Seminar 10

1. Let $f \in End_{\mathbb{R}}(\mathbb{R}^3)$ be defined by

$$f(x, y, z) = (x + y, y - z, 2x + y + z).$$

Determine the matrix $[f]_E$, where $E = (e_1, e_2, e_3)$ is the canonical basis for $\mathbb{R}^3$.

2. Let $f \in Hom_{\mathbb{R}}(\mathbb{R}^3, \mathbb{R}^2)$ be defined by

$$f(x, y, z) = (y, -x)$$

and consider the bases $B = (v_1, v_2, v_3) = ((1, 1, 0), (0, 1, 1), (1, 0, 1))$ of $\mathbb{R}^3$, $B' = (v_1', v_2') = ((1, 1), (1, -2))$ of $\mathbb{R}^2$ and let $E' = (e_1', e_2')$ be the canonical basis of $\mathbb{R}^2$. Determine the matrices $[f]_{BE'}$ and $[f]_{BB'}$.

3. Let $f \in Hom_{\mathbb{R}}(\mathbb{R}^3, \mathbb{R}^4)$ be defined by

$$f(e_1) = (1, 2, 3, 4), f(e_2) = (4, 3, 2, 1), f(e_3) = (-2, 1, 4, 1)$$

on the elements of the canonical basis of $\mathbb{R}^3$. Determine:
(i) $f(v)$ for every $v \in \mathbb{R}^3$.
(ii) the matrix of f in the canonical bases.
(iii) a basis and the dimension of $Ker\, f$ and $Im\, f$.

4. Let $f \in End_{\mathbb{R}}(\mathbb{R}^4)$ with the following matrix in the canonical basis E of $\mathbb{R}^4$:

$$[f]_E = \begin{pmatrix} 1 & 1 & -3 & 2 \\ -1 & 1 & 1 & 4 \\ 2 & 1 & -5 & 1 \\ 1 & 2 & -4 & 5 \end{pmatrix}.$$

(i) Show that $v = (1, 4, 1, -1) \in Ker\, f$ and $v' = (2, -2, 4, 2) \in Im\, f$.
(ii) Determine a basis and the dimension of $Ker\, f$ and $Im\, f$.
(iii) Define f.

5. Consider the real vector space $\mathbb{R}_2[X] = \{f \in \mathbb{R}[X] \mid degree(f) \leq 2\}$ and its bases $E = (1, X, X^2)$ and $B = (1, X - 1, X^2 + 1)$. Consider $\varphi \in End_{\mathbb{R}}(\mathbb{R}_2[X])$ defined by

$$\varphi(a_0 + a_1X + a_2X^2) = (a_0 + a_1) + (a_1 + a_2)X + (a_0 + a_2)X^2.$$

Determine the matrices $[\varphi]_E$ and $[\varphi]_B$.

6. In the real vector space $\mathbb{R}^2$ consider the bases $B = (v_1, v_2) = ((1, 2), (1, 3))$ and $B' = (v_1', v_2') = ((1, 0), (2, 1))$ and let $f, g \in End_{\mathbb{R}}(\mathbb{R}^2)$ having the matrices $[f]_B = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$ and $[g]_{B'} = \begin{pmatrix} -7 & -13 \\ 5 & 7 \end{pmatrix}$. Determine the matrices $[2f]_B$, $[f + g]_B$ and $[f \circ g]_{B'}$.

7. Consider the endomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$, defined by

$$f(x, y) = (x \cos\alpha - y \sin\alpha, x \sin\alpha + y \cos\alpha) \quad (\alpha \in \mathbb{R}).$$

Write its matrix in the canonical basis of $\mathbb{R}^2$ and show that f is an automorphism.

8. Let V be a vector space of dimension 2 over the field $K = \mathbb{Z}_2$. Determine $|V|$, $|End_K(V)|$ and $|Aut_K(V)|$.
[*Hint: use the isomorphism between* $End_K(V)$ *and* $M_n(K)$*, where* $dim_K(V) = n$.]