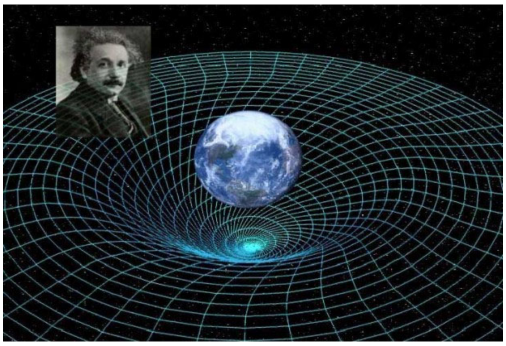
Chapter 12

Gravitation



楊 本 立 副教授

Outline

1. Newton's law of universal gravitation
2. Gravitational field
3. Gravitational potential energy
4. Kepler's laws of planetary motion

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2. Gravitational field
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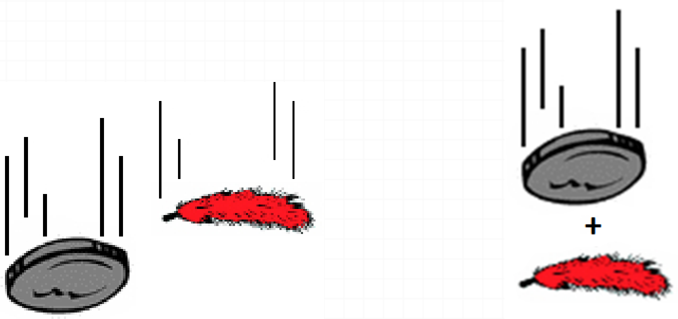
1. Newton's law of universal gravitation

Galileo

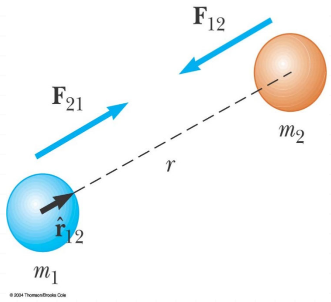


Faster or slower than the stone alone?

Newton's law of universal gravitation

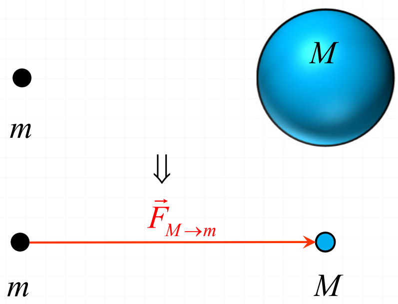
$$F_g = G\frac{m_1 m_2}{r^2}$$

$$\vec{F}_{12} = (-)G\frac{m_1 m_2}{r^2}\hat{r}_{12}, \quad \hat{r}_{12} = \hat{r}_2 - \hat{r}_1$$

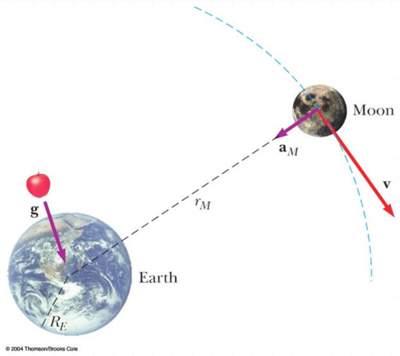


- Attractive, field force
- inverse-square law
- G: universal gravitational constant
- G=6.673×10^{-11}N·m^2/kg^2

- The gravitation force exerted by a finite-size, spherically symmetric mass distribution on a particle outside the distribution is the same as if the entire mass of the distribution were concentrated at the center.



Experimental proof of Newton's law:



(1) Newton's inverse square law

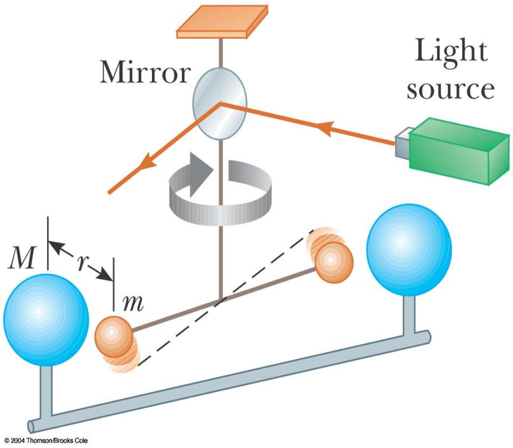
$$\begin{cases} a_M = \dfrac{F_M}{m_M} = G\dfrac{M_E}{r_M^2} \\ g = \dfrac{F_g}{m_A} = G\dfrac{M_E}{R_E^2} \end{cases} \qquad \frac{a_M}{g} = \frac{(1/r_M)^2}{(1/R_E)^2} \quad \Rightarrow a_M = \left(\frac{R_E}{r_M}\right)^2 g$$

(2) Centripetal acceleration

$$a_M = \frac{v^2}{r_M} = \frac{(2\pi r_M / T)^2}{r_M} = \frac{4\pi^2 r_M}{T^2}$$

Newton verified (1)=(2).

Cavendish apparatus for measuring *G*



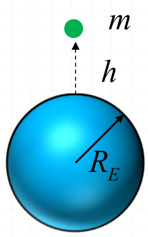
Free-Fall Acceleration

$$m\vec{a} = \vec{F}_g$$

$$\Rightarrow \quad mg = \frac{GM_E m}{(R_E + h)^2}$$

$$\Rightarrow \quad g = \frac{GM_E}{(R_E + h)^2}$$

$g \searrow$ as $h \nearrow$



1. Newton's law of universal gravitation
2. Gravitational field
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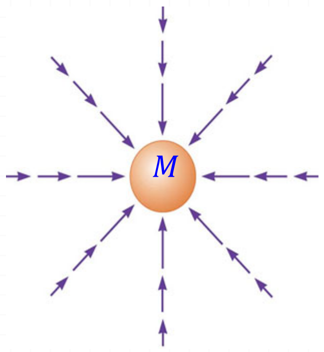
2. Gravitational field

The Gravitational Field



The inverse-square law can be understood by the "field" concept.

$$\vec{g} \equiv \frac{\vec{F}_g}{m} \qquad \vec{g}: \text{ gravitational field}$$

$$\vec{g} = -\frac{GM}{r^2}\hat{r}$$

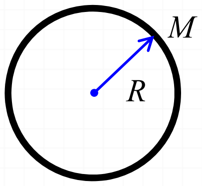
$|\vec{g}| \propto$ the densityof field lines (number of lines per unit area).

The number of field lines $N \propto M$.

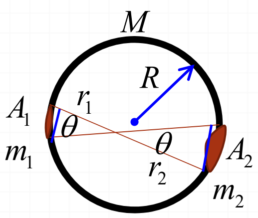
$$\Rightarrow \quad |\vec{g}| \propto \frac{N}{4\pi r^2} \propto \frac{M}{r^2}$$

- The Gravitational field g inside a Spherical Shell:

$$\begin{cases} \vec{g} = -\dfrac{GM}{r^2}\hat{r}, & r > R \\ \vec{g} = 0, & r < R \end{cases}$$



Proof:



$$\vec{g}_1 = -\frac{Gm_1}{r_1^2}\cos\theta\,\hat{r}_1 = -\frac{G\sigma A_1\cos\theta}{r_1^2}\hat{r}_1 = -G\sigma\Omega\hat{r}_1$$

$$\vec{g}_2 = -\frac{Gm_2}{r_2^2}\cos\theta\,\hat{r}_2 = -\frac{G\sigma A_2\cos\theta}{r_2^2}\hat{r}_2 = -G\sigma\Omega\hat{r}_2$$

$$\hat{r}_1 = -\hat{r}_2 \Rightarrow \quad \vec{g}_1 = -\vec{g}_2$$

$\Omega \equiv \dfrac{A_\perp}{r^2}$: solid angle

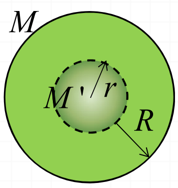
σ: surface density

So $\vec{g} = \sum_i \vec{g}_i = 0, \quad \text{for } r < R$

- The Gravitational field g inside a solid sphere:

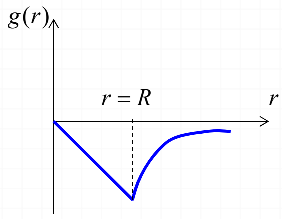
$$g(r<R)=\int_0^r \frac{\rho(r')}{(r-r')^2}dV'+\underbrace{\int_r^R \frac{\rho(r')}{(r-r')^2}dV'}_{=0} \Rightarrow \; g(r)=\frac{GM'}{r^2}$$

$$=\frac{GM}{r^2}\frac{r^3}{R^3}$$



$$M'=M\frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi R^3}=M\frac{r^3}{R^3}$$

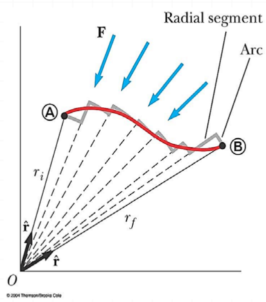
$$\Rightarrow \; \vec{g}(r)=-\frac{GM}{R^3}r\hat{r}, \text{ for } r<R.$$



😎 The above results can also be derived from the gravitational potential energy approach. Please refer to the textbook (Young, p. 401).

3. Gravitational potential energy

The gravitational force is "conservative".



$$\vec{F}(r) = F(r)\hat{r}$$

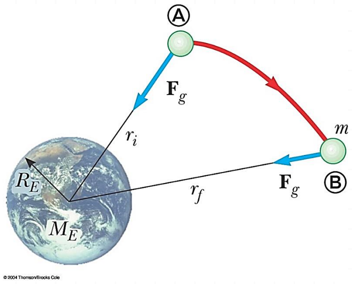
$$\begin{aligned} dW &= \vec{F}\cdot d\vec{\ell} \\ &= (\vec{F}\hat{r})\cdot(dr\hat{r} + rd\theta\hat{\theta} + r\sin\theta d\phi\hat{\phi}) \\ &= F(r)dr \end{aligned}$$

$\because W = \int_{r_i}^{r_f} F(r)dr = -\Delta U(r)$ depends only on the end points (indep. of θ and ϕ),

$\Rightarrow$ $F(r)$ is called "conservative force".

Any central force $\vec{F}(r) = F(r)\hat{r}$ is a conservative force.

Gravitational Potential Energy



$$\Delta U = U_f - U_i = -\int_{r_i}^{r_f} \vec{F}_g(r)\cdot d\vec{r}$$

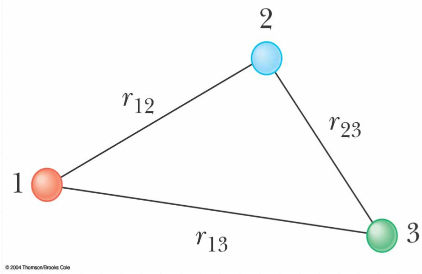
$$\vec{F}_g(r) = -\frac{GM_E m}{r^2}\hat{r}$$

$$\Rightarrow \quad \Delta U = \int_{r_i}^{r_f} \frac{GM_E m}{r^2} dr$$

$$= -GM_E m\left(\frac{1}{r_f} - \frac{1}{r_i}\right)$$

Set $r_i = \infty, \quad U_i = 0 \qquad \Rightarrow \quad U(r) = -\frac{GM_E m}{r}$

Ex.



$$U_{total} = U_{12} + U_{13} + U_{23}$$

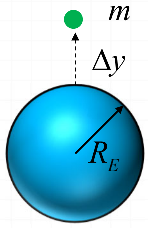
$$= -G\left(\frac{m_1 m_2}{r_{12}} + \frac{m_1 m_3}{r_{13}} + \frac{m_2 m_3}{r_{23}}\right)$$

For a small vertical distance Δy near the earth's surface,

$$\Delta U = -GM_E m\left(\frac{1}{r_f}-\frac{1}{r_i}\right)$$

$$= GM_E m\left(\frac{r_f - r_i}{r_i r_f}\right)$$

$$\approx \left(\frac{GM_E m}{R_E^2}\right)\Delta y$$

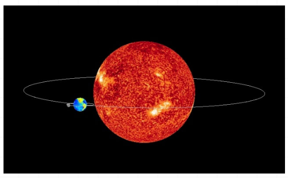


$$\Rightarrow \quad \Delta U = mg\Delta y$$

Total Energy in Planetary Motion

$$E = K + U = \frac{1}{2}mv^2 - \frac{GM_E m}{r}$$

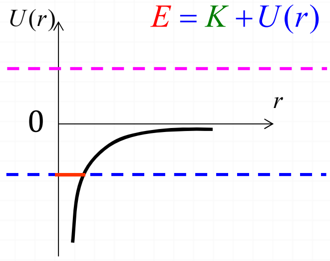
$$\frac{GM_E m}{r^2} = m\frac{v^2}{r} \quad \Rightarrow \quad \frac{1}{2}mv^2 = \frac{GM_E m}{2r}$$



$\Rightarrow$ Circular orbits: $E = -\dfrac{GM_E m}{2r}$

(Elliptical orbits: $E = -\dfrac{GM_E m}{2a}$, a : semi-major axis)

$$E = K + U(r)$$



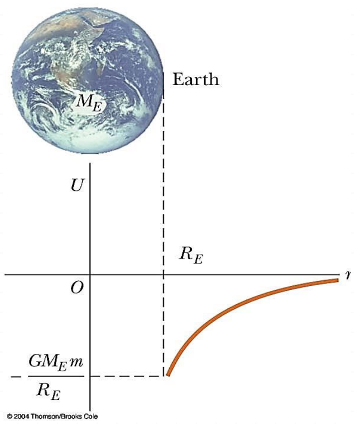
$E > 0$, unbound

$E < 0$, bound

$$U(r) = -\frac{GM_E m}{r}$$

$$K \geq 0 \Rightarrow \begin{cases} E < 0, & \text{bound} \\ E \geq 0, & \text{unbound} \end{cases}$$

Gravitational Potential Energy for the Earth



Binding energy:

$$U(\infty) - U(r) = 0 - (-\frac{GM_E m}{r}) = \frac{GM_E m}{r}$$

Escape Speed

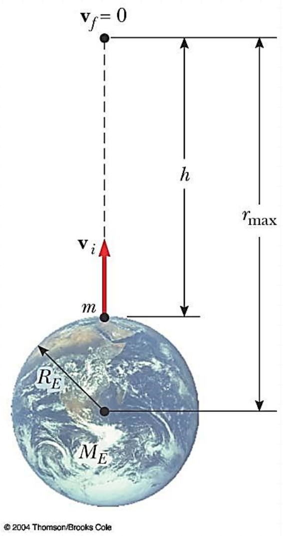
$$\frac{1}{2}mv_i^2 - \frac{GM_E m}{R_E} = \frac{1}{2}mv_f^2 - \frac{GM_E m}{r_{\max}}$$

$$\geq -\frac{GM_E m}{r_{\max}}$$

Escape, $r_{\max} \to \infty \quad \Rightarrow \quad v_i \geq \sqrt{\frac{2GM_E}{R_E}}$

Escape speed $\Rightarrow \quad v_{esc} = \sqrt{\frac{2GM_E}{R_E}} = \sqrt{2gR_E}$

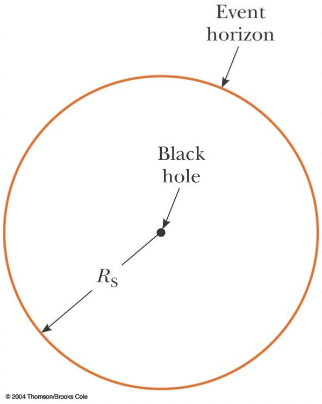
indep. of object's mass.



😎 Helium gas keeps escaping from the Earth.

$$\because v_{rms} = \sqrt{\frac{3k_B T}{m}} > v_{esc} = \sqrt{\frac{2GM_E}{R_E}} = 1.12 \times 10^4 \text{ m/s}$$

Black hole



Light can't escape from black holes if $v_{esc} < c$.

When $v_{esc} = \sqrt{\dfrac{2GM}{R_s}} = c,$

$$\Rightarrow \quad R_s = \frac{2GM}{c^2} \quad \text{Schwarzschild radius}$$

😎 Black hole doesn't have to be very massive, it is the density that determines the radius R_s to be > $R_{\text{block hole}}$.

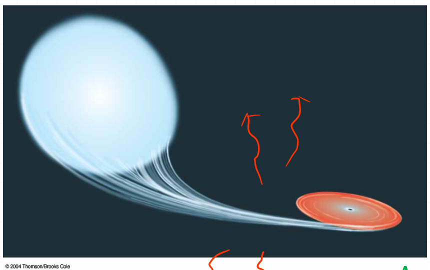
$$\text{Schwarzschild radius} \quad R_s = \frac{2GM}{c^2} = \frac{8\pi G}{3c^2} \rho R^3_{\text{Black hole}}$$

$$\Rightarrow \quad \rho \text{ has to be large so that } R_s > R_{\text{Black hole}}.$$

The formation of black holes

- "Supernova" – the catastrophic explosion of a heavy massive star.
- The core mass after explosion is m and the Sun mass is m_s
  - $m < 1.4\ m_s$, gradually cool down, white dwarf star.
  - $m > 1.4\ m_s$, collapse further, neutron star (radius ~10 km).
  - $m > 3\ m_s$, collapse into a black hole.

star



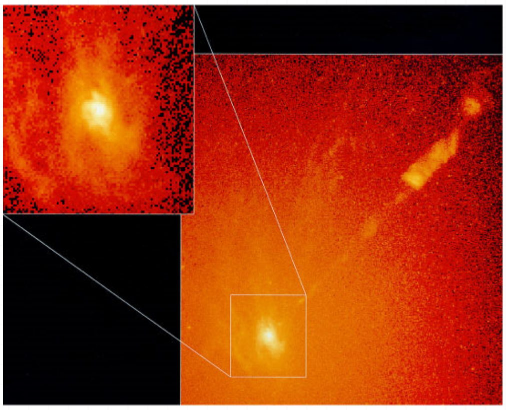
Black hole

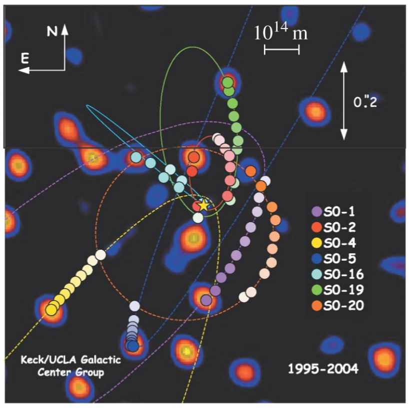
Accretion disk

X-ray

Hawking radiation
(thermal radiation)

Properties of black holes
- "X-rays" are characteristic of a black hole.
- Tidal force.
- Gravitational red shift, time dilation.





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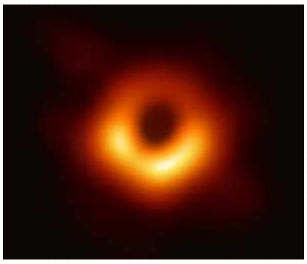
$$m_X = \frac{4\pi^2 a^3}{GT^2}$$

Supermassive black holes may exist at the center of galaxies.

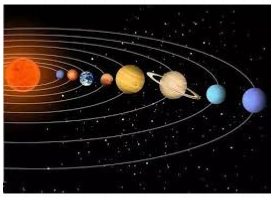
On April 10, 2019, the Event Horizon Telescope collaboration (eventhorizontelescope.org) released the first direct image of a black hole.

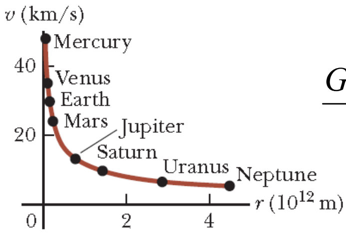


[https://www.aps.org/publications/apsnews/201906/black-hole.cfm]

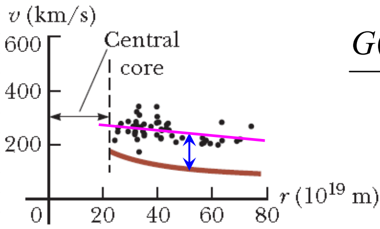
Dark matter

Luminous matters (stars) contribute major mass of the universe.
Non-luminous matters (planets, dust) are only a small fraction.





$$\frac{G(M_{\text{Sun}})m}{r^2} = m\frac{v^2}{r}$$

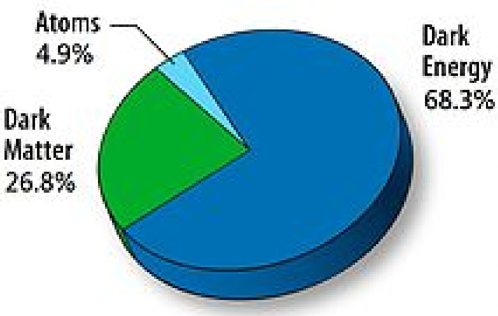


$$\frac{G(M_{\text{galaxy}})m}{r^2} \neq m\frac{v^2}{r}$$

$\Rightarrow$ Dark matter?

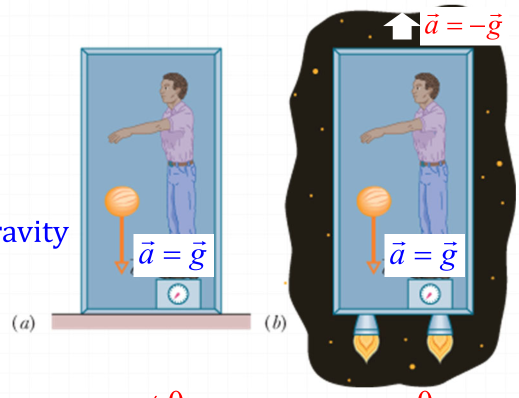
Dark energy

Dark energy is proposed to explain an increasing acceleration in the expansion of the Universe.



The General Theory of Relativity

Gravity

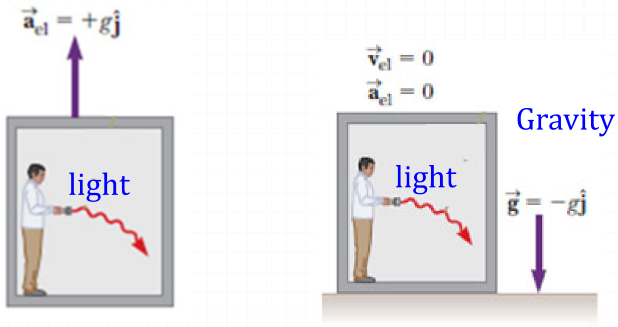


Acceleration

$g \neq 0$ $g = 0$

- Einstein proposed that *no* experiment, mechanical or otherwise, could distinguish between the two situations. Principle of equivalence: gravitation and acceleration are equivalent.

Acceleration

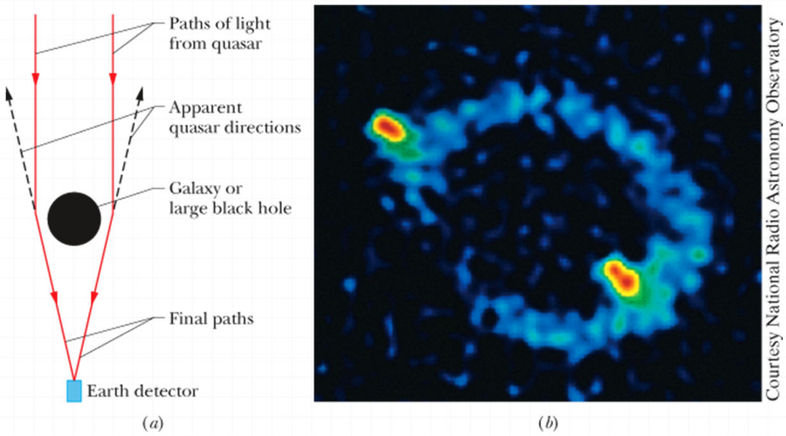


Gravity

$g = 0$ $g \neq 0$

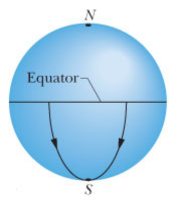
- Einstein assumes that all the laws of nature (including light) have the same form for observers in any reference frame, whether accelerated or not.

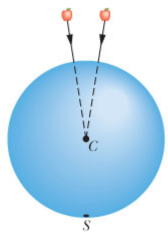
Gravitational lens

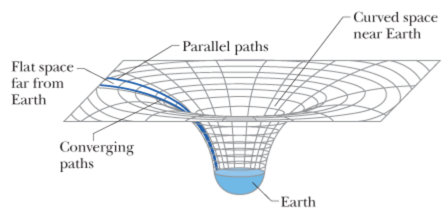


- Two postulates in Einstein's **general theory of relativity**
  - All the laws of nature have the same form for observers in any frame of reference, whether accelerated or not.
  - In the vicinity of any point, a gravitational field is equivalent to an accelerated frame of reference in gravity-free space (**the principle of equivalence**).

Gravitation $\Rightarrow$ Curvature of space and time







1. Newton's law of universal gravitation
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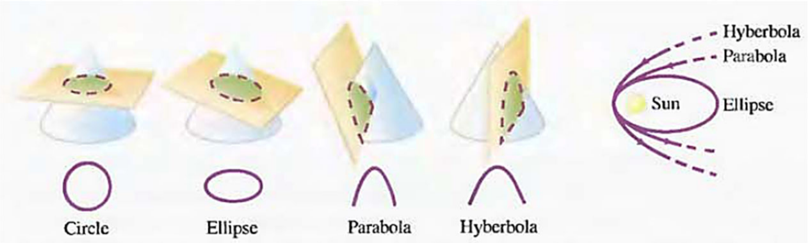
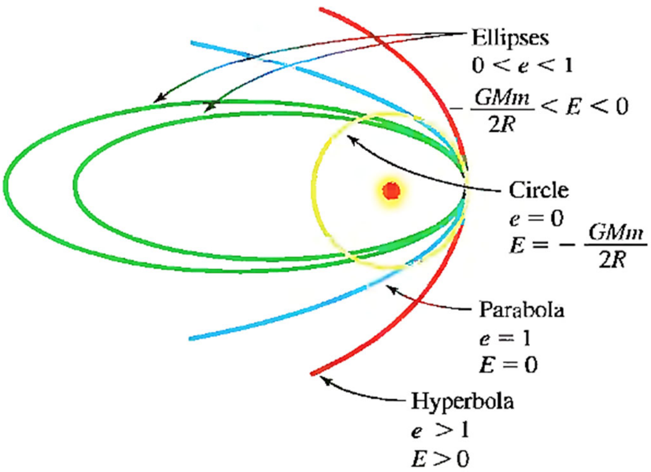
4. Kepler's laws of planetary motion

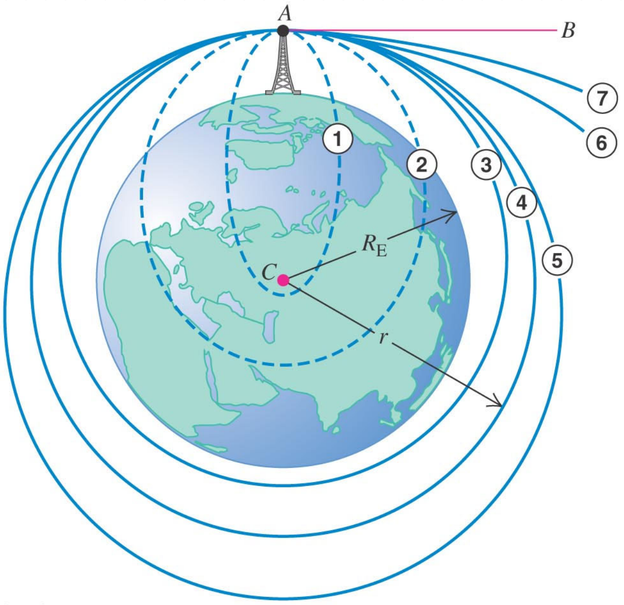
4. Kepler's laws of planetary motion

Kepler's 1st Law

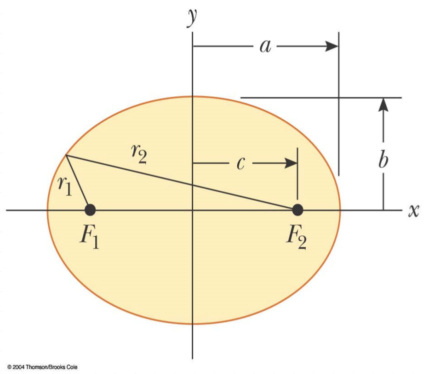
- All planets move in elliptical orbits with the Sun at one focus.
  - This is a direct result of the inverse-square of the Gravitational force.







A projectile is launched from A toward B. Trajectories (1) through (7) show the effect of increasing initial speed.



F_1, F_2: focus

a: semi-major axis

b: semi-minor axis

$$a^2 = b^2 + c^2$$

eccentricity: $e = c/a$

$$\begin{cases} e = 0, & \text{a circle} \\ 0 < e < 1, & \text{ellipse} \end{cases}$$

$\Rightarrow$ bound orbits,

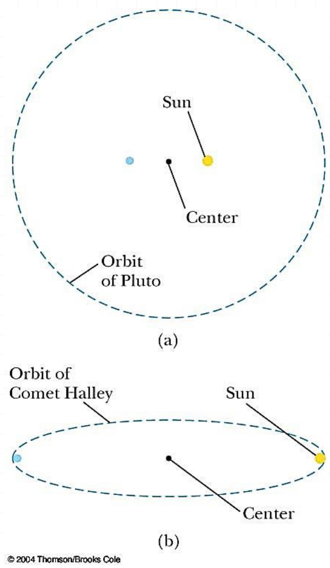
e.g., planets, asteroids, comets.

$$\begin{cases} e = 1, & \text{parabola} \\ e > 1, & \text{hyperbola} \end{cases}$$

$\Rightarrow$ unbound orbits,

e.g., meteorite.

- Sun is located at focus.
- Pluto has the highest eccentricity e_{Pluto} = **0.25** among the 8 planets in the solar system.
- Halley's comet has an orbit with high eccentricity $e_{\text{Halley's comet}}$ = **0.97**.



(Solar) planets 1930–2006								
1	2	3	4	5	6	7	8	9
Mercury ☿	Venus ♀	Earth ⊕	Mars ♂	Jupiter ♃	Saturn ♄	Uranus ♅	Neptune ♆	Pluto ♇

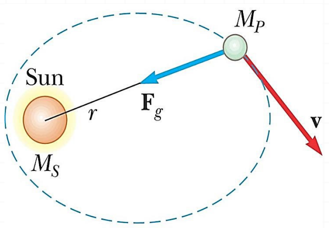
My Very Eager Mother Just Serves Us Nine Pizza!

Kepler's 2nd Law

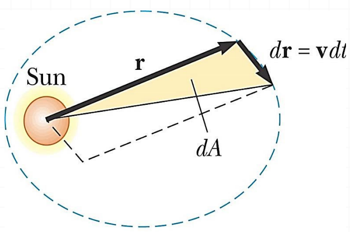
- The radius vector drawn from the Sun to a planet sweeps out equal areas in equal time intervals.

$$\Rightarrow \quad \frac{dA}{dt} = \frac{|\vec{L}|}{2M_P} = const.$$

  - This is a consequence of angular momentum conservation.
  - This law applies to any situation that involves a central force, whether inverse-square or not.
  - The orbit lies in the same plane.



(a)



(b)

$$\vec{\tau} = \vec{r} \times \vec{F} = \vec{r} \times F_g(r)\hat{r} = 0$$

$$\Rightarrow \quad \frac{d\vec{L}}{dt} = 0$$

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times M_P \vec{v} = const.$$

$$dA = \frac{1}{2}|\vec{r} \times d\vec{r}| = \frac{1}{2}|\vec{r} \times \vec{v}|dt$$

$$= \frac{|\vec{L}|}{2M_P} dt$$

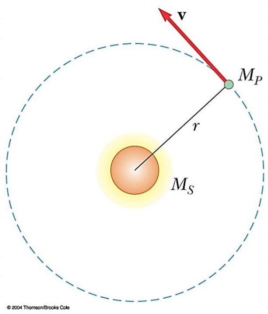
$$\Rightarrow \quad \frac{dA}{dt} = \frac{|\vec{L}|}{2M_P} = const.$$

Kepler's 3rd Law

- The square of the orbital period of any planet is proportional to the cube of the semi-major axis of the elliptical orbit.

$$T^2 = \left(\frac{4\pi^2}{GM_S}\right)a^3, \quad \text{for elliptical orbits.}$$

For circular orbit, $\vec{F}_g = m\vec{a}_c$



$$\frac{GM_S M_P}{r^2} = \frac{M_P v^2}{r}$$

$$\Rightarrow \quad \frac{GM_S}{r^2} = \frac{(2\pi r / T)^2}{r}$$

$$\Rightarrow \quad T^2 = \left(\frac{4\pi^2}{GM_S}\right)r^3 \equiv K_S r^3$$

In general,

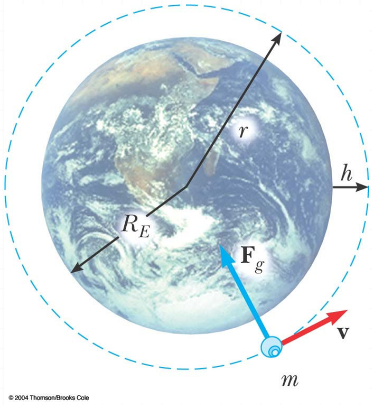
$$T^2 = \left(\frac{4\pi^2}{GM_S}\right)a^3, \quad \text{for elliptical orbits.}$$

indep. of planet's mass and eccentricity.

Ex. Geosynchronous Satellite

(1) $v = ?$ at h.

(2) Geosynchronous satellite $r = ?$ $v = ?$



Ans:

(1) $\vec{F} = m\vec{a}_c$

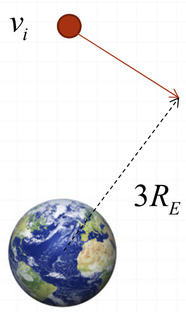
$$\frac{GM_E m}{r^2} = m\frac{v^2}{r} \quad \Rightarrow \quad v = \sqrt{\frac{GM_E}{R_E + h}}$$

(2)
$$T^2 = \left(\frac{4\pi^2}{GM_E}\right) r^3 = (24 \text{ hrs})^2$$

$$\Rightarrow \quad r = \ldots$$

$$v = \frac{2\pi r}{T} = \frac{2\pi r}{24 \text{ hrs}} \quad \Rightarrow \quad v = \ldots$$

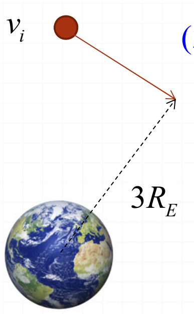
Ex: Astronomers detect a distant meteoroid moving along a straight line that, if extended, would pass at a distance $3R_E$ from the center of the Earth, where R_E is the radius of the Earth. What minimum speed must the meteoroid have if the Earth's gravitation is not to deflect the meteoroid to make it strike the Earth?



Not to strike $\Rightarrow$ $R \geq R_E$

(1) $L_i = L_f$ $\Rightarrow$ $mv_i(3R_E) = mv_f R$

$\Rightarrow$ $v_f = (3R_E / R)v_i$ $\Rightarrow$ $v_f \leq 3v_i$



(2) $E_i = E_f$

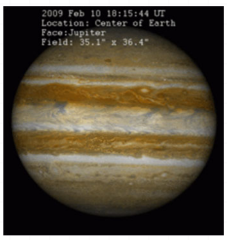
$$\Rightarrow \frac{1}{2}mv_i^2 = \frac{1}{2}mv_f^2 - \frac{GM_E m}{R}$$

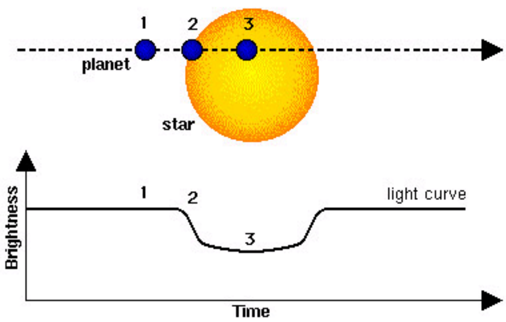
$$\Rightarrow v_f^2 = v_i^2 + \frac{2GM_E}{R}$$

So $v_i \geq \frac{1}{2}\sqrt{\frac{GM_E}{R_E}}$

https://exoplanets.nasa.gov/interactable/11/

Planetary transit





Negative mass?

$$\vec{F}_g = (-m)\vec{g} = (-m)\vec{a}$$

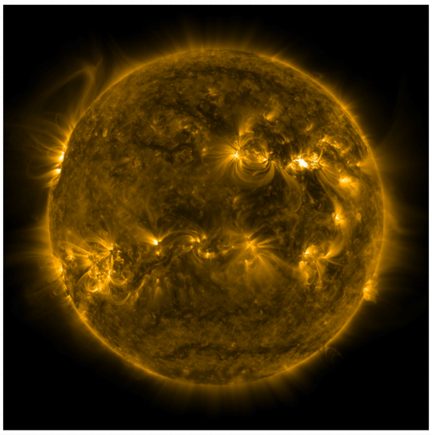
$\Rightarrow \vec{F}_g$ is opposite to $\vec{a}$ direction!

[“Negative mass can be positively amusing” by Richard H. Price,
American Journal of Physics 61, 216 (1993);
View online: https://doi.org/10.1119/1.17293]

😎 What would happen to the earth if the sun suddenly disappeared?

https://fb.watch/8RNhBCxTgE/

Significant Solar Flare Erupts From Sun



https://blogs.nasa.gov/solarcycle25/wp-content/uploads/sites/304/2022/03/FlareX13_20220330_171A_2048p30_Sq.gif