Chapter 9

Rotation of Rigid Bodies

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Outline

1. Angular position, velocity, acceleration
2. Rotational Kinematics
3. Rotational Kinetic Energy
4. Moments of Inertia

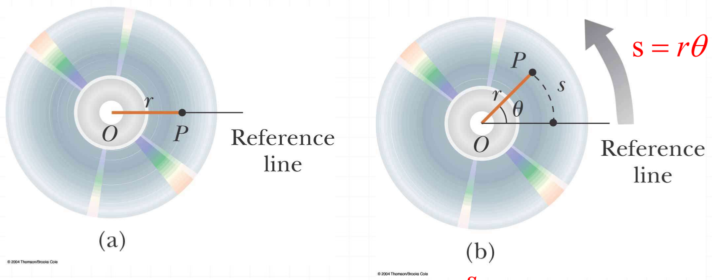
Useful Equations in Rotational and Linear Motion

Rotational Motion About a Fixed Axis	Linear Motion
Angular speed $\omega = d\theta/dt$	Linear speed $v = dx/dt$
Angular acceleration $\alpha = d\omega/dt$	Linear acceleration $a = dv/dt$
Net torque $\Sigma\tau = I\alpha$	Net force $\Sigma F = ma$
If $\alpha =$ constant $\begin{cases} \omega_f = \omega_i + \alpha t \\ \theta_f = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2 \\ \omega_f^2 = \omega_i^2 + 2\alpha(\theta_f - \theta_i) \end{cases}$	If $a =$ constant $\begin{cases} v_f = v_i + at \\ x_f = x_i + v_i t + \frac{1}{2}at^2 \\ v_f^2 = v_i^2 + 2a(x_f - x_i) \end{cases}$
Work $W = \int_{\theta_i}^{\theta_f} \tau \, d\theta$	Work $W = \int_{x_i}^{x_f} F_x \, dx$
Rotational kinetic energy $K_R = \frac{1}{2}I\omega^2$	Kinetic energy $K = \frac{1}{2}mv^2$
Power $\mathcal{P} = \tau\omega$	Power $\mathcal{P} = Fv$
Angular momentum $L = I\omega$	Linear momentum $p = mv$
Net torque $\Sigma\tau = dL/dt$	Net force $\Sigma F = dp/dt$

1. Angular position, velocity, acceleration
2. Rotational Kinematics
3. Rotational Kinetic Energy
4. Moments of Inertia

1. Angular position, velocity, acceleration

Angular Position



Polar coordinate

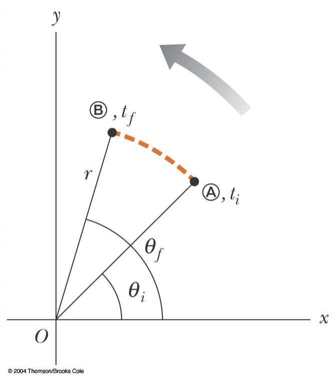
(r, θ)

$\theta = \frac{s}{r}$

"radian"

$\theta(\text{rad}) = \frac{\pi}{180^\circ}\theta(\text{deg})$

- "One radian" is the angle subtended by an arc length equal to the radius of the arc.
- Radian is dimensionless.



Angular displacement:

$$\Delta\theta = \theta_f - \theta_i \quad \text{(rad)}$$

Average angular speed:

$$\bar{\omega} = \frac{\theta_f - \theta_i}{t_f - t_i} = \frac{\Delta\theta}{\Delta t} \qquad \text{(rad/s)}$$

Instantaneous angular speed:

$$\omega \equiv \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

1 rev/sec = 2π rad/sec; rev = revolution

1 rev/min = 1 rpm = 2π/60 rad/sec; 1 rad/sec ≈ 10 rpm

Average angular acceleration:

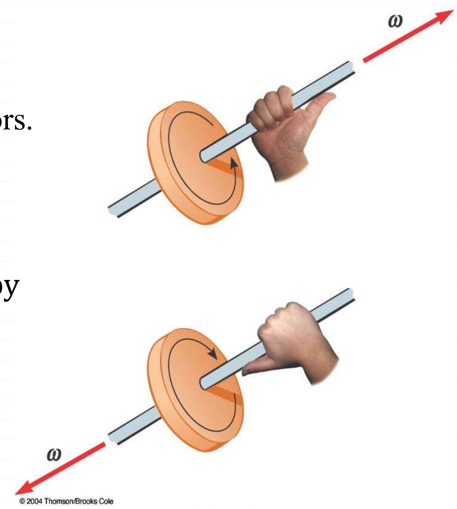
$$\bar{\alpha} = \frac{\omega_f - \omega_i}{t_f - t_i} = \frac{\Delta\omega}{\Delta t} \qquad (\mathrm{rad/s^2})$$

Instantaneous angular acceleration:

$$\alpha \equiv \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt}$$

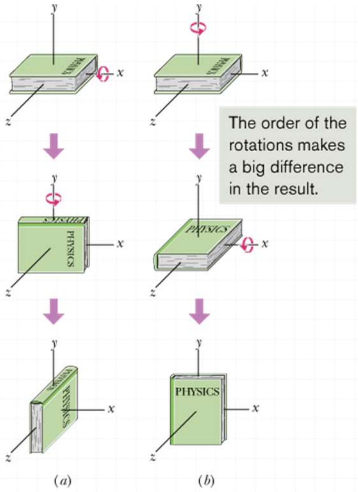
Instantaneous $\vec{\omega}$, $\vec{\alpha}$ are vectors.

Direction is determined by "right-hand" rule.



Note,

"Angular displacement" ($\Delta\theta$)can NOT be treated as a vector, because it does not satisfy the commutative rule.



$\Rightarrow$ Average angular velocity and acceleration are NOT vectors, but instantaneous angular velocity and acceleration are vectors.

2. Rotational kinematics

Rotational Kinematic Equations for constant angular acceleration

(1) $\omega_f = \omega_i + \alpha t$ $\Leftarrow \quad \alpha = \dfrac{d\omega}{dt}$

(2) $\theta_f = \theta_i + \omega_i t + \dfrac{1}{2}\alpha t^2$ $\Leftarrow \quad \alpha = \dfrac{d^2\theta}{dt^2}$

(3) $\omega_f^2 = \omega_i^2 + 2\alpha(\theta_f - \theta_i)$ $\Leftarrow$ (1), (2) ; remove "t"

(4) $\theta_f = \theta_i + \dfrac{1}{2}\left(\omega_i + \omega_f\right)t$ $\Leftarrow$ (1), (2) ; remove "α"

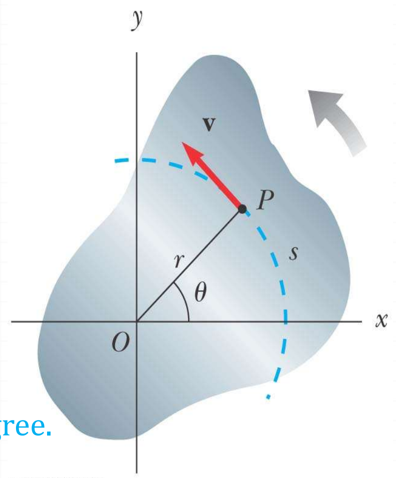
Kinematic Equations for Rotational and Linear Motion Under Constant Acceleration

Rotational Motion About Fixed Axis	Linear Motion
$\omega_f = \omega_i + \alpha t$	$v_f = v_i + at$
$\theta_f = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2$	$x_f = x_i + v_i t + \frac{1}{2}at^2$
$\omega_f^2 = \omega_i^2 + 2\alpha(\theta_f - \theta_i)$	$v_f^2 = v_i^2 + 2a(x_f - x_i)$
$\theta_f = \theta_i + \frac{1}{2}(\omega_i + \omega_f)t$	$x_f = x_i + \frac{1}{2}(v_i + v_f)t$

Relationship Between Angular and Linear Quantities

- The linear velocity is always tangent to the circular path, called the **tangential velocity**.

$$v = \frac{ds}{dt} = r\frac{d\theta}{dt} = r\omega$$



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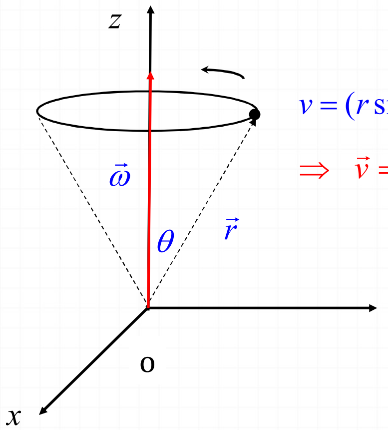
😎 θ is measured in "radian" instead of degree.



$$v = (r\sin\theta)\omega$$

$$\Rightarrow \quad \vec{v} = \vec{\omega} \times \vec{r}$$

😎Usually, both the origin and rotational axis have to be defined in rotational motion. The rotational axis doesn't have to be physically present.

😎 The origin and the rotational axis do not have to fix in space and they can move, too, as in rolling motion.

- The tangential acceleration is

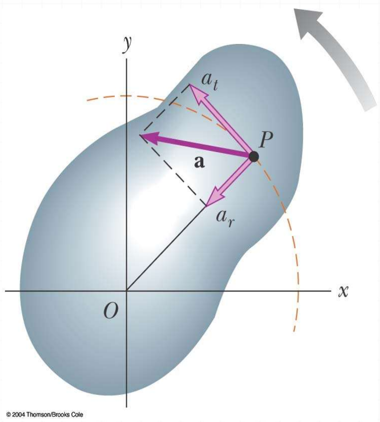
$$a_t = \frac{dv}{dt} = r\frac{d\omega}{dt} = r\alpha$$

$$\vec{a}_t = \vec{\alpha} \times \vec{r}$$

- The radial acceleration is the centripetal acceleration

$$a_r = a_C = \frac{v^2}{r} = r\omega^2$$

$$\vec{a}_r = \vec{\omega} \times \vec{v} \qquad \text{(See P9.64)}$$



- The total linear acceleration is

$$\vec{a} = \vec{a}_t + \vec{a}_r \qquad a = \sqrt{a_t^{\,2} + a_r^{\,2}} = \sqrt{r^2\alpha^2 + r^2\omega^4} = r\sqrt{\alpha^2 + \omega^4}$$

Angular and Linear Quantities

- Distance

  $s = \theta r$

- Speeds

  $v = \omega r$

- Accelerations

  $a = \alpha r$

- All points on the rigid object will have the same *angular speed*, but not the same *tangential speed*.
- All points on the rigid object will have the same *angular acceleration*, but not the same *tangential acceleration*.

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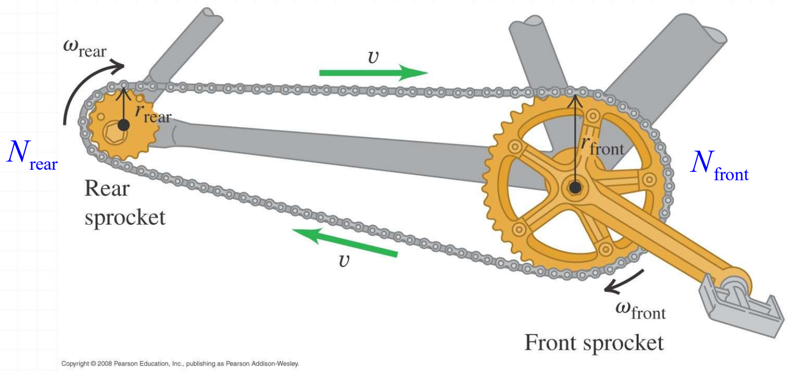
Ex.



$$v = r_{front}\omega_{front} = r_{rear}\omega_{rear}$$

$$\frac{2\pi r_{front}}{N_{front}} = \frac{2\pi r_{rear}}{N_{rear}} \quad \Rightarrow \quad \frac{\omega_{rear}}{\omega_{front}} = \frac{N_{front}}{N_{rear}}$$

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Ex. For a compact disc player to read a CD, the angular speed must vary to keep the tangential speed constant (v_t).

(A) Find the angular speed of the disc in revolution per minute when information is being read from the innermost first track r_1 and the outmost final track r_2.

$$\omega_1 = \frac{v_t}{r_1} \qquad \omega_2 = \frac{v_t}{r_2}$$

(B) The maximum playing time of a CD is t_0. How many revolutions does the disc make during that time?

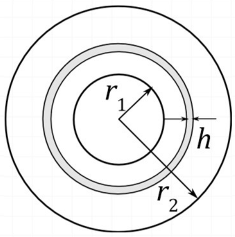
Assume α is constant,

$$\theta_f = \underbrace{\theta_i}_{=0} + \frac{1}{2}\left(\omega_i + \omega_f\right)t$$
$$= \frac{1}{2}\left(\frac{v_t}{r_1} + \frac{v_t}{r_2}\right)t_0 = \frac{1}{2}\left(\frac{1}{r_1} + \frac{1}{r_2}\right)v_t t_0$$

$$\Rightarrow n = \frac{\theta_f}{2\pi} = \frac{1}{4\pi}\left(\frac{1}{r_1} + \frac{1}{r_2}\right)v_t t_0$$

Another idea from a student:

Assume the track's width is h, then the area



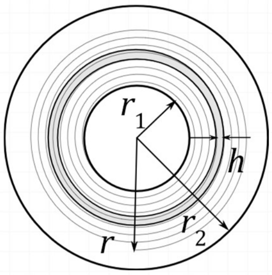
$$\sum_{i=1}^{n} h v_t \Delta t_i = h v_t t_0 = \pi(r_2^2 - r_1^2)$$

Δt_i : the time for the i-th revolution

Revolutions: $n = \dfrac{r_2 - r_1}{h} = \dfrac{v_t t_0}{\pi(r_1 + r_2)}$

(Serway, Problem 10.46)

$$r = r_1 + h\,(\theta/2\pi).$$



$$\omega = \frac{v_t}{r} \rightarrow \frac{d\theta}{dt} = \frac{v_t}{r_1 + (h\theta/2\pi)}$$

$$\theta(t) = 2\pi \frac{r_1}{h}\left(\sqrt{1+\frac{v_t h}{\pi r_1^2}t} - 1\right), \text{ depends on } h.$$

$$\text{Number of Revolution: } n = \frac{\theta_0}{2\pi} = \frac{r_1}{h}\left(\sqrt{1+\frac{v_t h}{\pi r_1^2}t_0} - 1\right), \text{ depends on } h.$$

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$$\begin{cases} r_2 = r_1 + h\,(\theta_0/2\pi) & (1) \\ \dfrac{\theta_0}{2\pi} = \dfrac{r_1}{h}\left(\sqrt{1+\dfrac{v_t h}{\pi r_1^2}t_0} - 1\right) & (2) \end{cases}$$

Solve for h:

$$\Rightarrow h = \frac{\pi(r_2^2 - r_1^2)}{v_t t_0} \text{ put it back to (1).}$$

$$\Rightarrow n = \frac{\theta_0}{2\pi} = \frac{v_t t_0}{\pi(r_1 + r_2)} \text{ different from previous method.}$$

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(c) What is the angular acceleration of the CD over the time interval? Assume that α is a constant.

$$\bar{\alpha} = \frac{\omega_2 - \omega_1}{t_0}$$

$$\alpha = \frac{d^2\theta}{dt^2} = -\frac{hv_t^2}{2\pi r_1^3\left(1+\dfrac{v_t h}{\pi r_1^2}t_0\right)^{3/2}}$$

(d) What total length of track moves past the objective lens during this time?

$$s = v_t t_0$$

1. Angular position, velocity, acceleration
2. Rotational Kinematics
3. Rotational Kinetic Energy
4. Moments of Inertia
5. Torque
6. Energy conservation in rotational motion
7. Rolling Motion

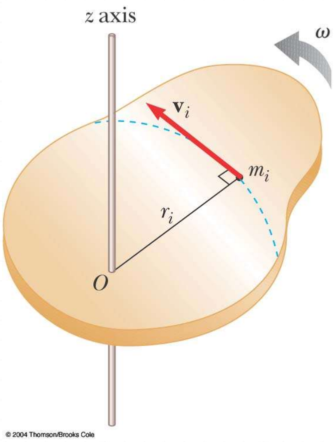
3. Rotational kinetic energy

Rotational Kinetic Energy

$$K_i = \frac{1}{2} m_i v_i^2 \qquad v_i = r_i \omega_i$$

$$K_R = \sum_i K_i = \sum_i \frac{1}{2} m_i r_i^2 \omega^2$$

$$K_R = \frac{1}{2}\left(\sum_i m_i r_i^2\right)\omega^2 \equiv \frac{1}{2} I \omega^2$$



Moment of Inertia:

$$I = \sum_i^n m_i r_i^2$$

4. Moments of inertia

Moment of Inertia (rotational inertial):

$$I = \sum_{i}^{n} m_i r_i^2 \quad \longleftarrow$$ Depends on the choice of axis.

Rotational Kinetic energy:

$$K_R = \frac{1}{2} I \omega^2$$

$(\omega : \text{rad/s})$

flywheel

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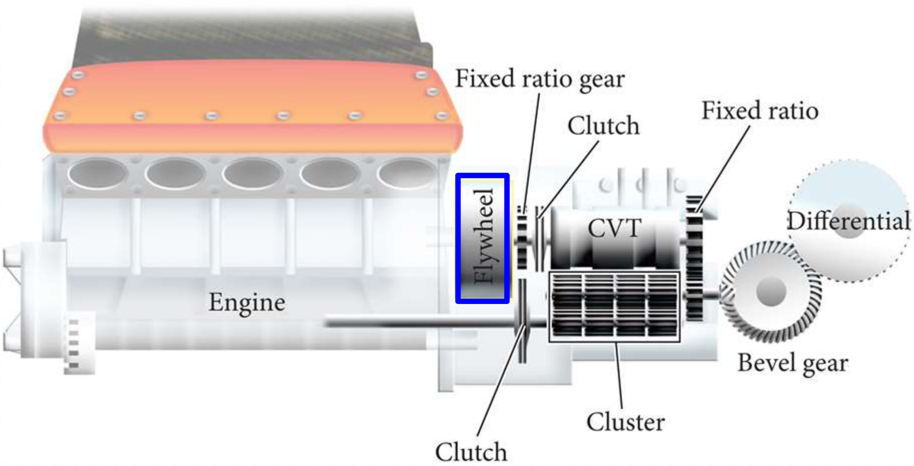

Continuously variable transmission (CVT)

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Linear		Rotational
m		$I = mr^2$
v	$\Leftrightarrow$	ω
$\frac{1}{2}mv^2$		$\frac{1}{2}I\omega^2$

Calculation of Moment of Inertia

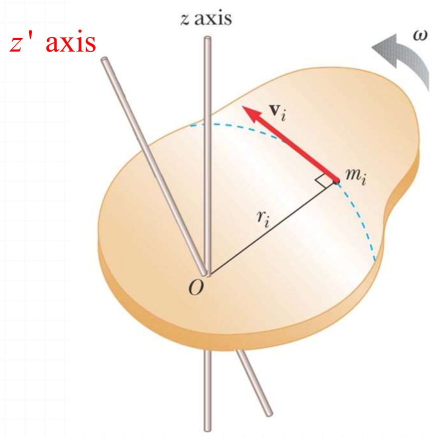
$$I = \sum_i r_i^2 m_i$$

$$I = \lim_{\Delta m_i \to 0} \sum_i r_i^2 \Delta m_i = \int r^2 dm$$

$$I = \int \rho r^2 dV \qquad dm = \rho dV \qquad \text{(volume)}$$

$$= \int \sigma r^2 dA \qquad = \sigma dA \qquad \text{(area)}$$

$$= \int \lambda r^2 d\ell \qquad = \lambda d\ell \qquad \text{(linear)}$$

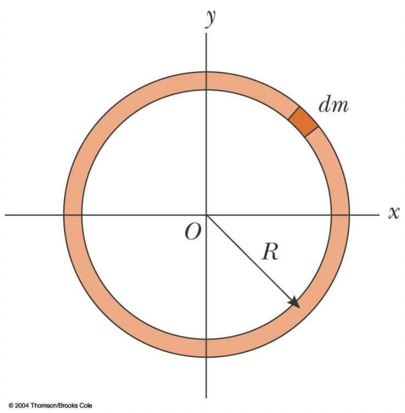


The moment of inertia for the rotation axis z' is not the same as that for the z axis.

Moment of Inertia of a Uniform Thin Hoop

$$I = \int r^2 dm = R^2 \int dm$$

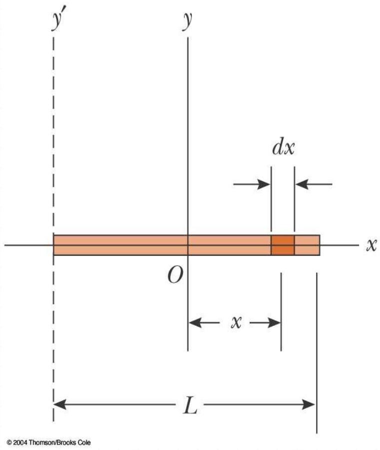
$$\Rightarrow \quad I = MR^2$$



Moment of Inertia of a Uniform Rigid Rod

$$I = \int r^2 dm = \int_{-L/2}^{L/2} x^2 \frac{M}{L}\, dx$$

$$\Rightarrow \quad I = \frac{1}{12} ML^2$$



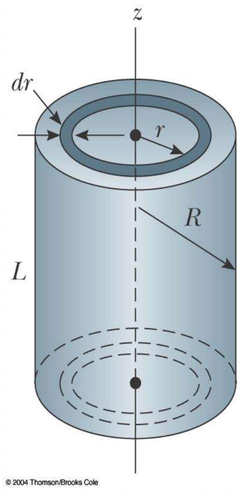
Moment of Inertia of a Uniform Solid Cylinder

$$I = \int r^2 dm = \int r^2 \rho dV$$

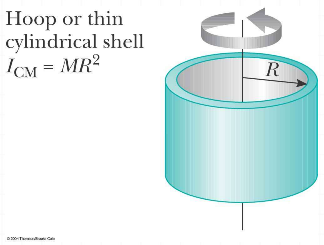
$$V = \pi r^2 L \quad \Rightarrow \quad dV = L 2\pi r dr$$

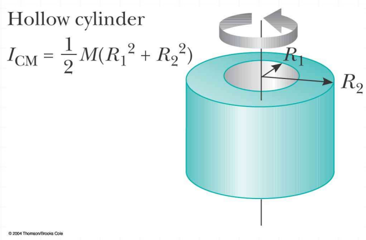
$$\Rightarrow \quad I_z = \int r^2 \rho \left(L 2\pi r dr \right)$$

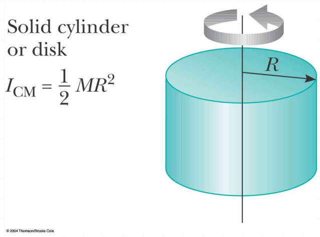
$$\Rightarrow \quad I_z = \frac{1}{2} MR^2$$



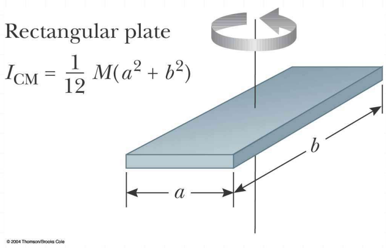
Moments of Inertia of Various Rigid Objects

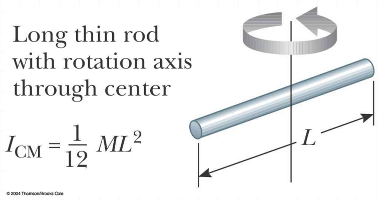


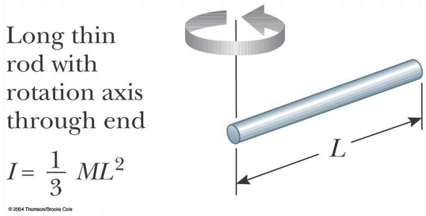




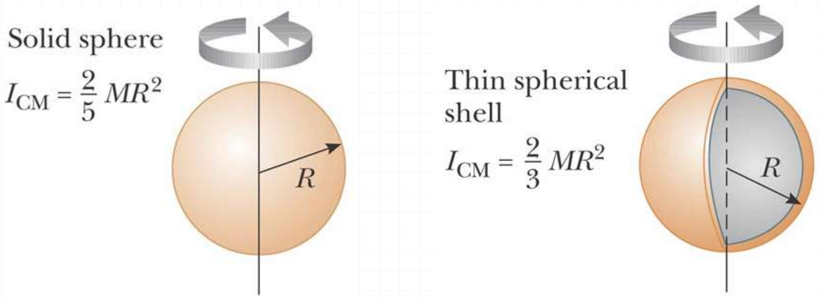
Moments of Inertia of Various Rigid Objects







Moments of Inertia of Various Rigid Objects



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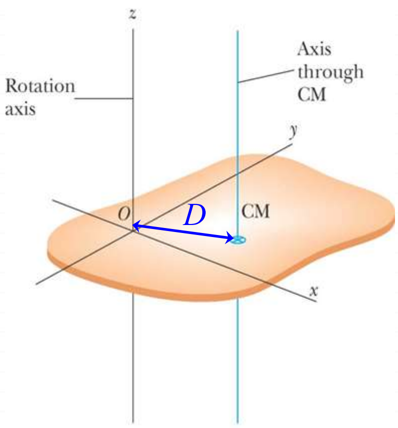
Parallel-Axis Theorem

- The moment of inertia about the axis through O would be

$$I_O = I_{CM} + MD^2$$

$$\Rightarrow \quad I_{\min} = I_{CM}$$



😎 The two axes have to be parallel to each other!

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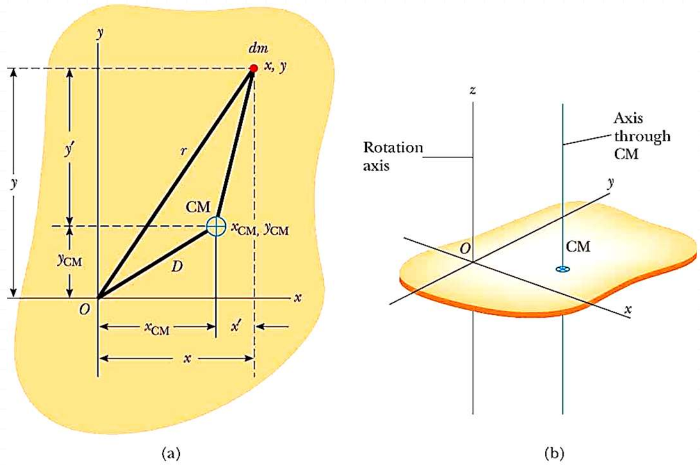


$$I_O = \int r^2 dm = \int (x^2 + y^2) dm$$

$$\begin{cases} x = x' + x_{CM} \\ y = y' + y_{CM} \end{cases}$$

$$I_O = \int r^2 dm = \int (x^2 + y^2) dm$$

$$\begin{cases} x = x' + x_{CM} \\ y = y' + y_{CM} \end{cases} \Rightarrow \quad = \int [(x' + x_{CM})^2 + (y' + y_{CM})^2] dm$$

$$= \underbrace{\int (x'^2 + y'^2) dm}_{I_{CM}} + 2x_{CM} \underbrace{\int x' dm}_{=0} + 2y_{CM} \underbrace{\int y' dm}_{=0}$$

$$\because \int (x - x_{CM}) dm = \int x dm - M x_{CM} = 0$$

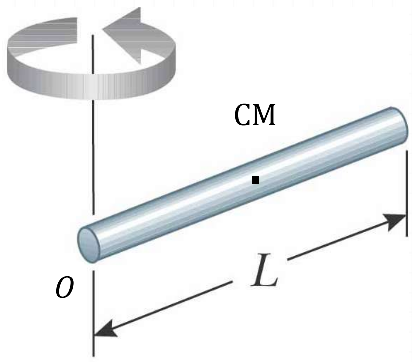
$$+ \underbrace{(x_{CM}^2 + y_{CM}^2)}_{=D^2} \underbrace{\int dm}_{=M}$$

$$\Rightarrow I_O = I_{CM} + MD^2$$

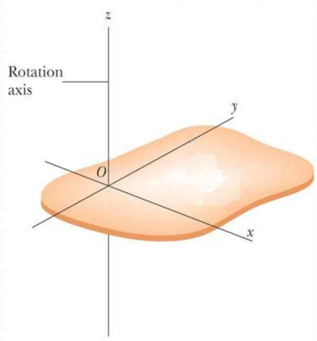
Ex. The moment of inertia of the rod about its center is $I_{CM} = \dfrac{1}{12}ML^2$.

$$I_O = I_{\mathrm{CM}} + MD^2$$

$$I_O = \frac{1}{12}ML^2 + M\left(\frac{L}{2}\right)^2 = \frac{1}{3}ML^2$$



Perpendicular axis theorem



$$I_O = I_x + I_y$$

$$I_O = \int r^2 dm = \int (x^2 + y^2) dm = \int x^2 dm + \int y^2 dm = I_x + I_y$$