Retake Exam

1. Draw the interior and the boundary of the set $\{(x,y) \in \mathbb{R}^2 \mid |x-1| + |y| < 1\}$. **1p**

2. Find and classify all the critical points of $f(x,y) = x^2 + xy - y^2$. **1p**

3. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = \sin(xy) + \cos(xy)$.

   (a) Find the first order Taylor polynomial for f around $(1, \pi/2)$. **1p**

   (b) In which direction does f decrease the most at $(1, \pi/2)$? **0.5p**

4. Find the extrema of $3x + 2y$ subject to $x^2 + y^2 = 1$. **1p**

5. Consider n distinct points (x_i, y_i), $i = \overline{1,n}$. Find the line of best fit $y = ax+b$ that minimizes the least squares error **1p**

$$E = \sum_{i=1}^{n} |y_i - (ax_i + b)|^2 \to \min.$$

6. Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix and let $b \in \mathbb{R}^n$ be a vector. Consider the function

$$f : \mathbb{R}^n \to \mathbb{R},\ f(x) = \|Ax - b\|^2 + \|x\|^2.$$

   Find the gradient of f. Prove that f has a unique minimum. **1p**

7. Consider the ellipse $\mathcal{E}$ given by the equation

$$\frac{x^2}{9} + \frac{y^2}{4} = 1.$$

   (a) Find the equation of the tangent line to the ellipse $\mathcal{E}$ at a point (x_0, y_0). **1p**

   (b) Using Lagrange multipliers, find the points on $\mathcal{E}$ the furthest away from the origin. **0.5p**

8. Compute the following integrals:

   (a) $\displaystyle\int_0^1 \int_x^1 e^{-y^2}\,\mathrm{d}y\,\mathrm{d}x$. **1p**

   (b) $\displaystyle\iint_D \sin(x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y$, where $D = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \le 4,\ y \ge 0\}$. **1p**

Time: 2h. Total number of points for the exam: 10p.