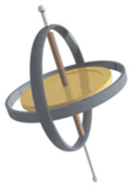
國立交通大學 電子物理系 NCTU Electrophysics



# Chapter 10
## Dynamics of Rotational Motion

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# Outline

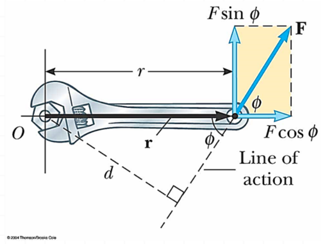
1. Torque
2. Energy conservation in rotational motion
3. Rolling motion
4. Angular momentum
5. Conservation of angular momentum
6. The motion of gyroscopes and tops
7. Angular momentum as a fundamental quantity

## 1. Torque

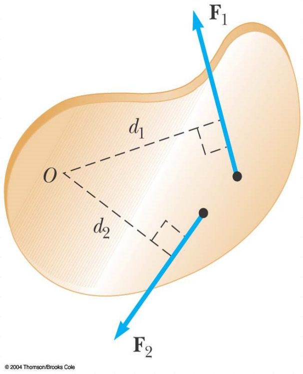


## Torque

- **Torque, $\tau$,** is the tendency of a force to rotate an object about some axis.
  - Torque is a vector.
  - $\boldsymbol{\tau = r\,(F \sin \phi) = F\,d}$ $\Rightarrow \vec{\tau} = \vec{r} \times \vec{F}$
    - **F** is the force.
    - $\phi$ is the angle the force makes with the horizontal.
    - $d$ is the **moment arm** (or **lever arm**).
- "Torque" and "work" are different quantities, though both have the same unit.



## Net Torque

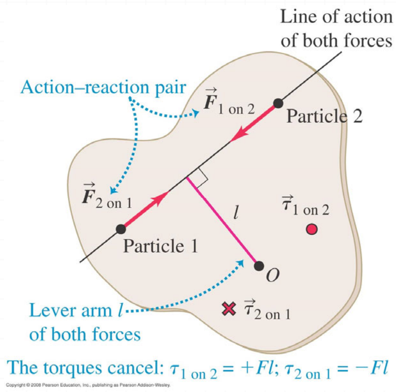



$$\tau = \tau_1 + \tau_2 = F_1 d_1 - F_2 d_2$$



😎 "Internal forces" do NOT contribute to the torque!

$$\begin{aligned}\vec{\tau}_{\text{int}} &= \vec{r}_1 \times \vec{F}_{21} + \vec{r}_2 \times \vec{F}_{12} = \vec{r}_1 \times \vec{F}_{21} + \vec{r}_2 \times (-\vec{F}_{21}) \\ &= (\vec{r}_1 - \vec{r}_2) \times \vec{F}_{21} \\ &= \vec{r}_{12} \times \vec{F}_{21} = 0 \\ &\because \vec{r}_{12} \parallel \vec{F}_{21}\end{aligned}$$



## Torque and Angular Acceleration

$$\vec{\tau} = \vec{r} \times \vec{F} = \vec{r} \times (\vec{F}_t + \vec{F}_r) = \vec{r} \times \vec{F}_t$$

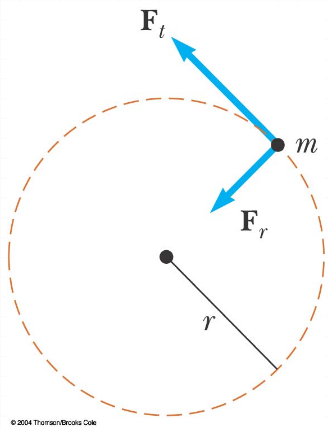
$$\tau = F_t r = (m a_t) r$$

$$a_t = r\alpha \quad \Rightarrow \tau = (mr^2)\alpha$$

$$= I\alpha$$

$$\Rightarrow \quad \vec{\tau} = I\vec{\alpha}$$

cp., $\quad \vec{F} = m\vec{a}$






For continuous objects,

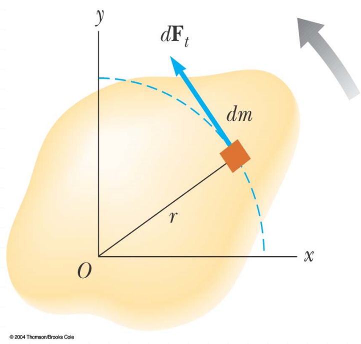
$$dF_t = (dm)\, a_t$$

$$d\tau = r dF_t = r(dm)\, a_t$$

$$= \alpha r^2 dm$$

$$\tau = \int \alpha r^2 dm = \alpha \int r^2 dm$$

$$\Rightarrow \quad \tau = I\alpha$$




😎 Notice that,

$\vec{\tau} = I\vec{\alpha}$ is valid for rigid bodies only,

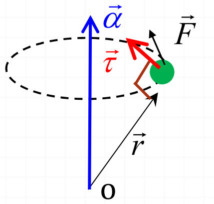
because $I$ may change for non-rigid objects such as water in a bucket.

😎 In fact, $\vec{\tau} = I\vec{\alpha} - m(\vec{r}\cdot\vec{\alpha})\vec{r} - m(\vec{\omega}\cdot\vec{r})\vec{v}$

$$\left[\begin{aligned}
\vec{\tau} &= \vec{r}\times\vec{F} = \vec{r}\times\left(m\frac{d\vec{v}}{dt}\right) = m\vec{r}\times\frac{d(\vec{\omega}\times\vec{r})}{dt} \\
&= m\vec{r}\times[\vec{\alpha}\times\vec{r} + \vec{\omega}\times\vec{v}] \\
&= m(\vec{r}\cdot\vec{r})\vec{\alpha} - m(\vec{r}\cdot\vec{\alpha})\vec{r} + m\vec{r}\times[\vec{\omega}\times(\vec{\omega}\times\vec{r})] \\
&= mr^2\vec{\alpha} - m(\vec{r}\cdot\vec{\alpha})\vec{r} + m\vec{r}\times[(\vec{\omega}\cdot\vec{r})\vec{\omega} - (\vec{\omega}\cdot\vec{\omega})\vec{r}] \\
&= mr^2\vec{\alpha} - m(\vec{r}\cdot\vec{\alpha})\vec{r} - m(\vec{\omega}\cdot\vec{r})\vec{v}
\end{aligned}\right]$$

Only if $\vec{r}\perp\vec{\alpha}$ and $\vec{r}\perp\vec{\omega}$, then $\vec{\tau} = I\vec{\alpha}$ is valid.

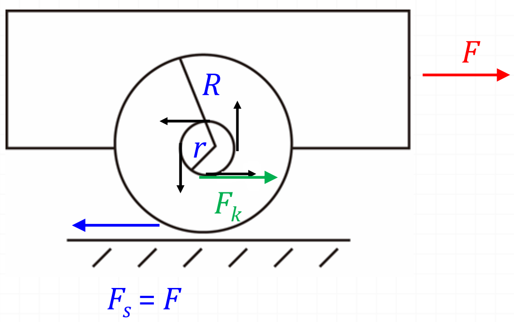
e.g.,




$\Rightarrow \quad \vec{\tau} \neq I\vec{\alpha}$ !



**Q**: Why wheels can save our effort on moving heavy objects?




**Ans:**

If no acceleration, $\quad F - F_s = ma = 0 \quad \Rightarrow \quad F_s = F$

$$F_sR - F_kr = I\alpha = 0 \quad \Rightarrow FR - F_kr = 0 \quad \Rightarrow F = \left(\frac{r}{R}\right)F_k < F_k$$

😎 In general, the total torque on a system does NOT equal to the torque on its center of mass.

$$\vec{\tau}_{total} = \sum_{i=1}^{n} \vec{r}_i \times \vec{F}_i = \sum_{i=1}^{n} \vec{r}_i \times (m_i \vec{a}_i)$$

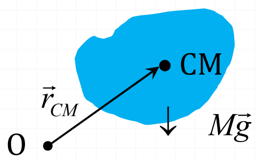
$$\vec{\tau}_{\text{for } CM} = \vec{r}_{CM} \times \vec{F}_{CM} = \frac{\sum_{i=1}^{n}(m_i \vec{r}_i)}{\sum_{i=1}^{n} m_i} \times \left[\sum_{i=1}^{n}(m_i \vec{a}_i)\right]$$

$$\Rightarrow \quad \vec{\tau}_{total} \neq \vec{\tau}_{\text{for } CM}$$



Eg.,

$$\vec{\tau}_{total} = \int \vec{r} \times d(m\vec{g}) = (\underbrace{\int \vec{r} dm}_{=\vec{r}_{CM} M}) \times \vec{g}$$



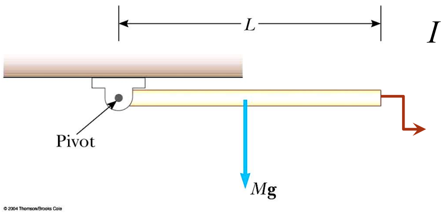

Only if $g$ is indep. of $\vec{r}$, then $\vec{\tau}_{total} = \vec{\tau}_{\text{for } CM} = \vec{r}_{CM} \times M\vec{g}$.

For $g = \text{const.}$,

$$\vec{\tau}_{total} = \vec{\tau}_{\text{for } CM} = \underbrace{\vec{r}_{CM}}_{=0} \times M\vec{g} = 0$$



**Ex.**




$$I = \frac{1}{3}ML^2$$

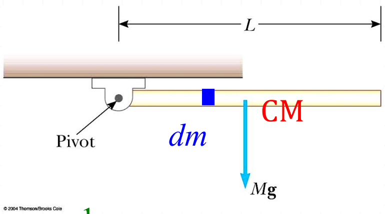
$$\alpha = ?$$

$$a_t = ?$$

Hint:

$$\begin{cases} \vec{F} = m\vec{a} \\ \tau = I\alpha \\ a = r\alpha \end{cases}$$

**Ans:**




$I = \frac{1}{3}ML^2$

① $\vec{\tau} = I\vec{\alpha} = \frac{1}{3}ML^2\vec{\alpha}$ $\qquad \vec{\tau} = \vec{r} \times \vec{F} = ?$

$d\vec{\tau} = \vec{r} \times d\vec{F} = \vec{r} \times dm\vec{g} \quad \Rightarrow \quad \vec{\tau} = \int (\vec{r}dm \times \vec{g})$

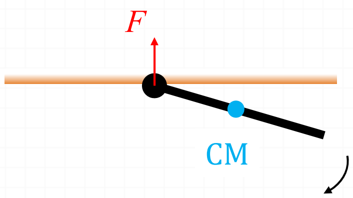
If $g$ is indep. of $\vec{r}$ $\quad = \left(\int \vec{r}dm\right) \times \vec{g} = M\vec{r}_{CM} \times \vec{g}$

$$\Rightarrow \quad \vec{\tau} = \frac{L}{2}Mg\sin\theta$$

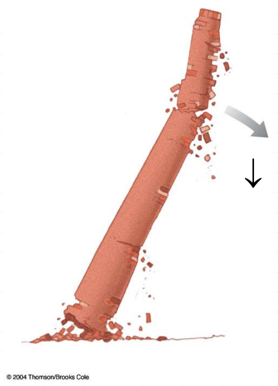
$$\theta = 90^\circ \quad \Rightarrow \quad \alpha = \frac{\tau}{I} = \frac{3g}{2L}$$



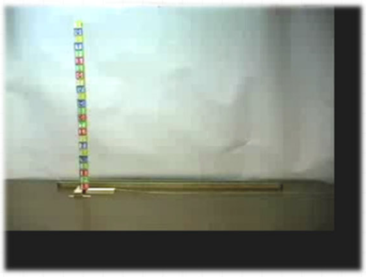
② $a_t = r\alpha = \frac{3}{2}g > g$




At the CM frame, it is the torque exerted by the hinge so that $a_t > g$.




$\downarrow \; ma_t > mg$





**Q:** In an emergent situation that you have to jump from 2F to the ground, will you jump directly as a free fall or grab a pole to jump?

**Ans:** free fall.

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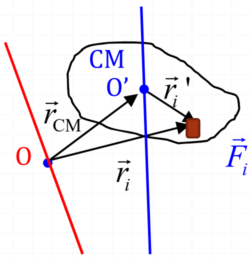
# Torque at different reference axis




$$\vec{\tau}_{o'\text{ at CM}} = \sum_i \vec{r}_i\,' \times \vec{F}_i$$

$$\vec{\tau}_o = \sum_i \vec{r}_i \times \vec{F}_i = \sum_i (\vec{r}_i\,' + \vec{r}_{\text{CM}}) \times \vec{F}_i$$

$$= \sum_i \vec{r}_i\,' \times \vec{F}_i + \sum_i \vec{r}_{\text{CM}} \times \vec{F}_i$$

$$\Rightarrow \quad \vec{\tau}_o = \vec{\tau}_{o'\text{ at CM}} + \vec{r}_{\text{CM}} \times \underbrace{\sum_i \vec{F}_i}_{=\vec{F}_{CM}}$$

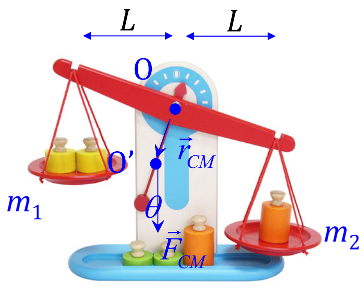
Note, $\vec{\tau}_o = \vec{\tau}_{o'}$, if

(1) $\vec{r}_{\text{CM}} = 0\ (o = o')$, or

(2) $\sum_i \vec{F}_i = 0$, or

(3) $\vec{r}_{\text{CM}} \parallel \sum_i \vec{F}_i$

**Q:** Why a balance in equilibrium is defined at a leveled position?




**Ans:**

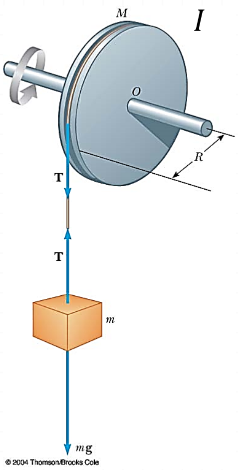
$$\vec{\tau}_o = \underbrace{\vec{\tau}_{o'\text{ at CM}}}_{=\underbrace{\vec{r}'_{CM}}_{=0}\times M\vec{g}=0} + \vec{r}_{\text{CM}} \times F_{CM}$$

$$\Rightarrow \quad m_2 gL - m_1 gL = r_{CM} Mg \sin\theta$$

When $\theta = 0, \Rightarrow m_1 = m_2$



**Ex.**




$\alpha = ?$

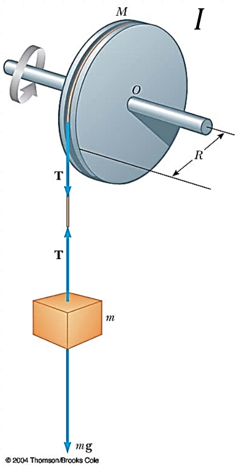
$a = ?$

$T = ?$

Hint:

$$\begin{cases} \vec{F} = m\vec{a} \\ \tau = I\alpha \\ a = r\alpha \end{cases}$$

**Ans:**




(1) $\Sigma\tau = I\alpha = TR \Rightarrow \alpha = \dfrac{TR}{I}$

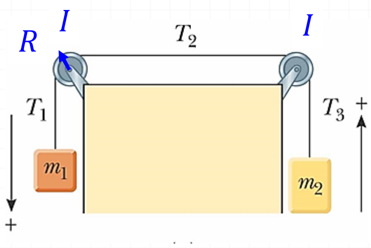
(2) $\Sigma F = ma = mg - T \quad \Rightarrow a = \dfrac{mg - T}{m}$

(3) $a = R\alpha$

$$\Rightarrow T = \frac{mg}{1 + (mR^2/I)}$$



**Ex.**



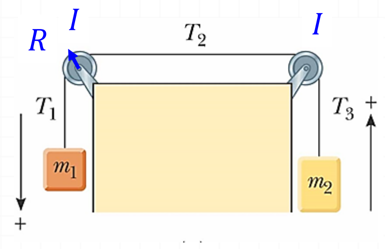

$a = ?$

$T_1,\ T_2,\ T_3 = ?$

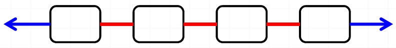
Hint:

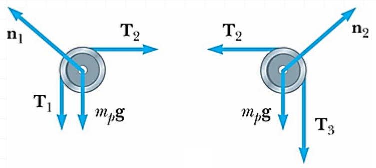
$$\begin{cases} \vec{F} = m\vec{a} \\ \tau = I\alpha \\ a = r\alpha \end{cases}$$

**Ans:**




Note: $T_1 \neq T_2 \neq T_3$. Similar to






$$(1)\quad \begin{cases} m_1 g - T_1 = m_1 a \\ T_3 - m_2 g = m_2 a \end{cases}$$

$$(2)\quad \begin{cases} (T_1 - T_2)R = I\alpha \\ (T_2 - T_3)R = I\alpha \end{cases}$$

$$(3)\quad a = R\alpha$$

$$\Rightarrow \begin{cases} a = \cdots \\ T_1 = \cdots \\ T_2 = \cdots \\ T_3 = \cdots \end{cases}$$



# Torque summary

1. $\vec{\tau} = \vec{r} \times \vec{F}, \quad \vec{F} = \vec{F}_{\text{external}}, \cancel{\vec{F}_{\text{internal}}}$
2. $\vec{\tau} = I\vec{\alpha}$ is only valid for rigid bodies

   and for rotation around a symmetry axis of the bodies!
3. $\vec{\tau}_{total} \neq \vec{\tau}_{\text{for } CM}$.
4. $\vec{\tau}_o = \vec{\tau}_{o' \text{ at CM}} + \vec{r}_{\text{CM}} \times \vec{F}_{CM}$

1. Torque
2. Energy conservation in rotational motion
3. Rolling motion
4. Angular momentum
5. Conservation of angular momentum
6. The motion of gyroscopes and tops
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## 2. Energy conservation in rotational motion



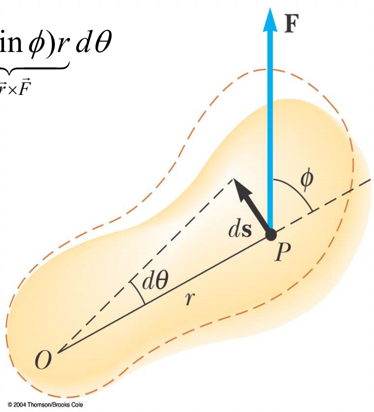
# Work, Energy, and Power in Rotational Motion

Work:

$$dW = \vec{F}\cdot d\vec{s} = F(rd\theta)\cos(90-\phi) = \underbrace{(F\sin\phi)r}_{=\vec{r}\times\vec{F}}\,d\theta$$

$$= \tau d\theta$$

$$\Rightarrow W = \int \tau d\theta$$




Instantaneous power:

$$P = \frac{dW}{dt} = \tau\frac{d\theta}{dt} = \tau\omega$$

## Work-Kinetic Energy Theorem in Rotational Motion

$$dW = \tau d\theta = I\alpha d\theta = I(\frac{d\omega}{dt})d\theta$$

$$\Rightarrow \quad dW = Id\omega(\frac{d\theta}{dt}) = I\omega d\omega$$

$$\Rightarrow W = \int_{\omega_i}^{\omega_f} I\omega d\omega = \frac{1}{2}I\omega_f^2 - \frac{1}{2}I\omega_i^2$$

$$K_R = \frac{1}{2}I\omega^2$$

Work-kinetic energy theorem: $W = \Delta K_R$

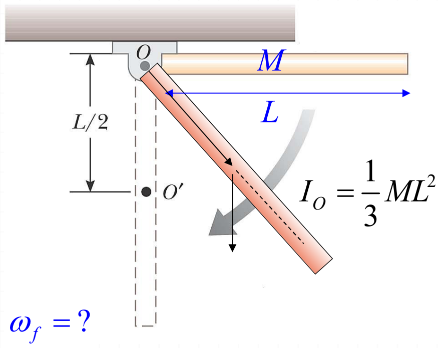


**Strategies for solving rotational problems**:

1. Define the system or use the center of mass.
2. Set the pivot.

   (For rolling, set at the CM or at the ground contact? )
3. For constant $\alpha$, use rotational kinematic equations.
4. By $F = ma$, $\tau = I\alpha$, $a_t = R\alpha$,
5. By conservation laws.

**Ex.**




$\omega_f = ?$

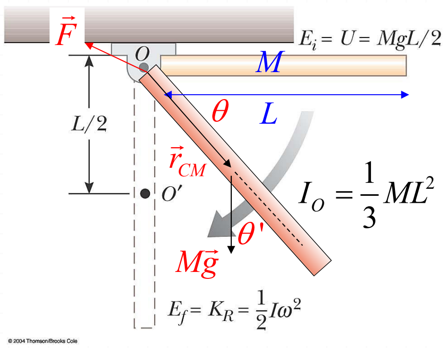
$v_{CM} = ?$



**Ans:**

Method #1

System = rod, pivot at $O$




$$W_{ext} = \Delta K_R$$

$$\Rightarrow \int_0^{\pi/2} \tau d\theta = \frac{1}{2} I_o \omega_f^2$$

$$\vec{F}_{ext} = M\vec{g} + \vec{F}$$

P.s.,

$\omega_f^2 = \omega_i^2 + 2\alpha\Delta\theta$ is not applicable. $\because \alpha$ is not constant!

$$W_{ext}=\int_0^{\pi/2}\tau d\theta=\int_0^{\pi/2}(\tau_g+\underbrace{\tau_F}_{=0\because r=0})d\theta=\int_0^{\pi/2}\left[\int\left|\vec{r}\times dm\vec{g}\right|\right]d\theta$$

$$=\int_0^{\pi/2}\left|\left[\int\vec{r}dm\right]\times\vec{g}\right|d\theta=\int_0^{\pi/2}\left|(M\vec{r}_{CM})\times\vec{g}\right|d\theta=\int_0^{\pi/2}M\frac{L}{2}g\sin\theta' d\theta$$

$$=\int_0^{\pi/2}M\frac{L}{2}g\sin(\frac{\pi}{2}-\theta)d\theta=\frac{L}{2}MgL$$

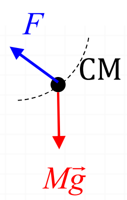
$$W_{ext}=\frac{1}{2}MgL=\frac{1}{2}I_o\omega_f^2-0$$

$$\Rightarrow\omega_f=\sqrt{\frac{3g}{L}},\quad v_{CM}=\frac{L}{2}\omega_f=\frac{\sqrt{3gL}}{2},$$



😎 If system = CM, the force from the hinge on the CM has to be included,

$$W_{CM}=W_{g,\,CM}+W_{F,\,CM}.$$




Otherwise,

by using $W_{CM}=\Delta K_{CM}$, $\underbrace{\frac{1}{2}MgL}_{W_{g,CM}}=\frac{1}{2}Mv_{CM}{}^2$

you will get the incorrect answer $v_{CM}=\sqrt{gL}$ , if without including $W_{F,CM}$!

$\because F$ is not $\perp$ the CM's path.

However, $F$ is undetermined. Since every point in the rod is known to have different acceleration, the force on each point has to be different so that the force exerting by the hinge is not along the rod or perpendicular to the CM’s path.

Method #2 System = rod + hinge + Earth

$$W = \Delta K + \Delta U = 0 \quad (\frac{1}{2}I\omega_f^2 - 0) + (0 - \frac{1}{2}MgL) = 0 \quad \Rightarrow \quad \frac{1}{2}(\frac{1}{3}ML^2)\omega_f^2 = \frac{1}{2}MgL$$

$$\omega_f = \sqrt{\frac{3g}{L}} \quad \Rightarrow \quad v_{CM} = \frac{L}{2}\omega_f = \frac{\sqrt{3gL}}{2}, \quad v = L\omega_f = \sqrt{3gL}$$

P.s., Only this equation $\Delta K + \Delta U = 0$ is valid here.

Such an equation $\Delta K_{CM} + \Delta U_{CM} = 0$ does not exist!

( $\because$ A point particle system doesn't have any potential energy.)

Therefore, $(\frac{1}{2}mv_{cm}^2 - 0) + (0 - \frac{1}{2}MgL) = 0$ is wrong!

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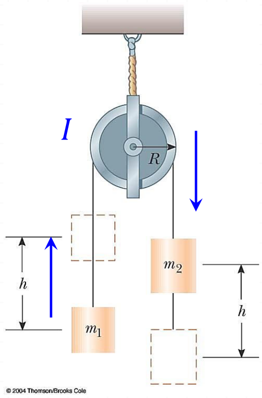
**Ex.**




$v_f = ?$

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Method #1 System = $m_1$ + $m_2$ + pulley + Earth

(1) $W = \Delta K + \Delta U = 0$

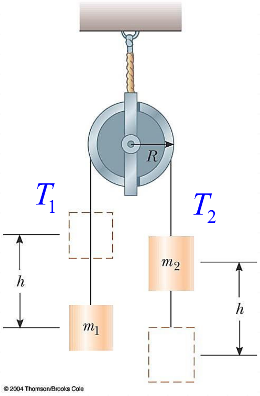
$$\Rightarrow \left(\frac{1}{2}m_1 v_{1f}^2 + \frac{1}{2}m_2 v_{2f}^2 + \frac{1}{2}I\omega_f^2\right) + \left(m_1 gh - m_2 gh\right) = 0$$

(2) $v_f = R\omega_f$

$$\Rightarrow v_f = \left[\frac{2(m_2 - m_1)gh}{m_1 + m_2 + I/R^2}\right]^{1/2}$$



Method #2 System = $m_1$ + $m_2$ + pulley




(1) $W = \Delta K$

$T_1, T_2$ are internal forces $\Rightarrow$ no contribution to $W$.

$$\Rightarrow -m_1 gh + m_2 gh = \frac{1}{2}m_1 v_{1f}^2 + \frac{1}{2}m_2 v_{2f}^2 + \frac{1}{2}I\omega_f^2$$

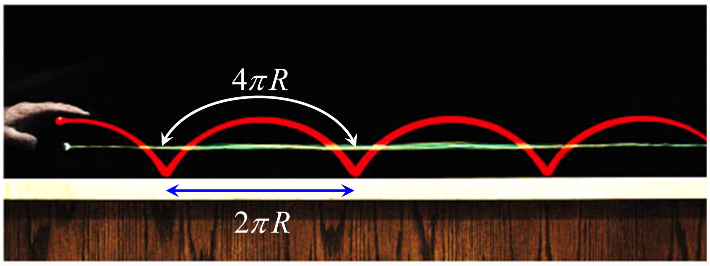
(2) $v_f = R\omega_f$

$$\Rightarrow v_f = \left[\frac{2(m_2 - m_1)gh}{m_1 + m_2 + I/R^2}\right]^{1/2}$$

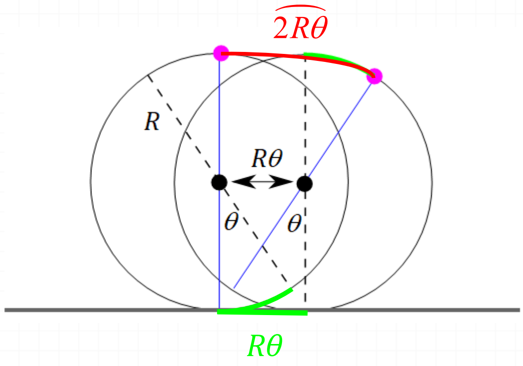
## 3. Rolling motion



## Rolling motion




- Rolling motion = translation + rotation




The pink dot moves $2R\theta$ and the CM moves $R\theta$.



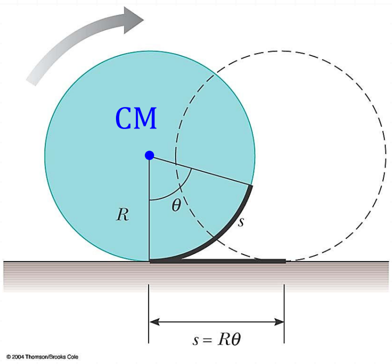
## Pure Rolling Motion, Center of Mass

- The velocity of the center of mass is

$$v_{\mathrm{CM}} = \frac{ds}{dt} = R\frac{d\theta}{dt} = R\omega$$

- The acceleration of the center of mass is

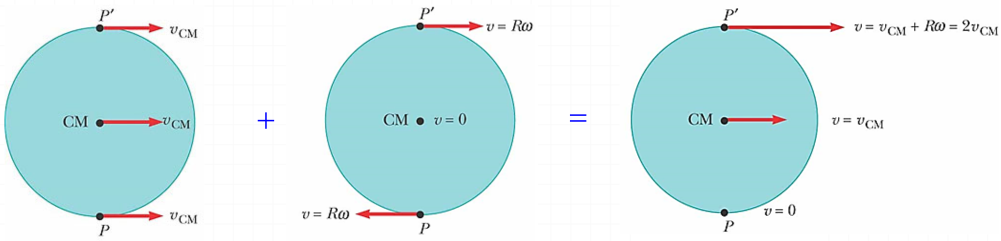
$$a_{\mathrm{CM}} = \frac{dv_{CM}}{dt} = R\frac{d\omega}{dt} = R\alpha$$






$$\begin{cases} R\omega = v_{CM}, & \text{no slipping} \\ R\omega > v_{CM}, & \text{when accelerating} \\ R\omega < v_{CM}, & \text{when breaking} \end{cases}$$






(a) Pure translation (b) Pure rotation **Rolling motion**

$$\vec{v}_{P\text{-ground}} = \vec{v}_{P\text{-CM}} + \vec{v}_{\text{CM-ground}}$$

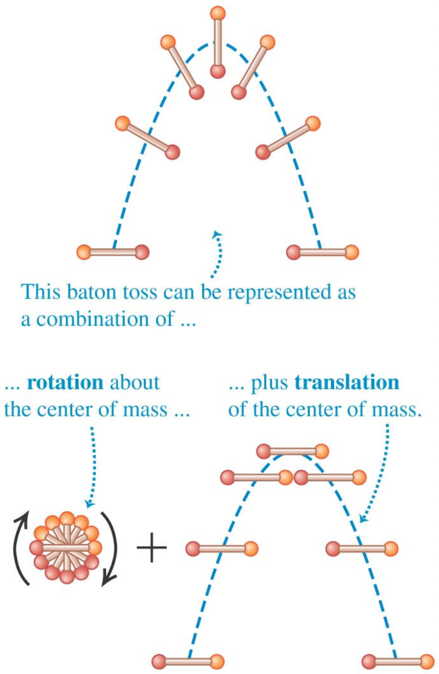
$$K_{\text{total}} = K_{\text{translation}} + K_{\text{rotation}} \quad \Rightarrow \quad K_{\text{total}} = \frac{1}{2}Mv_{CM}^2 + \frac{1}{2}I_{CM}\omega^2$$

$$v_{CM} = R\omega \quad \Rightarrow \quad K_{\text{total}} = \frac{1}{2}MR^2\omega^2 + \frac{1}{2}I_{CM}\omega^2$$

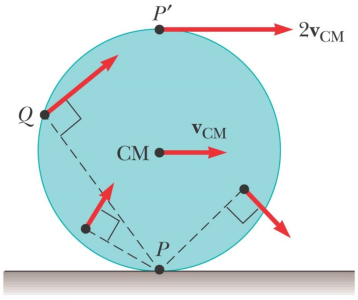
$$\Rightarrow \quad \frac{K_{\text{rotation}}}{K_{\text{total}}} = \frac{\frac{1}{2}I_{CM}\omega^2}{\frac{1}{2}MR^2\omega^2 + \frac{1}{2}I_{CM}\omega^2} = \frac{I_{CM}}{MR^2 + I_{CM}}$$

Equivalent to $K = K_{CM} + K_{rel}$.





Rolling motion can be also modeled as a pure rotation about the contact point *P*.




Total kinetic energy:

$$K_{total} = \frac{1}{2} I_P \omega^2,$$

$$I_P = I_{CM} + MR^2$$

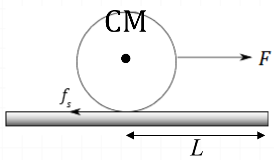
$$K_{total} = \frac{1}{2} I_{CM} \omega^2 + \frac{1}{2} MR^2 \omega^2$$

same as previous one.

Courtesy Alice Halliday



😎 In rotation motion, whether the static friction $f_s$ does work or not depends on the system! It can do work via torque and the work converts to rotational kinetic energy.




(a) If system = CM,

$$W_{CM} = \Delta K_{CM}, \quad \Rightarrow \underbrace{(F - f_s)}_{=F_{CM}=F_{\text{Total}}} L = \frac{1}{2} m v_{CM}^2$$

$$\Rightarrow FL = \frac{1}{2} m v_{CM}^2 + \boxed{f_s L}$$

(b) If system = Disk,

(i) Set pivot at ground contact

$$W = \Delta K, \quad \tau_F \theta = (RF)(\frac{L}{R}) = \frac{1}{2} I \omega^2 = \frac{1}{2}(I_{CM} + MR^2)\omega^2$$

$$\Rightarrow \quad FL = \frac{1}{2} m v_{CM}^2 + \boxed{\frac{1}{2} I_{CM} \omega^2}$$

$$\text{Compare (a), (b)} \Rightarrow \quad \frac{1}{2} I_{CM} \omega^2 = f_s L \left[ = \tau_{f_s} = (R f_s)(\frac{L}{R}) \right]$$

(ii) Set pivot at CM

$$W = \Delta K, \quad W_{translation} + W_{rotation} = \Delta K_{translation} + \Delta K_{rotation}$$

$$(F - f_s)L + (f_s R)(\frac{L}{R}) = \frac{1}{2} m v_{CM}^2 + \frac{1}{2} I_{CM} \omega^2$$



(c) If system = Disk +ground,

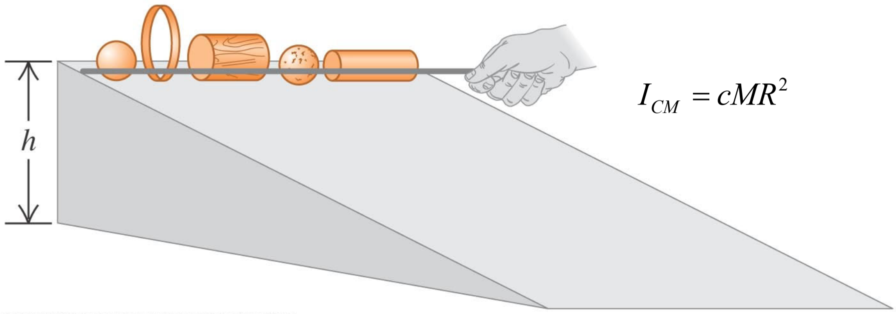
"$f_s$" is an internal force. ⇒ NO contribution to work

Also, no slipping on the contact point ⇒ No contribution to internal energy.

$$W_{ext} = \Delta K + \Delta U + \underbrace{\cancel{\Delta E_{int}}}_{\text{For } f_k \text{, NOT } f_s}$$

$$\Rightarrow \quad FL = \frac{1}{2} m v_{CM}^2 + \frac{1}{2} I \omega^2$$

**Q.** For the same mass objects, which one rolls slowest ?





$$I_{CM} = cMR^2$$



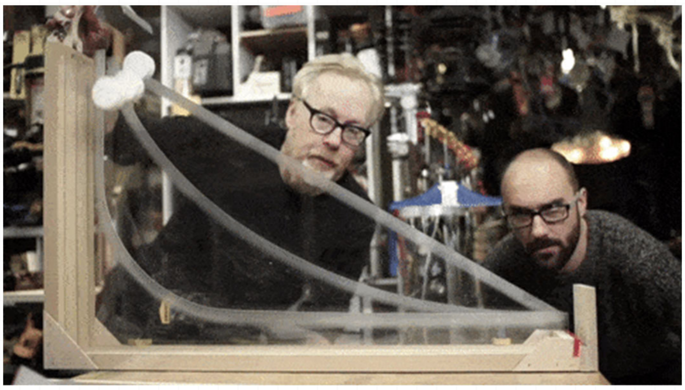
**Ans:**

$$\Delta K + \Delta U = 0$$

$$[\frac{1}{2}Mv_{CM}^2 + \frac{1}{2}I_{CM}(\frac{v_{CM}}{R})^2 - 0] + (0 - Mgh) = 0$$

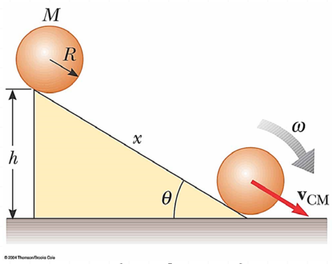
$$\Rightarrow v_{CM} = \left(\frac{2gh}{1+c}\right)^{1/2} \qquad \text{indep. of } R.$$





The shortest path doesn't mean the least time.

**Ex.**




(No slipping)

$v_{CM} = ?$

$a_{CM} = ?$



**Ans:** Method I: System=Ball+Earth+"Ramp"

(1) $\Delta K + \Delta U = 0$ ($f_s$ does no work!)

$$[\frac{1}{2}Mv_{CM}^2 + \frac{1}{2}I_{CM}(\frac{v_{CM}}{R})^2 - 0] + (0 - Mgh) = 0$$

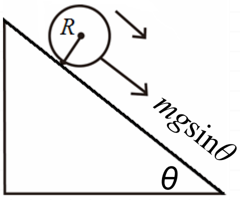
$$\Rightarrow v_{CM} = \left[\frac{2gh}{1+(I_{CM}/MR^2)}\right]^{1/2}$$

(2) $v_f^2 = v_i^2 + 2a\Delta x$

$$a_{CM} = \frac{\dfrac{2gx\sin\theta}{1+(I_{CM}/MR^2)} - 0}{2x} = \frac{g\sin\theta}{\left[1+(I_{CM}/MR^2)\right]}$$

Method II: System = Ball

Set the pivot at the contact point.



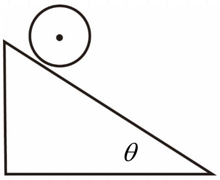

$$\tau = I\alpha$$

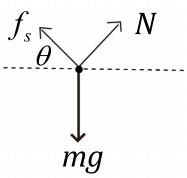
$$\Rightarrow \quad Mg\sin\theta\ R = (I_{CM} + MR^2)\alpha$$

$$\Rightarrow \quad a_{CM} = R\alpha = \frac{g\sin\theta}{1+\dfrac{I_{CM}}{MR^2}}$$



Method III: System = Ball Set the pivot at the CM.




(1) $\Sigma F = ma \quad \Rightarrow \quad mg\sin\theta - f_s = ma_{CM} \quad \Rightarrow \quad a_{CM} = \dfrac{g\sin\theta}{1+{I_{CM}}/{mR^2}}$

(2) $\Sigma\tau = I\alpha \quad \Rightarrow \quad f_s R = I_{CM}\alpha$

(3) $a_{CM} = R\alpha \Rightarrow \quad f_s = mg\sin\theta - ma_{CM} = \dfrac{mg\sin\theta}{1+{mR^2}/{I_{CM}}}$

p.s.,
The condition for the ball to roll without slipping is

$$f_s \le \mu_s N = \mu_s mg\cos\theta \quad \Rightarrow \quad \tan\theta \le \mu_s (1+{mR^2}/{I_{CM}})$$

Note,

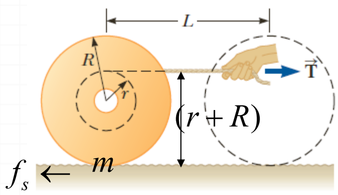
1. There is a static friction force so that $a_{CM} \neq g \sin\theta$.

2. Static friction does NOT do work if the system = Ball+Earth+Ramp.

😎 When the ball rolls upward, in what direction will be the frictional force?



**Ex.**




(1) $v_{CM} = ?$

(2) $f_s = ?$

(Note that $T \neq f_s$, unlike in translational case.)

**Ans:**

(1) System = Disk
(Set the pivot at ground contact)

$$W=\Delta K \qquad \theta=\frac{L}{R}$$

$$W=\int \tau d\theta = T\underbrace{(R+r)}_{\text{lever arm}}\theta \left[\text{or}=\int F ds = T\underbrace{(R+r)\theta}_{\text{displacement}}\right]=T(L+r\frac{L}{R})$$

$$\Rightarrow\ T(L+r\frac{L}{R})=\frac{1}{2}I_p\omega^2=\frac{1}{2}(I_{CM}+MR^2)\omega^2, \qquad v_{CM}=R\omega$$

$$\Rightarrow\ v_{CM}=\sqrt{\frac{2TL(1+r/R)}{m(1+I/mR^2)}}$$

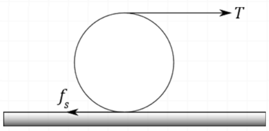


(2) System = CM only
(In this case, $f_s$ is an external force.)

$$T-f_s=ma_{CM}$$
$$v_{CM}^2=0^2+2a_{CM}L$$
$$\Rightarrow\ f_s=T\left(1-\frac{1+r/R}{1+I/mR^2}\right)$$

Alternative method: $W_{CM}=\frac{1}{2}mv_{CM}^2$

$$\Rightarrow\ (T-f_s)L=\frac{1}{2}mv_{CM}^2$$
$$\Rightarrow\ f_s=T\left(1-\frac{1+r/R}{1+I/mR^2}\right)$$

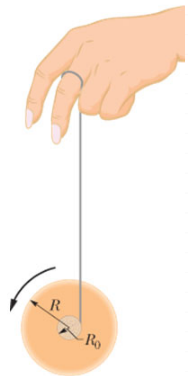



In this case, $f_s$ does work on CM in the rolling motion.

**Ex.**

# The Yo-Yo

(The rope is not moving.)




$a_{CM} = ?$

(Similar to the inversed pulley problem.)

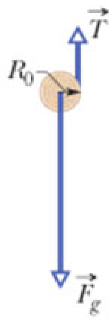


**Ans:**

Method #1 (rotation)

System = Yo-Yo

Set rotation at CM.




$$\begin{cases} F_g - T = ma_{CM} \\ \tau = I\alpha \;\Rightarrow\; R_0 T = I_{CM} \dfrac{a_{CM}}{\underbrace{R_0}_{\text{NOT } R}} \end{cases}$$

$$\Rightarrow \; a_{CM} = \frac{g}{1 + I_{CM} / MR_0^2}$$

Method #2 (rolling) System = Yo-Yo

Set rotation at the contact point.

$\tau = I\alpha$

$\because g$ is assumed constant, $\therefore \tau = \tau_{CM} = r_{CM} = R_0 Mg.$


(Parallel-axis theorem) $I = I_{CM} + MR_0^2$

$$\Rightarrow \quad R_0 Mg = (I_{CM} + MR_0^2)\alpha$$

$$\Rightarrow \quad \alpha = \frac{MgR_0}{I_{CM} + MR_0^2}$$

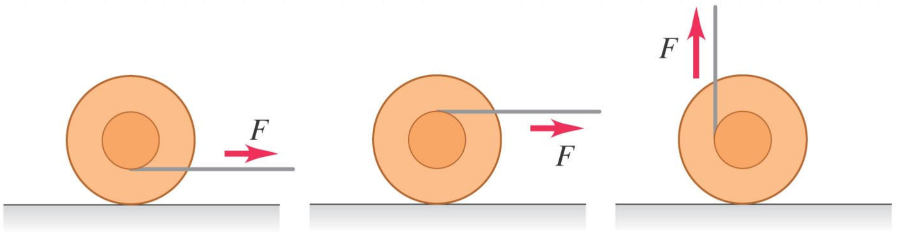
Therefore, $a_{CM} = R_0\alpha = \dfrac{g}{1 + I_{CM} / MR_0^2}$



1. How long does it take the yo-yo to hit the floor?
2. What is the CM velocity?

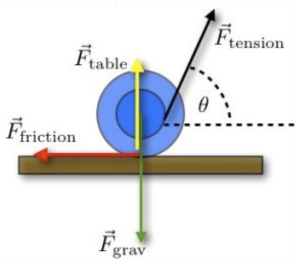
(Problem 10.71)

**Q**: In what direction will each yo-yo rotate?








Depending upon the angle of applied force, a yo-yo can be made to roll forwards, backwards or simply slide without rotating.

https://www.youtube.com/watch?v=tFHd8_h1QU&t=37s

https://www.wired.com/2010/01/yo-yo-rolling-sliding-pulling/

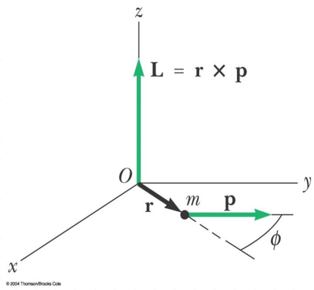
# 4. Angular momentum



# Angular Momentum

$$\vec{\tau} = \vec{r} \times \vec{F} = \vec{r} \times \frac{d\vec{p}}{dt} + \underbrace{\underbrace{\frac{d\vec{r}}{dt}}_{=\vec{v}} \times \underbrace{\vec{p}}_{=m\vec{v}}}_{=0}$$

$$\Rightarrow \quad \vec{\tau} = \frac{d(\vec{r} \times \vec{p})}{dt}$$




Angular momentum: $\vec{L} = \vec{r} \times \vec{p}$

$$\Rightarrow \quad \vec{\tau} = \frac{d\vec{L}}{dt}$$

p.s., $\vec{\tau}$ and $\vec{L}$ must be measured about the same origin, fixed in an inertial frame.

## Angular Momentum of a System of Particles

- The total angular momentum of a system of particles is

$$\boldsymbol{L}_{\text{tot}} = \boldsymbol{L}_1 + \boldsymbol{L}_2 + \ldots + \boldsymbol{L}_n = \Sigma \boldsymbol{L}_i$$

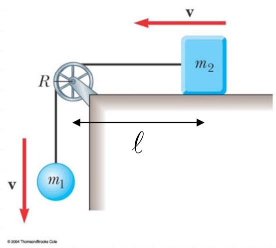
- Differentiating with respect to time

$$\frac{d\vec{L}_{\text{tot}}}{dt} = \sum_i \frac{d\vec{L}_i}{dt} = \sum_i \vec{\tau}_i = \sum_i \vec{\tau}_{i,\text{ext}} + \underbrace{\sum_i \vec{\tau}_{i,\text{int}}}_{=0}$$

$$\frac{d\vec{L}_{\text{tot}}}{dt} = \sum_i \vec{\tau}_{i,\text{ext}}$$

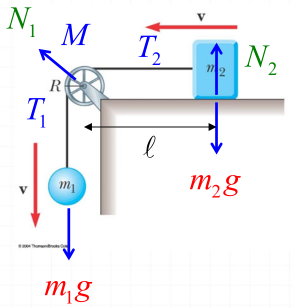


**Ex.**




$a = ?$ using $\vec{L}$ and $\vec{\tau}$.

**Ans:**




Set system=$m_1$+$M$+$m_2$

Set pivot at wheel axis

$$L = Rm_1v + Rm_2v + RMv$$

$$\tau_{ext} = \frac{dL}{dt} = R(m_1 + m_2 + M)\frac{dv}{dt}$$

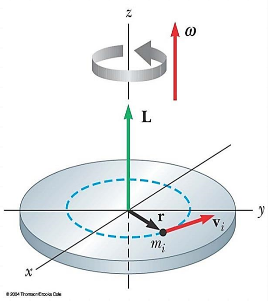
$$\Rightarrow \; m_1gR + \underbrace{0 \times N_1}_{0} + \underbrace{\ell(N_2 - m_2g)}_{=0} = R(m_1 + m_2 + M)a$$

[$T_1$, $T_2$ are internal forces, NO contribution to $\vec{\tau}$.]

$$\Rightarrow \; a = \frac{m_1g}{m_1 + m_2 + M}$$



## Angular Momentum of a Rotating Rigid Object




$$\vec{L}_i = \vec{r}_i \times \vec{p}_i = r_i m_i v_i \hat{z}$$

$$\Rightarrow \; \vec{L}_i = m_i r_i^2 \omega \hat{z}$$

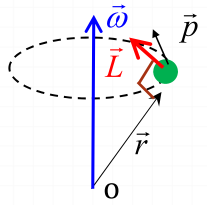
$$L_z = \sum_i L_i = \sum_i m_i r_i^2 \omega = \underbrace{\left(\sum_i m_i r_i^2\right)}_{=I}\omega$$

$$\Rightarrow \; L_z = I\omega$$

$$\Rightarrow \; \frac{dL_z}{dt} = I\frac{d\omega}{dt} = I\alpha$$

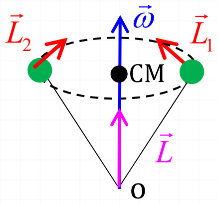
$$\Rightarrow \; \tau_z = I\alpha$$

Note, $\vec{L} = I\vec{\omega}$ is not always correct!




$\Rightarrow$ $\vec{L}$ is not parallel to $\vec{\omega}$

In general, $\vec{L} = I\vec{\omega}$, only if
the rotation is about a symmetry axis and through the CM.




$\Rightarrow$ $\vec{L} = I\vec{\omega}$



🤓 In fact,

$$\begin{aligned}
\vec{L} &= \vec{r} \times \vec{p} = \vec{r} \times (m\vec{v}) = m\vec{r} \times (\vec{\omega} \times \vec{r}) \\
&= m\vec{\omega}(\vec{r} \cdot \vec{r}) - m\vec{r}(\vec{r} \cdot \vec{\omega}) \\
&= mr^2\vec{\omega} - m(\vec{r} \cdot \vec{\omega})\vec{r} \\
&= I\vec{\omega} - m(\vec{r} \cdot \vec{\omega})\vec{r}
\end{aligned}$$

If $\vec{r} \perp \vec{\omega}$, $\Rightarrow$ $\vec{L} = I\vec{\omega}$

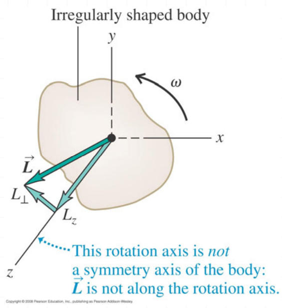
🤔 If the rotation axis is not a symmetry axis,

$\Rightarrow$ $\vec{L}$ will not be || to the rotation axis.

$\vec{L}$ keeps changing direction $\Rightarrow$ $\vec{\tau} \neq 0$.

This will cause force on the pivot!

e.g., spinning top, wheel balancing, washing machine






**Useful Equations in Rotational and Linear Motion**

| Rotational Motion About a Fixed Axis | Linear Motion |
|---|---|
| Angular speed $\omega = d\theta/dt$ | Linear speed $v = dx/dt$ |
| Angular acceleration $\alpha = d\omega/dt$ | Linear acceleration $a = dv/dt$ |
| Net torque $\Sigma\tau = I\alpha$ | Net force $\Sigma F = ma$ |
| If $\alpha$ = constant $\begin{cases} \omega_f = \omega_i + \alpha_t \\ \theta_f = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2 \\ \omega_f^2 = \omega_i^2 + 2\alpha(\theta_f - \theta_i) \end{cases}$ | If $a$ = constant $\begin{cases} v_f = v_i + at \\ x_f = x_i + v_i t + \frac{1}{2}at^2 \\ v_f^2 = v_i^2 + 2a(x_f - x_i) \end{cases}$ |
| Work $W = \int_{\theta_i}^{\theta_f} \tau\, d\theta$ | Work $W = \int_{x_i}^{x_f} F_x\, dx$ |
| Rotational kinetic energy $K_R = \frac{1}{2}I\omega^2$ | Kinetic energy $K = \frac{1}{2}mv^2$ |
| Power $\mathcal{P} = \tau\omega$ | Power $\mathcal{P} = Fv$ |
| Angular momentum $L = I\omega$ | Linear momentum $p = mv$ |
| Net torque $\Sigma\tau = dL/dt$ | Net force $\Sigma F = dp/dt$ |

1. Torque
2. Energy conservation in rotational motion
3. Rolling motion
4. Angular momentum
5. Conservation of angular momentum
6. The motion of gyroscopes and tops
7. Angular momentum as a fundamental quantity

## 5. Conservation of Angular momentum



## Conservation of Angular Momentum

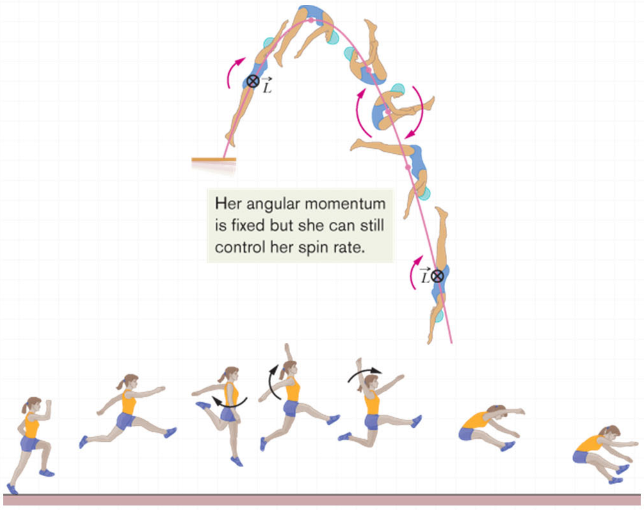
If $\sum \vec{\tau}_{ext} = \frac{d\vec{L}_{total}}{dt} = 0 \quad \Rightarrow \quad \vec{L}_{total} = const.$

$$\vec{L}_i = \vec{L}_f$$

For rotation about a **fixed axis** and through **the CM** of a moving system,

$$\vec{L} = I\vec{\omega}$$

$$\Rightarrow I_i\omega_i = I_f\omega_f$$

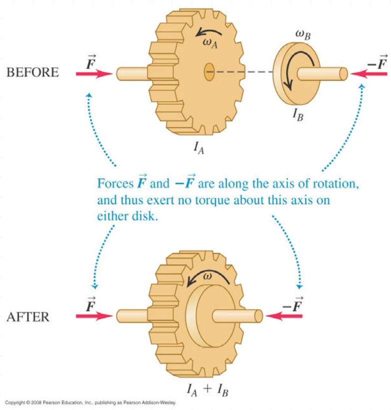



Set pivot at CM.

If "$g$" is constant. $\Rightarrow \quad \vec{\tau}_{total} = \vec{\tau}_{CM} = \underbrace{\vec{r}_{CM}}_{=0} \times M\vec{g} = 0 \quad \Rightarrow \quad \vec{L} = const.$



“Rotational” collision

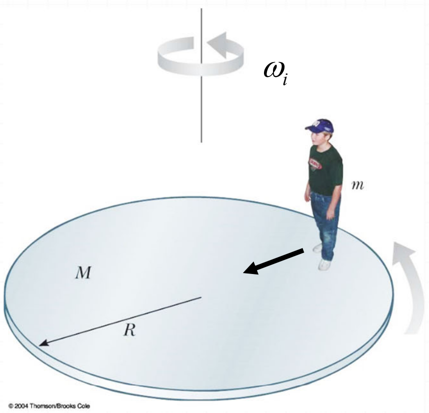



$$I_A\omega_A + I_B\omega_B = (I_A + I_B)\omega$$

$$\Rightarrow \quad \omega = \frac{I_A\omega_A + I_B\omega_B}{I_A + I_B}$$



**Ex.**




$\omega_f = ?$

**Ans:**

System = disk + person

$$\vec{\tau}_{ext} = 0 \quad \Rightarrow \quad I_i \omega_i = I_f \omega_f$$

$$(\frac{1}{2}MR^2 + mR^2)\omega_i = (\frac{1}{2}MR^2 + mr^2)\omega_f$$

$$\Rightarrow \quad \omega_f = \left( \frac{\frac{1}{2}MR^2 + mR^2}{\frac{1}{2}MR^2 + mr^2} \right) \omega_i$$



Note,
1. The kinetic energy is not conserved!

Similar to rocket motion, $K_i = 0$,
but $K_f \neq 0$.

2. If the person and turntable are considered as the system, the "momentum" is not conserved, because the ground exerts force on turntable while the person is walking. [Serway, Problem 11.23(b)]

## Conservation Laws Summary

For an isolated system,

(1) Conservation of Energy:

$E_i = E_f$

(2) Conservation of Linear Momentum:

$\mathbf{p}_i = \mathbf{p}_f$

(3) Conservation of Angular Momentum:

$\mathbf{L}_i = \mathbf{L}_f$

p.s., These quantities do not have to be conserved simultaneously.



**Ex.**


Initially, only the wheel is rotating.

After flipping the wheel,

(1) Will $|\boldsymbol{L}_\text{i}|$ change?

(2) What will be $\boldsymbol{L}_\text{person+stool}$?

**Ans:**

(1) $|\boldsymbol{L}_i|$ will not change; $\because \sum \boldsymbol{\tau}_{\text{by hand}} = 0$ about the "spinning axis" of the wheel.

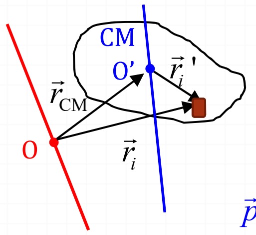
(2) $\vec{L}_{\text{total},i} = \underbrace{\vec{L}_i}_{\text{wheel}}$

$$\vec{L}_{\text{total},f} = \vec{L}_{\text{person+stool}} + \underbrace{(-\vec{L}_i)}_{\text{wheel}}$$

$$\sum \vec{\tau}_{\text{ext}} = 0 \Rightarrow \vec{L}_{\text{total},i} = \vec{L}_{\text{total},f} \quad \Rightarrow \vec{L}_{\text{person+stool}} = 2\vec{L}_i$$



# Angular momentum at different reference axis




$$\vec{L}_{o'\ \mathrm{at\ CM}} = \sum_i \vec{r}_i\,' \times \vec{p}_i\,'$$

$$\vec{r}_i\,' = \vec{r}_i - \vec{r}_{CM}$$

$$\vec{p}_i\,' = \vec{p}_i - m_i\vec{v}_{CM}$$

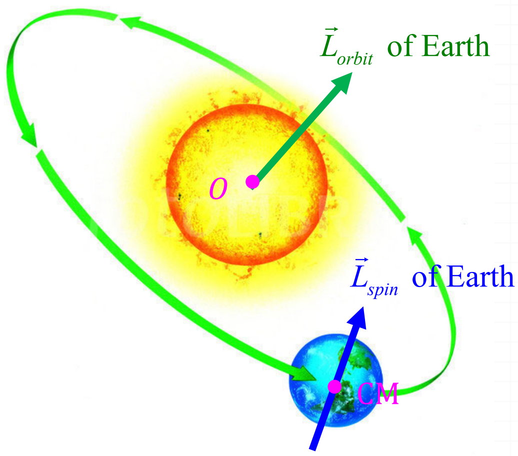
$$\vec{L}_o = \sum_i \vec{r}_i \times \vec{p}_i = \sum_i (\vec{r}_i\,' + \vec{r}_{\mathrm{CM}}) \times (\vec{p}_i\,' + m_i\vec{v}_{CM})$$

$$= \sum_i \vec{r}_i\,' \times \vec{p}_i\,' + \sum_i \vec{r}_{\mathrm{CM}} \times \vec{p}_i\,' + \sum_i m_i\vec{r}\,'_i \times \vec{v}_{CM} + \sum_i m_i\vec{r}_{CM} \times \vec{v}_{CM}$$

$$= \vec{L}_{o'\ \mathrm{at\ CM}} + \vec{r}_{\mathrm{CM}} \times \underbrace{\sum_i \vec{p}_i\,'}_{=0} + \underbrace{(\sum_i m_i\vec{r}_i\,')}_{=0} \times \vec{v}_{CM} + M\vec{r}_{CM} \times \vec{v}_{CM}$$

$$\Rightarrow \quad \vec{L}_o = \vec{L}_{o'\ \mathrm{at\ CM}} + \vec{r}_{\mathrm{CM}} \times \vec{p}_{CM}$$





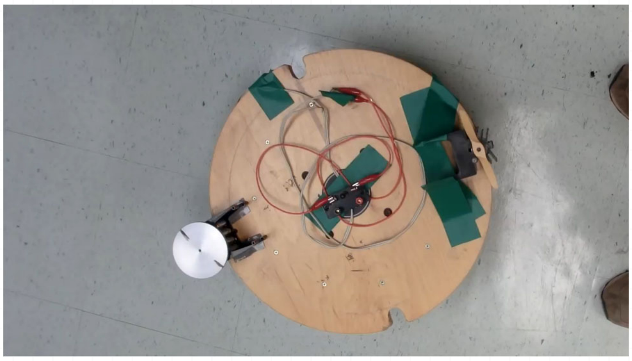

$$\vec{L}(\mathrm{Earth})_{\text{with reference at Sun}} = \vec{L}_{spin} + \vec{L}_{orbit}$$

$$\vec{L}_o = \vec{L}_{o' \text{ at CM}} + \vec{r}_{\text{CM}} \times \vec{p}_{CM}$$

$$\vec{\tau}_o = \vec{\tau}_{o' \text{ at CM}} + \vec{r}_{\text{CM}} \times \vec{F}_{CM}$$

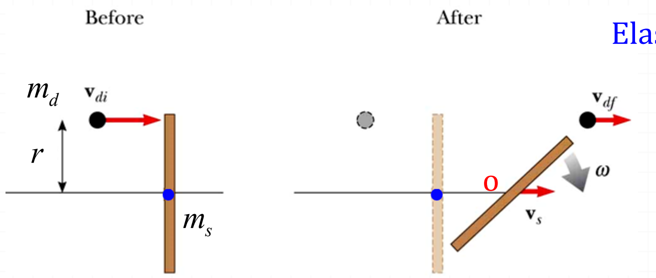
$$\Rightarrow \begin{cases} \vec{\tau}_o = \dfrac{d\vec{L}_o}{dt} \\ \vec{\tau}_{o' \text{ at CM}} = \dfrac{d\vec{L}_{o'\text{at CM}}}{dt} \\ \vec{r}_{\text{CM}} \times \vec{F}_{CM} = \dfrac{d\left(\vec{r}_{\text{CM}} \times \vec{p}_{\text{CM}}\right)}{dt} \end{cases}$$





https://www.wired.com/story/what-is-angular-momentum/
https://www.youtube.com/watch?v=DtxxmTdQ2Jc&feature=youtu.be

**Ex.**




Elastic collision

$v_{df} = ?$

$\omega = ?$

$v_s = ?$

😎 The middle of the stick moves on a straight line. Why?



**Ans:**

(1) $\vec{p}_i = \vec{p}_f \quad \Rightarrow \quad m_d v_{di} = m_d v_{df} + m_s v_s$

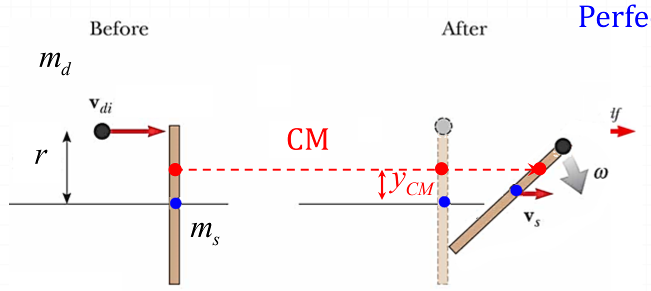
(2) $K_i = K_f \quad \Rightarrow \quad \frac{1}{2} m_d v_{di}^2 = \frac{1}{2} m_d v_{df}^2 + \frac{1}{2} m_s v_s^2 + \frac{1}{2} I\omega^2$

(3) $\vec{L}_i = \vec{L}_f$ $\Leftarrow$ Set the middle of the stick as the rotation reference, although it is moving (but without acceleration.)

$$\Rightarrow \quad r m_d v_{di} = r m_d v_{df} + I\omega$$

**Ex.**

Perfectly inelastic collision




$v_{CM} = ?$

$\omega = ?$

😎 The middle of the stick doesn't move on a straight line. Why?



**Ans:**

(1) $P_i = P_f \quad \Rightarrow \quad m_d v_{di} = (m_d + m_s) v_{CM} \quad \Rightarrow \quad v_{CM} = \dfrac{m_d v_{di}}{m_d + m_s}$

(2) $K_i \neq K_f$

(3) $L_i = L_f$ $\Leftarrow$ set the CM as the rotational reference point

$$\Rightarrow \quad (r - y_{CM}) m_d v_{di} = I_d \omega + I_s \omega$$

$$I_d = m_d (r - y_{CM})^2$$

$$I_s = I_{CM\ \text{of stick}} + MD^2 = I_{CM\ \text{of stick}} + m_s y_{CM}^2$$

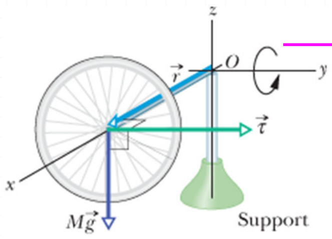
$$\Rightarrow \quad \omega = \ldots$$

## 6. The motion of gyroscopes and tops



## Motion of Gyroscopes and Tops

Without spin, $\vec{L}_{Spin} = 0$

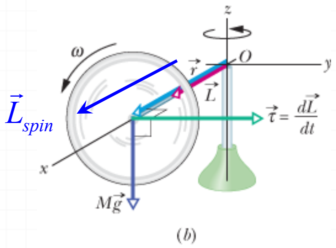


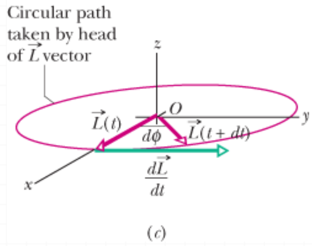

(a)

$\vec{L} = L_y \hat{y}$

With spin, $\vec{L}_{Spin} \neq 0$

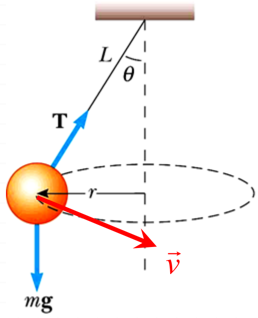
$\uparrow$ $\vec{L}_{Precession} = L_z \hat{z}$



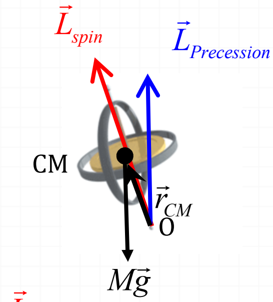



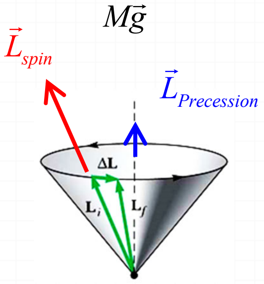



Analog to this case:




When the ball has circular speed, then it will be lifted.
Otherwise, it stays in the bottom.




(1) Set the pivot at the CM:

$$\vec{\tau}_{\text{origin at CM}} = \underbrace{\vec{r}}_{=0} \times \underbrace{\vec{F}}_{=-mg\hat{z}} = 0$$

$$\frac{d\vec{L}_{Spin}}{dt} = 0 \quad \Rightarrow \quad \vec{L}_{Spin} \text{ is constant.}$$

(2) Set the pivot at the tip:

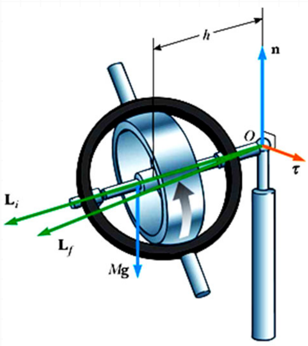
$$\vec{\tau}_0 = \vec{r}_{CM} \times M\vec{g} = \frac{d\vec{L}}{dt} \neq 0$$

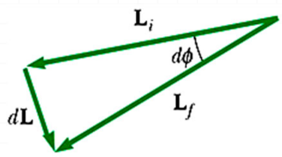
$\Rightarrow$ $\vec{\tau}_0$ changes $\vec{L}_{Spin}$ direction,

but not its magnitude.

$$\vec{L} = \vec{L}_{Spin} + \vec{L}_{Precession}$$






Assume $\left|\vec{L}_{Spin}\right| \gg \left|\vec{L}_{Precession}\right|$ so that $\vec{L} = \vec{L}_{Spin}$.

(1) $\left|\vec{L}_i\right| = \left|\vec{L}_f\right|$

(2) $\vec{\tau} = \dfrac{d\vec{L}}{dt} \quad \Rightarrow \quad \vec{\tau} \parallel d\vec{L} \quad \Rightarrow$ This is why the gyroscope doesn't fall.

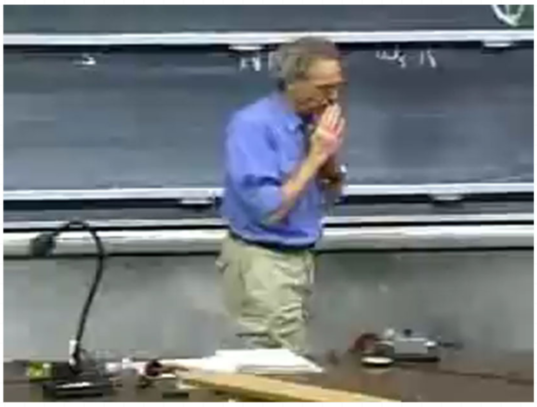
(3) $\tau dt = dL \quad \Rightarrow \quad (mgh)dt = (L)d\phi = (I\omega)d\phi$

Precessional frequency $\Rightarrow \quad \omega_p = \dfrac{d\phi}{dt} = \dfrac{Mgh}{I\omega}$

If $\left|\vec{L}_{Spin}\right|$ is not $\gg \left|\vec{L}_{Precession}\right|$, "nutation" will occur in addition to precession.


https://www.youtube.com/watch?v=5Sn2J1Vn4zU





https://www.youtube.com/watch?v=ekzwbt3hu2k


https://www.youtube.com/watch?v=hVsx4XWafXg



## Applications of Gyroscopes

Gyrocompasses, Ship stabilisers, Segway Scooter, etc.

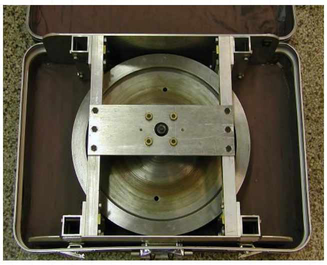
http://www.gyroscopes.org/

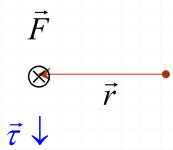
## Seakeeper


https://www.youtube.com/watch?v=KNDpsBlxUIY

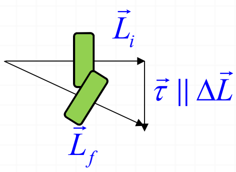


## Gyro suitcase



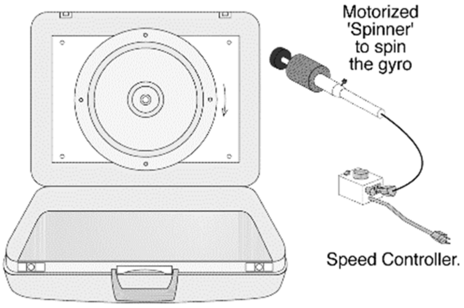


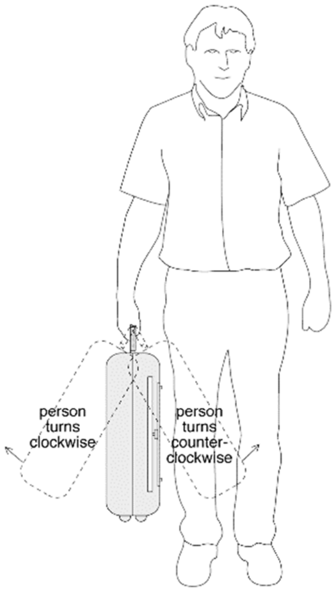

$$\vec{\tau}=\frac{d\vec{L}}{dt}=\vec{r}\times\vec{F}\quad\Rightarrow\quad\tau \parallel d\vec{L}$$



## ANGULAR MOMENTUM.

Suitcase Gyroscope.



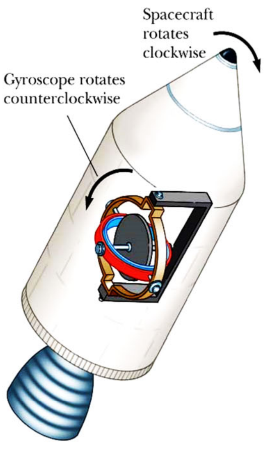




A gyro is mounted in a suitcase. The gyro is a large aluminum disk mounted on ball bearings. The gyro is spun up using a motorized 'spinner', and the suitcase is then closed. When a person holds the suitcase in their right hand (with the mounting bolts towards their right leg), then turning clockwise causes the suitcase to rise up and away from them. Turning counter-clockwise causes the suitcase to rise up towards them, running into the right leg.



(a)






(b)

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**Euler's Rigid Rotators, Jacobi Elliptic Functions, and the Dzhanibekov or Tennis Racket Effect**

**C. Peterson and W. Schwalm**
**University of North Dakota**

American Journal of Physics **89**, 349 (2021);
https://doi.org/10.1119/10.0003372





## 7. Angular momentum as a fundamental quantity

## Angular Momentum as a Fundamental Quantity

- Angular momentum has discrete values. The fundamental unit of angular momentum is $\hbar$, called Planck's constant.

$$\hbar = \frac{h}{2\pi} = 1.054 \times 10^{-34} \frac{\text{kg}\cdot\text{m}^2}{\text{s}}$$

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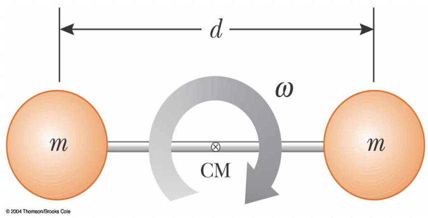


$$L = I\omega = n\hbar$$

$$\Rightarrow \quad \omega = \frac{n\hbar}{I}, \quad n = 1, 2, 3, \ldots$$

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