ADDITIONAL PROBLEMS FOR THE FINAL EXAM

PARK, BAE JUN

Math222 Section345 & 349

(1) Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be vectors in dimension three. Does the following expression make sense? For this problem you do not need to provide any explanation - just write "Yes" if the expression makes sense and "No" if it doesn't

- $(2\mathbf{u} - \mathbf{v}) \times \mathbf{w}$ Y
- $|\mathbf{u} - \mathbf{v}| \times \mathbf{w}$ N
- $|\mathbf{u}| - \mathbf{v} \cdot \mathbf{w}$ Y
- $|\mathbf{u}| - \mathbf{v} \times \mathbf{w}$ N
- $|\mathbf{u}|\mathbf{v} - \mathbf{w}$ Y
- $|\mathbf{u} \times \mathbf{v}| - \mathbf{w} \cdot \mathbf{w}$ Y
- $\mathbf{u} \times (\mathbf{v} \cdot \mathbf{w})$ N
- $\mathbf{u} \times \mathbf{v} - \mathbf{w}$ Y
- $\mathbf{u} \times |\mathbf{v}| - \mathbf{w}$ N
- $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$ Y

(2) Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be vectors in dimension three. Circle T or F for True or False.

T F $\mathbf{b} \times \mathbf{a} = -(\mathbf{a} \times \mathbf{b})$

T F $\mathbf{b} \cdot \mathbf{a} = \mathbf{a} \cdot \mathbf{b}$

T F $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c})$

T F $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \cdot \mathbf{b}) + (\mathbf{a} \cdot \mathbf{c})$

T F $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$

T F $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$

T F If $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$ and $\mathbf{a} \neq \mathbf{0}$, then $\mathbf{b} = \mathbf{c}$

T F $(\mathbf{a} - \mathrm{proj}_{\mathbf{b}}\mathbf{a}) \cdot \mathbf{b} = 0$

(3) $|\mathbf{a}| = 5$, $|\mathbf{b}| = 6$, and $|\mathbf{a} - \mathbf{b}| = 7$.
What is $\mathbf{a} \cdot \mathbf{b}$?

$$|a-b|^2 = (a-b)\cdot(a-b)$$
$$= a\cdot a - a\cdot b - b\cdot a + b\cdot b$$
$$= |a|^2 - 2a\cdot b + |b|^2$$

$\Rightarrow 49 = 25+36-2a\cdot b$

$\Rightarrow 2a\cdot b = 12$

$\Rightarrow a\cdot b = 6$

(4) $|\mathbf{a} + \mathbf{b}| = 3$, $|\mathbf{a} - \mathbf{b}| = 2$.
What is $\mathbf{a} \cdot \mathbf{b}$?

$$|a+b|^2 = (a+b)\cdot(a+b)$$
$$= a\cdot a + a\cdot b + b\cdot a + b\cdot b$$
$$= |a|^2 + 2a\cdot b + |b|^2$$
$$|a-b|^2 = |a|^2 - 2a\cdot b + |b|^2$$

$\Rightarrow \underbrace{|a+b|^2 - |a-b|^2}_{= 9-4 = 5} = 4a\cdot b \quad \Rightarrow a\cdot b = \frac{5}{4}$

(5) $|\mathbf{a}+\mathbf{b}| = 4,\ |\mathbf{a}| = 3,\ |\mathbf{b}| = 2$
What is $\mathbf{a}\cdot\mathbf{b}$?

$$|a+b|^2 = (a+b)\cdot(a+b)$$
$$= a\cdot a + 2a\cdot b + b\cdot b$$
$$= |a|^2 + 2a\cdot b + |b|^2$$
$$\Rightarrow 16 = 9+4+2a\cdot b$$
$$\Rightarrow 2a\cdot b = 3 \quad \Rightarrow \quad a\cdot b = \frac{3}{2}$$

(6) $|\mathbf{a}-\mathbf{b}| = 3,\ |\mathbf{a}| = 4,\ |\mathbf{b}| = 3$
What is $|\mathbf{a}\times\mathbf{b}|$?

$$|a\cdot b| = |a||b||\cos\theta|$$
$$|a\times b| = |a||b|\sin\theta.$$
$$\Rightarrow |a\times b|^2 + |a\cdot b|^2 = |a|^2|b|^2(\sin^2\theta + \cos^2\theta)$$
$$= |a|^2|b|^2$$
$$\therefore |a\times b|^2 + |a\cdot b|^2 = |a|^2|b|^2$$

$$|a-b|^2 = (a-b)\cdot(a-b)$$
$$= |a|^2 - 2a\cdot b + |b|^2$$
$$\Rightarrow 9 = 16 + 9 - 2a\cdot b$$
$$\Rightarrow 2a\cdot b = 16 \quad \Rightarrow a\cdot b = 8$$

$$|a\times b|^2 + 64 = 144 \Rightarrow |a\times b|^2 = 80$$
$$\Rightarrow |a\times b| = \sqrt{80} = 4\sqrt{5}.$$

(7) $|\mathbf{a}+\mathbf{b}| = 3,\ |\mathbf{a}| = 2, |\mathbf{b}| = 2$
What is $|\mathbf{a}\times\mathbf{b}|$?

From (6),
$$|a\times b|^2 + |a\cdot b|^2 = |a|^2|b|^2.$$

$$|a+b|^2 = (a+b)\cdot(a+b)$$
$$= |a|^2 + 2a\cdot b + |b|^2$$
$$\rightarrow 9 = 4+4+2a\cdot b$$
$$\Rightarrow a\cdot b = \frac{1}{2}$$

$$\Rightarrow |a\times b|^2 + \frac{1}{4} = 16$$
$$\Rightarrow |a\times b|^2 = \frac{63}{4}$$
$$\Rightarrow |a\times b| = \frac{\sqrt{63}}{2}$$

(8) Let $\mathbf{u} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{v} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$

(a) What is $|\mathbf{v}|$?

$$|v| = \sqrt{1+4+4} = 3$$

(b) What is $\cos\theta$, if θ is the angle between $\mathbf{u}$ and $\mathbf{v}$?

$$\cos\theta = \frac{u \cdot v}{|u||v|} = \frac{2-2+2}{\sqrt{4+1+1}\,\sqrt{1+4+4}} = \frac{2}{3\sqrt{6}}$$

(c) What is the scalar component of $\mathbf{u}$ in the direction of $\mathbf{v}$?

$$|u| \cdot \cos\theta = \sqrt{6} \cdot \frac{2}{3\sqrt{6}} = \frac{2}{3}$$

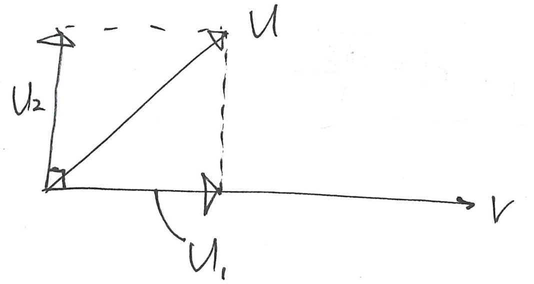
(d) What is $\text{proj}_{\mathbf{v}}\mathbf{u}$?

$$\text{proj}_v u = |u| \cdot \cos\theta \, \frac{v}{|v|} = \frac{2}{9}(i - 2j + 2k)$$

(e) Find two vectors $\mathbf{u_1}$ and $\mathbf{u_2}$ such that
(i)$\mathbf{u} = \mathbf{u_1} + \mathbf{u_2}$,
(ii)$\mathbf{u_1}$ is parallel to $\mathbf{v}$,
(iii)$\mathbf{u_2}$ is orthogonal to $\mathbf{v}$.

$$\begin{cases} \cdot\ u_1 = \text{proj}_v u = \frac{2}{9}(i - 2j + 2k) \\ \cdot\ u_2 = u - \text{proj}_v u = (2i + j + k) - \frac{2}{9}(i - 2j + 2k) \end{cases}$$

$$= \frac{16}{9}i + \frac{13}{9}j + \frac{5}{9}k$$



(9) $\mathbf{u} =< 2, 1, -1 >, \mathbf{v} =< 1, 0, -1 >$.
Find $|\mathbf{u}|$, $|\mathbf{v}|$, $\mathbf{u} \cdot \mathbf{v}$, $\mathbf{u} \times \mathbf{v}$, $\text{proj}_{\mathbf{u}}\mathbf{v}$ and the angle between $\mathbf{u}$ and $\mathbf{v}$

- $|u| = \sqrt{4+1+1} = \sqrt{6}$
- $|v| = \sqrt{1+0+1} = \sqrt{2}$
- $u \cdot v = 2+1 = 3$
- $$u \times v = \begin{vmatrix} i & j & k \\ 2 & 1 & -1 \\ 1 & 0 & -1 \end{vmatrix} = \underline{-i + j - k}$$
- $$\text{proj}_u v = \frac{u \cdot v}{|u|^2} u = \frac{3}{6} \langle 2, 1, -1 \rangle = \langle 1, \tfrac{1}{2}, -\tfrac{1}{2} \rangle$$
- $$\cos\theta = \frac{u \cdot v}{|u||v|} = \frac{3}{\sqrt{6}\sqrt{2}} = \frac{\sqrt{3}}{2} \qquad 0 \le \theta \le \pi$$

$$\Rightarrow \underline{\theta = \frac{\pi}{6}}$$

(10) $\mathbf{u} =< 0, 1, 1 >$, $\mathbf{v} =< 1, 1, 2 >$, $\mathbf{w} =< 1, 3, 1 >$.

(a) Find the angle between $\mathbf{u}$ and $\mathbf{v}$, and determine $\text{proj}_{\mathbf{v}}\mathbf{u}$.

$$\cos\theta = \frac{u \cdot v}{|u||v|} = \frac{1+2}{\sqrt{2}\sqrt{6}} = \frac{\sqrt{3}}{2} \quad 0 \le \theta \le \pi .$$

$$\Rightarrow \theta = \frac{\pi}{6}$$

$$\text{proj}_v u = \frac{u \cdot v}{|v|^2} v \left(= |u| \cos\theta \frac{v}{|v|} \right)$$

$$= \frac{3}{6} v = \frac{1}{2} \langle 1, 1, 2 \rangle = \langle \frac{1}{2}, \frac{1}{2}, 1 \rangle.$$

(b) Compute the triple scalar product $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$

$$(u \times v) \cdot w = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 3 & 1 \end{vmatrix} = 0 + 2 + 3 - 1 - 0 - 1 = 3.$$

(11) $\mathbf{u} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$, $\mathbf{v} = -\mathbf{j} + 2\mathbf{k}$, $P(1, 1, 0)$, and $Q(1, 2, 3)$.

(a) Compute $\mathbf{u} \cdot \mathbf{v}$.

$$u \cdot v = 0 - 3 - 2 = -5.$$

(b) Compute the unit vector $\mathbf{n}$ in the direction of $\mathbf{u} \times \mathbf{v}$.

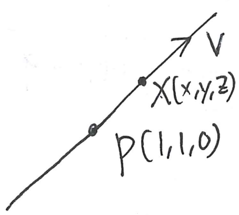
$$u \times v = \begin{vmatrix} i & j & k \\ 2 & 3 & -1 \\ 0 & -1 & 2 \end{vmatrix} = 5i - 4j - 2k.$$

$$\Rightarrow \frac{u \times v}{|u \times v|} = \frac{1}{\sqrt{25+16+4}}(5i - 4j - 2k)$$

$$= \frac{1}{3\sqrt{5}}(5i - 4j - 2k)$$

(c) Find parametric equations for the line parallel to $\mathbf{v}$ and passing through P.

X: any point on the line

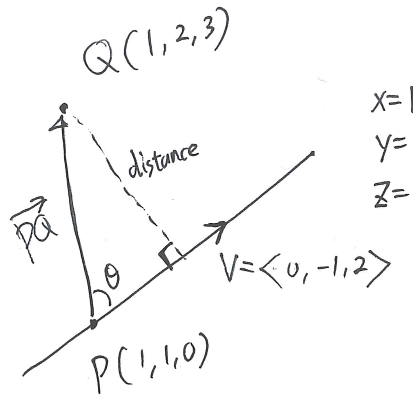


$$\Rightarrow \overrightarrow{PX} \parallel v$$

$$\Rightarrow \langle x-1, y-1, z \rangle \parallel \langle 0, -1, 2 \rangle$$

$$\Rightarrow \begin{cases} x - 1 = 0 \cdot t = 0 \\ y - 1 = -t \\ z = 2t \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = 1 - t \\ z = 2t \end{cases}$$

(d) Find the distance between the point Q and your line in (c).



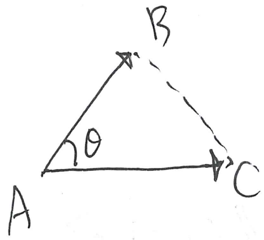
$x=1$
$y=1-t$
$z=2t$

$$\text{distance} = |\vec{PQ}|\sin\theta = |\vec{PQ}|\frac{|\vec{PQ}\times V|}{|\vec{PQ}||V|} = \frac{|\vec{PQ}\times V|}{|V|} = \frac{5}{\sqrt{1+4}} = \sqrt{5}$$

$$\vec{PQ}\times V = \begin{vmatrix} i & j & k \\ 0 & 1 & 3 \\ 0 & -1 & 2 \end{vmatrix} = 5i$$

$\Rightarrow |\vec{PQ}\times V| = 5.$

(12) Find the area of the triangle ABC with vertices $A(1,2,0)$, $B(2,3,1)$, $C(1,5,4)$.



$$\text{Area} = \frac{1}{2}|\vec{AC}|\cdot|\vec{AB}|\cdot\sin\theta = \frac{1}{2}|\vec{AC}\times\vec{AB}| = \frac{1}{2}\sqrt{1+16+9} = \frac{\sqrt{26}}{2}$$

$$\left(\vec{AC}\times\vec{AB} = \begin{vmatrix} i & j & k \\ 0 & 3 & 4 \\ 1 & 1 & 1 \end{vmatrix} = -i+4j-3k.\right)$$

(13) Given points $P(1,2,1)$, $Q(3,5,2)$, and $R(1,4,2)$ in 3-space.

(a) Find $\overrightarrow{PQ} \times \overrightarrow{PR}$ and the area of $\triangle PQR$.

$\overrightarrow{PQ} = \langle 2, 3, 1 \rangle$

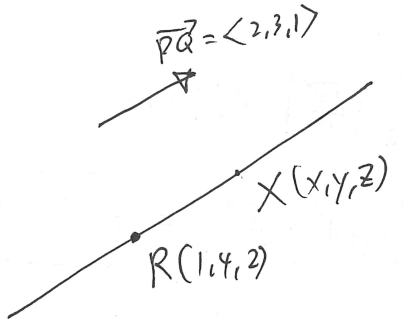
$\overrightarrow{PR} = \langle 0, 2, 1 \rangle$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ 2 & 3 & 1 \\ 0 & 2 & 1 \end{vmatrix} = \underline{i - 2j + 4k}.$$

$$\triangle PQR = \frac{1}{2}\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right| = \frac{1}{2}\sqrt{1+4+16} = \frac{1}{2}\sqrt{21}$$

(b) Find the equation of the line through point R and parallel to $\overrightarrow{PQ}$

$\overrightarrow{PQ} = \langle 2, 3, 1 \rangle$



Let X be any points on the line.

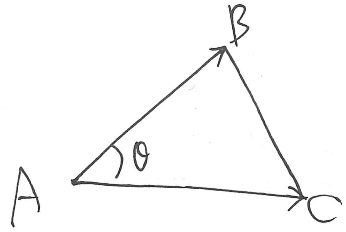
Then $\overrightarrow{RX} \parallel \overrightarrow{PQ}$

$\Rightarrow \langle x-1, y-4, z-2 \rangle \parallel \langle 2,3,1 \rangle$

$$\Rightarrow \begin{cases} x-1 = 2t \\ y-4 = 3t \\ z-2 = t \end{cases} \quad \Rightarrow \begin{cases} x = 1+2t \\ y = 4+3t \\ z = 2+t \end{cases}$$

(14) Let $A = (2, -1, 4)$, $B = (-3, 5, 2)$, and $C = (-1, -3, 5)$ be three points in space.

(a) The points A, B, and C are the vertices of a triangle. Find the angle between the side AB and the side AC.(You may leave your answer in terms of inverse trigonometric functions.)



$$\overrightarrow{AB} = \langle -5, 6, -2 \rangle$$

$$\overrightarrow{AC} = \langle -3, -2, 1 \rangle$$

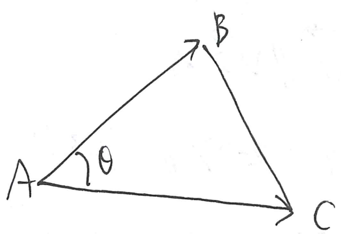
$$\overrightarrow{AB} \cdot \overrightarrow{AC} = |\overrightarrow{AB}||\overrightarrow{AC}| \cos\theta$$

$$\Rightarrow \cos\theta = \frac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{|\overrightarrow{AB}|\,|\overrightarrow{AC}|} = \frac{15-12-2}{\sqrt{25+36+4}\,\sqrt{9+4+1}}$$

$$= \frac{1}{\sqrt{65}\,\sqrt{14}}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{1}{\sqrt{65}\,\sqrt{14}}\right)$$

(b) Find the area of the triangle $\triangle ABC$

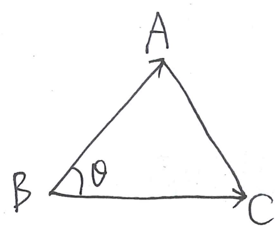


$$\text{Area} = \frac{1}{2}|\overrightarrow{AB}||\overrightarrow{AC}| \sin\theta$$

$$= \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|. = \frac{1}{2}\sqrt{4+121+28^2}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} i & j & k \\ -5 & 6 & -2 \\ -3 & -2 & 1 \end{vmatrix} = 2i + 11j + 28k$$

(15) (a) Find the area of the triangle with vertices $A(1,2,3), B(1,-1,4), C(2,1,1)$.



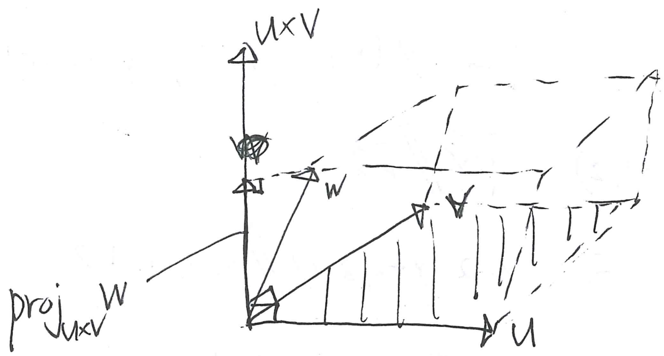
$\overrightarrow{BA} = \langle 0, 3, -1 \rangle$

$\overrightarrow{BC} = \langle 1, 2, -3 \rangle$

$$\overrightarrow{BA} \times \overrightarrow{BC} = \begin{vmatrix} i & j & k \\ 0 & 3 & -1 \\ 1 & 2 & -3 \end{vmatrix} = -7i + j - 3k.$$

$$\frac{1}{2}\left|\overrightarrow{BA} \times \overrightarrow{BC}\right| = \frac{1}{2}\sqrt{49+1+9} = \frac{1}{2}\sqrt{59}.$$

(b) Find the volume of the parallelpiped determined by the vectors
$\mathbf{u} = \mathbf{i}+\mathbf{j}+\mathbf{k}, \mathbf{v} = \mathbf{i}+2\mathbf{j}+\mathbf{k}, \mathbf{w} = \mathbf{i}+\mathbf{j}+3\mathbf{k}$.



$V =$ base area $\times$ height.

base area $= |u\times v|$

height $= \left|\text{proj}_{u\times v} w\right| = \left|\dfrac{(u\times v)\cdot w}{|u\times v|^2}\, u\times v\right| = \dfrac{|(u\times v)\cdot w|}{|u\times v|^2}|u\times v|$

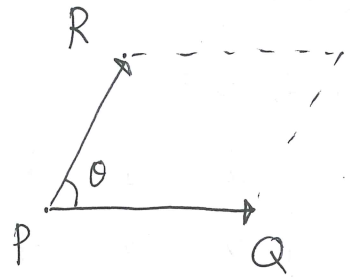
$= \dfrac{|(u\times v)\cdot w|}{|u\times v|}$

$$\Rightarrow V = |u\times v| \cdot \frac{|(u\times v)\cdot w|}{|u\times v|} = |(u\times v)\cdot w| = \left|\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{vmatrix}\right| = 6+1+1-2-3-1$$

$= 2$

(16) $P(2,-2,1)$, $Q(3,-1,2)$, $R(3,-1,1)$

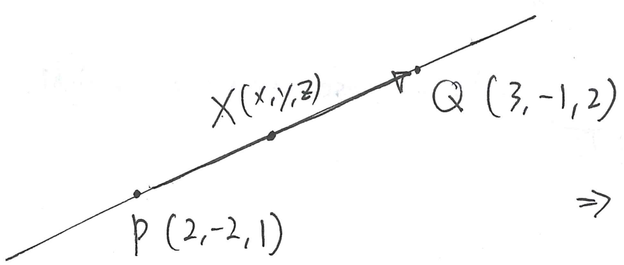
(a) Find the area of the parallelogram determined by the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$.



$$\text{Area} = |\overrightarrow{PQ}| \cdot |\overrightarrow{PR}| \cdot \sin\theta = |\overrightarrow{PQ} \times \overrightarrow{PR}| = \underline{\sqrt{2}}$$

$$\begin{cases} \overrightarrow{PQ} = \langle 1, 1, 1 \rangle \\ \overrightarrow{PR} = \langle 1, 1, 0 \rangle \end{cases} \qquad \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -i + j$$

(b) Find an equation for the line through P and Q.



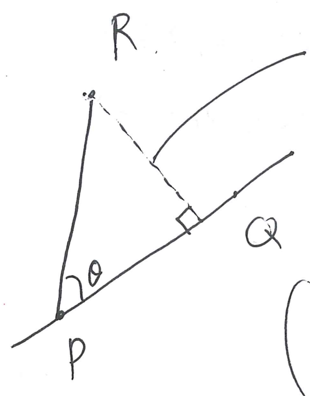
Let $X(x,y,z)$ be any points on the line.

$\overrightarrow{PX} \parallel \overrightarrow{PQ}$

$\Rightarrow \langle x-2, y+2, z-1 \rangle \parallel \langle 1, 1, 1 \rangle$

$$\Rightarrow \begin{cases} x-2 = t \\ y+2 = t \\ z-1 = t \end{cases} \Rightarrow \underline{\begin{pmatrix} x = 2+t \\ y = -2+t \\ z = 1+t \end{pmatrix}}$$

(c) Find the distance between R and your line in (b).



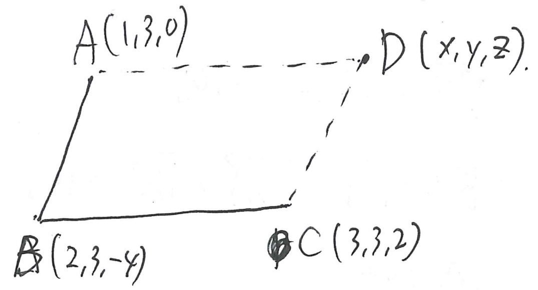
$$\text{distance} = |\overrightarrow{PR}| \cdot \sin\theta$$

$$= |\overrightarrow{PR}| \frac{|\overrightarrow{PR} \times \overrightarrow{PQ}|}{|\overrightarrow{PR}|\,|\overrightarrow{PQ}|} = \frac{|\overrightarrow{PR} \times \overrightarrow{PQ}|}{|\overrightarrow{PQ}|}.$$

$$\left(\overrightarrow{PR} \times \overrightarrow{PQ} = i - j \text{ from (a)}\right) \Rightarrow = \frac{\sqrt{2}}{\sqrt{3}} = \frac{\sqrt{6}}{3}$$

(17) You are given the points $A(1, 3, 0), B(2, 3, -4), C(3, 3, 2)$

(a) Find a point D so that A, B, C, D are the vertices of a parallelogram.



Then $\overrightarrow{BA} = \overrightarrow{CD}$

$\Rightarrow \langle -1, 0, 4\rangle = \langle x-3, y-3, z-2\rangle$

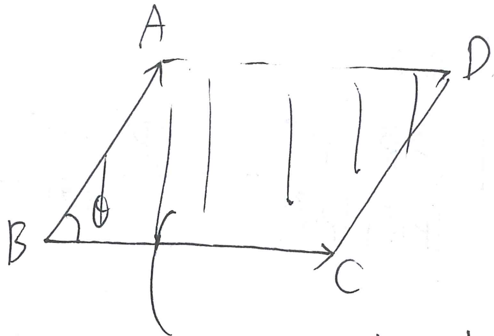
$\Rightarrow x = 2,\ y = 3,\ z = 6$ — $\boxed{\therefore D(2,3,6)}$

or $\overrightarrow{AD} = \overrightarrow{BC}$

$\Rightarrow \langle x-1, y-3, z\rangle = \langle 1, 0, 6\rangle$

$\Rightarrow x = 2,\ y = 3,\ z = 6$

(b) Find the area of this parallelogram.



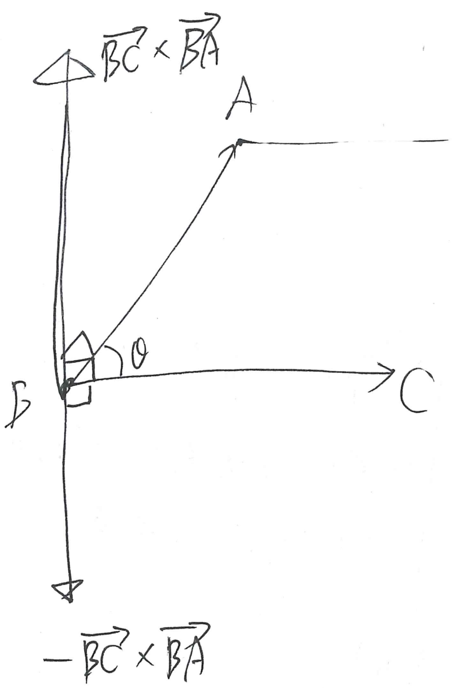
$\overrightarrow{BA} = \langle -1, 0, 4 \rangle$

$\overrightarrow{BC} = \langle 1, 0, 6 \rangle$

$$\text{Area} = |\overrightarrow{BA}|\,|\overrightarrow{BC}| \sin\theta$$

$$= |\overrightarrow{BA} \times \overrightarrow{BC}| = \begin{vmatrix} i & j & k \\ -1 & 0 & 4 \\ 1 & 0 & 6 \end{vmatrix} = |10j| = 10.$$

(c) Find two unit vectors perpendicular to the plane ABC.



$\overrightarrow{BC} \times \overrightarrow{BA} = -10j$

$-\overrightarrow{BC} \times \overrightarrow{BA} = \overrightarrow{BA} \times \overrightarrow{BC} = 10j$

from (b).

unit vector

$$\frac{\overrightarrow{BC} \times \overrightarrow{BA}}{|\overrightarrow{BC} \times \overrightarrow{BA}|} = -j = \langle 0, -1, 0 \rangle$$

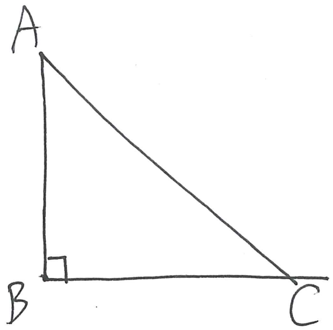
$$-\frac{\overrightarrow{BC} \times \overrightarrow{BA}}{|\overrightarrow{BC} \times \overrightarrow{BA}|} = j = \langle 0, 1, 0 \rangle.$$

(18) Consider the points:

$$A(1,0,-2),\ B(-1,1,-1),\ C(2,-1,q)$$

, where q is a constant.

(a) For which value of q is the triangle ABC a right triangle (with B being the right angle)?



$\Leftrightarrow \overrightarrow{BA} \perp \overrightarrow{BC} \Leftrightarrow \overrightarrow{BA} \cdot \overrightarrow{BC} = 0.$

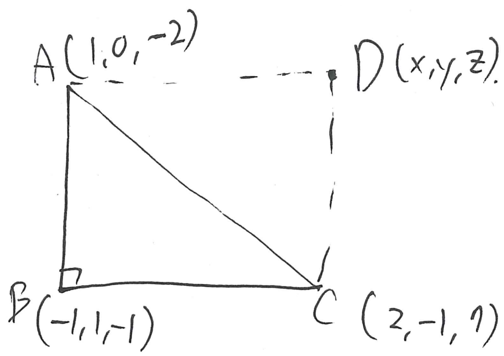
$\overrightarrow{BA} = \langle 2,-1,-1\rangle$

$\overrightarrow{BC} = \langle 3,-2,q+1\rangle$

$\Rightarrow \overrightarrow{BA}\cdot\overrightarrow{BC} = 6+2-(q+1)=0$

$\Rightarrow 7 = q.$

(b) Find the coordinates of the point D for which $ABCD$ is a parallelogram if q is the value you find in (a).



$\overrightarrow{BC} = \overrightarrow{AD}$

$\Rightarrow \langle 3,-2,8\rangle = \langle x-1, y, z+2\rangle$

$\Rightarrow \begin{cases} x=4 \\ y=-2 \\ z=6 \end{cases}$ $\therefore D(4,-2,6)$

Or. $\overrightarrow{AB} = \overrightarrow{DC}$

$\langle -2,1,1\rangle = \langle 2-x,-1-y,7-z\rangle$

$\Rightarrow \begin{cases} x=4 \\ y=-2 \\ z=6 \end{cases}$ $\therefore D(4,-2,6)$