BOOLEAN FUNCTIONS - GROUP 917, 1011

THEORETICAL ASPECTS

(applicable for all problems) $f:(B_2)^n \to B_2$, $n \in \mathbb{N}^*$ Boolean fn of n vars

MONOM = conjunction of variables eg $x_1 \wedge x_2$

MINTERM = conjunction of n variables $x_1 \wedge x_2 \wedge \overline{x}_3$ minterm of 3 variables

$x_1^{\alpha_1} \wedge x_2^{\alpha_2} \wedge x_3^{\alpha_3} \wedge \ldots \wedge x_n^{\alpha_n}$, $\alpha_i \in B_2$

MAXTERM = disjunction containing all n variables

$x_1^{\alpha_1} \vee x_2^{\alpha_2} \vee \ldots \vee x_n^{\alpha_n}$, $\alpha_i \in B_2$

→ index of the standard notation of a minterm → convert into decimal the binary number composed of the digits representing the powers of all n variables in the minterm expression

→ maxterms: analogous, but duals of the digits

$$m_4 = m_{100_{(2)}} = x_1^1 \wedge x_2^0 \wedge x_3^0 = x_1 \overline{x}_2 \overline{x}_3$$

$$M_4 = M_{100_{(2)}} = x_1^{\overline{1}} \vee x_2^{\overline{0}} \vee x_3^{\overline{0}} = x_1^0 \vee x_2^1 \vee x_3^1 = \overline{x}_1 \vee x_2 \vee x_3$$

CCF = CONJUNCTIVE CANONICAL FORM

→ conjunction of the maxterms corresponding to the values 0 of the function

DCF = DISJUNCTIVE CANONICAL FORM

→ disjunction of the minterms corresponding to the values 1 of the function

1.1. For the following Boolean fns of 3 variables, given by their truth tables, write the corresponding DCF and CCF. Using Karnaugh diagrams simplify both DCF and CCF.

x	y	z	f_1	f_7
0	0	0	0 M_0	0 M_0
0	0	1	1 m_1	0 M_1
0	1	0	0 M_2	1 m_2
0	1	1	1 m_3	0 M_3
1	0	0	1 m_4	1 m_4
1	0	1	0 M_5	1 m_5
1	1	0	0 M_6	0 M_6
1	1	1	1 m_7	1 m_7

I DCF SIMPLIFICATION

a) $f_1(x,y,z) = m_1 \vee m_3 \vee m_4 \vee m_7$ ← DCF, obtained from the truth table

x \ yz	00	01	11	10
0		m_1	m_3	
1	m_4		m_7	

→ obtain maximal monoms by applying factorization; first try triple factorization, then double factorization, then simple factorization

triple because $n=3$, f_1 fn of 3 variables

$m_1 \vee m_3 = \bar{x}_1\bar{y}_2 z_3 \vee \bar{x}_1 y_2 z_3 = \bar{x}_1 z_3 = max_1$

$m_3 \vee m_7 = \bar{x}_1 y_2 z_3 \vee x_1 y_2 z_3 = y_2 z_3 = max_2$

$m_4 = x_1\bar{y}_2\bar{z}_3 = max_3$ (isolated minterm)

$M(f_1) = \{max_1, max_2, max_3\}$ = set of maximal monoms

$C(f_1) = \{max_1, max_2, max_3\}$

a maximal monom is a central monom if its corresponding group contains at least 1 minterm circled only once

max_1: contains m_1

max_2: contains m_7

max_3 is an isolated minterm

⇒ $M(f_1) = C(f_1)$ ⇒ 1st case of the simplification algorithm

⇒ unique disjunctive simplified form of f_1, obtained as the disjunction of all central monoms

$f_1^S(x,y,z) = max_1 \vee max_2 \vee max_3$

$= \bar{x}_1 z \vee yz \vee x\bar{y}\bar{z}$

b) $f_7(x,y,z) = m_2 \vee m_4 \vee m_5 \vee m_7$ → DCF

x \ yz	00	01	11	10
0				m_2
1	m_4	m_5	m_7	

$max_1 = \dots$

$max_2 = \dots$

$max_3 = \dots$

II CCF simplification

→ dual simplification algorithm

→ similar steps applied : obtain maximal an central disjunction for a function

a) ~~$f_7(x,y,z) =$~~

$$CCF(f_7) = M_0 \wedge M_1 \wedge M_3 \wedge M_6$$

$$M_0 = M_{000_{(2)}} = x^{\bar{0}} \vee y^{\bar{0}} \vee z^{\bar{0}} = x \vee y \vee z$$

$$M_1 = M_{001_{(2)}} = x^{\bar{0}} \vee y^{\bar{0}} \vee z^{\bar{1}} = x \vee y \vee \bar{z}$$

$$M_3 = M_{011_{(2)}} = x^{\bar{0}} \vee y^{\bar{1}} \vee z^{\bar{1}} = x \vee \bar{y} \vee \bar{z}$$

$$M_6 = M_{110_{(2)}} = x^{\bar{1}} \vee y^{\bar{1}} \vee z^{\bar{0}} = \bar{x} \vee \bar{y} \vee z$$

x \ yz	00	01	11	10
0	M_0	M_1	M_3	
1				M_6

→ in the diagram, headers for rows and columns are used to express maxterm indices; they represent duals of the powers of the variables from the maxterm expressions

→ apply dual factorization to obtain maximal disjunctions of f

$maxd_1 = M_0 \wedge M_1 = (x \vee y \vee z) \wedge (x \vee y \vee \bar{z}) = x \vee y$ (common part of the maxterms)

$maxd_2 = M_1 \wedge M_3 = (x \vee y \vee \bar{z}) \wedge (x \vee \bar{y} \vee \bar{z}) = x \vee \bar{z}$

$maxd_3 = M_6$ (isolated maxterm)

$Md(f_7) = \{maxd_1, maxd_2, maxd_3\}$

$Cd(f_7) = \{maxd_1, maxd_2, maxd_3\}$

↳ each has a maxterm circled only once

$\Rightarrow Md(f_7) = Cd(f_7) \Rightarrow$

$\Rightarrow$ 1st case of dual simplification algorithm

$\Rightarrow$ unique conjunctive simplified form

$$f_7^{CS}(x,y,z) = maxd_1 \wedge maxd_2 \wedge maxd_3 = (x \vee y) \wedge (x \vee \bar{z}) \wedge (\bar{x} \vee \bar{y} \vee z)$$

b) $CCF(f_1) = M_0 \wedge M_2 \wedge M_5 \wedge M_6$

x \ yz	00	01	11	10
0	M_0			M_2
1		M_5		M_6

$maxd_1 = \dots$

$maxd_2 = \dots$

$maxd_3 = \dots$

$\boxed{\text{EX2}}$ Simplify the following Boolean fns of 4 variables using Karnaugh diagrams.

2.1. $f_1(x_1,x_2,x_3,x_4) = \overset{1}{x_1}\overset{1}{x_2}\overset{0}{\bar{x}_3}\overset{1}{x_4} \vee \overset{1}{x_1}\overset{1}{x_2}\overset{1}{x_3}\overset{0}{\bar{x}_4} \vee \overset{1}{x_1}\overset{1}{x_2}\overset{0}{\bar{x}_3}\overset{0}{\bar{x}_4} \vee \overset{1}{x_1}\overset{0}{\bar{x}_2}\overset{1}{x_3}\overset{0}{\bar{x}_4}$
$\vee \overset{0}{\bar{x}_1}\overset{0}{\bar{x}_2}\overset{0}{\bar{x}_3}\overset{0}{\bar{x}_4} \vee \overset{0}{\bar{x}_1}\overset{0}{\bar{x}_2}\overset{1}{x_3}\overset{0}{\bar{x}_4} \vee \overset{1}{x_1}\overset{0}{\bar{x}_2}\overset{1}{x_3}\overset{1}{x_4} \vee \overset{0}{\bar{x}_1}\overset{0}{\bar{x}_2}\overset{0}{\bar{x}_3}\overset{1}{x_4} \vee$
$\vee \overset{0}{\bar{x}_1}\overset{0}{\bar{x}_2}\overset{1}{x_3}\overset{1}{x_4}$

$DCF(f_1) = \cancel{m_{13}} \vee \cancel{m_{14}} \vee \cancel{m_{12}} \vee \cancel{m_{10}} \vee \cancel{m_0} \vee \cancel{m_2} \vee \cancel{m_{11}} \vee \cancel{m_1} \vee \cancel{m_3}$

x_1x_2 \ x_3x_4	00	01	11	10
00	m_0	m_1	m_3	m_2
01				
11	m_{12}	m_{13}		m_{14}
10			m_{11}	m_{10}

→ we first do double factorizations

$max_1 = m_0 \vee m_1 \vee m_2 \vee m_3 = \bar{x}_1\bar{x}_2$

$max_2 = m_2 \vee m_3 \vee m_{10} \vee m_{11} = \bar{x}_2 x_3$

→ then simple factorizations:

$max_3 = m_{12} \vee m_{13} = x_1 x_2 \bar{x}_3$

$max_4 = m_{14} \vee m_{10} = x_1 x_3 \bar{x}_4$

$max_5 = m_{12} \vee m_{14} = x_1 x_2 \bar{x}_4$

↗ set of maximal monoms

$M(f_1) = \{max_1, \dots, max_5\}$

$C(f_1) = \{max_1, max_2, max_3\}$

↘ set of central monoms

$\Rightarrow M(f_1) \neq C(f_1)$, $C(f_1) \neq \emptyset$ $\Rightarrow$ 2nd case of the simplification algorithm $\Rightarrow$

$\Rightarrow$ We denote by $g = max_1 \vee max_2 \vee max_3$ the disjunction of all central monoms, belonging to all simplified forms of f

$\Rightarrow$ the covered minterms are shaded in the Karnaugh diagram

$\Rightarrow m_{14}$ not covered $\Rightarrow$ must be covered in all possible ways by a minimum number of unused maximal monoms and with a minimum number of overlaps

$\Rightarrow$ we can cover m_{14} with max_4 or max_5 with the same number of overlaps $\Rightarrow$ 2 disjunctive simplified forms

$$f^S_{1_1}(x_1,x_2,x_3,x_4) = g \vee max_4 = \bar{x}_1\bar{x}_2 \vee \bar{x}_2 x_3 \vee x_1 x_2 \bar{x}_3 \vee x_1 x_3 \bar{x}_4$$

$$f^S_{1_2}(x_1,x_2,x_3,x_4) = g \vee max_5 = \bar{x}_1\bar{x}_2 \vee \bar{x}_2 x_3 \vee x_1 x_2 \bar{x}_3 \vee x_1 x_2 \bar{x}_4$$

2.7. $f_7(x_1,x_2,x_3,x_4) = \overset{1}{x_1}\overset{1}{x_2}\overset{1}{x_3}\overset{1}{x_4} \vee \overset{0}{\bar{x}_1}\overset{1}{x_2}\overset{0}{\bar{x}_3}\overset{1}{x_4} \vee \overset{0}{\bar{x}_1}\overset{1}{x_2}\overset{1}{x_3}\overset{1}{x_4} \vee \overset{1}{x_1}\overset{1}{x_2}\overset{0}{\bar{x}_3}\overset{0}{\bar{x}_4}$
$\vee \overset{1}{x_1}\overset{0}{\bar{x}_2}\overset{1}{x_3}\overset{0}{\bar{x}_4} \vee \overset{1}{x_1}\overset{0}{\bar{x}_2}\overset{0}{\bar{x}_3}\overset{0}{\bar{x}_4} \vee \overset{1}{x_1}\overset{0}{\bar{x}_2}\overset{1}{x_3}\overset{1}{x_4} \vee \overset{0}{\bar{x}_1}\overset{0}{\bar{x}_2}\overset{0}{\bar{x}_3}\overset{1}{x_4}$
$\vee \overset{0}{\bar{x}_1}\overset{0}{\bar{x}_2}\overset{1}{x_3}\overset{1}{x_4}$

$DCF(f_7) = \cancel{m_{15}} \vee \cancel{m_5} \vee \cancel{m_7} \vee \cancel{m_{12}} \vee \cancel{m_{10}} \vee \cancel{m_8} \vee \cancel{m_{11}} \vee \cancel{m_1} \vee \cancel{m_3}$

x_1x_2 \ x_3x_4	00	01	11	10
00		m_1	m_3	
01		m_5	m_7	
11	m_{12}		m_{15}	
10	m_8		m_{11}	m_{10}

$max_1 = ...$
$max_2 = ...$
$max_3 = ...$
$max_4 = ...$
$max_5 = ...$

$\boxed{\text{EX3}}$ Using Veitch diagrams, simplify the following functions:

3.7. $f_7(x_1,x_2,x_3) = \bar{x}_1(x_2 \downarrow x_3) \vee x_1\bar{x}_2 x_3 \vee \overline{(x_1 \vee (x_2 \uparrow x_3))} \vee x_1 \bar{x}_2 \bar{x}_3$
$= \bar{x}_1 \bar{x}_2 \bar{x}_3 \vee x_1 \bar{x}_2 x_3 \vee \overline{x_1 \vee \bar{x}_2 \vee \bar{x}_3} \vee x_1 \bar{x}_2 \bar{x}_3$
$= \overset{0}{\bar{x}_1}\overset{0}{\bar{x}_2}\overset{0}{\bar{x}_3} \vee \overset{1}{x_1}\overset{0}{\bar{x}_2}\overset{1}{x_3} \vee \overset{0}{\bar{x}_1}\overset{1}{x_2}\overset{1}{x_3} \vee \overset{1}{x_1}\overset{0}{\bar{x}_2}\overset{0}{\bar{x}_3}$

$x \downarrow y = \overline{x \vee y} = \bar{x}\bar{y}$
$x \uparrow y = \overline{x \wedge y} = \bar{x} \vee \bar{y}$

$DCF(f_7) = m_0 \vee m_5 \vee m_3 \vee m_4$

	x_1		x_1	
x_2				m_3
$\bar{x}_2$	m_5	m_4	m_0	
	x_3	$\bar{x}_3$	$\bar{x}_3$	x_3

$max_1 = m_4 \vee m_5 = x_1\bar{x}_2$
$max_2 = m_4 \vee m_0 = \bar{x}_2\bar{x}_3$
$max_3 = m_3$ (isolated minterm)

$M(f_7) = \{max_1, max_2, max_3\}$
$C(f_7) = \{max_1, max_2, max_3\}$
$\Rightarrow M(f_7) = C(f_7) \Rightarrow 1^{st}$ case of simplification $\Rightarrow$

$\Rightarrow$ unique disjunctive simplified form

$$f_7^S(x_1,x_2,x_3) = max_1 \vee max_2 \vee max_3 = x_1\bar{x}_2 \vee \bar{x}_2\bar{x}_3 \vee \bar{x}_1x_2x_3$$

3.1 $f_1(x_1,x_2,x_3) = \bar{x}_1(x_2 \downarrow x_3) \vee \bar{x}_1\bar{x}_2x_3 \vee \overline{(x_1 \vee (x_2 \uparrow x_3))} \vee x_1x_2\bar{x}_3$

$= \bar{x}_1\bar{x}_2\bar{x}_3 \vee \bar{x}_1\bar{x}_2x_3 \vee \overline{x_1 \vee \bar{x}_2 \vee \bar{x}_3} \vee x_1x_2\bar{x}_3$

$= \overset{0}{\bar{x}_1}\overset{0}{\bar{x}_2}\overset{0}{\bar{x}_3} \vee \overset{0}{\bar{x}_1}\overset{0}{\bar{x}_2}\overset{1}{x_3} \vee \overset{0}{\bar{x}_1}\overset{1}{x_2}\overset{1}{x_3} \wedge \overset{1}{x_1}\overset{1}{x_2}\overset{0}{\bar{x}_3}$

$x \downarrow y = \overline{x \vee y} = \bar{x}\bar{y}$
$x \uparrow y = \overline{x \wedge y} = \bar{x} \vee \bar{y}$

$DCF(f_1) = m_0 \vee m_1 \vee m_3 \vee m_6$

	x_1		$\bar{x}_1$	
x_2		m_6		m_3
$\bar{x}_2$			m_0	m_1
	x_3	$\bar{x}_3$	$\bar{x}_3$	x_3

$\boxed{EX4}$ Simplify the following Boolean fns of 4 vars using Veitch diagrams:

4.1. $f_1(x_1,x_2,x_3,x_4) = x_1\bar{x}_4 \vee \underbrace{\overset{1}{x_1}\overset{1}{x_2}\overset{0}{\bar{x}_3}\overset{1}{x_4}}_{\cancel{m_{13}}} \vee \bar{x}_1x_2x_4 \vee \bar{x}_1x_3 \vee x_3\bar{x}_4$

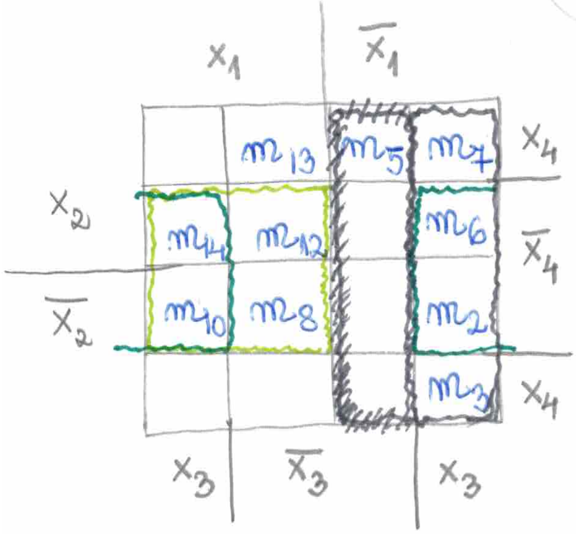
→ monom $x_1\bar{x}_4$ covers 4 minterms obtained through adding the missing variables:

- $x_1\bar{x}_4$: $x_1\bar{x}_2\bar{x}_3\bar{x}_4$ $\cancel{m_8}$
  $x_1\bar{x}_2x_3\bar{x}_4$ $\cancel{m_{10}}$
  $x_1x_2\bar{x}_3\bar{x}_4$ $\cancel{m_{12}}$
  $x_1x_2x_3\bar{x}_4$ $\cancel{m_{14}}$

$\bar{x}_1 x_2 x_4$: $\bar{x}_1 x_2 \bar{x}_3 x_4$ ~~m_5~~
$\bar{x}_1 x_2 x_3 x_4$ ~~m_7~~

$\bar{x}_1 x_3$: $\bar{x}_1 \bar{x}_2 x_3 \bar{x}_4$ ~~m_2~~
$\bar{x}_1 \bar{x}_2 x_3 x_4$ ~~m_3~~
$\bar{x}_1 x_2 x_3 \bar{x}_4$ ~~m_6~~
$\bar{x}_1 x_2 x_3 x_4$ ~~m_7~~

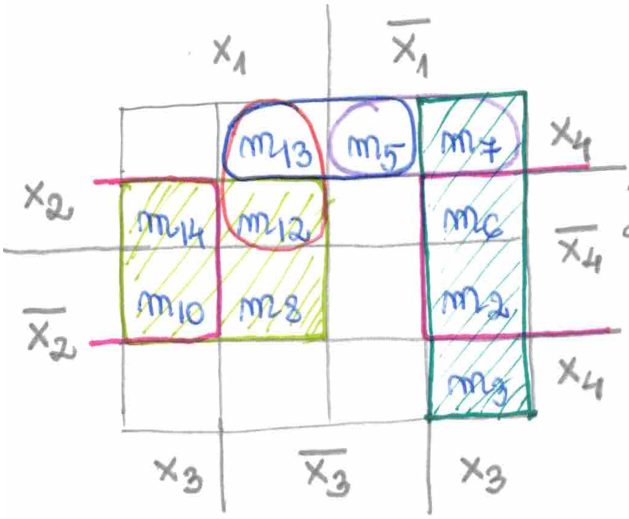
$x_3 \bar{x}_4$: $\bar{x}_1 \bar{x}_2 x_3 \bar{x}_4$ ~~m_2~~
$\bar{x}_1 x_2 x_3 \bar{x}_4$ ~~m_6~~
$x_1 \bar{x}_2 x_3 \bar{x}_4$ ~~m_{10}~~
$x_1 x_2 x_3 \bar{x}_4$ ~~m_{14}~~



the surrounding lines with this color mark the groups of cells where we have $\bar{x}_1$ and x_3 (~~~) and x_3 and $\bar{x}_4$ (~~~) ⇒ can fill them in directly in the diagram

$DCF(f_1) = m_8 \vee m_{10} \vee m_{12} \vee m_{14} \vee m_{13} \vee m_5 \vee m_7 \vee m_2 \vee m_3 \vee m_6$

can we simplify further?



these are what we started with; but

double factorizations:

$max_1 = m_{14} \vee m_{10} \vee m_6 \vee m_2 = x_3 \bar{x}_4$

$max_2 = m_{14} \vee m_{12} \vee m_{10} \vee m_8 = x_1 \bar{x}_4$

$max_3 = m_7 \vee m_6 \vee m_2 \vee m_3 = \bar{x}_1 x_3$

simple factorizations:

$max_4 = m_5 \vee m_7 = \bar{x}_1 x_2 x_4$

$max_5 = m_{13} \vee m_5 = x_2 \bar{x}_3 x_4$

$max_6 = m_{13} \vee m_{12} = x_1 x_2 \bar{x}_3$

$\left.\begin{array}{l} M(f_1) = \{max_1, \dots, max_6\} \\ C(f) = \{max_2, max_3\} \end{array}\right\} \Rightarrow$ ~~case 2~~ $\left.\begin{array}{l} M(f) \neq C(f) \\ C(f) \neq \emptyset \end{array}\right\} \Rightarrow 2^{nd}$ case

of the simplification algorithm ⇒

⇒ $g = max_2 \vee max_3$ (see ex. 2)

⇒ m_{13}, m_5 uncovered ⇒ max_5 covers both with minimum number of overlaps

$\boxed{\text{Ex5}}$ Using **Quine's** method, simplify the following Boolean fns given by their values 0.

x_1	x_2	x_3	f_7	
0	0	0	1	m_0
0	0	1	1	m_1
0	1	0	1	m_2
0	1	1	0	
1	0	0	0	
1	0	1	1	m_5
1	1	0	1	m_6
1	1	1	0	

5.7. $f_7(0,1,1) = f_7(1,0,0) = f_7(1,1,1) = 0$

$\Rightarrow$ DCF obtained from the table

$DCF(f_7) = m_0 \vee m_1 \vee m_2 \vee m_5 \vee m_6$

QUINE

STEP 1: Sort the support set of f

$S_{f_7} = \{(x_1,x_2,x_3) \mid f_7(x_1,x_2,x_3) = 1\}$

in ascending/descending order with respect ~~of~~ to number of values 1 in each n-tuple.

$$S_{f_7} = \{\underbrace{(0,0,0)}_{\text{I}}, \underbrace{(0,0,1),(0,1,0)}_{\text{II}}, \underbrace{(1,0,1),(1,1,0)}_{\text{III}}\}$$

STEP 2:

REPRESENTATION

GROUP	x_1	x_2	x_3	
I	0	0	0	m_0 ✓
II	0	0	1	m_1 ✓
	0	1	0	m_2 ✓
III	1	0	1	m_5 ✓
	1	1	0	m_6 ✓
IV = I + II	0	0	—	$m_0 \vee m_1 = \bar{x}_1\bar{x}_2 = max_1$
	0	—	0	$m_0 \vee m_2 = \bar{x}_1 \bar{x}_3 = max_2$
V = II + III	—	0	1	$m_1 \vee m_5 = \bar{x}_2 x_3 = max_3$
	—	1	0	$m_2 \vee m_6 = x_2\bar{x}_3 = max_4$

$\Rightarrow M(f) = \{max_1, max_2, max_3, max_4\}$

$\to$ only simple factorizations can be applied

only neighboring groups may contain two adjacent/neighbor monoms which factorise

STEP 3:

→ build minterms-maximal monoms correspondence tableaux

	max_1	max_2	max_3	max_4
m_0	*	*		
m_1	*		*	
m_2		*		*
m_5			*	
m_6				*

→ circle '*' symbols which are unique on their row

⇒ max_3 and max_4 are central monoms

$C(f_7) = \{max_3, max_4\}$

$g = max_3 \vee max_4$

→ m_1 and m_5 are covered by max_3
m_2 and m_6 are covered by max_4 } ⇒ m_0 uncovered. ⇒

⇒ can be covered in 2 ways, with the same number of overlaps: by max_1 or max_2 ⇒ 2 disjunctive simplified forms

$$\Rightarrow f^S_{7_1}(x_1,x_2,x_3) = g \vee max_1 = max_1 \vee max_3 \vee max_4 = \overline{x}_1\overline{x}_2 \vee \overline{x}_2 x_3 \vee x_2\overline{x}_3$$

$$f^S_{7_2}(x_1,x_2,x_3) = g \vee max_2 = max_2 \vee max_3 \vee max_4 = \overline{x}_1 x_2 \vee \overline{x}_2 x_3 \vee x_2\overline{x}_3$$

$\boxed{\text{EX6}}$ Using Quine's method, simplify the following Boolean fns given by their DCF.

6.1. $f_1(x_1,x_2,x_3) = m_0 \vee m_3 \vee m_4 \vee m_5 \vee m_6 \vee m_7$

→ follow the steps detailed for EX5

$$S_{f_1} = \{\underbrace{(0,0,0)}_{\mathrm{I}}, \underbrace{(1,0,0)}_{\mathrm{II}}, \underbrace{(0,1,1),(1,0,1),(1,1,0)}_{\mathrm{III}}, \underbrace{(1,1,1)}_{\mathrm{IV}}\}$$

6.7. $f_7(x_1,x_2,x_3) = m_0 \vee m_1 \vee m_2 \vee m_3 \vee m_4 \vee m_7$

$$S_{f_7} = \{\underbrace{(0,0,0)}_{\mathrm{I}}, \underbrace{(0,0,1),(0,1,0),(1,0,0)}_{\mathrm{II}}, \underbrace{(0,1,1)}_{\mathrm{III}}, \underbrace{(1,1,1)}_{\mathrm{IV}}\}$$

f_1

	x_1	x_2	x_3	
I	0	0	0	m_0 ✓
II	1	0	0	m_4 ✓
III	0	1	1	m_3 ✓
	1	0	1	m_5 ✓
	1	1	0	m_6 ✓
IV	1	1	1	m_7 ✓
V=I+II	–	0	0	$m_0 \vee m_4 = \overline{x}_2\overline{x}_3 = max_1$
VI=II+III	1	0	–	$m_4 \vee m_5 = x_1\overline{x}_2$ ✓
	1	–	0	$m_4 \vee m_6 = x_1\overline{x}_3$ ✓
VII=III+IV	–	1	1	$m_3 \vee m_7 = x_2x_3 = max_2$
	1	–	1	$m_5 \vee m_7 = x_1x_3$ ✓
	1	1	–	$m_6 \vee m_7 = x_1x_2$ ✓
VIII=VI+VII	1	–	–	$m_4 \vee m_5 \vee m_6 \vee m_7 = x_1 = max_3$
	1	–	–	$m_4 \vee m_6 \vee m_5 \vee m_7 = x_1$

REPRESENTATION

SIMPLE FACTORIZATIONS

DOUBLE FACTORIZATIONS

the symbol '–' cannot be combined with anything else
$\Rightarrow$ beginning with double factorizations, only rows from two neighboring groups having '–' on the same column will be combined

$M(f) = \{max_1, max_2, max_3\}$

	max_1	max_2	max_3
m_0	*		
m_3		*	
m_4	*		*
m_5			*
m_6			*
m_7		*	*

$C(f) = \{max_1, max_2, max_3\}$

$\Rightarrow M(f) = C(f) \Rightarrow 1^{st}$ case of simplification
$\Rightarrow$ unique simplified form

$$f_1^S(x_1, x_2, x_3) = max_1 \vee max_2 \vee max_3$$
$$= x_1 \vee \overline{x}_2\overline{x}_3 \vee x_2x_3$$

f_7

	x_1	x_2	x_3	
I	0	0	0	m_0 ✓
II	0	0	1	m_1 ✓
	0	1	0	m_2 ✓
	1	0	0	m_4 ✓
III	0	1	1	m_3 ✓
IV	1	1	1	m_7 ✓
V = I + II	0	0	–	$m_0 \vee m_1 = \bar{x}_1\bar{x}_2$ ✓
	0	–	0	$m_0 \vee m_2 = \bar{x}_1\bar{x}_3$ ✓
	–	0	0	$m_0 \vee m_4 = \bar{x}_2\bar{x}_3 = max_1$
VI = II + III	0	–	1	$m_1 \vee m_3 = \bar{x}_1 x_3$ ✓
	0	1	–	$m_2 \vee m_3 = \bar{x}_1 x_2$ ✓
VII = III + IV	–	1	1	$m_3 \vee m_7 = x_2 x_3 = max_2$
	0	–	–	$m_0 \vee m_1 \vee m_2 \vee m_3 = \bar{x}_1 = max_3$
	0	–	–	$m_0 \vee m_2 \vee m_1 \vee m_3 = \bar{x}_1$

REPRESENTATION

SIMPLE FACTORIZATIONS

DOUBLE FACTORIZATIONS

$M(f_7) = \{max_1, max_2, max_3\}$

	max_1	max_2	max_3
m_0	*		*
m_1			*
m_2			*
m_3		*	*
m_4	*		
m_7		*	

$C(f_7) = \{max_1, max_2, max_3\} \Rightarrow \ldots$

$\boxed{\text{EX 7}}$ Simplify the following functions of 4 variables given by their value 1, using Quine's method.

7.1. $f_1(x_1,x_2,x_3,x_4) = f_1\overset{m_{13}}{(1,1,0,1)} = f_1\overset{m_7}{(0,1,1,1)} =$
$= f_1\overset{m_{12}}{(1,1,0,0)} = f_1\overset{m_4}{(0,1,0,0)} = f_1\overset{m_0}{(0,0,0,0)} =$
$= f_1\overset{m_1}{(0,0,0,1)} = f_1\overset{m_3}{(0,0,1,1)} = 1$

$\Rightarrow DCF(f_1) = m_0 \vee m_1 \vee m_3 \vee m_4 \vee m_7 \vee m_{12} \vee m_{13} \vee m_{15}$

$S_{f_1} = \{(0,0,0,0), (0,0,0,1), (0,1,0,0), (0,0,1,1), (1,1,0,0), (1,1,0,1), (0,1,1,1), (1,1,1,1)\}$

REPRESENTATION

	x_1	x_2	x_3	x_4	
$\overline{I}$	0	0	0	0	m_0 ✓
$\overline{II}$	0	0	0	1	m_1 ✓
	0	1	0	0	m_4 ✓
$\overline{III}$	0	0	1	1	m_3 ✓
	1	1	0	0	m_{12} ✓
$\overline{IV}$	1	1	0	1	m_{13} ✓
	0	1	1	1	m_7 ✓
$\overline{V}$	1	1	1	1	m_{15} ✓
$\overline{VI} = \overline{I}+\overline{II}$	0	0	0	–	$m_0 \vee m_1 = \overline{x}_1\overline{x}_2\overline{x}_3 = max_1$
	0	–	0	0	$m_0 \vee m_4 = \overline{x}_1 x_3 x_4 = max_2$
$\overline{VII} = \overline{II}+\overline{III}$	0	0	–	1	$m_1 \vee m_3 = \overline{x}_1\overline{x}_2 x_4 = max_3$
	–	1	0	0	$m_4 \vee m_{12} = x_2\overline{x}_3\overline{x}_4 = max_4$
$\overline{VIII} = \overline{III}+\overline{IV}$	0	–	1	1	$m_3 \vee m_7 = \overline{x}_1 x_3 x_4 = max_5$
	1	1	0	–	$m_{12} \vee m_{13} = x_1 x_2 \overline{x}_3 = max_6$
$\overline{IX} = \overline{IV}+\overline{V}$	1	1	–	1	$m_{13} \vee m_{15} = x_1 x_2 x_4 = max_7$
	–	1	1	1	$m_7 \vee m_{15} = x_2 x_3 x_4 = max_8$

SIMPLE FACTORIZATIONS

$M(f_1) = \{max_1, max_2, \ldots, max_8\}$

	max_1	max_2	max_3	max_4	max_5	max_6	max_7	max_8
m_0	*	*						
m_1	*		*					
m_3			*		*			
m_4		*		*				
m_7					*			*
m_{12}				*		*		
m_{13}						*	*	
m_{15}							*	*

There is no symbol '*' that is unique on ~~its~~ a row. ⇒
⇒ $C(f_1) = \emptyset$ ⇒ 3^{rd} case of the simplification algorithm

⇒ 2 simplified forms of f_{1} obtained by covering all the minterms with a minimum number of maximal monoms, without overlaps

////// $f^S_{1_1}(x_1,x_2,x_3,x_4) = max_1 \vee max_4 \vee max_5 \vee max_7$
$= \bar{x}_1\bar{x}_2\bar{x}_3 \vee x_2\bar{x}_3\bar{x}_4 \vee \bar{x}_1x_3x_4 \vee x_1x_2x_4$

////// $f^S_{1_2}(x_1,x_2,x_3,x_4) = max_2 \vee max_3 \vee max_6 \vee max_8$
$= \bar{x}_1x_3x_4 \vee \bar{x}_1\bar{x}_2x_4 \vee x_1x_2\bar{x}_3 \vee x_2x_3x_4$

7.7. $f_7(\overset{m_{15}}{1,1,1,1}) = f_7(\overset{m_{13}}{1,1,0,1}) = f_7(\overset{m_5}{0,1,0,1}) = f_7(\overset{m_4}{0,1,0,0}) =$
$= f_7(\overset{m_6}{0,1,1,0}) = f_7(\overset{m_2}{0,0,1,0}) = f_7(\overset{m_{11}}{1,0,1,1}) =$
$= f_7(\overset{m_3}{0,0,1,1})$

$DCF(f_7) = m_2 \vee m_3 \vee m_4 \vee m_5 \vee m_6 \vee m_{11} \vee m_{13} \vee m_{15}$

.....

⇒ should also obtain 3^{rd} case of simplification

$\boxed{\text{Ex8}}$ Using Moisil's method, simplify the following fns of 3 variables:

MOISIL'S METHOD

→ uses propositional logic to obtain the simplified forms of a Boolean function from the set of maximal monoms

→ the set of maximal monoms can be computed ~~as~~ using any of the methods previously discussed (Veitch, Karnaugh, Quine)

8.7. $f_7(x_1,x_2,x_3) = m_1 \vee m_2 \vee m_4 \vee m_5 \vee m_6$

$x_1 \backslash x_2x_3$	00	01	11	10
0		m_1		m_2
1	m_4	m_5		m_6

$max_1 = m_2 \vee m_6 = x_2\overline{x_3}$

$max_2 = m_1 \vee m_5 = \overline{x_2}x_3$

$max_3 = m_4 \vee m_5 = x_1\overline{x_2}$

$max_4 = m_4 \vee m_6 = x_1\overline{x_3}$

We consider the following propositional sentences:

p_i: "max_i belongs to a simplified form of f_7"

$i = 1,2,3,4$

Each minterm must be covered by a maximal monom in a simplified form =>

"m_1 is covered by max_2" : $p_2 \equiv T$

"m_2 —"— —"— max_1" : $p_1 \equiv T$

"m_4 —"— —"— max_3 or max_4" : $p_3 \vee p_4 \equiv T$

"m_5 —"— —"— max_2 or max_3" : $p_2 \vee p_3 \equiv T$

"m_6 —"— —"— max_1 or max_4" : $p_1 \vee p_4 \equiv T$

To obtain a simplified form of f, all the minterms from the DCF must be covered by a minimum number of maximal monoms, with a minimum number of overlaps.

$\Rightarrow p_2 \wedge p_1 \wedge (p_3 \vee p_4) \wedge \cancel{(p_2 \vee p_3)} \wedge \cancel{(p_1 \vee p_4)} \equiv T$ (ABSORPTION) $U \wedge (U \vee V) \equiv U$

$p_2 \wedge p_1 \wedge (p_3 \vee p_4) \equiv T$ (DISTRIBUTIVITY)

$(p_2 \wedge p_1 \wedge p_3) \vee (p_2 \wedge p_1 \wedge p_4) \equiv T$ DNF with 2 cubes

· A DNF is T if at least 1 cube is true.

I $p_2 \wedge p_1 \wedge p_3 \equiv T \Rightarrow f^S_{F_1}(x_1, x_2, x_3) = max_1 \vee max_2 \vee max_3$

$= x_2\overline{x}_3 \vee \overline{x}_2 x_3 \vee x_1 \overline{x}_2$

$\underline{\mathrm{II}}$ $p_2 \wedge p_1 \wedge p_4 \equiv T \Rightarrow f^S_{F_2}(x_1, x_2, x_3) = max_1 \vee max_2 \vee max_4$

$= x_2\overline{x}_3 \vee \overline{x}_2 x_3 \vee x_1 \overline{x}_3$