Topic 4: Expectation and Variance

Lecture Outline

- Expectation (期望值)
- Variance (變異數)
- Expected Value Rule
  或稱 Law of the Unconscious Statistician (LOTUS)

Reading: Textbook 2.4

Review

- A random variable X is a ***function*** defined on the sample space of an experiment
- The PMF of a discrete RV X: You need to find $p_X(x) = P(X = x)$ for all x
- Given a random variable X and a function $g(.)$, we can define a new random variable $Y = g(X)$, i.e., $Y(\omega) = g(X(\omega))$ is a mapping from ω to a real number

  The PMF of Y can be computed from the PMF of X:

$$p_Y(y) = \sum_{x:g(x)=y} p_X(x)$$

Expectation

Definition

The ***expected value*** E**[**X**]** (also called the ***average value***, ***expectation,*** or ***mean*** 期望值、平均值) of a random variable X, with PMF $p_X(x)$, is defined by

$$E[X] = \sum_x x \cdot p_X(x)$$

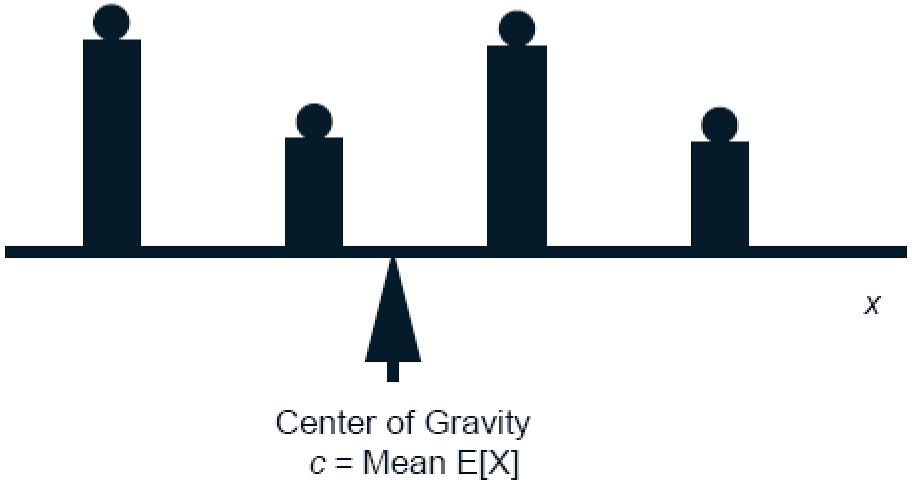
Example

What is the expected value of rolling a fair dice?

物理意義:

The expected value can be regarded as the representative sample value (**代表值**) of an R.V. (e.g. the average score in a class). It needs not be exactly identical to a possible value of the R.V.

Analogy to Center of Gravity



$$\sum_x (x - c) p_X(x) = 0$$

$$c = \sum_x x p_X(x)$$

Elementary Properties of Expectation

- If $X \geq 0$, then $\mathrm{E}[X] \geq 0$
- If $\mathrm{a} \leq X \leq b$, then $\mathrm{a} \leq E[X] \leq b$
- If c is a constant, $E[c] = c$

Variance

Definition

The *variance*（*變異數*）of X is defined as *the expected value* of the random variable $(X - E[X])^2$

$$\mathrm{var}(X) \triangleq E\left[(X - E[X])^2\right]$$

- ***物理意義***:
  The variance provides a *measure of dispersion (散亂程度)* of X around its mean.

- Standard deviation *(標準差)* is defined as the square root of the variance

$$\sigma_X = \sqrt{\mathrm{var}(X)}$$

- 實務上，怎麼計算variance呢?

如何計算變異數 Variance

- 土法煉鋼法： $E[(X-E[X])^2]$

  令 $Y=(X-E[X])^2$，找出 Y的PMF，則可用期望值的定義計算$E[Y]$，即是var(X)。

- 進階作法: 使用 **Expected Value Rule**

  Expected Value Rule (或稱 Law of the unconscious statistician, LOTUS)
  Let X be a random variable with PMF $p_X(x)$, and let $g(X)$ be a real-valued function of X. Then

$$E[g(X)] = \sum_x g(x) p_X(x)$$

  - 令 $g(X) = (X-E[X])^2$ 代入上式求解。
  - 上述LOTUS 好用之處: 可直接使用 X 的PMF來計算期望值，不需要 $g(X)$ 的PMF

Expectation for Functions of Random Variables

LOTUS

Let X be a random variable with PMF $p_X(x)$, and let $g(X)$ be a real-valued function of X. Then,

$$E[g(X)] = \sum_x g(x) p_X(x)$$

- This result is *extremely useful*. It tells us, to find **$E[g(X)]$**, we don't actually need the PMF of $g(X)$. Instead, knowing the PMF of X can do the trick.

  - Using the expected value rule and letting $g(X)=(X - E[X])^2$, we have the variance of X

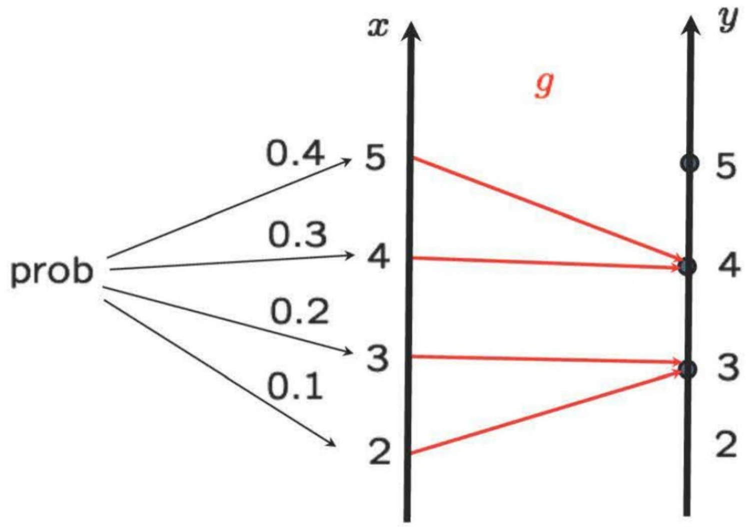
$$E\left[(X - E[X])^2\right] = \sum_x (x - E[X])^2 p_X(x)$$

  - Similarly, the nth moment of X is

$$E[X^n] = \sum_x x^n p_X(x)$$

  - Check the proof of **LOTUS** on page 85 of the textbook

Example of LOTUS



More on Variance

- Variance of X can be expressed in terms of its first 2 moments[1]

$$E\left[(X - E[X])^2\right] = E[X^2] - (E[X])^2$$

這個式子為計算variance最常用的公式。

- Variance of X is always nonnegative

$$E\left[(X - E[X])^2\right] \geq 0,$$

which yields the fact that

$$E[X^2] \geq (E[X])^2$$

[1]**Definition for Moments**

Sometimes we are interested in finding $E[X^n]$, which is called the n-th *moment* of the R.V. X. *也就是* $E[X^2]$ 是 2^{nd} moment， $E[X^7]$是 7^{th} moment

Mean and Variance of a Linear Function of X

Let X be a random variable and let

$$Y = a \cdot X + b$$

where *a* and *b* are given (non-random) scalars. Then,

$$E[Y] = aE[X] + b$$

$$\text{var}(Y) = a^2 \text{var}(X)$$

Remark:

Expectation is a linear operator, but variance is not.

Example: the best fit to a random variable

What is the constant α that minimizes the *mean square error* (MSE, 最小均方差) between a random variable *X* and α. That is, please find the constant α such that $E[(X-\alpha)^2]$ is minimized.

Any intuition before you start?

Mean and Variance of Bernoulli

Bernoulli Random Variable

$$p_X(x) = \begin{cases} p, & x = 1, \\ 1 - p, & x = 0. \end{cases}$$

$$E[X] = 0 \cdot (1 - p) + 1 \cdot p = p$$

$$\text{var}(X) = E[X^2] - (E[X])^2 = p - p^2$$

Mean and Variance of Binomial

Binomial Random Variables

Binomial random variable X with parameter (n,p) has PMF

$$p_X(x) = \binom{n}{x} p^x (1-p)^{n-x} \quad x = 0, 1, \cdots, n.$$

Its mean and variance are

$$E[X] = np$$

$$\mathrm{var}(X) = np(1-p)$$

This is left as an exercise for you.

Hint: Use the identity $i\binom{n}{i} = n\binom{n-1}{i-1}$

Mean and Variance of Uniform

Uniform Random Variables

X takes on values in {1,2,…,n} *equally likely*.

$$p_X(x) = \frac{1}{n}, \quad x = 1, 2, \cdots, n.$$

$$\begin{aligned} E[X] &= \frac{1}{n}(1 + 2 + \cdots + n) \\ &= \frac{n+1}{2} \end{aligned}$$

$$\begin{aligned} E[X^2] &= \frac{1}{n}\sum_{k=1}^{n} k^2 \\ &= \frac{1}{6}(n+1)(2n+1) \end{aligned}$$

$$\begin{aligned} \text{var}(X) &= E[X^2] - (E[X])^2 \\ &= \frac{n^2-1}{12} \end{aligned}$$

Mean and Variance of Uniform

General Uniform Random Variables

Suppose X takes on values in $\{a, a+1, \ldots, b\}$ equally likely.

$$p_X(x) = \frac{1}{b-a+1}, \quad x = a, a+1, \cdots, b.$$

$$E[X] = \frac{a+b}{2}$$

$$\begin{aligned} \mathrm{var}(X) &= E[X^2] - (E[X])^2 \\ &= \frac{(b-a)(b-a+2)}{12} \end{aligned}$$

Mean and Variance of Geometric

Geometric Random Variables

The PMF of Geometric X is given by $p_X(k) = (1-p)^{k-1}p, \quad k = 1, 2, \cdots$

Its mean is $E[X] = \frac{1}{p},$

這期望值的結果符合直覺嗎？

And, its variance is $\text{var}(X) = \frac{1-p}{p^2}$

Mean and Variance of Poisson

Poisson Random Variable

A Poisson random variable X with parameter λ has PMF

$$p_X(k) = e^{-\lambda}\frac{\lambda^k}{k!}, \quad k = 0, 1, 2, \cdots$$

Its mean and variance are both equal to the parameter λ.