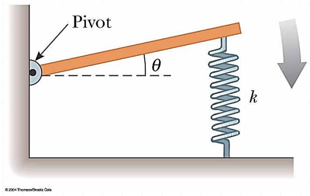
$$\Rightarrow \quad \frac{d^2\theta}{dt^2} = -\frac{\kappa}{I}\theta = -\omega^2\theta$$

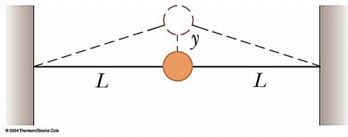
$$\omega = \sqrt{\frac{\kappa}{I}}$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{\kappa}}$$

Other SHMs.







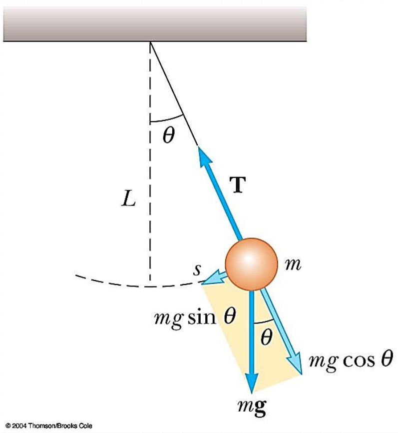
😎 Whether the motion is SHM or not can be judged by $F = ma$.

If the force is not easy to express, then use the equation of energy conservation and try to express it as $\frac{dx^2}{dt^2} + \omega^2 x = 0$.

The period then is also obtained.

4. The pendulum

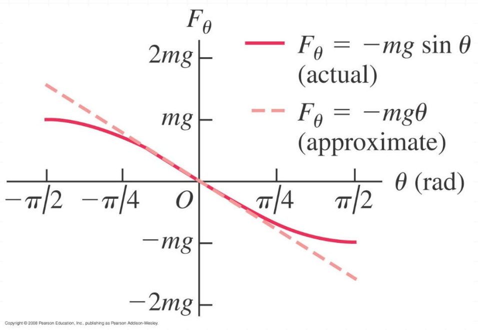
The Pendulum



$$F_t = -mg\sin\theta = m\frac{d^2 s}{dt^2}$$

$$s = L\theta$$

$$\Rightarrow \frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta$$



For small θ, $\sin\theta \approx \theta$

$$\Rightarrow \frac{d^2\theta}{dt^2} = -\frac{g}{L}\theta = -\omega^2\theta$$

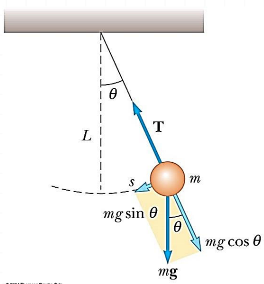
$$\omega = \sqrt{\frac{g}{L}}$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}}$$

$$\theta(t) = \theta_{\max}\cos(\omega t + \phi)$$

😎 For L = 1m and g =9.8 m/s, the period $T \approx 2$ s.

Pendulum oscillates at a larger angle



$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta$$

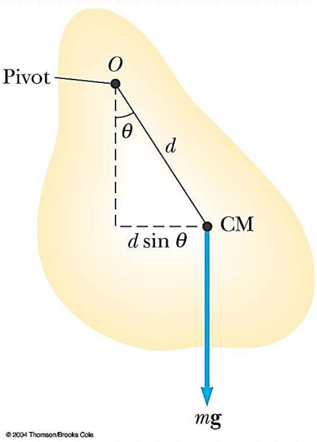
$$T_0 = 2\pi\sqrt{\frac{L}{g}}$$

For not small angle θ, $T = T_0\left[1 + \frac{1}{2^2}\sin^2\frac{1}{2}\theta + \frac{1}{2^2}\left(\frac{3}{4}\right)^2\sin^4\frac{1}{2}\theta + \cdots\right]$

$$\Rightarrow T > T_0 = 2\pi\sqrt{\frac{L}{g}}, \quad \frac{T - T_0}{T} \le 0.5\% \quad \text{for} \quad \theta \le 15°.$$

https://www.facebook.com/watch/?v=306222764371599&extid=NS-UNK-UNK-UNK-AN_GK0T-GK1C&ref=sharing

Physical Pendulum



$$\tau = I\alpha = I\frac{d^2\theta}{dt^2}$$

$$\Rightarrow \quad -mgd\sin\theta = I\frac{d^2\theta}{dt^2}$$

For small θ, $\sin\theta \approx \theta$

$$\Rightarrow \quad \frac{d^2\theta}{dt^2} = -\frac{mgd}{I}\theta = -\omega^2\theta$$

$$\omega = \sqrt{\frac{mgd}{I}}$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{mgd}} = 2\pi\sqrt{\frac{I_{CM} + md^2}{mgd}} \geq 2\pi\sqrt{\frac{d}{g}}$$

1. It depends on how the mass is distributed!
2. Can be experimentally used to determine the moment of inertia of an object.

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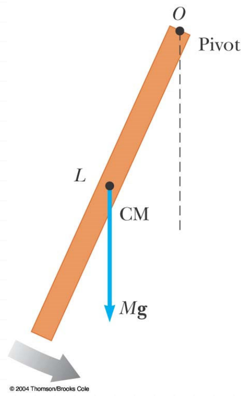
Ex. A uniform rod of mass M and length L is pivoted about one end and oscillated in a vertical plane. Find the period of oscillation if the amplitude of the motion is small.

$$(I = \frac{1}{3}ML^2)$$



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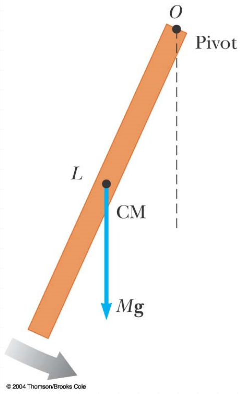
Ans:

$$T = 2\pi\sqrt{\frac{\frac{1}{3}ML^2}{Mg(L/2)}} = 2\pi\sqrt{\frac{2L}{3g}}$$



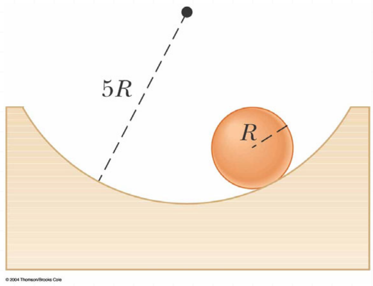
Note,

$$T \neq 2\pi\sqrt{\frac{(L/2)}{g}}$$

Dividing the rod into tiny pieces, the average T of these pieces does not equal to the true T.

$$\bar{T} = \frac{\int_0^L 2\pi\sqrt{\frac{\ell}{g}}d\ell}{\int_0^L d\ell} = \frac{2\pi\sqrt{\frac{1}{g}}\frac{2}{3}L^{3/2}}{L} = \frac{4\pi}{3}\sqrt{\frac{L}{g}} \neq 2\pi\sqrt{\frac{2L}{3g}}$$

Q:



T = ?

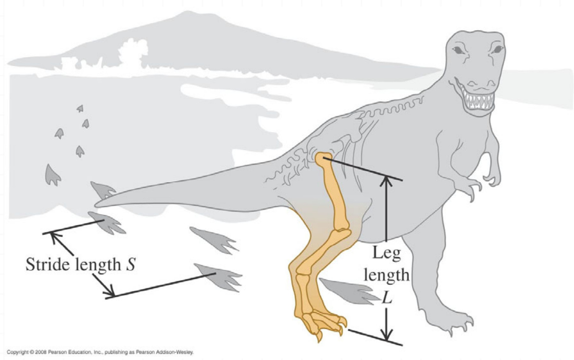
Hint:

Without rotation, the motion is the same as a pendulum.

With rotation, the equation of motion can be obtained from energy conservation. Then the period can be obtained.

[Problem 15.56 (Serway, 6th)]

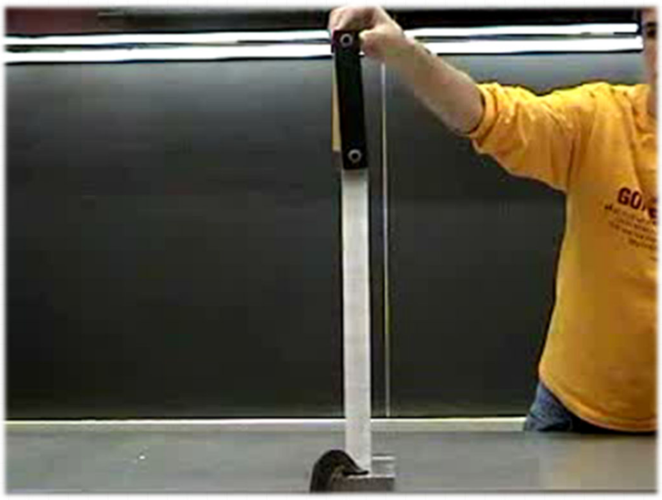
Ex. Estimate the walking speed of rex.

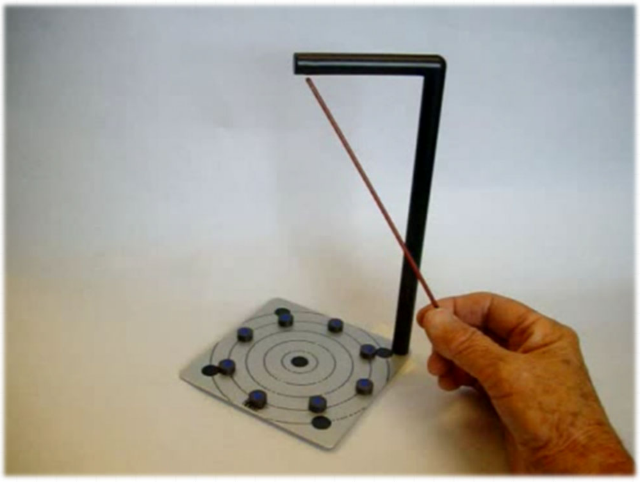


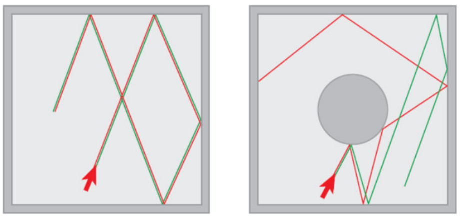
Ans:

$$T = 2\pi\sqrt{\frac{2L}{3g}} \quad \Rightarrow \quad v = S / T = \frac{S}{2\pi}\sqrt{\frac{3g}{2L}}$$

Non-linear effect: Chaos





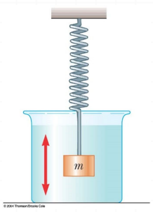


(a) (b)

The solution of nonlinear dynamics or chaotic motion
is sensitive dependent on initial conditions.

5. The damped oscillation and forced oscillation

Damped Oscillations



$$\sum \vec{F} = m\vec{a}$$

$$\Rightarrow \quad -kx - bv = ma$$

Retarding force: $-bv$

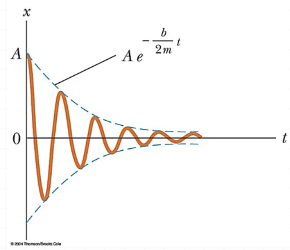
"b" : damping coefficient

$$\Rightarrow \quad -kx - b\frac{dx}{dt} = m\frac{d^2x}{dt^2}$$

When "b" is small, i.e., retarding force $\ll$ restoring force

$$x(t) = Ae^{-\frac{b}{2m}t}\cos(\omega t + \phi)$$

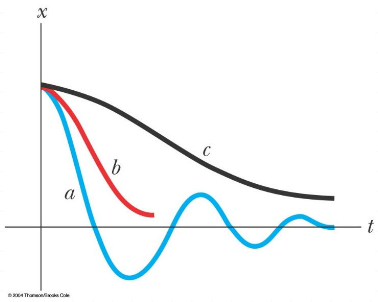
$$\omega = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}$$



Damped oscillation.

$$\omega = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2} = \sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2}$$

$$\omega_0 = \sqrt{\frac{k}{m}}, \quad \text{nature frequency}$$

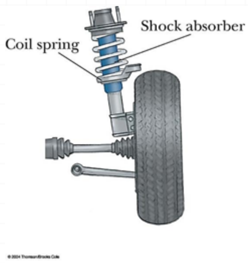


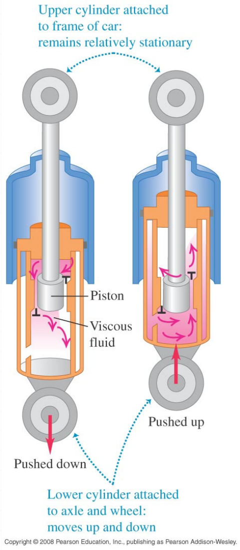
(a) Underdamped, $\omega_0 > \frac{b}{2m}$

(b) Critically damped, $\omega_0 = \frac{b}{2m}$

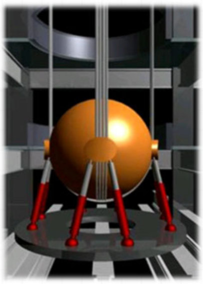
(c) Overdamped, $\omega_0 < \frac{b}{2m}$

Q: An automotive suspension system consists of a combination of springs and shock absorbers, as shown in Figure. If you were an automotive engineer, would you design a suspension system that was (a) underdamped (b) critically damped (c) overdamped?





Taipei 101



從九十二樓懸掛到八十八樓之世界最大被動式風阻尼器能抵銷自然力造成的擺動，確保大樓內人員平穩舒適。

調質阻尼器TMD（Tuned Mass Damper）

Energy loss in damped oscillation:

$$\begin{cases} \vec{F}_r = -b\vec{v} \\ \text{Power loss} = \vec{F}_r \cdot \vec{v} \end{cases} \Rightarrow \quad \text{Power loss} = -bv^2$$

Forced Oscillations

Apply an external force, $F(t) = F_0 \sin \omega t$

$$\sum \vec{F} = m\vec{a}$$

$F \rightarrow$

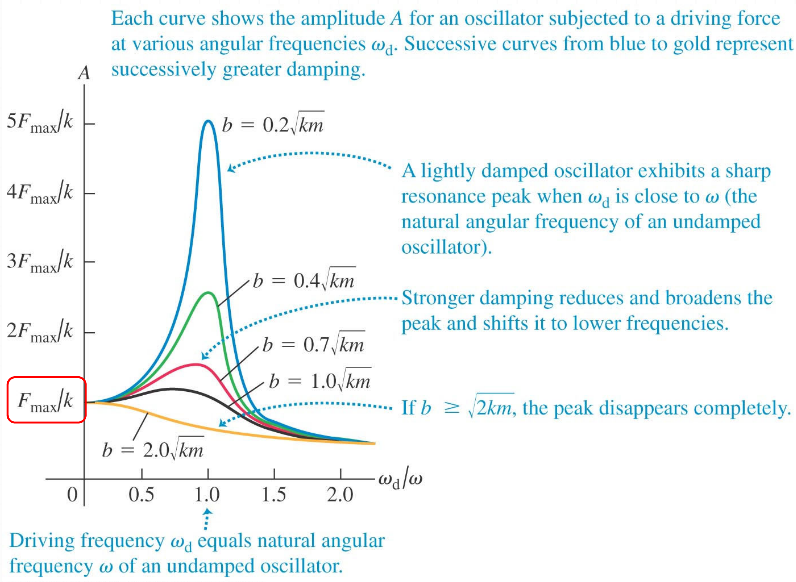
$$F_0 \sin \omega t - bv - kx = ma$$

$$\Rightarrow \quad F_0 \sin \omega t - b\frac{dx}{dt} - kx = m\frac{d^2x}{dt^2}$$

$$x(t) = A\cos(\omega t + \phi)$$

$$A = \frac{F_0 / m}{\sqrt{(\omega^2 - \omega_0^2)^2 + (\frac{b\omega}{m})^2}}$$

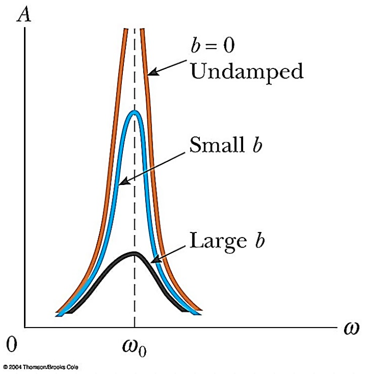
$\omega_0 = \sqrt{\frac{k}{m}}$, resonance frequency



6. Resonance

Resonance

- When near the natural frequency, $\omega \approx \omega_0$, an increase in amplitude occurs, called ***resonance.***
- At resonance, the applied force is in phase with the velocity and the power transferred to the oscillator is a maximum.
  - The power delivered is $\boldsymbol{F} \cdot \boldsymbol{v}$
  - This is a maximum when **F** and **v** are in phase

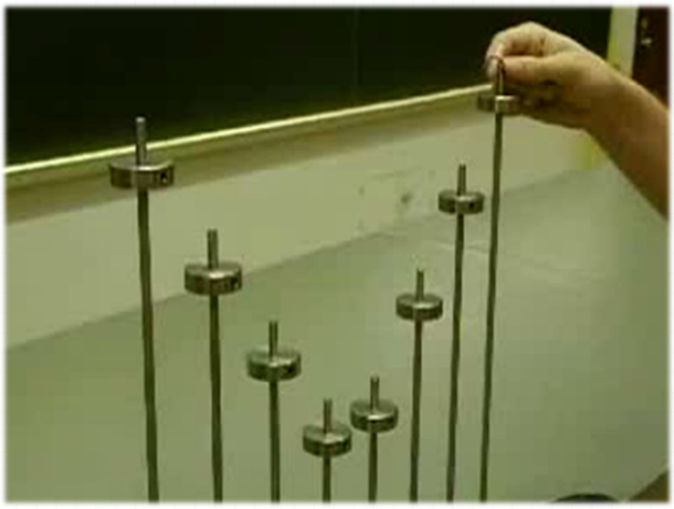


$\omega < \omega_0$

$F \rightarrow$

$\omega \approx \omega_0$

$\omega > \omega_0$



https://fb.watch/v/1KsXBMHLq/

[https://www.facebook.com/theactionlabofficial/videos/354291762265077]


池口研究室
NONLINEAR DYNAMICAL SYSTEM
メトロノーム同期 (32個)
Synchronization of thirty two metronomes
2012年09月14日，池口研究室前廊下にて撮影
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