

Exercises on Chapter 3: Ordering Relations

1. In the zoo, the animals are fed according to a fixed schedule:
 - (a) The giraffes are fed before the zebras, but after the apes.
 - (b) The bears are fed immediately after the apes.
 - (c) The lions are fed after the zebras.Can you recover the full schedule from this information?
2. The relation *IsDivisorOf* on the set $P := \{1, 2, 3, \dots\} = \mathbb{N} \setminus \{0\}$ is defined by
$$x \text{ IsDivisorOf } y \quad \text{if and only if} \quad y = a \cdot x \text{ for some } a \in P.$$
 - (a) Show that *IsDivisorOf* is a partial order on U .
 - (b) Show that *IsDivisorOf* is not a total order on U .
 - (c) Let A be the set of all common divisors of the numbers 12 and 30. Show that A has a largest element in the order defined by *IsDivisorOf*.
 - (d) Let B be the set of all common multiples of 12 and 30. Show that B has a smallest element in the order defined by *IsDivisorOf*.
3. Consider the ordering relation *IsDivisorOf* on the set $A := \{2, 3, 6, 10, 15, 30\}$.
 - (a) Construct the Hasse diagram of the relation *IsDivisorOf* on the set A using the algorithm that you have learned. Explicitly write down the sets G_x and H_x obtained in the construction and draw the Hasse diagram.
 - (b) Does the set A have a smallest element? If so, please give this smallest element. If not, please list all the minimal elements of A .
 - (c) Does the set A have a largest element? If so, please give this largest element. If not, please list all the maximal elements of A .
4. Suppose that \leq is a total order on a set V , and that $p \in V$ is a maximal element of V in this order. Argue that p is then in fact the largest element of V .
5. Let $Alphabet := \{a, b, c, \dots, z\}$. In this exercise we study the lexicographic order and the Cartesian order on the set $Alphabet^4$ of all four-letter words, induced by the usual alphabetic order on $\{a, b, c, \dots, z\}$.
 - (a) Let A be the set of all four-letter words of which both the second and the third letter is one of the letters b or c. Give a largest and a smallest element of A in the lexicographic order on $Alphabet^4$, or explain why such an element does not exist.
 - (b) Let B be the set of all four-letter words of which the second letter is a b or the third letter is a c (or both). Give a largest and a smallest element of B in the Cartesian order on $Alphabet^4$, or explain why such an element does not exist.

6. Let V be the subset of $\{a, b, c, d\}^2$ defined by

$$V := \{\langle a, a \rangle, \langle a, b \rangle, \langle b, a \rangle, \langle b, b \rangle, \langle c, c \rangle, \langle c, d \rangle, \langle d, c \rangle, \langle d, d \rangle\}.$$

We can now order the elements of V by the Cartesian ordering relation induced by the alphabetic order on $\{a, b, c, d\}$.

- (a) Construct and draw the Hasse diagram of this Cartesian ordering relation in the set V using the algorithm that you have learned. Explicitly write down the sets G_x and H_x obtained in the construction. To simplify the notation, you can write ab instead of $\langle a, b \rangle$, et cetera.
- (b) Let A be the following subset of V :

$$A := \{\langle a, b \rangle, \langle b, b \rangle, \langle c, c \rangle, \langle c, d \rangle, \langle d, c \rangle\}$$

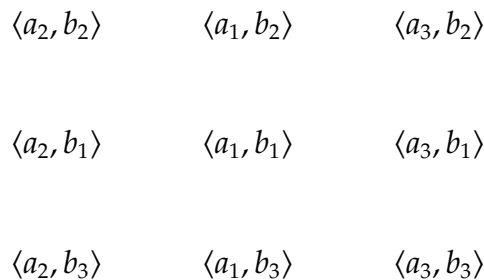
Does the set A have a smallest element? If so, please give this smallest element, if not, please list all the minimal elements of A .

- (c) Does the set A have a largest element? If so, please give this largest element, if not, please list all the maximal elements of A .

7. Consider the sets $A := \{a_1, a_2, a_3\}$ and $B := \{b_1, b_2, b_3\}$, with partial orders \leq_A and \leq_B on these respective sets described by the following Hasse diagrams:



- (a) Draw the Hasse diagram of the Cartesian ordering relation on $A \times B$ induced by \leq_A and \leq_B , by adding arrows to the picture below (you can use the algorithm you have learned, but you are not obliged to do so):



- (b) Determine and list all the maximal and all the minimal elements of $A \times B$ in this Cartesian ordering. Is there a maximum? Is there a minimum?
- (c) The lexicographic order on $A \times B$ induced by \leq_A and \leq_B is not total. Give an example of two elements in $A \times B$ that are not comparable in the lexicographic order.