Topic 8: Derived Distributions

Lecture Outline

- Functions of One Continuous Random Variable

  Given the PDF of X, find the PDF of $Y=g(X)$
- Function of Two Continuous Random Variable $W=g(X,Y)$

  Given the PDF of X and Y, find the PDF of $W=g(X,Y)$
- Joint PDF of Multiple Functions of Two Random Variables

$$\begin{aligned} V &= g(X,Y) \\ W &= h(X,Y) \end{aligned}$$

References:

1. Textbook: Section 4.1 for functions of one RV
2. H. Stark and J. Woods, ***Probability and Random Process with Applications to Signal Processing***, 3rd ed., 2002, (pp. 152 ~ 161)
3. S. Ghahramani, **Fundamentals of Probability with Stochastic Processes**, 4th ed., 2019 (Sec. 8.4)

Review

Discrete	Continuous
$p_X(x)$	$f_X(x)$
$p_{X,Y}(x,y)$	$f_{X,Y}(x,y)$
$p_{X\|Y}(x\|y) = \dfrac{p_{X,Y}(x,y)}{p_Y(y)}$	$f_{X\|Y}(x\|y) = \dfrac{f_{X,Y}(x,y)}{f_Y(y)}$
$p_X(x) = \sum_y p_{X,Y}(x,y)$	$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy$

$$F_X(x) = \mathbf{P}(X \leq x)$$

$$\mathbf{E}[X],\ \mathrm{var}(X)$$

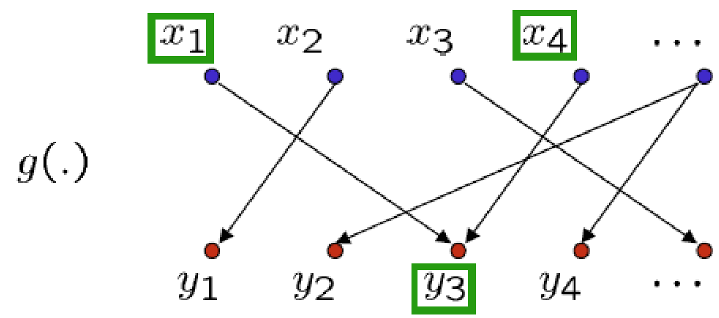
Review: Derived PMF

Recall derived PMF

Given $p_X(x)$ and a new random variable $Y = g(X)$, derive new PMF $p_Y(y)$ by

$$p_Y(y) = \sum_{x:g(x)=y} p_X(x)$$

Pick up a y value, find all x's that give $g(x)=y$. We sum over all x's that give $g(x)=y$.



Note:

The above method cannot be immediately extended to deriving PDFs since X is continuous. PDFs are *not* probabilities, and we cannot add probabilities of $\{X=x\}$ for continuous X

PDF of Y=g(X) for Continuous X

Two steps to derive

1. First, calculate the CDF $F_Y(y)$ of Y=g(X) using the formula

$$F_Y(y) = P(Y \leq y) = \int_{x:g(x)\leq y} f_X(x)dx$$

2. Differentiate to obtain the PDF of Y:

$$f_Y(y) = \frac{dF_Y(y)}{dy}$$

Example 4.3

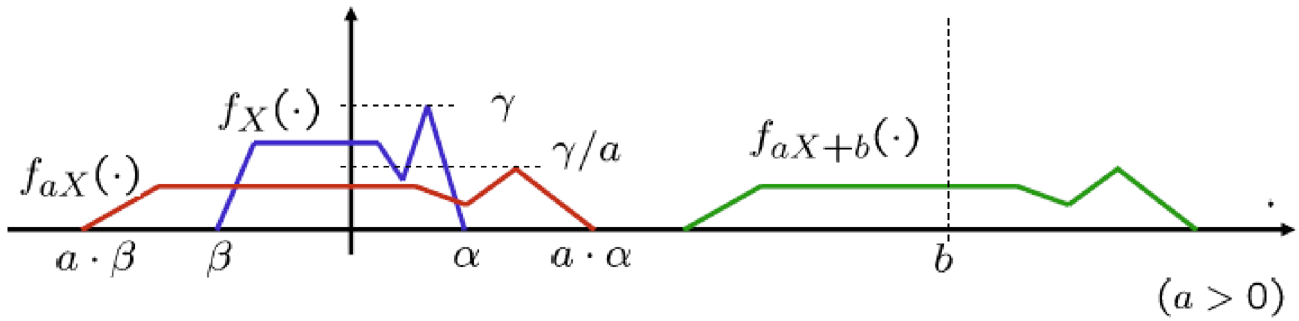
Let $Y = g(X) = X^2$, where X is a uniform random variable over [-2,2]. Find the PDF of Y.

(Sol) $f_Y(y) = \frac{1}{4} y^{-1/2} \text{ for } 0 \le y \le 4$

The Linear Case

Let X be a continuous random variable with PDF $f_X(x)$, and let $Y = aX + b$, for some scalars $a \neq 0$ and b. Then,

$$\boxed{f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)}$$



<u>Two steps</u>: **Scaling**: $z=ax$; **Shift**: $y=z+b$

<u>Example:</u>

1. <u>***A linear function of a normal random variable is normal***</u>
2. A linear function of an exponential random variable is **NOT** necessarily exponential

Example 4.4: A liner function of an exponential random variable

- Let X be an exponential random variable with parameter λ.
  We have the PDF

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

where λ is a positive parameter. Let $Y = aX + b$. Then,

$$f_Y(y) = \begin{cases} \dfrac{\lambda}{|a|} e^{-\lambda(y-b)/a} & \text{if } (y-b)/a \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Note that if $b = 0$ and $a > 0$, then Y is an exponential random variable with parameter λ/a. In general, however, Y need not be exponential. For example, if $a < 0$ and $b = 0$, then the range of Y is the negative real axis.

Monotonic Function of a Continuous Random Variable

Suppose that g is strictly monotonic and that for some function h and all x in the range of X we have

$$y = g(x) \quad \text{if and only if} \quad x = h(y)$$

Assume that h is differentiable. Then, the PDF of $Y=g(X)$ in the region is given by

$$f_Y(y) = f_X\left(h(y)\right) \left| \frac{dh}{dy}(y) \right|$$

This can be justified by first finding the CDF of Y, then taking the derivative.

- Monotonically increasing: $g(x) < g(x')$ for all x, x' satisfying $x < x'$
- Monotonically decreasing
- 上式絕對值的物理意義?

Illustration (page 210)

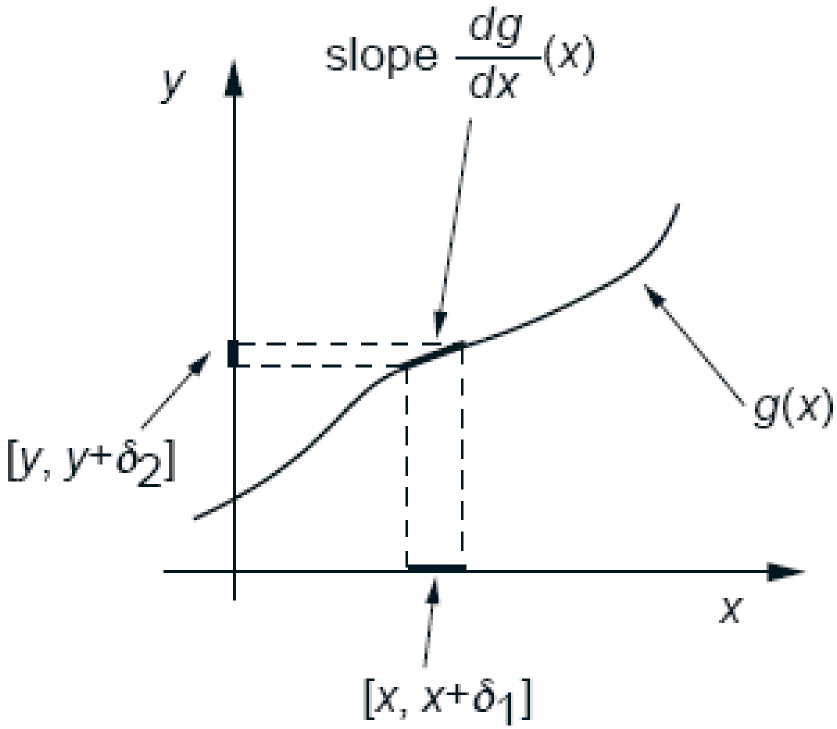


Fig. 4.4: Consider an interval, $[x, x+\delta_1]$, where δ_1 is a small number. Under the mapping *g*, the image of this interval is another interval $[y, y+\delta_2]$. Since (dg/dx) is the slope of *g*, we have $\frac{\delta_2}{\delta_1} \approx \frac{dg}{dx}(x)$

Or in terms of the inverse function,

$$\frac{\delta_1}{\delta_2} \approx \frac{dh}{dy}(y), \quad h = g^{-1}$$

- We now note the event $\{x \le X \le x+\delta_1\}$ is the same as the event $\{y \le Y \le y+\delta_2\}$. Thus,

  $f_Y(y)\delta_2 \approx Pr(y \le Y \le y + \delta_2)$

  $= Pr(x \le X \le x + \delta_1) \approx f_X(x)\delta_1$

Examples

- Let $Y = g(X) = X^2$, where X is a uniform random variable on [0,2]. Find the PDF of Y.
- Sol:

$$f_Y(y) = \begin{cases} \frac{1}{4}y^{-\frac{1}{2}}, & 0 \leq y \leq 4, \\ 0, & \text{otherwise.} \end{cases}$$

- Let $Y = g(X) = X^2$, where X is a uniform random variable on [-2,2]. Find the PDF of Y. <Non-monotonic>
- Sol:

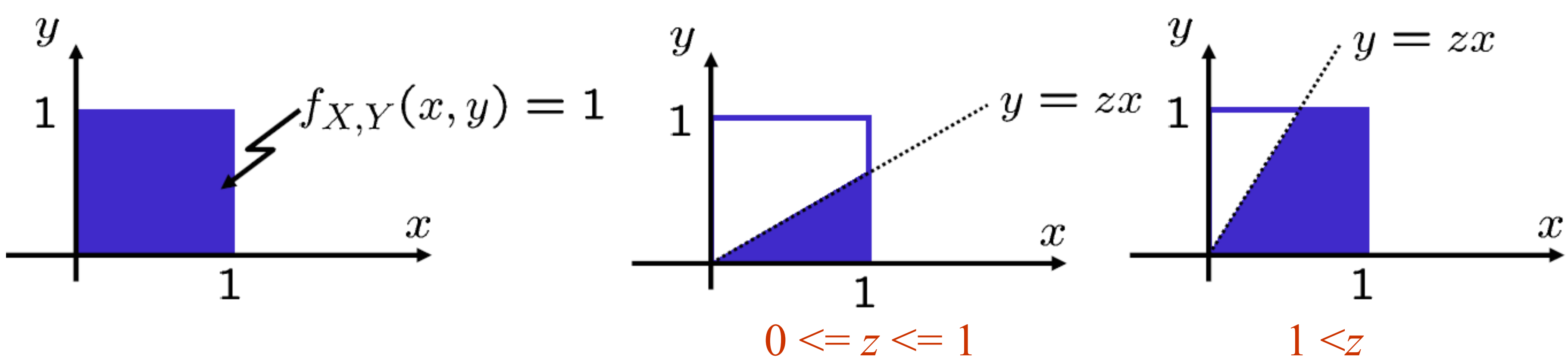
Functions of Two Random Variables

<u>Two-step procedure</u>

1. Calculate the CDF $F_Z(z)$ of Z=$g(X,Y)$
2. Differentiate to obtain the PDF of Z

<u>Example</u>: (4.8) (p.211)

Let X and Y be two independent uniform (continuous) random variables over [0,1]. Find the PDF of Z=Y/X.

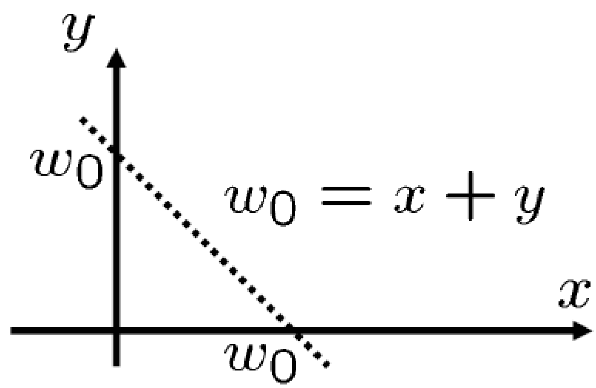


$$F_Z(z) = \quad z/2 \qquad 0 \le z \le 1$$

$$F_Z(z) = \quad 1 - 1/2z \qquad z \ge 1$$

The Distribution of X+Y

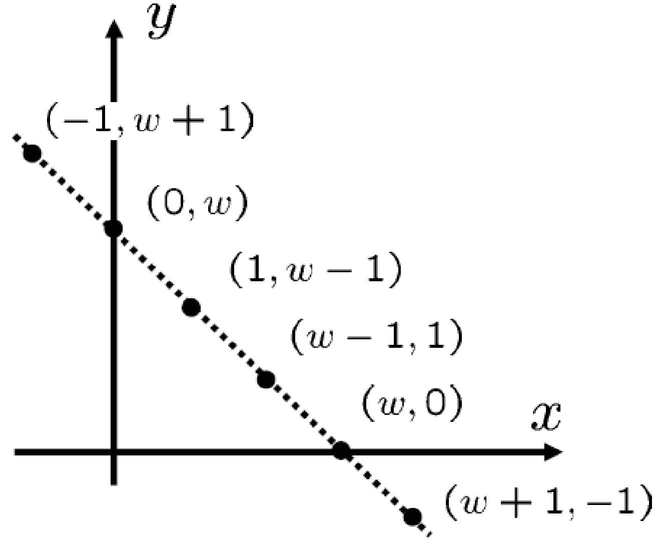
- Let X, Y be two RVs, and let $W = X + Y$. Find the PMF (*if* X, Y *discrete*) or PDF of W (*if* X, Y *continuous*).
- Points where the value $W = w_0$ is some constant lie on the following line



- *Idea*
  - Discrete case: add probabilities of all points on this line.
  - Continuous case: integrate the joint density on this line.

X+Y: Independent Discrete Integers

- Let X, Y be discrete, independent RVs.
- Then $W = X + Y$ is also integer-valued.



- Thus,

$$p_W(w) = Pr(X+Y=w) = \sum_x Pr(X=x)Pr(Y=w-x)$$

$$= \sum_x p_X(x)p_Y(w-x)$$

This operation is called discrete convolution (摺積、迴旋積)

Obtaining $p_W(w)$ by Discrete Convolution

- Problem10 (p. 246): Let X, Y be two indep. RVs with PMFs

$$p_X(x) = \begin{cases} 1/3, & \text{if } x = 1, 2, 3, \\ 0, & \text{otherwise.} \end{cases} \qquad p_Y(y) = \begin{cases} 1/2, & \text{if } y = 0, \\ 1/3, & \text{if } y = 1, \\ 1/6, & \text{if } y = 2, \\ 0, & \text{otherwise.} \end{cases}$$

Find the PMF of $W = X + Y$.

$$p_W(w) = \sum_x p_X(x) p_Y(w - x)$$

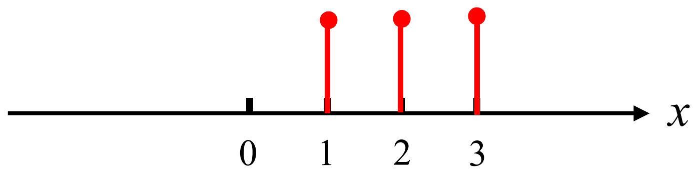
Discrete Convolution

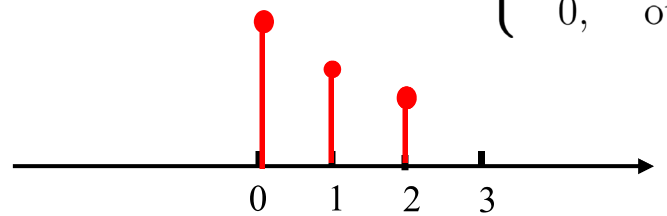
$$p_W(w) = \sum_x p_X(x) p_Y(w-x)$$

$$p_X(x) = \begin{cases} 1/3, & \text{if } x = 1,2,3, \\ 0, & \text{otherwise.} \end{cases}$$

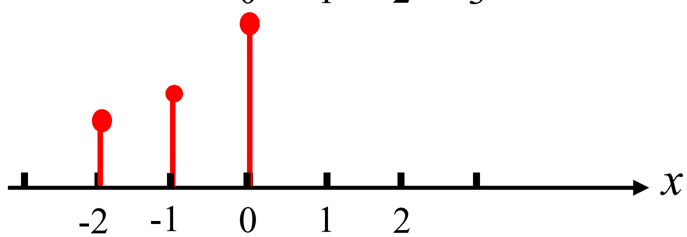
$$p_Y(y) = \begin{cases} 1/2, & \text{if } y = 0, \\ 1/3, & \text{if } y = 1, \\ 1/6, & \text{if } y = 2, \\ 0, & \text{otherwise.} \end{cases}$$

$p_X(x)$



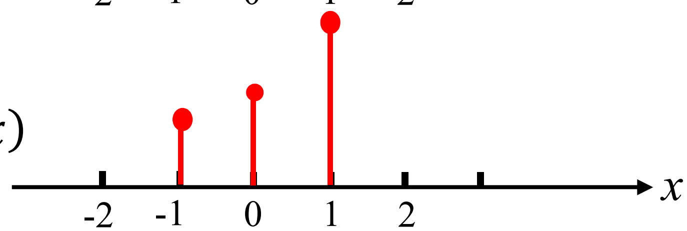


$p_Y(-x)$



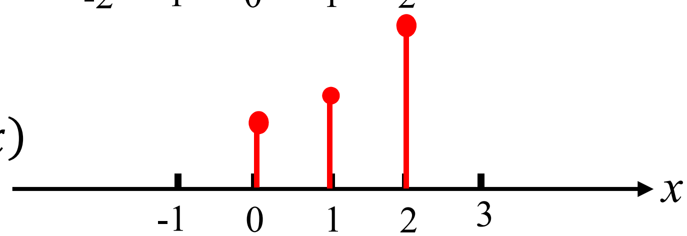
$w=0$: $\sum_x p_X(x)p_Y(-x) = 0$

$p_Y(1-x)$



$w=1$: $\sum_x p_X(x)p_Y(1-x) = 1/6$

$p_Y(2-x)$



$w=2$: $\sum_x p_X(x)p_Y(2-x) = \frac{1}{3}\times\frac{1}{3}+\frac{1}{3}\times\frac{1}{2} = \frac{5}{18}$

X+Y: Convolution Integral

- Let X, Y be independent, continuous RVs.
- Then the PDF of $W = X + Y$ is given by

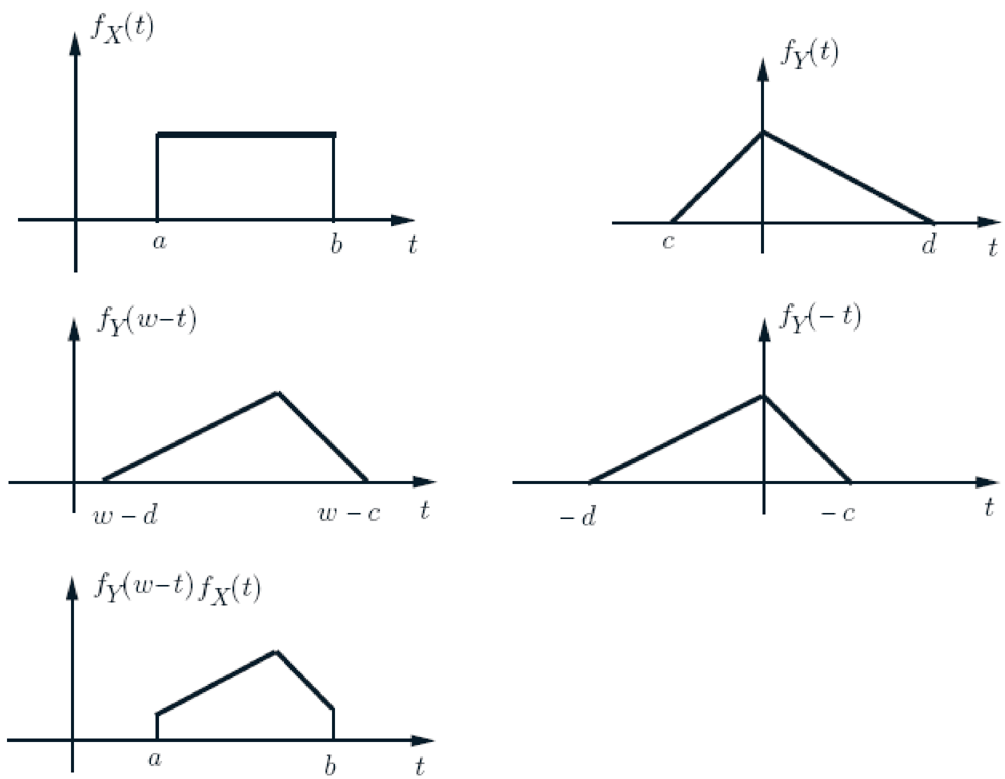
$$f_W(w) = \int_{-\infty}^{\infty} f_X(x) f_Y(w-x)\,dx$$

> This integral is called (continuous) **convolution（摺疊積分）**

Graphical Understanding of Continuous Convolutions

Convolution can be perceived with the help of graphical illustration.

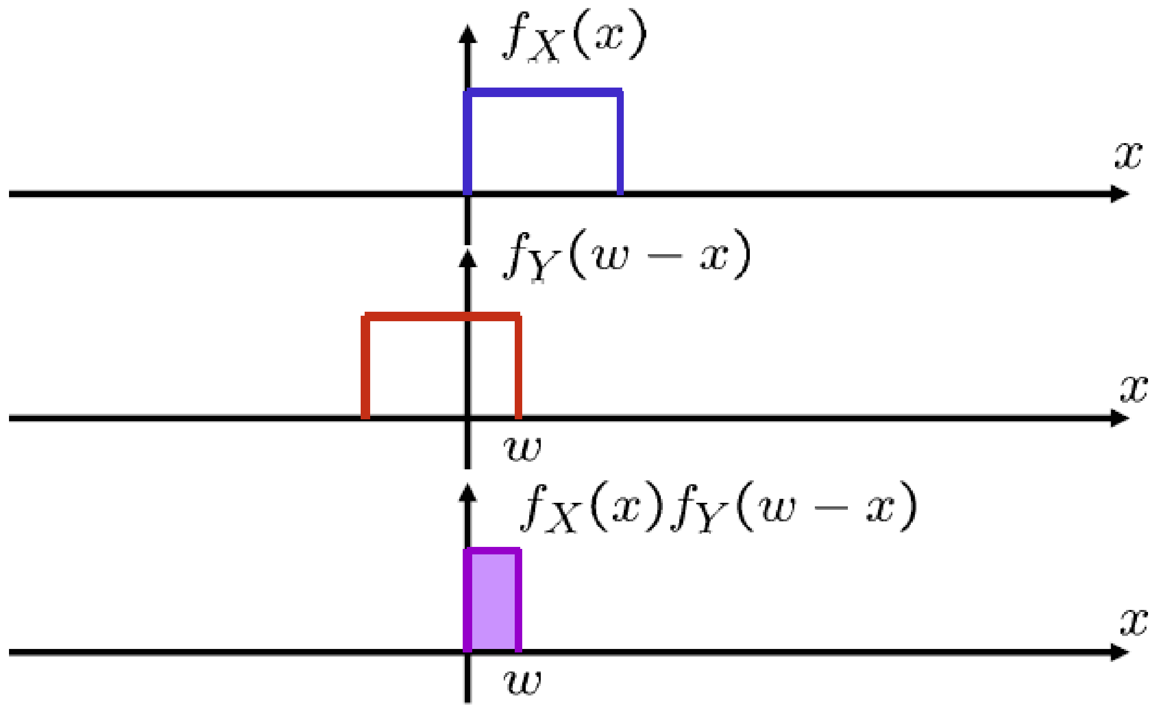
- Key step: how to draw $f_Y(w\text{-}t)$ from $f_Y(t)$ for a fixed w and variable t



$X+Y$ Example: Independent Continuous

- Let X, Y be independent, uniform on [0,1].
  Find the PDF of $W = X+Y$.
- Convolution idea applies:

$$f_W(w) = \int_{-\infty}^{\infty} f_X(x) f_Y(w-x)\,dx$$



Sum of Independent Normals

- Let X, Y be *independent*, normal RVs.

$$X \sim N(0, \sigma_x^2) \quad Y \sim N\left(0, \sigma_y^2\right)$$

- What is the PDF of $W = X + Y$?

$$f_W(w) = \int_{-\infty}^{\infty} f_X(x) f_Y(w-x) dx$$

$$= \frac{1}{2\pi\sigma_x \sigma_y} \int_{-\infty}^{\infty} \exp(-x^2/2\sigma_x^2) \exp[-(w-x)^2/2\sigma_y^2)\, dx$$

$$= C \cdot \exp(-w^2/2(\sigma_x^2 + \sigma_y^2)), \quad C = 1/\left(2\pi(\sigma_x^2 + \sigma_y^2)\right)^{1/2}$$

- 上式積分的結果很 neat，但推導有些繁瑣。推導關鍵有兩點：1) 配方法；2) $\int_{-\infty}^{\infty} f_X(x) dx = 1$ for any PDF $f_X(x)$.
- Hence, W is normal with mean = 0, and $\sigma_w^2 = \sigma_x^2 + \sigma_y^2$
- *Note*: More generally, $aX+bY$ is also normal for any two constants a and b
- 此結果若將 Independent 的條件移除則未必成立

Review

Review: Monotonic Function of a Continuous RV

Suppose that *g* is strictly monotonic and that for some function *h* and all x in the range of X we have

$$y = g(x) \quad \text{if and only if} \quad x = h(y)$$

Assume that h is differentiable. Then, the PDF of Y=*g*(X) in the region is given by

$$f_Y(y) = f_X\left(h(y)\right)\left|\frac{dh}{dy}(y)\right|$$

這個式子不需要背，其實本質上就是先找 Y的CDF，再取微分即可！

Joint Density of Two Functions of Two RVs

Problem:

Given joint PDF $f_{X,Y}(x,y)$ of two continuous random variables X and Y.

We now have two newly defined random variables V and W

$$\begin{aligned} V &= g(X,Y) \\ W &= h(X,Y) \end{aligned}$$

where $g(.)$ and $h(.)$ are two real-valued functions. How do we compute the joint PDF $f_{VW}(v,w)$ from $f_{X,Y}(x,y)$?

Important Remarks:

1. In general, $g(.)$ and $h(.)$ can be any 2 real-valued functions
2. In this topic, we will consider a special case of $g(.)$ and $h(.)$. We assume that the system of two equations in two unknowns has a unique solution $x=p(v,w)$ and $y=q(v,w)$ for x and y in terms of v and w, similar to the monotonic case mentioned on the previous page.

Basic Concept

- By properly selecting the values of *dv, dw, dx,* and *dy* , we have

  [probability before mapping] = [probability after mapping].

$$P(v \leq V \leq v + dv,\ w \leq W \leq w + dw) = P(x \leq X \leq +dx,\ y \leq Y \leq y + dy)$$

When *dv, dw, dx,* and *dy* are very small,

$$\begin{aligned} P(v \leq V \leq v + dv,\ w \leq W \leq w + dw) &\approx f_{VW}(v, w)(dvdw) \\ P(x \leq X \leq x + dx,\ y \leq Y \leq y + dy) &\approx f_{XY}(x, y)(dxdy) \end{aligned}$$

- The above amounts to

$$\boxed{f_{VW}(v, w) = f_{XY}(x, y) \cdot \underbrace{\left(\frac{dxdy}{dvdw}\right)}_{\text{ratio of two areas}}}$$

So, the question now is how do we find the ratio?

$\Rightarrow$ Use change of variable theorem in calculus

Formula to the Joint Density

$$f_{VW}(v,w) = f_{XY}(x,y) \cdot \underbrace{\left(\frac{dxdy}{dvdw}\right)}_{\text{ratio of two areas}}$$

Using change of variable theorem in calculus,

$$\boxed{f_{VW}(v,w) = f_{XY}(x,y) \cdot |J|^{-1}}$$ (evaluated at $x=p(v,w)$ and $y=q(v,w)$)

where J is the Jacobian representing the (inverse) ratio of the areas and is the determinant

$$J = \begin{vmatrix} \frac{\partial g(x,y)}{\partial x} & \frac{\partial g(x,y)}{\partial y} \\ \frac{\partial h(x,y)}{\partial x} & \frac{\partial h(x,y)}{\partial y} \end{vmatrix} = \frac{\partial g(x,y)}{\partial x} \cdot \frac{\partial h(x,y)}{\partial y} - \frac{\partial g(x,y)}{\partial y} \cdot \frac{\partial h(x,y)}{\partial x}$$

- H. Stark and J. Woods, ***Probability and Random Process with Applications to Signal Processing***, 3rd ed., 2002, (pp. 152 ~ 161)

Example

- Let $V=X+Y$, and $W=X-Y$. Find the joint PDF of V and W in terms of the joint PDF of X and Y. (S&W, p.153)

- **Solutions:**

  We use the formula $f_{VW}(v,w) = f_{XY}(x,y) \cdot |J|^{-1}$

  First, we solve for x and y in terms of v and w. It is clear that

$$
\begin{aligned}
x &= \frac{1}{2}(v+w) \\
y &= \frac{1}{2}(v-w)
\end{aligned}
$$

  The Jacobian is

$$
J = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2
$$

  Thus, $f_{VW}(v,w) = \frac{1}{2} f_{XY}\left(\frac{v+w}{2}, \frac{v-w}{2}\right)$ ∎