Final Exam #1

1. Let $u, v \in \mathbb{R}^n$ be two vectors such that $\|u\| = \|v\| = \|u - v\| = 1$.
   (a) Compute $\|u + v\|$. **1p**
   (b) For $n \in \{2, 3\}$, find the angle between u and v. **0.5p**
2. Find the second order Taylor polynomial for $f(x, y) = e^{x^2+y}$ around $(1, 0)$. **2p**
3. Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable at a local extremum $x \in \mathbb{R}^n$. Show that $\nabla f(x) = 0$. **0.5p**
4. Let $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ and define the function $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x) = x^T A x$, for $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.
   (a) Find the direction of steepest descent for f at the point $(1, 0)$. **0.5p**
   (b) Find and classify the critical points of f. **1p**
   (c) Find the minimum of f over the unit circle: $\min\limits_{x \in \mathbb{R}^2} f(x)$ subject to $\|x\|^2 = 1$. **1p**
   (d) For a symmetric matrix $B \in \mathbb{R}^{n \times n}$, find $\max\limits_{x \in \mathbb{R}^n} x^T B x$ subject to $\|x\|^2 = 1$. **0.5p**
5. (a) Compute $\displaystyle\int_0^\infty e^{-x} x^n \, \mathrm{d}x$, for $n \in \mathbb{N}$. **1p**
   (b) Compute $\displaystyle\iint_R \max\{x, y\} \, \mathrm{d}x \, \mathrm{d}y$, where $R = [0, 1] \times [0, 1]$. **1p**
6. Find the volume of the domain $D = \{(x, y, z) \mid z \geq 0,\ x^2 + y^2 \leq 1,\ x^2 + y^2 + z^2 \leq 4\}$. **1p**

Time: 2h. Total number of points: 10p.