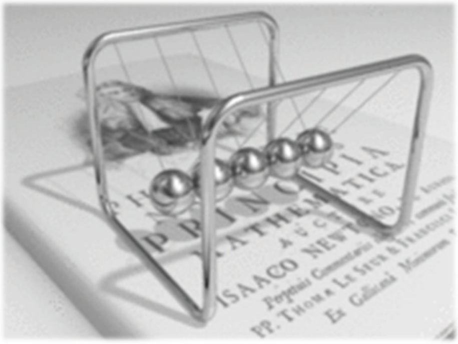
國立交通大學 電子物理系
NCTU Electrophysics

Chapter 8

Momentum, Impulse, and Collisions



楊 本 立 副教授

Outline

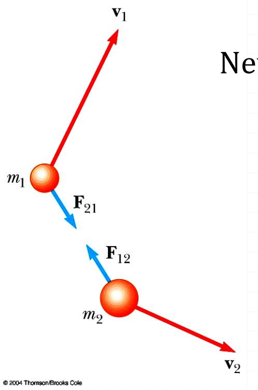
1. Linear momentum, Impulse, and Momentum conservation
2. Problems of collisions
   - 1D head-on elastic collisions
   - 1D head-on perfect inelastic collisions
   - 2D elastic glancing collisions
3. The center of mass
4. Rocket propulsion

1. Linear momentum, Impulse, and Momentum conservation

- Motion: $\vec{r},\ \vec{v},\ \vec{a},\ t$
- Force:
  - Newton's 2nd law $\Rightarrow$ $\vec{a} = \vec{F} / m$
- Energy:
  - Work-Kinetic energy theorem: $\underbrace{\vec{F}\cdot\Delta\vec{r}}_{\equiv W} = m\underbrace{(\vec{a}\cdot\Delta\vec{r})}_{=(v_f^2 - v_i^2)/2} = \Delta K$, a scalar
  - Energy conservation: $[W + Q + T_{\text{others}} = \Delta K + \Delta U + \Delta E_{\text{int}}]$
- Momentum: $\vec{F}\Delta t = m\underbrace{(\vec{a}\Delta t)}_{=\vec{v}_f - \vec{v}_i} \equiv \Delta\vec{P}$, a vector
  - Newton's 3rd law $\Rightarrow$ Momentum conservation

Linear Momentum

An isolated system: m_1+m_2



Newton's 3rd law: $\vec{F}_{12} = -\vec{F}_{21}$

$$\Rightarrow \vec{F}_{21} + \vec{F}_{12} = 0$$

$$\Rightarrow m_1\vec{a}_1 + m_2\vec{a}_2 = 0$$

$$\Rightarrow m_1\frac{d\vec{v}_1}{dt} + m_2\frac{d\vec{v}_2}{dt} = 0$$

$$m_1, m_2 \text{ constant } \Rightarrow \frac{d\left(m_1\vec{v}_1 + m_2\vec{v}_2\right)}{dt} = 0$$

For an isolated system, $\left(m_1\vec{v}_1 + m_2\vec{v}_2\right)$ is conserved!

- "Linear momentum" is defined as $\vec{P} \equiv m\vec{v}$.
- Conservation of momentum:

  For an isolated system, i.e., no external force, the linear momentum is conserved.

- Alternative form of Newton's 2nd law

$$\Sigma\mathbf{F} = m\mathbf{a} = m\frac{d\mathbf{v}}{dt} = \frac{d(m\mathbf{v})}{dt} = \frac{d\mathbf{p}}{dt}$$

- The time rate change of the linear momentum is equal to the net force.

Impulse

$$\vec{F} = \frac{d\vec{P}}{dt} \quad \Rightarrow \quad d\vec{P} = \vec{F}dt$$

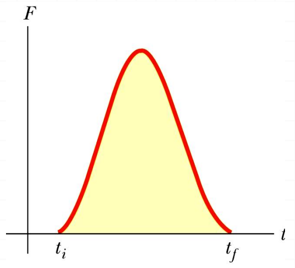
$$\Rightarrow \quad \Delta\vec{P} = \vec{P}_f - \vec{P}_i = \int_{t_i}^{t_f} \vec{F}dt$$

$$\text{Impulse: } I \equiv \Delta\vec{P} = \int_{t_i}^{t_f} \vec{F}dt$$

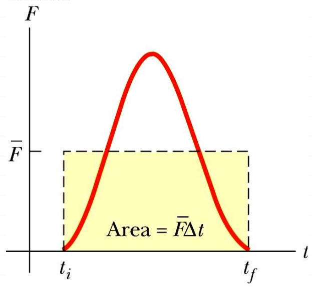
The impulse of the force acting on a particle equals the change in the momentum of the particle.

Impulse:

$$\vec{I} = \int_{t_i}^{t_f} \vec{F} dt$$
$$= \bar{\vec{F}} \Delta t$$



(a)



(b)

$$\bar{\vec{F}} = \frac{1}{\Delta t}\int_{t_i}^{t_f} \vec{F} dt = \frac{\Delta \vec{P}}{\Delta t}$$

The average force within Δt is called impulse force.

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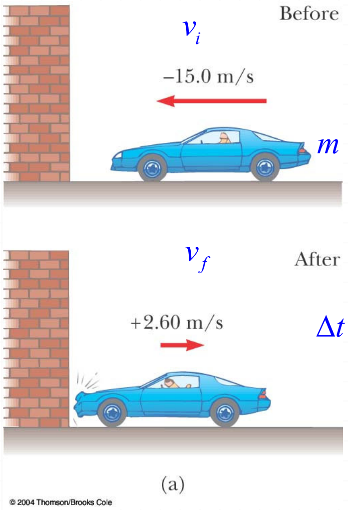
[https://tw.nextmgz.com/realtimenews/news/40719358]

$$I = \Delta P = \left(\bar{F}\downarrow\right)\left(\Delta t\uparrow\right)$$

潰縮式車體 → 以柔克剛

Ex.



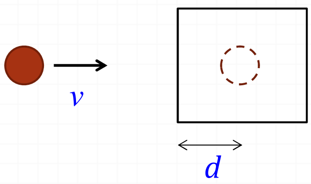
Impulse=?

$\bar{\vec{F}} = ?$

$$\vec{I} = \Delta\vec{P} = m\vec{v}_f - m\vec{v}_i$$

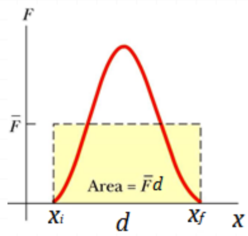
$$\bar{\vec{F}} = \frac{\Delta\vec{I}}{\Delta t} = \frac{\Delta\vec{P}}{\Delta t}$$

Ex.



(A) What is the impulse?

(B) What is the impulse force?



(A) $\vec{I} = \Delta\vec{P} = 0 - m\vec{v}_i = -mv\hat{i}$

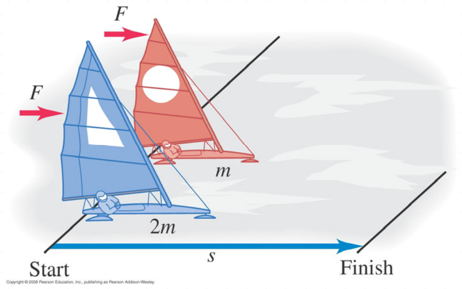
(B) $\bar{\vec{F}} = \dfrac{\vec{I}}{\Delta t}$ $\quad \Delta t = \dfrac{d}{\bar{v}} = \dfrac{d}{\frac{1}{2}(0+v)} = \dfrac{2d}{v}$

$\Rightarrow \bar{\vec{F}} = -\dfrac{mv^2}{2d}\hat{i}$

p.s., From work,

$$\bar{F}d = \frac{1}{2}mv^2 \Rightarrow \bar{F} = \frac{mv^2}{2d}$$

Ex: Which boat crosses the finish line with greater momentum?



$$Fs = \Delta K = \frac{p^2}{2m} \quad \Rightarrow \quad p \propto \sqrt{m}$$

Q: Which could potentially cause the greater injury:

1. being tackled by a light-weight, fast moving football player, or
2. being tackled by a player with double the mass but moving at half the speed?

https://news.tvbs.com.tw/world/1377938

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2020

一根細針若達光速　衝撞地球恐引大爆炸

記者 綜合 報導　2020/08/31 15:53

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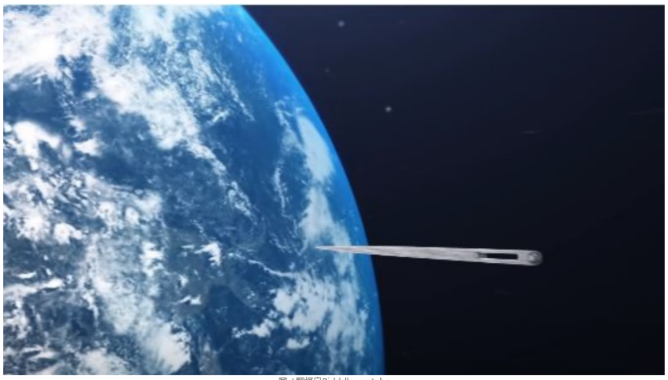

圖 / 翻攝自Ridddle youtube

物理學家告訴我們，任何有質量的物體都不可能達到光速，但如果今天真的有一根針從外太空以光速射向地球，後果恐怕不堪設想，但也有物理學家認為，這樣的假設根本不必擔心，因為慣性會讓這根針眨眼之間就離開地球。

天空中劃過的流星，或是偶爾出現在空中的火球，它們都是從外太空衝進地球的物體，多數時候他們會在大氣層燃燒殆盡。然而，你能想像一根針高速衝向地球，會帶來什麼後果嗎？

太空科技影片：「如果太空中有一根針加速到光速，接著撞上我們的星球。」

如果這樣的假設成真，光是小小一根針，甚至一粒沙，動能的結果可是超乎想像。

太空科技影片：「舉例來說，一粒沙子加速到近乎光速，它的動能等同100公噸的物體，從15層樓高的建築下墜。」

這樣的動能可是一整座自由女神像往下倒塌的強大能量。這對地球表面而言，恐怕是難以預料的災難。

太空科技影片：「這些動能會轉變成爆炸的力量，釋放出的能量等同核子彈爆炸，或是將近43千噸的TNT炸藥這很可觀。」

就連丟在日本長崎的原子彈「胖子」，都只有21千噸的威力，一根細細的針卻有如此威力，就連遠在4公里以外的建築窗戶都會震碎，一靠近爆炸中心，更會受到重傷，不過，倒是有另一派人這樣推測。

太空科技影片：「有的物理學家認為，因為太過強大的動能以及慣性，這根針只會穿過地球，而這一切會快到沒有人發覺。」

這派物理學家甚至還說，身體並不會完整吸收這樣的能量，因而爆炸，無論如何，根據物理學的相對論，任何有質量的物體不可能以光速移動，這樣的猜測自然也不必太過擔心。

水刀（Water jet cutter）

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2. Problems of collisions

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Problems of Collisions in one dimension

- **Collision**: strong interaction between bodies that lasts a relatively short time.
- For isolated systems, P is conserved after collision.
- K is not necessary conserved.
  - K_{total} conserved $\Rightarrow$ elastic collision
  - K_{total} not conserved $\Rightarrow$ inelastic collision

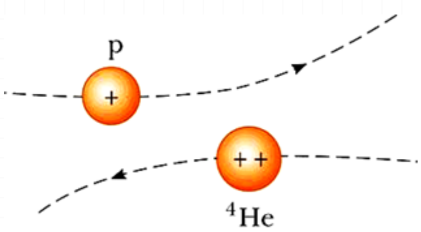
😎 For isolated systems,

Mechanical energy is conserved when the internal forces are conservative, but

Momentum conservation is valid even when the internal forces are not conservative.

Elastic collision:

- K is conserved.
- Truly elastic collisions occur between atomic and subatomic particles.



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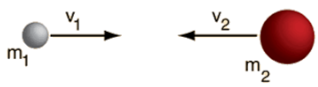
2. Problems of collisions

- 1D head-on elastic collisions

1-D head-on elastic collisions



(1) *P* conservation

(2) *K* conservation

$$\begin{cases} m_1\vec{v}_{1i}+m_2\vec{v}_{2i}=m_1\vec{v}_{1f}+m_2\vec{v}_{2f} & (1) \\ \frac{1}{2}m_1v_{1i}^2+\frac{1}{2}m_2v_{2i}^2=\frac{1}{2}m_1v_{1f}^2+\frac{1}{2}m_2v_{2f}^2 & (2) \end{cases}$$

$$\Rightarrow \begin{cases} m_1\left(\vec{v}_{1i}-\vec{v}_{1f}\right)=-m_2\left(\vec{v}_{2i}-\vec{v}_{2f}\right) \\ \frac{1}{2}m_1\left(v_{1i}^2-v_{1f}^2\right)=-\frac{1}{2}m_2\left(v_{2i}^2-v_{2f}^2\right) \end{cases}$$

$$\Rightarrow \vec{v}_{1i}-\vec{v}_{2i}=-(\vec{v}_{1f}-\vec{v}_{2f}) \quad (\text{i.e., } \boxed{\vec{v}_{12,i}=-\vec{v}_{12,f}}) \qquad (3)$$

$$(1),\ (3)\ \Rightarrow \begin{cases} \vec{v}_{1f} = \left(\dfrac{m_1 - m_2}{m_1 + m_2}\right)\vec{v}_{1i} + \left(\dfrac{2m_2}{m_1 + m_2}\right)\vec{v}_{2i} = 2\vec{v}_{CM} - \vec{v}_{1i} \\ \vec{v}_{2f} = \left(\dfrac{2m_1}{m_1 + m_2}\right)\vec{v}_{1i} - \left(\dfrac{m_1 - m_2}{m_1 + m_2}\right)\vec{v}_{2i} = 2\vec{v}_{CM} - \vec{v}_{2i} \end{cases}$$

$$\Rightarrow \vec{v}_{CM} = \frac{m_1\vec{v}_{1i} + m_2\vec{v}_{2i}}{m_1 + m_2} = \underbrace{\frac{\vec{v}_{1i} + \vec{v}_{1f}}{2}}_{\substack{\text{as if } \bar{\vec{v}}_1 \text{ in} \\ \text{constant acceleration.}}} = \underbrace{\frac{\vec{v}_{2i} + \vec{v}_{2f}}{2}}_{\bar{\vec{v}}_2}$$

$$(1)\ m_1 = m_2 \ \Rightarrow \begin{cases} \vec{v}_{1f} = \vec{v}_{2i} \\ \vec{v}_{2f} = \vec{v}_{1i} \end{cases}$$

$$(2)\ \vec{v}_{2i} = 0 \ \Rightarrow \begin{cases} \vec{v}_{1f} = \left(\dfrac{m_1 - m_2}{m_1 + m_2}\right)\vec{v}_{1i} \\ \vec{v}_{2f} = \left(\dfrac{2m_1}{m_1 + m_2}\right)\vec{v}_{1i} \end{cases}$$

$$a.\ m_1 \gg m_2 \ \Rightarrow \begin{cases} \vec{v}_{1f} \approx \vec{v}_{1i} \\ \vec{v}_{2f} \approx 2\vec{v}_{1i} \end{cases};\ \frac{P_{2f}}{P_{1i}} \approx 0\ ;\ \frac{K_{2f}}{K_{1i}} \approx 0 \qquad \Rightarrow v_{2f} \text{ is max.}$$

$$b.\ m_1 \ll m_2 \ \Rightarrow \begin{cases} \vec{v}_{1f} \approx -\vec{v}_{1i} \\ \vec{v}_{2f} \approx 0 \end{cases};\ \frac{P_{2f}}{P_{1i}} \approx 2\ ;\ \frac{K_{2f}}{K_{1i}} \approx 0 \qquad \Rightarrow P_{2f} \text{ is max.}$$

$$c.\ m_1 = m_2 \ \Rightarrow \begin{cases} \vec{v}_{1f} = 0 \\ \vec{v}_{2f} = \vec{v}_{1i} \end{cases};\ \frac{P_{2f}}{P_{1i}} = 1\ ;\ \frac{K_{2f}}{K_{1i}} = 1 \qquad \Rightarrow K_{2f} \text{ is max.}$$

Momentum and energy transfer for 1-D head-on elastic collisions

$$v_{2i} = 0 \;\Rightarrow\; \begin{cases} v_{1f} = \left(\dfrac{m_1 - m_2}{m_1 + m_2}\right) v_{1i} \\ v_{2f} = \left(\dfrac{2m_1}{m_1 + m_2}\right) v_{1i} \end{cases}$$

$$\begin{aligned} \vec{P}_{1i} &= m_1 \vec{v}_{1i} \\ \vec{P}_{1f} &= m_1 \vec{v}_{1f} \\ \vec{P}_{2f} &= m_2 \vec{v}_{2f} \end{aligned} \;\Rightarrow\; \begin{cases} \dfrac{P_{1f}}{P_{1i}} = \dfrac{m_1 - m_2}{m_1 + m_2} \\ \dfrac{P_{2f}}{P_{1i}} = \dfrac{2m_2}{m_1 + m_2} \end{cases}$$

$$\begin{aligned} K_{1i} &= \frac{1}{2} m_1 v_{1i}^2 \\ K_{1f} &= \frac{1}{2} m_1 v_{1f}^2 \\ K_{2f} &= \frac{1}{2} m_2 v_{2f}^2 \end{aligned} \;\Rightarrow\; \begin{cases} \dfrac{K_{1f}}{K_{1i}} = \left(\dfrac{m_1 - m_2}{m_1 + m_2}\right)^2 \\ \dfrac{K_{2f}}{K_{1i}} = \dfrac{4m_1 m_2}{(m_1 + m_2)^2} \end{cases}$$

Ex: Moderator in a nuclear reactor

Neutron collides elastically with Carbon at rest.

$$\frac{K_{1f}}{K_{1i}} = \left(\frac{m_n - m_C}{m_n + m_C}\right)^2$$

n collisions: $K_{nf} = \left(\dfrac{m_n - m_C}{m_n + m_C}\right)^{2n} K_{ni}$

Q: Which head-on collision could potentially cause the greater injury?

1. Two identical cars are moving at the same speed v toward each other.

2. One car is moving at speed $2v$ toward a stationary car.

(Assuming the same collision time.)

2. Problems of collisions

- 1D head-on perfectly inelastic collisions

Inelastic collision:

- K is NOT conserved.
- Inelastic collisions can also occur in atomic level, e.g., e^- collides with atoms which sometimes can be excited to a higher energy level.

"Perfectly" inelastic collision:

Why called "perfectly inelastic" collision?

$$\Delta P = 0 \quad \Rightarrow \quad P_{1i} + P_{2i} = P_{1f} + P_{2f}$$

$$\Delta K + \Delta E_{\text{int}} = 0 \quad \Rightarrow \quad \left(\frac{P_{1f}^2}{2m_1} - \frac{P_{1i}^2}{2m_1}\right) + \left(\frac{P_{2f}^2}{2m_2} - \frac{P_{2i}^2}{2m_2}\right) + \Delta E_{\text{int}} = 0$$

$$\Delta E_{\text{int}} = \frac{P_{1i}^2}{2m_1} - \frac{P_{1f}^2}{2m_1} + \frac{P_{2i}^2}{2m_2} - \frac{\left(P_{1i} + P_{2i} - P_{1f}\right)^2}{2m_2}$$

$$\text{Max } \Delta E_{\text{int}} \quad \Rightarrow \quad \frac{d\Delta E_{\text{int}}}{dP_{1f}} = -\frac{P_{1f}}{m_1} - \frac{1}{m_2}\left(P_{1i} + P_{2i} - P_{1f}\right)(-1) = 0$$

$$\Rightarrow \quad -\frac{P_{1f}}{m_1} + \frac{P_{2f}}{m_2} = 0 \qquad \Rightarrow \quad v_{1f} = v_{2f}$$

Therefore, after collision, both objects move in the same final velocity, if it is "perfectly inelastic".

Perfectly Inelastic Collisions

$$m_1\vec{v}_{1i} + m_2\vec{v}_{2i} = \left(m_1 + m_2\right)\vec{v}_f$$

$$\vec{v}_f = \frac{m_1\vec{v}_{1i} + m_2\vec{v}_{2i}}{m_1 + m_2}$$

p.s., If $m_1\vec{v}_{1i} + m_2\vec{v}_{2i} = 0$, i.e., $\vec{P}_{1i} = -\vec{P}_{2i}$

then all the kinetic energy is lost to other energy such as internal energy.

- The coefficient of Restitution,

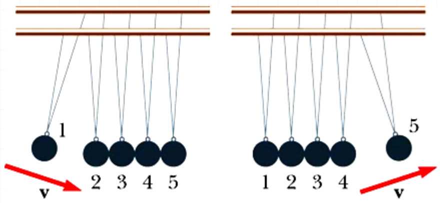
$$e \equiv \frac{\left|\vec{v}_{1f} - \vec{v}_{2f}\right|}{\left|\vec{v}_{1i} - \vec{v}_{2i}\right|} = \frac{\left|\vec{v}_{12,f}\right|}{\left|\vec{v}_{12,i}\right|}$$

- $0 \le e \le 1$

  e = 1, elastic collision

  e = 0, perfectly inelastic collision.

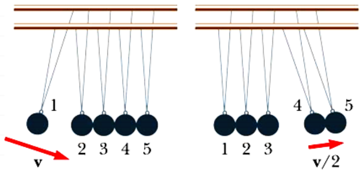
Newton's cradle



(Assuming elastic collision)

$$\begin{cases} m_1 v_1 = m_5 v_5 \\ \frac{1}{2} m_1 v_1^2 = \frac{1}{2} m_5 v_5^2 \end{cases} \quad \because \begin{matrix} m_1 = m_5 \\ v_1 = v_5 \end{matrix} \quad \Rightarrow \begin{cases} P_i = P_f \\ K_i = K_f \end{cases} \text{ is satisfied.}$$

Q: Can this happen?

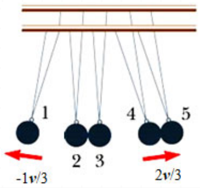


If m_4 and m_5 are moving together,

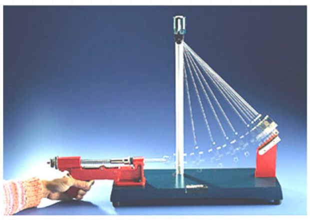
$$\begin{cases} m_1 v_1 = m_4 v_4 + m_5 v_5 \\ \frac{1}{2} m_1 v_1^2 = \frac{1}{2} m_4 v_4^2 + \frac{1}{2} m_5 v_5^2 \end{cases} \text{ with } \begin{cases} m_1 = m_4 = m_5 \\ v_1 = v,\ v_4 = v_5 = \frac{v}{2} \end{cases} \Rightarrow \begin{cases} P_i = P_f \text{ is satisfied, but} \\ K_i \neq K_f \text{ is not satisfied.} \end{cases}$$

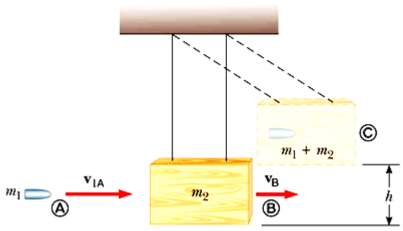
$$\begin{cases} m_1 v_{1i} = m_1 v_{1f} + (m_4 v_4 + m_5 v_5) \\ \frac{1}{2} m_1 v_{1i}^2 = \frac{1}{2} m_1 v_{1f}^2 + (\frac{1}{2} m_4 v_4^2 + \frac{1}{2} m_5 v_5^2) \end{cases} \text{ with } \begin{cases} m_1 = m_4 = m_5 \\ v_{1i} = v,\ v_4 = v_5 \end{cases}$$

$$\Rightarrow v_4 = v_5 = \frac{2}{3} v \text{ , } v_{1f} = -\frac{1}{3} v$$



Ex.





How to measure v_{1A} without using a timer?

$$\begin{cases} K_A = \frac{1}{2} m_1 v_{1A}^2 \\ K_B = \frac{1}{2} (m_1 + m_2) v_B^2 \\ U_C = (m_1 + m_2) gh \end{cases}$$

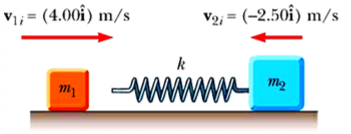
(1) $P_A = P_B$ Note: $K_A \neq K_B$

$$m_1 v_{1A} = (m_1 + m_2) v_B \Rightarrow v_{1A} = \frac{m_1 + m_2}{m_1} v_B$$

(2) $K_B = U_C$ $\frac{1}{2}(m_1 + m_2) v_B^2 = (m_1 + m_2) gh$

$$\Rightarrow v_{1A} = \frac{m_1 + m_2}{m_1} \sqrt{2gh}$$

Ex.

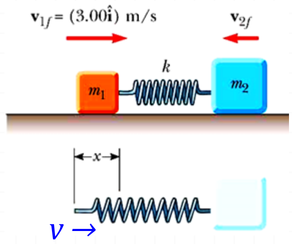


(1) $v_{1f} = ?\quad v_{2f} = ?$

1-D head-on elastic collision:

$$\begin{cases} m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f} \\ v_{1i} - v_{2i} = -(v_{1f} - v_{2f}) \end{cases}$$

$$\Rightarrow \begin{cases} v_{1f} = \left(\dfrac{m_1 - m_2}{m_1 + m_2}\right) v_{1i} + \left(\dfrac{2m_2}{m_1 + m_2}\right) v_{2i} \\ v_{2f} = \left(\dfrac{2m_1}{m_1 + m_2}\right) v_{1i} - \left(\dfrac{m_1 - m_2}{m_1 + m_2}\right) v_{2i} \end{cases}$$



(2) $v_{1f} = v\quad v_{2f} = ?\quad x = ?$

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f} \quad \Rightarrow v_{2f} = \frac{(m_1 v_{1i} + m_2 v_{2i} - m_1 v)}{m_2}$$

System= m_1+spring+m_2 $\quad \Delta K + \Delta U = 0$

$$\left(\frac{1}{2} m_1 v_{1i}^2 + \frac{1}{2} m_2 v_{2i}^2\right) + 0 = \left(\frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2\right) + \frac{1}{2} k x^2$$

$$x = \sqrt{\frac{\left(m_1 v_{1i}^2 + m_1 v_{2i}^2\right) - \left(m_1 v^2 + m_1 v_{2f}^2\right)}{k}}$$

(3) $x_{\max} = ?$

$x_{\max} \Rightarrow$ Perfectly inelastic

$$m_1\vec{v}_{1i} + m_2\vec{v}_{2i} = (m_1 + m_2)\vec{v}_f$$

$$\vec{v}_f = \frac{m_1\vec{v}_{1i} + m_2\vec{v}_{2i}}{m_1 + m_2}$$

System= m_1+spring+m_2 $\quad \Delta K + \Delta U = 0$

$$\left(\frac{1}{2}m_1v_{1i}^2 + \frac{1}{2}m_2v_{2i}^2\right) + 0 = \frac{1}{2}(m_1 + m_2)v_f^2 + \frac{1}{2}kx_{\max}^2$$

$$\Rightarrow x_{\max} = \sqrt{\frac{(m_1v_{1i}^2 + m_2v_{2i}^2) - (m_1 + m_2)v_f^2}{k}}$$

Ex. When two atoms combine to form a H_2 molecule, the potential energy is $-\Delta$, $\Delta>0$ called the binding energy. Show that a collision of two H atoms cannot form a H_2 molecule.

In an inertial frame,

$$\begin{cases} mv_{1i} + mv_{2i} = 2mv_f \\ \frac{1}{2}mv_{1i}^2 + \frac{1}{2}mv_{2i}^2 = \frac{1}{2}2mv_f^2 - \Delta \end{cases}$$

$$\frac{1}{2}mv_{1i}^2 + \frac{1}{2}mv_{2i}^2 - m\left(\frac{v_{1i} + v_{2i}}{2}\right)^2 = -\Delta$$

$$\Rightarrow \quad \Delta = -\frac{1}{4}m(v_{1i} - v_{2i})^2 \le 0 \quad \text{violates } \Delta > 0.$$

In the CM frame,

$$\frac{1}{2}m(v_{1i} - v_{CM})^2 + \frac{1}{2}m(v_{2i} - v_{CM})^2 = \underbrace{\frac{1}{2}m(v_{1f} - v_{CM})^2 + \frac{1}{2}m(v_{2f} - v_{CM})^2}_{=0 \quad \because v_{1i}=v_{2i}=v_{CM}} - \Delta$$

$$\Rightarrow \quad \Delta = -\left[\frac{1}{2}m(v_{1i} - v_{CM})^2 + \frac{1}{2}m(v_{2i} - v_{CM})^2\right] \le 0 \quad \text{violates } \Delta > 0.$$

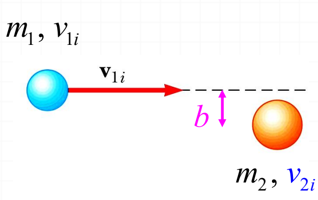
1. Linear momentum, Impulse, and Momentum Conservation
2. Problems of collisions
   - 1D head-on elastic collisions
   - 1D head-on perfect inelastic collisions
   - 2D elastic glancing collisions
3. The center of mass
4. Rocket propulsion

3. Problems of collisions

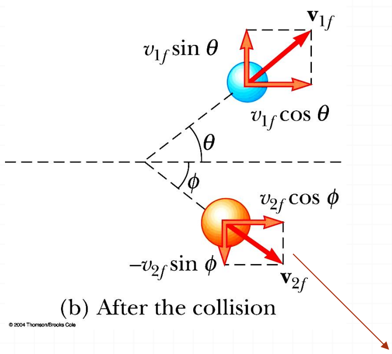
- 2D elastic glancing collisions

Problems of Collisions in two dimension

2-D elastic glancing collisions



(a) Before the collision



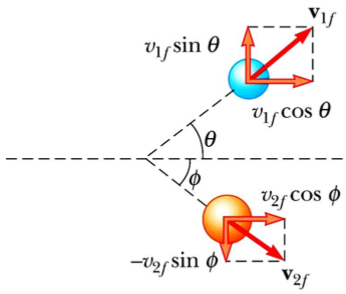
(b) After the collision

b : impact parameter

$$P_i = P_f \qquad \begin{cases} m_1 v_{1i} = m_1 v_{1f}\cos\theta + m_2 v_{2f}\cos\phi \\ 0 = m_1 v_{1f}\sin\theta - m_2 v_{2f}\sin\phi \end{cases}$$

$$K_i = K_f \qquad \frac{1}{2}m_1 v_{1i}^2 = \frac{1}{2}m_1 v_{1f}^2 + \frac{1}{2}m_2 v_{2f}^2$$

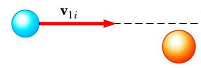
$$\Rightarrow \quad (v_{1f},\ v_{2f},\ \theta,\ \phi)$$



- Not enough information to solve
- Need to know the impact parameter;

(using angular momentum conservation)

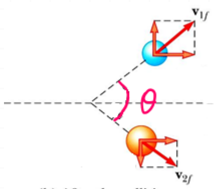
The dependance of θ on m_1 and m_2



$v_{2i} = 0$

(a) Before the collision

$$\begin{cases} \vec{P}_{1i} = \vec{P}_{1f} + \vec{P}_{2f} \\ \dfrac{P_{1i}^2}{2m_1} = \dfrac{P_{1f}^2}{2m_1} + \dfrac{P_{2f}^2}{2m_2} \end{cases}$$



(b) After the collision

$$\frac{1}{2m_1}\vec{P}_{1i}\cdot\vec{P}_{1i} = \frac{1}{2m_1}\left(\vec{P}_{1f}+\vec{P}_{2f}\right)\cdot\left(\vec{P}_{1f}+\vec{P}_{2f}\right) = \frac{P_{1f}^2}{2m_1} + \frac{P_{2f}^2}{2m_2}$$

$$\Rightarrow \cancel{\frac{P_{1f}^2}{2m_1}} + \frac{P_{2f}^2}{2m_1} + \frac{2}{2m_1}\vec{P}_{1f}\cdot\vec{P}_{2f} = \cancel{\frac{P_{1f}^2}{2m_1}} + \frac{P_{2f}^2}{2m_2}$$

$$\Rightarrow P_{2f}^2 + 2\vec{P}_{1f}\cdot\vec{P}_{2f} = \frac{m_1}{m_2}P_{2f}^2$$

$$\Rightarrow \frac{m_1}{m_2} = 1 + 2\frac{\left|\vec{P}_{1f}\right|}{\left|\vec{P}_{2f}\right|}\cos\theta$$

$$\Rightarrow \begin{cases} m_1 = m_2 & \Rightarrow \quad \theta = 90^0 \\ m_1 > m_2 & \Rightarrow \quad \theta < 90^0 \\ m_1 < m_2 & \Rightarrow \quad \theta > 90^0 \end{cases}$$

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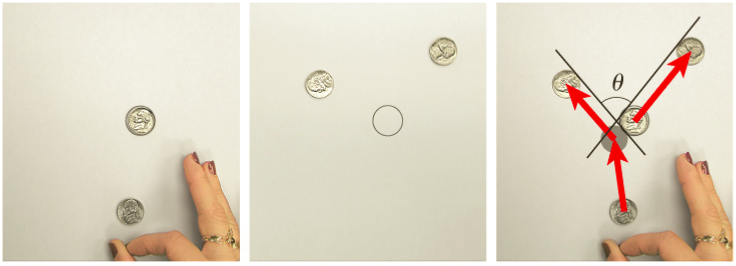

(a) (b) (c)

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1. Linear momentum, Impulse, and Momentum Conservation
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4. Rocket propulsion

3. The center of mass

The center of mass

- The system moves as if the resultant external force were applied to a single particle of mass located at the center of mass (CM). The motion of CM is independent of other components' motion, such as vibration, rotation, etc.
  - It is the mass-weighted average position of the system's mass.
  - If the force is applied at CM, the system moves in the direction of the force without rotating.
- The CM of any symmetric object lies on an axis of symmetry and on any plane of symmetry, but not necessary within the object.

(A) Discrete object:

$$\vec{r}_{CM} \equiv \frac{\sum_i m_i \vec{r}_i}{M}, \quad M = \sum_i m_i$$

(B) Continuous object:

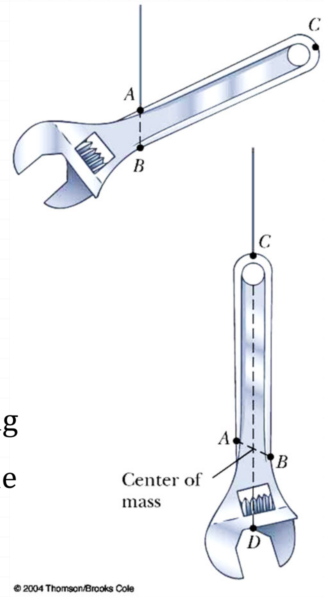
$$\vec{r}_{CM} \equiv \frac{\int \vec{r} dm}{M}, \quad M = \int dm, \quad m = m(\vec{r})$$

Center of Gravity

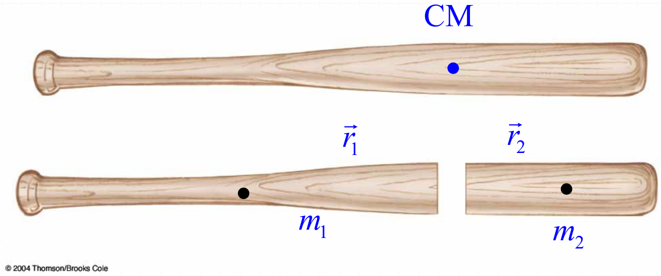
- The net force is equivalent to the effect of a single force Mg acting through a special point, called the center of gravity.
- If g is constant over the mass distribution, the center of gravity coincides with the center of mass.
- If an object is pivoted at its center of gravity, it balances in any direction.

Determine the CM

The CM aligns with the hanging direction in order to reduce the system potential energy.



Ex.



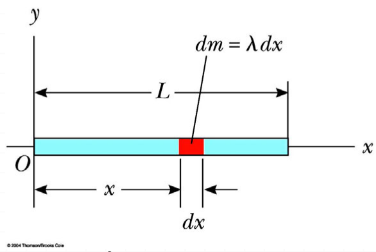
$$m_1 \overset{>}{\underset{<}{=}} m_2\ ?$$

$$\vec r_{CM} = \frac{m_1\vec r_1 + m_2\vec r_2}{m_1+m_2}$$

Set $\vec r_{CM} = 0 \quad\Rightarrow\quad 0 = m_1\vec r_1 + m_2\vec r_2$

$$\Rightarrow\quad \frac{m_1}{m_2} = \frac{|\vec r_2|}{|\vec r_1|} < 1 \qquad \text{so } m_1 < m_2$$

Ex.



Find CM=?

(1) $\lambda = \dfrac{M}{L}$ (λ : mass density)

(2) $\lambda = \lambda(x) = \alpha x$

(1) $$x_{CM} = \frac{\int x\,dm}{M} = \frac{1}{M}\int_0^L x\lambda dx = \frac{\lambda}{2}\frac{x^2}{M}\Big|_0^L = \frac{\lambda}{2}\frac{L^2}{M}$$

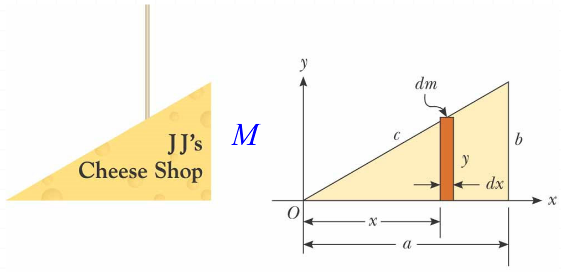
$$\lambda = \frac{M}{L} \Rightarrow x_{CM} = \frac{L}{2}$$

(2) $$x_{CM} = \frac{\int_0^L x(\alpha x)dx}{M} = \frac{\alpha x^3}{3M}\Big|_0^L = \frac{\alpha L^3}{3M}$$

$$M = \int dm = \int_0^L \alpha x dx = \frac{\alpha L^2}{2}$$

$$\Rightarrow\quad x_{CM} = \frac{2}{3}L$$

Ex.



(a) (b)

CM=?

$$x_{CM} = \frac{1}{M}\int x dm = \frac{1}{M}\iint x\sigma dxdy$$

$$= \frac{1}{M}\int_{x=0}^{a}\sigma x dx\left(\int_{y=0}^{y=\frac{b}{a}x} dy\right)$$

$$= \frac{1}{M}\int_{0}^{a} x\left(\frac{2Mx}{a^2}\right)dx = \frac{2}{3}a$$

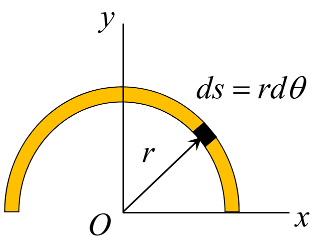
$$\sigma = \frac{M}{\frac{1}{2}ab}$$

$$\text{Slope} = \frac{y}{x} = \frac{b}{a} \Rightarrow y = \frac{b}{a}x$$

Indep. of b

😎 CM can be outside of the object.

e.g.,



$$M\vec{r}_{CM} = \int \vec{r} dm = \int (x\hat{i} + y\hat{j})dm$$

$$= \int (r\cos\theta\hat{i} + r\sin\theta\hat{j})dm$$

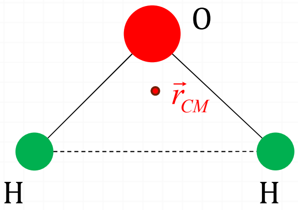
$$dm = \lambda ds = \lambda r d\theta \qquad = \int_0^{\pi}(r\cos\theta\hat{i} + r\sin\theta\hat{j})\lambda r d\theta$$

$$\vec{r}_{CM} = \frac{r^2}{M}\int_0^{\pi}(\cos\theta\hat{i} + \sin\theta\hat{j})\lambda d\theta \quad \Rightarrow \quad \vec{r}_{CM} = \frac{2r}{\pi}\hat{j}$$

$$\lambda = M/(\pi r)$$

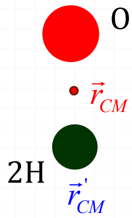
Ex.

H_2O

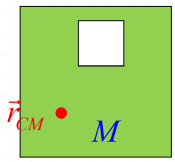


$$M\vec{r}_{CM} = \underbrace{m_H\vec{r}_1 + m_H\vec{r}_2} + m_O\vec{r}_O$$

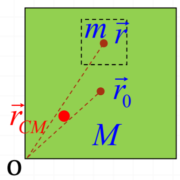
$$= (2m_H)\vec{r}\,'_{CM} + m_O\vec{r}_O$$



Ex.

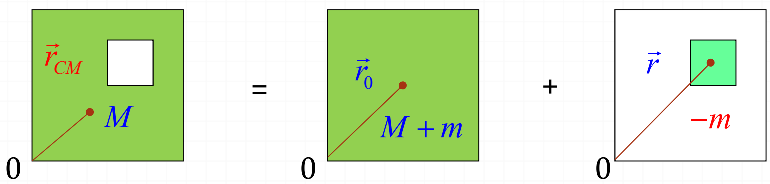


CM=?



$$\Rightarrow \vec{r}_0 = \frac{M\vec{r}_{CM} + m\vec{r}}{M+m}$$

$$\Rightarrow \quad \vec{r}_{CM} = \frac{(M+m)\vec{r}_0 + (-m)\vec{r}}{(M+m)+(-m)}$$



Motion of multiple particles

(1) Position: $\vec{r}_{CM} \equiv \dfrac{\sum_i m_i \vec{r}_i}{M}, \quad M = \sum_i m_i$

(2) Velocity: $\vec{v}_{CM} = \dfrac{d\vec{r}_{CM}}{dt} = \dfrac{1}{M}\sum_i m_i \left(\dfrac{d\vec{r}_i}{dt}\right) = \dfrac{\sum_i m_i \vec{v}_i}{M}$

(3) Momentum: $\Rightarrow \quad M\vec{v}_{CM} = \sum_i m_i \vec{v}_i = \sum_i \vec{P}_i = \vec{P}_{tatal}$

$\Rightarrow$ Total linear momentum of a system equals the total mass multiplied by the CM velocity.

(4) Acceleration: $\vec{a}_{CM} = \dfrac{d\vec{v}_{CM}}{dt} = \dfrac{1}{M}\sum_i m_i \left(\dfrac{d\vec{v}_i}{dt}\right) = \dfrac{1}{M}\sum_i \underbrace{m_i \vec{a}_i}_{\vec{F}_i}$

$$M\vec{a}_{CM} = \sum_i \vec{F}_i = \sum_i \left(\vec{F}_{i,\text{int}} + \vec{F}_{i,\text{ext}}\right)$$

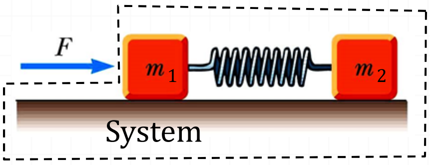
$$= \sum_i \Big(\sum_{\substack{j \\ j\neq i}} \vec{F}_{ji}\Big) + \sum_i \vec{F}_{i,\text{ext}}$$

$$= \frac{1}{2}\sum_i \Big(\sum_{\substack{j \\ i\neq j}} \underbrace{\left(\vec{F}_{ij} + \vec{F}_{ji}\right)}_{\substack{=0, \\ \because \vec{F}_{ji} = -\vec{F}_{ij} \\ \text{Newton's 3rd law}}}\Big) + \sum_i \vec{F}_{i,\text{ext}}$$

(5) Force: $\Rightarrow \quad \vec{F}_{\text{total}} = \sum_i \left(\vec{F}_{i,\text{ext}}\right) = M\vec{a}_{CM} = \vec{F}_{CM}$

$\Rightarrow$ The net external force on a system equals the total mass multiplied by the acceleration of CM.

Q:



$$\vec{F}_{CM} = ?$$

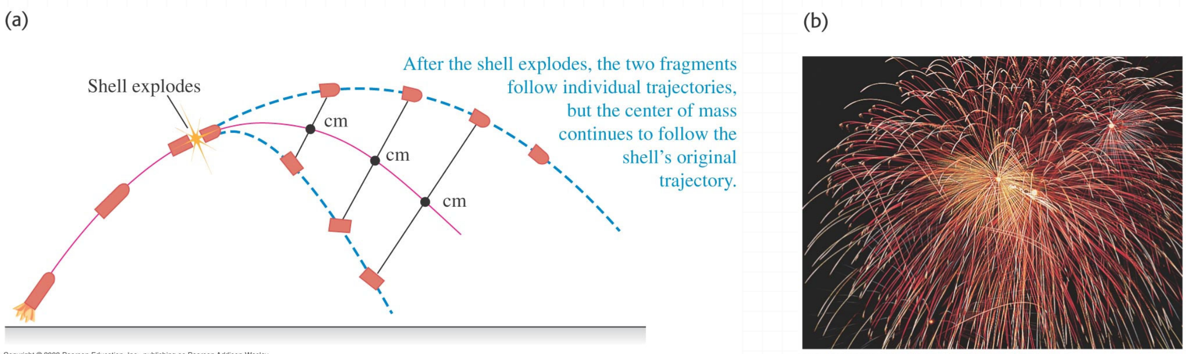
$$\vec{F}_{total} = ?$$

$$\vec{F}_{total} = \sum_{i=1}^{n} \vec{F}_{i,ext} = \underbrace{\vec{F}_{1,ext}}_{=\vec{F}} + \underbrace{\vec{F}_{2,ext}}_{=0} = \vec{F}$$

😎 The force on m_2 is an “internal” force.

$$\Rightarrow \vec{F}_{CM} = \vec{F}_{total} = \vec{F}$$

Q: Will the CM continue on the same parabolic trajectory even after one of the fragments hits the ground?



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(6) Kinetic energy of a system:

$$K = \sum_i K_i = \sum_i \frac{1}{2} m_i v_i^2 = \sum_i \frac{1}{2} m_i \left(\vec{v}_i \cdot \vec{v}_i\right)$$

Define $\vec{u}_i \equiv \vec{v}_i - \vec{v}_{CM}$

$$\begin{aligned} K &= \sum_i \frac{1}{2} m_i \left(\vec{v}_i \cdot \vec{v}_i\right) \\ &= \sum_i \frac{1}{2} m_i \left(\vec{v}_{CM} + \vec{u}_i\right) \cdot \left(\vec{v}_{CM} + \vec{u}_i\right) \\ &= \sum_i \frac{1}{2} m_i v_{CM}^2 + \sum_i \frac{1}{2} m_i u_i^2 + \vec{v}_{CM} \cdot \underbrace{\sum_i m_i \vec{u}_i}_{\substack{=0 \\ \because \sum m_i (\vec{v}_i - \vec{v}_{CM}) = 0}} \end{aligned}$$

$$\begin{aligned} K &= \frac{1}{2} M v_{CM}^2 + \sum_i \frac{1}{2} m_i u_i^2 \\ &= K_{CM} + K_{rel} \end{aligned}$$

$\Rightarrow$ Total kinetic energy of a system equals the kinetic energy of the CM plus the kinetic energy relative to the CM.

(7) Work-Kinetic energy theorem:

$$W_{CM} = \Delta K_{CM}$$

$$\vec{F}_{CM} \cdot \Delta \vec{r}_{CM} = \Delta K_{CM} = \Delta\left(\frac{P_{CM}^2}{2M}\right)$$

If $\vec{F}_{CM} = 0 \quad \Rightarrow \quad \vec{P}_{CM} = const.$ in consistant with

$\vec{F}_{total} = \vec{F}_{CM} = 0 \quad \& \quad \vec{P}_{total} = \vec{P}_{CM} = const.$

If $\sum_i \vec{F}_{ext} = 0, \quad \Rightarrow \vec{P}_{CM}$ is conserved!

$\Rightarrow \frac{1}{2} M v_{CM}^2 = \frac{P_{CM}^2}{2M}$ =constant;

$\Rightarrow K_{CM}$ is conserved, but K_{total} is not necessary conserved;

$\Rightarrow K_{rel}$ can change in an isolated system.

For example,

In perfectly inelastic collisions, K_{rel} converts into E_{int};

after collision, the total kinetic energy is then K_{CM} only.

(7) Gravitational potential energy of a system:

$$U_g = \sum_i m_i g_i y_i$$

If $g_i = g \Rightarrow \; = g\sum_i m_i y_i$

$$= gMy_{CM} = U_{g,CM}$$

$\Rightarrow$ If g is independent of position, the gravitational potential energy of a system is the same as if all the mass were concentrated at CM.

(8) Work done by external forces on a system:

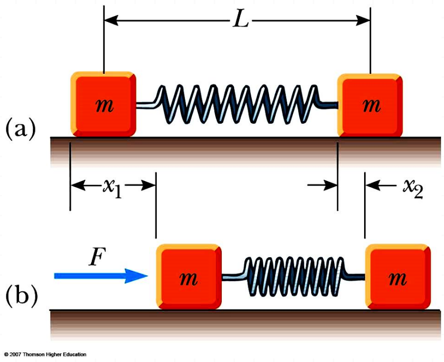
$$\boxed{W_{CM} = \left(\sum_{i=1}^{n} \vec{F}_{i,ext}\right) \cdot \Delta\vec{r}_{CM} = \Delta K_{CM}}$$

$$W_{total,ext} = \left[\left(\sum_{i=1}^{n} \vec{F}_{i,ext} \cdot \Delta\vec{r}_i\right) = \Delta K_{CM} + \Delta K_{rel} + \Delta U + \Delta E_{\text{int}}\right]$$

$$\Rightarrow W_{CM} \neq W_{total,ext}$$

The work done by an external force on a system is NOT necessary equal to the work done on the CM of the system.

Ex.



(1) $v_{CM} = ?$

(2) $E_{\text{vibration}}$ associated with the vibration relative to the CM?

(1) Method#1:

$$F\Delta t = \Delta P_{CM} = (m+m)v_{CM} - 0 = 2mv_{CM}$$

$$v_{CM} = \frac{F\Delta t}{2m}$$

$$\Delta t = \frac{\Delta x_{CM}}{\bar{v}_{CM}} = \frac{\Delta x_{CM}}{\frac{1}{2}(0+v_{CM})} = \frac{2\Delta x_{CM}}{v_{CM}}$$

$$\Rightarrow \quad v_{CM} = \sqrt{\frac{F\Delta x_{CM}}{m}}$$

$$\Delta x_{CM} = x_{CM,f} - x_{CM,i} = \frac{m_1 \Delta x_1 + m_2 \Delta x_2}{M}$$

$$= \frac{m x_{1f} + m x_{2f}}{2m} - \frac{m x_{1i} + m x_{2i}}{2m}$$

$$= \frac{(x_{1f} - x_{1i}) + (x_{2f} - x_{2i})}{2} = \frac{x_1 + x_2}{2}$$

$$\Rightarrow \quad v_{CM} = \sqrt{\frac{F(x_1 + x_2)}{2m}}$$

Method#2:

$$\text{By } W_{CM} = \underbrace{\vec{F}_{CM}}_{=\left(\sum_i \vec{F}_{i,ext}\right)} \cdot \Delta \vec{r}_{CM} = \Delta K_{CM}$$

$$\Delta x_{CM} = \frac{m_1 \Delta x_1 + m_2 \Delta x_2}{M} = \frac{x_1 + x_2}{2}$$

$$F\left(\frac{x_1 + x_2}{2}\right) = \frac{1}{2}(m + m) v_{CM}^2$$

$$\Rightarrow \quad v_{CM,f} = \sqrt{\frac{F(x_1 + x_2)}{2m}}$$

Method#3:

$$v_{CM,f}^2 = \underbrace{v_{CM,i}^2}_{=0} + 2a_{CM}\Delta x_{CM}$$

$$\vec{F} = M\vec{a}_{CM} \quad \Rightarrow \quad a_{CM} = \frac{F}{2m}$$

$$\Delta x_{CM} = \frac{x_1 + x_2}{2}$$

$$\Rightarrow \quad v_{CM,f} = \sqrt{\frac{F(x_1 + x_2)}{2m}}$$

(2) $$W_{total} = \Delta K + \Delta U$$

$$\underbrace{Fx_1}_{\neq F\Delta x_{CM}} = \underbrace{\Delta K_{CM}}_{=F\Delta x_{CM}} + \underbrace{\Delta K_{rel} + \Delta U}_{=E_{vibration}}$$

$$\Rightarrow \quad E_{vibration} = Fx_1 - F(\frac{x_1 + x_2}{2})$$

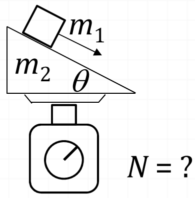
$$= F\frac{(x_1 - x_2)}{2}$$

Note: $W_{total} = Fx_1$ does not equal to

$$W_{CM} = F\Delta x_{CM} = F\frac{1}{2}(x_1 + x_2).$$

😎 Work on the system ≠ Work on CM!

Ex.



Method #1 (Center of mass)

System = m_1+m_2

(1) $\sum \vec{F} = m\vec{a} \Rightarrow \; N\hat{j} - (m_1+m_2)g\hat{j} = -Ma_{CM}\hat{j} = -(m_1+m_2)a_{CM}\hat{j}$

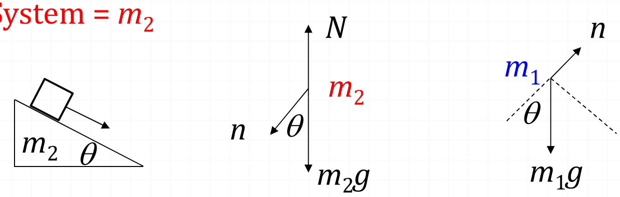
(2) $-Ma_{CM}\hat{j} = -m_1 a_1 \hat{j} + m_2 a_2 \hat{j} = -m_1\left[(g\sin\theta)\sin\theta\right] + 0$

$$\Rightarrow \; a_{CM} = \frac{m_1}{m_1+m_2} g\sin^2\theta$$

So $N = (m_1+m_2)g - m_1 g\sin^2\theta < (m_1+m_2)g$

Method#2

System = m_2



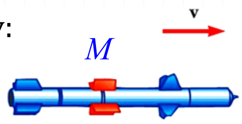
$-m_2 g\hat{j} + (-n\cos\theta)\hat{j} + N\hat{j} = 0 \qquad n = m_1 g\cos\theta$

$$\Rightarrow \; N = m_2 g + m_1 g\cos^2\theta = (m_1+m_2)g - m_1 g\sin^2\theta$$

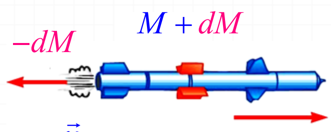
4. Rocket propulsion

Rocket propulsion

1. No gravity:



t



$\vec{u}$ $t+dt$ $\vec{v}+d\vec{v}$

The process is like the reverse of a perfectly inelastic collision.

Kinetic energy is not conserved.

P conservation: $M\vec{v} = -dM\vec{u} + (M + dM)(\vec{v} + d\vec{v})$

$\vec{v}_{ex} \equiv \vec{u} - \vec{v}$, fuel velocity relative to rocket

$$M\vec{v} = -dM(\vec{v}_{ex} + \vec{v}) + (M + dM)(\vec{v} + d\vec{v})$$

$$\Rightarrow \vec{v}_{ex}dM = Md\vec{v} + \underbrace{dMd\vec{v}}_{\text{much smaller than the other terms, can be ignored.}}$$

Therefore, $\vec{v}_{ex}dM = Md\vec{v}$

(1)

$$\Rightarrow \quad -v_{ex}dM = Mdv$$

$$\Rightarrow \quad \int_{v_i}^{v_f} dv = -v_{ex}\int_{M_i}^{M_f} \frac{dM}{M}$$

$$\Rightarrow \quad v_f - v_i = v_{ex}\ln\left(\frac{M_i}{M_f}\right)$$

(2)

$$\text{Thrust} = \left|M\frac{d\vec{v}}{dt}\right| = \left|\vec{v}_{ex}\frac{dM}{dt}\right|$$

2. With gravity:

$$\frac{d\vec{P}_{rocket}(t)}{dt} = \frac{d}{dt}(M\vec{v}) = M\dot{\vec{v}} + \dot{M}\vec{v}$$

$$\frac{d\vec{P}_{exhaust}(t)}{dt} = \frac{(-dM)\vec{u}}{dt} = -(\vec{v}+\vec{v}_{ex})\dot{M} \qquad \vec{v}_{ex} = \vec{u} - \vec{v}$$

$$\vec{F}_{ext} = \frac{d(\vec{P}_{rocket}+\vec{P}_{exhaust})}{dt} = M\dot{\vec{v}} + \dot{M}\vec{v} - (\vec{v}+\vec{v}_{ex})\dot{M}$$

$$= M\dot{\vec{v}} - \vec{v}_{ex}\dot{M}$$

$$\Rightarrow \quad (\vec{F}_{ext} + \underbrace{\vec{v}_{ex}\dot{M}}_{\text{thrust force}}) = M\dot{\vec{v}}$$

Or $$\frac{d\vec{P}_{rocket}}{dt} = \lim_{\Delta t\to 0}\frac{\vec{P}_{rocket}(t+\Delta t)-\vec{P}_{rocket}(t)}{\Delta t}$$

$$= \lim_{\Delta t\to 0}\frac{(M+\Delta M)(\vec{v}+\Delta\vec{v})-M\vec{v}}{\Delta t} = \lim_{\Delta t\to 0}\frac{\vec{v}\Delta M + M\Delta\vec{v}+\Delta\vec{v}\Delta M}{\Delta t}$$

$$= \dot{M}\vec{v} + M\dot{\vec{v}} \qquad (\text{ps.,}\ \lim_{\Delta t\to 0}\frac{\Delta\vec{v}\Delta M}{\Delta t} = 0)$$

$$\frac{d\vec{P}_{exhaust}}{dt} = \lim_{\Delta t\to 0}\frac{(-\Delta M)\vec{u}-0}{\Delta t} = -(\vec{v}+\vec{v}_{ex})\dot{M}$$

$$\vec{u} = \vec{v}_{ex} + \vec{v}$$

$$\vec{F}_{ext} = \frac{d(\vec{P}_{rocket}+\vec{P}_{exhaust})}{dt} = M\dot{\vec{v}} + \dot{M}\vec{v} - (\vec{v}+\vec{v}_{ex})\dot{M}$$

$$= M\dot{\vec{v}} - \vec{v}_{ex}\dot{M}$$

$$\Rightarrow (\vec{F}_{ext} + \underbrace{\vec{v}_{ex}\dot{M}}_{\text{thrust force}}) = M\dot{\vec{v}}$$

Q:

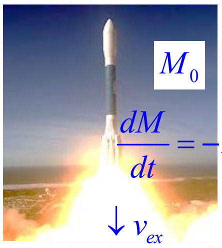
(a) If a rocket in gravity-free outer space has the same thrust at all times, is its acceleration constant, increasing, or decreasing?

(b) If the rocket has the same acceleration at all times, is the thrust constant, increasing, or decreasing?

- Rocket propulsion: $\vec{F}_{\text{ext}} + \underbrace{\frac{dM}{dt}\vec{v}_{ex}}_{\text{Thrust}} = M\frac{d\vec{v}}{dt}$

Ex:

$v_y = ?$



$$M\vec{g} + \frac{dM}{dt}\vec{v}_{\text{ex}} = M\frac{d\vec{v}}{dt}, \qquad \frac{dM}{dt} = -R$$

$$-Mg\hat{j} + (-R)(-v_{\text{ex}}\hat{j}) = M\frac{dv_y}{dt}\hat{j}$$

$$\frac{dv_y}{dt} = \frac{Rv_{\text{ex}}}{M} - g = \frac{Rv_{\text{ex}}}{M_0 - Rt} - g$$

$$\Rightarrow \quad v_y = v_{\text{ex}} \ln\left(\frac{M_0}{M_0 - Rt}\right) - gt$$

Not a uniform acceleration motion!

Blowing into your own Sail

https://www.youtube.com/watch?v=gKzWrumXS7E

https://www.youtube.com/watch?v=n9cdfUYkrLY

Ex.

$$\frac{dM}{dt} = 60\text{kg / s}$$

$$v_{ex} = ?$$

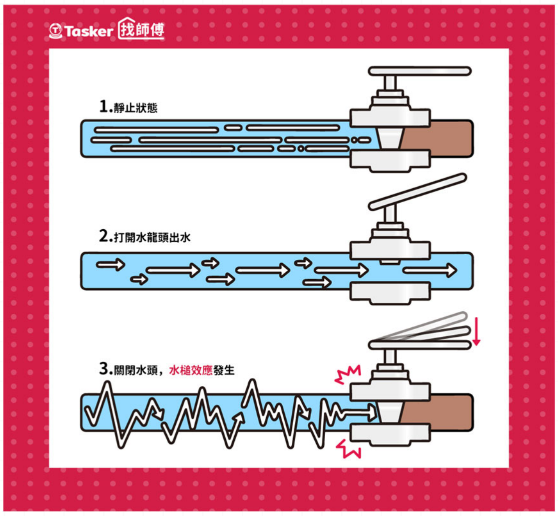
$$F = 600N$$

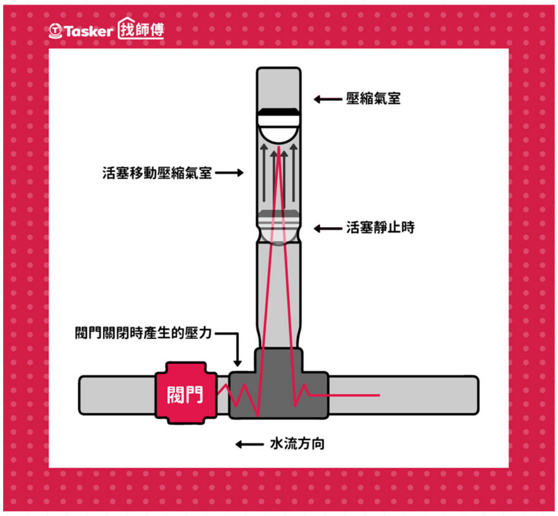
$$\text{Thrust} = \left| v_{ex} \frac{dM}{dt} \right|$$

$$600N = \left| v_{ex} (60\text{kg/s}) \right|$$

$$v_{ex} = 10\text{m/s}$$

水錘效應(Water Hammer)



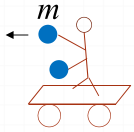


[https://www.tasker.com.tw/articles/detail/218]

Ex.

$\vec{u}$: throwing speed relative to man

m



$v_i = 0$

M = man + cart

(A) V_f = ? After throwing 1st ball.

(B) V_f = ? After throwing 2nd ball.

(C) What if throwing two balls together?

(1) Throw the 1st ball:

$$\vec{P}_{1i} = 0$$

$$\vec{P}_{1f} = (M+m)\vec{v}_{1f} + m\vec{v}_b, \quad \vec{u} = \vec{v}_b - \vec{v}_{1f}$$

$$\vec{P}_{1i} = \vec{P}_{1f} \quad \Rightarrow \quad (M+m)\vec{v}_{1f} + m\vec{u} + m\vec{v}_{1f} = 0,$$

$$\Rightarrow \quad \vec{v}_{1f} = \frac{-m\vec{u}}{M+2m}$$

(2) Throw the 2nd ball:

$$\vec{P}_{2i} = (M+m)(\frac{-m\vec{u}}{M+2m})$$

$$\vec{P}_{2f} = M\vec{v}_{2f} + m\vec{v}_b = M\vec{v}_{2f} + m(\vec{u} + \vec{v}_{2f})$$

$$\vec{P}_{2i} = \vec{P}_{2f} \quad \Rightarrow \quad \vec{v}_{2f} = \frac{-m\vec{u}}{M+2m}(2 - \frac{m}{M+2m})$$

(3) Throw two balls together

$$\vec{P}_i = 0$$

$$\vec{P}_f = M\vec{v}_f + 2m\vec{v}_b, \quad \vec{u} = \vec{v}_b - \vec{v}_f$$

$$\vec{P}_i = \vec{P}_f \quad \Rightarrow \quad M\vec{v}_f + 2m\vec{u} + 2m\vec{v}_f = 0$$

$$\Rightarrow \quad \vec{v}_f = \frac{-2m\vec{u}}{M+2m}$$

Note that v_f is larger than the previous case, which can be understood by the thrust $F \propto dM/dt$.

Alternative,

One by one:

1st throw: $\Delta P_m = \Delta P_{M+m} \quad \Rightarrow \quad F\Delta t = (M+m)(v_f' - 0)$

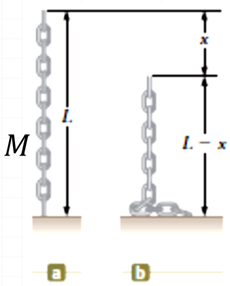
2nd throw: $\Delta P_m = \Delta P_M \quad \Rightarrow \quad F\Delta t = M(v_{1f} - v_f')$

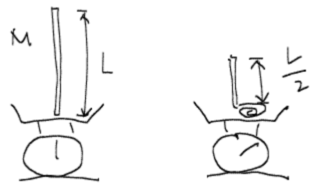
$$\Rightarrow 2F\Delta t = Mv_{1f} + mv_f'$$

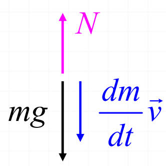
Two at once: $\quad 2F\Delta t = M(v_{2f} - 0)$

$$\Rightarrow v_{2f} > v_{1f}$$

Ex.







$$N\hat{j} - \frac{dm}{dt}\vec{v} - mg\hat{j} = 0 \qquad \frac{dm}{dt} = \underbrace{\frac{dm}{d\ell}}_{=\frac{M}{L}}\frac{d\ell}{dt} = \frac{M}{L}v,$$

$$\Rightarrow N = mg + \frac{M}{L}v^2$$

$$\text{At } \frac{L}{2},\ m = \frac{M}{2} \quad v^2 = v_0^2 - 2g\left(\frac{-L}{2}\right) = gL \quad \Rightarrow \quad N = \frac{3}{2}Mg$$

https://fb.watch/7CyaJ9-en5/

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