Exercises on Chapter 6: Induction and Recursion

1. Consider a sequence $(t_n)_{n=0}^{\infty}$ of real-valued numbers defined recursively by

$$t_0 := 0, \qquad t_{n+1} := t_n + 2n + 3.$$

   (a) Calculate the terms t_1, t_2, t_3, t_4 of this sequence.
   (b) Prove by mathematical induction that for all $n \geq 0$,

$$t_n = n(n+2).$$

2. Consider a sequence $(t_n)_{n=1}^{\infty}$ of real-valued numbers defined recursively by

$$t_1 := 2, \qquad t_{n+1} := 2t_n - 2n + 3.$$

   (a) Calculate the terms t_2, t_3, t_4, t_5 of this sequence.
   (b) Prove by mathematical induction that for all $n \geq 1$,

$$t_n = 2^{n-1} + 2n - 1.$$

3. Consider a sequence $(t_n)_{n=1}^{\infty}$ of real-valued numbers defined recursively by

$$t_1 := 1, \qquad t_{n+1} := t_n + 3n + 1.$$

   (a) Calculate the terms t_2, t_3, t_4, t_5 of this sequence.
   (b) Prove by mathematical induction that for all $n \geq 1$,

$$t_n = \tfrac{1}{2}n(3n-1).$$

4. Prove each of the following statements by mathematical induction in n:
   (a) For all $n \geq 1$, $\sum_{k=1}^{n} \frac{1}{k\cdot(k+1)} = \frac{n}{n+1}$, that is,

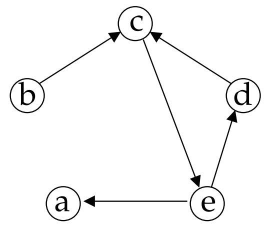
$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n\cdot(n+1)} = \frac{n}{n+1}.$$

   (b) For all $n \geq 1$, $\sum_{k=1}^{n}(k+1)\cdot 2^k = n\cdot 2^{n+1}$, that is,

$$2\cdot 2^1 + 3\cdot 2^2 + 4\cdot 2^3 + \cdots + (n+1)\cdot 2^n = n\cdot 2^{n+1}.$$

5. Prove each of the following statements by mathematical induction in n:
   (a) For all $n \geq 4$ we have that $n^2 \leq 2^n$.
   (b) For all $n \geq 1$, the number $7^n + 3^{n+1}$ is divisible by 4.
   (c) For all $n \geq 1$, the number $3^{2n+1} + 2^{n-1}$ is divisible by 7.

6. Draw a directed graph representation of the transitive closure of the relation R in the set $\{a, b, c, d, e\}$ that is depicted by the following directed graph:



7. Determine the transitive closures of the following relations:
   (a) The relation R in $\mathbb{N}$ with description "is a direct predecessor of", that is, the relation R defined by $x\,R\,y \iff x + 1 = y$.
   (b) The relation R in the set of all non-empty strings of bits 0 and 1, described as follows: every string of bits u that ends with a 0 relates to exactly two other strings of bits, namely to the string obtained by replacing the last 0 of u by 10, and the string obtained by replacing the last 0 of u by 1. That is,

$$\begin{array}{cccc} 0\,R\,10 & 10\,R\,110 & 00\,R\,010 & 110\,R\,1110 \\ 0\,R\,1 & 10\,R\,11 & 00\,R\,01 & 110\,R\,111 \end{array}$$

   and so on.