Topic 3: Discrete Random Variables

Topic Outline

- Random Variables (RV, 隨機變數)
  - ➢ *Discrete* Random Variables 離散隨機變數 (Topic 3，textbook Chapter 2)
  - ➢ *Continuous* Random Variables 連續隨機變數 (Topic 6, textbook Chapter 3)
- Probability Mass Functions (PMF) of Discrete RVs
- Typical Discrete Random Variables:
  - ➢ Uniform
  - ➢ Bernoulli and Binomial
  - ➢ Geometric
  - ➢ Poisson
- Functions of Random Variables

Reading: Textbook 2.1 – 2.3

What is Random Variable (RV)?

Definition

Mathematically:

A random variable is an assignment (i.e., a mapping or, a function) of a *real number* to each *sample point* (i.e., outcome) in the sample space, or a *function* that maps sample space into the real line

簡言之，隨機變數是一個**函數**。

Example:

Flip a coin. We can define a random variable X, in a way that $X(\mathrm{H})=1$ and $X(\mathrm{T})=0$

What is Random Variable?

隨機變數是一個函數，它將樣本空間中的每一個 outcome 對應到某一個特定的實數

Concept:

- It's not merely a variable; it has a sense of *randomness* in it

  "Randomness" lies in the sample space in the sense that, ***before*** conducting an experiment, we are *uncertain* which outcome the experiment will produce.

- More precisely, it's a *function*

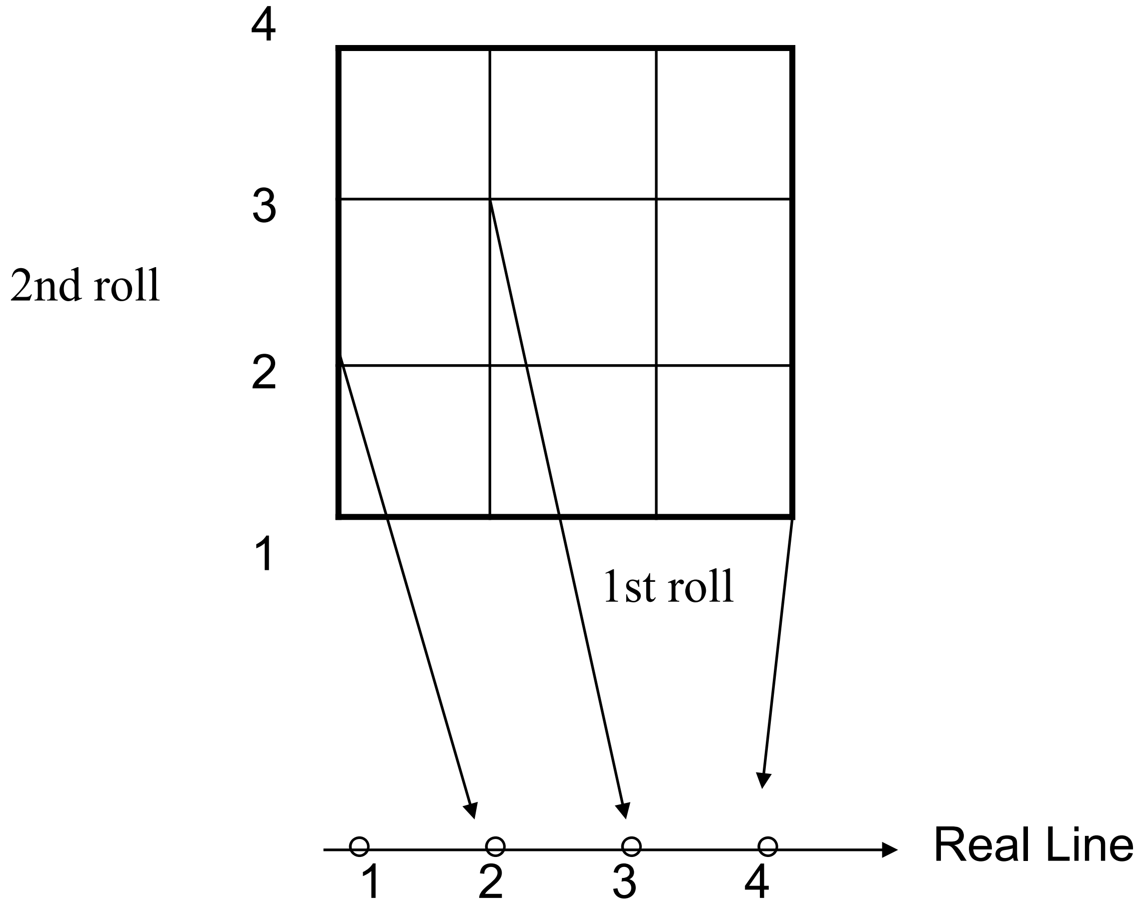
  Performing an experiment yields a specific sample point (outcome) ω, which produces a *sample value,* say $x = X(\omega)$, by means of the random variable.
  (Be aware of the difference between the capital letter X and the small letter x)

  一般我們用***斜體大寫***(X)表示隨機變數，***斜體小寫***(x)表示這個隨機變數的一個可能實數結果

Example:

Experiment: roll two 4-sided dice
The maximum of the two throws can be considered as a random variable,

$X(a,b)=\max(a,b)$, where a and b are the outcome of the first and second dice.



Why the Notion of "Random Variable"?

- For mathematical convenience
  - We can describe complicated events using simple math expressions by means of random variables (化繁為簡)
  - Particularly useful when outcomes of the considered experiment do not involve with any numerical values, *e.g.* coin flips (Heads, Tails)

Example:

Flip a coin 3 times. Define the random variable X_i=1 if the ith flip comes Heads, and X_i=0 if Tails.

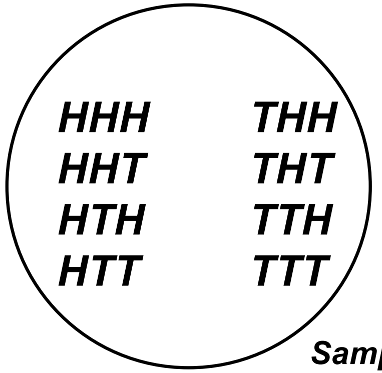
→F={Two heads in 3 flips}

⇨ F= $\{(X_1, X_2, X_3): X_1 + X_2 + X_3 = 2, X_i = 0 \text{ or } 1\}$;

更簡潔一點的寫法是 F={ $X_1+X_2+X_3=2$ }

→G={1st flip is a head, 2nd and 3rd flips have different results}

⇨ G={ $X_1=1$, $X_2 \neq X_3$ }



Sample space

→Every event has its particular physical meanings, and can be described precisely and elegantly by properly designed random variables (functions)

Examples

Use the previous sample space of two rolls of a four sided dice:

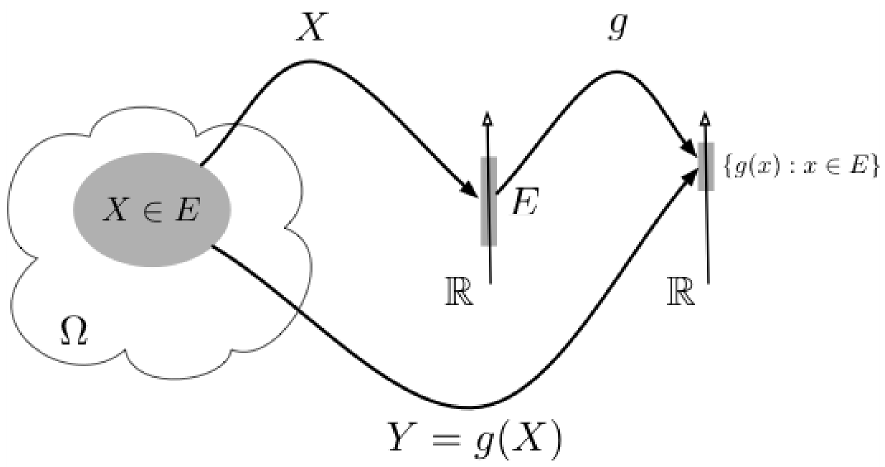
$\boldsymbol{\omega} = (\omega_1, \omega_2)$, $\omega_i =$ value of the ith toss

The following functions are all random variables:

- R = outcome of the first toss, $R(\boldsymbol{\omega}) = \omega_1$
- S = outcome of the second toss, $S(\boldsymbol{\omega}) = \omega_2$
- X = product of two faces, $X(\boldsymbol{\omega}) = \omega_1 \times \omega_2$
- Y = (difference)^2, $Y(\boldsymbol{\omega}) = (\omega_1 - \omega_2)^2$

對隨機變數取函數運算（Function of An RV）

- A function of a random variable defines another random variable
  - Since a random variable (RV) is a just function。再取一次函數，則其定義域為原樣本空間，值仍為一實數
  - 下圖例：X 是一個隨機變數，$Y=g(X)$ 也是一個隨機變數，只是將樣本空間內元素對應到不同的值！



- It is to the designer's convenience to define different RVs on a single experiment
  - R and S in the previous example are the most straightforward
  - We can use R and S to describe the other random variables, e.g., $X=RS$

隨機變數 與 事件

隨機變數與事件之間的關係

- When a *random variable* takes on a specific real value or belongs to certain set, it constitutes an *event.*
  - Example 1: **X=4** in the previous example (X=RS) means the ***event*** that the product of two rolls is 4

$$\{X=4\}=\{(\omega_1,\omega_2):\omega_1\omega_2=4,(\omega_1,\omega_2)\in\Omega\}$$
$$=\{(\omega_1,\omega_2)=(1,4),(2,2),(4,1)\}$$

這是一個事件

- For a random variable X and a specific value x, $\{X=x\}$ is an event. Thus, we can find the probability associated with this event.
  - *Conditional probability and concept of independence starts to kick in with RVs:*

    Ex: **P**(R=1|X=4)=?

- 更廣義而言，一個或多個隨機變數間的運算(等式、不等式、屬於)結果均會對應到一個事件
  - Ex: (1) $R+S=6$ 所對應到的事件為何? (2) $R\in\{2,4,6\}$ 表示結果為偶數的事件

隨機變數的分類

隨機變數可分為兩類: **<u>離散</u>** 與 **<u>連續</u>**

- <u>**Discrete**</u> random variables (**離散隨機變數**, Chapter 2)
  - Discrete random variables are defined over *discrete experiments*
  - Ex 1: 丟銅板: X(正面)=1、 X(反面)=0
  - Ex 2: 下棋比賽: X(Type 1對手)=0、 X(Type 2對手)=1、 X(Type 3對手)=2

離散 RV值域裡的值一定是「可數的」。
個數可以是有限個，或是「可數的」無窮多個 (如整數一般)。

- <u>**Continuous**</u> random variables (**連續隨機變數**, Chapter 3)
  - Ex 1: 幸運旋轉盤指針所指結果 X 可以是0到1之間的任何實數
  - Ex 2: 羅密歐與茱麗葉的遲到時間 X 和 Y 可以是0到1之間的任何實數
  - Ex 3: 雷達系統接收到由飛機反射回來的訊號 Y 可以是任意實數
  - Ex 4: 手機收到的訊號 Y 可以是任意實數

連續 RV值域裡的值一定是「不可數的」。個數一定是無窮多個 (如實數一般)。

離散隨機變數

- 重點回顧
  - 隨機變數的定義
  - 為什麼引進隨機變數的概念?
  - 隨機變數 與 事件 的連結
  - 隨機變數的分類

    → 離散隨機變數、連續隨機變數

- Next, 先談談*離散*隨機變數
  - Probability mass function
  - 一些常見的離散隨機變數

    →Uniform

    →Bernoulli 和 Binomial

    →Geometric

    →Poisson

Probability Mass Function (PMF)

- A random variable is called discrete if its range is finite or at most countably infinite
- A discrete random variable is characterized through the probabilities of the values it can take

- **Probability Mass Function** (PMF) of a discrete RV X: 機率質量函數

If x is any possible value of a discrete RV X, the probability mass of x, denoted $p_X(x)$, is the probability of the event $\{X = x\}$ consisting of all outcomes that give rise to X equal to x:

$$p_X(x) = \underbrace{P(\{X = x\})}_{\text{Probability of the event } \{X=x\}} = P(\{\omega: X(\omega) = x\})$$

Ex: 投擲公正骰子，X 為其出現點數 $p_X(6)$=?

- We will simply use $P(X = x)$ to represent $P(\{X = x\})$
- 機率質量函數和離散隨機變數是綁在一起的

Remarks:

- Comments on the notation $p_X(x) = P(X = x)$
  - Subscript X is the random variable of interest, it's capital and italic
  - The argument x is the real numerical value that the random variable X may take

- With PMF, we can use the third axiom of probability (additivity) to compute the probability of any event involving the random variable

$$P(\underbrace{X \in S}_{\text{This is an event!}}) = \sum_{x \in S} p_X(x)$$

General Rule of Finding PMF

- Collect all the sample points in the original sample space that give rise to the mapping $X=x$ for a fixed x. Find the set in the original sample space

$$S_x=\{\ \omega: X(\omega) = x\ \}$$

- Compute the probability of this set using the original probability measure

$$p_X(x) = P(\ \{\ \omega: X(\omega) = x\ \}\ ) = \sum_{\omega\in S_x} P(\ \{\omega\}\ )$$

- If $p_X(x)$ is a PMF, then it must satisfy
  - $0 \leqq p_X(x) \leqq 1$ for all x
  - $\sum_x p_X(x) = 1$

上述兩條件可被用來驗證一個 PMF 是否被正確計算出來，合乎規範 (legitimate)

Example

Let X be the number of heads obtained in two independent tosses of a fair coin. What is the probability of at least one head?

Uniform PMF

A *uniform* PMF puts equal probability on all sample points. Suppose a random variable takes on N possible values 0,1,…,N-1, equally likely. Then,

$$p_X(x) = \frac{1}{N}, \quad x = 0, 1, \ldots, N-1.$$

A uniform PMF provides a good model in experiments where there is no reason to suspect that any outcome is more or less likely than any other.

Common examples:

1. Flipping a fair coin
2. Rolling a fair dice

Bernoulli PMFs

Bernoulli Random Variable:

Let X take values only on {0,1} with $P(X=1)=p$. The probability mass function is called Bernoulli PMF and is given by

$$p_X(x) = \begin{cases} p, & x = 1, \\ 1-p, & x = 0. \end{cases}$$

Bernoulli 的應用

Bernoulli RV 可用來描述某件事情成功或失敗，這種單純的二元事件。**如:**

1. 銅板(正面、反面): p_X**(正面)=0.5**
2. 子瑜是否答應晚餐邀約: p_X**(答應)=0.3**、 p_X**(拒絕)=0.7**
3. 無線通訊網路中資料傳輸成功與否: p_X**(成功)=0.9**、 p_X**(失敗)=0.1**

Questions ?

Is Bernoulli PMF a legitimate PMF? 檢查 Is it nonnegative (obvious) 、Does it sum to 1?

Binomial Random Variable:

Let X = number of heads in n independent coin tosses. $P(\text{正面}) = p$ (bias of the coin)

What is the PMF $p_X(x)$ of X ?

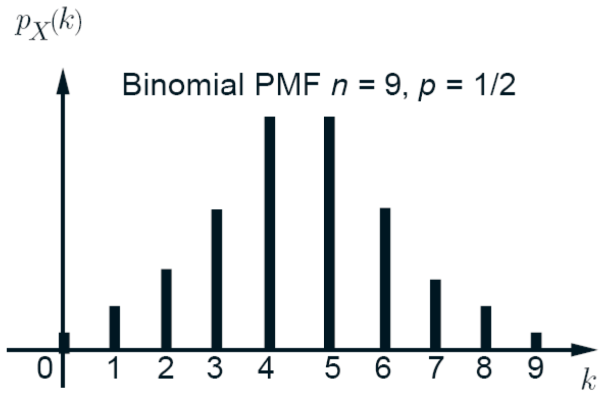
Note that for x = 2, n = 4

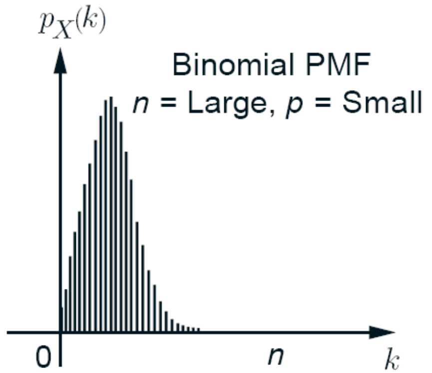
$$p_X(2) = P(HHTT) + P(HTHT) + P(HTTH) + P(THHT) + P(THTH) + P(TTHH)$$
$$= 6p^2(1-p)^2$$

Binomial PMF

In general, binomial X with **parameters** (n,p) has PMF

$$p_X(x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, \ldots, n.$$





Binomial PMF

- **Binomial 的應用**

  Binomial RV 可用來描述某件事在 n 次的試驗中成功的次數，參數為 n 和 p。 例如:

  1. 邀約子瑜晚餐10次中成功6次的機率? (n=10, p=0.3)

  2. 資料傳100次中成功95次的機率? (n=100, p=0.9)

- We can describe a binomial random variable using Bernoulli random variables!!

  假設以Bernoulli X_i 表示第 i 次試驗的結果， X_i(成功)=1 ，X_i(失敗)=0，且 $P(X_i=1)=p$ 。則binomial RV Y with parameters (n,p) 可表示成

$$Y = \sum_{i=1}^{n} X_i$$

Geometric Random Variable

Geometric Random Variable:

Flip a coin with a bias p until you get the first head. The random variable X defined as the number of flips required is called geometric random variable.

$$p_X(1) = P(H_1) = p$$
$$p_X(2) = P(T_1H_2) = (1-p)\,p$$
$$p_X(3) = P(T_1T_2H_3) = (1-p)^2p$$

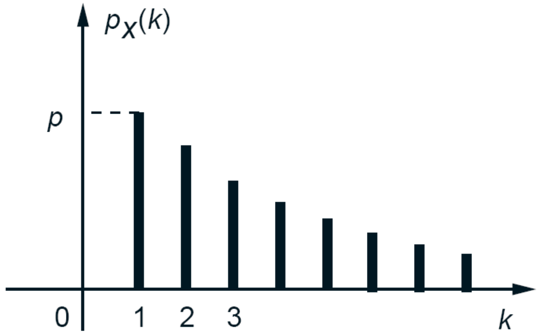
Geometric PMF:

In general, geometric RV 可用來描述嘗到第一次成功所需要實驗的次數。

the PMF is given by $p_X(k) = (1-p)^{k-1}p$, for $k=1, 2, 3, \ldots$

Question:

Is it a legitimate PMF?



Poisson Random Variable**

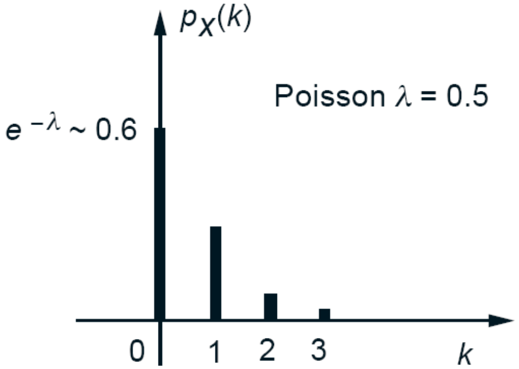
Poisson PMF:

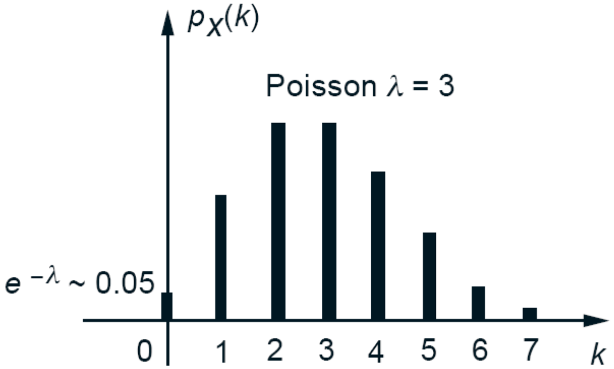
A discrete random variable X taking on nonnegative integers is called Poisson random variable with parameter λ if its PMF is given by

$$p_X(k) = e^{-\lambda}\frac{\lambda^k}{k!}, \qquad k = 0, 1, 2, \cdots$$

Questions:

1. Is it a legitimate PMF?
2. What are the applications of Poisson random variables?





Poisson PMF 怎麼得來的?

Poisson is good approximation for binomial with n large, p small and np a moderate fixed value λ.

- Introduced by French mathematician Simeon-Denis Poisson in 1837
- Consider a binomial random variable with (n, p) such that
  1) $n \to \infty$
  2) $p \to 0$
  3) $np = \lambda$

$$
\begin{aligned}
p_X(x) &= \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, \ldots, n \\
&= \frac{n!}{(n-k)!k!} p^k (1-p)^{n-k} \\
&= \frac{n(n-1)\cdots(n-k+1)}{n^k} \cdot \frac{\lambda^k}{k!} \cdot \left(1 - \frac{\lambda}{n}\right)^{n-k} \\
&\to
\end{aligned}
$$

$$
e^x \triangleq \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n
$$

Applications of Poisson Random Variables

Poisson random variable is typically used to model the number of occurrences of rare events, e.g.

- number of typos in a book
- number of phone calls arriving in an interval of T seconds
- number of customers being served by a bank teller in an hour
- number of packets in a network received at a router in T seconds

We will see more applications of Poisson in later chapters (Chapter 6).

Example

假設每位進入巨城的顧客有0.01的機會是陽明交大的學生。請問100位客人中恰有5位陽明交大學生的機率為何?
(Sol.)

1. From binomial

$$\binom{100}{5}(0.01)^5(0.99)^{95} \approx 0.00290$$

2. Approximated by Poisson ($\lambda = np = 100 \times 0.01 = 1$)

$$e^{-1}\frac{1}{5!} \approx 0.00306$$

Poisson 的常用型

Poisson R.V. X_t can often be used to describe ***the number*** of a certain event occurred within a time period $[0, t]$. This X_t is called a Poisson process.

The PMF of X_t with parameter α is given by

$$
\begin{aligned}
p_{X_t}(k) &\triangleq P(X_t = k) \\
&= e^{-\alpha t}\frac{(\alpha t)^k}{k!}, \quad k = 0, 1, 2, \ldots
\end{aligned}
$$

Example:

You get an email according to a Poisson process with parameter α=0.2. What is the probability of finding 0 and 1 new email within an hour?

$$
\begin{aligned}
P(X_1 = 0) &= e^{-0.2} = 0.819 \\
p(X_1 = 1) &= 0.2 \cdot e^{-0.2} = 0.164
\end{aligned}
$$

Functions of Random Variables (Revisited)

Function of a random variable is also a random variable.

Consider a random variable X. We define a function $Y = g(X)$ of the random variable X, such as X^2, $|X|$, $\cos(X)$, or e^{3x}

Then, as discussed earlier, these are also random variables defined on the original experiment by

$$Y(\omega) = g(X(\omega)).$$

問題是，若已知 X 的PMF，如何求得 Y 的PMF呢?

PMF of Functions of Random Variables

Given the PMF of X, $p_X(x)$, the PMF of $Y=g(X)$ is

$$p_Y(y) = \sum_{x:g(x)=y} p_X(x)$$

Example:

Recall the random variable X = maximum of two rolls of a four-sided die.

Define a new random variable $Y = g\,(X)$ by

$$g(X) = \begin{cases} 1, & X \geq 3, \\ 0, & \text{otherwise.} \end{cases}$$

Find the PMF for Y.

$$p_Y(y) = \sum_{x:g(x)=y} p_X(x)$$

So,

$$p_Y(1) = \sum_{x:x\geq 3} p_X(x)$$

$$= \frac{7}{16} + \frac{5}{16} = \frac{3}{4}$$

$$p_Y(0) = 1 - p_Y(1) = \frac{1}{4}.$$