Exercises on Chapter 5: Functions

1. In this exercise, we consider the following relations defined in the set of all people living in strictly monogamous countries:

$$\begin{aligned} \mathit{IsMotherOf} &:= \{\, \langle x, y\rangle \colon x \text{ is biological mother of } y \,\} \\ \mathit{IsUncleOf} &:= \{\, \langle x, y\rangle \colon x \text{ is uncle of } y \,\} \\ \mathit{IsMarriedTo} &:= \{\, \langle x, y\rangle \colon x \text{ is lawfully married to } y \,\} \end{aligned}$$

   (a) For each of these three relations, determine whether or not the relation is a function. Explain your answers.
   (b) Now consider the inverse of each relation, and determine whether or not this inverse is a function. Explain your answers.

2. For each of the functions described below, determine whether the function is one-one (injective) or many-one (not injective). Explain your answers.
   (a) The function $f\colon A \to \mathbb{N}$, where $A := \{10, 11, 12, \dots, 99\}$ and $f(n)$ is the sum of the digits of n in the decimal representation (so that, for example, $f(29) = 2 + 9 = 11$).
   (b) The function $f\colon W \to W$, where W is the set of all words of at least one lower case letter, and f is the function that converts such a word into a new word by replacing every letter of the word by the next letter of the alphabet (and every `z` by `a`): $f($`this`$) =$ `uijt`, $f($`zero`$) =$ `afsp`, and so on.
   (c) The function f described by the following table:

x	0	1	2	3	4	5	6	7	8	9	10
$f(x)$	0	1	3	5	9	11	18	17	14	5	2

3. For each of the functions below, determine whether or not the function is injective (one-one), and whether or not it is surjective. Explain your answers.
   (a) The function $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) := x$.
   (b) The function $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) := x^2$.
   (c) The function $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) := x^3$.
   (d) The function $f\colon \{x\colon x \geq 0\} \to \mathbb{R}$ given by $f(x) := x^2$.

4. For each of the injective functions below, give a description of its inverse function (that is, give an explicit formula for $f^{-1}(x)$), and determine the domain of definition and image (range) of this inverse function.
   (a) The function $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) := x + 3$.
   (b) The function $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) := 3 \cdot x$.
   (c) The function $f\colon \{x\colon x \geq 0\} \to \mathbb{R}$ given by $f(x) := x^2$.
   (d) The function $f\colon \{x\colon x \leq 0\} \to \mathbb{R}$ given by $f(x) := x^2$.

5. Let W_2 be the set of all words of exactly two lower case letters, and let W_{2+} be the set of all words of at least two lower case letters. Let $tr\colon W_{2+} \to W_2$ be the function that truncates a word to its first two letters, so that, for example, $tr(\texttt{tree}) = \texttt{tr}$. Furthermore, let $index\colon W_2 \to \mathbb{N}$ be the function that assign to a word of two letters its position (index) in the lexicographically sorted list of all two-letter words, so that, for example, $index(\texttt{aa}) = 1$, $index(\texttt{af}) = 6$, $index(\texttt{ba}) = 27$, and so on. Let $f\colon W_{2+} \to \mathbb{N}$ be defined as $f := index \circ tr$.
   (a) Briefly describe what the function f does.
   (b) Show that the function f is total.
   (c) Determine whether or not f is surjective. Explain your answer.

6. In this exercise, we consider the function $sqrt\colon \{x\colon x \geq 0\} \to \{y\colon y \geq 0\}$ and, for every $a \neq 0$, the functions $mul_a\colon \mathbb{R} \to \mathbb{R}$ and $add_a\colon \mathbb{R} \to \mathbb{R}$ defined by

$$sqrt(x) := \sqrt{x}, \qquad mul_a(x) := a \cdot x, \qquad add_a(x) := x + a.$$

   (a) Give descriptions (i.e. formulas for $f(x)$ and $g(x)$) of the two functions

$$f := sqrt \circ add_1 \circ mul_2$$
$$g := add_2^{-1} \circ sqrt^{-1} \circ mul_5$$

   and determine their domains of definition and images (ranges).
   (b) Consider the inverse functions of the functions f and g below. Give descriptions (i.e. formulas for $f^{-1}(x)$ and $g^{-1}(x)$) of these inverse functions, and determine their domains of definition and images (ranges).

$$f := mul_5^{-1} \circ add_2 \circ mul_3$$
$$g := sqrt^{-1} \circ add_1$$

   (c) Express the functions $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) := 2\sqrt{x-3}$$
$$g(x) := \left(3\sqrt{x} + 2\right)^2$$

   as a composition of the functions mul_a, add_a, $sqrt$ and their inverses.

7. For each pair of sets A and B given below, your task is to prove that A and B have the same cardinality by finding a bijection $f\colon A \to B$. In each case, describe your bijection by giving an explicit formula for $f(x)$.
   (a) $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and $B = \{6, 8, 10, 12, 14, 16, 18, 20\}$.
   (b) $A = \mathbb{N}$ and $B = \{1, 4, 9, 16, 25, \dots\}$ the set of perfect (positive) squares.