Retake Final Exam #1

1. Study the convergence and the absolute convergence of the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n(n+1)}}$. **1p**

2. (a) Draw the interior and the boundary of the set $\{(x, y) \in \mathbb{R}^2 \mid |x-1| + |y+1| < 1\}$. **1p**
   (b) Let $x, y \in \mathbb{R}^n$ with $\|x\| = \|y\| = 1$. Prove that $\|x+y\|^2 + \|x-y\|^2 = 4$. **1p**

3. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = \sin(xy) + \cos(xy)$.
   (a) In which direction does f decrease the most at $(1, \pi/2)$? **0.5p**
   (b) In which direction is f constant (zero directional derivative) at $(1, \pi/2)$? **0.5p**
   (c) Find the first order Taylor polynomial for f around $(1, \pi/2)$. **0.5p**

4. Find and classify all the critical points of $f(x, y) = 3x^2 + 2y^3 - 6xy$. **1p**

5. Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix and let $b \in \mathbb{R}^n$ be a vector. Consider the function
$$f : \mathbb{R}^n \to \mathbb{R},\ f(x) = \|Ax - b\|^2.$$
   (a) Find the gradient of f. Prove that a critical point x^* satisfies $A^2x^* = Ab$. **1p**
   (b) Find the Hessian of f. Give a condition for A s.t. f has a unique minimum. **0.5p**

6. Consider the ellipse $\mathcal{E}$ given by the equation
$$\frac{x^2}{3} + \frac{y^2}{4} = 1.$$
   Using Lagrange multipliers, find the points on $\mathcal{E}$ the furthest away from the origin. **1p**

7. (a) Let $D_1 = \{(x, y) \in \mathbb{R}^2 \mid 0 \le x \le 1,\ x^2 \le y \le \sqrt{x}\}$. Find the area of D_1, for example by using a double integral. **1p**
   (b) Let $D_2 = \{(x, y) \in \mathbb{R}^2 \mid 0 \le x \le 1,\ x \le y \le \sqrt{x}\}$. Compute $\iint_{D_2} \frac{e^y}{y}\,\mathrm{d}x\mathrm{d}y$ by changing the order of integration. **1p**

Time: 2h. The marks in the final exam add up to **10p**.