Logic Exercises 2

1. Investigate which of the following semantic entailments hold:

   (a) $p \vee (q \to \neg p) \models p \vee \neg q$

   (b) $p \vee \neg q \models p \vee (q \to \neg p)$

   (c) $\models p \vee (q \to \neg p)$

   (d) $p \vee (q \to p) \models p \vee \neg q$

   (e) $p \vee \neg q \models p \vee (q \to p)$

   (f) $(p \to q) \to r \models p \to (q \to r)$

   (g) $p \to (q \to r) \models (p \to q) \to r$

   (h) $p \to (\neg q \vee r), \neg r \models q \to \neg p$

2. For each of the following specifications, try to find a formula ϕ which satisfies it. If no such ϕ exists, then explain why.

   (a) A formula ϕ such that $p \vDash \phi$ and $\neg p \vDash \phi$.

   (b) A formula ϕ such that $p \vDash \phi$ and $p \vDash \neg\phi$.

   (c) A formula ϕ such that $\phi \vDash p$ and $\phi \vDash \neg p$.

3. A binary relation $\mathcal{R}$ on a set S of elements is called:

   - *reflexive* if $s \mathcal{R} s$ for all $s \in S$;
   - *symmetric* if $s \mathcal{R} t$ implies $t \mathcal{R} s$, for all $s, t \in S$;
   - *transitive* if $s \mathcal{R} t$ and $t \mathcal{R} u$ imply $s \mathcal{R} u$, for all $s, t, u \in S$.

   (a) Is the semantic entailment relation $\models$ reflexive? Is it symmetric? Is it transitive?

   (b) Is semantic equivalence $\equiv$ reflexive? Is it symmetric? Is it transitive?

4. Addition $+$ and multiplication $\cdot$ in arithmetic are (just as conjunction and disjunction in propositional logic) *commutative* and *associative*. Also a *distributivity* law holds for these two operators.

   (a) Write down the corresponding five laws for addition and multiplication: two commutative laws, two associative laws, and one distributivity law.

   (b) Which distributivity law for addition and multiplication does not hold?

5. Which of the following statements are true? Prove if true, disprove if not true.

   (a) If $\phi \vDash \psi_1 \vee \psi_2$ holds, then $\phi \vDash \psi_1$ holds or $\phi \vDash \psi_2$ holds.

   (b) If $\phi \vDash \psi_1 \wedge \psi_2$ holds, then $\phi \vDash \psi_1$ holds and $\phi \vDash \psi_2$ holds.

6. Let $(\phi \to \psi_1) \vDash \chi$.

   (a) Suppose that $\psi_1 \vDash \psi_2$. Can we then conclude that $(\phi \to \psi_2) \vDash \chi$?

   (b) Suppose that $\psi_2 \vDash \psi_1$. Can we then conclude that $(\phi \to \psi_2) \vDash \chi$?

7. On the island of liars and truth speakers, everybody is either a liar (who always lies) or a truth speaker (who always speaks the truth). During a walk on this island you meet islanders A and B, and ask whether the road to the harbour is straight ahead.

   - A says: "Yes".
   - B says: "No".
   - "At least one of us is a liar", concludes B.

   (a) Formalize these three claims in propositional logic.

   (b) Determine for both A and B by means of a truth table whether this islander is a truth speaker or a liar.

   (c) Is the road to the harbour straight ahead?

8. On the island of liars and truth speakers, you meet three islanders A, B and C.

   - A says: "B or C is a liar."
   - B says: "C is a liar."
   - C says: "A is a liar."

   Determine by means of logical reasoning which of these three islanders speak the truth and which ones lie. Also check explicitly that the claims by the three islanders are in line with your conclusion whether they are truth speakers or liars.

9. Suppose you ask the following question to an islander hurrying by on a T-crossing to find the way to the harbour:

   Is the declarative sentence "*you are a truth speaker* $\leftrightarrow$ *the harbour is to the right*" true?

   Either explain that this question suffices to determine to way to the harbour, or explain why this is not the case.

Three Gods Puzzle

Have some fun with your fellow students trying to solve the *Three Gods Puzzle* by Raymond Smullyan. If you can't find a solution, don't be disappointed, as it has been called the hardest logic puzzle ever.

> Three gods are called True, False, and Random. True always speaks the truth, False always lies, and Random randomly speaks the truth or lies.
>
> Determine the identities of the three gods by asking three yes-no questions. (Each question must be put to one god.)
>
> The gods understand English, but answer in their own language, in which the words for 'yes' and 'no' are *strmpf* and *brzt*, in some order. You do not know which word means which.

Hint: Let your first question rule out one of the gods from being Random. Then ask that god a question to find out which of the other two gods is Random. With your final question, figure out who is True and who is False.

For a solution and background reading, see `https://en.wikipedia.org/wiki/The_Hardest_Logic_Puzzle_Ever`.