Exercises on Chapter 3: Ordering Relations

1. In the zoo, the animals are fed according to a fixed schedule:
   (a) The giraffes are fed before the zebras, but after the apes.
   (b) The bears are fed immediately after the apes.
   (c) The lions are fed after the zebras.
   Can you recover the full schedule from this information?

2. The relation *IsDivisorOf* on the set $P := \{1, 2, 3, \dots\} = \mathbb{N} \setminus \{0\}$ is defined by

$$x \textit{ IsDivisorOf } y \qquad \text{if and only if} \qquad y = a \cdot x \text{ for some } a \in P.$$

   (a) Show that *IsDivisorOf* is a partial order on U.
   (b) Show that *IsDivisorOf* is not a total order on U.
   (c) Let A be the set of all common divisors of the numbers 12 and 30. Show that A has a largest element in the order defined by *IsDivisorOf*.
   (d) Let B be the set of all common multiples of 12 and 30. Show that B has a smallest element in the order defined by *IsDivisorOf*.

3. Consider the ordering relation *IsDivisorOf* on the set $A := \{2, 3, 6, 10, 15, 30\}$.
   (a) Construct the Hasse diagram of the relation *IsDivisorOf* on the set A using the algorithm that you have learned. Explicitly write down the sets G_x and H_x obtained in the construction and draw the Hasse diagram.
   (b) Does the set A have a smallest element? If so, please give this smallest element. If not, please list all the minimal elements of A.
   (c) Does the set A have a largest element? If so, please give this largest element. If not, please list all the maximal elements of A.

4. Suppose that $\leq$ is a total order on a set V, and that $p \in V$ is a maximal element of V in this order. Argue that p is then in fact the largest element of V.

5. Let *Alphabet* $:= \{\mathrm{a}, \mathrm{b}, \mathrm{c}, \dots, \mathrm{z}\}$. In this exercise we study the lexicographic order and the Cartesian order on the set *Alphabet*^{4} of all four-letter words, induced by the usual alphabetic order on $\{\mathrm{a}, \mathrm{b}, \mathrm{c}, \dots, \mathrm{z}\}$.
   (a) Let A be the set of all four-letter words of which both the second and the third letter is one of the letters b or c. Give a largest and a smallest element of A in the lexicographic order on *Alphabet*^{4}, or explain why such an element does not exist.
   (b) Let B be the set of all four-letter words of which the second letter is a b or the third letter is a c (or both). Give a largest and a smallest element of B in the Cartesian order on *Alphabet*^{4}, or explain why such an element does not exist.

6. Let V be the subset of $\{\mathrm{a},\mathrm{b},\mathrm{c},\mathrm{d}\}^2$ defined by

$$V := \{\langle \mathrm{a},\mathrm{a}\rangle, \langle \mathrm{a},\mathrm{b}\rangle, \langle \mathrm{b},\mathrm{a}\rangle, \langle \mathrm{b},\mathrm{b}\rangle, \langle \mathrm{c},\mathrm{c}\rangle, \langle \mathrm{c},\mathrm{d}\rangle, \langle \mathrm{d},\mathrm{c}\rangle, \langle \mathrm{d},\mathrm{d}\rangle\}.$$

We can now order the elements of V by the Cartesian ordering relation induced by the alphabetic order on $\{\mathrm{a},\mathrm{b},\mathrm{c},\mathrm{d}\}$.

(a) Construct and draw the Hasse diagram of this Cartesian ordering relation in the set V using the algorithm that you have learned. Explicitly write down the sets G_x and H_x obtained in the construction. To simplify the notation, you can write ab instead of $\langle \mathrm{a},\mathrm{b}\rangle$, et cetera.

(b) Let A be the following subset of V:

$$A := \{\langle \mathrm{a},\mathrm{b}\rangle, \langle \mathrm{b},\mathrm{b}\rangle, \langle \mathrm{c},\mathrm{c}\rangle, \langle \mathrm{c},\mathrm{d}\rangle, \langle \mathrm{d},\mathrm{c}\rangle\}$$

Does the set A have a smallest element? If so, please give this smallest element, if not, please list all the minimal elements of A.

(c) Does the set A have a largest element? If so, please give this largest element, if not, please list all the maximal elements of A.

7. Consider the sets $A := \{a_1, a_2, a_3\}$ and $B := \{b_1, b_2, b_3\}$, with partial orders $\leq_A$ and $\leq_B$ on these respective sets described by the following Hasse diagrams:

$$\begin{tikzcd} a_2 & & a_3 & & & b_1 & \\ & a_1 \arrow[ul] \arrow[ur] & & & b_3 \arrow[ur] & & b_2 \arrow[ul] \end{tikzcd}$$

(a) Draw the Hasse diagram of the Cartesian ordering relation on $A \times B$ induced by $\leq_A$ and $\leq_B$, by adding arrows to the picture below (you can use the algorithm you have learned, but you are not obliged to do so):

$$\begin{array}{ccc} \langle a_2, b_2\rangle & \langle a_1, b_2\rangle & \langle a_3, b_2\rangle \\ \\ \langle a_2, b_1\rangle & \langle a_1, b_1\rangle & \langle a_3, b_1\rangle \\ \\ \langle a_2, b_3\rangle & \langle a_1, b_3\rangle & \langle a_3, b_3\rangle \end{array}$$

(b) Determine and list all the maximal and all the minimal elements of $A \times B$ in this Cartesian ordering. Is there a maximum? Is there a minimum?

(c) The lexicographic order on $A \times B$ induced by $\leq_A$ and $\leq_B$ is not total. Give an example of two elements in $A \times B$ that are not comparable in the lexicographic order.