Chapter 7

Potential Energy and Energy Conservation

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Outline

1. Potential energy
2. Mechanical energy
3. Internal energy
4. Energy in isolated and nonisolated systems
5. Conservation of energy
6. Conservative and nonconservative forces
7. Energy diagram and equilibrium

1. Potential energy

Potential Energy

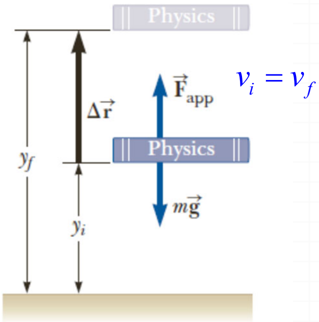
- Potential energy is the energy associated with the configuration of a system of objects that exert internal forces on each other.
  - Always associated with a system of two or more interacting objects.

Types of Potential Energy

- There are many forms of potential energy, including:
  - Mechanical (e.g., gravitational, elastic.)
  - Electromagnetic
  - Chemical
  - Nuclear
- One form of energy in a system can be converted into another.

1-1 Gravitational Potential

$\boxed{\text{System = book + Earth}}$



External force $\vec{F}_{\text{app}}$: $\vec{F}_{\text{app}} + m\vec{g} = 0$ for $v_i = v_f$

$$\vec{F}_{\text{app}} = -m\vec{g}$$

$$W_{\text{ext}} = \underbrace{W_{\text{on book}}}_{F_{app}\Delta r = mg(y_f - y_i)} + \underbrace{W_{\text{on Earth}}}_{=0} \neq 0 = \underbrace{\Delta K_{\text{Book}}}_{=0\ (\because v_i = v_f)} + \underbrace{\Delta K_{\text{Earth}}}_{=0}$$

$$\Rightarrow W_{\text{ext}} \neq \Delta K \ ?$$

Define Gravitational potential energy

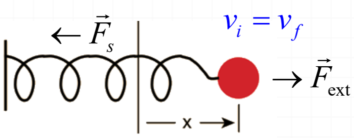
$$U_g \equiv mgy \qquad \text{(assuming } g \text{ is a constant)}$$

such that $W_{\text{ext}} = \Delta K + \Delta U_g$ instead of $W_{\text{ext}} = \Delta K$.

$$\text{i.e., } \underbrace{\vec{F} \cdot (\vec{r}_f - \vec{r}_i)}_{\text{External input}} = \underbrace{\left(\frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2\right) + \left(mgy_f - mgy_i\right)}_{\text{System energy change}}$$

- Note that K and U_g depend on the choice of the reference frame, but the equation doesn't.
- The gravitational potential is stored between the two objects.
- Work only by external forces is considered. Internal forces are excluded.

1-2 Elastic Potential



$\boxed{\text{System = ball + spring}}$

$$\vec{F}_{\text{ext}} = -\vec{F}_s = -(-kx)\hat{i} = kx\hat{i}$$

$$W_{\text{ext}} = \int_{x_i}^{x_f} F_{\text{ext}}\,dx = \int_{x_i}^{x_f} kx\,dx = \frac{1}{2}kx_f^2 - \frac{1}{2}kx_i^2 \quad \neq \quad 0 = \Delta K$$

$$\Rightarrow \quad W_{\text{ext}} \neq \Delta K\ ?$$

Define elastic potential energy

$$U_s \equiv \frac{1}{2}kx^2$$

such that $W_{\text{ext}} = \Delta K + \Delta U_s$

$$\text{i.e., } \underbrace{\vec{F}\cdot(\vec{r}_f - \vec{r}_i)}_{\text{External input}} = \underbrace{\left(\frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2\right) + \left(\frac{1}{2}kx_f^2 - \frac{1}{2}kx_i^2\right)}_{\text{System energy change}}$$

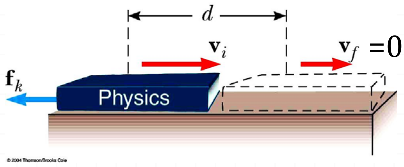
2. Mechanical Energy

Mechanical energy

- **Mechanical energy** of a system is the sum of kinetic energy and potential energy.

$$E_{mech} = K + U$$

3. Internal Energy



(1) System = Book

$$\Delta K_{\text{Book}} = -\frac{1}{2}mv_i^2 = -f_k d$$

Where does the kinetic energy go?

(2) System = Book + Table

Book: $\Delta K_{\text{Book}} = -\frac{1}{2}mv_i^2$

Table: $\Delta K_{\text{Table}} = 0$

$$\Rightarrow \Delta K = \Delta K_{\text{Book}} + \Delta K_{\text{Table}} = \underbrace{-\frac{1}{2}mv_i^2}_{=-f_k d} \neq 0 = W_{\text{ext}}$$

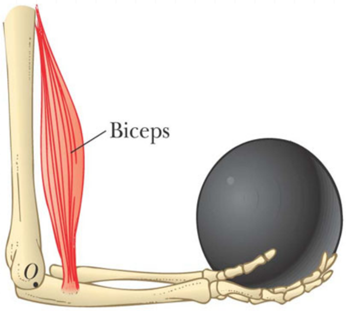
Why $\Delta K \neq W_{\text{ext}}$? $\Rightarrow$ The kinetic energy is transferred to internal energy.

$$\Rightarrow \Delta E_{\text{int}} = f_k d \quad \text{such that } W_{\text{ext}} = \Delta K + \Delta E_{\text{int}}$$

$$\text{i.e., } \underbrace{\vec{F}\cdot(\vec{r}_f - \vec{r}_i)}_{\text{External input}} = \underbrace{\left(\frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2\right) + f_k d}_{\text{System energy change}}$$

😎 Internal energy exists only in systems with more than one object.

Why does it take energy to hold a weight in position?



https://www.facebook.com/theactionlabofficial
/posts/958254638316176

4. Energy in isolated and nonisolated systems

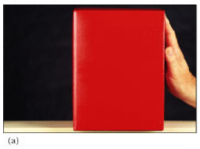
A system's energy can be stored in various forms:

1. Kinetic energy (motion).
2. Potential energy (elastic, electric, gravitational, chemical, nuclear; by conservative forces).
3. Internal energy (microscopic kinetic and potential energies relative to system's center of mass, related to temperature).

Ways to Transfer Energy

- *Work* – transfers by applying a force and causing a displacement of the point of application of the force.
- *Mechanical Waves* – allow a disturbance to propagate through a medium.
- *Heat* – is driven by a temperature difference between two regions in space.
- *Matter Transfer* – matter physically crosses the boundary of the system, carrying energy with it.
- *Electrical Transmission* – transfer is by electric current.
- *Electromagnetic Radiation* – energy is transferred by electromagnetic waves.

a) Work



d) Matter transfer

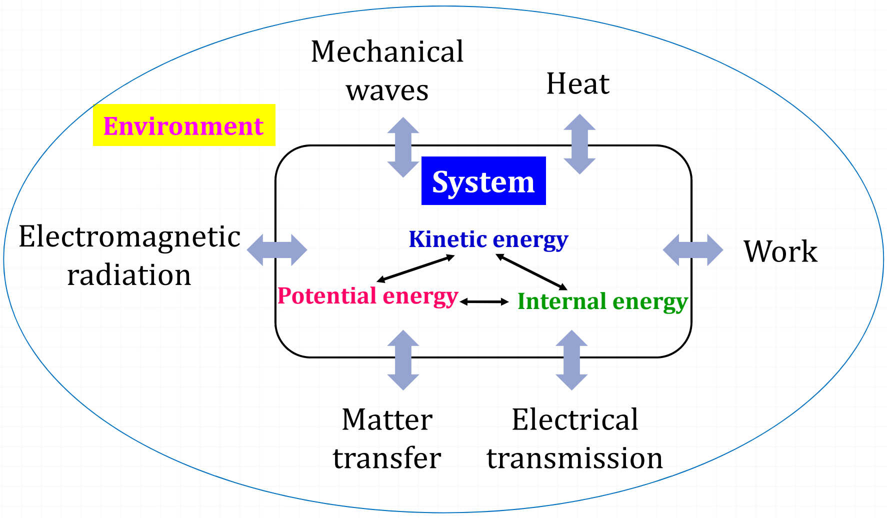
b) Mechanical Waves

e) Electrical Transmission

c) Heat

f) Electromagnetic radiation

Universe



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5. Conservation of energy

Conservation of Energy

- Energy is conserved; we can neither create nor destroy energy.
- For an isolated system, $\Delta E_{\text{system}} = 0$
- For a non-isolated system,

$$\underbrace{\Delta E_{\text{system}}}_{\text{Energy changed in a system}} = \underbrace{\sum T}_{\text{Energy transferred across the system boundary}}$$

e.g., $\Delta K + \Delta U + \Delta E_{\text{int}} = W + Q + T_{\text{others}}$

Q : heat

T_{others} : energy transfer via mechancial wave, matter transfer, electromagnetic radiation, etc.

😎

1. Conservation of energy can be rigorously proven by Noether's theorem as a consequence of continuous time translation **symmetry**; that is, from the fact that the laws of physics do not change over time.

2. Energy at each fixed time can in principle be exactly measured without any trade-off in precision forced by the time-energy uncertainty relations. Thus the conservation of energy in time is a well defined concept even in quantum mechanics.

https://en.wikipedia.org/wiki/Conservation_of_energy

Work-Kinetic Energy Theorem

- When work is done on a system and if the only change in the system is in its speed, the net work done on the system equals the change in kinetic energy of the system. (Also valid for rotational motion)

$$W_{\text{external}} = K_f - K_i = \Delta K$$

Always valid for a “one-particle” system.

Conservation of mechanical energy

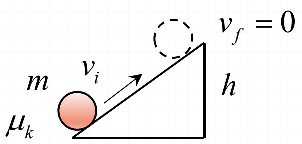
- **Mechanical energy**: $E_{\text{mech}} = K + U$
- For an isolated system with no nonconservative forces acting,

  $\Rightarrow$ the mechanical energy is conserved, $\Delta E_{\text{mech}} = 0$.

Ex. Which one loses smaller mechanical energy?

(1)

(2)

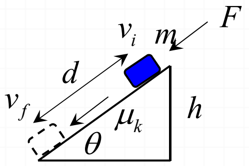


$$\Delta E_{mech,1} = \Delta K_1 = -\frac{1}{2} m v_i^2$$

$$\Delta E_{mech,2} = \Delta K_2 + \Delta U = -\frac{1}{2} m v_i^2 + mgh$$

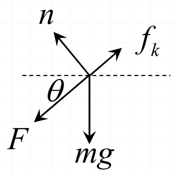
$$\Rightarrow \left|\Delta E_{mech,2}\right| < \left|\Delta E_{mech,1}\right|$$

Case 1: set system = block



$$W_{\text{ext}} = \Delta K + \cancel{\Delta U_g} + \cancel{\Delta E_{\text{int}}}$$

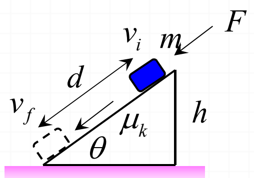
∵The system has only one object, so
no potential energy and internal energy.



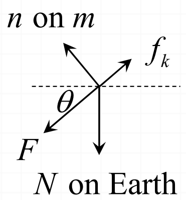
Work-kinetic theorem

$$\Rightarrow \left(F + mg\sin\theta - f_k\right)d = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$$

Case 2: set system = block + Earth

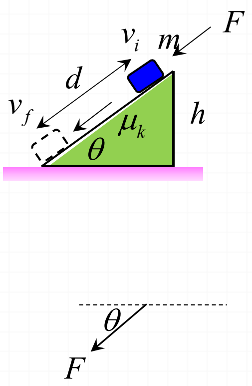


$$W_{\text{ext}} = Fd - f_k d = \Delta K + \Delta U_g$$



$$\Rightarrow Fd - f_k d = \left(\frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2\right) - mgd\sin\theta$$

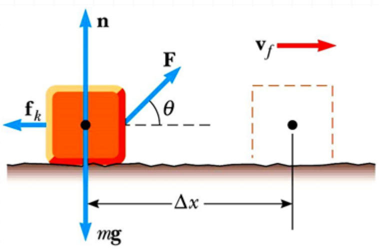
Case 3: set system = block + Earth+ Ramp



$$W_{\text{ext}} = Fd = \Delta K + \Delta U_g + \Delta E_{\text{int}}$$

$$\Rightarrow \quad Fd = \left(\frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2\right) - mgd\sin\theta + f_k d$$

Ex: At what angle should the block achieve the largest speed?



System = block

$$W_{\text{ext}} = \Delta K$$

$$\begin{aligned} W_{\text{ext}} &= F\cos\theta\Delta x - f_k\Delta x \\ &= F\cos\theta\Delta x - \mu_k N\Delta x \\ &= F\cos\theta\Delta x - \mu_k(mg - F\sin\theta)\Delta x \end{aligned}$$

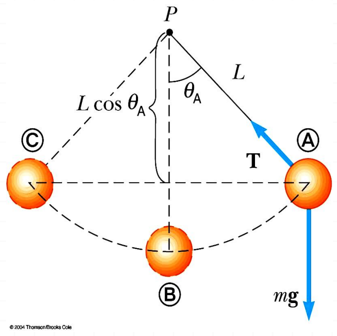
$$\Delta K = \frac{1}{2}mv_f^2 = F\cos\theta\Delta x - \mu_k(mg - F\sin\theta)\Delta x$$

$$v_f \ \ \text{max} \ \ \Rightarrow \ \ \Delta K \ \ \text{max}$$

$$\frac{d(\Delta K)}{d\theta} = 0 \ \ \Rightarrow \ \ -F\sin\theta\Delta x - \mu_k(0 - F\cos\theta)\Delta x = 0$$

$$\Rightarrow \theta = \tan^{-1}\mu_k$$

Ex:



$v_B = ?$

$T_B = ?$

System = ball + Earth

$$W = \Delta K + \Delta U$$

$$(1)\quad \underbrace{\int \vec{T}\cdot d\vec{r}}_{=0,\ \because \vec{T}\perp d\vec{r}} = (\frac{1}{2}mv_B^2 - 0)$$

$$+[(-mgL)-(-mgL\cos\theta_A)]$$

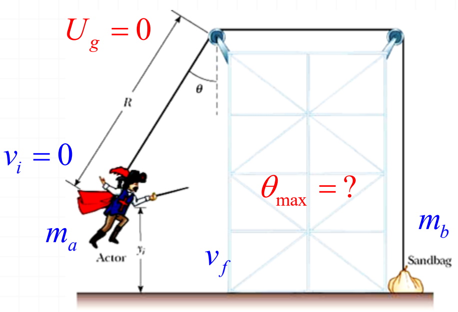
$$\Rightarrow \quad v_B = \sqrt{2gL(1-\cos\theta_A)}$$

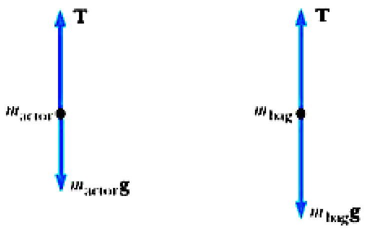
$$(2)\quad \sum \vec{F} = m\vec{a}$$

$$T_B - mg = ma_c = m\frac{v_B^2}{r}$$

$$\Rightarrow \quad T_B = mg(3-2\cos\theta_A)$$

Ex:





$$m_b g \geq T$$

$$\sum \vec{F} = m\vec{a} \quad \Rightarrow T - m_a g = m_a\frac{v_f^2}{R}$$

System = actor + Earth

$$\Delta K + \Delta U_g = 0$$

$$\Rightarrow (\frac{1}{2}mv_f^2 - 0) + m_a g[(-R)-(-R\cos\theta)] = 0$$

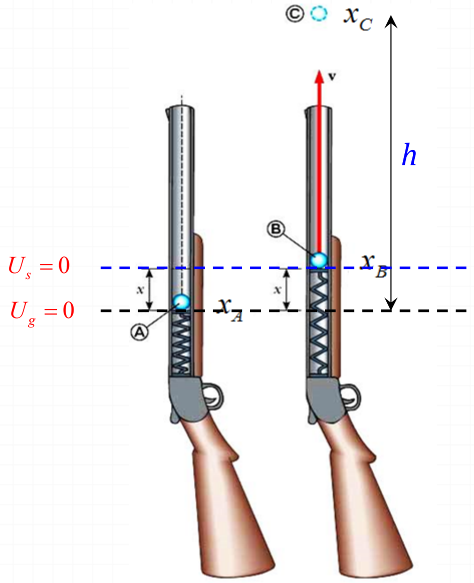
$$\Rightarrow \quad T = 3m_a g - 2m_a g\cos\theta$$

$$m_b g \geq T = 3m_a g - 2m_a g\cos\theta$$

$$\Rightarrow \quad \cos\theta \geq \frac{3m_a - m_b}{2m_a}$$

$$\Rightarrow \quad \theta \leq \cos^{-1}\left(\frac{3m_a - m_b}{2m_a}\right)$$

Ex:



(A) The spring constant k = ?

(B) v_B = ?

System = bullet + spring + Earth

$$\text{(A)}\ W = \Delta K + \Delta U_s + \Delta U_g$$

$$0 = (\underbrace{K_C}_{=0} - \underbrace{K_A}_{=0}) + (\underbrace{U_{s,C}}_{=0} - \underbrace{U_{s,A}}_{=\frac{1}{2}kx^2}) + (\underbrace{U_{g,C}}_{=mgh} - \underbrace{U_{g,A}}_{=0}) = 0$$

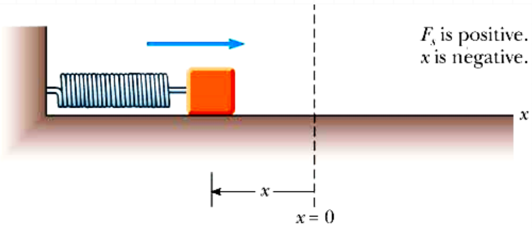
$$\Rightarrow \quad k = \frac{2mgh}{x^2}$$

(B)

$$0 = (\underbrace{K_B}_{=\frac{1}{2}mv_B^2} - \underbrace{K_A}_{=0}) + (\underbrace{U_{s,B}}_{=0} - \underbrace{U_{s,A}}_{=\frac{1}{2}kx^2}) + (\underbrace{U_{g,B}}_{=0} - \underbrace{U_{g,A}}_{=-mgx}) = 0$$

$$\Rightarrow \quad v_B = \sqrt{\frac{kx^2}{m} - 2gx}$$

Ex:



(1) No friction

(2) With friction f_k

$v_i(-x) = 0$ $\qquad v_f(x=0) = ?$

System = block+spring

$$W_{\text{ext}} = \Delta K + \Delta U_s$$

$$(1)\quad 0 = \left(\frac{1}{2}mv_f^2 - 0\right) + \left(0 - \frac{1}{2}kx^2\right) \quad \Rightarrow \quad v_f = \sqrt{\frac{k}{m}}x$$

$$(2)\quad -f_k x = \left(\frac{1}{2}mv_f^2 - 0\right) + \left(0 - \frac{1}{2}kx^2\right) \quad \Rightarrow \quad v_f = \sqrt{\frac{x}{m}(kx - f_k)}$$

Alternative method:

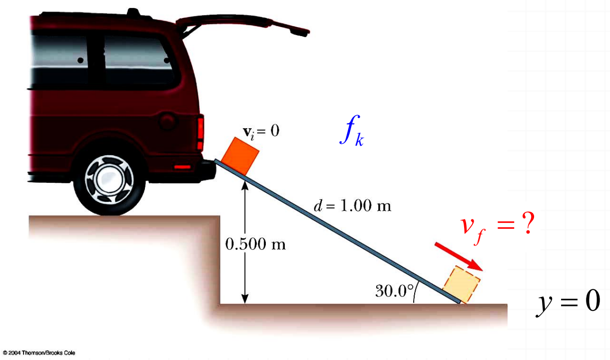
System = block+spring+Earth

$$W_{\text{ext}} = \Delta K + \Delta U_s + \Delta E_{\text{int}}$$

$$(2)\quad 0 = \left(\frac{1}{2}mv_f^2 - 0\right) + \left(0 - \frac{1}{2}kx^2\right) + f_k x$$

$$\Rightarrow \quad v_f = \sqrt{\frac{x}{m}\left(kx - f_k\right)}$$

Ex:

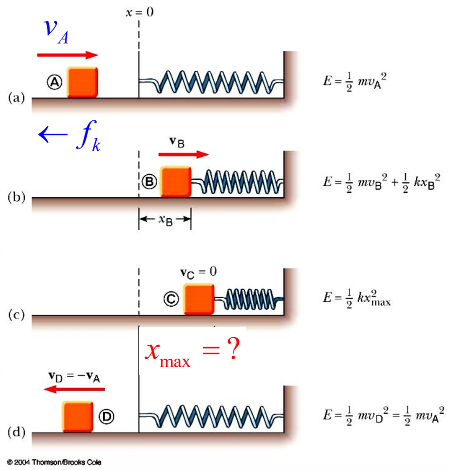


System = block+Earth

$$W_{ext} = \Delta K + \Delta U_g$$

$$-f_k d = \left(\frac{1}{2}mv_f^2 - 0\right) + \left(0 - mgy_i\right) \quad \Rightarrow \quad v_f = \sqrt{2(mgy_i - f_k d)/m}$$

Ex:



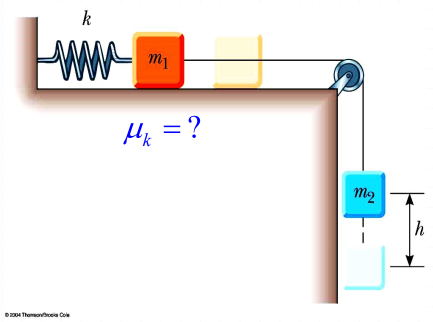
<u>System = block+Spring</u>

$$W_{ext} = \Delta K + \Delta U_s$$

$$-f_k x_{max} = \left(0 - \frac{1}{2} m v_A^2\right) + \left(\frac{1}{2} k x_{max}^2 - 0\right)$$

$$\Rightarrow x_{max} = \frac{-f_k + \sqrt{f_k^2 + mkv_A^2}}{k}$$

Ex:



<u>System = block+Earth+Spring</u>

$$W_{ext} = \Delta K + \Delta U_g + \Delta U_s$$

$$-f_k h = (0-0) + (-m_2 g h - 0) + \left(\frac{1}{2} k h^2 - 0\right)$$

$$-\mu_k m_1 g h = -m_2 g h + \frac{1}{2} k h^2$$

$$\Rightarrow \mu_k = \frac{m_2 g - \frac{1}{2} k h}{m_1 g}$$

https://fb.watch/8S0_2aYAoJ/

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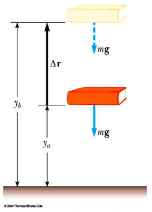
6. Conservative and nonconservative forces

Conservative Forces

- The work done by **a conservative force** on a particle moving between any two points has the following features:
  1. Independent of the path; dependent on the starting and ending points.
  2. The work done through any closed path is zero.
  3. Reversible.
  4. Can be expressed as the difference between the initial and final values of a potential energy function.

Conservative Forces and Potential Energy

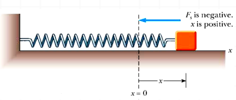
- The work done by a "conservative force" equals the decrease in the "potential energy" of the system.



$$W_g = \int_{r_i}^{r_f} \vec{F}_g \cdot d\vec{r} = \int_{r_i}^{r_f} \left(-mg\,\hat{j}\right)\cdot\left(dx\hat{i} + dy\hat{j} + dz\hat{k}\right)$$

$$= \int_{y_i}^{y_f} -mgdy = -mg\left(y_f - y_i\right)$$

$$\boxed{\Rightarrow \quad W_g = -\Delta U_g}$$



$$W_s = \int_{x_i}^{x_f} (-kx)\cdot dx = -\left(\frac{1}{2}kx_f^2 - \frac{1}{2}kx_i^2\right)$$

$$\boxed{\Rightarrow \quad W_s = -\Delta U_s}$$

- The conservative force is related to the potential energy function through $F_c = -\dfrac{dU(x)}{dx}$.

$$W_c = \int_{x_i}^{x_f} F_c dx = -\Delta U = U_i - U_f = \int_{x_i}^{x_f} (-) dU$$

$$\Rightarrow \quad U_f(x) = -\int_{x_i}^{x_f} F_c dx + U_i \quad \Rightarrow \quad dU(x) = -F_c dx$$

$$\Rightarrow \quad F_c = -\frac{dU(x)}{dx}$$

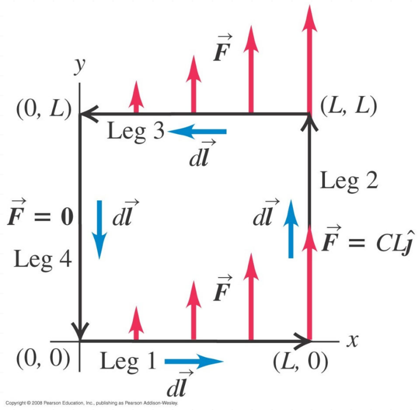
e.g., $U_s = \dfrac{1}{2}kx^2 \quad \Rightarrow \quad F_s = -\dfrac{dU_s}{dx} = -kx$

$U_g = mgy \quad \Rightarrow \quad F_g = -\dfrac{dU_g}{dy} = -mg$

For 3 dimensions,

$$\vec{F}_c = -\left(\frac{\partial U(x,y,z)}{\partial x}\hat{i} + \frac{\partial U(x,y,z)}{\partial y}\hat{j} + \frac{\partial U(x,y,z)}{\partial z}\hat{k} \right)$$
$$= -\nabla U(x,y,z)$$

Ex. $\vec{F} = Cx\hat{j}$, C is a constant. Work = ? Is F a conservative force?



$$W = \oint \vec{F} \cdot d\vec{\ell}$$

$$= \int_{(0,0)}^{(L,0)} \vec{F} \cdot d\vec{\ell} + \int_{(L,0)}^{(L,L)} \vec{F} \cdot d\vec{\ell} + \int_{(L,L)}^{(0,L)} \vec{F} \cdot d\vec{\ell} + \int_{(0,L)}^{(0,0)} \vec{F} \cdot d\vec{\ell}$$

$$= \int_0^L Cx\hat{j} \cdot dx\hat{i} + \int_0^L CL\hat{j} \cdot dy\hat{j} + \int_L^0 Cx\hat{j} \cdot dx\hat{i} + \int_L^0 C0\,\hat{j} \cdot dy\hat{j}$$

$$= 0 + CL^2 + 0 + 0$$

$$= CL^2$$

$W = \oint \vec{F} \cdot d\vec{\ell} \neq 0$, NOT conservative!

Nonconservative Forces

- A **nonconservative force** does not satisfy the conditions of conservative forces.
- Nonconservative forces acting in a system cause a *change* in the mechanical energy of the system
  - E.g., friction force: $W_{f_k} = -f_k d = -\Delta E_{\text{int}}$
- If the force acting on objects "within" a system is not "conservative", then the mechanical energy of the system is NOT conserved.

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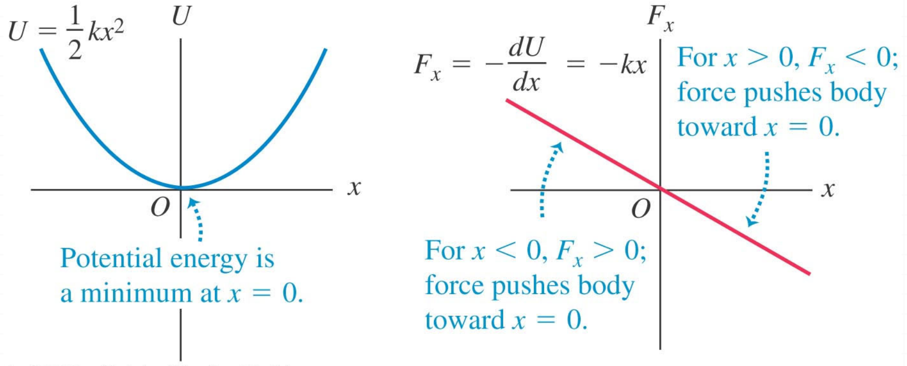
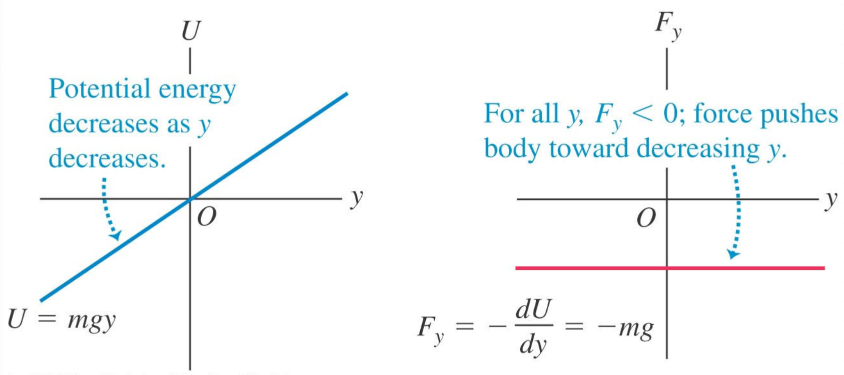
(a) Spring potential energy and force as functions of x



(b) Gravitational potential energy and force as function of y



😎 A conservative always acts to push the system toward lower potential energy!

- Stable equilibrium:

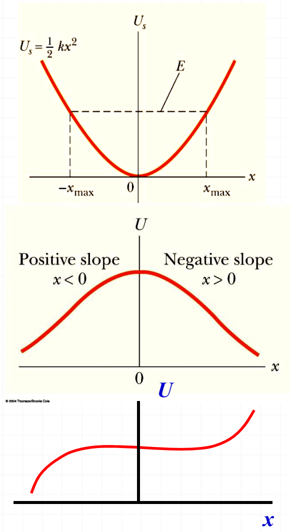
  → $U(x)$ has a minimum.

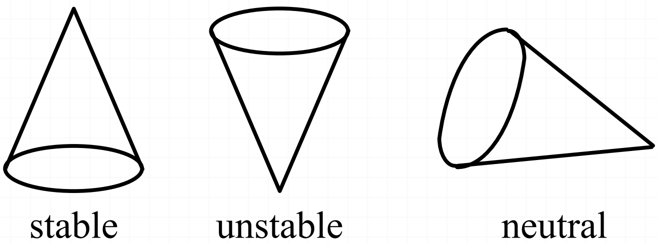
- Unstable equilibrium:

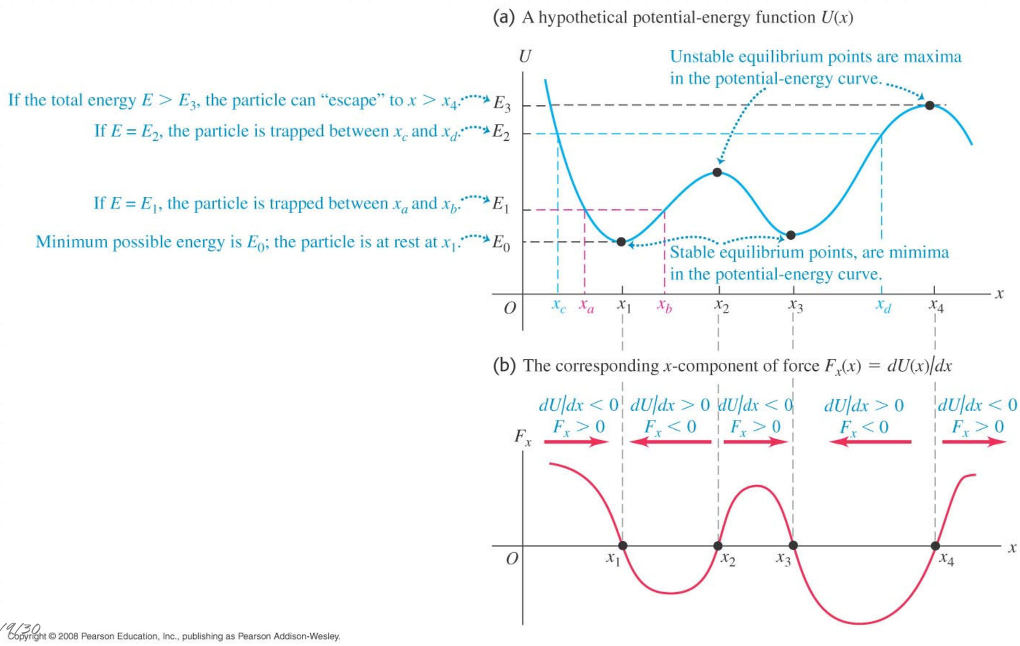
  → U(x) has a maximum.

- Neutral equilibrium:

  → U(x) is constant over a small regime.



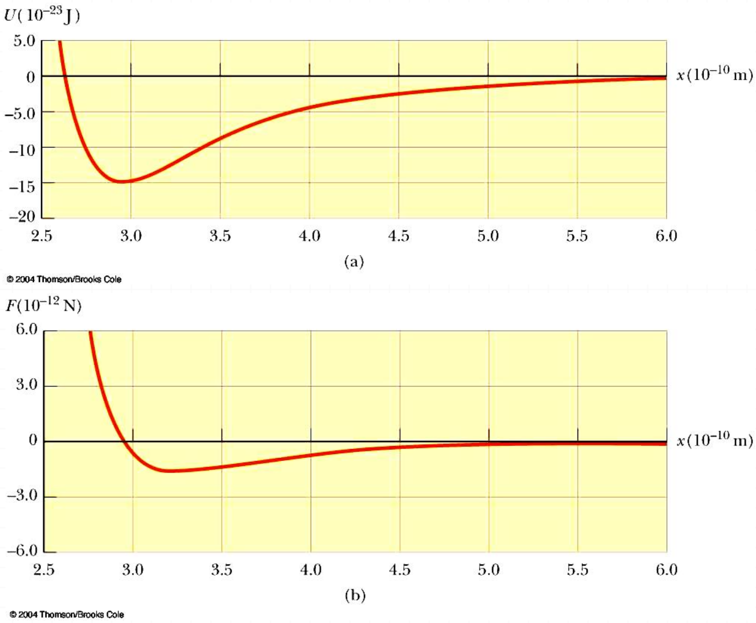




Potential Energy in Molecules

- There is potential energy associated with the force between two neutral atoms in a molecule which can be modeled by the *Lennard-Jones* function

$$U(x) = 4\varepsilon\left[\left(\frac{\sigma}{x}\right)^{12} - \left(\frac{\sigma}{x}\right)^{6}\right]$$



$$U(x) = 4\varepsilon\left[\left(\frac{\sigma}{x}\right)^{12} - \left(\frac{\sigma}{x}\right)^{6}\right]$$

$$\frac{dU(x)}{dx} = 4\varepsilon\frac{d}{dx}\left[\left(\frac{\sigma}{x}\right)^{12} - \left(\frac{\sigma}{x}\right)^{6}\right] = 4\varepsilon\left[\frac{-12\sigma^{12}}{x^{13}} + \frac{6\sigma^{6}}{x^{7}}\right]$$

$$\text{Energy minimum: } 4\varepsilon\left[\frac{-12\sigma^{12}}{x^{13}} + \frac{6\sigma^{6}}{x^{7}}\right] = 0$$

$$\Rightarrow x_{eq} = 2^{1/6}\sigma$$