Exercises on Chapter 4: Equivalence Relations

1. On the set $A := \{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$, let the relation R be defined by the description

$$x \, R \, y \qquad \Longleftrightarrow \qquad |x| = |y|$$

where $|x|$ denotes the absolute value of x.

(a) Show that R is an equivalence relation.
(b) Write down the equivalence classes $[0]$ and $[2]$.
(c) Determine the number of different equivalence classes, and give a complete system of representatives for the equivalence relation R on A.

2. Let W be the set of all words of at least one and at most six lower case letters. Let the relation R on W be defined as follows: two words w_1 and w_2 in W satisfy $w_1 \, R \, w_2$ if and only if w_1 and w_2 have the same number of letters, and either both words start with a vowel, or both start with a consonant.
(a) Show that R is an equivalence relation in W.
(b) How many different equivalence classes are represented by the ten words listed below?

yes	indeed	some	ai	li
no	not	none	cs	imm

(c) Give a complete system of representatives for R.

3. Suppose a turn in a game consists of three independent throws of a fair coin. An outcome such as *Heads*, *Tails*, *Tails* (in that order) is denoted briefly as *HTT*. We define an equivalence relation R on these outcomes as follows:

$$x \, R \, y \qquad \Longleftrightarrow \qquad x \text{ and } y \text{ have the same total number of heads}$$

(a) For every possible outcome x of a turn in the game (*HHH*, *HHT*, *HTH*, and so on), write down the corresponding equivalence class $[x]$.
(b) How many different equivalence classes are there?
(c) Give a complete system of representatives for the equivalence relation R.

4. For each of the relations below, determine whether it is an equivalence relation, and if it is, try to find a complete system of representatives:
(a) The relation R in the set $\mathbb{N}$ defined by the description

$$x \, R \, y \qquad \Longleftrightarrow \qquad x = y$$

(b) The relation R in the set $\mathbb{N} \times \mathbb{N}$ defined by the description

$$\langle a, b \rangle \, R \, \langle c, d \rangle \qquad \Longleftrightarrow \qquad a + d = b + c$$

5. The set $\mathbb{R}$ can be partitioned into the sets $E_z := \{x\colon \lfloor x \rfloor = z\}$ with $z \in \mathbb{Z}$, where $\lfloor x \rfloor$ is the *integer part* of $x \in \mathbb{R}$, that is, $\lfloor x \rfloor$ is the greatest integer less than or equal to x. Let R be the equivalence relation in $\mathbb{R}$ that has the sets E_z ($z \in \mathbb{Z}$) as its equivalence classes.
   (a) Give a convenient definition of R by means of a description (i.e. similarly to how we defined the relations R in previous exercises).
   (b) Give a complete system of representatives for R.

6. Since $\{1, 2, \ldots, 6\}$ is the set of all possible outcomes of a throw with a regular die, the set of all possible outcomes of a throw with two dice is

$$\mathit{Throws} := \{1, 2, \ldots, 6\} \times \{1, 2, \ldots, 6\}.$$

   We define eleven subsets $P_2, P_3, \ldots, P_{12}$ of *Throws* as follows:

$$P_k := \{\langle m, n \rangle\colon m + n = k\} \qquad \text{for } k \in \{2, 3, \ldots, 12\}.$$

   For example, P_3 is the set of all outcomes for which the sum of the two numbers of dots thrown is 3.
   (a) Show that the sets $P_2, P_3, \ldots, P_{12}$ form a partition of the set *Throws*.
   (b) Let R be the equivalence relation on *Throws* that has $P_2, P_3, \ldots, P_{12}$ as its equivalence classes. Give a definition of R by means of a description.
   (c) Give a complete system of representatives for the equivalence relation R.

7. The set $\mathbb{R}^2$ can be partitioned into the sets

$$C_r := \left\{\langle x, y \rangle\colon x^2 + y^2 = r^2\right\}$$

   with $r \geq 0$ a real number. Note that the elements of C_r are the points on a circle of radius r around the origin (except in the case $r = 0$, for which C_r contains only the origin $\langle 0, 0 \rangle$). Let R be the equivalence relation on $\mathbb{R}^2$ that has the sets C_r ($r \geq 0$) as its equivalence classes.
   (a) Give a definition of R by means of a description.
   (b) Give a complete system of representatives for R.